Borcherds Products on Unitary Groups

Eric Hofmann

TU Darmstadt

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   - Automorphic forms on $\mathrm{U}(L)$
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Setup

Let $\mathbb{F}$ be an imaginary quadratic number field, $F = \mathbb{Q}(\sqrt{d})$, $d < 0$, with discriminant $D_{\mathbb{F}}$, different δ and ring of integers $\mathcal{O}_{\mathbb{F}}$.

Let L be a hermitian lattice of signature $(1, q)$ over $\mathcal{O}_{\mathbb{F}}$, with $\langle \cdot, \cdot \rangle$ a nondegenerate hermitian form

$$\langle av, bw \rangle = a\bar{b} \langle v, w \rangle = \overline{\langle bw, av \rangle},$$

extended to the $\mathbb{F}$-vectorspace $V = L \otimes \mathbb{F}$ and to the complex space $V_{\mathbb{R}} = V \otimes_{\mathbb{F}} \mathbb{R}$.

Assume L to be *integral* and *even*, that is $\mathrm{tr}_{\mathbb{F}/\mathbb{Q}} \langle \lambda, \mu \rangle \in \mathbb{Z}$ and $\mathrm{tr}_{\mathbb{F}/\mathbb{Q}} \langle \lambda, \lambda \rangle \in 2\mathbb{Z}$ for all $\lambda, \mu \in L$.

Denote by L' the $\mathbb{Z}$-dual of L,

$$L' = \left\{ v \in V \; ; \; \mathrm{tr}_{\mathbb{F}/\mathbb{Q}} \langle \lambda, v \rangle \in \mathbb{Z}, \quad \text{for all} \quad \lambda \in L \right\}.$$

The symmetric domain

Denote by $\mathrm{U}(V)$ the unitary group of $V, \langle\cdot,\cdot\rangle$ and by $\mathrm{U}(V)(\mathbb{R})$ its real points. The isometries of L, form an arithmetic subgroup, $\mathrm{U}(L)$.

Symmetric domain

The symmetric domain attached to $\mathrm{U}(V)$ can be obtained as the Grassmannian

$$\mathrm{Gr}_U = \mathrm{U}(V)(\mathbb{R})/\mathscr{C}, \quad \text{with } \mathscr{C} \subset \mathrm{U}(V)(\mathbb{R}),$$

$\mathscr{C}$ a maximal compact subgroup isomorphic to $\mathrm{U}(1) \times \mathrm{U}(q)$ under $V_{\mathbb{R}} \to \mathbb{C}^{1,q}$. Gr_U is isomorphic to the projective cone

$$\mathcal{K}_U = \{[z] \in \mathbb{P}(V_{\mathbb{R}})\,;\ \langle z, z\rangle > 0\}\,.$$

We will give an affine model for this later on.

Modified Witt basis

We assume that L splits off a hyperbolic plane over $\mathcal{O}_{\mathbb{F}}$, so that

$$L = H \oplus D, \quad \text{with} \quad \langle H, D\rangle = 0,$$

H of signature $(1,1)$ and D negative definite.

Choose lattice vectors ℓ, ℓ' with $\ell \in L$ primitive isotropic and $\ell' \in L'$ also isotropic, such that $\langle \ell, \ell' \rangle \neq 0$. Then,

$$L = \left(\mathfrak{a}\ell + \mathfrak{b}\ell'\right) \oplus D,$$

with $\mathfrak{a},\mathfrak{b}$ (fractional) $\mathcal{O}_{\mathbb{F}}$-ideals.

We call an $\mathbb{F}$-basis of V of the form ℓ, ℓ', $v_1, \ldots, v_{q-1} \in L'$ with ℓ, ℓ' as above a *modified Witt basis*.

A Siegel domain model

Now, for the elements of $\mathcal{K}_U$, consider representatives $z \in V$ of the form

$$z = \ell' - \tau \langle \ell', \ell \rangle \delta\ell + \sigma, \quad \text{with} \quad \sigma \in D \otimes_{\mathbb{F}} \mathbb{R}.$$

The condition $\langle z, z \rangle > 0$ yields the following description.

