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Geometric Liftings for Unitary Groups

Contents

0. Introduction

Around the mid 1990s, Borcherds developed the theory of his famous Borcherds products. In [2], he gave a construction for them using a singular theta lift for the dual reductive pair $\mathrm{O}(p,2) \times \mathrm{SL}_2(\mathbb{R})$, also extending a number of previously known liftings along the way. The singular theta lift of this type allows the construction of automorphic functions with take their zeros and poles along algebraically defined divisors prescribed by the Fourier expansion of the input function, a weakly holomorphic elliptic modular form. It has been generalized in a number of ways and has found numerous applications.

For instance, Borcherds used his lifting to obtain a modularity result for a generating series of divisors, see [3]. His argument is fairly algebraic in nature, using Serre duality between different spaces of formal generating series.

At about this time, Kudla initiated what became known as the Kudla program [41]. It asserts the modularity of certain generating series of 'special cycles' in the cohomology on (integral models of) Shimura varieties for orthogonal and unitary groups. In a way, this program can be viewed as an extension and a considerable refinement of his joint work with Millson during the 1980s, where in a series of papers [44, 45, 46] they used a theta lift to show the modularity of certain generating series in the cohomology on symmetric spaces for orthogonal and unitary groups.

At the intersection of algebraic geometry and number theory, the Kudla program has yielded many key insights into both fields. Speaking in broad terms, in this program special cycles are interpreted as elements in the arithmetic Chow group of the Shimura variety, using suitable Green currents for the complex points of these cycles, see [43] for an overview. Hence, the construction of such currents is of key importance. Indeed, in [41, 42], Kudla constructed one type of Green functions for special cycles on the hermitian spaces of orthogonal groups $\mathrm{O}(p,2)$ via an exponential integral.

In [5] Bruinier extended the singular theta lift of Borcherds by taking Maass Poincaré series as inputs, and obtained a lifting into the cohomology. This lifting can then be utilized to construct a further type of Green functions [5], [11], [10].

Note that Maass Poincaré series span the space of weak harmonic Maass form, the importance of which as a space of inputs was first recognized by Bruinier and Funke, who in [10] used the singular theta lift to construct Green functions for the special divisors. They clarified the relationship between the singular theta lift of Borcherds type and the more classical theta lift of Kudla and Millson, in terms of the geometric currents which can be associated to both. The underlying structure here is the dual reductive pair $\mathrm{O}(p,q) \times \mathrm{SL}_2(\mathbb{R})$.

The singular theta lift for unitary groups $\mathrm{U}(p,1)$ was first studied in detail by the author in his doctoral thesis [33] and subsequently in [34, 35], with weakly holomorphic inputs. We note that while the construction from [33] uses an embedding $\mathrm{U}(p,1) \hookrightarrow \mathrm{O}(2p,2)$, the dual reductive pair $\mathrm{U}(p,1) \times \mathrm{U}(1,1)$ is accessible via the singular theta lift of Borcherds type, as $\mathrm{U}(1,1)$ is isomorphic to $\mathrm{SL}_2(\mathbb{R})$. The extension of this lift to weak harmonic Maass forms has been utilized to again construct Green functions for the special cycles in [14] and [15, 16].

Also, quite recently, Ehlen and Sankaran uncovered a fairly subtle relationship between the difference of the two Green functions (i.e. those of 'Kudla type' and those of 'Bruinier type'). They showed, both for $\mathrm{O}(p,2)$ and $\mathrm{U}(p,1)$, that the difference of their generating series can be interpreted as a smooth modular form, of weight $\frac{p}{2}+1$ and $p+1$, respectively.

In [9], Bruinier studied the case of the Hilbert modular group. Here, faced with the problem of the non-existence of weak harmonic Maass forms, he replaced them with 'Whittaker-forms' as input functions for the lift, again yielding Green functions for the special divisors.

In the joint work of the author with Funke [25], we carried out the construction of Green currents, actually Green functions, for the dual pair $\mathrm{U}(p,q) \times \mathrm{U}(1,1)$. On the one hand, since $\mathrm{U}(1,1) \simeq \mathrm{SL}_2(\mathbb{R})$ this case is still accessible through a singular theta lift of Borcherds type. On the other hand, the special cycles have codimension q and hence for $q > 1$ are no longer divisors.

Let us briefly mention some other results on the construction of Green forms for cycles of higher codimension in the context of the Kudla program. Since Kudla's original [41] and the work of Liu [47], further progress has been made only quite recently by Bruinier and Yang [12], who used star-products to construct Green forms for cycles of higher codimension for $\mathrm{O}(p,2)$ and $\mathrm{U}(p,1)$, and by Garcia and Sankaran [29], who utilized Quillen's theory of supercongruences. Actually, in this manner, they were able to construct Green forms for cycles in any codimension in $\mathrm{U}(p,q)$.

In the context of the Kudla program and the developments described above, the author's own work has revolved around two main focal points, the first being Borcherds products for $\mathrm{U}(p,1)$ and related constructions such as local Borcherds products, with some applications to modularity results for generating series [33, 34, 35, 37], and second, more recently, the construction of Green functions and the singular theta lift for $\mathrm{U}(p,q)$ [25].

It is the aim of the present postdoctoral thesis, on the one hand to present and summarize this work in a uniform setting and notation, on the other hand, beyond this, to present an explicit calculation of the Fourier-Jacobi expansion of the singular theta lift of a weak harmonic Maass form in any signature $\mathrm{U}(p,q)$.

Structure The main text is organized as follows, a detailed overview of the results presented in the Chapters 2,3 and 4 is given further below:

Chapter 1 provides the common setup and notation, for unitary groups and their Lie algebras, hermitian lattices and special cycles. We give a brief review of the Schrödinger model for Weil representation for the dual pair $\mathrm{U}(p,q) \times \mathrm{U}(1,1)$ and introduce the finite Weil representation via the transformation behavior of theta functions. Finally, we introduce some spaces of vector valued modular forms.

In Chapter 2, we relate the author's previous work in the setting of unitary groups $\mathrm{U}(p,1)$, from [33, 34, 35] and [37], the main focus of which is Borcherds products as well as, in [37], local Borcherds products and applications thereof.

Chapter 3 treats the joint work of the author and Funke [25], in which we construct Green functions and introduce a singular theta lift for the dual pair $\mathrm{U}(p,q) \times \mathrm{U}(1,1)$. Two types of Green functions are constructed, one through a 'singular' Schwartz form and another via the singular theta lift. We prove the analogue of the results of [10], show the modularity of the difference of the generating series, along the lines of [19], and finally consider a further kind of Green object, which we relate to the results of [53].

In Chapter 4, we explicitly calculate a form of the Fourier-Jacobi expansion for the singular theta lift of a weak harmonic Maass form, adapting a fairly recent method for the evaluation of the theta integral introduced by Kudla [40].

Appendix A contains some results from representation theory used in Chapters 3 and 4.

Appendix B gathers some useful formulas for special functions and their integral representations, and for some Fourier transforms, mainly used in Chapter 4.

Overview of results Now, we give an overview of the results presented in the main text.

Let V be a complex hermitian space with a hermitian form $(\cdot,\cdot)$ of signature (p,q). The associated symmetric domain $\mathbb{D}$ can be considered as the Grassmannian of negative definite q-planes. Let $\mathbb{F}$ be an imaginary quadratic number field, which we will view as a subfield of $\mathbb{C}$. Denote by $\mathcal{O}_\mathbb{F}$ the ring of integers and by $\mathcal{D}_\mathbb{F}^{-1}$ the inverse different ideal in $\mathbb{F}$. Let L be an even (hermitian) lattice of full rank in V, i.e. a projective module over $\mathcal{O}_\mathbb{F}$ for the which the restriction of $(\cdot,\cdot)$ is $\mathcal{O}_\mathbb{F}$-valued, and with $V = L \otimes_{\mathcal{O}_\mathbb{F}} \mathbb{C}$. Here, in the introduction, we assume L to be unimodular. Denote by Γ a finite-index subgroup in the stabilizer[1] of L in $\mathrm{U}(V)$. We define a quasi-projective variety of dimension pq by setting $X = \Gamma\backslash\mathbb{D}$.

For a vector of positive norm, define the subsymmetric space

$$\mathbb{D}(x) = \{z \in \mathbb{D};\ z \perp x\}.$$

Let Γ_x be the stabilizer of x in Γ and define the cycle $Z(x)$ as the image of $\Gamma_x\backslash\mathbb{D}(x)$ in X. Note that these cycles have codimension q. Further, for $n > 0$ set

$$Z(n) = \sum_{\substack{x \in L \\ (x,x)=n}} Z(x) \quad \in \mathrm{H}^{q,q}(X)$$

and $Z(n) = \emptyset$ for $n < 0$. Also, we let $Z(0) = c_q$, the q-th Chern form on $\mathbb{D}$. Finally, we denote by $\mathbb{D}(n)$ the preimage of $Z(n)$ in $\mathbb{D}$.

[1] In the main text, we consider finite-index subgroups of $\mathrm{Fix}(L^\sharp/L) \subset \mathrm{U}(V)$, where $L^\sharp$ is the dual lattice.

Chapter 2 In this chapter, we study the case of signature $(p,1)$. It is based on the author's work on Borcherds products [33], [35] and [34] and local Borcherds products [37].

In this signature, $\mathbb{D}$ consists of 1-dimensional negative definite lines. Let $\ell, \ell' \in L$ be two isotropic lattice vectors with $(\ell, \ell') \neq 0$. Denote by D the definite lattice $L \cap \ell^\perp \cap \ell'^\perp$ and set $W = D \otimes_{\mathcal{O}_\mathbb{F}} \mathbb{C}$. Then, the Siegel domain model for $\mathbb{D}$ is given by the set

$$\mathcal{H}_{\ell,\ell'} = \left\{ (\tau, \sigma) \in \mathbb{C} \times W;\ 2|\delta| \Im\tau |(\ell, \ell')|^2 - (\sigma, \sigma) > 0 \right\}.$$

To each element $(\tau, \sigma) \in \mathcal{H}_{\ell,\ell'}$, we associated a negative line $\mathfrak{z}$ with $\mathbb{C}\mathfrak{z} \in \mathbb{D}$ by setting $\mathfrak{z}(\tau, \sigma) = \ell' + \tau\delta(\ell', \ell)\ell + \sigma$. The isotropic line $[\ell] = \mathbb{C}\ell$ corresponds to the cusp at infinity of $\mathcal{H}_{\ell,\ell'}$. We remark that here the special cycles $\mathbb{D}(n)$ are usually called Heegner divisors.

In Section 2.5 we give a version of the main theorem on Borcherds products [see 35, Theorem 4] based on [5] see Theorem 2.29 below. The construction is realized through an embedding $\varepsilon : \mathrm{U}(p,1) \hookrightarrow \mathrm{O}(2p,2)$ and the induced embedding of $\mathcal{H}_{\ell,\ell'}$ into the symmetric domain for the orthogonal group. The embedding is developed in Section 2.3, the necessary background on orthogonal groups is provided in Section 2.2.

Here, we reproduce the following simpler version of Theorem 2.29 for unimodular latices [cf. 35, Corollary 1].

Theorem 0.1 (Corollary 2.30)**.** *Given a weakly holomorphic modular form f for $\mathrm{SL}_2(\mathbb{Z})$ of weight $1-p$, with a Fourier expansion of the form $\sum_{n \gg -\infty} a(n)\, e(n\tau)$. Assume that f has integer coefficients in its principal part. Then, there is a meromorphic function $\Psi_f(\tau, \sigma)$ on $\mathcal{H}_{\ell,\ell'}$ with the following properties*

1. *$\Psi_f(\tau, \sigma)$ is an automorphic form of weight $a(0)/2$ for $\mathrm{U}(L)$.*

2. *The zeros and poles of Ψ_f lie on divisors of the form*

$$\operatorname{div}\big(\Psi_f\big) = \frac{1}{2} \sum_{\substack{n<0 \\ a(n)\neq 0}} a(n)\, \mathbb{D}(n),$$

   with the special cycles $\mathbb{D}(n)$ introduced above.

3. *Near the cusp corresponding to ℓ, the function $\Psi_f(\tau, \sigma)$ has an absolutely converging infinite product expansion, for every Weyl chamber W, of the form*

$$\Psi_f(\tau, \sigma) = Ce\left(\frac{(\rho_f(W), \mathfrak{z})}{(\ell, \ell')} \right) \prod_{\substack{\mu \in K \\ (\mu, \varepsilon_K(W))_\mathbb{R} > 0}} \left(1 - e\left(\frac{(\mu, \mathfrak{z})}{(\ell, \ell')} \right) \right),$$

   where, as above, $\mathfrak{z} = \ell' + \tau\delta(\ell, \ell')\ell + \sigma$, C is a constant of absolute value 1 and $\rho_f(W)$ is the Weyl vector attached to W; K denotes a $\mathbb{Z}$-submodule of L. Here, K can be written in the form $K = \mathbb{Z}\zeta\ell \oplus \mathbb{Z}\ell' \oplus D$, with D positive definite. The positivity condition $(\mu, \varepsilon_K(W))_\mathbb{R} > 0$ depends on the Weyl chamber.

The Weyl chambers occurring here are connected components of $\mathbb{D}$ defined by inequalities depending on the principal part of f, and $\rho_f(W)$ is the Weyl-vector attached to f and W, see Section 2.5 for details.

Further, following [35, Sec. 9] we determine the value of Ψ_f at the cusp, i.e. in the limit $\tau \to i\infty$.

Theorem 0.2 (Theorem 2.35)**.** *Let W be a Weyl chamber such that the cusp corresponding to ℓ is contained in the closure of W (viewed as a subset of $\mathbb{D}$). If the cusp is neither a zero nor a pole of Ψ_f, then the limit $\lim_{r\to\infty} \Psi_f(ir,\sigma)$ is given by*

$$\lim_{r\to\infty} \Psi_f(ir,\sigma) = Ce\left(\overline{\rho_f(W)_\ell}\right) \prod_{\substack{\mu \in K^\sharp \\ \mu = a\kappa_{\mathbb{F}}\ell \\ a\in\mathbb{Q}_{<0}}} \left(1 - e\left(a\bar{\kappa}_{\mathbb{F}}\right)\right)^{a(\mu,0)}.$$

Finally, in Section 2.6, we give a modularity statement for generating series of Heegner divisors in the style of [3] from [35, Sec. 10].

Let $\mathrm{CH}^1(X)$ be the first Chow group of the modular variety X. Recall that $\mathrm{CH}^1(X)$ is isomorphic to the Picard group $\mathrm{Pic}(X)$. Now, let $\pi : \tilde{X} \to X$ be a desingularization and let $\mathcal{B} = \mathcal{B}(\tilde{X})$ the group of boundary divisors of $\tilde{X}$. We consider a modified Chow group, the quotient $\mathrm{CH}^1(\tilde{X})/\mathcal{B}$. Put $(\mathrm{CH}^1(\tilde{X})/\mathcal{B})_{\mathbb{Q}} = (\mathrm{CH}^1(\tilde{X})/\mathcal{B}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Denote by $\mathcal{L}_k$ the sheaf of meromorphic automorphic forms on X. By the theory of Baily-Borel, there is a positive integer $n(\Gamma)$, such that if k is a positive integer divisible by $n(\Gamma)$, the sheaf $\mathcal{L}_k$ is an algebraic line bundle and thus defines an element in $\mathrm{Pic}(X)$. The pullback of $\mathcal{L}_k$ to $\tilde{X}$ defines a class in $\mathrm{CH}^1(\tilde{X})/\mathcal{B}$, which we denote $c_1(\mathcal{L}_k)$. More generally, if k is rational, we choose an integer n such that nk is a positive integer divisible by $n(\Gamma)$ and put $c_1(\mathcal{L}_k) = \frac{1}{n} c_1(\mathcal{L}_{nk}) \in (\mathrm{CH}^1(\tilde{X})/\mathcal{B})_{\mathbb{Q}}$.

As the Heegner divisors are $\mathbb{Q}$-Cartier on X, their pullbacks define elements in the modified Chow group $(\mathrm{CH}^1(X)/\mathcal{B})_{\mathbb{Q}}$.

Theorem 0.3 (Theorem 2.37)**.** *The generating series in $\mathbb{Q}[L'/L][[q]] \otimes (\mathrm{CH}^1(\tilde{X})/\mathcal{B})_{\mathbb{Q}}$ given by*

$$A(\tau) = c_1(\mathcal{L}_{-1/2}) + \sum_{\substack{n\in\mathbb{Z} \\ n>0}} \pi^*\big(\mathbb{D}(n)\big)\, q^n$$

is a modular form in M_{1+p} with values in $(\mathrm{CH}^1(\tilde{X})/\mathcal{B})_{\mathbb{Q}}$, i.e. $A(\tau)$ is contained in $\mathrm{M}_{1+p} \otimes (\mathrm{CH}^1(\tilde{X})/\mathcal{B})_{\mathbb{Q}}$.

Local Borcherds products In Section 2.7 we review [37], where, inspired by the work of Bruinier and Freitag [7], the author used 'local' Borcherds products to describe the position of (local) Heegner divisors in the cohomology. The term *local* in this context refers to a neighborhood of the boundary component.

Denote by $X^*_{\Gamma,BB}$ the Baily-Borel compactification of X. The local Picard group $\mathrm{Pic}(X_\Gamma, \ell)$ is defined as the direct limit of the Picard groups of the regular loci of the open neighborhoods $U_\epsilon(\ell)$ of this cup,

$$\mathrm{Pic}(X_\Gamma, \ell) = \varinjlim \mathrm{Pic}\left(U_\epsilon^{reg}(\ell)\right).$$

For the definition for $U_\epsilon(\ell)$ and a brief review of the compactification theory see Section 2.1.2.

Consider a lattice vector $\lambda \in L$ of positive norm with $\lambda \perp \ell$. Then, a local Heegner divisor attached to λ is defined by setting

$$\mathbb{D}(\lambda)_\ell = \sum_{\alpha \in \mathcal{D}_{\mathbb{F}}^{-1}} \mathbb{D}(\lambda + \alpha\ell).$$

Note that Γ contains a Heisenberg group stabilizing ℓ, which acts with only finitely many orbits on the set $\{\lambda + \alpha\ell;\ \alpha \in \mathcal{D}_{\mathbb{F}}^{-1}\}$, hence this definition. A local version of the cycles $\mathbb{D}(n)$ for $n > 0$ is then given by

$$\mathbb{D}(n)_\ell = \sum_{\substack{\delta \in D \\ (\delta,\delta)=n}} \mathbb{D}(\delta)_\ell.$$

Now, the local Borcherds product $\Psi_\lambda(\mathfrak{z})$ attached to λ and $\mathbb{D}(\lambda)_\ell$ is defined as

$$\Psi_\lambda(\mathfrak{z}) = \prod_{\alpha \in \mathcal{O}_{\mathbb{F}}} \left(1 - e\left(\sigma\left(\Im\alpha\right)\left[(\lambda, \mathfrak{z}) + \frac{\alpha}{|\delta_{\mathbb{F}}|^2}\right]\right)\right),$$

where $\sigma(\Im\alpha) \in \{\pm 1\}$ is $+1$ if $\Im\alpha \geq 0$ and -1 otherwise. Note that $\Psi_\lambda(\mathfrak{z})$ is an absolutely convergent product with $\mathbb{D}(\lambda)$ as its divisor.

Denote by N the Heisenberg group in Γ. We note that N can be parameterized by pairs $[w, r]$ with $r \in \mathbb{Q}$ and w from a sublattice $D_\Gamma \subset D$. The local Borcherds product is only invariant under the action of the center of N, otherwise there is a non-trivial automorphy factor, which can be utilized to determine the Chern class of $\mathbb{D}(\lambda)_\ell$ in the local Picard group. In this manner and with a torsion criterion from local cohomology theory, one obtains the following theorem [cf. 37, Theorem 4.1])

Theorem 0.4 (Theorem 2.44). *Let* $\mathbf{D}$ *be finite linear combination of local Heegner divisors of the form*

$$\mathbf{D} = \frac{1}{2}\sum_{\substack{n \in \mathbb{Z} \\ n<0}} a(n)\mathbb{D}(-n)_\ell,$$

with integer coefficients $a(n)$*. Then* $\mathbf{D}$ *is a torsion element in the Picard group* $\operatorname{Pic}\left(H_\ell \backslash U_\epsilon(\ell)\right)$ *if and only if for all* $w, w' \in D_\Gamma$ *the equation*

$$\sum_{\substack{m \in \mathbb{Z} \\ n<0}} a(n) \sum_{\substack{\lambda \in D \\ Q(\lambda)=-n}} \left[F_\lambda(w, w') - \frac{Q(\lambda)}{p-1}(w, w')\right] = 0 \tag{0.0.1}$$

holds. Further, as a necessary condition for this to be the case, the following equation must hold for the bilinear form $B_\lambda(\cdot,\cdot) = (\lambda,\cdot)(\lambda,\cdot)$*:*

$$\sum_{h \in \mathcal{L}} \sum_{\substack{m \in \mathbb{Z}+Q(h) \\ n<0}} a(h, n) \sum_{\substack{\lambda \in D^\sharp \\ \lambda + D \equiv \pi(h) \\ Q(\lambda)=-n}} \operatorname{trace} B_\lambda = 0.$$

Finally, as an application of Theorem 0.4 one can prove an obstruction statement in terms of certain spaces of cusp forms [37, Theorem 5.1].

Theorem 0.5 (Theorem 2.45)**.** *A finite linear combination of local Heegner divisors of the form*

$$\mathbf{D} = \frac{1}{2}\sum_{\substack{n\in\mathbb{Z}\\ n<0}} a(n)\mathbb{D}(n)_\ell$$

with integer coefficients $a(n)$ is a torsion element in the local Picard group $\operatorname{Pic}(H_\ell\backslash U_\epsilon(\ell))$ *if any only if*

$$\sum_{\substack{n\in\mathbb{Z}\\ n<0}} a(n)b(-n) = 0 \tag{0.0.2}$$

for every cusp form $g = \sum_{n\in\mathbb{Z}} b(n)e(n\tau)$ in S_l^Θ, where S_l^Θ denotes a space of cusp forms spanned by certain definite theta series (see Section 2.7.4 for the precise definition).

Chapter 3 In this chapter, we follow closely the joint work of the author and Jens Funke [25]. The starting point for our considerations is the Kudla-Millson Schwartz form introduced in [44, 45, 46]

$$\varphi_{KM} \in \left[\mathcal{S}(V)\otimes\mathcal{A}^{q,q}(\mathbb{D})\right]^G,$$

which takes values in the closed differential (q,q)-forms in $\mathbb{D}$. Under the action of the Weil representation of $\mathrm{SO}(2)\subset\mathrm{SL}_2(\mathbb{R})\simeq\mathrm{SU}(1,1)$ it is an eigenfunction of weight $p+q$. Then, the associated theta series $\theta(z,\tau,\varphi_{KM})$ to L ($\tau = u+iv\in\mathbb{H}$) is a (non-holomorphic) modular form of weight $p+q$ for a congruence subgroup $\Gamma'\subseteq\mathrm{SL}_2(\mathbb{Z})$ with values in the closed differential (q,q)-forms in X. Furthermore, in the cohomology we have

$$[\theta(z,\tau,\varphi_{KM})] = \sum_{m\geq 0}[Z(m)]q^m \qquad (q = e^{2\pi i\tau}).$$

The key observation for our construction, see Section 3.3 is

Theorem 0.6 (Theorem 3.3)**.** *There exists a Schwartz form*

$$\psi \in \left[\mathcal{S}(V)\otimes\mathcal{A}^{q-1,q-1}(\mathbb{D})\right]^G$$

such that

$$\omega(L)\,\varphi_{KM} = dd^c\,\psi. \tag{0.0.3}$$

Here $\omega(L)$ is the Weil representation action of $L = \frac{1}{2}\left(\begin{smallmatrix}1 & -i\\ -i & -1\end{smallmatrix}\right)\in\mathfrak{sl}_2(\mathbb{C})\simeq\mathfrak{su}(1,1)(\mathbb{C})$ which corresponds to the Maass lowering operator $L = L_{p+q}$ for forms on the upper half plane, and d and d^c are the standard exterior derivatives acting on $\mathcal{A}^\bullet(\mathbb{D})$. Furthermore, ψ has weight $p+q-2$ under the action of $\mathrm{SO}(2)$.

Note that the solution to the equation $\omega(L)\varphi_{KM} = d\psi'$ was already constructed in [46], in fact, more generally for the dual pairs $\mathrm{O}(p,q)\times\mathrm{Sp}(n)$ and $\mathrm{U}(p,q)\times\mathrm{U}(n,n)$.

We explicitly construct ψ and establish its properties using the Fock model of the Weil representation (see Section 3.3.1), a brief review of the Fock model for the dual pair $\mathrm{U}(p,q) \times \mathrm{U}(1,1)$ is given below in Appendix A.2.2. In [25, Appendix B] the formulas for the Weil representation are developed in greater generality for the dual pair $\mathrm{U}(p,q) \times \mathrm{U}(n,n)$. We remark that similarly to [46], our form ψ can be used to solve the higher rank equations for $\mathrm{U}(p,q) \times \mathrm{U}(n,n)$, too.

We then define the *Green form of Kudla type* by setting

$$\Psi^0(x,z) := -\int_1^\infty \psi(\sqrt{t}x,z)e^{\pi t(x,x)}\frac{dt}{t}$$

for nonzero x, and then for $n \in \mathbb{Q}$ and $w > 0$ setting

$$\Xi^K(m,w)(z) := \sum_{\substack{\lambda \in L, \lambda \neq 0 \\ (\lambda,\lambda)=n}} \Psi^0(\sqrt{2w}\lambda, z),$$

which defines a $(q-1,q-1)$-form on X with singularities along the cycles $Z(n)$ for $n > 0$. For $n \leq 0$, the forms are smooth. Note that the principle of this construction and its properties for the form ψ' mentioned above were already implicit in [10] in the case $\mathrm{O}(p,2)$ and also have been outlined in [26]. Garcia and Sankaran [29] follow similar lines, too, while using supercongruences to solve (0.0.3) for $\mathrm{U}(p,q) \times \mathrm{U}(n,n)$.

