Holomorphic Derivatives of Siegel Modular Forms

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In this paper we give a negative answer to the question of whether on the Siegel half-space of genus $g \geq 2$ a holomorphic derivative exists, following a criterion of E. Yang and L. Yin [4].[1]

1 Introduction

Let

$$P(z) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) e^{2\pi i n z}$$

be the "nearly modular" Eisenstein series of weight 2 on $\Gamma_1 := \mathrm{SL}_2(\mathbb{Z})$. Here z is in the complex upper half-plane $\mathbb{H}$ and $\sigma_1(n) = \sum_{d|n} d$. Then, it is well-known that the "Serre-derivative"

$$f \longmapsto \partial_k f := \frac{12}{2\pi i} f' - kPf$$

sends holomorphic modular forms of weight k on Γ_1 to those of weight $k + 2$. The derivative ∂_k plays an important role in the classical theory of modular forms in one variable as well as in the corresponding p-adic theory. It is therefore a very natural and important question to ask whether ∂_k has a generalization to Siegel modular forms of genus $g \geq 2$.

In their nice paper [4], E. Yang and L. Yin introduced the concept of a "modular" connection on $\mathbb{H}$ and showed that ∂_k can be obtained from the unique holomorphic Γ_1-modular connection on $\mathbb{H}$. Furthermore, they generalized the latter notion to the Siegel upper half-space $\mathbb{H}_g$ of arbitrary genus $g \geq 1$ (using the $\mathrm{Sp}_g(\mathbb{R})$-invariant metric on $\mathbb{H}_g$). Let $\Gamma \subset \Gamma_g := \mathrm{Sp}_g(\mathbb{Z})$ be a congruence subgroup. Then they showed that any symmetric matrix G of size g of smooth functions on $\mathbb{H}_g$ with values in $\mathbb{C}$ satisfying a certain transformation law under Γ gives rise to a Γ-modular connection and therefore – as follows from their theory – to a derivative sending $\mathcal{C}^\infty$-Siegel modular forms of weight k on Γ to $\mathcal{C}^\infty$-Siegel modular forms of weight $gk + 2$ on Γ.

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We will recall some of the details in section 2. The natural question then is if a holomorphic function G with the required properties exists which then would lead to a derivative preserving holomorphicity.

In this short note, we will give a negative answer to the above question. The idea of proof, roughly speaking, is the following. We will first use Witt operators (i.e. restrictions to the "diagonal") to reduce the problem to the case $g = 2$. Then, exploiting the well-known transformation formula for $\Im(Z)$ ($Z \in \mathbb{H}_2$) under Γ_2 we will arrive at a vector-valued holomorphic modular form on Γ for the symmetric square representation sym^2 of $\mathrm{GL}_2(\mathbb{C})$. However, by results proven in [2] (which basically follow from a vanishing theorem of Weissauer [3] regarding vector-valued holomorphic Siegel modular forms), such a function must be identically zero, and we obtain a contradiction. We would like to point out that for our argument the (deeper) full classification of nearly holomorphic vector-valued Siegel modular forms of genus 2 as given in [2] and in [1] for arbitary genus is not needed.

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2 Holomorphic derivatives and statement of result

Recall that

$$\mathbb{H}_g = \left\{ Z \in \mathrm{M}_g(\mathbb{C}) \mid Z' = Z, \Im(Z) > 0 \right\}$$

denotes the Siegel upper half-plane of genus g. We denote the components of $Z \in \mathbb{H}_g$ by $Z_{\mu\nu}$ ($1 \le \mu, \nu \le g$), The real symplectic group $\mathrm{Sp}_g(\mathbb{R}) \subset \mathrm{GL}_{2g}(\mathbb{R})$ operates on $\mathbb{H}_g$ in the usual manner by

$$Z \longmapsto \gamma \circ Z = (AZ + B)\,(CZ + D)^{-1}$$

where $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_g(\mathbb{R})$. We let $\Gamma_g = \mathrm{Sp}_g(\mathbb{Z})$ be the Siegel modular group and denote by $\Gamma \subset \Gamma_g$ a congruence subgroup.

Let G be a symmetric matrix of size g of $\mathcal{C}^\infty$-functions on $\mathbb{H}_g$ with values in $\mathbb{C}$ satisfying the transformation law

$$(CZ + D)^{-1}\,G\,(\gamma \circ Z) = G(Z)\,(CZ + D)' + 2C^t \qquad (Z \in \mathbb{H}_g) \tag{1}$$

for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$. Let f be a $\mathcal{C}^\infty$-Siegel modular form of weight k on Γ. Then, according to [4] the function

$$\det\left(\left[\frac{\partial}{\partial Z} - \frac{k}{2}G\right] f\right)$$

is a $\mathcal{C}^\infty$-Siegel modular form of weight $gk + 2$ on Γ. Here $\frac{\partial}{\partial Z}$ is the matrix whose components on the diagonal are $\frac{\partial}{\partial Z_{\mu\mu}}$ and those off the diagonal are $\frac{1}{2}\frac{\partial}{\partial Z_{\mu\nu}}$. Furthermore, the notation Gf means the matrix whose general component is $G_{\mu\nu}f$ ($1 \le \mu, \nu \le g$).

We shall prove

Theorem. *If $g \geq 2$, there is no* holomorphic *symmetric matrix G of size g satisfying* (1).