Siegel domain

The following subset of $\mathbb{C}^q$ is an affine model for Gr_U:

$$\mathcal{H}_U = \left\{ (\tau, \sigma) \subset \mathbb{C} \times (D \otimes_{\mathbb{F}} \mathbb{R}) \,;\, |\langle \ell', \ell \rangle|^2 \Im\tau > -\langle \sigma, \sigma \rangle \right\}.$$

Note that $\mathcal{H}_U$ is not a tube domain, unless $q = 1$, in which case $\mathcal{H}_U \simeq \mathbb{H}$, the upper half plane in $\mathbb{C}$.

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Automorphic forms

Let $\Gamma \subseteq \mathrm{U}(L)$ be an arithmetic subgroup. Then, Γ acts on $\mathcal{H}_U$ with an automorphy factor $j : \Gamma \times \mathcal{H}_U \to \mathbb{C}^\times$.

Definition

An automorphic form of weight k on Γ is a meromorphic function $f : \mathcal{H}_U \to \bar{\mathbb{C}}$ satisfying

$$f(\gamma(\tau, \sigma)) = j(\gamma; \tau, \sigma)^{-k} f(\tau, \sigma) \quad \text{for all } \gamma \in \Gamma.$$

If in addition f is holomorphic and regular at the cusps, f is called a *modular form*.

Fourier-Jacobi expansion

A holomorphic automorphic form can be partially expanded as a Fourier series:

$$f(\tau, \sigma) = \sum_{n \in \mathbb{Z}} a_n(\sigma) e\left(\frac{n\tau}{N} + b\right), \quad \text{with } N \in \mathbb{Z},\ b \in \mathbb{Q}.$$

The coefficients $a_n(\sigma)$ transform as Jacobi-forms.

- a_0 is constant, we have $\lim_{\tau \to i\infty} f(\tau, \sigma) = a_0$.
- For $q > 1$ the *Koecher principle* holds: If f is holomorphic, $a_n = 0$ for $n < 0$, so f is regular at the cusps.
- If $q = 1$, f is an elliptic modular form and all the a_n are constant.

The Heisenberg Group

The arithmetic Heisenberg group in in $\mathrm{U}(L)$ consists of pairs $[h, t]$, with $h \in \frac{1}{N}\mathbb{Z}$ for some integer N, t from a sublattice of D'. The group law is given by

$$[h, t] \circ [h', t'] = [h + h' + \frac{\Im \langle t', t \rangle}{|\delta|}, t + t'].$$

On $\mathcal{H}_U$, the Heisenberg group acts according to

$$[h, 0](\tau, \sigma) = (\tau + h, \sigma)$$
$$[0, t](\tau, \sigma) = \left(\tau + \frac{\langle \sigma, t \rangle}{\delta \langle \ell', \ell \rangle} + \frac{1}{2}\frac{\langle t, t \rangle}{\delta}, \sigma + \langle \ell', \ell \rangle t\right).$$

V as a quadratic space

The hermitian space V also carries a structure as a quadratic space over $\mathbb{Q}$:

The following is a non-degenerate symmetric form, bilinear of signature $(2,2q)$ on V as a $\mathbb{Q}$-vectorspace

$$(\cdot,\cdot) = \mathrm{tr}_{\mathbb{F}/\mathbb{Q}} \langle \cdot,\cdot \rangle = 2\Re \langle \cdot,\cdot \rangle .$$

Attached to this is the quadratic form $q(\cdot) = \langle \cdot,\cdot \rangle$.

By extension of scalars, $(\cdot,\cdot)$ is a real bilinear form on $V_{\mathbb{R}}$ as a vector space over $\mathbb{R}$.

