Borcherds Products for Unitary Groups

27th Automorphic Forms Workshop, UCD Dublin

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1. Hermitian spaces and unitary groups $\mathrm{U}(1, q)$.
2. An embedding into $\mathrm{SO}(2, 2q)$.
3. Main Theorem: The Borcherds lift.
4. Modularity of divisors.

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Setup 1: A hermitian space

- $\mathbb{F} = \mathbb{Q}(\sqrt{d})$, $d \in \mathbb{Z}_{<0}$, d square-free.
- $D_{\mathbb{F}}$ the discriminant, $\delta = \sqrt{D_{\mathbb{F}}}$ the different of $\mathbb{F}$.
- $\mathcal{O}_{\mathbb{F}}$ ring of integers, $\mathcal{D}_{\mathbb{F}}^{-1}$ inverse different ideal of $\mathbb{F}$

$$\mathcal{O}_{\mathbb{F}} = \mathbb{Z} \oplus \zeta\mathbb{Z}, \quad \mathcal{D}_{\mathbb{F}}^{-1} = \delta^{-1}\mathcal{O}_{\mathbb{F}}$$

- $V_{\mathbb{F}}, \langle \cdot, \cdot \rangle$ a hermitian space over $\mathbb{F}$, of dimension $q+1$
- $\langle \cdot, \cdot \rangle$ a non-degenerate, indefinite hermitian form, of signature $(1, q)$.
- $L \subset V$ an even hermitian lattice, $L \otimes_{\mathcal{O}_{\mathbb{F}}} \mathbb{F} = V_{\mathbb{F}}$, with dual L' Fix two vectors u, u' with $u \in L$ primitive, $u' \in L'$ and

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- Let $\mathrm{U}(V)$ be the unitary group of $V_{\mathbb{F}}$
- $\Gamma \subset \mathrm{U}(L) = Stab(L) \subset \mathrm{U}(V)$
  (In particular, $\Gamma = \Gamma_L$ the discriminant kernel)

Symmetric domain $\mathrm{U}(V)(\mathbb{R})/\mathcal{C} \simeq \mathcal{K}_{\mathrm{U}}$ (with $\mathcal{C}$ max. compact)

$$\mathcal{K}_{\mathrm{U}} = \{[v]\,;\, \langle v, v\rangle > 0\} \subset \mathbb{P}^1\mathbb{C}.$$

The modular variety $X_\Gamma = \Gamma\backslash\mathcal{K}_{\mathrm{U}}$ is called a *ball-quotient*.

For each $[z] \in \mathcal{K}_{\mathrm{U}}$ fix a representative of the form
$z = u' - \tau\delta\langle u', u\rangle u + \sigma$.
Then, with $\langle z, z\rangle > 0$, we get the following affine model

$$\mathcal{H}_{\mathrm{U}} = \{(\tau,\sigma) \in \mathbb{C} \times \mathbb{C}^{q-1}\,;\, 2\Im\tau|\delta||\langle u, u'\rangle|^2 > -\langle\sigma,\sigma\rangle\} \simeq \mathcal{K}_{\mathrm{U}}.$$

(the *Siegel domain* model)

Setup 2: Unitary group

- Let $\mathrm{U}(V)$ be the unitary group of $V_{\mathbb{F}}$
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Example $q = 1$

Let $q = 1$, i.e. $V_{\mathbb{F}} \simeq \mathbb{F}^2$ and $\langle \cdot, \cdot \rangle$ has signature $(1, 1)$, then

- $\mathcal{H}_{\mathrm{U}}$ is just the usual complex upper half-plane,

$$\mathbb{H} = \{\tau \in \mathbb{C}\,;\, \Im\tau > 0\}.$$

- $\mathrm{SU}(V)(\mathbb{R}) \simeq \mathrm{SL}_2(\mathbb{R})$.

In $V_{\mathbb{F}}$, consider the lattice $L = \mathcal{O}_{\mathbb{F}} \oplus \mathcal{D}_{\mathbb{F}}^{-1}$:

- L is unimodular (i.e. $L' = L$).
- $\Gamma_L = \mathrm{SU}(L)$ is isomorphic to $\mathrm{SL}_2(\mathbb{Z})$.

Unitary modular forms

Definition

Let Γ be of finite index in Γ_L, and $k \in \mathbb{Z}$. Then, $f : \mathcal{H}_{\mathrm{U}} \to \mathbb{C}$ is a *unitary modular form* (for Γ with weight k), if

1. f is holomorphic on $\mathcal{H}_{\mathrm{U}}$.
2. For all $\gamma \in \Gamma$, $f\left(\gamma(\tau,\sigma)\right) = j(\gamma;\tau,\sigma)^k f(\tau,\sigma)$.
3. f is entire at the cusps of $\mathcal{H}_{\mathrm{U}}$.
   (For $q > 1$ this follows from the Köcher-principle.)