On the other hand, we define a singular theta lift (of Borcherds type) using the theta series $\theta(z,\tau,\psi)$ as integral kernel. Namely, for f, a harmonic Maass form of weight $k = 2-p-q$, we set

$$\Phi(z,f) := \int_{\Gamma' \backslash \mathbb{H}}^{reg} f(z)\theta(z,\tau,\psi)d\mu(\tau).$$

Here the regularization follows the by now standard procedure introduced by Harvey and Moore [32] and Borcherds [2]. We then define for $n > 0$ the Green form of Bruinier type by

$$\mathcal{G}^B(n)(z) := \Phi(z,F_n).$$

Here $F_n(\tau)$ denotes the Maass Poincaré series of weight k which has principal part q^{-n} and 'shadow' $\xi_k(F_n) = P_n$, the holomorphic Poincaré series for Γ' of index n and weight $2-k = p+q$. Here $\xi_k = 2iv^k\overline{\frac{\partial}{\partial\bar{\tau}}} = v^{k-2}\overline{L_k}$ is the differential operator mapping forms of weight k to weight $2-k$. For $n \leq 0$, we set $\mathcal{G}^B(n)(z) = 0$. We show

Theorem 0.7 (Theorems 3.14, 3.18)**.** *The forms $\Xi^K(n,w)$ and $\mathcal{G}^B(n)$ both define Green currents for the cycle $Z(m)$. More precisely, as currents we have*

$$dd^c[\Xi^K(n,w)] + (-i)^q\delta_{Z(n)} = [\varphi^0_{KM}(n,w)],$$
$$dd^c[\mathcal{G}^B(n)] + (-i)^q\delta_{Z(m)} = [dd^c\Phi(F_n)].$$

Here $\varphi^0_{KM}(n,w) = \sum_{\lambda \in L, (\lambda,\lambda)=n} \varphi_{KM}(\sqrt{2w}x)e^{2\pi n w}$.

The proof employs the same Lie-theoretic set-up as in [10] and [26] for the orthogonal case, see Section 1.1.2 and Appendix A.1. We first consider the analogous question for $\Psi^0(x)$, and as a consequence we obtain the Green property for $\Xi^K(n,w)$. We then show that $\mathcal{G}^B(n)$ has the same singularities as $\Xi^K(n,w)$, and thus yields the same residue.

We can identify the term $dd^c\Phi(z,f)$ in the previous theorem explicitly as follows:

Theorem 0.8 (Theorem 3.19)**.** *Let f be a harmonic weak Maass form for Γ' of weight $k = 2-p-q$ with holomorphic constant term a_0^+, and let $\xi_k(f)$ be its shadow, a cusp form of weight $p+q$. Then*

$$dd^c\Phi(z,f) = \left(\Theta(\cdot,z,\varphi_{KM}),\xi_k(f)\right)_{p+q} + a^+(0,0)c_q$$

as differential (q,q)-forms on X. Here $(\alpha,\beta)_\ell$ denotes the Petersson inner product in weight ℓ. In particular, $dd^c\Phi(z,f)$ extends to a smooth closed (q,q)-form of moderate growth and $dd^c\Phi(z,f) = a^+(0,0)c_q$ for f weakly holomorphic.

This result is the analogue of the main result in [10], and the proof is fairly similar. It can be viewed as an adjointness result between the Kudla-Millson lift and the singular theta lift associated to ψ.

Following ideas of Ehlen and Sankaran [19] we then compare the two Green forms in a different way. We show

Theorem 0.9 (Theorem 3.23)**.** *Assume $p+q>2$. Then for each $z \in \mathbb{D}$, the generating series*

$$F(\tau) = -\log(v)\psi(0)(z) - \sum_{n\in\mathbb{Q}} \left(\Xi^K(n,v) - \mathcal{G}^B(n)\right)(z)\, q^n$$

transforms like a smooth modular form of weight $p+q$. In addition, F is orthogonal to cusp forms and satisfies $L_{p+q}F(\tau) = -\theta(\tau,\psi)$.

Finally, we define a different Green object $\mathcal{G}^B_s(n)(z)$ $(n>0)$ depending on a complex parameter s. It is given essentially[2] as $\Phi(F_n(s),z)$, where $F_n(\tau,s)$ is the Hejhal Poincaré series of weight k with complex parameter s (at $s = s_0 = 1-k/2$ this is the weak Maass form F_n introduced above). We show

Theorem 0.10 (Theoremen 3.29, Corollary 3.32)**.** *Let Δ be the Laplace operator acting on differential forms on X. Then*

$$\Delta\mathcal{G}^B_s(n) = \left((2s-1)^2 - (2s_0-1)^2\right)\mathcal{G}^B_s(n).$$

Furthermore, $\mathcal{G}^B_s(n)$ agrees (up to a multiplicative constant) with the Green form constructed for $n>0$ by Oda and Tsuzuki [53].

[2] Due to slightly different regularization process $\mathcal{G}^B_{s_0}(n)(z)$ differs from $\Phi(z,F_n)$ by a smooth form.

Chapter 4 In this chapter we calculate an explicit form of the Fourier-Jacobi expansion for the lift $\Phi(z,f)$ for a harmonic Maass form f of weight $k = 2-p-q$. We adapt a method of evaluating the theta integral introduced by Kudla [40] and work in the mixed model of the Weil representation, for the setup of which see Appendix A.2.1.

Let ℓ, ℓ' be isotropic lattice vectors with $(\ell,\ell') = 1$. For a vector $x \in V$ write x in the form $x = \alpha\ell + x_0 + \beta\ell'$, with $x_0 \in W$. To pass to the mixed model, one has to carry out a partial Fourier transform in the variable α. We denote the new variable by β' and set $\eta = [\beta, \beta']$.

Now, the main idea for the evaluation of the regularized integral is that by invariance under $\mathrm{SL}_2(\mathbb{Z})$ the regularized integral can be decomposed by systems of representatives of $\mathrm{SL}_2(\mathbb{Z})$-orbits of η (viewed as a rational 2×2-matrix). Thus, we set

$$\Phi(z,f,\psi) = \sum_{i=0}^{2} \Phi_i(z,f,\psi), \quad \text{with}$$

$$\Phi_i(z,f,\psi) := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{reg} \sum_{\substack{\eta=[\beta,\beta']/\sim \\ \mathrm{rank}(\eta)=i}} \sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})_\eta\backslash\mathrm{SL}_2(\mathbb{Z})} \left\langle f(\gamma\tau), \overline{\theta_\eta(\gamma\tau,z)}\right\rangle_L v^{-2} du\, dv.$$

Moreover, due to rapid decay of the integrand, the integrals can be evaluated for each term separately, with fixed η, and summed up later, as

$$\Phi_i(z,f,\psi) =: \sum_{\substack{\eta=[\beta,\beta']/\sim \\ \mathrm{rank}(\eta)=i}} \sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})_\eta\backslash\mathrm{SL}_2(\mathbb{Z})} \sum_{n\gg-\infty} \left(\hat{a}^+(n)\phi_i(n,\eta)^+ + \hat{a}^-(n)\phi_i(n,\eta)^-\right),$$

for $i = 0,1,2$. Here, $\hat{a}^\pm(n)$ denotes the Fourier coefficients of f in the mixed model, with $\hat{a}^+(n)$ from the holomorphic part f^+ and $\hat{a}^-(n)$ from the non-holomorphic part f^-. The terms $\phi_i(n,\eta)^+$ and $\phi_i(n,\eta)^-$ are given by

$$\phi_i(n,\eta)^+ = \int_{\mathrm{SL}_2(\mathbb{Z})_\eta\backslash\mathbb{H}}^{reg} e^{2\pi i n u}\theta_\eta(\tau,z)d\mu(\tau), \quad \phi_i(0,\eta)^- = \int_{\mathrm{SL}_2(\mathbb{Z})_\eta\backslash\mathbb{H}}^{reg} \theta_\eta(\tau,z)v^{1-k}d\mu(\tau),$$

$$\text{and} \quad \phi_i(n,\eta)^- = \int_{\mathrm{SL}_2(\mathbb{Z})_\eta\backslash\mathbb{H}}^{reg} \Gamma(1-k,4\pi|n|)e^{2\pi i n u}\theta_\eta(\tau,z)d\mu(\tau) \qquad (n\neq 0).$$

The domain of integration is the upper half plane $\mathbb{H}$ for $i=2$, is given by $\mathrm{SL}_2(\mathbb{Z})_\infty\backslash\mathbb{H}$ for $i=1$ and is the usual fundamental domain $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ for $i=0$.

We remark that in general, we will not determine the 'rank 0' (i.e. $i=0$) term, which is given by the convolution integral of an indefinite theta series and contributes only an additive constant to the lift. However, it falls out in signature $(1,q)$ (an example we treat prominently, see Example 4.12 and Corollary 4.16) and in signature $(p,1)$, where the theta series is definite, it can be worked out the methods of Borcherds [2], [see 40].

To facilitate calculation somewhat we first evaluate all terms at the base point $z_0 \in \mathbb{D}$, the results are given in Theorem 4.9. Then, we apply the group action of $G = \mathrm{U}(V)$ to obtain the Fourier-Jacobi expansion, see Theorem 4.14. As coordinates we use the group elements of G in terms of the the Levi decomposition $G = NAM$, where $M \simeq \mathrm{SU}(p-1, q-1)$, $A \simeq \mathrm{GL}([\ell])$ and N is the Heisenberg group.

Since the notation is very involved, rather than reproducing Theorems 4.9 and 4.14 here, we will just consider a (simple) example:

If the signature of V is $(p, 1)$, the Schwartz form ψ is essentially the Gaussian, as $\psi = 2i\varphi_0$. In the mixed model and using the coordinates introduced above, it is given by

$$\widehat{\psi}_{p,1}(\sqrt{2}(\eta, x_0), \tau) = 2i \exp\left(-\frac{2\pi}{v}\left[|\beta' + \bar{\tau}\beta|^2 + 2v\Im\left(\beta'\bar{\beta}\right)\right]\right) e^{2\pi i \tau (x_0, x_0)} \otimes 1.$$

Thus, the resulting contributions to the lift of a weak harmonic Maass form take a fairly simple form, for example one has for fixed η and n

$$\phi_2(n, \eta)^+(z_0) = \frac{\sqrt{2}}{2}|\beta|{\mathbf{B}_\eta}^{-\frac{1}{2}} \exp\left(-\frac{2\pi}{|\beta|^2}|\mathbf{A}_\eta|\left(\tfrac{1}{2}\mathbf{B}_\eta\right)^{\frac{1}{2}}\right) e^{-2\pi i \mathbf{C}_\eta((x_0, x_0)+n)},$$

where $\mathbf{A}_\eta, \mathbf{B}_\eta$ and $\mathbf{C}$ are defined as

$$\mathbf{A}_\eta(n, x_0) = n + 2|\beta|^2 + (x_0, x_0), \quad \mathbf{C}_\eta = \frac{\mathrm{Re}(\beta'\bar{\beta})}{|\beta|^2} \quad \text{and} \quad \mathbf{B}_\eta = 2\Im\left(\beta'\bar{\beta}\right)^2.$$

The other contributions $\phi_2(n, \eta)^-$ and $\phi_1(n, \eta)^\pm$ can be found in Example 3.5.

For the Fourier-Jacobi expansion, we use the following coordinates. For $z \in \mathbb{D}$, let $g(z) \in G$ be an element with $g(z)z_0 = z$. And let $g(z) = n(w, r)a(t)\mu$ with $n(w, r) \in N$ ($w \in W, r \in \mathbb{R}$), $a(t) \in A$ ($t > 0$) and $\mu \in M$ be the decomposition of $g(z)$. Using w, t and $\tau_\ell := r + it^2$ as coordinates (and omitting μ), the lift of a weak harmonic Maass form $f \in \mathrm{H}_k^+$ with Fourier coefficients $a(n)^\pm$ takes the following form (see Corollary 4.15):

$$\frac{1}{2i}\Phi(z, f, \psi) = c_0(t, w) + \sum_{\kappa \in \mathbb{Q}^\times} c_\kappa(t, w) e^{2\pi i \kappa \operatorname{Re} \tau_\ell},$$

where the constant term $c_0(t, w)$ is given by

$$\begin{aligned} c_0(t, w) = 4\pi I_0 + t^2 \sum_{\beta' = (a,b)} \Bigg[& a^-(0)\frac{1}{(2\pi t^2|\beta'|)^{p+1}}\Gamma(p+1) + \\ & \sum_{n \neq 0}\Bigg(a^+(\lambda, n)\frac{1}{2\pi t^2|\beta'|^2} + a^-(n)(p-1)!\sum_{r=0}^{p-1}\frac{\pi^r}{r!}(4|n|)^{\frac{r}{2}-\frac{1}{4}}t^{r-\frac{1}{2}}|\beta'|^{r-\frac{1}{2}} \\ & \cdot h_r\left(\frac{1}{4\pi t|\beta'||n|^{\frac{1}{2}}}\right) e^{-4\pi t|\beta'||n|^{\frac{1}{2}}}\Bigg)\Bigg] \cdot e\Big(-\operatorname{Re}\left(\beta'(x_0, w)\right)\Big), \end{aligned}$$

with a rational constant I_0, which can be evaluated using the methods of [2], see [40]. The coefficients $c_\kappa(t, w)$ ($\kappa > 0$) take the form

$$c_\kappa(t, w) = \sum_{a,b}\sum_m A_\kappa(n, [\beta, \beta'])(t, w),$$

wherein

$$A_\kappa(n,[\beta,\beta'])(t,w) = \left(a^+(n)\frac{\sqrt{2}}{2}\frac{t|\beta|}{\mathbf{B}_\eta^{\frac{1}{2}}} + a^-(n)A_n^-(\eta;t,w)\right)$$
$$\cdot \exp\left(-2\frac{\pi}{|\beta|^2}|\mathbf{A}_{t\eta}(x_0-\beta w)|\left(\tfrac{1}{2}\mathbf{B}_\eta\right)^{\frac{1}{2}} - 2\pi i\big[\mathbf{C}_\eta\left((x_0,x_0)+n\right) + 2\alpha\Im(x_0,w)\big]\right),$$

with a term $A_n^-(\eta;t,w)$ from the contribution of the non-holomorphic part f^-, given by

$$\sqrt{2}(p-1)!\sum_{r=0}^{p-1}\frac{(4|n|\pi)^r}{r!}t^{2r+1+2}|\beta|\mathbf{B}_\eta^{\frac{r-1}{2}}\left(\tfrac{1}{2}\mathbf{A}_{t\eta}^2(x_0-\beta w) - 2t^2|\beta|^2\left(2|n|-n\right)\right)^{-\frac{r}{2}}$$
$$\cdot h_{\max\{0,r-1\}}\left(\frac{|\beta|^2}{2\pi}\left(\left(\tfrac{1}{2}\mathbf{A}_{t\eta}(x_0-\beta w) + 2t^2|\beta|^2(2|n|-n)\right)\mathbf{B}_\eta\right)^{-\frac{1}{2}}\right).$$

We remark that for $f \in \mathrm{M}_k^!$, like in [40], it is possible to obtain another form of the Borcherds product expansion from this Fourier-Jacobi expansion.

The singular theta lift for $\mathrm{U}(p,q)\times\mathrm{U}(1,1)$ offers many possibilities for future research. For example, it should be of considerable interest, both in its own right and in view of applications in the Kudla program, to consider suitable integrals of the singular theta lift $\Phi(z,f)$, say, along the lines of [42] or [11].

Also, it should be quite interesting to analyze the behavior of the singular theta lift at the boundary components of suitable toroidal compactifications of X, in terms of the Fourier-Jacobi expansion.

We hope to come back to these questions in the near future.

Finally, I would like to thank the many colleagues how have encouraged and inspired me over the years. And especially, I would like to thank Jan Bruinier, Stephan Ehlen and my coauthor Jens Funke for many helpful and interesting discussions.

1. Setup

In the present chapter, we establish the setup and notation used. We introduce the unitary group over an indefinite complex hermitian space V, its symmetric space and its Lie algebra, following [25]. Further, we describe the setup of the Schrödinger model of the Weil representation and the construction of theta functions. Through their transformation behavior, we define the finite Weil representation ρ_L associated to a hermitian lattice $L \subset V$. We then review the definitions of some spaces of vector valued modular forms transforming under ρ_L and their properties. In the last section of this chapter we introduce a regularized pairing and define regularized theta integrals, through the standard regularization procedure used for singular theta lifts of Borcherds type.

1.1. The unitary group

1.1.1. The symmetric space

In the following let $(V, (\cdot,\cdot))$ be a complex space of dimension m with a non-degenerate indefinite hermitian form $(\cdot,\cdot)$ of signature (p,q), with $p, q \geq 1$. We assume that $(\cdot,\cdot)$ is $\mathbb{C}$-linear in the second argument and conjugate linear in the first.

We pick standard orthogonal basis elements $v_1, \ldots, v_m$ with $(v_\alpha, v_\alpha) = 1$ for $\alpha = 1, \ldots, p$ and $(v_\mu, v_\mu) = -1$ for $\mu = p+1, \ldots, m$, respectively[1]. We let z_α and z_μ denote the corresponding coordinate functions, so that for $x \in V$,

$$x = \sum_\alpha z_\alpha v_\alpha + \sum_\mu z_\mu v_\mu, \quad \text{and} \quad (x,x) = \sum_{\alpha=1}^{p} |z_\alpha|^2 + \sum_{\mu=p+1}^{m} |z_\mu|^2.$$

This choice of basis also gives rise to an orthogonal decomposition $V = V_+ \oplus V_-$ into definite subspaces.

Further, we will sometimes consider V as a real quadratic space together with the bilinear form $(\cdot,\cdot)_\mathbb{R} = \mathrm{Re}(\cdot,\cdot) = \frac{1}{2}\,\mathrm{trace}_{\mathbb{C}/\mathbb{R}}(\cdot,\cdot)$. We denote this space by $V_\mathbb{R}$. Then, an orthogonal basis of $V_\mathbb{R}$ is formed by $\{v_\alpha, iv_\alpha, v_\mu, iv_\mu\}_{\alpha,\mu}$.

We let $G = \mathrm{U}(V)$ be the unitary group of V and let $\mathbb{D} = G/\mathcal{K}$ be the associated symmetric space of complex dimension pq. Here $\mathcal{K} \simeq \mathrm{U}(p) \times \mathrm{U}(q)$ is the maximal compact subgroup corresponding to the basis of V chosen above. We realize the symmetric space as the Grassmannian of negative definite q-planes in V:

$$\mathbb{D} \simeq \{z \subset V : \dim(z) = q,\ (\cdot,\cdot)|_z < 0\},$$

[1] Throughout, we follow [46] in using 'early' Greek letters to denote indices ranging from 1 to p and 'late' Greek letters for indices from $p+1$ to m.

and fix $z_0 := \operatorname{span}_{\mathbb{C}}\{z_\mu;\, \mu = p+1, \dots, m\}$ as the base-point of $\mathbb{D}$.

Naturally, isotropic subspaces correspond to to boundary components of $\mathbb{D}$. Thus beside the standard basis $\{v_j\}_{j=1,\dots,m}$ introduced above we will also use basis containing an isotropic vector, denoted by ℓ and a second vector ℓ' with $(\ell, \ell') \neq 0$. Unless stated otherwise, ℓ' is assumed to be isotropic, too.

Usually we take $\operatorname{span}_{\mathbb{C}}\{\ell, \ell'\} = \operatorname{span}_{\mathbb{C}}\{v_1, v_m\}$, so that the set $\{\ell, v_2, \dots, v_{m-1}, \ell'\}$ is a basis of V. For example, we can set

$$\ell = \frac{v_1 + v_m}{\sqrt{2}} \quad \text{and} \quad \ell' = \frac{v_1 - v_m}{\sqrt{2}}.$$

Given $z \in \mathbb{D}$, the standard majorant $(x, x)_z$ is given by

$$(x, x)_z = (x_{z^\perp}, x_{z^\perp}) - (x_z, x_z),$$

where $x = x_z + x_{z^\perp}$, using the orthogonal decomposition $V = z \oplus z^\perp$. We also set

$$R(x, z) := -(x_z, x_z).$$

Note that $R(x, z) \geq 0$ with $R(x, z) = 0$ if and only if $x \in z^\perp$. In particular for $z = z_0$ we have the decomposition $z_0 \oplus z_0^\perp = V_- \oplus V_+$, hence $R(x, z_0) = -(x_-, x_-)$, with $x_- \in V_-$.

When x has positive norm, let $\mathbb{D}(x)$ denote the codimension q sub-Grassmannian

$$\mathbb{D}(x) := \{z \in \mathbb{D} :\, z \perp x\} = \{z \in \mathbb{D} :\, R(x, z) = 0\}. \tag{1.1.1}$$

Also, for convenience, if x is non-positive, set $\mathbb{D}(x) = \emptyset$.

Hermitian lattices Let $\mathbb{F} = \mathbb{Q}\left(\sqrt{D_{\mathbb{F}}}\right)$ be an imaginary quadratic number field with discriminant $D_{\mathbb{F}}$, for which we fix of $\mathbb{F}$ into $\mathbb{C}$. We denote by $\mathcal{O}_{\mathbb{F}}$ the ring of integers and let $\delta_{\mathbb{F}}$ be the square of the discriminant, with the principal branch of the complex square root function. Further, denote by $\mathcal{D}_{\mathbb{F}}^{-1}$ the inverse different ideal, given by $\delta_{\mathbb{F}}^{-1}\mathcal{O}_{\mathbb{F}}$. Finally, we denote by $\kappa_{\mathbb{F}}$ a generator of $\mathcal{O}_{\mathbb{F}}$ as a $\mathbb{Z}$-module with $\Im \kappa_{\mathbb{F}} = \frac{1}{2}\delta_{\mathbb{F}}$, e.g. $\kappa_{\mathbb{F}} = \frac{1}{2}(D_{\mathbb{F}} + \delta_{\mathbb{F}})$.

Let $L \subset V$ be an even hermitian lattice, i.e. a projective module over the ring of integers $\mathcal{O}_{\mathbb{F}}$ of $\mathbb{F}$, on which the restriction of $(\cdot, \cdot)$ is $\mathcal{O}_{\mathbb{F}}$-valued. If L has full rank, i.e. if $V = L \otimes_{\mathcal{O}_{\mathbb{F}}} \mathbb{C}$, we denote by $V_{\mathbb{F}}$ the space $L \otimes_{\mathcal{O}_{\mathbb{F}}} \mathbb{F}$, understood as a hermitian space with the restriction of $(\cdot, \cdot)$. The dual lattice $L^\sharp$ is given by

$$L^\sharp = \{x \in V; (x, \lambda) \in \mathcal{D}_{\mathbb{F}}^{-1},\, \forall \lambda \in L\} = \{x \in V; \operatorname{trace}_{\mathbb{F}/\mathbb{Q}}(x, \lambda) \in \mathbb{Z},\, \forall \lambda \in L\}.$$

Note that $L \subset L^\sharp$, hence L is integral. The quotient $L^\sharp/L$ is called the discriminant group of L.

Remark 1.1. Beside $L^\sharp$, more precisely called the $\mathbb{Z}$-dual of L, one can also introduce an $\mathcal{O}_{\mathbb{F}}$-dual $L^\sharp_{\mathcal{O}_{\mathbb{F}}}$, defined as

$$L^\sharp_{\mathcal{O}_{\mathbb{F}}} := \{x \in V; (x, \lambda) \in \mathcal{O}_{\mathbb{F}}\}.$$

Then, $L^\sharp = \delta_{\mathbb{F}}^{-1} L^\sharp_{\mathcal{O}_{\mathbb{F}}}$ (cf. [14, Section 4.1]).

We also note that since L is even, $(\lambda, \lambda) \in \mathbb{Z}$ for all $\lambda \in L$. Thus, L can be considered as an even, integral lattice over $\mathbb{Z}$, too. Indeed, equipped with the quadratic form $Q(\lambda) := (\lambda, \lambda)$, the $\mathbb{Z}$-module L is an even and integral lattice contained in the rational space $L \otimes_{\mathbb{Z}} \mathbb{Q} = V_{\mathbb{Q}}$, which of course is isomorphic to $V_{\mathbb{F}}$, considered as a vector space over $\mathbb{Q}$. Hence, we extend this quadratic form to V by setting $Q(x) = (x, x) = (x, x)_{\mathbb{R}}$ for $x \in V$. However, note that $Q(\cdot)$ is more commonly associated to a bilinear form[2] which coincides with $\mathrm{trace}_{\mathbb{C}/\mathbb{R}}(\cdot, \cdot) = 2(\cdot, \cdot)_{\mathbb{R}}$ (see [33, 34, 35, 37]).

For $n \in \mathbb{Q}$ and $h \in L^\sharp/L$, we define the special cycle $\mathbb{D}(n, h)$ in $\mathbb{D}$ by

$$\mathbb{D}(n, h) = \sum_{\substack{\lambda \in L+h \\ (\lambda,\lambda)=n}} \mathbb{D}(\lambda). \tag{1.1.2}$$

Note that $\mathbb{D}(n, h)$ is locally finite. We let $\Gamma_L = \mathrm{Fix}(L^\#/L) \subset G$ and write

$$X = \Gamma_L \backslash \mathbb{D}$$

for the resulting quasi-projective variety. Further, we let $Z(x)$ and $Z(n, h)$ be the image in X of $\mathbb{D}(\lambda)$ and $\mathbb{D}(n, h)$, respectively.

1.1.2. The unitary Lie algebra

We let $\mathfrak{g}_0 = \mathfrak{u}(V)$ be the Lie algebra of G. We define the $\mathbb{R}$-linear surjective map

$$\phi_V : \bigwedge\nolimits^2_{\mathbb{R}} V \longrightarrow \mathfrak{u}(V)$$

by

$$\phi_V(v \wedge \tilde{v})(x) = (v, x)\tilde{v} - (\tilde{v}, x)v.$$

Note that we have

$$\phi_V(iv \wedge \tilde{v}) = \phi_V(v \wedge -i\tilde{v}).$$

In the following we will abuse notation and drop ϕ_V and just write $v \wedge \tilde{v} \in \mathfrak{u}(V)$. Note that in this way we realize $\mathfrak{u}(V)$ as a quotient of $\bigwedge^2_{\mathbb{R}} V$ by the relation $iv \wedge \tilde{v} + v \wedge i\tilde{v} = 0$. We have

$$\mathfrak{g}_0 = \mathrm{span}_{\mathbb{R}}\{v_r \wedge v_s, iv_r \wedge v_s\}.$$

We put

$$X_{rs} = v_r \wedge v_s \qquad \text{and} \qquad Y_{rs} = iv_r \wedge v_s.$$

In the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ with $\mathfrak{k}_0 = \mathrm{Lie}(K) = \mathfrak{u}(p) \times \mathfrak{u}(q)$, we note that

$$\mathfrak{p}_0 = \mathrm{span}_{\mathbb{R}}\{X_{\alpha\mu}, Y_{\alpha\mu}; 1 \leq \alpha \leq p, p+1 \leq \mu \leq m\}.$$

We let $\{\omega_{\alpha\mu}, \omega'_{\beta\nu}\}$ be the corresponding dual basis for $\mathfrak{p}_0^*$. Furthermore, the natural complex structure on $\mathfrak{p}_0$ is given by $X_{\alpha\mu} \mapsto Y_{\alpha\mu}$, $Y_{\alpha\mu} \mapsto -X_{\alpha\mu}$.