Weil-representation

The double cover of $\mathrm{SL}_2(\mathbb{R})$, the metaplectic group $\mathrm{Mp}_2(\mathbb{R})$ consists of elements

$$(M, \phi(\tau)), \quad M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \ \phi(\tau)^2 = c\tau + d.$$

The subgroup $\mathrm{Mp}_2(\mathbb{Z})$ has two generators,

$$T = \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right), \qquad S = \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right).$$

It has a representation ρ_L on the group algebra $\mathbb{C}[L/L']$.

$$\rho_L(T)\mathfrak{e}_\gamma = e\big(q(\gamma)\big)\mathfrak{e}_\gamma, \ \rho_L(S)\mathfrak{e}_\gamma = \frac{\sqrt{i}^{q-1}}{\sqrt{L'/L}} \sum_{\delta \in L'/L} e\big(-(\gamma, \delta)\big)\mathfrak{e}_\delta.$$

Vector-values modular forms

There is also a 'dual' representation of $\mathrm{Mp}_2(\mathbb{Z})$ on the vector-valued functions $\mathbb{H} \longrightarrow \mathbb{C}[L'/L]$,

$$\left(f \mid_\kappa^* (M,\phi)\right)(\tau) = \phi(\tau)^{-2\kappa} \rho_L^{-1}(M,\phi)\, f(M\tau).$$

Definition

The space $\mathcal{M}^!_\kappa(\rho_L)$ of nearly holomorphic modular forms consists of functions $f : \mathbb{H} \longrightarrow \mathbb{C}[L'/L]$ satisfying

- $f \mid_\kappa^* (M,\phi) = f$, for all $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$.
- f is holomorphic on $\mathbb{H}$.
- f is meromorphic at the cusp $i\infty$.

Such a modular form has a Fourier expansion

$$f(\tau) = \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} - q(\gamma) \\ n \gg -\infty}} c(\gamma, n) e(n\tau) \mathfrak{e}_\gamma.$$

Borcherds products for $\mathrm{U}(1, q)$

Theorem

Given $f \in \mathcal{M}^{!}_{q-1}(\rho_L)$ with Fourier coefficients satisfying $c(\gamma, n) \in \mathbb{Z}$ for $n < 1$, there is a meromorphic function $\Xi_f : \mathcal{H}_U \to \bar{\mathbb{C}}$ with the following properties:

- *Ξ_f is an automorphic form of weight $c(0,0)/2$.*
- *Its poles and zeros lie on generalized Heegner divisors.*
- *On each Weyl chamber W of $\mathcal{H}_U$, Ξ_f has an absolutely converging infinite product expansion around the cusps,*

$$\Xi_f^W(z) = Ce^{\rho} \prod_{\substack{\lambda \in K' \\ \lambda \in L^+(W)}} \left(1 - e\left(\frac{\langle z, \lambda\rangle}{\langle \ell', \ell\rangle}\right)\right)^{c(\lambda, \langle\lambda,\lambda\rangle)}.$$

- The product converges absolutely when $\langle z, z\rangle \gg 0$.
- *Heegner dvisors* in $\mathcal{K}_U$ respectively $\mathcal{H}_U$ are given by unions of *primitive* Heegner divisors, sets of the form

$$H_\lambda = \{[z] \in \mathcal{K}_U\,;\, \langle z, \lambda\rangle = 0\}, \quad \text{for a } \lambda \in L',\ \langle \lambda, \lambda\rangle < 0.$$

- For $c(\gamma, n)$ with $n < 0$, the Heegner divisors H_γ dissect $\mathcal{K}_U$ respectively $\mathcal{H}_U$ into components called *Weyl chambers*.
- Considering L as a lattice over $\mathbb{Z}$, K is a $\mathbb{Z}$-sublattice of rank $2q$ of the form $K = (\mathbb{Z}a + \mathbb{Z}b) \oplus D$, with $a, b \in H$.
- The positivity condition $\lambda \in L^+(W)$ means that

$$\Im\left[\langle z, \lambda\rangle \langle \ell', \ell\rangle^{-1}\right] > 0 \quad \text{for all } z \in W \subset \mathcal{K}_U.$$

- C is a constant, while $\rho = 2\pi i \left\langle z, \rho_W^f\right\rangle \langle \ell', \ell\rangle^{-1}$, where ρ_W^f is the *Weyl vector* attached to f and W.

