Local Borcherds Products for Unitary Groups

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Abstract

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0 Introduction and statement of results

A local Borcherds product is a holomorphic function, which, like a Borcherds form has an absolutely convergent infinite product expansion and an arithmetically defined divisor,
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Local Borcherds Products [draft version]

called a local Heegner divisor. Here, ‘local’ refers to boundary components of a modular variety. Such products were first introduced by Bruinier and Freitag, who, in [3] studied the local divisor class groups of generic boundary components for the modular varieties of indefinite orthogonal groups $O(2, l), \ l \geq 3$. Since then, local Borcherds products have appeared in several places in the literature, for example in [5], for the Hilbert modular group, and in [6], where they are introduced to study a rather specific problem in the geometry of Siegel three folds.

The aim of the present paper is to develop a theory similar to that of Bruinier and Freitag for unitary groups of signature $(1, n + 1), \ n \geq 1$.

Let $k = \mathbb{Q}(\sqrt{D_k})$ be an imaginary quadratic number field with discriminant $D_k$, which we consider as a subset of $\mathbb{C}$. Denote by $\mathcal{O}_k$ the ring of integers in $k$, by $\delta_k^{-1}$ the inverse different ideal and by $\delta_k$ the square-root of $D_k$, with the principal branch of the complex square-root.

Let $V$ be an indefinite hermitian vector space over $k$ of signature $(1, n + 1)$, equipped with a non-degenerate hermitian form $\langle \cdot, \cdot \rangle$. Let $L$ be a lattice in $V$. We assume that $L$ has full rank as an $\mathcal{O}_k$-module, so $L \otimes \mathcal{O}_k k = V$, and that $L$ is an even (and integral) lattice, hence $\langle \lambda, \lambda \rangle \in \mathbb{Z}$ for all $\lambda \in L$. Only in this introductory section, we additionally assume that $L$ is unimodular (over $\mathbb{Z}$), i.e. $L = L' = \{ \mu \in V; \langle \lambda, \mu \rangle \in \delta_k^{-1}, \forall \lambda \in L \}$.

We denote by $U(V)$ the unitary group of $V$ and by $U(L) \subset U(V)$ the isometry group of $L$. Subgroups of finite index in $U(L)$ are called unitary modular groups.

We consider $U(V)$ as an algebraic group defined over $\mathbb{Q}$. Its set of real points, denoted $U(V)(\mathbb{R})$, is the unitary group of the complex hermitian space $V \otimes_k \mathbb{C}$. A symmetric domain for the operation of this group is given by the quotient $H = U(V)(\mathbb{R})/K$, where $K$ is a maximal compact subgroup of $U(V)(\mathbb{R})$.

If $\Gamma \subset U(L)$ is a unitary modular group, we denote by $X_{\Gamma}$ the modular variety given by the quotient $\Gamma \backslash H$.

The boundary points of $H$ correspond one-to-one to the elements of the set of one-dimensional isotropic subspaces of $V$, denoted $\text{Iso}(V)$. In section 1.4 we recall how using a toroidal compactification, $X_{\Gamma}$, which is non-compact, can be turned into a normal complex space without singularities at the cusps. In this compactification, loosely speaking, for each element of $I \in \text{Iso}(V)$ a suitable open neighbourhood $U_{\epsilon}(I)$ of the respective cusp is glued to $X_{\Gamma}$. We want to study the local Picard groups of such small open neighbourhoods.

Since the construction we carry out is local in nature, it suffices to examine only one fixed cusp. For this purpose, we choose a primitive isotropic lattice vector $\ell \in L$. The stabilizer of $\ell$ in $\Gamma$ contains a Heisenberg group, denoted $\Gamma_{\ell}$, the elements of which can be written as pairs $[h, t]$, with $h \in \mathbb{Q}$ and $t$ from a definite lattice $D_{\ell, \Gamma}$ of rank $n$. These lattice vectors act as translations. The local Picard group for a neighbourhood $U_{\epsilon}(\ell)$ of the cusp $\ell$ is given by $\text{Pic}(\Gamma_{\ell} \backslash U_{\epsilon}(\ell))$.

For a lattice vector of negative norm, $\lambda \in L$, with $\langle \lambda, \lambda \rangle \in \mathbb{Z}_{<0}$, a primitive Heegner divisor $H(\lambda)$ is defined by the complement with respect to $\langle \cdot, \cdot \rangle$ of $\lambda$ in $H$. If $\ell$ lies in that
complement, the primitive local Heegner divisor attached to \( \lambda \) is obtained by summing over the the orbit of \( \lambda \) under the translations in \( \Gamma_\ell \), hence \( \mathbf{H}_\infty(\lambda) = \sum_{t \in D_\ell, \Gamma} \mathbf{H}([0, t] \lambda) \).

A Heegner divisor of \( \mathcal{H} \) is a \( \Gamma \)-invariant finite linear combination of primitive Heegner divisors and the pre-image under the canonical projection of a divisor on \( X_{\Gamma} \). By a \textit{local Heegner divisor}, we mean a finite linear combination of primitive local Heegner divisors, which corresponds to the pre-image of an element of the divisor group \( \text{Div}(\Gamma_\ell \backslash U_{\ell}(t)) \), see section 3.1 below for details.

Our goal is to describe the position of local Heegner divisors in the Picard group. This is where local Borcherds products come into play: For a negative-norm lattice vector \( \lambda \) we define the a local Borcherds product \( \Psi_{\lambda}(z) \) as follows (see p. 16f for a more explicit definition):

\[
\Psi_{\lambda}(z) := \prod_{\mu \in D_\ell, \Gamma \lambda} (1 - e(\sigma(\mu) \langle z, \mu \rangle)).
\]

Here, \( \mu \) runs over the orbits of \( \lambda \) under the operation of \( \Gamma_\ell \), and \( \sigma(\mu) \) is a sign introduced to assure absolute convergence. The product has divisor \( \mathbf{H}_\infty(\lambda) \). However, because of the sign \( \sigma \), it is not invariant under \( \Gamma_\ell \). Instead, there is a non-trivial automorphy factor.

This is actually a desirable situation: By calculating the automorphy factor, we are able to determine the Chern class \( [c_{\lambda}] \) of \( \mathbf{H}_\infty(\lambda) \) in the cohomology group \( H^2(\Gamma_\ell, \mathbb{Z}) \), see sections 3.2 and 3.3. It turns out that \( [c_{\lambda}] \) is given by the image in the cohomology of a bilinear form:

\[
c_{\lambda}([h, t], [h', t']) = \Re \left[ (\delta_k + b) F_{\lambda}(t, t') \right] \quad \text{(for } [h, t], [h', t'] \in \Gamma_\ell),
\]

with \( F_{\lambda}(x, y) = \langle x, \lambda \rangle \langle y, \lambda \rangle + \langle \lambda, x \rangle \langle y, \lambda \rangle \) (1)

and \( z \in \mathcal{H} \). Through this, we know the Chern class of every local Heegner divisor as a finite linear combination, and we can describe its position in the cohomology.

By applying some results derived in section 2 on cocycles in \( H^2(\Gamma_\ell, \mathbb{Z}) \) obtained from bilinear forms, we obtain an if-and-only-if condition for the Chern class of Heegner divisors to be a torsion element, in Lemma 3.1. Some further, necessary conditions come out as a by-product, see Corollary 3.1. Finally, our main result, Theorem 3.1 describes exactly when Heegner divisors are torsion elements in the local Picard group. For a unimodular lattice \( L \), the theorem can be formulated as follows, for the general version, see Theorem 3.1 on p. 21.

**Theorem 0.1.** A linear combination of local Heegner divisors of the form

\[
H = \frac{1}{2} \sum_{m \in \mathbb{Z}} c(m) \mathbf{H}_\ell(m),
\]

with coefficients \( c(m) \in \mathbb{Z} \), is a torsion element in the local Picard group \( \text{Pic}(\Gamma_\ell \backslash U_{\ell}(t)) \), if and only if the equation

\[
\sum_{m \in \mathbb{Z}} c(m) \sum_{\lambda \in D \atop q(\lambda) = m} \left[ \delta_k (b + \delta_k) F_{\lambda}(t, t') - \frac{D_k}{n} \langle \lambda, \lambda \rangle \langle t', t \rangle \right]
\]

holds for all \( t, t' \in D_\ell, \Gamma \).
Here, $D$ is a definite lattice with $D = L \cap \ell \perp / \ell$, while $b = 0$ if the discriminant $D_k$ is even and 1 if $D_k$ is odd. The form $F_\lambda$ is as in [1] above and $D_{\ell, \Gamma} \subset D$ is the lattice of translations $[0, t] \in \Gamma_\ell$.

In the proof of this theorem, one implication, if $H$ torsion, follows directly from Lemma 3.1 while the converse is showed constructively.

As an application of the theorem, we study the (local) obstructions to realizing Heegner divisors through local Borcherds products. These turn out to be given by certain spaces of cusp forms spanned by theta-series. For the following, assume that $D_k$ is even (more precisely in the previous theorem statement let $b = 0$), see Theorem 4.1 in section 4 for details. Also, we set $k = n + 2$ and write $G = SL_2(\mathbb{Z})$:

**Theorem 0.2.** A finite linear combination of Heegner divisors

$$H = \frac{1}{2} \sum_{m \in \mathbb{Z}} c(m) H_\ell(m)$$

is a torsion element in $\text{Pic}(\Gamma_\ell \backslash U_\ell(\ell))$ if and only if

$$\sum_{m \in \mathbb{Z}, m < 0} c(m)a(-m) = 0$$

for all cusp forms $f \in S_{k,Q}(G)$ with Fourier coefficients $a(m)$. Here, $S_{k,Q}(G) \subset S_k(G)$ denotes the space of cusp forms spanned by theta-series with coefficients given by values of the harmonious polynomial $Q(u, v) = 2(\Re(u, v))^2 - \frac{1}{n} \langle u, u \rangle \langle v, v \rangle$, with $u$ ranging over lattice vectors $\lambda \in D$ and $v$ varying through $D \otimes \mathbb{O}_k \mathbb{C}$.

Related results for orthogonal groups are [3], Proposition 5.2 and Theorem 5.4. Theorem 0.2 can be seen a local version of an obstruction theory along the lines of that developed by Borcherds using Serre-duality, see [1] Theorem 3.1, a unitary version of which can be found in [8], section 5.

The paper is structured as follows: In the first section, we present the set-up and notation used throughout. We introduce a Siegel domain model of the symmetric domain, with the fixed isotropic lattice vector $\ell$ corresponding to the cusp at infinity. We then describe the stabilizer of that cusp and define the Heisenberg group $\Gamma_\ell$. Also, we sketch the construction of the compactification used for $X_\Gamma$.

In section 2 we study the cohomology of the Heisenberg group $\Gamma_\ell$ and derive criteria describing when certain two-cocycles obtained from bilinear forms are torsion elements in the group cohomology $H^2(\Gamma_\ell, \mathbb{Z})$. These results are used in the following section 3 as the main part of the paper: Here, we study Heegner divisors, introduce the local Borcherds products and determine their Chern classes. Using the results of the second section we get an if and only if condition, when a linear combination of Heegner divisors is a torsion element in the cohomology, this is Lemma 2.1 on p. 9. Finally, as our main result, we derive Theorem 3.1.

The last section closes with an application to modular forms: A description of spaces of obstruction to local Borcherds products is given in Theorem 4.1 and, in a weaker, but somewhat more general version, in Proposition 4.1.
1 Hermitian lattices and symmetric domains

1.1 Hermitian spaces and lattices

Let \( k = \mathbb{Q}(D_k) \) be an imaginary quadratic number field of discriminant \( D_k \), with \( D_k \) a square-free negative integer. Let \( \mathcal{O}_k \subset k \) be the ring of integers in \( k \). Denote by \( \mathfrak{d}_k \) the different ideal and by \( \mathfrak{d}_k^{-1} \) the inverse different ideal.

We shall consider \( k \) as a subset of the complex numbers \( \mathbb{C} \) and denote by \( \delta_k \) the square-root of the discriminant, with the usual choice of the complex square-root. Then, \( \mathfrak{d}_k \) is given by \( \delta_k \mathcal{O}_k \) and \( \mathfrak{d}_k^{-1} \) by \( \delta_k^{-1} \mathcal{O}_k \).