[2] Denoting this form by $B(x, y)$. One sets $Q(x) = \frac{1}{2}B(x, x)$, see for example [5]. In this case, one has the polarization identity $B(x, y) = Q(x + y) - Q(x) - Q(y)$.

We let $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ be the complexification of $\mathfrak{g}_0$, which we view as a right $\mathbb{C}$-vector space. We define

$$Z'_{rs} = \frac{1}{2}(X_{rs} - Y_{rs}i) \qquad \text{and} \qquad Z''_{rs} = \frac{1}{2}(X_{rs} + Y_{rs}i).$$

Note that $Z''_{rs} = -Z'_{sr}$. In the Harish-Chandra decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-,$$

we see that

$$\mathfrak{k} = \operatorname{span}_{\mathbb{C}}\{Z'_{\alpha\beta}, Z'_{\mu\nu}\}, \qquad \mathfrak{p}^+ = \operatorname{span}_{\mathbb{C}}\{Z'_{\alpha\mu}\}, \qquad \mathfrak{p}^- = \operatorname{span}_{\mathbb{C}}\{Z''_{\alpha\mu}\}.$$

We let $\{\xi'_{\alpha\mu}\}$ and $\{\xi''_{\alpha\mu}\}$ be the corresponding dual basis of $\mathfrak{p}^+$ and $\mathfrak{p}^-$.

We let $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. We view $V_{\mathbb{C}}$ as a *right* complex vector space of dimension $2m$ and hence write vi for $v \otimes i$. Note that iv (internal multiplication of the left $\mathbb{C}$-vector space V) is not equal to vi. We decompose $V_{\mathbb{C}} = V' \oplus V''$ into the $+i$ and $-i$ eigenspaces under left multiplication by i. The maps

$$v \longmapsto v - ivi \qquad \text{and} \qquad v \longmapsto v + ivi$$

realize a $\mathbb{C}$-linear isomorphism of (the left $\mathbb{C}$-vector space) V with (the right $\mathbb{C}$-vector space) V' and a $\mathbb{C}$-anti-linear isomorphism with V and V''. Hence we can view $V'' \simeq V^*$ as $\mathbb{C}$-vector spaces. We denote the natural bases of V' and V'' by

$$v'_r := v_r - iv_ri \qquad \text{and} \qquad v''_r := v_r + iv_ri,$$

respectively. Furthermore, we obtain decompositions $V' = V'_+ \oplus V'_-$ and $V'' = V''_+ \oplus V''_-$ in the natural way. We have

$$Z'_{rs}(v'_t) = -(v_s, v_t)v'_r \qquad \text{and} \qquad Z'_{rs}(v''_t) = (v_r, v_t)v''_s,$$

and we note that this realizes the isomorphism $\mathfrak{g} \simeq \mathfrak{gl}_m(\mathbb{C})$ by the action of $\mathfrak{g}$ on V'. More precisely, we obtain

$$\mathfrak{k} \simeq \operatorname{Hom}(V'_+, V'_+) \oplus \operatorname{Hom}(V'_-, V'_-), \qquad \mathfrak{p}^+ \simeq \operatorname{Hom}(V'_-, V'_+), \qquad \mathfrak{p}^- \simeq \operatorname{Hom}(V'_+, V'_-).$$

Correspondingly, the action of $\mathfrak{g}$ on V'' realizes the dual of the standard representation of $\mathfrak{g}$.

Recall how we consider V as a real quadratic space $(V_{\mathbb{R}}, (\cdot,\cdot)_{\mathbb{R}})$, with an orthogonal basis given by $\{v_\alpha, iv_\alpha, v_\mu, iv_\mu\}_{\alpha,\mu}$. We let $\mathfrak{o}_{V_{\mathbb{R}}}$ be the Lie algebra of the orthogonal group $\mathrm{O}(V_{\mathbb{R}})$. We now have the isomorphism

$$\phi_{V_{\mathbb{R}}} : \bigwedge^2 V_{\mathbb{R}} \simeq \mathfrak{o}(V_{\mathbb{R}})$$

given by

$$\phi_{V_{\mathbb{R}}}(v \wedge \tilde{v})(x) = (v, x)_{\mathbb{R}}\tilde{v} - (\tilde{v}, x)_{\mathbb{R}}v.$$

We let $\iota : \mathfrak{g}_0 = \mathfrak{u}(V) \mapsto \mathfrak{o}(V_{\mathbb{R}})$ be the natural embedding. We easily see

$$\iota(\phi_V(v \wedge \tilde{v})) = \phi_{V_{\mathbb{R}}}(v \wedge \tilde{v}) + \phi_{V_{\mathbb{R}}}(iv \wedge i\tilde{v}).$$

Note this realizes $\mathfrak{u}(V)$ as the subspace of $\bigwedge^2_{\mathbb{R}} V$ which is fixed by (left)-multiplication with i in both factors.

1.2. Schwartz forms and Weil representation

Let $\mathcal{S}(V)$ be the Schwartz space of V. We now describe the setup of the Weil representation for the dual reductive pair $\mathrm{U}(1,1) \times \mathrm{U}(V)$ in the Schrödinger model, acting on $\mathcal{S}(V)$. We denote by ψ an additive character of $\mathbb{R}$ and by (ω, ψ) the associated Weil representation. Recall that all such additive characters are given by $\psi_\alpha(t) = e(\alpha t)$ with $\alpha \in \mathbb{R}$, where $e(t) = e^{2\pi i t}$, as usual.

Let $\phi \in \mathcal{S}(V)$ be Schwartz form. The unitary group $G = \mathrm{U}(V)$ acts linearly, through

$$\omega(g)\phi(x) = \phi(g^{-1}x).$$

The action of $G' = \mathrm{U}(1,1) \simeq \mathrm{SL}_2(\mathbb{R})$ is given as follows.

$$\begin{aligned} \omega\left((n(b))\right)\phi(x) &= \psi_\alpha\left(\tfrac{1}{2}b(x,x)\right)\phi(x) && \text{for} \quad n(b) = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right), \\ \omega\left(m(a)\right)\phi(x) &= a^m\phi(ax) && \text{for} \quad m(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \quad \text{with } a > 0, \\ \omega\left(S\right) &= i^{p-q}\hat{\phi}(x) && \text{for} \quad S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \end{aligned} \tag{1.2.1}$$

where $\hat{\phi}(x) = \alpha^m \int_V \phi(y) e\left(-(x,y)_{\mathbb{R}}\right) dy$ denotes the Fourier transform of $\phi(x)$. Here, we identify V and $\mathbb{R}^{2m}$ and dy is the usual Lebesgue measure.

Note that for $\alpha > 0$ the representations (ω, ψ_α) are all isomorphic. Explicitly, the intertwiner between (ω, ψ_1) and (ω, ψ_α) is given by $\phi(x) \mapsto \phi\left(\sqrt{\alpha}x\right)$.

For the time being, let $\alpha = 1$, so that the additive character is given by $t \mapsto \psi_1 = e(t)$. We say that ϕ has weight $r \in \mathbb{Z}$ if $\omega(k'_\theta)\phi = e^{ri\theta}\phi$ for $k'_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$ in $K' = \mathrm{U}(1) \simeq \mathrm{SO}_2(\mathbb{R})$. Now, the standard Gaussian

$$\varphi_0(x,z) := e^{-\pi(x,x)_z}$$

has weight $r = p - q$. Sometimes, we will write $\varphi_0^{p,q}$ for φ_0 to emphasize dependence on the signature.

For $\tau = u + iv \in \mathbb{H}$, let g'_τ be an element of $\mathrm{SL}_2(\mathbb{R})$ mapping i to τ, e.g. $g'_\tau = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)\begin{pmatrix} v^{\frac{1}{2}} & 0 \\ 0 & v^{-\frac{1}{2}} \end{pmatrix}$. Then, for a Schwartz form ϕ of weight r, we set

$$\phi(x,\tau) := v^{-\frac{r}{2}}\omega\left(g'_\tau\right)\phi(x) = v^{-\frac{r}{2}+\frac{p+q}{2}}\phi^0(\sqrt{v}x)e^{\pi i(x,x)\tau},$$

where we have set $\phi^0(x) = e^{\pi(x,x)}\phi(x)$.

Remark 1.2. We note that using ψ_1 as an additive character here and the definition of φ_0 allow us to maintain consistency with [25] and ultimately with the convention used by Kudla and Millson in their construction of Schwartz forms in [45].

However, in dealing with theta functions, this will cause some technical difficulties, as we shall see presently. In this context, the additive character ψ_2, $t \mapsto e(2t)$ presents a better choice. Hence, we will use ψ_1 as an additive character in local calculations and switch to ψ_2 when dealing with global objects like theta functions.

1.2.1. Theta functions and the finite Weil representation

Let L be an even hermitian lattice with the hermitan form $(\cdot,\cdot)$. For $\phi \in \mathcal{S}(V)$, and $h \in L^\sharp/L$, we define a theta function

$$\theta(g', z, \phi, h) := \sum_{\lambda \in h+L} \omega(g', \psi_\alpha)\phi(\lambda, z).$$

Further, if ϕ has weight $r \in \mathbb{Z}$, we get a function on the upper half-plane by setting

$$\theta(z, \tau, \phi, h) = v^{\frac{r}{2}} \sum_{\lambda \in h+L} \omega(g'_\tau, \psi_\alpha)\phi(\lambda, z) = \sum_{\lambda \in h+L} \phi(\lambda, \tau, z).$$

However, since (λ,λ) is an integer, but not necessarily even, for $\alpha = 1$, the function $\phi(z,\tau,\lambda) = v^{\frac{1}{2}(-r+p+q)}\phi^0(\sqrt{v}\lambda, z)e^{\pi i(\lambda,\lambda)\tau}$ does not give rise to a q-expansion. Hence, we introduce a factor of $\sqrt{2}$, which amounts to switching from $\omega = (\omega, \psi_1)$ to (ω, ψ_α), with $\alpha = 2$. Thus, we get the theta function

$$\theta(z, \tau, \phi)_h := \sum_{\lambda \in h+L} \phi(\sqrt{2}\lambda, \tau, z),$$

which under the operation of the generators S and $T = n(1)$ of $\mathrm{SL}_2(\mathbb{Z}) = \Gamma'$ transforms as

$$\begin{aligned} \omega(T)\,\theta(\tau, z, \phi)_h &= e\left((h,h)\right)\theta(\tau, z, \phi)_h, \\ \omega(S)\,\theta(\tau, z, \phi)_h &= \frac{i^{p-q}}{\sqrt{|L^\sharp/L|}} \sum_{\mu \in L^\sharp/L} e\left(-2(\mu, h)_{\mathbb{R}}\right)\theta(\tau, z, \phi)_\mu. \end{aligned} \tag{1.2.2}$$

This can be used to define a representation of Γ', the finite Weil representation ρ_L, through which Γ' acts on the group algebra $\mathbb{C}[L^\sharp/L]$ (cf. [10]). We denote by $\{\mathfrak{e}_h\}_{h \in L^\sharp/L}$ the standard basis elements of $\mathbb{C}[L^\sharp/L]$. Now, introducing the vector

$$\Theta(\tau, z; \phi)_L = \left(\theta(\tau, z, \phi)_h\right)_{h \in L^\sharp/L} = \sum_{h \in L^\sharp/L} \theta(\tau, z, \phi)_h \mathfrak{e}_h,$$

we may define ρ_L by setting

$$\Theta(\gamma\tau, z; \phi)_L = \rho_L(\gamma)\Theta(\tau, z; \phi)_L \qquad \text{for all} \quad \gamma \in \Gamma'.$$

Hence, by (1.2.2) the generators of Γ' act as follows

$$\rho_L(T)\mathfrak{e}_h = e\left((h,h)\right)\mathfrak{e}_h, \quad \rho_L(S)\mathfrak{e}_h = \frac{i^{p-q}}{\sqrt{|L^\sharp/L|}} \sum_{\mu \in L^\sharp/L} e\left(-2(\mu,h)_{\mathbb{R}}\right)\mathfrak{e}_\mu.$$

We denote by $\rho_L^\vee \simeq \bar{\rho}_L$ the dual representation.

Remark. We note that ρ_L is is essentially the finite Weil representation associated with the quadratic module $(L, Q(\cdot))$, since the factor $\sqrt{2}$ introduced above amounts to replacing the bilinear form $(\cdot,\cdot)_{\mathbb{R}}$ with $2(\cdot,\cdot)_{\mathbb{R}}$.

Now, let L^- be the same $\mathcal{O}_\mathbb{F}$ module as L but with the hermitian form $-(\cdot,\cdot)$. Similarly to L, there is a finite Weil representation ρ_{L^-} through which Γ' acts on the group algebra of L^-. Note that $\rho_{L^-} \simeq \rho_L^\vee$.

Finally, we introduce the hermitian pairing on $\mathbb{C}[L^\sharp/L]$

$$\langle\, ,\, \rangle_L : \mathbb{C}[L^\sharp/L] \times \mathbb{C}[L^\sharp/L] \to \mathbb{C}, \quad \text{by setting } \langle \mathfrak{e}_\mu, \mathfrak{e}_\nu \rangle_L = \delta_{\mu,\nu} \quad (\mu, \nu \in L^\sharp/L).$$

A similar definition is made for the lattice L^-.

Example 1.3. For the standard Gaussian, φ_0, the theta series is given by

$$\Theta(\tau, z; \varphi_0)_L = \sum_{h \in L^\sharp/L} \sum_{\lambda \in L+h} \varphi_0(\lambda, \tau, z)\mathfrak{e}_h,$$

with

$$\varphi_0(\lambda, \tau, z) = v^q \exp\left(2\pi i \left((\lambda,\lambda)u + (\lambda,\lambda)_z iv\right)\right) = v^q e\left((\lambda_{z^\perp}, \lambda_{z^\perp})\tau + (\lambda_z, \lambda_z)\bar{\tau}\right).$$

Essentially, this is the Siegel theta function for the lattice L; we will discuss this example in somewhat more detail in the next chapter, see Section 2.4.

Remark 1.4. Consider $\mathrm{O}(V_\mathbb{R})$, the orthogonal group of the quadratic space $V_\mathbb{R}$. For the dual reductive pair $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{O}(V_\mathbb{R})$ the Schrödinger model for the Weil representation is isomorphic to (ω, ψ_1), and the finite Weil representation of the lattice L (as a $\mathbb{Z}$-module with the quadratic form $Q(\cdot)$) is isomorphic to ρ_L.

Somewhat more generally, if V is a real quadratic space of signature (p, q), there is a representation of the metaplectic double cover $\mathrm{Mp}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$, from which, if M is an even lattice in V, one obtains a finite Weil representation of the preimage of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{Mp}_2(\mathbb{R})$, acting on $\mathbb{C}[M^\sharp/M]$. For both representations, the action of the generators is similar to ω and ρ_L, except that in (1.2.1) and (1.2.2), p and q are replaced by $\frac{p}{2}$ and $\frac{q}{2}$, see [10] and [5] for details.

1.3. Some spaces of vector valued modular forms

For $k \in \mathbb{Z}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, define the weight k slash-operation on functions $\mathbb{C}[L^\sharp/L] \to \mathbb{C}$ as

$$f\mid_{k,L} \gamma = (c\tau + d)^{-k} \rho_L(\gamma)^{-1} f(\gamma\tau).$$

The slash-operation for the dual representation $\rho_L^\vee \simeq \rho_{L^-}$ is defined similarly.

The following definitions for vector-valued modular forms transforming under ρ_L are well-known.

Definition 1.5. For $k \in \mathbb{Z}$, let $\mathrm{S}_{k,L}$, $\mathrm{M}_{k,L}$ and $\mathrm{M}^!_{k,L}$ be the spaces of holomorphic functions $f : \mathbb{H} \to \mathbb{C}[L^\sharp/L]$ which satisfy

1. $f\mid_{k,L} (\gamma) = f$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

2. If f is in $\mathrm{M}^!_{k,L}$, then f is meromorphic at the cusp ∞, while if $f \in \mathrm{M}_{k,L}$, it is holomorphic at the cusp. Finally if $f \in \mathrm{S}_{k,L}$ it vanishes at ∞.

Clearly, $\mathrm{S}_{k,L} \subset \mathrm{M}_{k,L} \subset \mathrm{M}^!_{k,L}$. The elements of these spaces are called cusp forms, (holomorphic) modular forms and weakly holomorphic modular forms, respectively.

As an example for cusp forms, we have the holomorphic Poincaré series.

Definition 1.6. Given $h \in L^\sharp/L$ and $n \in \mathbb{Z} + Q(h)$ with $n > 0$. Let $\kappa = (p+q) - 2$. A holomorphic Poincaré series $P_{n,h} \in \mathrm{S}_{\kappa,L}$ of index (n,h) is defined as follows [cf. 5, Section 1.2.1]

$$P_{n,h} := \sum_{A \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} e(n\tau)\mathfrak{e}_h \mid_{L,\kappa} A.$$

Next, following [10], we introduce harmonic weak Maass form.

Definition 1.7 ([see 10, Section 3])**.** For $k \in \mathbb{Z}$, let $\mathrm{H}_{k,L}$ be the space of twice continuously differentiable functions $f : \mathbb{H} \to \mathbb{C}[L^\sharp/L]$, which satisfy

1. $f \mid_{k,L} (\gamma) = f$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.
2. There exists a constant $C > 0$ such that $f(\tau) = O(e^{Cv})$ as $v \to \infty$.
3. $\Delta_k f = 0$.

The elements of $\mathrm{H}_{k,L}$ are called harmonic weak Maass forms. Any such form f has a decomposition $f(\tau) = f^+(\tau) + f^-(\tau)$ into a holomorphic and a non-holomorphic part, where the Fourier expansion of the holomorphic part is

$$f^+(\tau) = \sum_{h \in L^\sharp/L} \sum_{n \in \mathbb{Q}} a^+(h,n) e(n\tau)\, \mathfrak{e}_h,$$

whilst that of the non-holomorphic part is

$$f^-(\tau) = \sum_{h \in L^\sharp/L} \Big(a^-(h,0) v^{1-k} + \sum_{\substack{n \in \mathbb{Q} \\ n \neq 0}} a^-(h,n) \Gamma(1-k, 4\pi n v) e(nu) \Big)\, \mathfrak{e}_h.$$

We denote by $P(f)$ the principal part of f, i.e. the Fourier polynomial

$$P(f)(\tau) = P(f^+)(\tau) = \sum_{h \in L^\sharp/L} \sum_{\substack{n \in \mathbb{Q} \\ n < 0}} a^+(h,n) e(n\tau)\mathfrak{e}_h. \tag{1.3.1}$$

Note that $\mathrm{H}_{k,L}$ contains the spaces of weakly holomorphic modular forms $\mathrm{M}^!_{k,L}$ and holomorphic modular forms $\mathrm{M}_{k,L}$, with $\mathrm{H}_{k,L} \supset \mathrm{M}^!_{k,L} \supset \mathrm{M}_{k,L}$.

The Maass differential operators on smooth functions $\mathbb{H} \to \mathbb{C}[L^\sharp/L]$ are defined as

$$R_k = 2i \frac{\partial}{\partial \tau} + \frac{k}{v}, \qquad L_k = 2iv^2 \frac{\partial}{\partial \bar{\tau}}. \tag{1.3.2}$$

For any smooth function $f : \mathbb{H} \to \mathbb{C}[L^\sharp/L]$ and $\gamma' \in \mathrm{SL}_2(\mathbb{Z})$ one has

$$\begin{aligned}(R_k f) \mid_{k+2} \gamma &= R_k\big(f \mid_k \gamma\big),\\ (L_k f) \mid_{k-2} \gamma &= L_k\big(f \mid_k \gamma\big).\end{aligned}$$

Hence, R_k and L_k are called the Maass raising and lowering operator, respectively.

Closely related to L_k, the operator ξ_k introduced in [10] affords an antilinear mapping given by

$$\xi_k : \mathrm{H}_{k,L} \longrightarrow \mathrm{M}^!_{2-k,L^-}, \qquad f(\tau) \longmapsto 2iv^k \overline{\frac{\partial f(\tau)}{\partial \bar\tau}} = v^{k-2}\overline{L_k f(\tau)}. \tag{1.3.3}$$

Now, the space $\mathrm{H}^+_{k,L}$ is defined as the inverse image of the cusp forms S_{2-k,L^-}. It follows immediately from this definition that for $f \in \mathrm{H}^+_{k,L}$,

$$f(\tau) - P(f)(\tau) = \mathbf{O}(e^{-Cv}),$$

as $v \to \infty$ for some constant $C > 0$. Further, by [10, Corollary 3.8], there are exact sequences

$$0 \longrightarrow \mathrm{M}^!_{k,L} \longrightarrow \mathrm{H}_{k,L} \xrightarrow{\xi_k} \mathrm{M}^!_{2-k,L^-} \longrightarrow 0,$$

and

$$0 \longrightarrow \mathrm{M}^!_{k,L} \longrightarrow \mathrm{H}^+_{k,L} \xrightarrow{\xi_k} \mathrm{S}_{2-k,L^-} \longrightarrow 0.$$

Remark 1.8. For a harmonic weak Maass form $f \in \mathrm{H}^+_{k,L^-}$, the image under ξ_k is sometimes called its 'shadow', while its holomorphic part f^+ is considered as a 'mock-modular form'. This terminology comes from the Zwegers' seminal thesis [59].

We now introduce a pairing between the spaces M_{2-k,L^-} and $\mathrm{H}^+_{k,L}$ as follows. For $g \in \mathrm{M}_{2-k,L^-}$ with Fourier expansion $g = \sum_{h,n} b(h,n)e(n\tau)\mathfrak{e}_h$ and $f \in \mathrm{H}^+_{k,L}$ with $f^+ = \sum_{h,n} a^+(h,n)e(n\tau)$, put (see [10, (3.15) on p. 62])

$$\begin{aligned}\{g,f\}' :&= \sum_{h\in L^\sharp/L}\sum_{n<0} a^+(h,n)b(h,-n)\\ &= (h, \xi_k(f))_{2-k,L} - \sum_{h\in L^\sharp/L} a^+(h,0)b(h,0).\end{aligned}$$

Note that the induced pairing between $\mathrm{H}^+_{k,L}/\mathrm{M}^!_{k,L}$ and S_{2-k,L^-} is non-degenerate [see 10, Corollary 3.9].

Remark 1.9. For cusp forms, the pairing $\{\cdot,\cdot\}'$ is essentially the residue pairing, see Section 2.6 below.

For the following result, see [5, Theorem 1.17], also [2, Theorem 10.3].

Proposition 1.10. *A Fourier polynomial in* $\mathbb{C}[q^{-1}] \otimes \mathbb{C}[L^\sharp/L]$

$$\sum_{h \in L^\sharp/L} \sum_{\substack{n \in \mathbb{Z}+Q(h) \\ n<0}} a(h,n) e(n\tau) \mathfrak{e}_h,$$

with $a(h,n) = (-1)^{k+(q-p)} a(-h,n)$*, is the principal part of a weakly holomorphic modular form* $f \in \mathrm{M}^!_{k,L}$ *if and only if the functional*

$$\mathrm{S}_{2-k,L^-} \ni g = \sum_{\substack{h,n \\ n>0}} b(h,n) e(n\tau) \longmapsto \sum_{h \in L^\sharp/L} \sum_{\substack{n \in \mathbb{Z}+Q(h) \\ n<0}} a(h,n) b(h,-n)$$

is zero on S_{2-k,L^-}*.*

Further, by [10, Proposition 3.11], for any such Fourier polynomial Q, there is weak harmonic Maass form $f \in \mathrm{H}^+_{k,L}$ with principal part $P(f) = Q + c$ with some T-invariant constant $c \in \mathbb{C}[L^\sharp/L]$. (The proof uses Proposition 1.10 and the non-degeneracy of the pairing $\{\cdot,\cdot\}'$.)

Following Ehlen and Sankaran [19], we generalize the setup by introducing two further spaces of modular forms, A^{mod}_{k,L^-} and $\mathrm{A}^!_{k,L^-}$. For the former space, we use the following, slightly modified definition from [4, Definition 3.2]:

Definition 1.11. Let $\mathrm{A}^{mod}_k(\rho_L^\vee) = \mathrm{A}^{mod}_{k,L^-}$ denote the space of $\mathcal{C}^\infty$-functions $f : \mathbb{H} \to \mathbb{C}[L^\sharp/L]$ satisfying

1. $f \mid_{k,L^-} (\gamma) = f$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

2. For all $a, b \in \mathbb{Z}_{\geq 0}$, there is an $r \in \mathbb{Z}$ such that $\frac{\partial^a}{\partial^a u} \frac{\partial^b}{\partial^b v} f(\tau) = \mathbf{O}(v^r)$ as $v \to \infty$.

3. If $f = \sum_{m \in \mathbb{Q}} c(m,v) e(m\tau)$ denotes the Fourier expansion of f, then the integral
$$\int_1^\infty c(0,t) t^{-2-s} dv,$$
has a meromorphic continuation to a half-plane $\mathrm{Re}(s) > -\epsilon$ for some $\epsilon > 0$. (The integral converges for sufficiently large $\mathrm{Re}(s) \gg 0$, since by 2., f is of polynomial growth as $v \to \infty$.)