Remark. *As is easy to see [cf. 4, Lemma 5] if $g = 1$, the function $\frac{i\pi}{3}P$ is the unique holomorphic function G satisfying* (1). *Note that in [4] there is a typo regarding the appropriate normalization of the Eisenstein series of weight 2.*

3 Proof of Theorem

We will first reduce to the case $g = 2$, by restricting to the diagonal and using induction. This process is of course well-known, however for the reader's convenience we will carry out some details. We note that a similar reduction process is used in [1], however in the language of Fourier-Jacobi expansions.

Suppose that a holomorphic matrix G in genus $g \geq 2$ satisfying (1) exists. We will then show that such a matrix also exists in genus $g - 1$.

To be precise, decompose $Z \in \mathbb{H}_g$ as $Z = \left(\begin{smallmatrix} z & \zeta \\ \zeta' & \tau \end{smallmatrix}\right)$ with $z \in \mathbb{H}_{g-1}$, $\tau \in \mathbb{H}$, $\zeta' \in \mathbb{C}^{g-1}$ and put $\zeta' = (0, \ldots, 0)$. Accordingly, we write the matrix G in the form

$$G(Z) = \begin{pmatrix} G_1(Z) & G_2(Z) \\ G_2'(Z) & G_4(Z) \end{pmatrix},$$

with $G_4(Z)$ a $\mathcal{C}^\infty$-function on $\mathbb{H}_g$, $G_1(Z)$ a symmetric $(g-1, g-1)$-matrix of $\mathcal{C}^\infty$-functions on $\mathbb{H}_g$ and $G_2(Z)$ a column vector of such functions.

Now, we consider the action of elements $\gamma \in \Gamma_g$ of the form

$$\gamma = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} = \begin{pmatrix} A & & B & \\ & 1 & & 0 \\ C & & D & \\ & 0 & & 1 \end{pmatrix}, \quad \text{with } \delta := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{g-1},$$

and where all omitted entries are zeros. Immediately, we have:

$$\tilde{C}Z + \tilde{D} = \begin{pmatrix} Cz + D & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and}$$

$$\left(\tilde{A}Z + \tilde{B}\right)\left(\tilde{C}Z + \tilde{D}\right)^{-1} = \begin{pmatrix} (Az + B)(Cz + D)^{-1} & 0 \\ 0 & \tau \end{pmatrix}.$$

Thus, from (1), we get

$$\begin{pmatrix} (Cz + D)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} G_1(\gamma \circ Z) & G_2(\gamma \circ Z) \\ G_2(\gamma \circ Z)' & G_4(\gamma \circ Z) \end{pmatrix} =$$

$$\begin{pmatrix} G_1(Z) & G_2(Z) \\ G_2(Z)' & G_4(Z) \end{pmatrix} \begin{pmatrix} (Cz + D)' & 0 \\ 0 & 1 \end{pmatrix} + 2 \begin{pmatrix} C' & 0 \\ 0 & 0 \end{pmatrix}.$$

By inspecting the upper left blocks, we find that for any $\tau \in \mathbb{H}$, and any $z \in \mathbb{H}_{g-1}$,

$$(Cz+D)^{-1} G_1 \begin{pmatrix} \delta \circ z & \\ & \tau \end{pmatrix} = G_1 \begin{pmatrix} z & \\ & \tau \end{pmatrix} (Cz+D)^t + 2C'.$$

Since $\delta = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is an arbitrary element in Γ_{g-1}, we conclude that for fixed τ, the function $G_1 \left(\begin{smallmatrix} z & \\ & \tau \end{smallmatrix}\right)$, viewed as a (matrix-valued) function on $\mathbb{H}_{g-1}$ satisfies (1), with g replaced by $g-1$. Further, the entries of G_1 are holomorphic functions on $\mathbb{H}_{g-1}$.

By induction, we see that if a holomorphic function on $\mathbb{H}_g$ with $g \geq 2$ satisfying (1) exists, then there also must be such a function G on $\mathbb{H}_2$.

Now, recall the well-known transformation formula

$$\Im\,(\gamma \circ Z) = (CZ+D)^{-1}\,\Im\,(Z)\,(CZ+D)^{-1'} \qquad \left(z \in \mathbb{H}_2, \gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_2\right),$$

from which we immediately deduce that

$$(CZ+D)^{-1}\, i\,\Im(\gamma \circ Z)^{-1} = i\Im(Z)^{-1}\,(CZ+D)' + 2C', \tag{2}$$

see e.g. [4, Lemma 9]. Define

$$F(Z) := G(Z) - i\Im(Z)^{-1} \qquad (Z \in \mathbb{H}_2)\,.$$

Then F is a $(2,2)$-matrix of nearly holomorphic functions, as its entries are polynomials in the entries of $\Im(Z)^{-1}$ with holomorphic coefficients. Also, as follows from (1) and (2) we have

$$F(\gamma \circ Z) = (CZ+D)\,F(Z)\,(CZ+D)'$$

for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$. Thus F is a nearly holomorphic vector-valued modular form for Γ with respect to the representation sym^2 of $\mathrm{GL}_2(\mathbb{C})$. However, according to Lemma 4.20 (see also Lemma 4.21) and Remark 4.31 in [2], such a function must be identically zero, and we obtain a contradiction to our hypothesis that G is holomorphic. This finishes the proof of the theorem.

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