Modular forms on $\mathcal{H}_{\mathrm{U}}$ can be developed as Fourier-Jacobi series

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The embedding: Overview

Let $(\cdot,\cdot) := \mathrm{Tr}_{\mathbb{F}/\mathbb{Q}}\langle\cdot,\cdot\rangle$. Then, $V_{\mathbb{F}}$ becomes a quadratic space $(V_{\mathbb{Q}},(\cdot,\cdot))$ over $\mathbb{Q}$ of signature $(2,2q)$. Hence, there is an embedding

$$\mathrm{U}(V)(\mathbb{R}) \hookrightarrow \mathrm{SO}(V)(\mathbb{R})^{+},$$

which, in turn, induces an embedding of the symmetric domains

$$\alpha\,:\,\mathcal{H}_{\mathrm{U}} \hookrightarrow \mathcal{H}_{\mathrm{O}}.$$

The Borcherds lift can then be transferred to $\mathrm{U}(1,q)$ by pull-back under α.

$$\Xi(\tau,\sigma;f) := \alpha^{*}(\Psi_L(f))(\tau,\sigma) \longleftarrow \Psi_L(Z;f).$$

Difficulties to surmount:

1. Different complex structures on the symmetric domains of $\mathrm{SO}(V)(\mathbb{R})$ and $\mathrm{U}(V)(\mathbb{R})$.
2. Choice of basis for L as module over $\mathbb{Z}$ and over $\mathcal{O}_{\mathbb{F}}$.
3. Compatible choice of cusps, geometry of boundary components.

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The embedding on the Grassmannian

The symmetric domain of $\mathrm{SO}(V)(\mathbb{R}) \simeq \mathrm{SO}(2,2q)$ is given by

$$\mathrm{SO}(V)(\mathbb{R})/C_{\mathrm{SO}} \simeq \mathrm{SO}(2,2q)/\left(\mathrm{SO}(2)\times\mathrm{SO}(2q)\right) \simeq \mathrm{Gr}_{\mathrm{O}},$$

a Grassmannian of two dimensional positive definite subspaces, complex structure through 'spin-orientation'.

$$\mathrm{Gr}_{\mathrm{O}} = \{v \subset V_{\mathbb{Q}}(\mathbb{R});\ \dim(v) = 2, (\cdot,\cdot)\mid_v > 0\}.$$

The embedding

Image of $[z] \in \mathcal{K}_{\mathrm{U}}$ under α: $[z] \mapsto \mathbb{R}X_L + \mathbb{R}Y_L \in \mathrm{Gr}_{\mathrm{O}}$, with

$$X_L = \frac{1}{2\langle u', u\rangle} z, \quad Y_L = \frac{-i}{2\langle u', u\rangle} z.$$

Note: $X_L \perp Y_L$, $X_L^2 = Y_L^2 > 0$ and $(X_L, u) = 1$.

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The embedding in coordinates

With suitable basis vectors $u = e_1$, $e_3 \in L$, $e_2, e_4 \in L'$, satisfying $e_i^2 = 0$, $(e_1, e_2) = (e_3, e_4) = 1$, $\{e_1, e_2\} \perp \{e_3, e_4\}$, we get

$$Z_L(\tau,\sigma) = X_L(\tau,\sigma) + iY_L(\tau,\sigma) = (q(Z), 1, Z)$$
$$Z = (\tau, -\bar{\zeta}, \mathfrak{z}(\sigma)) \in \mathcal{H}_{\mathrm{O}},$$

where $\mathcal{H}_{\mathrm{O}} \simeq \mathrm{Gr}_{\mathrm{O}}$ is the *tube-domain model*.

Example

For $\mathrm{SO}(2,2)$, $\mathcal{H}_{\mathrm{O}} \simeq \mathbb{H} \times \mathbb{H}$, and the embedding takes the form

$$\mathbb{H} \hookrightarrow \mathbb{H} \times \mathbb{H}, \quad \tau \mapsto (\tau, -\bar{\zeta}).$$

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Main theorem: The Borchers lift

$$\Xi(z; f, W) = Ce\left(\frac{\langle z,\rho\rangle}{\langle \ell',\ell\rangle}\right) \prod_{\substack{\lambda\in K' \\ (\lambda,W)>0}} \prod_{\substack{\gamma\in L_0'/L \\ p(\gamma)=\lambda+K}} \left(1 - e\left(\frac{\langle z,\lambda\rangle}{\langle \ell',\ell\rangle}\right)\right)^{c(\langle\lambda,\lambda\rangle,\gamma)}$$

$$\uparrow$$

$$f(\tau) = \sum_{\gamma\in L'/L} \sum_{\substack{n\in\mathbb{Z}-q(\gamma) \\ n\gg-\infty}} c(n,\gamma)e(n\tau)\mathfrak{e}_\gamma \in \mathcal{M}^!_{1-q,\rho_L}(\Gamma(1))$$

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1. Meromorphic modular form for Γ_L on $\mathcal{H}_{\mathrm{U}}$ of weight $c(0,0)/2$.
2. Zeros and poles lie on *Heegner-divisors*
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W a Weyl-chamber, $\rho = \rho(f, W)$ Weyl-vector