Definition 1.12 ([see 19, Definition 2.8])**.** Let $\mathrm{A}^!_k(\rho_L^\vee) = \mathrm{A}^!_{k,L^-}$ denote the space of $\mathcal{C}^\infty$-functions $f : \mathbb{H} \to \mathbb{C}[L^\sharp/L]^\vee$ satisfying

1. $f \mid_{k,L^-} (\gamma) = f$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

2. There exists a constant $C > 0$ such that $f(\tau) = \mathbf{O}(e^{Cv})$ as $v \to \infty$.

3. $L_k(f) \in \mathrm{A}^{mod}_{k-2}(\rho_L^\vee)$.

Hejhal Poincaré series We now introduce non-holomorphic Poincaré series of negative weight, which span the space of weakly harmonic Maass forms, a good reference is [5, Chapter 1.2].

Definition 1.13. Let k be a negative integer. The Hejhal Poincaré series, also known as Maass Poincaré series[3], of weight k and index (n,h) with $h \in L^\sharp/L$ and $n \in \mathbb{Z}$ is defined as follows, for $\tau \in \mathbb{H}$, $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$:

$$F_{n,h}(\tau,s) := \frac{1}{4\Gamma(2s)} \sum_{A \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \mathcal{M}_s(4\pi|n|v) e^{2\pi i n u} \mathfrak{e}_h \mid_{k,L^-} A, \tag{1.3.4}$$

where $\mathcal{M}_s(t) = t^{-\frac{k}{2}} M_{-\frac{k}{2}, s-\frac{1}{2}}(t)$, with the usual M-Whittaker function $M_{\nu,\mu}(t)$ [see 56, Chap. 13]. Note that this definition differs from [5, Definition 1.8] by a factor of $\frac{1}{2}$.

Set $s_0 = 1 - \frac{k}{2}$. For fixed $s = s_0$ the Poincaré series $F_{n,h}(\tau, s_0)$ are weak Maass forms $F_{n,h}(\tau)$. They have principal parts

$$e(n\tau)\mathfrak{e}_h + (-1)^{k+(p-q)} e(n\tau)\mathfrak{e}_{-h}$$

and span the space H^+_{k,L^-}, see [5, Proposition 1.12] and [10, Remark 3.10].

For example, let $f \in \mathrm{M}^!_{k,L^-}$ be a weakly holomorphic modular form of negative weight k with Fourier coefficients $a(h,n)$ $(h \in L^\sharp/L, n \in \mathbb{Z} - Q(h))$. Then,

$$f(\tau) = \sum_{h \in L^\sharp/L} \sum_{\substack{n \in \mathbb{Z}-Q(h) \\ n<0}} a(h,n) F_{n,h}(\tau, s_0). \tag{1.3.5}$$

Further, note that the 'shadow' of $F_{n,h}(\tau)$, i.e. the image of $F_{n,h}(\tau, s_0)$ under the operator ξ_k is given by (see [10])

$$\xi_k(F_{n,h}) = P_{-n,h},$$

where $P_{-n,h}$ is a holomorphic, cuspidal Poincaré series of index $(-n,h)$ and weight $2-k$ as defined above, see Definition 1.6.

1.4. Regularized integrals

We now introduce a regularized version of the Petersson pairing using the by now standard regularization procedure originally introduced by Harvey and Moore [32].

The (regularized) Petersson pairing For two modular forms, $f, g \in \mathrm{M}_{k,L}$, of which at least one is a cusp form, the weight k Petersson pairing is defined as usual,

$$\langle f, g \rangle_{L,k} = \int_{\mathcal{F}} \langle f, g \rangle_L v^k d\mu,$$

[3] We will use both names interchangeably.

where $\mathcal{F} \subset \mathbb{H}$ is a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ and $d\mu = v^{-2}dudv$. Similarly, for two modular forms $f \in \mathrm{M}_{k,L^-}$ and $g \in \mathrm{M}_{-k,L}$ we define the Petersson pairing

$$\langle f, g\rangle_{L^-} = \int_{\mathcal{F}} \langle f, \bar{g}\rangle_L \, d\mu,$$

whenever the integral converges absolutely.

More generally, we introduce a regularized pairing as follows. Denote by $\mathcal{F}_t$, for $t \in \mathbb{R}_{>0}$ the truncated fundamental domain given by

$$\mathcal{F}_t := \left\{\tau = u + iv; |\tau| > 1, -\tfrac{1}{2} < u < \tfrac{1}{2}, 0 < v \leq t\right\}.$$

For $f \in \mathrm{H}^+_{k,L^-}$ be a weak harmonic Maass form, and g transforming as a modular form of weight κ under ρ_L, for the regularized pairing of f and g, we set

$$\begin{aligned}(f, g)^{reg}_{L^-} &:= \int^{reg}_{\mathrm{SL}_2(\mathbb{Z})\mathbb{H}} \langle f, \bar{g}\rangle_L \, d\mu \\ &:= \underset{s=0}{\mathrm{CT}} \left[\lim_{t\to\infty} \int_{\mathcal{F}_t} \langle f, \bar{g}\rangle_L \, v^{-s} \, d\mu\right],\end{aligned} \tag{1.4.1}$$

where the notation $\mathrm{CT}_{s=0}$ denotes the constant term at $s = 0$ of the meromorphic continuation of the limit[4] We say that *the pairing exists* if for sufficiently large $\mathrm{Re}(s)$ the limit $t \to \infty$ defines a holomorphic function in s for which a meromorphic continuation to some $\mathrm{Re}(s) < 0$ exists, so that the constant term of the Laurent expansion around $s = 0$ can be evaluated.

Regularized theta integrals Given a theta-function $\Theta(\tau, z) = \Theta(\tau, z; \phi)_L$, with a Schwartz form ϕ of weight $r \in \mathbb{Z}$, and a weak harmonic Maass form $f \in \mathrm{H}^+_{-r,L^-}$, of weight $-r$, the regularized pairing

$$(f, \Theta(\cdot, z))^{reg}_{L^-} = \underset{s=0}{\mathrm{CT}} \left[\lim_{t\to\infty} \int_{\mathcal{F}_t} \left\langle f(\tau), \overline{\Theta(\tau, z)}\right\rangle_L v^{-s} \, d\mu\right], \tag{1.4.2}$$

is called a *regularized theta integral.*

Remark 1.14. If in the Fourier expansion of f the term $a^+(0,0)$ is zero, in (1.4.1) a somewhat simpler regularization procedure can be used: simply taking the limit $t \to \infty$ for the integrand with $s = 0$.

Regularized integrals of this type can be used to realize singular theta lifts (of Borcherds type), see [32, 2, 5, 10]. In the following we will be studying such theta lifts and further related topics. In Chapter 2 we will treat the case where the signature of V is $(p, 1)$, in which case $V_\mathbb{R}$ is a quadratic space of signature $(2p, 2)$. In this setting, consider the Gaussian $\varphi_0^{p,1}$, which gives rise to the Siegel theta function from Example 1.3. Now, for $V_\mathbb{R}$ and the reductive pair $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{O}(V_\mathbb{R})$ (see Remark 1.4) we have essentially the

[4] If 0 happens to be a pole, a slight variation of this recipe is required, see [5].

same Siegel theta function, and this kind of theta function[5] was used by Borcherds in his seminal construction of Borcherds products in [2, Sec. 13]. One can exploit this fact to obtain the theta lift and, in particular, the Borcherds products by using an embedding between the symmetric domain $\mathbb{D}$ of $\mathrm{U}(V)$ and the symmetric domain of the orthogonal group $\mathrm{O}(V_{\mathbb{R}})$, see [33, 35]. We will explore all this in Chapter 2.

In Chapter 3 we will carry out the construction of a regularized theta lift in arbitary signature (p,q) using a Schwartz form ψ which coincides with $\varphi_0^{p,1}$ in the special case of signature $(p,1)$. The construction of ψ and the theta-lift $\Theta(\tau,z;\psi)_L$ is joint work of the author and Jens Funke [25] and is based on previous work by Kudla and Millson, who, somewhat implicitly, constructed ψ in [46], and also on work of Bruinier and Funke [10], who carried out a similar construction for orthogonal groups, i.e. for the dual reductive pair $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{O}(p,q)$.

Finally, in Chapter 4, we will explicitly calculate the lift of a weak harmonic Maass form using the regularized theta integral against $\Theta(\tau,z;\psi)_L$ and determine its Fourier-Jacobi expansion, up to an additive constant.

[5] Of course Borcherds' construction of a theta lift in [2] is much more general, and covers a considerable numbers of previously known liftings for indefinite orthogonal groups. However, Borcherds products require the symmetric domain of the orthogonal group to bear a hermitian structure, hence restricting to the case where the quadratic space has signature $(p,2)$.

2. The case of signature $(p, 1)$ and Borcherds products

In the present chapter, we focus on the case where the signature (p, q) of V is $(p, 1)$ with $p \geq 1$. References for this case are [33, 35, 37]. As in Chapter 1, denote by $\mathbb{D}$ the symmetric domain for the operation of $G = \mathrm{U}(V)$ given by $\{z \subset V;\ \dim z = q,\ v \text{ neg. definite}\}$. In the present case, $\mathbb{D}$ can be identified with the projective cone

$$\mathcal{C}_{\mathrm{U}} = \left\{[\mathfrak{z}] \in \mathbb{P}^1 V;\ (\mathfrak{z}, \mathfrak{z}) < 0\right\},$$

where $\mathbb{P}^1 V$ denotes the projective space of V, and the elements of $\mathcal{C}_{\mathrm{U}}$ are just the negative definite lines in V.

First, in Section 2.1, we construct an affine model for $\mathbb{D}$ by choosing a representative for each negative line z. After that, in Section 2.2, we will review the construction of the symmetric domain for orthogonal groups for signature $(2p, 2)$. In Section 2.3 we then introduce an embedding, originally from [33], between the symmetric domains for $\mathrm{U}(V)$ and $\mathrm{O}(V_{\mathbb{R}})$, which, after a brief review of Bruinier's construction of the Borcherds products (Section 2.4) is then used in Section 2.5 to construct Borcherds products for the unitary group, in a slight variation on [33, 35].

In Section 2.6 the existence of Borcherds products is used (following [35]) to prove a modularity statement for certain generating series, along the lines of [3]. At the end of the chapter, in Section 2.7 we will briefly present the construction of 'local' Borcherds products from [37], inspired by the work of Bruinier and Freitag [7].

2.1. The Siegel domain model

Let L be an even lattice in V, with dual $L^\sharp$. Denote by $\mathrm{Iso}_{\mathbb{F}}(V)$ the set of one-dimensional isotropic subspaces of $V_{\mathbb{F}}$. Its elements are in one-to-one correspondence with the rational boundary components of the symmetric domain. We fix an element I_ℓ in $\mathrm{Iso}_{\mathbb{F}}(V)$ by choosing a primitive isotropic lattice vector $\ell \in L$, and setting $I_\ell = \mathbb{F}\ell$. Further, we choose a primitive vector ℓ' in $L^\sharp$ with $(\ell, \ell') \neq 0$.

We denote by D the lattice $L \cap \ell^\perp \cap \ell'^\perp$, where, naturally, the complement is taken with respect to $(\cdot, \cdot)$. Equipped with the restriction of $(\cdot, \cdot)$, it is an even definite hermitian lattice, with signature $(p-1, 0)$. Set $W_{\mathbb{F}} := D \otimes_{\mathcal{O}_{\mathbb{F}}} \mathbb{F}$ and $W := W_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{C}$. Hence, W is a definite $p-1$ dimensional complex hermitian space.

Denote by $v_2, \ldots, v_p$ the standard orthogonal basis[1] of W. Let δ be a purely imaginary

[1]Strictly speaking, with the notation from Chapter 1, we are assuming here that $\ell, \ell' \in \mathrm{span}_{\mathbb{C}}\{v_1, v_m\}$, which, of course, is possible without any loss of generality.

constant with $\delta \in i\mathbb{R}_{>0}$, for example $\delta = \delta_{\mathbb{F}}$. For z in $\mathbb{D}$, choose a vector $\mathfrak{z} \in z$, and write $\mathfrak{z}$ in the form

$$\mathfrak{z} = \mathfrak{z}(\tau,\sigma) = \ell' + \tau\delta(\ell,\ell')\ell + \sigma, \tag{2.1.1}$$

with $\sigma \in W$. Then, since z is negative definite, we have

$$0 > (\mathfrak{z},\mathfrak{z}) = 2\,\mathrm{Re}\,(\tau\delta(\ell,\ell')(\ell',\ell)) + (\sigma,\sigma) + (\ell',\ell') = -2\Im\tau|\delta||(\ell,\ell')|^2 + (\sigma,\sigma) + (\ell',\ell'),$$

whence

$$\Im\tau > \frac{(\sigma,\sigma)+(\ell',\ell')}{2|\delta||(\ell,\ell')|^2}.$$

We can thus define an affine model for $\mathbb{D}$ as follows.

Definition 2.1. The *Siegel domain model* $\mathcal{H}_{\ell,\ell'}$ is the generalized upper half-plane given by

$$\mathcal{H}_{\ell,\ell'} := \left\{(\tau_\ell,\sigma) \in \mathbb{C}\times W;\quad 2|\delta||(\ell',\ell)|^2\Im\tau - (\sigma,\sigma) - (\ell',\ell') > 0\right\}.$$

An isomorphism between $\mathcal{H}_{\ell,\ell'}$ and $\mathbb{D}$ is given by

$$\begin{aligned} \mathcal{H}_{\ell,\ell'} &\longrightarrow \mathbb{D}, & (\tau,\sigma) &\longmapsto \mathbb{C}\mathfrak{z}(\tau,\sigma) \\ \text{and}\quad \mathbb{D} &\longrightarrow \mathcal{H}_{\ell,\ell'}, & z &\longmapsto (\tau,\sigma), \end{aligned}$$

where $\mathfrak{z}(\tau,\sigma) = \ell' + \tau\delta(\ell,\ell')\ell + \sigma$, and the converse map in the last line is the composition of $z \mapsto \mathbb{C}\mathfrak{z}$ (assigning to z a representative of the form (2.1.1)) with $\mathfrak{z} \mapsto (\tau,\sigma)$. We note that the isotopic line $I_{\ell,\mathbb{C}} = \mathbb{C}\ell = [\ell]$ corresponds to the cusp at infinity of the generalized upper-half plane $\mathcal{H}_{\ell,\ell'}$.

In the following, we will *assume that ℓ', too, is an isotropic vector.* Note that this is a non-trivial assumption about the hermitan lattice L and its dual.

Remark 2.2. We note that $\mathcal{H}_{\ell,\ell'}$ is not a tube-domain. This means for example, that functions $\mathcal{H}_{\ell,\ell'} \to \mathbb{C}$ which are translation invariant do not posses an expansion as Fourier series, i.e. with constant coefficients, instead, there is a Fourier-Jacobi expansion with coefficients depending on σ, see Section 2.1.3.

Finally, we note that the action of G on $\mathcal{H}_{\ell,\ell'}$ gives rise to a non-trivial automorphy factor, explicitly given by the ℓ'-component of $\mathfrak{z}(\tau,\sigma)$:

$$j:\, G\times\mathcal{H}_{\ell,\ell'} \longrightarrow \mathbb{C}^\times, (g,(\tau,\sigma)) \longmapsto j(g,(\tau,\sigma)) = \frac{(\ell, g\circ\mathfrak{z}(\tau,\sigma))}{(\ell,z)}, \tag{2.1.2}$$

where g operates on the vector $\mathfrak{z}(\tau,\sigma)$ through matrix multiplication.

2.1.1. Operation of the parabolic group

Denote by P_ℓ the stabilizer of the cusp $[\ell]$ i.e. of $I_{\ell,\mathbb{C}} = I_\ell \otimes \mathbb{C}$ in $G = \mathrm{U}(V)$. We denote by N the unipotent radical of P_ℓ. Consider the Levi-decomposition $G = NAM$, where the Levi-factor is given by the direct product of the groups $M = \mathrm{SU}(W) \simeq \mathrm{SU}(p-1)$ and $A = \mathrm{GL}([\ell])$.

Written as matrices in the basis $\ell, v_2, \dots, v_p, \ell'$ the elements of A and M take the form

$$a(t) = \begin{pmatrix} t & & \\ & 1_{p-1} & \\ & & t^{-1} \end{pmatrix} \quad \big(t \in \mathbb{R}_{>0}\big), \qquad \mu = \begin{pmatrix} 1 & & \\ & \mu' & \\ & & 1 \end{pmatrix} \quad \big(\mu' \in \mathrm{SU}(W)\big),$$

while the elements of N are parameterized by pairs (w, r) with $w \in W$ and $r \in \mathbb{R}$. We will denote them either using the notation $[w, r]$, or as matrices $n(w, r) = n(w, 0) \cdot n(0, r)$. In matrix notation, we have

$$n(0, r) = \begin{pmatrix} 1 & 0 & \delta(\ell, \ell')r \\ & 1_{p-1} & \\ & & 1 \end{pmatrix}$$
$$n(w, 0) = \begin{pmatrix} 1 & -\bar{w}^t & -\frac{1}{2}(\ell, \ell')\bar{w}^t\bar{w} \\ & 1_{p-1} & (\ell, \ell')w \\ & & 1 \end{pmatrix}.$$

The operation on $\mathcal{H}_{\ell,\ell'}$ is given as follows (recall that $w, \sigma \in W$ both have positive norm)

$$\begin{aligned} [0, r]: &\quad (\tau, \sigma) \longmapsto (\tau + r, \sigma), \\ [w, 0]: &\quad (\tau, \sigma) \longmapsto \left(\tau - \frac{(w, \sigma)}{\delta(\ell, \ell')} - \frac{(w, w)}{2\delta}, \sigma + (\ell, \ell')w\right). \end{aligned}$$

Note that the unipotent radical N has the structure of a Heisenberg group, the center of which is formed by the transformations of type $[0, r]$. The transformations of the type $[w, 0]$ are also known as *Eichler elements*.

By direct calculation, we derive the group law[2] for N,

$$[w, r] \circ [w', r'] = \big[w + w', r + r' - \frac{\Im(w, w')}{|\delta|}\big], \tag{2.1.3}$$

and the commutation relation

$$[w, 0] \circ [w', 0] = \left[w + w', -\frac{\Im(w, w')}{|\delta|}\right]. \tag{2.1.4}$$

Remark 2.3. We remark that the for transformation in N and M the automorphy factor $j(g, z)$ from (2.1.2) is trivial, while for the elements of A one has $j\left(a(t), z\right) = t^{-1}$.

[2] The notational convention for concatenation used here is $(g' \circ g)(v) = g'(g(v))$, consistent with matrix multiplication.

If Γ is a discrete subgroup of G, e.g. the discriminant kernel Γ_L, we denote the discrete Heisenberg group $N \cap \Gamma$ by $\mathrm{Heis}(\Gamma)$. The stabilizer in Γ of the cusp $I_\ell = \mathbb{F}\ell$ is given by the semi-direct product

$$\Gamma_\ell := \mathrm{stab}_\Gamma(\ell) = \underbrace{\mathrm{Heis}(\Gamma)}_{N\cap\Gamma} \ltimes \underbrace{(\mathrm{U}(W) \cap \Gamma)}_{M\cap\Gamma} = P_\ell \cap \Gamma.$$

Remark 2.4. Naturally, the description of N, A and K using matrices generalizes to other signatures (p, q), with $q > 1$. We will come back to this later, see Sections A.2.1, 4.1.1 and 4.4.

The following Lemma is well-known:

Lemma 2.5. *Let Γ be a discrete subgroup of $\mathrm{U}(V)$, commensurable with Γ_L. Then, there is a positive integer $N_{\Gamma,\ell}$ and a lattice D_Γ of finite index in D, such that $n(w, r)$ is an element of $\mathrm{Heis}(\Gamma)$ for all $w \in D_\Gamma$ and all $r \in N_{\Gamma,\ell}$. Also,*

$$\frac{\Im(w, w')}{|\delta|} \in \mathbb{Z}N_{\Gamma,\ell} \quad \textit{for all } w, w' \in D_\Gamma.$$

2.1.2. Boundary components and compactification

Recall from Chapter 1 the definition of the discriminant kernel Γ_L as the subgroup $\mathrm{Fix}_G(L^\sharp/L) \subset G$, and of the modular variety $X = \Gamma_L\backslash\mathbb{D}$. Somewhat more generally, let Γ be discrete subgroup commensurable with Γ_L and denote by X_Γ the variety $\Gamma\backslash\mathbb{D}$, which is non-compact quasi-projective. In this section, we want to discuss two compactifications which can be applied to modular varieties of tis type, namely the Baily-Borel compactification and the toroidal compactification see [33, 34, 37] or [14]. For further background on the toroidal compactification see [38, Section 3.3].

With the affine model $\mathcal{H}_{\ell,\ell'}$, we have the following isomorphisms

$$X_\Gamma = \Gamma\backslash\mathbb{D} \simeq \Gamma\backslash\mathrm{SU}(V)(\mathbb{R})/\left(\mathrm{SU}(W)(\mathbb{R}) \times \mathrm{SU}(W^\perp)(\mathbb{R})\right) \simeq \Gamma\backslash\mathcal{H}_{\ell,\ell'}.$$

The Baily-Borel compactification, denoted $X^*_{\Gamma,BB}$, is obtained by forming the union of X_Γ with $I_\mathbb{C}$ for all rational isotropic subspaces $I \in \mathrm{Iso}(V)$ and defining a topology and complex structure on the quotient

$$\Gamma\backslash\left(\mathcal{H}_{\ell,\ell'} \cup \{I_\mathbb{C};\ I \in \mathrm{Iso}(V)\}\right).$$

Since compactification is a local process, it suffices for us to sketch the construction for the istropic line $\mathbb{C}\ell$, which we denote $I_{\ell,\mathbb{C}}$ here. It corresponds to the cusp at infinity $[\ell]$ of $\mathcal{H}_{\ell,\ell'}$. A system of neighborhoods of this cusp is given by

$$\begin{aligned} U_\epsilon(\ell) &= \left\{[z] \in \mathcal{C}_\mathrm{U};\ \frac{(\mathfrak{z},\mathfrak{z})}{|(\ell,\mathfrak{z})|^2}|(\ell,\ell')|^2 > C = \frac{1}{\epsilon}\right\} \\ &\simeq \left\{(\tau,\sigma) \in \mathcal{H}_{\ell,\ell'};\ 2\Im\tau|\delta||(\ell,\ell')|^2 - (\sigma,\sigma) > C = \frac{1}{\epsilon}\right\} \quad (\epsilon > 0) \end{aligned} \tag{2.1.5}$$

A subset V of $\mathcal{C}_{\mathrm{U}} \cup I_{\ell,\mathbb{C}}$ is called open, if $V \cap \mathcal{C}_{\mathrm{U}}$ is open in the usual sense and if further $I_{\ell,\mathbb{C}} \in V$ implies $U_\epsilon(\ell) \subset V$ for some $\epsilon > 0$. By repeating this construction for every $I \in \operatorname{Iso}(V)$, one obtains a topology on $\mathcal{C}_{\mathrm{U}}^* = \mathcal{C}_{\mathrm{U}} \cup \{I_{\mathbb{C}};\ I \in \operatorname{Iso}(V)\}$. Moreover, the quotient topology yields a topology on $\Gamma \backslash \mathcal{C}_{\mathrm{U}}^*$.

The complex structure is defined through pullback under the canonical projection $\mathcal{C}_{\mathrm{U}}^* = \mathcal{C}_{\mathrm{U}} \cup \{I_{\mathbb{C}};\ I \in \operatorname{Iso}(V)\} \to X^*_{\Gamma,BB}$, locally for each cusp. We give a brief sketch, for more details see [33, p. 30] and [35, Sec. 1].

Denote by pr the canonical projection $\mathrm{pr} : \mathcal{C}_{\mathrm{U}}^* \to \Gamma \backslash \mathcal{C}_{\mathrm{U}}^*$. For an open set $U \subset \Gamma \backslash \mathcal{C}_{\mathrm{U}}^*$, let $U' \in \mathcal{C}_{\mathrm{U}}^*$ be the inverse image under pr and let U'' be the inverse image of U' in $\mathcal{C}_{\mathrm{U}}$, as depicted in the following diagram

$$\begin{tikzcd} \mathcal{C}_{\mathrm{U}} \arrow[r] & \mathcal{C}_{\mathrm{U}}^* \arrow[r, "\mathrm{pr}"] & \Gamma \backslash \mathcal{C}_{\mathrm{U}}^* \\ U'' \arrow[u, hook] \arrow[r] & U' \arrow[u, hook] \arrow[r] & U. \arrow[u, hook] \end{tikzcd}$$

Now, define $\mathcal{O}(U)$ as the ring of continuous functions $f : U \to \mathbb{C}$, which have holomorphic pullback $\mathrm{pr}^*(f)$ to U' (and to U''). With the usual methods of algebraic geometry this defines the sheaf $\mathcal{O}$ of holomorphic functions on $X^*_{\Gamma,BB}$. Hence, this construction yields the structure of a normal complex space on $X^*_{\Gamma,BB}$. Its drawback however, is that in general, there are still singularities at the boundary points.

The toroidal compactification[3], denoted $X^*_{\Gamma,tor}$, presents an alternative. In sketching this construction, we can again restrict our attention to the boundary component $I_{\ell,\mathbb{C}}$. For more details see [14, Section 4.3]. In the following, identify the sets $U_\epsilon(\ell) \subset \mathcal{C}_{\mathrm{U}}$ with the corresponding sets of representatives in $\mathcal{H}_{\ell,\ell'}$ from (2.1.5). Clearly, the Heisenberg group N operates on $U_\epsilon(\ell)$. For sufficiently small ϵ, there is an open immersion

$$\operatorname{Heis}(\Gamma) \backslash U_\epsilon(\ell) \to X_\Gamma.$$

Let $C_\ell(\Gamma_L)$ denote the center of $\operatorname{Heis}(\Gamma)$, i.e. $C_\ell(\Gamma_L) = \{n(0,r) \in N \cap \Gamma\}$. Recall that $C_\ell(\Gamma) \simeq \mathbb{Z}N_{\Gamma,\ell}$, and set $q_\ell := \exp(2\pi i \tau / N_{\Gamma,\ell})$.