Sketch of the proof

Consider $V_{\mathbb{R}}$ with the form $(\cdot,\cdot)$ as a real quadratic space. The orthogonal group $\mathrm{O}(V)(\mathbb{R})$ is isomorphic to $\mathrm{O}(2,2q)$.
Since $(\cdot,\cdot) = 2\Re\langle\cdot,\cdot\rangle$, we have an embedding of groups

$$\mathrm{U}(V)(\mathbb{R}) \hookrightarrow \mathrm{O}(V)(\mathbb{R}),$$

This induces an embedding of the attached symmetric domains

$$\alpha : \mathcal{H}_U \hookrightarrow \mathcal{H}_O,$$

permitting us to pull back the automorphic products constructed by Borcherds to the unitary side,

$$\Xi_f(z(\tau,\sigma)) = \alpha^*(\Psi(f;Z))(\tau,\sigma).$$

Some remarks on the embedding:

- Since $End_{\mathbb{C}}(V_{\mathbb{R}}) \subsetneq End_{\mathbb{R}}(V_{\mathbb{R}})$, a complex scalar defines an endomorphism of the real space $V_{\mathbb{R}}$, $(\cdot,\cdot)$.
- $\mathrm{U}(L)$ is embedded into the orthogonal group $\mathrm{O}(L)$ of L (as a $\mathbb{Z}$-lattice). The discriminant kernel is mapped to the discriminant kernel.
- Subgroups of finite index remain subgroups of finite index. (Similarly for commensurable subgroups.)
- The Heisenberg group in $\mathrm{U}(V)$ is mapped into the Heisenberg group of $\mathrm{O}(V)$.

A symmetric domain for $\mathrm{O}(V)$

The symmetric domain for $\mathrm{O}(V)$ can be obtained as a Grassmannian

$$\mathrm{Gr}_O = \mathrm{O}(V)(\mathbb{R})/\mathscr{K},$$

where $\mathscr{K} \simeq \mathrm{O}(2) \times \mathrm{O}(2q)$ is a maximal compact subgroup.
The points of Gr_O are 2-dimensional positive definite subspaces of $V_{\mathbb{R}}$.
Choose a continuously varying orientation on these and for each $v \in \mathrm{Gr}_O$ an orthogonal basis

$$X_L, Y_L, \quad \text{with} \quad X_L^2 = Y_L^2,\ X_L \perp Y_L.$$

With this, we introduce a complex structure on Gr_O.

A positive cone in $V_{\mathbb{C}}$.

The real quadratic space $V_{\mathbb{R}}$ is complexified to $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and $(\cdot,\cdot)$ is extended to a complex bilinear form.
Now, consider the maps

$$\begin{aligned} X_L, Y_L &\longmapsto \quad Z_L = X_L + iY_L \in V_{\mathbb{C}}, \\ \text{and} \quad \mathrm{Gr}_O \ni v &\longmapsto \qquad\qquad [Z_L] \in \mathbb{P}(V_{\mathbb{C}}). \end{aligned}$$

This sends Gr_O to either of the two components of the set

$$\mathcal{K}_O = \{[Z_L] \in \mathbb{P}(V_{\mathbb{C}})\,;\, (Z_L, Z_L) = 0, \big(Z_L, \bar{Z}_L\big) > 0\}.$$

Choose one component, $\mathcal{K}_O^+$. It is isomorphic to Gr_O.

Construction of the tube domain

For the construction of the tube domain, we require a primitive vector $e_1 \in L$ and an $e_2 \in L'$ satisfying

$$e_1^2 = 0, \qquad (e_1, e_2) = 1.$$

Normalize X_L and Y_L as follows

$$(X_L, e_1) = 1, \quad (Y_L, e_1) = 0.$$

Set $K = L \cap e_1^\perp \cap e_2^\perp$ and denote by X and Y the projections of X_L and Y_L to the subspace $K \otimes \mathbb{R}$.

The *tube domain model* $\mathcal{H}_O$ is defined as the image of $\mathcal{K}_O^+$ under $[Z_L] \mapsto Z \in K \otimes \mathbb{C}$:

$$\mathcal{H}_O \subsetneq \left\{ Z = X + iY \in K \otimes \mathbb{C}\, ;\ Y^2 > 0 \right\}.$$

Matching basis

We already have a $\mathbb{C}$-basis ℓ,ℓ' consisting of lattice vectors for the hyperbolic space $H \otimes_{\mathbb{F}} \mathbb{R}$, $\langle \cdot, \cdot \rangle$.
To fix the embedding $\mathcal{H}_U \hookrightarrow \mathcal{H}_O$ induced by α, we need to choose an $\mathbb{R}$-basis of $H \otimes_{\mathbb{Q}} \mathbb{R}$, $(\cdot, \cdot)$, as well.
Since ℓ corresponds to the cusp it should be part of this new basis.