Let \( V = V(k) \) be an indefinite hermitian space over \( k \) of signature \((1, n+1)\), endowed with a non-degenerate hermitian form \( \langle \cdot, \cdot \rangle \), linear in the left and conjugate linear in the right argument. A complex hermitian space \( V(\mathbb{C}) = V \otimes_k \mathbb{C} \) is obtained by extension of scalars. We denote by \( V(\mathbb{Q}) \) the \( \mathbb{Q} \)-vector space underlying \( V \), which bears the structure of a quadratic space of signature \((2, 2n+2)\) with the quadratic form \( q(\cdot) \) defined by \( q(x) := \langle x, x \rangle \). Similarly, the real quadratic space underlying \( V(\mathbb{C}) \) is denoted \( V(\mathbb{R}) \). We have \( V(\mathbb{R}) = V(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R} \).

Let \( L \) be a hermitian lattice in \( V \), with \( L \otimes_k k = V \), and with as hermitian form the restriction of \( \langle \cdot, \cdot \rangle \) to \( L \). We denote by \( L' \) the \( \mathbb{Z} \)-dual of \( L \), defined as

\[
L' = \{ x \in V; \langle x, y \rangle \in \mathfrak{d}_k^{-1} \text{ for all } y \in L \} = \{ x \in V; \ Tr_{k/\mathbb{Q}}(x, y) \in \mathbb{Z} \text{ for all } y \in L \}.
\]

Naturally, \( L' \) is a lattice in \( V \), too. If \( L' = L \), then \( L \) is called unimodular. If \( L \subseteq L' \) the lattice is called integral. It is called even, if further, for all \( x \in L \), \( \langle x, x \rangle \in \mathbb{Z} \). The quotient \( L'/L \) is referred to as the discriminant group of \( L \).

More generally in the context of this paper, by a hermitian lattice we mean a discrete subgroup \( M \) of \( V \), for which the ring of multipliers \( \mathcal{O}(M) \) is an order in \( k \). (A multiplier of \( M \) is a complex number \( \alpha \) with \( \alpha M \subset M \).) Most lattices will occur here as sublattices of a fixed lattice \( L \), with \( L \) as above, of full rank, hermitian and even.

Denote by \( U(V) \) the unitary group of \( V \), and by \( SU(V) \) the special unitary group. The isometry group of a lattice \( L \) in \( U(V) \) is denoted \( U(L) \), similarly for \( SU(L) \). The discriminant kernel \( \Gamma_L \) is the subgroup of finite index in \( SU(L) \) which acts trivially on the discriminant group of \( L \). We refer to subgroups of finite index in \( \Gamma_L \) as unitary modular groups. In the following, \( \Gamma \) will always denote a unitary modular group.

1.2 A symmetric domain

Viewing \( U(V) \) as an algebraic group, its set of real points, denoted \( U(V)(\mathbb{R}) \), is the unitary group of \( V(\mathbb{C}) \). We briefly describe the construction of a symmetric domain for the action of \( U(V)(\mathbb{R}) \) on \( V(\mathbb{C}) \). Let \( PV(\mathbb{C}) \) denote the projective space of \( V(\mathbb{C}) \). Now, a projective model is given by the positive cone

\[
\mathcal{C} = \{ [v] \in PV(\mathbb{C}); \langle v, v \rangle > 0 \}.
\]

Denote by \( \text{Iso}(V) \) the set of one-dimensional isotropic subspaces of \( V(k) \). Its elements are in one-to-one correspondence with the rational boundary components of the symmetric
domain. In particular, we fix an element \( I \in \text{Iso}(V) \) by choosing a primitive isotropic lattice vector \( \ell \in L \) and setting \( I = k\ell \). Further, we choose a primitive vector \( \ell' \in L' \) such that \( \langle \ell, \ell' \rangle \neq 0 \). We shall assume that \( \ell' \) is isotropic, too. Note that this is a non-trivial assumption about the hermitian lattice \( L \) and its dual.

For \( a \in V \), we denote by \( a^\perp \) the complement with respect to \( \langle \cdot, \cdot \rangle \). We set \( D := L \cap \ell^\perp \cap \ell'^\perp \). Equipped with the restriction of \( \langle \cdot, \cdot \rangle \), \( D \) is a definite hermitian lattice of signature \((0, n)\). Denote by \( W = W(k) \) the subspace \( D \otimes \mathbb{Q}_k \), and let \( W(\mathbb{C}) = W \otimes \mathbb{C} \).

An affine model for the symmetric domain of \( U(V) (\mathbb{R}) \), called the Siegel domain model, is given by the following generalized upper-half-plane:

\[
\mathcal{H}_{\ell, \ell'} = \left\{ (\tau, \sigma) \in \mathbb{C} \times W(\mathbb{C}) ; \quad 2\Im(\tau)\delta_k||\langle \ell, \ell' \rangle|^2 > -\langle \sigma, \sigma \rangle - \langle \ell', \ell' \rangle \right\}.
\]

For \( (\tau, \sigma) \in \mathcal{H}_{\ell, \ell'} \), we set

\[
z = z(\tau, \sigma) := \ell' - \delta_k \tau \langle \ell, \ell' \rangle \ell + \sigma.
\]

Clearly, under the canonical projection \( \pi_V : V(\mathbb{C}) \to \mathbb{P}V(\mathbb{C}) \), we have \( \pi_V(z) \in \mathbb{C} \) for all \( (\tau, \sigma) \in \mathcal{H}_{\ell, \ell'} \). Conversely, every \( [v] \in \mathbb{C} \) contains a representative of the form \( z(\tau, \sigma) \) for some pair \( (\tau, \sigma) \in \mathcal{H}_{\ell, \ell'} \). Usually, in the following, since \( \ell \) and \( \ell' \) are fixed, we shall simply write \( \mathcal{H} = \mathcal{H}_{\ell, \ell'} \).

The isotropic line \( I_\mathbb{R} = I \otimes_\mathbb{Q} \mathbb{R} = [\ell] \) corresponds to the cusp at infinity of \( \mathcal{H} \).

### 1.3 Stabilizer of the cusp

Next, we will describe the stabilizer in \( \Gamma \) of the cusp \([\ell]\). Consider the following transformations corresponding to elements of \( SU(V) \):

\[
[h, 0] : v \mapsto v - \langle v, \ell \rangle \delta_k h \ell \quad \text{for} \quad h \in \mathbb{Q}, \quad (2)
\]

\[
[0, t] : v \mapsto v + \langle v, \ell \rangle t - \langle v, t \rangle \ell - \frac{1}{2} \langle v, \ell \rangle \langle t, \ell \rangle \ell \quad \text{for} \quad t \in W. \quad (3)
\]

Clearly, these transformations stabilize the isotropic subspace \( k\ell \). Their action on \( \mathcal{H} \) is given by

\[
[h, 0] : (\tau, \sigma) \mapsto (\tau + h, \sigma), \quad [0, t] : (\tau, \sigma) \mapsto \left( \tau + \frac{\langle \sigma, t \rangle}{\delta_k \langle \ell', \ell \rangle} + \frac{1}{2} \frac{\langle t, t \rangle}{\delta_k} \sigma + \langle \ell', \ell \rangle t \right).
\]

The \textit{Heisenberg group} attached to \( \ell \), denoted \( \text{Heis}_\ell \), is the set of pairs \([h, t]\) with group law given by

\[
[h, t] \circ [h', t'] = [h + h' + \frac{3}{\delta_k} \langle t', t \rangle, t + t'].
\] (4)

Here, we follow the convention that \( ([h, t] \circ [h', t'])(v) = ([h, t][h', t'](v)) \) for \( v \in V(k) \).

The center of \( \text{Heis}_\ell \) consists of transformations of type \([2] \). Let \( \Gamma_\ell \) be the subgroup \( \Gamma \cap \text{Heis}_\ell \). We denote its center by \( \Gamma_{\ell, T} \). The full stabilizer of the cusp is given by the direct product \( \Gamma_\ell \times (U(W) \cap \Gamma) \).

A description of the elements of \( \Gamma_\ell \) can be given as follows (this is well-known):
Remark 1.1. Let $\Gamma$ be a subgroup of finite index in $U(L)$, and let $\Gamma_{\ell} = \Gamma \cap \text{Heis}_{\ell}$. Then, there exist a positive rational number $N_{\ell, \Gamma}$ and a lattice $D_{\ell, \Gamma}$ of finite index in $D$, such that $[h, t] \in \Gamma_{\ell}$ for all $h \in N_{\ell, \Gamma}\mathbb{Z}$, $t \in D_{\ell, \Gamma}$, and that $\Im(t', t)\delta_k^{-1} \in N_{\ell, \Gamma}\mathbb{Z}$ for all $t, t' \in D_{\ell, \Gamma}$.

1.4 Boundary components

The modular variety $X_{\Gamma}$ is given by the quotient

$$\Gamma \backslash SU(V)(\mathbb{R})/(SU(W)(\mathbb{R}) \times SU(W^\perp)(\mathbb{R})) \simeq \Gamma \backslash \mathcal{H}.$$  

Note that $X_{\Gamma}$ is non-compact. The usual Baily-Borel compactification $X_{\Gamma, BB}^*$ is obtained by introducing a topology and a complex structure on the quotient

$$\Gamma \backslash (\mathcal{H} \cup \{I_B; I \in \text{Iso}(V)\}).$$  

We sketch this for the cusp at infinity, defined by $[\ell]$: The following sets constitute a system of neighbourhoods of the cusp

$$U_\epsilon(\ell) = \left\{ [z] \in \mathcal{C}; \frac{\langle z, z \rangle}{|\langle z, \ell \rangle|^2} |\langle \ell', \ell \rangle|^2 > \frac{1}{\epsilon} \right\} \quad (\epsilon > 0).$$

A subset $V$ of $\mathcal{C} \cup \{[\ell]\}$ is called open if $V \cap \mathcal{C}$ is open in the usual sense and further if $[\ell] \in V$ implies $U_\epsilon(\ell) \subset V$ for some $\epsilon > 0$.

Through the quotient topology, this construction yields a topology on $\Gamma \backslash (\mathcal{C} \cup \{[\ell]\})$. The complex structure is defined though the pullback under the canonical projection $\mathcal{C} \cap \{I_B; I \in \text{Iso}(V)\} \to X_{\Gamma, BB}^*$, locally for each cusp, see [7] for details. This way, one gets the structure of normal complex space on $X_{\Gamma, BB}^*$. In general, however, there are still singularities at the boundary points.

This difficulty can be avoided by using a toroidal compactification, instead. We recall the construction briefly, see [7], §1.1.5 and, in particular [4] §4.3, for more details. In the following, identify the sets $U_\epsilon(\ell) \subset \mathcal{C}$ with the corresponding sets of representatives in $\mathcal{H}_{\ell, \ell'}$. Clearly, the Heisenberg group $\Gamma_{\ell}$ operates on $U_\epsilon(\ell)$. For sufficiently small $\epsilon$, there is an open immersion

$$\Gamma_{\ell} \backslash U_\epsilon(\ell) \to X_{\Gamma}.$$  

Recall that for the center $C(\Gamma_{\ell}) = \Gamma_{\ell, T}$, we have $\Gamma_{\ell, T} \simeq \mathbb{Z}N_{\ell, \Gamma}$. Set $q_{\ell} := \exp(2\pi i \tau/N_{\ell, \Gamma})$.