The quotient $C_\ell(\Gamma) \backslash U_\epsilon(\ell)$ can now be viewed as bundle of punctured disks over W:

$$V_\epsilon(\ell) := C_\ell(\Gamma) \backslash U_\epsilon(\ell) \simeq \left\{ (q_\ell, \sigma);\ 0 < |q_\ell| < \exp\left(\frac{\pi(\sigma,\sigma) + \epsilon^{-1}}{\delta |(\ell,\ell')|^2} \right) \right\}.$$

Adding the center to each disk, we get the disk bundle

$$\widetilde{V}_\epsilon(\ell) := \left\{ (q_\ell, \sigma);\ |q_\ell| < \exp\left(\frac{\pi(\sigma,\sigma) + \epsilon^{-1}}{|\delta|^2 |(\ell',\ell)|^2} \right) \right\}.$$

The action of $\operatorname{Heis}(\Gamma)$ is well-defined at each center, leaving the divisor $q = 0$ fixed.

Also, if Γ is sufficiently small, the operation is free, hence we get an open immersion

$$\operatorname{Heis}(\Gamma) \backslash U_\epsilon(\ell) \to \left(\operatorname{Heis}(\Gamma) / C_\ell(\Gamma)\right) \backslash \widetilde{V}_\epsilon(\ell), \tag{2.1.6}$$

[3]This type of compactification can also be approached as a resolution of the remaining singularities. This point of view is taken in [33, Sec. 1.1.5].

by which the right hand side can be glued to X_Γ, yielding a partial compactification. For a point $(0, \sigma_0) \in \widetilde{V}_\epsilon(\ell)$, we define a system of open sets

$$B_\delta(0, \sigma_0) = \left\{ (q_\ell, \sigma) \in \widetilde{V}_\epsilon(\ell)\, ;\, (\sigma - \sigma_0, \sigma - \sigma_0) < \delta, |q_\ell| < \delta \right\} \qquad (\delta > 0).$$

Under the immersion (2.1.6) the images of these sets form a system of open neighborhoods for the boundary point at $(0, \sigma_0)$.

Repeating this construction and the gluing procedure for every Γ-equivalence class of isotropic spaces in $\mathrm{Iso}(V)$ yields a compactification of X_Γ, which we denote $X^*_{\Gamma,tor}$.

2.1.3. Modular forms

For $\gamma \in G$ and $(\tau, \sigma) \in \mathcal{H}_{\ell,\ell'}$ denote by $j(g, (\tau, \sigma))$ the automorphy factor for the operation of g on $\mathcal{H}_{\ell,\ell'}$, as introduced in (2.1.2).

Definition 2.6. Let Γ be a unitary modular group, i.e. commensurable with Γ_L, χ a character of Γ and k an integer. A *holomorphic automorphic form* of weight k and with character χ for Γ is a holomorphic function $f : \mathcal{H}_{\ell,\ell'} \to \mathbb{C}$ which satisfies

$$f\left(\gamma \circ (\tau, \sigma)\right) = j\left(\gamma, (\tau, \sigma)\right)^k \chi(\gamma) f(\tau, \sigma) \quad \text{for all } \gamma \in \Gamma.$$

A holomorphic modular form which is regular at every cusp is called a *modular form.*

Meromorphic automorphic forms are defined similarly. Note that by the Koecher-principle, if $p > 1$, holomorphicity on $\mathcal{H}_{\ell,\ell'}$ automatically entails regularity at all cusps, and hence any holomorphic automorphic form is already a modular form. (See [33, p. 33] for a simple proof in the present signature.)

Immediately from the transformation behavior, it follows that an automorphic form is invariant under the operation of the translations $[0, r] \in N$. Hence, a holomorphic automorphic form f with character χ has a Fourier-Jacobi expansion of the form

$$f(\tau, \sigma) = \sum_{\kappa \in \mathbb{Q}} c_\kappa(\sigma) e^{2\pi i \kappa \tau}, \tag{2.1.7}$$

where κ in $N^{-1}_{\Gamma,\ell}(\mathbb{Z} + r_0)$ with $N_{\Gamma,\ell}$ from Lemma 2.5 and non-negative constant $r_0 \in \mathbb{Z} \geq 0$. Further, if f is a modular form, $\kappa \geq 0$. The coefficients $c_\kappa(\sigma)$ exhibit the transformation behavior of theta functions, i.e.

$$c_\kappa([w, 0] \circ \sigma) = \chi([w, 0]) \cdot c_\kappa(\sigma) \cdot e\left(+\kappa\left(\frac{(w, \sigma)}{\delta(\ell, \ell')} + \frac{(w, w)}{2\delta}\right)\right)$$

Hence, the coefficient $c_0 = c_0(\sigma)$ is a constant.

Remark 2.7. In Chapter 4, where we allow V to have arbitrary signature (p, q), and in the absence of the tube-domain coordinate τ, we will use a slightly different (and more general) notion of the Fourier-Jacobi expansion, based on the operation of the translations $[0, r]$ in the Heisenberg group from Section 4.4.2. In the present case, for holomorphic automorphic forms on $\mathcal{H}_{\ell,\ell'}$, basically just consider (2.1.7) as an expansion in $\mathrm{Re}\,\tau$ with coefficients $c_\kappa(\sigma, \Im\tau) = c_\kappa(\sigma) \exp\left(-2\pi\kappa\Im\tau\right)$.

2.2. Orthogonal groups

In his seminal paper [2], Borcherds constructed a regularized lifting for the dual reductive pair consisting of $\mathrm{SL}_2(\mathbb{R})$ and an indefinite real orthogonal group $\mathrm{O}(b_+, b_-)$ of signature[4] (b_+, b_-), for which he used a (slightly generalized) Siegel theta function. If $b_- = 2$, the symmetric domain for $\mathrm{O}(b_+, 2)$ carries the structure of a hermitian symmetric domain. In this setting[5], Borcherds' lifting gives rise to his celebrated Borcherds products and a multiplicative lift.

A construction of Borcherds products for unitary groups of signature $(p, 1)$ was developed by the author in [33], [35] and, for the case of signature $(1, 1)$ [34], using an embedding from unitary into orthogonal groups. With a view to this embedding, we will mainly consider the case in which the quadratic space is a real vector space $V_\mathbb{R}$ underlying a complex hermitian space V as in Chapter 1 and hence has signature $(b_+, b_-) = (2p, 2q)$. Mostly, we will have $q = 1$.

Later in this chapter, we will review the theory of Borcherds products for orthogonal groups using a generalization of Borcherds' work due to Bruinier [5].

2.2.1. Orthogonal groups and their symmetric domains

In this section, in view of the embedding from [33, 34], we use the following setup for orthogonal groups. References for this, besides [33, Chapter 1.2] and [36], can be found in [22] and [5, Chapter 3]. Consider $V_\mathbb{R}$ as a quadratic space of signature $(2p, 2q)$ with the bilinear form $(\cdot, \cdot)_\mathbb{R}$. Let L be an even lattice in V (as in Section 2.1). Thus, L is also an even lattice in $V_\mathbb{R}$, with the quadratic form $Q(\cdot)$.

We denote by $\mathrm{O}(V_\mathbb{R}) \simeq \mathrm{O}(2p, 2q)$ the orthogonal group of $V_\mathbb{R}$ and by $\mathrm{SO}(V_\mathbb{R})$ the special orthogonal group. Further, we denote by $\mathbb{D}_\mathrm{O}$ a symmetric domain for the operation of $\mathrm{O}(V_\mathbb{R})$, given as follows

$$\begin{aligned}\mathbb{D}_\mathrm{O} &= \mathrm{O}(V_\mathbb{R})/\mathcal{K}_\mathrm{O} \simeq \mathrm{O}(2p, 2q)/\left(\mathrm{O}(2p) \times \mathrm{O}(2q)\right)\\ &\simeq \left\{v \subset V_\mathbb{R} \ \dim(v) = 2q, (\cdot, \cdot)_\mathbb{R}\,|_v < 0\right\},\end{aligned}$$

wherein $\mathcal{K}_\mathrm{O}$ is a maximal compact subgroup. This set of two-dimensional definite subspaces is also called a Grassmannian model for the symmetric domain.

Remark 2.8. We note that the same Grassmannian can also be viewed as a symmetric domain for the operation of $\mathrm{SO}(V_\mathbb{R})$, since the group quotients are isomorphic

$$\mathbb{D}_\mathrm{O} \simeq \mathrm{SO}(V_\mathbb{R})/\mathcal{K}_\mathrm{SO} \simeq \mathrm{SO}(2p, 2q)/\mathrm{S}\left(\mathrm{O}(2p) \times \mathrm{O}(2q)\right).$$

Also note that $\mathbb{D}_\mathrm{O}$ is path-connected. If, alternatively, one considers the quotient of $\mathrm{SO}(V_\mathbb{R})$ by a maximal *path-connected* and compact subgroup,

$$\begin{aligned}\mathbb{D}_\mathrm{SO}^{\circlearrowleft} :=& \mathrm{SO}(V_\mathbb{R})/\left\{\text{max. path-connected}\right\} \simeq \mathrm{SO}(2p, 2q)/\left(\mathrm{SO}(2p) \times \mathrm{SO}(2q)\right)\\ \simeq& \left\{v \subset V_\mathbb{R};\ \dim(v) = 2q,\ (\cdot, \cdot)_\mathbb{R}\,|_v < 0,\ v \text{ oriented}\right\},\end{aligned}$$

[4] We will use this notation in the present chapter for the signature of an (arbitrary) indefinite orthogonal group and reserve (p, q) for the signature of a hermitian space V and its unitary group.

[5] Strictly speaking, this is the opposite of Borcherds' convention for the signature in [2], but we prefer the present setup to maintain consistency with [25] and [46]

one obtains a realization of a symmetric domain for $\mathrm{SO}(V_{\mathbb{R}})$ as a Grassmannian of oriented two-dimensional negative definite subspaces, which has two connected components, since it contains spaces of both orientations.

Denote by $\mathrm{SO}^+(V_{\mathbb{R}})$ the subgroup of orientation preserving transformations in $\mathrm{SO}(V_{\mathbb{R}})$. In the following exact sequence,

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}(V_{\mathbb{R}}) \longrightarrow \mathrm{SO}(V_{\mathbb{R}}) \xrightarrow{\theta} \mathbb{R}^\times/\mathbb{R}^{\times 2},$$

$\mathrm{SO}^+(V_{\mathbb{R}})$ is the image of the spin group $\mathrm{Spin}(V_{\mathbb{R}})$ in $\mathrm{SO}(V_{\mathbb{R}})$ and hence the kernel of the map θ. It is thus called the *spinor kernel* (θ is known as the spinor norm), and is the connected component of the identity in $\mathrm{SO}(V_{\mathbb{R}})$.

We denote by $\mathrm{O}(L)$ the isometry group with respect to $(\cdot,\cdot)_{\mathbb{R}}$ of L, and by $\mathrm{SO}^+(L)$ the subgroup $\mathrm{O}(L) \cap \mathrm{SO}^+(V_{\mathbb{R}})$. By the *discriminant kernel* of L we understand the subgroup $\Gamma_L^{\mathrm{O}} := \mathrm{Fix}_{\mathrm{SO}^+(L)}(L^\sharp/L)$. We shall consider subgroups of finite index in the discriminant kernel as *(orthogonal) modular groups*. We remark here that Γ_L, the discriminant kernel in $\mathrm{U}(L)$, see Section 1.1.1, is contained as a subgroup in Γ_L^{O}.

Now, if $q = 1$, $\mathbb{D}_{\mathrm{O}}$ is a hermitian space. It can be equipped with a complex structure by a 'spin-orientation', a continuously varying choice of oriented basis. Indeed, let $v \in \mathbb{D}_{\mathrm{O}}$. After setting

$$v = \mathrm{span}_{\mathbb{R}}\left(X_L, Y_L\right), \quad \text{with } X_L \perp Y_L, \quad (X_L, X_L)_{\mathbb{R}} = (Y_L, Y_L)_{\mathbb{R}} < 0, \tag{2.2.1}$$

the two real vectors X_L and Y_L can be interpreted as the real and the imaginary part of a complex vector. Denote by $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ the complexifcation of $V_{\mathbb{R}}$ and extend $(\cdot,\cdot)_{\mathbb{R}}$ to a complex valued bilinear form on $V_{\mathbb{C}}$. Note that $V_{\mathbb{C}}$ has real dimension $2m$. Now, setting

$$Z_L := X_L + iY_L,$$

we have

$$(Z_L, Z_L)_{\mathbb{R}} = 0 \quad \text{and} \quad \left(Z_L, \bar{Z}_L\right)_{\mathbb{R}} = (X_L, X_L)_{\mathbb{R}} + (Y_L, Y_L)_{\mathbb{R}} < 0.$$

Thus, associating to $v \in \mathbb{D}_{\mathrm{O}}$ the line $\mathbb{C}Z_L \subset V_{\mathbb{C}}$ affords an isomorphism of $\mathbb{D}_{\mathrm{O}}$ to (a connected component of) a negative-definite cone in a null-quadric in the projective space $\mathbb{P}^1 V_{\mathbb{C}}$ of $V_{\mathbb{C}}$. For an element $[Z] \in \mathbb{P}^1 V_{\mathbb{C}}$, denote by Z the preimage under the canonical projection $\pi_{\mathbb{C}} : V_{\mathbb{C}} \longmapsto \mathbb{P}^1 V_{\mathbb{C}}$. Then, the null-quadric and the negative cone are given by

$$\begin{aligned} \mathcal{N} &= \{[Z];\ (Z, Z)_{\mathbb{R}} = 0\} \subset \mathbb{P}^1(V_{\mathbb{C}}) \\ \text{and} \qquad \mathcal{C} &= \left\{[Z] \in \mathcal{N};\ \left(Z, \bar{Z}\right)_{\mathbb{R}} < 0\right\}, \end{aligned}$$

respectively. We note that $\mathcal{C}$ has two connected components, which we write as $\mathcal{C} = \mathcal{C}^+ \uplus \mathcal{C}^-$. Either of these may be used as complex projective model for the symmetric domain. Hence we choose one of the two and denote it by $\mathcal{C}_{\mathrm{O}}$.

To construct an affine model, we take a rational hyperbolic plane in V, spanned by two lattice vectors $e_1, e_2 \in L$, both assumed to be isotropic and with $(e_1, e_2)_{\mathbb{R}} \neq 0$. This amounts to fixing a boundary component of $\mathbb{D}_{\mathrm{O}}$. Denote by K the lattice $L \cap e_1^{\perp} \cap e_2^{\perp}$, equipped with the restriction of the bilinear form. Now we can write Z_L in the form

$$Z_L = e_2 + be_1 + Z \qquad \text{with} \quad Z \in K \otimes_{\mathbb{Z}} \mathbb{C}.$$

In the following, assume that $(e_1, e_2)_{\mathbb{R}} = 1$. Then,

$$\begin{aligned} Z_L = e_2 - \tfrac{1}{2}(Z,Z)_{\mathbb{R}} e_1 + Z = (-\tfrac{1}{2}(Z,Z)_{\mathbb{R}}, 1, Z), \\ X_L = (-\tfrac{1}{2}(X,X)_{\mathbb{R}} + \tfrac{1}{2}(Y,Y)_{\mathbb{R}}, 1, X), \qquad Y_L = (-(X,Y)_{\mathbb{R}}, 0, Y). \end{aligned} \tag{2.2.2}$$

Note that $Q(Y) = Q(Y_L) = Q(X_L)$, since the projection $p_K : x \mapsto x_K = x - (x, e_1)_{\mathbb{R}} e_2$ is an isometry for any $x \in V_{\mathbb{R}}$ with $(x, e_1)_{\mathbb{R}} = 0$.

Also, we remark that for this construction, the assumption that e_2 is isotropic is not essential; dropping it one merely needs to add $-(e_2, e_2)_{\mathbb{R}} e_1$ to Z_L.

Now, denote by $\mathcal{H}^{\pm}$ the two connected components of the set

$$\mathcal{H} = \{Z = X + iY,\ X, Y \in K \otimes_{\mathbb{Z}} \mathbb{R},\ (Y,Y)_{\mathbb{R}} < 0\}.$$

We remark that the connected components are stabilized by the action of $\mathrm{SO}^+(V_{\mathbb{R}})$ and interchanged by the action of $\mathrm{SO}(V_{\mathbb{R}})/\mathrm{SO}^+(V_{\mathbb{R}})$.

The *tube-domain model* is defined as the connected component of $\mathcal{H}$ mapped to $\mathcal{C}_{\mathrm{O}}$ by the biholomorphic map $Z \mapsto [Z_L]$,

$$\mathcal{H}_p := \left\{Z = X + iY;\ X, Y \in K \otimes_{\mathbb{Z}} \mathbb{R},\ [(-\tfrac{1}{2}(Z,Z)_{\mathbb{R}}, 1, Z)] \in \mathcal{C}_{\mathrm{O}}\right\}.$$

Remark 2.9. Naturally, if we start out with the special orthogonal group and a Grassmannian model $\mathbb{D}_{\mathrm{SO}}^{\circlearrowleft}$ with two connected components for its symmetric domain, see Remark 2.8, picking one component of $\mathcal{C}^{\pm}$ and of $\mathcal{H}^{\pm}$ is not necessary, as $\mathbb{D}_{\mathrm{SO}}^{\circlearrowleft}$ is isomorphic to the entire projective cone. In this case, $\mathcal{H}^+ \uplus \mathcal{H}^-$ is an affine model.

Remark 2.10. The spin-orientations on the Grassmannian play the role of conjugacy classes under complex conjugation. If for $v \in \mathbb{D}_{\mathrm{O}}$ we have (X_L, Y_L) as an oriented basis, a basis with the opposite orientation is given by $(X_L, -Y_L)$, which obviously corresponds to $\overline{Z_L} \in V_{\mathbb{C}}$. Thus $\mathcal{H}^+$ and $\mathcal{H}^-$ play the role of upper and lower half-spaces. Indeed

$$\mathcal{H} = K \otimes \mathbb{R} + i\{Y \in K \otimes_{\mathbb{Z}} \mathbb{R};\ (Y,Y)_{\mathbb{R}} < 0\},$$

and the set of imaginary parts is a quadratic cone with two connected components switched by complex conjugation.

We note that the operation of $\mathrm{SO}^+(V_{\mathbb{R}})$ on the tube-domain model gives rize to a non-trivial automorphy factor, which we denote by $J(g, Z)$ for $g \in \mathrm{SO}^+(V_{\mathbb{R}})$, $Z \in \mathcal{H}_p$. Similarly to the unitary case (see (2.1.2)), the automorphy factor is explicitly given by

$$\mathrm{SO}^+(V_{\mathbb{R}}) \times \mathcal{H}_p \to \mathbb{C}, \qquad (g, Z) \mapsto J(g, Z) = (gZ_L, e_2)_{\mathbb{R}}, \tag{2.2.3}$$

where, naturally, Z_L is the element of $\pi_{\mathbb{C}}^{-1}(\mathcal{C}_{\mathrm{O}})$ corresponding to $Z \in \mathcal{H}_p$.

Coordinates for the Grassmannian We remark that beside the coordinates X_L and Y_L related to the tube-domain model, another system of coordinates is often used for the elements of the Grassmannian (see [2], [5] or [54]).

For $v \in \mathbb{D}_{\mathrm{O}}$, let $v^{\perp}$ be the orthogonal complement, and denote by e_v and $e_{v^{\perp}}$ the orthogonal projection (with repect to $(\cdot,\cdot)_{\mathbb{R}}$) of e_1 to v and $v^{\perp}$, respectively. Define a subspace w through the orthogonal decomposition $v = w \oplus \mathbb{R}e_v$. Then, $V_{\mathbb{R}}$ can be written as

$$V_{\mathbb{R}} = v \oplus v^{\perp} = (w \oplus \mathbb{R}e_v) \oplus (w^{\perp} \oplus \mathbb{R}e_{v^{\perp}}).$$

Consider the vector defined by

$$\mu := -e_2 + \frac{e_v}{(e_v, e_v)_{\mathbb{R}}} + \frac{e_{v^{\perp}}}{(e_{v^{\perp}}, e_{v^{\perp}})_{\mathbb{R}}} = -e_2 + \frac{e_v}{(e_v, e_v)_{\mathbb{R}}} - \frac{e_1}{2(e_v, e_v)_{\mathbb{R}}}. \tag{2.2.4}$$

We can express v through w and $\mu_v = e_{2,v} + \frac{1}{2}e_v(e_v, e_v)_{\mathbb{R}}^{-1}$, since

$$v = w + \mathbb{R}\mu_v \quad \text{and} \quad v^{\perp} = w^{\perp} + \mathbb{R}\mu_{v^{\perp}}.$$

Using w and μ or μ_v as coordinates for the Grassmannian has the advantage of being independent of the definition of a complex structure. In fact, the use of such Grassmannian coordinates is not limited to the present signature $(2p, 2)$. Here, they are related to the tube domain coordinates in the following manner

$$\mu = \left(-\frac{1}{2}(X, X)_{\mathbb{R}}, 0, X\right), \quad \mu_K = X, \quad w = \mathbb{R}Y_L \text{ and } w_K = \mathbb{R}Y.$$

Remark 2.11. The definition of μ follows [2] and [5]. The following version [see 54, Theorem 3.3.11 on p. 111] can be used without requiring $(e_1, e_2)_{\mathbb{R}} = 1$:

$$\mu = -e_2 + \frac{(e_{v^{\perp}}, e_2)_{\mathbb{R}}}{(e_v, e_v)_{\mathbb{R}}} e_v + \frac{(e_v, e_2)_{\mathbb{R}}}{(e_{v^{\perp}}, e_{v^{\perp}})_{\mathbb{R}}} e_{v^{\perp}}.$$

2.2.2. Boundary components

Next, we will briefly describe the boundary components of $\mathcal{H}_p$, introducing a more refined version of the coordinates $Z = X + iY$ along the way (see e.g. [7]). Further, similarly to 2.1.1, we describe the action of the stabilizer of a boundary component.

The boundary components Let F denote a (totally) isotropic subspace of $V_{\mathbb{R}}$, and $F_{\mathbb{C}} := F \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. Isotropic subspaces of $V_{\mathbb{R}}$ define boundary components of $\mathcal{H} = \mathcal{H}_p \uplus \overline{\mathcal{H}_p}$, which can be described as follows (see [7]):

(i) Let F be a one-dimensional isotropic subspace of $V_{\mathbb{R}}$. Then, F represents a boundary-point of $\mathcal{H}$. A boundary point of this type is called *special*. A zero-dimensional boundary component is a set consisting of one special boundary point. Boundary points, which are not special, are called *generic*.

(ii) Let F be a two-dimensional totally isotropic subspace of $V_{\mathbb{R}}$. Then, the set of all *generic* boundary points, which can be represented by elements of $F_{\mathbb{C}}$ is called a one-dimensional boundary component.

Further, there is a one-to-one correspondence between boundary components and isotropic subspaces of $V_{\mathbb{R}}$ of the corresponding dimensions. The boundary of $\mathcal{H}$ is the disjoint union of all zero- and one-dimensional boundary components.

A boundary component is called *rational* if the corresponding isotropic subspace F is defined over $\mathbb{Q}$. *Cusps* are defined by equivalence classes of rational one-dimensional isotropic subspace under the operation of an orthogonal modular group.

Following [7], we give a brief description of the neighborhoods in $\mathcal{H}_p$ of te cusp defined by e_1: Let $F_{\mathbb{Q}}$ be a rational totally isotropic subspace in $L \otimes \mathbb{Q}$, and let $F'_{\mathbb{Q}}$ be second such (i.e. rational and isotropic) subspace. (For example, a lattice of signature $(l, 2)$ with $l > 5$ splits two hyperbolic planes. So, if we assume $p > 2$, we have two two-dimensional isotropic subspaces spanned by lattice vectors.)

We may assume $e_1 \in F_{\mathbb{Q}}$ and $e_2 \in F'_{\mathbb{Q}}$. Then, there are vectors e_3 and e_2, with the properties

$$\begin{aligned} F_{\mathbb{Q}} = \operatorname{span}_{\mathbb{Q}}\{e_1, e_3\}, \qquad & F'_{\mathbb{Q}} = \operatorname{span}_{\mathbb{Q}}\{e_2, e_4\} \\ \text{and} \quad (e_j, e_k)_{\mathbb{R}} = \delta_{j1}\delta_{k2} + \delta_{j3}\delta_{k4} \quad & (j \leq k), \end{aligned}$$

where δ_{jl} denotes the usual Kronecker symbol. Using this basis, we can write $Z \in \mathcal{H}_p$ in the form

$$Z = z_1 e_3 + z_2 e_4 + \mathfrak{Z} = (z_1, z_2, \mathfrak{Z}),$$

with $\mathfrak{Z} = \mathfrak{X} + i\mathfrak{Y}$ contained in the definite space $V_{\mathbb{R}} \otimes \mathbb{C} \cap F^{\perp} \cap F'^{\perp}$. Then,

$$Z_L = -\left(z_1 z_2 + \tfrac{1}{2}(\mathfrak{Z}, \mathfrak{Z})_{\mathbb{R}}\right) e_1 + e_2 + z_1 e_3 + z_2 e_4 + \mathfrak{Z}. \tag{2.2.5}$$

Since $0 > (Y, Y)_{\mathbb{R}} = y_1 y_2 + (\mathfrak{Y}, \mathfrak{Y})_{\mathbb{R}}$, one can define an embedding of two complex upper half-planes $\mathbb{H} \times \mathbb{H}$ into the boundary of the projective cone $\mathcal{C}$ via

$$(z_1, z_2) \mapsto [(z_1 z_2) e_1 + e_2 + z_1 e_3 - z_2 e_4] \in \mathbb{P}^1 V_{\mathbb{C}}. \tag{2.2.6}$$

Setting $z_2 = it$ and taking the limit $z_2 \to i\infty$ we get

$$\lim_{t \to \infty} [z_1 it, 1, z_1, -it] = [z_1, 0, 0, -1].$$

Thus, we have showed to first statment of the following remark.

Remark 2.12. A one-dimensional boundary component can be identified with a copy of a complex upper half-plane.

Quite similarly, see [22, 23], the set of all boundary points attached to $F_{\mathbb{C}} = F \otimes_{\mathbb{R}} \mathbb{C}$ (both special and generic) can be identified with $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$.

Now, we can use this observation to define a system of neighborhoods of the cusp of $\mathcal{H}_p$ attached to e_1. Let F be the rational isotropic subspace spanned by e_1 and e_3. For $F_{\mathbb{C}}$, we use (2.2.6), to identify $F_{\mathbb{C}} \cap \pi_{\mathbb{C}}^{-1}(\partial \mathcal{C}^+)$ with the complex upper half-plane $\mathbb{H}$.