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Heegner-divisors

The divisor of $\Xi(f)$

$$\mathrm{div}(\Xi(f)) = \frac{1}{2} \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n<0}} c(n,\gamma)\mathbf{H}(n,\gamma)$$

The *Heegner-divisor* of index (n,γ), $\mathbf{H}(n,\gamma)$ is defined as follows:

- For$\lambda \in L'$ with$\langle \lambda, \lambda \rangle = n$, $n \in \mathbb{Z}_{<0}$ define

$$\mathbf{H}(\lambda) := \{(\tau,\sigma) \in \mathcal{H}_{\mathrm{U}};\ \langle z(\tau,\sigma), \lambda \rangle = 0\}.$$

- For an index (n,γ), with $\gamma \in L'/L$ and $n \in \mathbb{Z}_{<0}$, set

$$\mathbf{H}(n,\gamma) := \sum_{\substack{\lambda \in \gamma+L \\ \langle \lambda,\lambda \rangle = n}} \mathbf{H}(\lambda).$$

Values around the cusps

Theorem

If $\Xi(z;f)$ is regular at the cusp $[u]$ of $\mathcal{H}_{\mathrm{U}}$ and not a cusp form, then

$$\lim_{\tau \to i\infty} \Xi(z;f) = e(\bar{\rho}_u) \prod_{\substack{\lambda = \kappa\zeta u \in K \\ \kappa \in \mathbb{Q}_+}} \left(1 - e(-\kappa\bar{\zeta})\right)^{c(0,\lambda)},$$

with $\zeta = \delta$ for $\mathcal{D}_{\mathbb{F}} \equiv 0 \pmod{2}$ and $\zeta = \frac{1}{2}(1+\delta)$ otherwise.

Example: $\mathrm{U}(1,1)$

Let $V_{\mathbb{F}} = \mathbb{F}^2$ and $L = \mathcal{O}_{\mathbb{F}} \oplus \mathcal{D}_{\mathbb{F}}^{-1}$, $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$.
Let $J_m(\tau) = q^{-m} + \mathbf{O}(q) \in \mathcal{M}_0^!(\Gamma(1))$, for $m \in \mathbb{Z}_{>0}$.
Then, $\Xi(\tau; J_m)$ is a meromorphic modular form on $\mathbb{H}$ with product expansion (absolutely convergent for $\Im\tau > 2m|\delta|^{-1}$) given by

$$\Xi(\tau; J_m, W) = e(-\sigma_m \tau) \prod_{\substack{k,l\in\mathbb{Z} \\ l \geq -km}} \left(1 - e\left(k\tau - l\bar{\zeta}\right)\right)^{c(kl)},$$

where $\sigma_m = \sum_{d|m} d$ and $\zeta \in \mathbb{F}$, with $\mathcal{O}_{\mathbb{F}} = \mathbb{Z} + \zeta\mathbb{Z}$.

(Here, the Weyl-chamber W is defined through $2\Im\tau > |\delta| m$.)

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Modularity of divisors

Let $X_\Gamma = \Gamma_L \backslash \mathcal{H}_{\mathrm{U}}$. Consider the first Chow-group $\mathrm{CH}^1(X_\Gamma) \simeq \mathrm{Pic}(X_\Gamma)$.

We introduce a modified Chow-group as follows:

- Let $\pi : \tilde{X}_\Gamma \to X_\Gamma$ be a desingularization.
- Denote by $\mathcal{B}$ the boundary divisors of of $\tilde{X}_\Gamma$ and introduce the modified Chow-group $\mathrm{CH}^1(\tilde{X}_\Gamma)/\mathcal{B}$.
- Denote by $\mathcal{L}_k$ the line bdl. of meromorphic automorphic forms of weight k on X_Γ, and by $c_1(\mathcal{L}_k)$ its class in $\left(\mathrm{CH}^1(\tilde{X}_\Gamma)/\mathcal{B}\right)_{\mathbb{Q}}$.

Theorem: Modularity of divisors

Lemma (Borcherds)

A power series $g \in \mathbb{C}[L'/L][[q]]$ is contained in $\mathcal{M}_{1+q,\rho_L^*}$ iff $\{f, g\} = 0$ for every $f \in \mathcal{M}^!_{1-q,\rho_L}$.

Theorem

The generating series $A(\tau)$ in $\mathbb{Q}[L'/L][[q]] \otimes (\mathrm{CH}^1(\tilde{X}_\Gamma)/\mathcal{B})_{\mathbb{Q}}$ given by

$$A(\tau) = c_1(\mathcal{L}_{-1/2}) + \sum_{\beta \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\beta) \\ n>0}} \pi^* \left(\mathbf{H}(-n, \beta)\right) q^n \mathfrak{e}_\beta,$$

is a modular form contained in $\mathcal{M}_{1+q,\rho_L^*} \otimes (\mathrm{CH}^1(\tilde{X}_\Gamma)/\mathcal{B})_{\mathbb{Q}}$.

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Thank you!