The quotient $\Gamma_{\ell, T} \backslash U_\epsilon(\ell)$ can now be viewed as bundle of punctured discs over $W(\mathbb{C})$:

$$V_\epsilon(\ell) := \Gamma_{\ell, T} \backslash U_\epsilon(\ell) \simeq \left\{ (q_{\ell}, \sigma); \ 0 < |q_{\ell}| < \exp \left( \frac{\pi |\sigma, \sigma| + \epsilon^{-1}}{|\delta_k|^2 |\langle \ell', \ell \rangle|^2} \right) \right\}.$$  

Adding the midpoint to each disk, we get the bundle of discs

$$\tilde{V}_\epsilon(\ell) := \left\{ (q_{\ell}, \sigma); \ |q_{\ell}| < \exp \left( \frac{\pi |\sigma, \sigma| + \epsilon^{-1}}{|\delta_k|^2 |\langle \ell', \ell \rangle|^2} \right) \right\}.$$
The action of $\Gamma_\ell$ is well defined at the 'midpoints', leaving the divisor $q = 0$ fixed. Also, if $\Gamma$ is sufficiently small, the operation is free, hence we get an open immersion

$$\Gamma_\ell\backslash U_\ell(\ell) \to (\Gamma_\ell/\Gamma_{\ell,T})\backslash \tilde{V}_\ell(\ell),$$

by which the right hand side can be glued to $X_\Gamma$, yielding a partial compactification. For a point $(0, \sigma_0) \in \tilde{V}_\ell(\ell)$, we define a system of open sets

$$B_\delta(0, \sigma_0) = \left\{ (q_\ell, \sigma) \in \tilde{V}_\ell(\ell) : \langle \sigma - \sigma_0, \sigma - \sigma_0 \rangle < \delta, |q_\ell| < \delta \right\} \quad (\delta > 0).$$

Under the immersion (6), the images of these sets form a system of open neighborhoods for the boundary point at $(0, \sigma_0)$.

Repeating this construction and the gluing procedure for every $I \in \text{Iso}(V)$ yields a compactification of $X_{\Gamma}$, which we denote $X_{\Gamma}^\ast$.

2 The local cohomology group

In the following, let $\Gamma$ be unitary modular group and let $\Gamma_\ell \subset \Gamma$ be a Heisenberg group of the form $\Gamma_\ell = N_{\ell,\Gamma} \mathbb{Z} \rtimes D_{\ell,\Gamma}$ with $N_{\ell,\Gamma} \in \mathbb{Q}_{>0}$ and $D_{\ell,\Gamma} \subset P$ as introduced in Remark 1.1. We are interested in the cohomology of $\Gamma_\ell$, more specifically the second cohomology group $H^2(\Gamma_\ell, \mathbb{Z})$.

As usual, if $G$ is group acting on an abelian group $A$, the $n$th-cohomology group is defined as the quotient

$$H^n(G, A) = \frac{\ker(C^n(G, A) \xrightarrow{\partial} C^{n+1}(G, A))}{\text{im}(C^{n-1}(G, A) \xrightarrow{\partial} C^n(G, A))},$$

wherein $C^n$ is the set of $n$-cocycles, consisting of all functions $f : G^n \to A$, and $\partial$ is the coboundary operator. In the present setting, $G = \Gamma_\ell$, $A = \mathbb{Z}$ and the action of $G$ is trivial.

Let $U_\ell(\ell)$ be a neighborhood of the cusp of infinity, as defined in (5) above, with $\epsilon$ sufficiently small, so that the map in (6) is indeed an open immersion. Further, denote by $\mathcal{O}_\epsilon = \mathcal{O}_\epsilon(\ell)$ the sheaf of holomorphic functions on $U_\ell(\ell)$. The action of $\Gamma_\ell$ on $U_\ell(\ell)$ induces an action on $\mathcal{O}_\epsilon$, giving rise to the natural map

$$H^2(\Gamma_\ell, \mathbb{Z}) \to H^2(\Gamma_\ell, \mathcal{O}_\epsilon).$$

(7)

Definition 2.1. Denote by $\text{Bil}$ the set of $\mathbb{Z}$-valued bilinear maps $B : D_{\ell,\Gamma} \times D_{\ell,\Gamma} \to \mathbb{Z}$. To every element $B \in \text{Bil}$, we associate a two-cocycle by

$$B([h, t], [h', t']) := B(t, t') \quad ([h, t], [h', t'] \in \Gamma_\ell).$$

(8)

By the image of $B$ in the cohomology, we mean the class $[B]$ of this cocycle in $H^2(\Gamma_\ell, \mathbb{Z})$. Note that $[B]$ has a well-defined image under the map (7), yielding an element of $H^2(\Gamma_\ell, \mathcal{O}_\epsilon)$. Usually, by abuse of notation, for $B \in \text{Bil}$ we denote both the attached cocycle and its class the cohomology by $B$, too.
Note that since $D_{\ell,\Gamma}$ has full rank in $W(k)$, $B$ can also be considered as $(k$-valued) bilinear form on $W(k)$.

In this section, we examine the image of certain bilinear forms in the cohomology. All calculations here are carried out using the standard inhomogeneous complex of group cohomology (cf. [9] Chapter 8).

### 2.1 Criteria for torsion elements

Through definition 2.1, with the association $B \mapsto [B]$ we have a map $\text{Bil} \to H^2(\Gamma_\ell, \mathbb{Z})$, which can be composed with the natural map from (7):

$$\text{Bil} \rightarrow H^2(\Gamma_\ell, \mathbb{Z}) \rightarrow H^2(\Gamma_\ell, \mathcal{O}_\ell),$$

In general, neither of the two maps in this diagram is injective. If $B \in \text{Bil}$ lies in the kernel of the composition of both maps, the following lemma can be used to describe the position of $[B]$ in the cohomology $H^2(\Gamma_\ell, \mathbb{Z})$.

**Lemma 2.1.** Let $B : D_{\ell,\Gamma} \times D_{\ell,\Gamma} \to \mathbb{Z}$ be a $\mathbb{Z}$-valued bilinear form. Assume that the image of $[B]$ under the composition of the maps in (9) is zero in $H^2(\Gamma_\ell, \mathcal{O}_\ell)$. Then, $[B]$ is a torsion element in $H^2(\Gamma_\ell, \mathbb{Z})$ if and only if $B$ and $\frac{1}{|\delta_k|}\Im\langle \cdot, \cdot \rangle$ are linear dependent over $\mathbb{Z}$.

**Proof.** The proof uses a transgression map, which we introduce next.

The transgression map $\text{tg}$ is defined as the map for which the sequence

$$\mathbb{Z} \rightarrow H^2(D_{\ell,\Gamma}, \mathbb{Z}) \rightarrow H^2(\Gamma_\ell, \mathbb{Z}) \rightarrow H^2(\Gamma_\ell, \mathcal{O}_\ell) \rightarrow 0$$

becomes exact. Thus, the kernel of the map into $H^2(\Gamma_\ell, \mathbb{Z})$ is generated by the image of the identity map $1 : \mathbb{Z} \to \mathbb{Z}$ under $\text{tg}$. The image $\text{tg}(1)$ is represented by a cocycle $(t, t') \mapsto (\partial u)([\cdot, t], [\cdot, t'])$ with a map $u : \Gamma_\ell \to \mathbb{Z}$, which has to satisfy two conditions:

1. $u([N_{\ell,\Gamma}h, 0]) = h$ for all $h \in \mathbb{Z}$ and
2. $(\partial u)([h, t], [h', t'])$ does not depend on $h$ or $h'$.

A suitable map is obtained by setting $u([N_{\ell,\Gamma}h, t]) := h$. We get

$$(\partial u)([h, t], [h', t']) = [h, t]u([h', t']) - u([h, t][h', t']) + u([h, t]) = -\frac{1}{N_{\ell,\Gamma}} \Im\langle t', t \rangle.$$

Hence, $\text{tg}(1)$ is represented by the cocycle

$$(t, t') \mapsto |\delta_k|^{-1}\Im\langle t, t' \rangle,$$

and $[B]$ is a torsion element precisely if it satisfies a linear dependence relation over $\mathbb{Z}$ to this $\text{tg}(1)$. 

\[ \square \]
Hence, by Lemma 2.1, if a bilinear form $B \in \text{Bil}$ has vanishing image in $H^2(\Gamma, \mathcal{O}_\epsilon)$, $B$ defines a torsion element, precisely if there is a $Q \in \mathbb{Q}$ with
\[ B(t, t') + Q \frac{\Im \langle t, t' \rangle}{|\delta_k|} = 0 \quad \text{for all} \quad t, t' \in D_{\ell, \Gamma}. \tag{12} \]

Since the lattice $D_{\ell, \Gamma}$ has full rank in $W(k)$, by linear extension, the equation also holds for all pairs of vectors $(t, t')$ in $W(k)$. Now, by replacing the first vector $t$ by a purely imaginary multiple, say $\delta_k t$, we get
\[ B(\delta_k t, t') + Q \Re \langle t, t' \rangle = 0 \quad \text{for all} \quad t, t' \in W(k). \]

We will use this observation to determine the factor $Q$ for several specific bilinear forms in the next subsection and in section 3.

Note that rather than through linear extension, we could also argue using the fact that since the lattice $D_{\ell, \Gamma}$ is hermitian (cf. p. 5), there exist purely imaginary multipliers.

2.2 Cocycles originating from hermitian forms

In the following, let $H$ denote a hermitian form, $\mathbb{C}$-linear in the first and conjugate linear in the second argument, and assume that both $|\delta_k|^{-1} \Im H$ and $\Re H$ are $\mathbb{Z}$-valued on $D_{\ell, \Gamma}$.

The image of the imaginary part We have the following lemma

**Lemma 2.2.** Denote by $H$ a hermitian form on $W(k)$ with the property that $|\delta_k|^{-1} \Im H$ is $\mathbb{Z}$-valued on $D_{\ell, \Gamma}$. Then, the image in $H^2(\Gamma, \mathcal{O}_\epsilon)$ under the composition of the maps in (9) vanishes.

**Proof.** We claim that $\frac{1}{|\delta_k|} \Im H$ is trivialized by the following cochain
\[ u([h, t])(\tau, \sigma) = H(\sigma, t) \frac{\delta_k(t', \ell)}{2\delta_k}, \]
and hence is zero in the cohomology. Indeed,
\[ (\partial u) ([h, t], [h', t']) = [h, t]u([h', t']) - u([h, t][h', t']) + u([h, t]) = \frac{H(t, t') - H(t', t')}{2\delta_k} = \frac{1}{|\delta_k|} \Im H(t, t'). \]

With this result, we can give an example for the application of Lemma 2.1

**Corollary 2.1.** We use the notation of Lemma 2.2. Assume the trace of $H$, taken over a normal basis with respect to $\langle \cdot, \cdot \rangle$ is non-zero. Then, the antisymmetric bilinear form $|\delta_k|^{-1} \Im H$ defines a torsion element in $H^2(\Gamma, \mathbb{Z})$ precisely if
\[ \Im H(t, t') + \frac{\text{Tr} H}{n} \Im \langle t, t' \rangle = 0 \quad (\forall t, t' \in D_{\ell, \Gamma}), \tag{13} \]
(The trace $\text{Tr} H$ here, is taken over a basis $(e_l)_{l=1,\ldots,n}$ with $\langle e_l, e_l \rangle = -1$ for all $l$ and $\langle e_l, e_m \rangle = 0$ for $m \neq l$.)
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Proof. Clearly, since Tr $H \in \mathbb{Q}$, if (13) holds, the two forms $\Im H(\cdot)$ and $\Im \langle \cdot, \cdot \rangle$ are linear dependent over $\mathbb{Z}$ (after multiplication with a suitable integer). Hence, we have a torsion element in the cohomology by Lemma 2.1.

Conversely, assume that $|\delta_k|^{-1} \Im H$ is a torsion element. Then, by Lemma 2.1, we have

$$\Im H(t, t') + Q \cdot \Im \langle t, t' \rangle = 0 \quad (\forall t, t' \in D_{\ell, \Gamma}),$$

with a constant $Q \in \mathbb{Q}$. Now, we use the argument indicated above: Since $D_{\ell, \Gamma}$ has full rank in $W(k)$, by linear extension, this equation holds for all vectors in $W(k)$. In particular, any $t$ can be replaced by $\delta_k t$, yielding a similar equation with $\Im H(\cdot, \cdot)$ and $\Im \langle \cdot, \cdot \rangle$ replaced by $\Re H(\cdot, \cdot)$ and $\Re \langle \cdot, \cdot \rangle$. After adding the two equations, we have

$$H(t, t') + Q \cdot \langle t, t' \rangle = 0 \quad (\forall t, t' \in W(k)).$$

Taking the trace, we find that $Q = \text{Tr} H$. (Recall that $\text{Tr} \langle \cdot, \cdot \rangle = -n$.)