Thus, for $z' \in \mathbb{H}$, let $x = z'e_1 - e_3$. A fundamental system of neighborhoods of x is given by the sets $U(x,\epsilon) \cup V(x,\epsilon)$ $(\epsilon > 0)$, with

$$\begin{aligned} V(x,\epsilon) &= \{z'' \in \mathbb{H};\, |z'' - z'| < \epsilon\}, \\ U(x,\epsilon) &= \left\{(z_1, z_2, \mathfrak{Z}) \in \mathcal{H}_p;\ z_2 \in V(x,\epsilon), -y_1 y_2 - \tfrac{1}{2}(\mathfrak{Y},\mathfrak{Y})_{\mathbb{R}} > \epsilon^{-1}\right\}. \end{aligned} \tag{2.2.7}$$

Let Γ be an orthogonal modular group. Then, a modular variety X_Γ^{O} is given by the quotient

$$X_\Gamma^{\mathrm{O}} = \Gamma\backslash\mathcal{C}_{\mathrm{O}} \simeq \Gamma\backslash\mathcal{H}_p.$$

It has the structure of a projective variety, contained as a Zariski open subset in its Baily-Borel compactification[6] $X_\Gamma^{\mathrm{O}*} \simeq \Gamma\backslash\mathcal{C}_{\mathrm{O}}^*$, where $\mathcal{C}_{\mathrm{O}}^*$ is the union of $\mathcal{C}_{\mathrm{O}}$ with all its (rational) boundary components. For any point $x \in \mathcal{C}_{\mathrm{O}}^*$, with stabilizer Γ_x in Γ, there is an open embedding

$$\Gamma_x\backslash\mathcal{C}_{\mathrm{O}}^* \longrightarrow \Gamma\backslash\mathcal{C}_{\mathrm{O}}^*.$$

Now, if x is contained in a boundary component, it is easily seen that the stabilizer Γ_x of x is contained in the normalizer of that boundary component.

The normalizer of a boundary component We want to describe the centralizer and the normalizer of a one-dimensional boundary component. Thus, let $F_{\mathbb{Q}} = \mathrm{span}_{\mathbb{Q}}\{e_1, e_3\}$, we denote the centralizer of $F_{\mathbb{Q}}$ in $\mathrm{SO}^+(V_{\mathbb{Q}})$ by $C_F(\mathbb{Q})$ and by $N_F(\mathbb{Q})$ the normalizer. Thus,

$$C_F(\mathbb{Q}) = \{g \in \mathrm{SO}^+(V_{\mathbb{Q}});\, g\,|_{F_{\mathbb{Q}}} = \mathrm{Id}_{F_{\mathbb{Q}}}\} \quad \text{and} \quad N_F(\mathbb{Q}) = \{g \in \mathrm{SO}^+(V_{\mathbb{Q}});\, g \circ F_{\mathbb{Q}} = F_{\mathbb{Q}}\}.$$

For an explicit description of both, we use Eichler elements, defined as follows:

Definition 2.13. Let $u \in V_{\mathbb{R}}$ be an isotropic vector and $v \in V_{\mathbb{R}}$ a vector perpendicular to u. Then, the Eichler element $E(u,v)$ is a transformation defined by

$$E(u,v) : V_{\mathbb{R}} \to V_{\mathbb{R}}, \quad x \mapsto x - (x,u)_{\mathbb{R}} v + (x,v)_{\mathbb{R}} u - \tfrac{1}{2}(v,v)_{\mathbb{R}}(x,u)_{\mathbb{R}} u.$$

Obviously, for fixed u, Eichler elements are additive, with

$$E(u,v_1) \circ E(u,v_2) = E(u, v_1 + v_2).$$

Further, we note that Eichler elements are contained in $\mathrm{SO}^+(V_{\mathbb{R}})$ and that, if u and v are contained in an even lattice L, then $E(u,v)$ lies in the discriminant kernel of L.

Finally, if u, u' are isotropic with $u \perp u'$ and v, v' are both perpendicular to $\mathbb{R}u + \mathbb{R}u'$, then the following relations hold

$$\begin{aligned} E(u,u') &= E(u',u)^{-1}, \\ E(u,v) \circ E(u',v') &= E(u', v' + (v,v')_{\mathbb{R}} u) \circ E(u,v). \end{aligned}$$

[6] For a detailed account of the construction of the (Satake-)Baily-Borel compactification see [22].

Denote by D the definite lattice $K \cap e_3^\perp \cap e_4^\perp$, and consider Eichler elements of the form $E(e_3, \mu)$, with $\mu \in D \otimes_\mathbb{Z} \mathbb{Q}$. Clearly, their action on $\mathbb{Q}e_1$ is trivial. Their action on $\mathcal{H}_p$ is given by

$$\begin{aligned} &Z = (z_1, z_2, \mathfrak{z}) \mapsto \\ &\quad Z + ((\mu, \mathfrak{z})_\mathbb{R} - \tfrac{1}{2}(\mu,\mu)_\mathbb{R} z_2)e_3 - z_2\mu = (z_1 + (\mu,\mathfrak{z})_\mathbb{R} - \tfrac{1}{2}(\mu,\mu)_\mathbb{R} z_2, z_2, \mathfrak{z} - z_2\mu). \end{aligned}$$

Denote $[0, \mu, 0] := E(e_3, \mu)$, and set

$$[\lambda, 0, t] := E(e_3, te_1) \circ E(e_1, \lambda) \quad (t \in \mathbb{Q}, \lambda \in D \otimes_\mathbb{Z} \mathbb{Q}).$$

Note that transformations of the type $E(e_1, \lambda)$ act trivially on $\mathbb{Q}e_3$. Also, the two types of Eichler elements commute and hence form an additive group. On $\mathcal{H}_p$ the action of an element $[\lambda, 0, t]$ is given by

$$[\lambda, 0, t] : Z = (z_1, z_2, \mathfrak{z}) \longmapsto (z_1 + t, z_2, \mathfrak{z} - \lambda).$$

Further, one finds

$$[0, \mu, 0] \circ [\lambda, 0, t] = [\lambda, 0, t - (\mu, \lambda)_\mathbb{R}] \circ [0, \mu, 0].$$

Hence the triples $[\lambda, \mu, t]$, $\lambda, \mu \in D \otimes_\mathbb{Z} \mathbb{Q}$, $t \in \mathbb{Q}$ form a group, as a semi-direct product with the direct factor consisting of elements of the form $[0, 0, t]$. This algebraic group has the group law

$$[\lambda, \mu, t] \circ [\lambda', \mu', t'] = [\lambda + \lambda', \mu + \mu', t + t' - (\mu, \lambda')_\mathbb{R}],$$

and thus is a (rational) Heisenberg group, which we denote by $\mathrm{Heis}(D)$. Its set of real points $\mathrm{Heis}(D)(\mathbb{R})$ is given by $\{[\lambda, \mu, t]; \lambda, \mu \in D \otimes \mathbb{R}, t \in \mathbb{R}\}$, and it acts on $\mathcal{H}_p$ via

$$[\lambda, \mu, t] : \; Z = (z_1, z_2, \mathfrak{z}) \longmapsto \left(z_1 + t + (\mu, \mathfrak{z})_\mathbb{R} - \tfrac{1}{2}(\mu, \mu)_\mathbb{R} z_2, z_2, \mathfrak{z} - \lambda - \mu z_2\right).$$

It is easily seen that the centralizer $C_F(\mathbb{Q})$ of $F_\mathbb{Q}$ is given by the semi-direct product $C_F(\mathbb{Q}) = \mathrm{Heis}(D) \ltimes \mathrm{SO}^+(D \otimes \mathbb{Q})$, while the normalizer $N_F(\mathbb{Q})$ sits in the exact sequence

$$1 \longrightarrow C_F(\mathbb{Q}) \longrightarrow N_F(\mathbb{Q}) \longrightarrow \mathrm{GL}_2^+(F_\mathbb{Q}) \longrightarrow 1.$$

Now, assuming that e_3, like e_1, is contained in the even lattice L, those Eichler elements from $\mathrm{Heis}(D)$ for which $\lambda, \mu \in D$ and $t \in \mathbb{Z}$ are contained in the discriminant kernel Γ_L^{O} in $\mathrm{SO}^+(L)$.

More generally, since $u, v \in L$ implies that $E(u, v)$ lies in the discriminant kernel, for any orthogonal modular group Γ, there are lattices $\tilde{K} \subset K \otimes_\mathbb{Z} \mathbb{Q}$, $\tilde{D} \subset D \otimes_\mathbb{Z} \mathbb{Q}$ and an integer $M \in \mathbb{Z}$, such that $[\kappa, \nu, r] \in \Gamma$ for all $\kappa \in \tilde{K}$ all $\nu \in \tilde{D}$ and $r \in M\mathbb{Z}$.

Modular forms Automorphic forms and modular forms as complex-valued functions on the tube domain $\mathcal{H}_p$ are defined as usual, using the automorphy factor $J(g, Z)$ from (2.2.3).

Definition 2.14. Let Γ be an orthogonal modular group, χ a character of Γ and k an integer. A *holomorphic automorphic* form of weight k and multiplier system χ for Γ is a holomorphic function $F : \mathcal{H}_p \to \mathbb{C}$ which satisfies

$$F(\gamma \circ Z) = J(\gamma, Z)^k \chi(\gamma) F(Z) \quad \text{for all } \gamma \in \Gamma.$$

If additionally F is regular on the boundary of $\mathcal{H}_p^*$, it is called a *holomorphic modular form.*

If the signature of $V_\mathbb{R}$ is (b_+, b_-) with $b_+ \geq 3$ the Koecher principle implies that holomorphic modular forms are automatically regular on the boundary of $\mathcal{H}_p$ (see [22, Chapter IV.3] for a proof). Also note that in this case (i.e. $b_+ > 3$) multiplier systems are always of finite order by a result of Margulis [49]. In particular, this is the case for $V_\mathbb{R}$ with the real bilinear form $(\cdot, \cdot)_\mathbb{R}$ of signature $(2p, 2)$ if $p > 1$.

2.3. The embedding

In this section, following [33, 35, 34] we describe the setup of the embedding between the symmetric domains of $\mathrm{U}(p, 1)$ and $\mathrm{O}(2p, 2)$, which is used in Section 2.5 to construct Borcherds products (see Section 2.4) via pull-back.

The underlying quadratic space The identification of $V, (\cdot, \cdot)$ with the underlying rational quadratic $V_\mathbb{R}, (\cdot, \cdot)_\mathbb{R}$ induces am embedding of the unitary group $\mathrm{U}(V)$ into the special orthogonal group $\mathrm{SO}(V_\mathbb{R})$ associated with the bilinear form $(\cdot, \cdot)_\mathbb{R}$. Thus, we may consider $\mathrm{U}(V)$ as a subgroup of $\mathrm{SO}(V_\mathbb{R})$.

Similarly, an even hermitian lattice $L \subset V_\mathbb{F}$ is also an even lattice in $V_\mathbb{Q}$, with the quadratic form $Q(\cdot)$, and the $\mathbb{Z}$-dual $L^\sharp$ is the dual of L both as a hermitian lattice over $\mathcal{O}_\mathbb{F}$ and as a quadratic module over $\mathbb{Z}$. Thus, $L^\sharp/L$ is the discriminant group, either way, and the discriminant kernel $\Gamma_L \subset \mathrm{U}(L)$ is a subgroup of the discriminant kernel $\Gamma_L^{\mathrm{O}} \subset \mathrm{SO}^+(L)$.

We introduce the following notation for endomorphisms of the real space $V_\mathbb{R}$ induced from scalar multiplication on V (which are trivial for real scalars, of course):

Definition 2.15. Let α be in $\mathbb{C} \setminus \mathbb{R}$. We denote by $\hat{\alpha}$ the endomorphism of $V_\mathbb{R}, (\cdot, \cdot)_\mathbb{R}$ induced from the scalar multiplication with α in the complex hermitian space V, $(\cdot, \cdot)$. For typographic reasons, the endomorphism induced by the complex unit i is denoted $\hat{\imath}$. Note that $\hat{\imath} \in \mathrm{SO}(V_\mathbb{R})$.

We note that if $\alpha \in \mathbb{F}$, then $\hat{\alpha}$ is an endomorphism of $V_\mathbb{Q}$ defined over $\mathbb{Q}$.

Embedding of symmetric domains If $\mathcal{K}$ is a maximal compact subgroup of $\mathrm{U}(V)$, embedded into $\mathrm{SO}(V_{\mathbb{R}})$, there exists a maximal compact subgroup $\mathcal{K}_{\mathrm{SO}}$ of $\mathrm{SO}(V_{\mathbb{R}})$ with $\mathcal{K} \hookrightarrow \mathcal{K}_{\mathrm{SO}}$. Thus, we can embed the respective symmetric domains given by the group quotients

$$\varepsilon : \ \mathbb{D} = \mathrm{U}(V)/\mathcal{K} \hookrightarrow \mathrm{SO}(V_{\mathbb{R}})/\mathcal{K}_{\mathrm{SO}} = \mathbb{D}_{\mathrm{O}}. \tag{2.3.1}$$

Also, since $\mathbb{D} \simeq \mathcal{C}_{\mathrm{U}}$, through this embedding, we can identifying an element $[z] \in \mathcal{C}_{\mathrm{U}}$ with a subspace $v = \mathbb{R}X_L + \mathbb{R}Y_L$ contained in $\mathbb{D}_{\mathrm{O}}$. Further, the bijection between $\mathbb{D}_{\mathrm{O}}$ and $\mathcal{C}_{\mathrm{O}}$ yields an embedding of $\mathcal{C}_{\mathrm{U}}$ into $\mathcal{C}_{\mathrm{O}}$. Like the bijection between $\mathbb{D}_{\mathrm{O}}$ and $\mathcal{C}_{\mathrm{O}}$, this embedding is real analytic. Moreover, by carefully choosing the oriented basis vectors X_L, Y_L for the image of $[z]$ in $\mathcal{C}_{\mathrm{O}}$, it can be realized as a holomorphic embedding. Then, finally, with suitable coordinates, we also obtain an embedding of $\mathcal{H}_{\ell,\ell'}$ into the tube domain $\mathcal{H}_p$.

To summarize, the embedding in (2.3.1) induces embeddings between the different models, Grassmannian, projective cones or affine, for the symmetric domains of the unitary group in the left column and the orthogonal group in the right column of the following diagram:

$$\begin{tikzcd}
z \in \mathbb{D} \arrow[r, hook, "\varepsilon=\varepsilon_{Gr}"] \arrow[d, leftrightarrow] & \mathbb{D}_{\mathrm{O}} \ni v \arrow[d, leftrightarrow] \\
[\mathfrak{z}] \in \mathcal{C}_{\mathrm{U}} \arrow[r, hook, "\varepsilon_{pr}"] \arrow[d, leftrightarrow] & \mathcal{C}_{\mathrm{O}} \ni Z_L \arrow[d, leftrightarrow] \\
(\tau,\sigma) \in \mathcal{H}_{\ell,\ell'} \arrow[r, hook, "\varepsilon_{af}"] & \mathcal{H}_p \ni Z
\end{tikzcd} \tag{2.3.2}$$

To facilitate notation, we will usually denote all of these maps simply by ε, since its should usually be clear from the context which of them is the actual map under consideration.

Choice of cusp Assume that as in Section 2.1 we are given $\mathcal{H}_{\ell,\ell'}$ with a fixed choice of a primitive isotropic vector $\ell \in L$ and of $\ell' \in L^\sharp$ with $(\ell, \ell') \neq 0$. We want to explicitly describe the embedding of the Siegel domain model $\mathcal{H}_{\ell,\ell'}$ into the tube domain model $\mathcal{H}_p$. Since the cusp at infinity of $\mathcal{H}_p$ is given by a primitive isotropic lattice vector e_1, we set $e_1 = \ell$. Hence, ℓ corresponds to the cusp at infinity both for $\mathcal{H}_{\ell,\ell'}$ and for $\mathcal{H}_p$.

Remark 2.16. With this definition, the parabolic subgroup $P_\ell \subset \mathrm{U}(V)$ stabilizing ℓ is mapped into the stabilizer of e_1 in $\mathrm{SO}(V_{\mathbb{R}})$. In particular, the elements of the Heisenberg group $\mathrm{Heis}(\Gamma_L)$ are mapped to transformations generated by Eichler elements in $\mathrm{SO}^+(V)$. For example, it is easily verified that a translation of the forms $[0, r]$ can be identified with an Eichler element of the form $E(\ell, \frac{r}{2}\hat{\imath}\ell)$, see [33, Chapter 3.1.1].

Complex structure Now recall how on the one hand, for every projective line in $\mathcal{C}_{\mathrm{U}}$ represented by $\mathfrak{z} \in V$, the negative definite line $z = \mathbb{C}\mathfrak{z}$ (i.e. the corresponding element of $\mathbb{D}$) can be considered as a two-dimensional (real) subspace of V_R contained in $\mathbb{D}_{\mathrm{O}}$. While on the other hand, for an element of $\mathbb{D}_{\mathrm{O}}$, the choice of an oriented basis (X_L, Y_L), of the form (2.2.1), determines an element $[X_L + iY_L = Z_L]$ in $\mathcal{C}_{\mathrm{O}}$.

Thus, for each representative $\mathfrak{z}$ for $[\mathfrak{z}]$ of the form (2.1.1), i.e. $\mathfrak{z} = \mathfrak{z}(\tau, \sigma)$ we want to fix the choice of X_L and Y_L, in such a manner that the map $\varepsilon_{pr} : \mathcal{C}_{\mathrm{U}} \mapsto \mathcal{C}_{\mathrm{O}}$ extends to a homomorphism between the complex projective spaces $\mathbb{P}^1 V$ and $\mathbb{P}^1 V_{\mathbb{C}}$.

Consider the following diagram:

$$\begin{tikzcd}
\mathfrak{z} \arrow[r] \arrow[d, "i"] & (X_L, Y_L) \arrow[r] \arrow[d, "i"] & Z_L = X_L + iY_L \arrow[d, "i"] \\
i\mathfrak{z} \arrow[r] & (\hat{\imath} X_L, \hat{\imath} Y_L) \arrow[r] & iZ_L = -Y_L + iX_L
\end{tikzcd} \tag{2.3.3}$$

To the left and to the right of the diagram, the complex unit i acts as a scalar of the complex spaces $V, (\cdot,\cdot)$ and $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}, (\cdot,\cdot)_{\mathbb{R}}$, respectively. In the middle column of the diagram, by definition, it acts as the endomorphism $\hat{\imath}$ on the real space $V_{\mathbb{R}} \ni X_L, Y_L$. Note that all arrows in (2.3.3) represent $\mathbb{R}$-linear maps. Thus, if (2.3.3) commutes, the following diagram also commutes for every $\alpha \in \mathbb{C} \setminus \{0\}$

$$\begin{tikzcd}
{[\mathfrak{z}]} \arrow[r, "\varepsilon"] \arrow[d, equal] & {[Z_L]} \arrow[d, equal] \\
{[\alpha\mathfrak{z}]} \arrow[r, "\varepsilon"] & {[\alpha Z_L].}
\end{tikzcd}$$

Then, the embedding ε_{pr} between $\mathcal{C}_{\mathrm{U}} \subset \mathbb{P}^1 V$ and $\mathcal{C}_{\mathrm{O}} \subset \mathbb{P}^1 V_{\mathbb{C}}$ is indeed a homomorphism of complex projective spaces. Moreover, the induced embedding ε_{af} between the affine models $\mathcal{H}_{\ell,\ell'}$ and $\mathcal{H}_p$ in (2.3.2) is holomorphic, too.

Since X_L and Y_L are contained in $\mathbb{C}\mathfrak{z}$, we set $X_L = \hat{\psi}\mathfrak{z}$. Then, clearly (2.3.3) commutes exactly if $\hat{\imath} X_L = -Y_L$. Also, in this case, $X_L \perp Y_L$ and $X_L^2 = Y_L^2 = |\psi|^2(\mathfrak{z},\mathfrak{z}) > 0$, as required in (2.2.1).

Normalization with respect to e_1. Now, with $X_L = \hat{\psi}\mathfrak{z}$, $Y_L = -\hat{\imath}\hat{\psi}\mathfrak{z}$ the point $[Z_L]$ in $\mathbb{P}^1 V_{\mathbb{C}}$ with $Z_L = X_L + iY_L$ lies in the positive cone $\mathcal{C}$ of the zero quadric $\mathcal{N}$. We may also assume that it lies in the correct connected component $\mathcal{C}_{\mathrm{O}}$.

To complete the definition of the affine embedding $\varepsilon_{af} : \mathcal{H}_{\ell,\ell'} \to \mathcal{H}_p$, we should fix ψ. Recall that for each point in the image of the map $\mathcal{H}^{\pm} \to \mathcal{C}^{\pm}$, there is unique representative of the form (2.2.2). Thus, for $Z_L = X_L + iY_L$ to be of this form, we require

$$(X_L, e_1)_{\mathbb{R}} = \mathrm{Re}(\psi\mathfrak{z}, \ell) = 1 \quad \text{and} \quad (Y_L, e_1)_{\mathbb{R}} = \mathrm{Re}(-i\psi\mathfrak{z}, \ell) = 0.$$

Hence, we set $\psi = (\ell', \ell)^{-1}$.

To summarize, for each z with $z = \mathbb{C}\mathfrak{z}$, a suitable choice of basis vectors for its image v in $\mathbb{D}_{\mathrm{O}}$, is given by

$$X_L = \widehat{\left(\frac{1}{(\ell,\ell')}\right)}\mathfrak{z} \quad \text{and} \quad Y_L = \widehat{\left(\frac{-i}{(\ell,\ell')}\right)}\mathfrak{z}, \tag{2.3.4}$$

where $\mathfrak{z}$ is a representative for $[\mathfrak{z}]$ of the form (2.1.1).

Lemma 2.17. *The pullback of a holomorphic (meromorphic) automorphic form on $\mathcal{H}_p$ is a holomorphic (meromorphic) automorphic form on $\mathcal{H}_{\ell,\ell'}$.*

Proof. Denote by $\widetilde{\mathcal{C}_{\mathrm{O}}}$ and $\widetilde{\mathcal{C}_{\mathrm{U}}}$ the preimages of $\mathcal{C}_{\mathrm{O}}$ and $\mathcal{C}_{\mathrm{U}}$ under the canonical projections $\pi : V \to \mathbb{P}^1 V$ and $\pi_{\mathbb{C}} : V_{\mathbb{C}} \to \mathbb{P}^1 V_{\mathbb{C}}$, respectively. Denote by $\tilde{\varepsilon}$ the map $V \to V_{\mathbb{C}}$ induced from ε. Note that with the above choice, $\tilde{\varepsilon}$ is $\mathbb{C}$-linear.

Let $f : \mathcal{H}_p \to \mathbb{C}$ be a holomorphic automorphic form of weight k for some modular group $\Gamma \subset \Gamma_L^{\mathrm{O}}$. Then, f can be identified with a holomorphic function $\tilde{f} : \widetilde{\mathcal{C}_{\mathrm{O}}} \to \mathbb{C}$, which is $\mathbb{C}$-homogeneous of weight $-k$ and invariant under the operation of Γ on $V_{\mathbb{C}}$.

Consider the pullback $\tilde{\varepsilon}^* \tilde{f}$. Since $\tilde{\varepsilon}$ is $\mathbb{C}$-linear, the pullback is holomorphic and also $\mathbb{C}$-homogeneous of weight $-k$. Further, it is invariant under $\Gamma' := \Gamma \cap \mathrm{U}(L)$. Clearly Γ' has finite index in $\Gamma_L \subset \mathrm{SO}^+(L)$, as the index of Γ in Γ_L^{O} is finite and $\Gamma_L \subset \Gamma_L^{\mathrm{O}}$. Then, $\varepsilon^* f$, the attached function on $\mathcal{H}_{\ell,\ell'}$, is a holomorphic automorphic form of weight k on Γ'. The proof for meromorphic f is similar. $\square$

A choice of basis for the isotropic subspaces The four dimensional real subspace of $V_{\mathbb{R}}$ defined by the $\mathbb{C}$-span of ℓ and ℓ' in V can obviously be decomposed into two totally real isotropic subspaces F and F'. Recall that we have set $e_1 = \ell$, we want to find further (rational) basis vectors e_2, e_3 and e_4 such that $F = \operatorname{span}_{\mathbb{R}}\{e_1, e_3\}$ and $F' = \operatorname{span}_{\mathbb{R}}\{e_2, e_4\}$, and satisfying $(e_j, e_j)_{\mathbb{R}} = 0$ for $j \in \{1, \ldots, 4\}$ and $(e_j, e_k)_{\mathbb{R}} = 0$ for $j < k$, unless $(j,k) \in \{(1,2),(3,4)\}$.

In a basis with these properties, by (2.2.5), we can write a point Z_L in the image of the embedding $\varepsilon(\mathcal{H}_{\ell,\ell'})$ in the form

$$\begin{aligned} Z_L(\tau,\sigma) &= -z_1 z_2 e_1 + e_2 + z_1 e_3 + z_4 e_4 + \mathfrak{Z}(\sigma), \\ \text{with} \quad \mathfrak{Z} &= \mathfrak{X} + i\mathfrak{Y} = \left(\widehat{\frac{1}{(\ell,\ell')}}\right)\sigma + i\left(\widehat{\frac{-i}{(\ell,\ell')}}\right)\sigma. \end{aligned} \tag{2.3.5}$$

Note that since $\mathfrak{X}$ and $\mathfrak{Y}$ have the same norm and $\mathfrak{X} \perp \mathfrak{Y}$ (as $(x, \hat{\imath}x)_{\mathbb{R}} = 0$ for all $x \in V_{\mathbb{R}}$),

$$Q(\mathfrak{Z}) = Q(\mathfrak{X}) - Q(\mathfrak{Y}) = 0, \quad \text{whereas} \quad \left(\mathfrak{Z}, \overline{\mathfrak{Z}}\right)_{\mathbb{R}} = Q(\mathfrak{X}) + Q(\mathfrak{Y}) > 0.$$

To determine the basis vectors, we set

$$e_3 = \hat{\gamma}\ell, e_2 = \hat{a}\ell', e_4 = \hat{b}\ell',$$

with complex parameters $\gamma, a, b \in \mathbb{C}^\times$, with $\gamma, \frac{a}{b} \notin \mathbb{R}$, and find that a and b are given by

$$a = \frac{1}{(\ell,\ell')}\left(1 - i\frac{\operatorname{Re}\gamma}{\Im\gamma}\right), \quad b = \frac{i}{(\ell,\ell')\Im\gamma},$$

leaving γ as the only parameter, which we are free to choose.