Set $e_1 = \ell$, and choose lattice vectors $e_2, e_3, e_4 \in H'$ with the following properties

$$(e_1, e_2) = (e_3, e_4) = 1 \text{ and } (e_i, e_j) = 0 \text{ otherwise.}$$

Clearly, $e_1, \dots, e_4$ span $H \otimes \mathbb{R}$ as a real quadratic space.

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Clearly, $e_1, \ldots, e_4$ span $H \otimes \mathbb{R}$ as a real quadratic space.

The image of $[z] \in \mathcal{K}_U$ in Gr_O is given by

$$[z] \longmapsto \mathbb{R}X_L + \mathbb{R}Y_L, \quad X_L = \frac{z}{2\langle \ell', \ell \rangle}, Y_L = \frac{-iz}{2\langle \ell', \ell \rangle}.$$

It is quickly checked that the following statements hold:

1. The two complex structures are compatible. For example, we have $iz \mapsto -Y_L + iX_L = iZ_L$.
2. We have $(X_L, \ell) = 1$, $(Y_L, \ell) = 0$.
3. Further $X_L \perp Y_L$, and $X_L^2 = Y_L^2 > 0$.

Example: Choice of basis

Assume $D_{\mathbb{F}}$ to be even. Consider a lattice L of the form $L = \mathcal{O}_{\mathbb{F}} \oplus \delta^{-1}\mathcal{O}_{\mathbb{F}}$. Clearly $L = L'$.
Fix an isometry $L \otimes \mathbb{C}$ to $\mathbb{C}^{1,1}$. Set $\ell = (1, 0)$, $\ell' = (0, -\delta^{-1})$.
Then, $\langle \ell, \ell' \rangle = \delta^{-1}$ and $L = \mathcal{O}_{\mathbb{F}}\ell \oplus \mathcal{O}_{\mathbb{F}}\ell'$.
For L as the $\mathbb{Z}$-lattice $(\mathbb{Z} + \delta/2\mathbb{Z})\ell \oplus (\mathbb{Z} + \delta/2\mathbb{Z})\ell'$, a basis of the required form is given by

$$\ell, e_2 = \frac{\delta}{2}\ell', e_3 = -\frac{\delta}{2}\ell, e_4 = -\ell'.$$

With this, the representative $z = \ell' - \tau\ell$ maps to

$$v = \mathbb{R}X_L + \mathbb{R}Y_L, \quad \text{where } X_L = \frac{\delta}{2}z,\ Y_L = -2|\delta|z.$$

Finally, $\tau \in \mathcal{H}_U$ maps to $Z = \delta/2 e_4 + \tau e_3$ in $\mathcal{H}_O$.

Example: Liftings from $\mathcal{M}_0^!(\rho_L)$

Assume $D_{\mathbb{F}} \equiv 1(2)$, $L = \mathcal{O}_{\mathbb{F}} \oplus \delta^{-1}\mathcal{O}_{\mathbb{F}}$. Denote $\omega = \frac{1}{2}(1+\delta)$. Consider the function $J(\tau) \in \mathcal{M}_0^!(\rho_L)$ given by

$$J(\tau) = j(\tau) - 744 = q^{-1} + 196884q + \dots,$$

where $j(\tau)$ is the usual j-function.
We find that there are two Weyl chambers, $W_>$ and $W_<$, defined by $\Im\tau > |\delta|$ and $\Im\tau < |\delta|$.
On the $W_>$, the Weyl vector is found to be $\rho_{W_>}^J = -e_4$ and the constant C equals 1. Thus, the infinite product expansion of the lift Ξ_J takes the form

$$\Xi_J^{W_>}(\tau) = e(-\tau) \prod_{\substack{m,n\in\mathbb{Z} \\ m>0}} \left(1 - e\left(m\tau - n\bar{\omega}\right)\right)^{c(mn)}.$$