The real part of a hermitian form  As before, let $H$ be a hermitian form with real part contained in $\text{Bil}(\mathbb{Q})$. Similarly to Lemma 2.2 for the imaginary part, we get the following result for the real part:

**Lemma 2.3.** If $H$ is a hermitian form on $W(k)$ with the property that $\Re H$ a $\mathbb{Z}$-valued bilinear form on $D_{\ell, \Gamma}$ then, the image of $\Re H$ in $H^2(\Gamma_{\ell}, \mathcal{O}_\ell)$ vanishes.

**Proof.** A cochain which trivializes $\Re H$ is given by $u([h, t])(\tau, \sigma) := \frac{H(t, t)}{2}$. Indeed,

$$(\partial u) ([h, t], [h', t']) = [h, t] u([h', t']) - u([h, t][h', t']) + u([h, t])$$

$$= \frac{H(t, t)}{2} - \frac{H(t + t', t + t')}{2} + \frac{H(t', t')}{2} = \Re H(t, t').$$

Both results, Lemma 2.2 and Lemma 2.3 will be used in section 3 below, see p. 19.

### 2.3 The image of a symmetric bilinear form

Next, we study some cohomology classes obtained from symmetric bilinear forms.

Denote by $\mathcal{B}$ the set of symmetric $k$-valued bilinear forms $G$ on $W(k)$, for which both $|\delta_k|^{-1} \Im B$ and $\Re B$ are integer valued on $D_{\ell, \Gamma}$, hence contained in $\text{Bil}$. Similarly to Lemma 2.2 for the imaginary part, we get the following result for the real part:

**Lemma 2.4.** Let $G : W(k) \times W(k) \to k$ be a bilinear form with $G \in \mathcal{B}$. Then, the cocycle attached to $|\delta_k| \Im (G)$ vanishes in $H^2(\Gamma_{\ell}, \mathcal{O}_\ell)$.

The proof of this Lemma is somewhat elaborate and draws on methods developed in the author’s dissertation [7], see [7], Chapter 3, or [8], Section 4:

Recall that the $\mathbb{Q}$-vector space $V(\mathbb{Q})$ underlying $V(k)$ carries the structure of a rational quadratic space of signature $(2, 2n + 2)$, with the bilinear form $\langle \cdot, \cdot \rangle = \text{Tr}_{k/\mathbb{Q}} \langle \cdot, \cdot \rangle$, the attached quadratic form $q(x) = \langle x, x \rangle$. The orthogonal and special orthogonal groups
of \( V(\mathbb{Q}) \) are denoted \( \text{O}(V) \) and \( \text{SO}(V) \), respectively. Viewed as algebraic groups, the respective sets of real points are denoted \( \text{O}(V)(\mathbb{R}) \) and \( \text{SO}(V)(\mathbb{R}) \), acting on \( V(\mathbb{R}) \). We use similar notation for groups acting on the definite subspace \( W(\mathbb{Q}) = D \otimes_{\mathbb{Z}} \mathbb{Q} \) and on \( W(\mathbb{R}) = W(\mathbb{Q}) \otimes \mathbb{R} \).

A symmetric domain for the operation of \( \text{SO}(V)(\mathbb{R}) \) on \( V(\mathbb{R}) \) is given by the quotient

\[
\mathcal{D} \simeq \text{SO}(V)(\mathbb{R}) / \left( \text{SO}(W)(\mathbb{R}) \times \text{SO}(W^\perp)(\mathbb{R}) \right),
\]

which can be identified with the Grassmannian of two-dimensional positive definite subspaces. The orientation of these subspaces is used to define a complex structure on \( \mathcal{D} \), see for example [2], Section 13, or [2], Chapter 2. The boundary components of \( \mathcal{D} \) are given by isotropic subspaces, and in particular, rational one-dimensional boundary components correspond to maximal totally isotropic subspaces, see for example [3], Section 2.

Now, an isotropic \( k \)-subspace \( I \in \text{Iso}(V) \) of \( V(k) \) defines a two-dimensional (totally) isotropic subspace of \( V(\mathbb{Q}) \) and hence a one-dimensional rational boundary component.

In particular, let \( I = k\ell \in \text{Iso}(V) \). A complementary isotropic subspace is given by \( I' = k\ell^\perp \). Denote these rational subspaces by \( I_Q \) and \( I_Q' \), respectively. We define a basis of \( I_Q \cup I_Q' \) by setting \( e_1 = \ell \) and choosing vectors \( e_3 \in I_Q \) and \( e_2, e_4 \in I_Q' \), with \( e_3 \) primitive, and such that

\[
I_Q = \text{span} \{e_1, e_3\}, \quad I_Q' = \text{span} \{e_2, e_4\},
\]

and \( (e_1, e_2) = (e_3, e_4) = 1 \), \( (e_i, e_j) = 0 \) \( (\forall i \leq j; (i, j) \not\in \{(1, 2), (3, 4)\}) \).

For technical reasons, we also require that \( e_3 = \mu \ell \in V(k) \), with \( \mu \) purely imaginary.

The tube-domain model for \( \mathcal{D} \), which we denote \( \mathcal{H}_{2n+1} \), is defined as follows, see e.g. [3], Section 2: The bilinear form \( (\cdot, \cdot) \) is extended to a complexified space \( V(\mathbb{R})_C := V(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \). In the projective space \( \mathbb{P}V(\mathbb{R})_C \), consider the subset represented by vectors \( Z_L \) which satisfy \( (Z_L, Z_L) = 0 \) and \( (Z_L, Z_L') > 0 \). It has two connected components, we fix one of these and denote it by \( \mathcal{H}_{2n+1}' \). Then, \( \mathcal{H}_{2n+1}' \) is isomorphic to \( \mathcal{D} \) through a real-analytic mapping. Further, the set of vectors \( Z_L \) representing \( \mathcal{H}_{2n+1}' \), can be parametrized by a subset of the Lorentzian subspace \( V(\mathbb{R})_C \cap e_1^+ \cap e_2^+ \) (the complement being taking with respect to \( (\cdot, \cdot) \)).

This subset is the tube domain model \( \mathcal{H}_{2n+1} \). For \( Z \in \mathcal{H}_{2n+1} \), write \( Z = z_1e_3 + z_2e_4 + z_3 \), with \( z_3 \in D \otimes_{\mathbb{Z}} \mathbb{C} \). Then, \( \Im(z_1)\Im(z_2) + q(\Im(z_3)) > 0 \).

The one-dimensional boundary component given by \( I_Q \otimes \mathbb{C} \) can be described as a usual complex upper half-plane \( \{ \Im z_2 > 0 \} \simeq \mathbb{H} \), see [3], p. 4. For a point \( w \in I_Q \otimes \mathbb{C} \), an open \( \epsilon \)-neighborhood in \( \mathcal{H}_{2n+1} \cup (I_Q \otimes \mathbb{C}) \) is given by the union \( U_\epsilon(w) \cup V_\epsilon(w) \) with

\[
U_\epsilon(w) = \{ (z_1, z_2, z_3); \Im(z_1) \cdot \Im(z_2) + q(\Im(z_3)) > \frac{1}{\epsilon} \} \quad (\subset \mathcal{H}_{2n+1}),
\]

\[
V_\epsilon(w) = \{ z_2; |z_2 - w| < \epsilon \} \quad (\subset \mathbb{H}).
\]

Note that \( U_\epsilon(w) \) does not depend on \( w \). Thus, we set \( U_\epsilon(I) = U_\epsilon(w) \) (for any \( w \in I_Q \otimes \mathbb{C} \). We denote the sheaf of meromorphic functions on \( U_\epsilon(I) \) by \( R_\epsilon(I) \).
Finally, we denote by $H_I$ the Heisenberg group in the stabilizer of $I_Q$ in $SO(V)$, see \[3\], p. 5ff. The set of its real points is denoted by $H_I(\mathbb{R})$.

By the theory developed in \[7, 8\], the embedding of the symmetric domains $\mathcal{C} \hookrightarrow \mathcal{D}$ (which arises from the natural inclusion $SU(V) \hookrightarrow SO(V)$ due to $V(k) = V(\mathbb{Q}) \otimes k$) can be realized as an embedding of $\mathcal{H}_{\ell,\ell'}$ into $\mathcal{H}_{2n+1}$ which is compatible with the respective complex structures and well-behaved on the boundary. In particular, for all $\epsilon > 0$, under this embedding, $U_\epsilon(\ell)$ is mapped into $U_\epsilon(\ell)$ with $\epsilon' > 0$ see \[7\], Chapter 3.3. In fact, $\epsilon'/\epsilon$ is constant. Since the embedding is compatible with the complex structure, $O_\epsilon$ can be identified with a subspace of $R_\epsilon(I)$. In the following, to facilitate notation, we identify $\epsilon' = \epsilon$.

Further, $\Gamma_\ell$ is embedded into $H_I$ as subgroup. From the inclusions $\Gamma_\ell \hookrightarrow H_I$ and $O_\ell \hookrightarrow R_\ell(I)$, by pull-back under the second inclusion, we get a map

$$H^2(H_F, R_\ell(I)) \to H^2(\Gamma_\ell, O_\ell).$$

Indeed, let $[c] \in H^2(H_I, R_\ell(I))$ be a cocycle given by $c : (h, h') \mapsto c(h, h') (h, h' \in H_I)$, then its image in $H^2(\Gamma_\ell, O_{\ell})$ is given by $(g, g') \mapsto c(h(g), h(g'))$, where for $g, g' \in \Gamma_\ell$, we denote by $h(g), h(g')$ the images in $H_I$ under the embedding $\Gamma_\ell \hookrightarrow H_I$.

For our purposes, it suffices to describe the image of the transformation $[0, t] (t \in D_\ell, \Gamma)$. The following result from \[7\], p. 75, can be showed by direct calculation:

**Remark 2.1.** Let $[0, t] \in \text{Heis}_\ell$. Its image in $H_I(\mathbb{R})$ is given by

$$E(\ell, -\frac{1}{2}t) \circ E(i\ell, \frac{1}{2}it)$$

where the $E(\cdot, \cdot)$ denotes an Eichler transformation in $SO(V)$. For $u, v \in V(\mathbb{R})$, with $u$ isotropic and $(u, v) = 0$, $E(u, v)$ is defined as:

$$E(u, v)(a) = a - (a, u)v + (a, v)u - \langle v, v \rangle(a, u)u \quad (\forall a \in V(\mathbb{R})).$$

For $[0, t] \in \Gamma_\ell$ the Eichler transformations from (14) are contained in $H_I$. Using the identity $E(u, v) = E(cu, c^{-1}v)$, valid for $c \in \mathbb{R} \setminus \{0\}$, we can get a more explicit description.

Now, we complete the proof of Lemma 2.4

**Proof.** The real part of $G$, $\Re(G)$, viewed as a map on $W(\mathbb{R})$, is a symmetric bilinear form, and is non-degenerate if $G$ is. Similarly to \[8\], for $\Re(G)$ a 2-cocycle in $H^2(H_I, R_\ell(F))$, denoted $[\Re(G)]$, is defined by setting

$$\Re(G)([a, t, b], [a', t', b']) = \Re(G)(a, b') \quad \text{for} \quad [a, t, b], [a', t', b'] \in H_I,$$

where for elements of $H_I$, we use the notation of \[3\], see p. 10f for details. Now, by the above remark 2.1 the image of $[\Re(G)]$ under $H^2(H_I, R_\ell) \to H^2(\Gamma_\ell, O_\ell)$ is given by

$$([0, t], [0, t']) \mapsto \Re(G)(-r_2^{\frac{1}{2}}t, is|\delta_k|^{-1}\frac{1}{2}t') = rs\frac{1}{|\delta_k|} : \Im(G)(t, t'),$$

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with suitable \( r, s \in \mathbb{Q} \).