Since $Z_L = X_L + iY_L$ is of the form (2.3.5), using (2.3.4), one finds

$$Z_L = \left(-\theta\bar{\gamma}\tau, 1, \theta\tau, \bar{\gamma}, \mathfrak{Z}\right),$$

where we have set $\theta := \frac{|\delta|}{2\Im\gamma}$, with δ from (2.1.1).

If one now chooses γ with $\Im\gamma = \frac{1}{2}|\delta|$, then θ becomes 1. Also note that if $\gamma \in \mathcal{O}_{\mathbb{F}}$, we have $e_3 \in L$ and $e_4 \in L^\sharp$. Further, if $\delta = \delta_{\mathbb{F}}$, let $\kappa_{\mathbb{F}}$ be a generator $\kappa_{\mathbb{F}} \in \mathcal{O}_{\mathbb{F}}$ with $\mathcal{O}_{\mathbb{F}} = \mathbb{Z} + \kappa_{\mathbb{F}}\mathbb{Z}$ and $i\Im\kappa_{\mathbb{F}} = \frac{1}{2}\delta_{\mathbb{F}}$. Then, after setting $\gamma = \kappa_{\mathbb{F}}$, one has

$$Z_L = (-\bar{\kappa}_{\mathbb{F}}\tau, 1, \tau, \bar{\kappa}_{\mathbb{F}}, \mathfrak{Z})\,.$$

Further, in this case, the basis vectors e_j, $j = 1, \ldots, 4$ are all contained in $L^\sharp$:

$$e_1 = \ell,\; e_3 = \kappa_{\mathbb{F}}\ell,\; e_2 = \frac{2\kappa_{\mathbb{F}}}{\delta_{\mathbb{F}}(\ell,\ell')}\ell',\; e_4 = -\frac{2}{(\ell,\ell')\delta_{\mathbb{F}}}\ell', \tag{2.3.6}$$

wherein all complex factors are to be understood as scalars for the complex hermitian space V, thus acting as endomorphisms of $V_{\mathbb{R}}$.

The embedding on the boundary

Proposition 2.18. *Boundary points of $\mathcal{H}_{\ell,\ell'}$ are mapped to one-dimensional boundary components of $\mathcal{H}_p$. The boundary point attached to the primitive isotropic lattice vector ℓ is mapped into the boundary component attached to the rational isotropic subspace $F_{\mathbb{Q}} = \mathbb{Q}\ell \oplus \mathbb{Q}\hat{\kappa}_{\mathbb{F}}\ell \subset V_{\mathbb{Q}}$ of the quadratic space $V_{\mathbb{R}}, (\cdot,\cdot)_{\mathbb{R}}$.*

Proof. The boundary points of $\mathcal{H}_{\ell,\ell'}$ can be described by isotropic lines in V. Let $0 \neq x \in V$ with $(x,x) = 0$. Then, in the quadratic space $V_{\mathbb{R}}, (\cdot,\cdot)_{\mathbb{R}}$, the two vectors x and $\hat{\xi}x$, are isotropic and, for $\xi \notin \mathbb{R}$, linear independent. Hence, $F = \mathbb{R}x \oplus \mathbb{R}\hat{\xi}x$ is a two-dimensional isotropic subspace of $V_{\mathbb{R}}, (\cdot,\cdot)_{\mathbb{R}}$ and corresponds to a boundary component of $\mathcal{H}_p$.

Further, if $x \in V_{\mathbb{F}}$, the isotropic line $\mathbb{F}x \subset V_{\mathbb{F}}$ defines a two-dimensional rational (i.e. contained in $V_{\mathbb{Q}}$) isotopic subspace of $V_{\mathbb{R}}$, $(\cdot,\cdot)_{\mathbb{R}}$, and thus defines a rational boundary component. In particular, this is the case for $x = \ell$. □

We now examine how the neighborhood of a cusp behaves under the embedding ε. It suffices to consider the cusp at infinity, $[\ell]$.

Lemma 2.19. *Consider the boundary point at infinity of $\mathcal{H}_{\ell,\ell'}$, attached to ℓ. The inverse image of every open neighborhood in the closure of $\mathcal{C}_{\mathrm{O}}$ of this the boundary point contains an open neighborhood of infinity in $\mathcal{C}_{\mathrm{U}} \cup I_{\ell,\mathbb{C}}$.*

Proof. Consider the two-dimensional rational isotropic subspace $F_{\mathbb{Q}} = \mathbb{Q}e_1 \oplus \mathbb{Q}e_2$ of $V_{\mathbb{Q}}$, and let $F_{\mathbb{C}} = F \otimes_{\mathbb{Q}} \mathbb{C}$. Then $F_{\mathbb{C}}$ is the image of $I_{\ell,\mathbb{C}}$ under ε. Let $[x]$ be a point in the one-dimensional boundary component of $\mathcal{C}_{\mathrm{O}}$ defined by $F_{\mathbb{C}}$. Denote by $\pi_{\mathbb{C}}$ the canonical projection from $V_{\mathbb{C}}$ to $\mathbb{P}^1 V_{\mathbb{C}}$. In the zero quadric $\mathcal{N}$, a neighborhood of $[x]$ is a union of the form $U \cup V$, with U open in $\mathcal{C}_{\mathrm{O}}$ and V a subset of $\pi_{\mathbb{C}}(F_{\mathbb{C}}) \cap \partial\mathcal{C}_{\mathrm{O}}$, open with respect to the subset topology. For a more precise description, identify $F_{\mathbb{C}} \cap \pi_{\mathbb{C}}^{-1}(\partial\mathcal{C}_{\mathrm{O}})$ with the upper half-plane $\mathbb{H} \subset \mathbb{C}$ via (2.2.6). Recall the definition of the neighborhoods $V(x,\epsilon)$ and $U(x,\epsilon)$ of x from (2.2.7). Now,

$$\lim_{t\to\infty} \varepsilon(z(it,\sigma)) = \lim_{t\to\infty} [-it\bar{\kappa}_{\mathbb{F}}, 1, it, \bar{\kappa}_{\mathbb{F}}, \varepsilon(\sigma)] = [-\bar{\kappa}, 0, 1, 0, 0], \tag{2.3.7}$$

we have $x = \bar{\kappa}_{\mathbb{F}} e_1 - e_3$. Now, for every $Z \in \varepsilon(\mathcal{H}_{\ell,\ell'})$, clearly $z_2 = \bar{\kappa}_{\mathbb{F}} \in V(x,\epsilon)$ for all $\epsilon > 0$. For $Z \in \varepsilon(\mathcal{H}_{\ell,\ell'})$, the imaginary part Y is given by

$$Y = \Im\tau e_3 - \frac{|\delta|}{2} e_4 - (i\widehat{(\ell,\ell')})^{-1}\sigma$$

for some $(\tau,\sigma) \in \mathcal{H}_{\ell,\ell'}$. Thus, if $Z \in U(x,\epsilon) \cap \varepsilon(\mathcal{H}_{\ell,\ell'})$, we have

$$+\frac{|\delta|}{2}\Im\tau - \frac{(\sigma,\sigma)}{|(\ell',\ell)|^2} > \frac{1}{\epsilon}.$$

It follows that (τ,σ) is contained in one of the neighborhoods of infinity $U(\epsilon',\ell)$, as introduced in (2.1.5)

$$U(\epsilon',\ell) = \left\{(\tau,\sigma) \in \mathcal{H}_{\ell,\ell'};\, 2\Im\tau|\delta||(\ell,\ell')|^2 - (\sigma,\sigma) > C = \frac{1}{\epsilon'}\right\},$$

where, in this case, $C = 2|(\ell',\ell)|^2/\epsilon$. $\square$

2.4. Borcherds products for orthogonal groups

2.4.1. The work of Borcherds and Bruinier's take

In [2], Borcherds used a generalized Siegel theta function on $\mathbb{D}_{\mathrm{O}}$ to construct a lifting of weakly holomorphic modular forms to automorphic forms for the orthogonal group. In our overall setting and with the notation of the current section, one can use the following Siegel theta function. Let $v \in \mathbb{D}_{\mathrm{O}}$, $\tau \in \mathbb{H}$ and $r,t \in V_{\mathbb{R}}$ and set

$$\Theta(\tau,v;r,t)_L = \sum_{h\in L^\sharp/L}\sum_{\lambda\in L+h} e\left(\tau Q\left((\lambda+t)_{v^\perp}\right) + \bar\tau Q\left((\lambda+t)_v\right) - \left(\lambda+\tfrac{1}{2}t, r\right)_{\mathbb{R}}\right)\mathfrak{e}_h.$$

Now, we define the regularized theta lift of a weakly holomorphic modular form $f \in \mathrm{M}^!_{k,L^-}$ of weight $k = 1-p$ as

$$\Phi_f(v) = \int^{reg}_{\mathrm{SL}_2(\mathbb{R})\backslash\mathbb{H}} \left\langle f(\cdot), \overline{\Theta(\cdot,v)_L}\right\rangle_L d\mu = (f,\Theta(\cdot,z))^{reg}_{L^-}.$$

with the regularized pairing and the regularization procedure introduced in Section 1.4.

Remark 2.20. We note that in [2] and [5] a slightly different definition of the theta integral was used. In our signature convention (the opposite of that in loc. cit.) and for the theta function $\Theta(\tau,v)_L$ introduced above one could similarly define the regularized lift of a weakly holomorphic modular form $g \in \mathrm{M}^!_{p-1,L}$ transforming under ρ_L by setting

$$\Phi_g(v) = \int^{reg}_{\mathrm{SL}_2(\mathbb{R})\backslash\mathbb{H}} \langle g(\cdot), \Theta(\cdot,v)_L\rangle_L\, v^{p-1} d\mu.$$

Essentially, this is a regularized form of the weight k Petersson pairing for ρ_L defined analogously to $(\cdot,\cdot)^{reg}_{L^-}$. See also [10, Remark 5.3].

Borcherds' approach to evaluating this type of integral involved using a partial Poisson summation to express the integral in terms of Poincaré series for the lattice $K = L \cap e_1^\perp \cap e_2^\perp$ (a further reduction step also gives a contribution for the – in the present signature – definite lattice $K \cap e_3^\perp \cap e_4^\perp$). This yields an additive lifting, the Fourier expansion of which is essentially the logarithm of the absolute value of a function $\Psi(Z)$ ($Z \in \mathcal{H}_p$) (similar to (2.4.6) below), the multiplicative lift f. The function $\Psi(Z)$ is a meromorphic automorphic form taking its zeros and poles along special cycles and with an absolutely convergent infinite product expansion (see Theorem 2.24).

Bruinier generalized Borcherds' construction by using a regularized pairing of Hejhal Poincarè series with the Siegel theta function. Set

$$\Phi_{n,h}(v, s) := \lim_{t \to \infty} \int_{F_t} \langle F_{n,h}(\cdot, s), \Theta(\cdot, v)_L \rangle_L \, d\mu,$$

with the Hejhal Poincaré series of weight k and index (n, h) and let $s_0 = 1 - \frac{k}{2}$. Up to a constant term resulting from the slightly different regularization procedure, $\Phi_{n,h}(v, s_0)$ agrees with $\Phi_{F_{n,h}}(Z)$ see [5, Prop 2.11]. Since the Hejhal Poincaré series span the weak harmonic Maass forms, the lift of a weakly holomorphic modular form $f \in \mathrm{M}^!_{k,L^-}$, in particular, can be expressed in terms of the $\Phi_{n,h}$'s. Using these Poincaré series yields not only the terms contributing to the product expansion but also further contributions which allowed Bruinier to extend his lifting into the cohomology [see 5, Chap. 5]. In Chapter 3 we will use the pairing of Hejhal Poincaré series with the theta function associated to the Schwartz form ψ (which replaces φ_0 for unitary groups in signatures (p, q) with $q > 1$) in a somewhat similar fashion to construct Green functions, see Sections 3.6 and 3.7.

First we review some of the results on the construction of Borcherds products from [5].

2.4.2. Special cycles and Weyl chambers

v Similar to (1.1.1) and (1.1.2), one defines codimension one sub-Grassmannians in $\mathbb{D}_\mathrm{O}$ associated to lattice vectors by setting

$$\begin{aligned} \mathbb{D}_\mathrm{O}(x) &:= \{v \in \mathbb{D}_\mathrm{O}; v \perp x\} \quad (x \in L^\sharp, Q(x) > 0) \\ \text{and} \quad \mathbb{D}_\mathrm{O}(x) &= \emptyset \quad (x \in L^\sharp, Q(x) \leq 0). \end{aligned}$$

Further for an index (n, h) with $n \in \mathbb{Q}$ and $h \in L^\sharp/L$, one defines a special cycle, also known as a *Heegner divisor*, by setting

$$\mathbb{D}_\mathrm{O}(n, h) := \sum_{\substack{\lambda \in L+h \\ (\lambda,\lambda)=n}} \mathbb{D}_\mathrm{O}(\lambda). \tag{2.4.1}$$

As in the unitary case, special cycles of this type are invariant under the action of an orthogonal modular group hence can viewed as the preimage of a special cycle on the modular variety.

We will use the same notation, $\mathbb{D}_\mathrm{O}(x)$ and $\mathbb{D}_\mathrm{O}(n, h)$ to denote the corresponding special cycles in the complex cone $\mathcal{C}_\mathrm{O}$ and in the tube domain model $\mathcal{H}_p$.

Now, assume that $e_1 \in L$ is primitive. Define N_{e_1} as the unique positive integer such that $2(L, e_1)_{\mathbb{R}} = N_{e_1}\mathbb{Z}$. Now, define a $\mathbb{Z}$-submodule of the dual lattice $L^\sharp$ as follows:

$$L_0^\sharp := \left\{\lambda \in L^\sharp \, ; \, 2(\lambda, e_1)_{\mathbb{R}} \equiv 0 \mod N_{e_1}\right\}.$$

Clearly, $L \subseteq L_0^\sharp \subset L^\sharp$ and $L_0^\sharp/L \subseteq L^\sharp/L$.

Remark 2.21. We remark that if L and $L^\sharp$ are hermitian lattices in the sense defined in Chapter 1 (i.e. projective modules over the ring of integers $\mathcal{O}_{\mathbb{F}}$ in an imaginary quadratic number field $\mathbb{F}$), the lattice $L_0^\sharp$ is not, in general, hermitian in this sense, but the multiplier ideal of $L_0^\sharp$ is an order $\mathcal{O}$ of $\mathbb{F}$, with $N_{e_1}\mathcal{O}_{\mathbb{F}} \subset \mathcal{O} \subset \mathcal{O}_{\mathbb{F}}$.

The Lorentzian space and Weyl chambers Next, we define Weyl chambers for an index (n, h), where $n \in \mathbb{Q}$, $h \in L^\sharp/L$, as connected components of $\mathcal{H}_p$ (and the corresponding subsets of $\mathcal{C}_{\mathrm{O}}$ and $\mathbb{D}_{\mathrm{O}}$). For this, we need to take a look at the Lorentzian space $K \otimes \mathbb{R}$, where $K = L \cap e_1^\perp \cap e_2^\perp$, with the restriction of $(\cdot, \cdot)_{\mathbb{R}}$ to $K \otimes \mathbb{R} \subset V_{\mathbb{R}}$. The reason for this lies in the reduction process described above.

Denote by $\mathbb{D}_{\mathrm{O}}(K)$ the symmetric domain for the operation of $\mathrm{SO}(K \otimes \mathbb{R})$ on $K \otimes \mathbb{R}$. Beside the Grassmannian model, which in this case consists of one-dimensional negative definite subspaces, there is also a hyperboloid model and an upper half-space model, cf. [5, Chap. 3.1] for details.

Now as before, define special cycles with $\mathbb{D}_{\mathrm{O}}(x)$ for $x \in K \otimes \mathbb{R}$ as codimension one sub-Grassmannians, i.e. as codimension one hyperplanes. Further, define Heegner divisors $\mathbb{D}_{\mathrm{O}}(m, h)$ for $m \in \mathbb{Q}$ and $h \in K^\sharp/K$, as locally finite sums of special cycles $\mathbb{D}_{\mathrm{O}}(\kappa)$, with $\kappa \in K + h$ and $Q(\kappa) = n$, similarly to (2.4.1). We note that the discriminant group $K^\sharp/K$ has cardinality $\left|L^\sharp/L\right| = N_{e_1}^2 \cdot \left|K^\sharp/K\right|$.

For $n \in \mathbb{Q}$ and $h \in K^\sharp/K$, *Weyl chambers* of index (n, h) in $\mathbb{D}_{\mathrm{O}}(K)$ are defined as the connected components of $\mathbb{D}_{\mathrm{O}}(K) \setminus \mathbb{D}_{\mathrm{O}}(n, h)$ in the hyperboloid model. If W is a Weyl chamber of index (n, h), viewed as a subset of $K \otimes \mathbb{R}$, then for any $\lambda \in L^\sharp$, with $Q(\lambda) = n$ and $\lambda \equiv h \pmod{L}$, the inner product $(\lambda, x)_{\mathbb{R}}$ has constant sign on W, i.e. if $(\lambda, x_0)_{\mathbb{R}} > 0$ for one $x_0 \in W$, it is non-negative for every $x \in W$ [cf. 5, Lemma 3.2]. Hence, in this case, we write $(W, \lambda)_{\mathbb{R}} > 0$, and similarly $(W, \lambda)_{\mathbb{R}} < 0$ if $(x_0, \lambda)_{\mathbb{R}} < 0$ for any $x_0 \in W$.

Further, one defines a projection from $L_0^\sharp$ to $K^\sharp$ with the property that $p(L) = K$ as follows. If ν is a vector in L with $2(\nu, \nu)_{\mathbb{R}} = N_{e_1}$ and $\lambda \in L^\sharp$, denote by $\nu_K = p_K(\nu)$ and $\lambda_K = p_K(\lambda)$ the projections to the rational Lorentzian space $K \otimes \mathbb{Q}$. Then, the projection

$$p(\lambda) = \lambda_K - \frac{2(\lambda, e_1)_{\mathbb{R}}}{N_{e_1}}\nu_K \tag{2.4.2}$$

takes L to K and induces a surjective map from $L_0^\sharp/L$ to $K^\sharp/K$. Let $\kappa \in K^\sharp$. A system of representatives for $\beta \in L_0^\sharp/L$, with $p(\beta) = \kappa + K$ is given by $\beta - 2(\beta, \mu)_{\mathbb{R}}/N_{e_1} + be_1/N_{e_1}$, where b runs over a system of representatives modulo N_{e_1}.

Finally, one defines Weyl chambers in $\mathbb{D}_{\mathrm{O}}$ with index (n, h), with $n \in \mathbb{Q}$ and $h \in L^\sharp/L$, by considering the projection $p(h) \in K^\sharp/K$ and the Weyl chambers in $\mathbb{D}_{\mathrm{O}}(K)$ with index

$(m, p(h))$. Thus, if W is such a Weyl chamber in $\mathbb{D}_{\mathrm{O}}(K)$, the set

$$\left\{ Z = X + iY \in \mathcal{H}_p\,;\; \frac{Y}{|Y|} \in W \right\} \subset \mathcal{H}_p,$$

is called a Weyl chamber of $\mathcal{H}_p$, also denoted by W. The corresponding subsets of $\mathbb{D}_{\mathrm{O}}$ are similarly considered as Weyl chambers.

Then, for a lattice vector $\lambda \in L^\sharp$, the sign with respect to a given Weyl chamber W is defined as the sign of the projection $p(\lambda)$ with respect to the corresponding Weyl chamber of $\mathbb{D}_{\mathrm{O}}(K)$, i.e. as the sign of $(x, p(\lambda))_{\mathbb{R}}$ for any $x \in W \subset K \otimes \mathbb{R}$.

2.4.3. Decomposition of the lift

As indicated at the beginning of this section, the lift $\Phi_{n,h}(Z)$ of the Hejhal Poincaré $F_{n,h}$ series in [5] yields not only infinite product expansions of Borcherds type, but also terms bearing on a lifting into the cohomology. To separate the two, one decomposes the function $\Phi_{n,h}(Z)$ as follows, see [5, Definition 3.11],

$$\Phi_{n,h}(Z) = \psi_{n,h}(Z) + \xi_{n,h}(Z).$$

For $\gamma \in L^\sharp/L$ and $\ell \in \mathbb{Z} + Q(\gamma)$, denote by $b(\gamma, \ell) = b_{n.h}(\gamma, k)$ the Fourier coefficients of $F_{n,h}(\tau, 1 - \frac{k}{2})$. The first component function $\xi_{n,h} : \mathcal{H}_p \to \mathbb{R}$ is given by

$$\begin{aligned} \xi_{n,h} = \tfrac{1}{2}\sqrt{2}|Y|\,\xi^K_{n,h} - b(0,0)\log\left(Y^2\right) + \frac{2}{\sqrt{\pi}} \sum_{\substack{\lambda \in K^\sharp \\ Q(\lambda)>0}} \sum_{\substack{\delta \in L_0^\sharp/L \\ p(\delta)=\lambda+K}} b(\delta, -Q(\lambda)) \\ \cdot \sum_{\ell \geq 1} \frac{1}{\ell} \cdot e\left(\ell(\delta, e_2)_{\mathbb{R}} + \ell(\lambda, X)_{\mathbb{R}}\right) \cdot \mathcal{V}_{2-l}\big(\pi\ell|\lambda||Y|, \pi\ell(\lambda, Y)_{\mathbb{R}}\big), \end{aligned} \tag{2.4.3}$$

wherein $\xi^K_{n,h}$ is either zero, if $(h, z)_{\mathbb{R}} \not\equiv 0 \pmod{N_{e_1}}$, or, if $(h, z)_{\mathbb{R}} \equiv 0 \pmod{N_{e_1}}$, it is given by $\xi^K_{n,p(h)}$, defined in loc.cit. Definition 3.3. This is a part of the similar decomposition of the lift $\Phi^K_{n,p(h)} = \xi^K_{n,p(h)} + \psi^K_{n,p(h)}$, the second part of which, $\psi^K_{n,p(h)}$, in turn, contributes to $\psi_{n,h}$, the second component function of $\Phi_{n,h}$. It turns out (see loc. cit. Theorem 3.4), that $\psi^K_{n,p(h)}$ is piecewise linear on $\mathbb{D}_{\mathrm{O}}(K)$, with singularities along the Heegner divisor $\mathbb{D}_{\mathrm{O}}(n, p(h))$. Hence, for a Weyl chamber W of $\mathbb{D}_{\mathrm{O}}(K)$, following loc. cit. Definition 3.5, one defines a *Weyl vector* $\rho_{n,p(h)}(W)$ by setting (recall that our definition of $F_{n,h}$ differs from Bruinier's by a factor of $\frac{1}{2}$)

$$\psi^K_{n,p(h)}(v_1) =: 4\pi\sqrt{2}\big(v_1, \rho_{n,p(h)}(W)\big)_{\mathbb{R}}, \tag{2.4.4}$$

where v_1 is a vector of norm one contained in the hyperboloid model of $\mathbb{D}_{\mathrm{O}}(K)$ see loc. cit. p. 66f for details.

The Fourier expansion of $\psi_{n,h}$ can be written in the form (cf. loc. cit. (3.38))

$$\psi_{n,h}(Z) = C_{n,h} + 8\pi(\rho_{n,h}(W), Y)_{\mathbb{R}} - 4 \sum_{\substack{\lambda\in\pm p(h)+K\\ (\lambda,W)_{\mathbb{R}}>0\\ Q(\lambda)=-n}} \log|1 - e\left(\pm(h,e_2)_{\mathbb{R}} + (\lambda,Z)_{\mathbb{R}}\right)|$$
$$-4 \sum_{\substack{\lambda\in K\\ (\lambda,W)_{\mathbb{R}}>0\\ Q(\lambda)\leq 0}} \sum_{\substack{\delta\in L_0^\sharp/L\\ p(\delta)=\lambda+K}} b(\delta, -Q\,(Y)) \log|1 - e\left((\delta,e_2)_{\mathbb{R}} + (\lambda,Z)_{\mathbb{R}}\right)|,$$

where W is used to denote a Weyl chamber of $\mathbb{D}_{\mathrm{O}}$ and the corresponding Weyl chamber of $\mathbb{D}_{\mathrm{O}}(K)$, i.e. containing the normalized imaginary parts $\frac{Y}{|Y|}$ of $Z \in W \subset \mathbb{D}_{\mathrm{O}}$.

Hence, for $Z \in \mathcal{H}_p$ with $Q\,(Y) < |n|$, one defines

$$\Psi_{n,h}(Z) := e\left((\rho_{n,h}(W), Z)_{\mathbb{R}}\right) \prod_{\substack{\lambda\in\pm p(h)+K\\ (\lambda,W)_{\mathbb{R}}>0\\ Q(\lambda)=-n}} \left(1 - e\left(\pm(h,e_2)_{\mathbb{R}} + (\lambda,Z)_{\mathbb{R}}\right)\right)$$
$$\cdot \prod_{\substack{\lambda\in K\\ (\lambda,W)_{\mathbb{R}}>0\\ Q(\lambda)\leq 0}} \prod_{\substack{\delta\in L_0^\sharp/L\\ p(\delta)=\lambda+K}} \left(1 - e\left((\delta,e_2)_{\mathbb{R}} + (\lambda,Z)_{\mathbb{R}}\right)\right)^{b(\delta,-Q(\lambda))}. \tag{2.4.5}$$

The product expansion is absolutely convergent for $Q\,(Y) < n$ and, on the complement of $\mathbb{D}_{\mathrm{O}}(n,h)$, satisfies

$$\log|\Psi_{n,h}|(Z) = -\frac{1}{4}\left(\psi_{n,h} - C_{n,h}\right). \tag{2.4.6}$$

Further, $\Psi_{n,h}$ has a holomorphic continuation to $\mathcal{H}_p$ and (2.4.6) holds on $\mathcal{H}_p - \mathbb{D}_{\mathrm{O}}(n,h)$.