However, it turns out that \([\mathbb{R}(G)]\) vanishes in \( H^2(H_I, R_\epsilon(I))\) (cf. [3], p. 11), as indeed, it is trivialized by a cochain of the form

\[
\mathbb{R}(G)(b, z) - \frac{z^2}{2}\mathbb{R}(G)(b, b),
\]

where \( E(\epsilon_1, b) = [b, 0, 0] \in H_I\), with \( b \in D \). (Here, \( D \) is considered as a lattice in \( V(\mathbb{Q})\)). Hence its image in \( H^2(\Gamma_\ell, \mathcal{O}_\epsilon)\) vanishes, too. The proof of the Lemma is complete. \( \square \)

Finally, for later application we establish that, like the imaginary part, the real part of \( G \) also vanishes in the cohomology \( H^2(\Gamma_\ell, \mathcal{O}_\epsilon)\):

**Lemma 2.5.** Let \( G : W(k) \times W(k) \to k \) be a bilinear form with \( G \in \mathcal{B} \). Then, the image of \( \mathbb{R}(G) \) in \( H^2(\Gamma_\ell, \mathcal{O}_\epsilon) \) vanishes.

**Proof.** Denote by \([\mathbb{I}(G)]\) and \([\mathbb{R}(G)]\) the images of the real and imaginary parts of \( G \) in \( H^2(\Gamma_\ell, \mathcal{O}_\epsilon)\). We get a well-defined element of the cohomology group by setting \([G] := [\mathbb{R}(G)] + i[\mathbb{I}(G)]\).

However, \([G]\) is trivialized by the cochain \( u([h, t]) = -\frac{G(\sigma, t)}{\langle \ell', \ell \rangle} - G(t, t)\):

\[
(\partial u)([h, t][h', t']) = -\frac{G(\sigma + \langle \ell', \ell \rangle t, t')}{\langle \ell', \ell \rangle} + \frac{G(\sigma, t' + t)}{\langle \ell', \ell \rangle} - \frac{G(\sigma, t)}{\langle \ell', \ell \rangle} - G(t', t') + G(t + t', t + t') - G(t, t) = G(t', t).
\]

Obviously, \([\mathbb{R}(G)] = 0\], since by Lemma 2.4 \([\mathbb{I}(G)] = 0\]. \( \square \)

### 3 Local Heegner divisors and Borcherds products

This section is the main part of the paper: We will introduce local Borcherds products attached to primitive Heegner divisors and use them calculate Chern classes of linear combinations of Heegner divisors. Then, using the methods prepared in section 2, we will proceed to derive our main results.

#### 3.1 Local Heegner divisors

First, we recall the usual definition of Heegner divisors on \( \mathcal{H} \), cf. [8] section 6, and introduce local Heegner divisors in the neighborhoods \( U_\epsilon(\ell) \) of the cusp \( \ell \).

Let \( \lambda \in \mathcal{L}' \) be a lattice vector of negative norm, i.e. \( \langle \lambda, \lambda \rangle < 0 \). The (primitive) **Heegner divisor** \( \mathbf{H}(\lambda) \) attached to \( \lambda \) is a divisor on \( \mathcal{H} \) given by

\[
\mathbf{H}(\lambda) := \{ (\tau, \sigma) \in \mathcal{H}; \langle \lambda, z(\tau, \sigma) \rangle = 0 \},
\]

with \( z(\tau, \sigma) = \ell' - \tau \delta_k(\ell, \ell') \ell + \sigma \) (see section 1.2). Clearly, the divisor \( \mathbf{H}(\lambda) \) intersects \( U_\epsilon(\ell) \) for every \( \epsilon > 0 \), if and only if \( \langle \lambda, \ell \rangle = 0 \). In the following, we denote the complement of \( \ell \) with respect to \( \langle , \rangle \) as \( \ell^\perp \).
Thus, let $\lambda \in \ell^\perp$. Then, $\lambda = \lambda_\ell \ell + \lambda_D$ with $\lambda_D \in W(k)$ and $H(\lambda)$ is given by an equation of the form

$$\lambda_\ell \langle \ell, \ell' \rangle + \langle \lambda_D, \sigma \rangle = 0.$$ 

Consider the orbit of $\lambda$ under $\Gamma_\ell$. Since $\lambda \in \ell^\perp$, we have $[h, 0] \lambda = \lambda$ for all $h \in N_{\ell, \Gamma} \mathbb{Z}$, and $\Gamma_{\ell, T}$ acts trivially. Let $[0, t]$ with $t \in D_{\ell, \Gamma}$ be an Eichler element, then

$$[0, t] \lambda = \lambda - \langle \lambda_D, t \rangle \ell = (\lambda - \langle \lambda_D, t \rangle) \ell + \lambda_D.$$ 

Since $\Gamma$ is a modular group, $\Gamma$ is contained in $\Gamma_L$, and the Heisenberg group $\Gamma_{\ell}$ operates trivially on $L'/L$. Thus, $[h, 0] \lambda \equiv \lambda \pmod{L}$ for all $[h, 0] \in \Gamma_{\ell, T}$.

We denote by $\mathfrak{T} = \mathfrak{T}(\lambda)$ the set

$$\mathfrak{T}(\lambda) := \{\langle \lambda, t \rangle \mid t \in D_{\ell, \Gamma}\}.$$ 

Note that since $D_{\ell, \Gamma} \subseteq D$, $\mathfrak{T}(\lambda)$ is contained in $\mathfrak{a}_k^{-1}$ (as a fractional ideal).

The group $\Gamma_\ell$ operates on the set $\lambda + \mathfrak{T}$ with only finitely many orbits, hence the divisor

$$H_\infty(\lambda) := \sum_{\alpha \in \mathfrak{T}} H(\lambda + \alpha \ell)$$

is invariant under $\Gamma_{\ell}$ and defines an element of $\text{Div}(\Gamma_{\ell} \setminus U_{\epsilon}(\ell))$.

**Heegner divisors with index** Now, let $\beta \in L'/L$ be an element of discriminant group and $m$ a negative integer. Then, the Heegner divisor of index $(\beta, m)$, defined as the (locally finite) sum

$$H(\beta, m) = \sum_{\substack{\lambda \in L' \\ q(\lambda) = m \\ \lambda + k = \beta}} H(\lambda),$$

is a $\Gamma$-invariant divisor on $\mathcal{H}$. Under the canonical projection $H(\beta, m)$ is the inverse image of the a divisor on $X_{\Gamma}$.

Through the open immersion $\Gamma_{\ell} \setminus U_{\epsilon}(\ell) \hookrightarrow \Gamma \setminus \mathcal{H} = X_{\Gamma}$ from section [1.4], the inclusion $U_{\epsilon}(\ell) \subset \mathcal{H}$ and the projection maps, we get a commutative diagram

$$\begin{array}{ccc}
\text{Div}(X_{\Gamma}) & \longrightarrow & \text{Div}(\Gamma_{\ell} \setminus U_{\epsilon}(\ell)) \\
\downarrow & & \downarrow \\
\text{Div}(\mathcal{H}) & \longrightarrow & \text{Div}(U_{\epsilon}(\ell)).
\end{array}$$

We denote by $H_{\ell}(\beta, m)$ the image in $\text{Div}(\Gamma_{\ell} \setminus U_{\epsilon}(\ell))$ of the divisor $H(\beta, m) \in \text{Div}(X_{\Gamma})$. The corresponding $\Gamma_{\ell}$-invariant divisor in $\text{Div}(U_{\epsilon}(\ell))$ is also denoted by $H_{\ell}(\beta, m)$.

For sufficiently small $\epsilon$, the divisor $H_{\ell}(\beta, m)$ is given by the restriction to $U_{\epsilon}(\ell)$ of the sum on the right hand-side of (16). Then, only $\lambda$’s lying in the complement of $\ell$ contribute. In particular, $H_{\ell}(\beta, m)$ is non-zero, only if $\beta$ is contained in the subgroup

$$\mathcal{L} := \{\gamma \in L'/L; 2\Re \langle \gamma, \ell \rangle \equiv 0 \pmod{M_1} \text{ and } |\delta_k|\Im \langle \gamma, \ell \rangle \equiv 0 \pmod{M_2}\} \subseteq L'/L,$$
where $M_1, M_2$ are the unique integers given by $2\Re(L, \ell) = M_1\Z$ and by $|\delta_k|\Im(L, \ell) = M_2\Z$.

If $\beta \in \mathcal{L}$ the local divisor $H_\ell(\beta, m)$ can be written in the form

$$H_\ell(\beta, m) = \sum_{\kappa \in D} H_\infty(\kappa + \hat{\beta}).$$  \hspace{1cm} (17)

Here, we adopt the notation of [3] section 4, by which $\hat{\beta}$ denotes a representative of $\beta$ with $\hat{\beta} \in \mathcal{L}' \cap \ell^\perp$, fixed once and for all for every $\beta \in \mathcal{L}$. Note that a surjective homomorphism is given by

$$\pi : \mathcal{L} \longrightarrow \mathcal{D}' / \mathcal{D}, \quad \beta \mapsto \hat{\beta}_D,$$

where $\hat{\beta}_D$ denotes the definite part of $\hat{\beta}$.

### 3.2 Local Borcherds products

In this section, our aim is to use local Borcherds products to describe the position of Heegner divisors in the cohomology. Given a lattice vector $\lambda$ of negative norm with $\lambda \in \ell^\perp$ and a primitive Heegner divisor $H_\infty(\lambda)$ as in \([15]\), we can attach a product of the form

$$\prod_{\alpha \in \Sigma(\lambda)} \left[ 1 - e\left( \sigma(\delta \alpha) \left( \langle z, \lambda \rangle + \langle \ell', \ell \rangle \delta \alpha \right) \right) \right],$$

with signs $\sigma(\delta \alpha) \in \{ \pm 1 \}$ introduced to assure absolute convergence. Such products are not fully $\Gamma_\ell$-invariant. Indeed, the operation of Eichler transformations gives rise to a non-trivial automorphy factor, which can be used to determine the position of $H_\infty(\lambda)$ in the local Picard group.

We shall restrict our considerations to the following situation:

(I) We require that $\langle \ell, \ell' \rangle = \delta_k^{-1}$.

(II) We assume that $\Sigma = \Sigma(\lambda)$ is given by a fractional ideal of the form

$$\frac{1}{\delta_k} \left( a\Z \oplus \frac{b + \delta_k}{2} \Z \right)$$

with $a, b \in \Z, (a, D) = 1$.

For example, the inverse different ideal $\delta_k^{-1}$ is of the required form with $a = 1$ and $b \equiv D_k \mod 4$.

Now, the local Borcherds products are defined as follows:

**Definition 3.1.** Let $\lambda \in \mathcal{L}'$ be a negative norm lattice vector in the complement of $\ell$. The local Borcherds product $\Psi_\lambda(z)$ attached to $H_\infty(\lambda)$ is defined as

$$\Psi_\lambda(z) := \prod_{p \mod |D_k|} \prod_{q \in \Z} \left[ 1 - e\left( \sigma(q) \left( \langle z, \lambda \rangle + \frac{1}{D_k} \langle p + \bar{\zeta}_k q \rangle \right) \right) \right],$$
where
\[ \sigma(q) := \begin{cases} +1 & \text{if } q \geq 0, \\ -1 & \text{if } q < 0, \end{cases} \quad \text{and} \quad \zeta_b := \frac{b + \delta_k}{2}. \]

Note that \( \Psi_\lambda \) is invariant under translations in \( \Gamma_{\ell,T} \), while the operation of Eichler transformations, \([0,t] \) with \( t \in D_{\ell,T} \), gives rise to the (non-trivial) automorphy factor
\[ J_\lambda([h,t],z) = \frac{\Psi_\lambda([0,t]z)}{\Psi_\lambda(z)} \quad ([h,t] \in \Gamma_\ell). \]

**Proposition 3.1.** The automorphy factor \( J_\lambda \) attached to \( H_\infty(\lambda) \) takes the form
\[ J_\lambda([h,t],z) = e \left( \tilde{\zeta}_b \langle t, \lambda \rangle^2 + (t, \lambda) |D_k| \langle z, \lambda \rangle - \frac{1}{2} (t, \lambda) (1 + \tilde{\zeta}_b) \right). \]

**Proof.** Since \( \langle \ell', \ell \rangle = \delta_k^{-1} \), we have \( \langle [0,t]z, \lambda \rangle = \langle z, \lambda \rangle - \delta_k^{-1} \langle t, \lambda \rangle \). By assumption, \( \langle t, \lambda \rangle = \langle t, \lambda D \rangle \) can be written in the form \( \delta_k^{-1} (r + s\zeta_b) \), with \( r, s \) integers and \( a \mid r \).

Here, \( q \) runs over \( Z \) while \( p \) runs over a system of representatives modulo \( |D_k| \). We consider \( p - r \) modulo \( |D_k| \) and write \( q\tilde{\zeta}_b - s\zeta_b \) in the form \( (q + s)\tilde{\zeta}_b - bs \).

Thus, after permuting representatives modulo \( |D_k| \) and a shift in the index \( q \), the automorphy factor takes the form
\[ J_\lambda([h,t],z) = \prod_{p \mod |D_k|, q \in Z} \prod_{0 \leq q < s} 1 - e \left( \sigma(q - s) \left( \langle z, \lambda \rangle + D_k^{-1} (p + q\tilde{\zeta}_b) \right) \right), \quad (18) \]

Only factors with \( \sigma(q - s) \neq \sigma(q) \) contribute to the product. There are two cases: Either we have \( s > q \geq 0 \), or \( s \leq q < 0 \). We examine the first case. As in [3], section 4, we apply the elementary identity
\[ \frac{1 - e(z)}{1 - e(-z)} = -e(z). \]

We get
\[ J_\lambda([h,t],z) = \prod_{p \mod |D_k|, 0 \leq q < s} -e \left( \langle z, \lambda \rangle - \frac{1}{D_k} (p + q\tilde{\zeta}_b) \right) \]
\[ = \prod_{p \mod |D_k|} e \left( s \left( \frac{1}{2} - \langle z, \lambda \rangle - \frac{1}{D_k} \left( p + \frac{s - 1}{2} \zeta_b \right) \right) \right) \]
\[ = e \left( s|D_k| \left( \frac{1}{2} - \langle z, \lambda \rangle - \frac{1}{D_k} \left( \frac{|D_k| - 1}{2} + \frac{s - 1}{2} \zeta_b \right) \right) \right) \]
\[ = e \left( -s|D_k| \langle z, \lambda \rangle + \frac{s^2}{2} \zeta_b - \frac{s}{2} (\tilde{\zeta}_b + 1) + |D_k| s \right). \]

By definition, \( s = 2\Re(t, \lambda) \in \mathbb{Z} \). Hence, we have
\[ J_\lambda([h,t],z) = e \left( \tilde{\zeta}_b \langle t, \lambda \rangle^2 - (t, \lambda) \left( |D_k| \langle z, \lambda \rangle + \frac{1}{2} (1 + \tilde{\zeta}_b) \right) \right). \quad (19) \]
The second case ($s \leq q < 0$) can be treated similarly, yielding the same result for the automorphy factor $J_\lambda([h, t], z)$.

### 3.3 The Chern class of a primitive Heegner divisor

From the automorphy factor $J_\lambda$ we now determine a 2-cocycle representing the Chern class of the Heegner divisor $H_\infty(\lambda)$.

**Proposition 3.2.** The Chern class \( \delta(H_\infty(\lambda)) \in H^2(\Gamma_\ell, \mathbb{Z}) \) of the local Heegner divisor $H_\infty(\lambda)$ is given by the 2-cocycle

\[
[c_\lambda] : ([h, t], [h', t']) \mapsto -[\delta_k](t, \lambda)\Im(t', \lambda) + b \cdot (t, \lambda)\Re(t', \lambda).
\]

Set $F_\lambda(x, y) := \langle x, \lambda \rangle \langle y, \lambda \rangle + \langle \lambda, x \rangle \langle y, \lambda \rangle$. Then, $c_\lambda$ can be written in the form

\[
c_\lambda([h, t], [h', t']) = \Re\left( (\delta_k + b) F_\lambda(t, t') \right).
\]

Here, as usual, bilinear forms are identified with their image in the cohomology, as in section 2.

**Proof.** Let $A(g, z)$ be a holomorphic function satisfying $J_\lambda(g, z) = e(A(g, z))$ and set

\[
c(g, g') = A(gg', z) - A(g, g'z) - A(g', z) \quad \text{for all } g, g' \in \Gamma_{\ell}.
\]

Then, the 2-cocycle in question is given by the function $(g, g') \mapsto c(g, g')$, cf. [3], p. 15.

Clearly, it suffices to calculate $c(g, g')$ for Eichler transformations, thus let $g = [0, t]$, $g' = [0, t']$ with $t, t' \in D_{\ell, \Gamma}$. From (20), we can see that terms linear in $t$ cancel. We get

\[
A([0, t + t'], z) - A([0, t], [0, t']z) - A([0, t'], z) = \\
= \tilde{c}_b \left( (t + t', \lambda)^2 - (t, \lambda)^2 - (t', \lambda)^2 \right) + |D_k|(t, \lambda)\langle 0, t' \rangle z - z, \lambda) \\
= \tilde{c}_b(t, \lambda)(t', \lambda) - \frac{|D_k|}{\delta_k}(t, \lambda)\langle t', \lambda \rangle = \langle t, \lambda \rangle \left( 2\tilde{c}_b \Re\langle t', \lambda \rangle + \delta_k \langle t', \lambda \rangle \right) \\
= b(t, \lambda)\Re\langle t', \lambda \rangle - |\delta_k|(t, \lambda)\Im\langle t', \lambda \rangle,
\]

by definition of $\zeta_b$. (Note that $c(g, g') \in \mathbb{Z}$ for all $g, g' \in \Gamma_{\ell}$.) The rest of the statement follows, as

\[
F_\lambda(x, y) = \langle y, \lambda \rangle (\langle x, \lambda \rangle + \langle \lambda, x \rangle) = \langle y, \lambda \rangle(x, \lambda).
\]

### 3.4 Torsion criteria for Heegner divisors

Up to here, we have only worked on primitive Heegner divisors attached to individual lattice vectors, i.e. $H_\infty(\lambda)$, for $\lambda \in L'$ with $q(\lambda) < 0$. Next, we consider linear combinations of Heegner divisors. We will be mainly interested in the Heegner divisors $H_\ell(\beta, m)$.

For general linear combinations of primitive Heegner divisors, we have the following Lemma, where as in Proposition 3.2, $F_\lambda(x, y) = \langle x, \lambda \rangle \langle y, \lambda \rangle + \langle y, \lambda \rangle \langle \lambda, x \rangle$ for $\lambda \in L' \cap \ell^\perp$. Also, assume that for all $\lambda$, $\mathfrak{T}(\lambda)$ is of the form $a_\lambda \mathbb{Z} + \frac{1}{2} (b_\lambda + \delta_k) \mathbb{Z}$, with $a_\lambda, b_\lambda \in \mathbb{Z}$. 
Lemma 3.1. Let $H$ be a finite linear combination of Heegner divisors of the form
\[ H = \sum_{\lambda \in L' \cap \ell^\perp \{q \lambda < 0\}} a(\lambda)H_\infty(\lambda), \]
with $a(\lambda) \in \mathbb{Z}$, and denote by $[c(H)]$ the Chern class of $H$ in $H^2(\Gamma_\ell, \mathbb{Z})$. Then, $[c(H)]$ is a torsion element in $H^2(\Gamma_\ell, \mathbb{Z})$ if and only if for all $t, t' \in D_{t, \Gamma}$ the following equation holds
\[ \sum_{\lambda \in L' \cap \ell^\perp \{q \lambda < 0\}} a(\lambda) \left[ \delta_k (\delta_k + b_\lambda) F_\lambda(t, t') - D_k \frac{\langle \lambda, \lambda \rangle}{n} \langle t', t \rangle \right] = 0. \]

From the proof of this lemma we will also get the following:

Corollary 3.1. We use the notation of the Lemma. If $[c(H)]$ is a torsion element then the following equations hold
\[ \sum_{\lambda \in L' \cap \ell^\perp \{q \lambda < 0\}} a(\lambda) \text{Tr} B_\lambda = 0 \quad \text{and} \quad \sum_{\lambda \in L' \cap \ell^\perp \{q \lambda < 0\}} b_\lambda \cdot a(\lambda) \langle \lambda, \lambda \rangle = 0, \quad (21) \]
where $B_\lambda$ denotes the bilinear form $\langle x, \lambda \rangle \langle y, \lambda \rangle$ and the trace is taken over a normal basis with respect to $\langle \cdot, \cdot \rangle$.

Proof. The Chern class $[c(H)]$ is given by a linear combination of cocycles $[c_\lambda]$ in $H^2(\Gamma_\ell, \mathbb{Z})$. By Proposition 3.2, each $[c_\lambda]$ is represented by the two-cocycle
\[ (t, t') \mapsto \Re \left[ (\delta_k + b_\lambda) F_\lambda(t, t') \right]. \]

Now, $F_\lambda$ is the sum of a symmetric bilinear form, denoted $B_\lambda$, and a hermitian form, denoted $H_\lambda$. Note that $H_\lambda$ is linear in its second argument. By the results of section 2 (Lemmas 2.2, 2.3, 2.4 and 2.5), the image of these forms is zero in $H^2(\Gamma_\ell, \mathcal{O}_\ell)$, for every $\lambda$. Hence by Lemma 2.1, $[c(H)]$ is a torsion element if and only if there is a rational number $Q$, such that the equation
\[ \sum_{\lambda \in L' \cap \ell^\perp \{q \lambda < 0\}} a(\lambda) \cdot \Re \left( (\delta_k + b_\lambda) F_\lambda(t, t') \right) = Q \frac{\Im(t', t)}{\Im(\delta_k)} \]
holds for all $t, t' \in D_{t, \Gamma}$. (The order of the arguments for $\Im(\cdot, \cdot)$ used on the right hand side is a mere notational convenience.) Since $D_{t, \Gamma}$ has full rank in $W(k)$, by linear extension, the equation holds for all pairs of vectors in $W(k)$, and indeed in $W(\mathbb{C})$. Both sides of the equation are linear in $t'$. Thus replacing $t'$ with a purely imaginary multiple gives a second, equivalent equation:
\[ - \sum_{\lambda \in L' \cap \ell^\perp \{q \lambda < 0\}} a(\lambda) \cdot \Re \left( (\delta_k + b_\lambda) F_\lambda(t, t') \right) = Q \frac{\Re(t', t)}{\Im(\delta_k)} \]
By linear combination of the two equations, we get
\[ \sum_{\lambda \in L \cap \ell^\perp \left( \frac{q}{\delta_k} \right) < 0} a(\lambda) \cdot (\delta_k + b_\lambda) F_\lambda(t, t') = Q \frac{\langle t', t \rangle}{\delta_k}. \] (22)

To determine \( Q \), we take the trace of both sides of (22) over an orthogonal basis of \( W(k) \) with respect to \( \langle \cdot, \cdot \rangle \), say \( \{ e_l \}_{l=1}^n \) with \( \langle e_l, e_m \rangle = -\delta_{l,m} \). Now, \( \text{Tr} \langle \cdot, \cdot \rangle = -n \) and the trace of \( H_\lambda \) is \( -\langle \lambda, \lambda \rangle \), hence
\[ Q(-n) = \sum_{\lambda \in L \cap \ell^\perp \left( \frac{q}{\delta_k} \right) < 0} a(\lambda) \delta_k (\delta_k + b_\lambda) \left( -\langle \lambda, \lambda \rangle + \text{Tr}_{\langle \cdot, \cdot \rangle} F_\lambda \right). \] (23)

It turns out the trace of \( \Re B_\lambda \) does not contribute to \( Q \). Indeed, if we take the trace of (22) over a basis of \( W(C) \) obtained from \( \{ e_l \} \) by rescaling with \( i \), i.e. \( \{ ie_l \}_{l=1}^n \), the traces of the hermitian forms \( \langle \cdot, \cdot \rangle \) and \( H_\lambda \) remain unchanged while that of \( B_\lambda \) switches sign.

Comparing this result with (23) we obtain
\[ Q = \sum_{\lambda \in L \cap \ell^\perp \left( \frac{q}{\delta_k} \right) < 0} a(\lambda) \cdot Q_\lambda, \quad \text{with} \quad Q_\lambda = D_k \frac{\langle \lambda, \lambda \rangle}{n}. \]

Together with (22) the statement follows.

Further, since the contribution of \( B_\lambda \) to the trace vanishes and since \( Q \) being rational is, in particular, real, we get the following two necessary conditions:
\[ \sum_{\lambda \in L \cap \ell^\perp \left( \frac{q}{\delta_k} \right) < 0} a(\lambda) \text{Tr} B_\lambda = 0 \quad \text{and} \quad \sum_{\lambda \in L \cap \ell^\perp \left( \frac{q}{\delta_k} \right) < 0} b_\lambda \cdot a(\lambda) \langle \lambda, \lambda \rangle = 0. \]

This proves the corollary, as well. □

3.5 The Main Result

We can now turn to the object of our main interest, Heegner divisors of the form \( H_\ell(\beta, m) \). We want to describe their position in the local Picard group. Recall that the divisors \( H_\ell(\beta, m) \) can be written using primitive Heegner divisors by (17). Also, note that the Chern class of a primitive Heegner divisor \( H_\infty(\lambda) \) depends only on the projection \( \lambda_D \).

Thus, given a finite linear combination \( H \) of Heegner divisors \( H_\ell(\beta, m) \), we can write this as a locally finite sum of primitive Heegner divisors \( H_\infty(\lambda) \). Denote by \( \mathcal{T}(H) \) the union of \( \mathcal{T}(\lambda) \) over all \( \lambda \) occurring in this sum. Then, \( \mathcal{T}(H) \) is a fractional ideal in \( \mathfrak{d}_k^{-1} \) of the form \( \delta_k^{-1}(a \mathbb{Z} + \zeta_b \mathbb{Z}) \), with \( a, b \in \mathbb{Z}, (a, D_k) = 1 \), and where \( \zeta_b = \frac{1}{2} (b + \delta_k) \). With this notation, we formulate the following theorem.
Theorem 3.1. Consider a finite linear combination of local Heegner divisors of the form

$$H = \frac{1}{2} \sum_{\beta \in \mathcal{L}} \sum_{m \in \mathbb{Z} + q(\beta)} c(\beta, m) \mathcal{H}_\ell(\beta, m),$$

(24)

with integral coefficients $c(\beta, m)$, satisfying $c(\beta, m) = c(-\beta, m)$.

Then, $H$ is a torsion element in the local Picard group $\text{Pic}(\Gamma \backslash \mathcal{U}_\epsilon)$ if and only if for all $t, t' \in D_\ell, \Gamma$ the following equation holds

$$\sum_{\beta \in \mathcal{L}} \sum_{m \in \mathbb{Z} + q(\beta)} c(\beta, m) \sum_{\lambda \in D'} \left[ \delta_k (b + \delta_k) \mathcal{F}_\lambda(t, t') - D_k \frac{\langle \lambda, \lambda \rangle}{n} \langle t', t \rangle \right] = 0.$$  

(25)

Further, necessary conditions for this to be the case are

1. That for $B_\lambda(x, y) = \langle x, \lambda \rangle \langle y, \lambda \rangle$, we have

$$\sum_{\beta \in \mathcal{L}} \sum_{m \in \mathbb{Z} + q(\beta)} c(\beta, m) \sum_{\lambda \in D'} \langle \lambda, \lambda \rangle \langle t', t \rangle = 0,$$

(26)

where the trace is taken over a normal basis with respect to $\langle \cdot, \cdot \rangle$.

2. And, if $b \neq 0$, that

$$\sum_{\beta \in \mathcal{L}} \sum_{m \in \mathbb{Z} + q(\beta)} m \cdot c(\beta, m) \cdot \sharp \{ \lambda \in D'; \lambda + D \equiv \pi(\beta), q(\lambda) = m \} = 0.$$  

(27)

Proof. If $H$ is a torsion element, the equations follow from Lemma 3.1. Also, from Corollary 3.1 and the proof of the Lemma, it is clear that in this case, the identities (26) and (27) hold.

For the converse, assume that (25) holds for all $t, t' \in D_\ell, \Gamma$. We will show that $H$ is a torsion element in the Picard group. By linear continuation, the equation remains true for all $t, t' \in W(\mathbb{C})$.

Using (19), an automorphy factor describing $H$ in $\text{Pic}(\Gamma \backslash \mathcal{U}_\epsilon)$ is given by the following (finite) product (for $g = [h, t] \in \Gamma_\ell, z \in \mathcal{U}_\epsilon$):

$$J(g, z) = \prod_{\beta \in \mathcal{L}} \prod_{m \in \mathbb{Z} + q(\beta)} \prod_{\kappa \in D} J_{\kappa + \beta}(g, z)^{\frac{1}{2} c(\beta, m)}$$

$$\prod_{\beta, m, \kappa} e \left( \frac{\tilde{\gamma}_b}{2} (t, \kappa + \beta D)^2 - (t, \kappa + \beta D) \left( |D_k| \langle z, \kappa + \beta \rangle + \frac{\tilde{\gamma}_b + 1}{2} \right) \right)^{\frac{c(\beta, m)}{2}}.$$  

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Since \( c(\beta, m) = c(-\beta, m) \), terms which are linear in the \( \kappa + \hat{\beta}_D \) cancel. The remaining factors are of the form
\[
e\left(-|D_k|\langle z, \kappa + \hat{\beta}\rangle(t, \kappa + \hat{\beta}_D) + \frac{\bar{\zeta}_b}{2}(t, \kappa + \hat{\beta}_D)^2\right)^{\frac{c(\beta, m)}{2}}.
\]
Now, we replace \( \langle z, \kappa + \hat{\beta}\rangle = \langle z, \kappa + \hat{\beta}_D \rangle + \langle z, \hat{\beta} - \hat{\beta}_D \rangle \). Since \( \langle \hat{\beta}, \ell \rangle = 0 \), the second term depends only on the constant \( t' \)-component of \( z \). Thus, gathering all these terms gives a constant factor \( C(t) \) (depending on \( t \)). We get
\[
C(t) \cdot \prod_{\beta \in \mathcal{L}} \prod_{m \in \mathbb{Z}^+ q(\beta)q(\kappa + \beta) = m < 0} e\left(-|D_k|\langle \sigma, \kappa + \hat{\beta}_D \rangle(t, \kappa + \hat{\beta}_D) + \frac{\bar{\zeta}_b}{2}(t, \kappa + \hat{\beta}_D)^2\right)^{\frac{c(\beta, m)}{2}}.
\]
For the first term in the exponent, we apply \([25]\) with \( t' = \sigma \) and \( t = t \), while for the second term we apply \([25]\) with \( t' = t \) to the real part of \( F^\lambda \). Thus, we can rewrite each factor in the form
\[
e \left( D_k \frac{q(\kappa + \hat{\beta}_D)}{n}\left[-|D_k|\langle \sigma, t \rangle_{\delta_k(\delta_k + b)} + \bar{\zeta}_b q(t) \Re \left( \frac{1}{\delta_k(\delta_k + b)} \right) \right] \right) = e\left( D_k \frac{q(\kappa + \hat{\beta}_D)}{n}\left[ D_k \langle \sigma, t \rangle_{\delta_k(\delta_k + b)} + \bar{\zeta}_b q(t) D_k \right] \right) = e\left( \frac{q(\kappa + \hat{\beta}_D)}{n}\frac{D_k^2}{\delta_k(\delta_k + b)} \left[ \langle \sigma, t \rangle - \frac{1}{2\delta_k} q(t) \right] \right).
\]
Finally, we can write the automorphy factor \( J(g, z) \) in the following form,
\[
J(g, z) = C(t) \prod_{\beta \in \mathcal{L}} \prod_{m \in \mathbb{Z}^+ q(\beta)q(\kappa + \beta) = m < 0} e\left( \frac{q(\kappa + \hat{\beta}_D)}{n}\frac{D_k^2}{\delta_k(\delta_k + b)} \left[ \langle \sigma, t \rangle - \frac{1}{2\delta_k} q(t) \right] \right) (28)
\]
We claim that \( J(g, z) \) is, up to torsion, equivalent to a trivial automorphy factor. To see this, consider the invertible functions \( u_1(z) = e(c \tau) \) with \( c \in \mathbb{C}^* \) and \( u_2(z) = e(\langle \sigma, \mu \rangle) \) for some \( \mu \in W(k) \), which give rise to the following trivial automorphy factors:
\[
j_1([h, t], z) = \frac{u_1([h, t]z)}{u_1(z)} = e\left( c(-\langle \sigma, t \rangle + \frac{1}{2\delta_k}(t, t) + h) \right),
\]
\[
j_2([h, t], z) = \frac{u_2([h, t]z)}{u_2(z)} = e\left(-\delta_k^{-1}(t, \mu) \right).
\]
Now, as we compare this with \([28]\), we see that, in either case, \( C(t)^{-1}J(g, r) \) is equivalent to a product of appropriate rational rational powers of (finitely many) factors of the form
j_1 and j_2. It remains to consider the factor C(t). It consists of the exponential of a rational linear combination of terms of the form \langle ℓ, β⟩(t, λ) with λ = κ + β_D. Thus, we can find a vector μ ∈ W(k), such that after multiplying C(t) with an automorphy factor of type j_2 for μ, only a rational linear combination of \{|δ_k|3(t, λ)\} and \ℜ(t, λ) is left in the exponential. Thus, C(t) is equivalent to a torsion element, and the claim follows.

From this, in turn, it follows that H is a torsion element in Pic(Γ_i \setminus U_e).

**Remark 3.1.** The local Heegner divisors we consider here can be described as the restriction of local Heegner divisors on the boundary of the symmetric domain \mathcal{D} for the orthogonal group SO(V)(\mathbb{R}), which are studied in [3]: For λ ∈ D', \mathbf{H}_∞(λ) is the restriction of a primitive local Heegner divisor attached to λ and, similarly, \mathbf{H}_p(β, m) is the restriction of a composite local Heegner divisor with respect to a generic boundary component of \mathcal{D} defined by the rational two-dimensional isotropic subspace I_Q ≃ k_ℓ (compare the proof of Lemma 2.4 for the notation used here).

The relationship between Theorem 3.1 and results of [3] is most readily described in the following situation: Restrict to such Heegner divisors, for which in \mathfrak{H}(D), \zeta_0 = \frac{1}{2} δ_k.

For example, this occurs if the number field k has even discriminant \delta_k and \mathfrak{H} = \delta_k^{-1}. Then, taking the real part of both sides of (25) in Theorem 3.1, we get precisely the torsion condition from [3], Theorem 4.5. It follows that, in this case, the local Heegner divisor H from Theorem 3.1 is a torsion element in Pic(Γ_i \setminus U_e) if and only if the Heegner divisor

\[ \frac{1}{2} \sum_{β ∈ \mathcal{L}} \sum_{m ∈ \mathbb{Z} + q(β), m < 0} c(β, m)H_1(β, m), \]

is a torsion element in the local Picard group Pic(Γ_i \setminus U_e, I)), where H_1(β, m) denotes a local Heegner divisor which restricts to H_1(β, m), while the neighborhoods U_e(I) of the boundary and the Heisenberg group H_I are as in the proof of Lemma 2.4. For the definition of the local Picard group in this setting cf. [3].

**4 Application to modular forms**

In this section we study an application of Theorem 3.1 to the theory vector valued modular forms, and get a statement on modular forms as obstructions to Borcherds products. Related results, in the context of orthogonal rather than unitary groups were obtained by Bruinier and Freitag, see [3] Section 5. Indeed, we will loosely parallel their setup.

Let us briefly recall some standard facts about the Weil representation and vector valued modular forms. Denote by Mp_2(\mathbb{R}) the metaplectic cover of SL_2(\mathbb{R}). The elements of Mp_2(\mathbb{R}) can be written as pairs \((A, φ(A))\), with \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})\) and \(φ(A)\) a holomorphic function on \(\mathbb{H}\) with \(φ(τ)^2 = cr + d\). The group Mp_2(\mathbb{Z}) is defined as the inverse image of SL_2(\mathbb{Z}) under the covering map Mp_2(\mathbb{R}) → SL_2(\mathbb{R})

Consider the lattice D with the quadratic form \(q(x) = ⟨x, x⟩\), negative definite and non-degenerate, and its dual D'. There is a unitary representation \(q_D\) of Mp_2(\mathbb{Z}) on the group algebra \(\mathbb{C}[D'/D]\) (this is essentially the Weil representation attached to the
quadratic module \( D' / D \). It is defined by the action of the standard generators of \( \text{Mp}_2(\mathbb{Z}) \), \( T = \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \) and \( S = \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \), \( \sqrt{\tau} \):

\[
\varrho_D(T) \epsilon_\gamma = e(-q(\gamma)) \epsilon_\gamma, \\
\varrho_D(S) \epsilon_\gamma = \frac{\sqrt{i}^{-2n}}{\sqrt{|D'/D|}} \sum_{\delta \in D'/D} e(\langle \gamma, \delta \rangle) \epsilon_\delta,
\]

(29)

where \((\epsilon_\gamma)_{\gamma \in D'/D}\) is the standard basis for \( \mathbb{C}[D'/D] \).

For \( k \in \frac{1}{2} \mathbb{Z} \), a (vector valued) modular form of weight \( k \) with respect to \( \varrho_D \) is a holomorphic function \( f : \mathbb{H} \to \mathbb{C}[D'/D] \) with the following properties:

1. \( f(A\tau) = \phi(\tau)^{2k} \varrho_D(A, \phi) f(\tau) \) for all \( (A, \phi) \in \text{Mp}_2(\mathbb{Z}) \),
2. \( f \) is holomorphic at infinity.

The first condition (here \( A \in \text{SL}_2(\mathbb{Z}) \) acts on \( \mathbb{H} \) as usual) implies the existence of a Fourier expansion:

\[
f(\tau) = \sum_{\gamma \in \mathcal{L}' / \mathcal{L}} \sum_{n \in \mathbb{Z} - \langle q, \gamma \rangle} a(n, \gamma) e(n\tau) \epsilon_\gamma.
\]

The second condition means that all coefficients with \( n < 0 \) vanish. If \( a(n, \gamma) = 0 \) for all \( n \leq 0 \), then \( f \) is called a cusp form. We denote the space of modular forms of weight \( k \) with respect to \( \text{Mp}_2(\mathbb{Z}) \) by \( \mathcal{M}_k(\varrho_D) \) and the subspace of cusp forms by \( \mathcal{S}_k(\varrho_D) \).

In the following, set \( k = n + 2 \). Special \( \mathbb{C}[D'/D] \)-valued theta series are given by sums of the form,

\[
\Theta_p(\tau, v) = \sum_{\lambda \in D'} p(\lambda, v) e(-q(\lambda) \tau) \epsilon_\lambda,
\]

(30)

where \( v \in W \) is fixed and \( p \) a homogeneous polynomial of degree two in \( \lambda \). Further, if \( p(u, v) \) is harmonic in \( u \), then \( \Theta_p(\tau, v) \) is a cusp form contained in \( \mathcal{S}_k(\varrho_D) \). This can be seen through the usual Poisson summation argument used in the theory of theta functions, see for example [1], Theorem 4.1. Following [4], we denote by \( \mathcal{S}^\text{Sp}_{k,p}(\varrho_D) \) the space of cusp forms generated by theta-series of the type (30) with a harmonic polynomial \( p(u, v) \) for \( v \) varying through \( W(\mathbb{C}) \).

The condition in theorem 3.1 leads us to consider polynomials of the form

\[
Q_1(b; u, v, w) = \frac{1}{D_k} \Re \left[ \delta_k (\delta_k + b) F_u(v, w) - \frac{D_k}{n} \langle u, u \rangle \langle w, w \rangle \right] \\
= \Re F_u(v, w) + b \frac{\delta_k}{|\delta_k|} \Im F_u(v, w) - \frac{\langle u, u \rangle}{n} \Re \langle w, v \rangle
\]

(31)

\[
Q_2(b; u, v, w) = \frac{1}{D_k} \Im \left[ \delta_k (\delta_k + b) F_u(v, w) - \frac{D_k}{n} \langle u, u \rangle \langle w, w \rangle \right] \\
= - b \frac{\delta_k}{|\delta_k|} \Re F_u(v, w) + \Im F_u(v, w) - \frac{\langle u, u \rangle}{n} \Im \langle w, v \rangle,
\]

(32)
for \( u, v, w \in W(\mathbb{C}) \). Clearly, these polynomials are of degree two in \( u \). If \( b = 0 \), both \( Q_1(0; u, v, w) \) and \( Q_2(0; u, v, w) \) are harmonic in \( u \), for the Laplace operator of the \( 2n \)-dimensional quadratic space \( W(Q) \). Further, in this case, both polynomials can be expressed as (complex) linear combinations of a quadratic polynomial \( Q(u, v) \) given by

\[
Q(u, v) = 2(\mathbb{R}(u, v))^2 - \frac{(\bar{u}, u)}{n}(v, v),
\]

through the polarization identity. For example,

\[
Q_2(0; v, w) = -Q(u, v) - Q(u, -iw) + Q(u, v - iw).
\]

Thus, for \( b = 0 \), the complex vector spaces \( S_{\Theta, k, Q_1}(\rho_D) \) and \( S_{\Theta, k, Q_2}(\rho_D) \) are isomorphic to \( S_{\Theta, k, Q}(\rho_D) \).

For \( \gamma \in D'/D \) and \( m \in \mathbb{Z}, m < 0 \) set

\[
P_{\gamma, m}(v, w) := \sum_{\lambda \in D'/D} \left[ 2\mathbb{R}(v, \lambda)\mathbb{R}(w, \lambda) - \frac{q(\lambda)}{n}\mathbb{R}(w, v) \right] (v, w \in W(\mathbb{C})),
\]

Then, the theta-series \( \Theta_Q(\tau, v) \) can be written as

\[
\Theta_Q(\tau, v) = \sum_{\gamma \in D'/D} \sum_{m \in \mathbb{Z} + q(\gamma)} \sum_{m < 0} P_{\gamma, m}(v, v) \cdot e(-m\tau) e_{\gamma},
\]

and the theta-series with \( \gamma \)-components given by \( \sum_m P_{\gamma, m}(v_1, v_2) e(-m\tau) \) span the same space \( S_{\Theta, k, Q}(\rho_D) \), same as the \( \Theta_Q(\tau, v) \)'s. From these observations and Theorem 3.1 one can now derive a result on obstructions for torsion elements in the local Picard group.

Thus, assume that for a given linear combination of Heegner divisors \( H \) the set \( \Delta(H) \) is of the form \( \delta_k^{-1}(a\mathbb{Z} + \frac{1}{2}\delta_k\mathbb{Z}) \), i.e. that \( b = 0 \). For example, this the case if \( \Delta(H) = \delta_k^{-1} \) and \( k \) has even discriminant. Taking the real part of both sides of equation (25), we get:

\[
\sum_{\beta \in \Lambda} \sum_{m \in \mathbb{Z} + q(\beta)} c(\beta, m) \sum_{\lambda \in D'/D} \left[ \mathbb{R}(t', \lambda)(t, \lambda) - \frac{(\lambda, \lambda)}{n}\mathbb{R}(t', t) \right] = 0.
\]

Since the inner sum, \( P_{\pi(\beta), m}(t, t') \), is symmetric, it suffices to verify this condition for \( t = t' \). (Conversely, if (34) holds, (25) holds for all \( t, t' \).) We get the following theorem:

**Theorem 4.1.** Assume that for a finite linear combination of Heegner divisors

\[
H = \frac{1}{2} \sum_{\beta \in \Lambda} \sum_{m \in \mathbb{Z} + q(\beta)} c(\beta, m) H_\ell(\beta, m), \quad \text{with } c(\beta, m) = c(\beta, -m),
\]


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the ideal $\Sigma(H)$ is of the form $\delta_k^{-1}(a\mathbb{Z} + \frac{1}{2}d_k\mathbb{Z})$. Then, $H$ is a torsion element in the local Picard group $\text{Pic}(\Gamma \setminus U,)$ if and only if

$$\sum_{\beta \in \mathcal{L}} \sum_{m \in \mathbb{Z} + \langle \beta \rangle} c(\beta, m)a(\pi(\beta), -m) = 0$$

for all cusp forms $f \in S_{k,Q}^\Theta(\varrho_D)$, with Fourier coefficients $a(\gamma, m)$. (Where $k = n + 2$, and $Q(u,v)$ is the polynomial defined in (33).)

**Remark 4.1.** The polynomials $Q(u,v)$ and $P_{\gamma,s}(v,v)$ and the space $S_{k,Q}^\Theta(\varrho_D)$ generated by the attached theta-series have already been studied by Bruinier and Freitag in [3]. Indeed the above Proposition is quite analogous Proposition 5.1 from [3], according to which $S_{k,Q}^\Theta(\varrho_D)$ is the space of obstructions to finding torsion elements among the Heegner divisors in $\text{Pic}(H \setminus U, \ell)$. (See Remark 3.1)

If more generally, we consider $\Sigma(H)$ with $b \neq 0$, the polynomials $Q_1(b; \gamma, v, w)$ and $Q_2(b; \gamma, v, w)$ are harmonic in $u$ only for certain choices of $v$ and $w$. Indeed, using the harmonicity for $b = 0$, it is easily seen that for $b \neq 0$, the only case where $Q_1$ is harmonic, is if $\Im \langle w, v \rangle$ vanishes. Similarly $Q_2$, is harmonic only if $\Re \langle w, v \rangle = 0$. As a consequence our space of cusp forms thus not suffice to precisely describe obstruction for Heegner divisors as torsion elements in the local Picard group.

For example, consider the following harmonic polynomial

$$Q_1(b; \gamma, v, w) = Q(u, v) + b|\delta|^{-1}\Im [\langle v, u \rangle^2].$$

If a finite linear combination of Heegner divisors $H$ like in Theorem 3.1 is a torsion element, then (25) holds for all $t, t' \in D_t, \Gamma$, and in particular, (25) holds for the real parts with $t = t'$. Hence, we get

**Proposition 4.1.** If a finite linear combination of Heegner divisors of the form 

$$H = \frac{1}{2} \sum_{\beta \in \mathcal{L}} \sum_{m \in \mathbb{Z} + \langle \beta \rangle} c(\beta, m)H_\ell(\beta, m),$$

with $c(\beta, m) = c(\beta, -m)$, is a torsion element in the local Picard group $\text{Pic}(\Gamma \setminus U,)$, then

$$\sum_{\beta \in \mathcal{L}} \sum_{m \in \mathbb{Z} + \langle \beta \rangle} c(\beta, m)a(\pi(\beta), -m) = 0$$

for all cusp forms $f \in S_{k,R}^\Theta(\varrho_D)$ with Fourier coefficients $a(\gamma, m)$. Here $R$ denotes the polynomial $R(u,v) = Q_1(k; u, v, v)$.

**Remark 4.2.** Formally, Theorem 4.1 resembles a statement in the style of Borcherds’ paper [1], Theorem 3.1, where cusp forms of a certain type are identified as the space of obstructions for realizing Heegner divisors through Borcherds products. A version of this result for the unitary modular variety $X_\Gamma$ was proved by the author in [3], section 10.
Contrastingly, the construction presented in this paper is local in nature. It is carried out in the neighborhood of any given boundary point, corresponding to an isomorphism class of one-dimensional isotropic subspaces of $V$. (Of course, for convenience, we have only considered one fixed boundary point defined by $\ell$.)

Now say we are given a Heegner divisor $H$ of $X_\Gamma$ with the property that, for every boundary component, $H$ is trivial in the local Picard group of that boundary component. We say that $H$ is trivial at all boundary points.

It is an interesting question how far the two notions, that of being trivial at all boundary points of $H$ (locally), and that of being the divisor of a Borcherds product on $H$ are related.

In the setting of indefinite orthogonal groups studied by Bruinier and Freitag, who obtain a local result of a very similar flavour as Theorem 4.1, see [3], Proposition 5.1, it is possible, at least in some cases, to give a positive answer to the above question and go from the local results to the global statement. Indeed, here, if the definite lattice $D$, equipped with the quadratic form $q(\cdot)$, is unimodular, a result of Waldspurger from [10], may be used to show that a Heegner divisor is trivial at generic boundary points exactly if it belongs to a Borcherds product globally, see [3], Theorem 5.4.

Acknowledgements

References


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