The zero divisor of $\Psi_{n,h}$ can be described as follows [5, Theorem 3.16]: For a relatively compact open neighborhood $U \subset \mathcal{H}_p$, define the set

$$S_{n,h}(U) = \{\lambda \in h + L;\, Q\,(\lambda) = -n,\ (Z_L,\lambda)_{\mathbb{R}} = 0 \quad \text{for some } Z \in U\}. \tag{2.4.7}$$

Then, the zeros of $\Psi_{n,h}(Z)$ on U are located in $(Z_L,\lambda)_{\mathbb{R}}$ for $\lambda \in S_{n,h}(U)$, and their multiplicity is given by the product $\prod_{\lambda\in S_{n,h}(U)}(Z_L,\lambda)_{\mathbb{R}}$, in other words, a holomorphic and zero-free function on U is given by

$$\Psi_{n,h}(Z) \prod_{\lambda\in S_{n,h}(U)} (Z_L,\lambda)_{\mathbb{R}}^{-1}.$$

Remark 2.22. We note that $\Psi_{n,h}$ is not necessarily automorphic. However, by construction and through the properties of the theta-integral, the function

$$|\Psi_{n,h}|e^{-\frac{1}{4}\xi_{n,h}}$$

is invariant under the discriminant kernel Γ_L^{O}.

Remark 2.23. A twice continuously differentiable real function f on a domain $D \subset \mathbb{C}^p$ is called pluriharmonic if all mixed second derivatives vanish, i.e. if

$$\frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} = 0 \quad (1 \leq j, k \leq p).$$

If D is simply connected, a twice continuously differentiable function f is pluriharmonic if and only if there is a holomorphic function $h : D \to \mathbb{C}$ with $f = \mathrm{Re}(h)$ [see 31, Section IX.C].

Also since the components of $\mathcal{H}^{\pm}$ are convex, in particular, the multiplicative Cousin problem is universally solvable on $\mathcal{H}_p \setminus \mathbb{D}_{\mathrm{O}}(h, n)$ [see 30, Section V.2]. Thus, there exists a meromorpic function $g = \prod (Z_L, \lambda)_{\mathbb{R}}$ with the same divisor as $\Phi_{h,n}$. Then $\Phi_{n,h} - \log|g|$ extends to a pluriharmonic function on $\mathcal{H}_p$ and hence there is a holomorphic function h with $\mathrm{Re}(h) = \Psi_{n,h} - \log|g|$, and one can set $\Psi_{n,h} = e^h g$. (For a detailed version of this argument [see 5, p. 82ff] or [6, Lemma 6.6]).

2.4.4. Infinite product expansion

Next, we review the main result concerning Borcherds products for orthogonal groups of signature $(2p, 2)$, [2, Theorem 13.2] in the version by Bruinier [5, Theorem 3.22]. First, some further definitions.

For a weakly holomorphic modular form f transforming under the Weil representation ρ_{L^-} with principal part

$$P(f) = \sum_{h \in L^\sharp / L} \sum_{\substack{n \in \mathbb{Z} - Q(h) \\ n < 0}} a^+(h, n) e(n\tau) \mathfrak{e}_h,$$

the Weyl chambers of $\mathbb{D}_{\mathrm{O}}(K)$ with respect to f are defined as the connected components of

$$\mathbb{D}_{\mathrm{O}}(K) - \bigcup_{h \in L_0^\sharp / L} \bigcup_{\substack{n \in \mathbb{Z} - Q(h) \\ n < 0 \\ a^+(h,n) \neq 0}} \mathbb{D}_{\mathrm{O}}(-n, p(h)).$$

Let W be one of these Weyl chambers. Then, for every $h \in L_0^\sharp / L$, and $n \in \mathbb{Z} - Q(h)$ with $a^+(h, n) \neq 0$, there is a Weyl chamber $W_{n,h} \subset \mathbb{D}_{\mathrm{O}}(K)$ of index $(-n, p(h))$ with $W \subset W_{n,h}$ hence W can be written as the intersection

$$W = \bigcap_{h \in L_0^\sharp / L} \bigcap_{\substack{n \in \mathbb{Z} - Q(h) \\ n < 0 \\ a^+(h,n) \neq 0}} W_{n,h}. \tag{2.4.8}$$

Further, the Weyl vector attached to W and f is defined as

$$\rho_f(W) = \frac{1}{2} \sum_{h \in L_0^\sharp / L} \sum_{\substack{n \in \mathbb{Z} - Q(h) \\ n < 0}} a^+(h, n) \rho_{n, p(h)}(W_{n,h}), \tag{2.4.9}$$

where $\rho_{n,p(h)}(W_{n,h})$ denotes a Weyl vector for $W_{n,h}$ of the form (2.4.4). As before, the subset of $\mathbb{D}_{\mathrm{O}}$ attached to W is also considered as a Weyl chamber attached to f and denoted by W. In the tube domain, it consists of all $Z \in \mathcal{H}_p$, with $Y/|Y| \in W \subset \mathbb{D}_{\mathrm{O}}(K)$.

Theorem 2.24 (Borcherds-Bruinier)**.** *Let L be an even lattice of signature $(2p, 2)$, with $p \geq 2$ and $e_1 \in L$ a primitive isotropic vector, $e_2 \in L'$ with $(e_1, e_2)_{\mathbb{R}} = 1$ and $K = L \cap e_1^{\perp} \cap e_2^{\perp}$, and denote by p the projection defined in (2.4.2). Further, assume that K, too, contains an isotropic vector.*

Let $f \in \mathrm{M}^!_{L^-,1-p}$ be a weakly holomorphic modular form of weight $1-k$, the Fourier coefficients $a(h,n)$ of whose principal part $P(f)$ are integral for $n < 0$. Then,

$$\Psi(Z) = \prod_{h \in L^\sharp/L} \prod_{\substack{n \in \mathbb{Z}-Q(h) \\ n<0}} \Psi_{n,h}(Z)^{\frac{1}{2}a(h,n)}$$

is a meromorphic function on $\mathcal{H}_p$ with the following properties

1. *$\Psi(Z)$ is a meromorphic modular form of weight $\frac{1}{2}a(0,0)$ for Γ_L^{O} with some multiplier system of at most finite order. If $a(0,0)$ is even, the multiplier system is, in fact, a character.*

2. *The divisor of $\Psi(Z)$ is given by*

$$\frac{1}{2} \sum_{h \in L^\sharp/L} \sum_{\substack{n \in \mathbb{Z}-Q(h) \\ n<0}} a(h,n)\, \mathbb{D}_{\mathrm{O}}(-n,h).$$

3. *Let $W \subset \mathcal{H}_p$ be a Weyl chamber with respect to f and $n_0 = \min\{n \in \mathbb{Q}; a(h,n) \neq 0\}$. On the set of $Z \in \mathcal{H}_p$, which lie in the complement of the set of poles of $\Psi(Z)$ and which satisfy $Q(Y) < |n_0|$, there is a normally convergent product expansion*

$$\Psi(Z) = Ce\left((\rho_f(W), Z)_{\mathbb{R}}\right) \prod_{\substack{\mu \in K^\sharp \\ (\mu,W)_{\mathbb{R}}>0}} \prod_{\substack{h \in L_0^\sharp/L \\ p(h) \in \mu+K}} \left(1 - e\left((h,e_2)_{\mathbb{R}} + (\mu, Z)_{\mathbb{R}}\right)\right)^{a(h,-Q(\mu))},$$

where C is a constant of absolute value one and $\rho_f(W)$ is the Weyl vector attached to W and f.

Remark 2.25. By part 3. of Theorem 2.24, the infinite product expansion is absolutely convergent in a suitable neighborhood of the cusp at infinity of $\mathcal{H}_p$. In other words near the zero-dimensional boundary component attached to the isotropic vector e_1. Quite recently, Kudla [see 40] came up with a different infinite product expansion for $\Psi(Z)$, which puts *one-dimensional* boundary components into focus. It is based on a slightly different procedure for evaluating the singular theta integral. Interestingly enough, this also avoids the reduction step from $L \otimes_{\mathbb{Z}} \mathbb{R}$ to $K \otimes_{\mathbb{Z}} \mathbb{R}$ and hence, there is no immediate equivalent to the Weil-vector term, which in Borcherds' setup comes from the contribution of $\Phi^K(v_1)$.

We will use an adaptation of Kudla's method in Chapter 4 to evaluate the singular theta lift to be constructed in Chapter 3.

2.5. Borcherds products for $\mathrm{U}(p,1)$

Now, we will use the embedding ε to pull back the Borcherds products of Theorem 2.24 from $\mathcal{H}_p$ to $\mathbb{H}$. The main result of this section, Theorem 2.29 is a slight variation of the main result of [33] and [35], where Borcherds' original version [2, Theorem 13.3] was subject to the pull back, with some results from [5] used for the Weyl chambers and Weyl vector terms. First, we establish the analogue of Section 2.4.2 by studying the behavior of the special cycles and Weyl chambers under pull-back.

Behavior of special cycles and Weyl chambers under pull-back

Lemma 2.26. *Let $\lambda \in L^\sharp$ be a lattice vector with $(\lambda,\lambda) > 0$ and $\mathbb{D}(\lambda)$ the attached special cycle, and denote by $\varepsilon : \mathcal{H}_{\ell,\ell'} \hookrightarrow \mathcal{H}_p$ the embedding from Section 2.3. Then, the image of the special cycle under ε has the following properties:*

1. *It is given by the non-empty intersection $\varepsilon\left(\mathbb{D}(\lambda)\right) = \left(\varepsilon(\mathcal{H}_{\ell,\ell'}) \cap \mathbb{D}_{\mathrm{O}}(\lambda)\right) \subset \mathcal{H}_p$, where $\mathbb{D}_{\mathrm{O}}(\lambda) \subset \mathcal{H}_p$ is the special cycle in $\mathcal{H}_p$ attached to λ.*
2. *For all $u \in \mathcal{O}_{\mathbb{F}}^\times$, the special cycles given by $\mathbb{D}_{\mathrm{O}}(\hat{u}\lambda)$ intersect in $\varepsilon\left(\mathbb{D}(\lambda)\right) \subset \varepsilon\left(\mathcal{H}_{\ell,\ell'}\right)$.*

Further, if $\mathbb{D}_{\mathrm{O}}(n,h)$ is a Heegner divisor of index (n,h) in $\mathcal{H}_p$, we have $\mathbb{D}_{\mathrm{O}}(n,h) \cap \varepsilon\left(\mathcal{H}_{\ell,\ell'}\right) = \varepsilon\left(\mathbb{D}(n,h)\right)$, with $\mathbb{D}(n,h)$ the special cycle of index (n,h) in $\mathcal{H}_{\ell,\ell'}$.

Proof. To show the first statement, we use the isomorphisms between $\mathcal{C}_{\mathrm{O}}$ and $\mathcal{H}_p$ and $\mathcal{C}_{\mathrm{U}}$ and $\mathcal{H}_{\ell,\ell'}$ respectively. Thus, for $Z \in \mathcal{H}_p$, let $Z_L \in \mathrm{pr}_{\mathrm{O}}^{-1}(\mathcal{C}_{\mathrm{O}})$ be the attached vector in $V_{\mathbb{R}} \otimes \mathbb{C}$, and for $(\tau,\sigma) \in \mathcal{H}_{\ell,\ell'}$, let $\mathfrak{z} = \mathfrak{z}(\tau,\sigma)$ be the corresponding normalized representative for a negative definite line in V. Assume $Z_L \in \varepsilon(\mathcal{H}_{\ell,\ell'})$. Then

$$\begin{aligned} Z_L \in \mathbb{D}_{\mathrm{O}}(\lambda) \iff Z_L \perp \lambda \iff \left(\frac{\mathfrak{z}}{(\ell,\ell')} + i\frac{-\hat{\imath}\mathfrak{z}}{(\ell,\ell')}, \lambda\right)_{\mathbb{R}} &= 0 \\ \iff (\ell,\ell')(\mathfrak{z},\lambda) = 0 \iff \mathfrak{z} &\in \mathbb{D}(\lambda). \end{aligned}$$

Also, since ε is injective and both $\mathbb{D}(\lambda) \subset \mathcal{H}_{\ell,\ell'}$ and $\mathbb{D}_{\mathrm{O}}(\lambda) \subset \mathcal{H}_p$ are non-empty, the intersection $\varepsilon(\mathcal{H}_{\ell,\ell'}) \cap \mathbb{D}_{\mathrm{O}}(\lambda)$ is non-empty.

The second statement is immediate.

Since both kinds of special cycles are given by linear combinations of special cycles of the form $\mathbb{D}(\lambda)$ and $\mathbb{D}_{\mathrm{O}}(\lambda)$, and for both, the λ's satisfy the same conditions concerning their norm and their class in the discriminant group, the third statement follows from the first two. $\square$

Definition 2.27. Denote by $\mathcal{V}$ the collection of all Weyl chambers of $\mathbb{D}_{\mathrm{O}}(K)$ (which, of course correspond to Weyl chambers of $\mathbb{D}_{\mathrm{O}}$), and let ε_K be map given by

$$\varepsilon_K : \ (\tau,\sigma) \mapsto \mathbb{R}\,Y(\tau,\sigma) = \mathbb{R}\Big(-\hat{\imath}\widehat{(\ell,\ell')}^{-1}\mathfrak{z}(\tau,\sigma)\Big)$$

A Weyl chamber of $\mathcal{H}_{\ell,\ell'}$ is a connected subset W with $\varepsilon_K(W) = V \cap \varepsilon_K(\mathcal{H}_{\ell,\ell'})$ for some $V \in \mathcal{V}$. We say that the Weyl chamber $W \subset \mathcal{H}_{\ell,\ell'}$ has index (n,h) if the corresponding Weyl chamber $V \subset \mathcal{H}_p$ has this index.

Let V be a Weyl chamber of $\mathbb{D}_{\mathrm{O}}(K)$ with index $(n, p(h))$. Recall that W can be described through inequalities of the form $\pm(Y_L, \mu)_{\mathbb{R}} > 0$ with lattice vectors μ with $\mu \in p(h) + K$ and $Q(\mu) = n$.

Since for $Y_L = \varepsilon_K(\tau, \sigma)$ we have $(Y_L, \mu)_{\mathbb{R}} = (Y, \mu)_{\mathbb{R}}$, it follows that a Weyl chamber of index $(n, p(h))$ $(n \in \mathbb{Z}, h \in L^\sharp/L)$ in $\mathcal{H}_{\ell,\ell'}$ is defined geometrically (as a subset of $\mathrm{pr}^{-1}(\mathcal{C}_{\mathrm{U}})$) by a set of inequalities of the form

$$\pm\Im\left(\frac{(\mathfrak{z}(\tau,\sigma),\mu)}{(\ell,\ell')}\right) > 0$$

for $\mu \in p(h) + K$, $Q(\mu) = n$.

Since from the properties of Weyl chambers in $\mathbb{D}_{\mathrm{O}}(K)$, it follows that the sign is constant throughout the Weyl chamber, we will write these inequalities in the form

$$(\varepsilon_K(W), \mu)_{\mathbb{R}} > 0 \quad \text{and} \quad (\varepsilon_K(W), \mu)_{\mathbb{R}} < 0.$$

For a weakly holomorphic modular form f with principal part

$$P(f) = \sum_{h \in L^\sharp/L} \sum_{\substack{n \in \mathbb{Z} - Q(h) \\ n < 0}} a(h,n) e(n\tau) \mathfrak{e}_h,$$

by the above considerations, the Weyl chambers with respect to f in $\mathcal{H}_p$ define Weyl chambers in $\mathcal{H}_{\ell,\ell'}$ which can be described as an intersection (cf. (2.4.8)) of Weyl chambers of index $(-n, p(h))$ for $h \in L_0^\sharp/L$, $0 > n \in \mathbb{Z} - Q(h)$ with $a^+(h,n) \neq 0$. Thus, if $V_{n,h} \subset \mathcal{H}_p$ is one of these Weyl chambers in $\mathcal{H}_p$ from (2.4.8), denote by $W_{n,h} \subset \mathcal{H}_{\ell,\ell'}$ the Weyl chamber with index $(p(h), n)$ with $\varepsilon_K(W_{h,n}) \subset V_{h,n}$. Then, a Weyl chanber W with respect to f is given by

$$W = \bigcap_{h \in L_0^\sharp/L} \bigcap_{\substack{n \in \mathbb{Z} - Q(h) \\ n < 0 \\ a(h,n) \neq 0}} W_{n,h}.$$

2.5.1. Infinite products

We use the embedding to pull back the infinite products $\Psi_{n,h}$ from (2.4.5) on p. 53 associated to the lift of the Hejhal Poincaré series $F_{n,h}$ and get

Lemma 2.28. *For $h \in L^\sharp/L$ and $n \in \mathbb{Z}$, the infinite product*

$$\begin{aligned}\Psi_{n,h}(\mathfrak{z}) = e\left(\frac{(\rho_{h,n},\mathfrak{z})}{(\ell,\ell')}\right) \prod_{\substack{\lambda \in p(h)+K \\ (\varepsilon_K(W),\lambda)_{\mathbb{R}} > 0 \\ Q(\lambda) = -n}} \left(1 - e\left(\pm \operatorname{Re}\left(\frac{2(h,\ell')\kappa_{\mathbb{F}}}{\delta_{\mathbb{F}}(\ell,\ell')}\right) + \frac{(\lambda,\mathfrak{z})}{(\ell,\ell')}\right)\right) \\ \cdot \prod_{\substack{\mu \in K^\sharp \\ (\mu, \varepsilon_K(W))_{\mathbb{R}} > 0 \\ Q(\mu) \leq 0}} \prod_{\substack{h \in L_0^\sharp/L \\ p(h) = \mu + K}} \left(1 - e\left(\operatorname{Re}\left(\frac{2\kappa_{\mathbb{F}}(h,\ell')}{\delta_{\mathbb{F}}(\ell,\ell')}\right) + \frac{(\lambda,\mathfrak{z})}{(\ell,\ell')}\right)\right)^{b(h,-Q(\mu))}.\end{aligned} \tag{2.5.1}$$

is normally convergent for $\mathfrak{z}$ *with* $(\mathfrak{z},\mathfrak{z}) < 4|(\ell,\ell')|^2|n|$. *Here,* $\rho_{h,n}$ *is the Weyl-vector from* (2.4.4).

Proof. This is an immediate consequence of the results of Bruinier, see [5], Lemma 3.15 and Theorem 3.16, with the properties of the embedding ε with the choice of the basis vector $e_2 = \hat{a}\ell'$ with $a = (\delta_{\mathbb{F}}(\ell,\ell'))^{-1}\kappa_{\mathbb{F}}$. □

Now, we can formulate the main theorem of this section (cf. [33, Theorem 4.21] and [35, Theorem])

Theorem 2.29. *Let* L *be an even hermitian lattice of signature* $(p,1)$, *with* $p \geq 1$ *and* $\ell \in L$ *a primitive isotropic vector. Let* $\ell' \in L^\sharp$ *with* $(\ell',\ell) \neq 0$ *and assume that* ℓ' *is also isotropic. Given a weakly holomorphic modular form* $f \in \mathrm{M}^!_{L^-,1-p}$ *with principal part* $\sum_{h,n} a(h,n)$ *with* $a(h,n) \in \mathbb{Z}$ *for* $n<0$, *the function*

$$\Psi_f(\mathfrak{z}) = \prod_{h\in L_0^\sharp/L}\prod_{\substack{n\in\mathbb{Z}-Q(h)\\ n<0}} \Psi_{n,h}(\mathfrak{z})^{\frac{1}{2}a(h,n)},$$

with $\Psi_{n,h}$ *from* (2.5.1), *is a meromorphic function on* $\mathcal{H}_{\ell,\ell'}$ *with the following properties:*

1. Ψ_f *is meromorphic modular form of weight* $\frac{1}{2}a(0,0)$ *for* Γ_L *with some multiplier system* χ *of finite order. If* $a(0,0)$ *is even,* χ *is a character.*

2. *The zeros and poles of* Ψ_f *lie on the special cycles. The divisor in* $\mathcal{H}_{\ell,\ell'}$ *of* Ψ_f *is given by*

$$\operatorname{div}(\Psi_f) = \frac{1}{2}\sum_{h\in L^\sharp/L}\sum_{\substack{n\in\mathbb{Z}-Q(h)\\ n<0}} a(h,n)\mathbb{D}(-n,h).$$

   The multiplicity of $\mathbb{D}(-n,h)$ *is* 2 *if* $2h = 0 \in L_0^\sharp/L$ *and* 1 *otherwise. Note that* $a(h,-n) = a(-h,-n)$ *and that* $\mathbb{D}(-n,h) = \mathbb{D}(-n,uh)$ *for all* $u \in \mathcal{O}_{\mathbb{F}}^\times$, $h \neq 0$.

3. *Let* W *be a Weyl chamber of* $\mathcal{H}_{\ell,\ell'}$ *with respect to* f, *and* $\rho_f(W)$ *the attached Weyl-vector. Then,* $\Psi_f(\mathfrak{z})$ *has an infinite product expansion of the form*

$$\Psi_f(\mathfrak{z}) =$$
$$Ce\left(\frac{(\rho_f(W),\mathfrak{z})}{(\ell,\ell')}\right)\prod_{\substack{\mu\in K^\sharp\\ (\mu,\varepsilon_K(W))_{\mathbb{R}}>0}}\prod_{\substack{h\in L_0^\sharp/L\\ p(h)=\mu+K}}\left(1-e\left(\mathrm{Re}\left(\frac{2\kappa_{\mathbb{F}}(h,\ell')}{\delta_{\mathbb{F}}(\ell,\ell')}\right)+\frac{(\mu,\mathfrak{z})}{(\ell,\ell')}\right)\right)^{a(h,-Q(\mu))},$$

   where C *is a constant of absolute value one and* $\rho_f(W)$ *is the Weyl vector attached to* W. *The product converges normally on the set of* $\mathfrak{z}$ *in the complement of the set of poles of* $\Psi_f(\mathfrak{z})$ *and satisfying* $(\mathfrak{z},\mathfrak{z}) < 4|(\ell,\ell')|^2|n_0|$ *with* $n_0 = \min\{n\in\mathbb{Z}; a(h,n)\neq 0\}$.

Proof. In principle, it is only necessary to apply the pull-back under the embedding ε to Theorem 2.24 and keep in mind our remarks concerning Weyl chambers and Weyl vectors. However, we will reproduce some of the argumentation for the proof of Theorem 2.24 from [5, proof of Theorem 3.22].

Let $f = \sum_h \sum_n a(h,n) F_{n,h}(\tau, \frac{p-1}{2})$ be the decomposition of $f \in \mathrm{M}^!_{L^-,1-p}$ in terms of Maass-Poincaré series. Set

$$\Phi(\mathfrak{z}) = \sum_{h,n} a(h,n)\varepsilon^*\left(\Phi_{h,n}(Z)\right), \qquad \xi(\mathfrak{z}) = \sum_{h,n} a(h,n)\varepsilon^*\left(\xi_{h,n}(Z)\right).$$

From (2.4.3) we get

$$\begin{aligned}
\xi(\mathfrak{z}) = &\sum_{h\in L^\sharp/L}\ \sum_{\substack{n\in\mathbb{Z}-Q(h)\\ n<0}} a(h,n)\Bigg[\frac{(\mathfrak{z},\mathfrak{z})^{\frac{1}{2}}}{\sqrt{2}|(\ell,\ell')|}\varepsilon_K^*\left(\xi_{h,n}^K\left(\frac{-\hat{\imath}\mathfrak{z}}{|(\mathfrak{z},\mathfrak{z})|^{\frac{1}{2}}}\frac{(\ell,\ell')}{|(\ell,\ell')|}\right)\right)\\
&\qquad - b_{h,n}(0,0)\log\left(\frac{|(\mathfrak{z},\mathfrak{z})|}{|(\ell,\ell')|}\right)\Bigg]\\
&- \frac{1}{\sqrt{\pi}\cdot\Gamma(p)}\sum_{\substack{\mu\in K^\sharp\\ Q(\mu)\le 0}}\ \sum_{\substack{\delta\in L_0^\sharp/L\\ p(\delta)=\mu+K}}\ \sum_{\substack{h\in L^\sharp/L\\ n\in\mathbb{Z}-Q(h)\\ n<0}} a(h,n)\cdot p_{\delta,Q(\mu)}(h,-n)\\
&\cdot\sum_{m\ge 1}\frac{1}{m}e\left(m\,\mathrm{Re}\left[\frac{\kappa_{\mathbb{F}}(\delta,\ell')}{\delta_{\mathbb{F}}(\ell,\ell')}\right] + \mathrm{Re}\,\frac{(\mu,\mathfrak{z})}{(\ell,\ell')}\right)\mathcal{V}_{p+1}\left(\pi m|\mu|\frac{(\mathfrak{z},\mathfrak{z})^{\frac{1}{2}}}{|(\ell',\ell)|}, \pi m\,\mathrm{Re}\left(\frac{(\mu,-i\mathfrak{z})}{(\ell,\ell')}\right)\right)
\end{aligned}$$

Here, in the second line, the Fourier coefficients $b_{h,n}(\delta, -Q(\mu))$ of the Maass-Poincaré have been expressed in terms of the coefficients $p_{\delta,Q(\mu)}$ of a holomorphic Poincaré series via [see 5, Prop. 1.16]

$$b_{h,n}(\delta, -Q(\mu)) = -\frac{1}{\Gamma(p)}\cdot p_{\delta,Q(\mu)}(h,-n),$$

an explicit formula for these coefficients is given in [5, Theorem 1.4]. Since f is in $\mathrm{M}^!_{1-p,L^-}$ and the Poincaré series are contained in $\mathrm{M}^!_{p+1,L}$, by the duality result in Proposition 1.10, one finds that the sum $\sum_{h,n} a(h,n)p_{\delta,Q(\mu)}(h,-n)$ vanishes. Thus, only the first line in the above expression for $\xi(\mathfrak{z})$ remains. By similar arguments, one can show that $\sum_{h,n} a(h,n)\xi^K_{n,h}$ is a rational function, which then implies, see [2, Theorem 10.3], [5, Theorem 3.6], that the sum is identically zero. Thus, $\xi(\mathfrak{z})$ is given by the only remaining term,

$$\xi(\mathfrak{z}) = -a(0,0)\log\frac{|(\mathfrak{z},\mathfrak{z})|}{|(\ell,\ell')|^2}.$$

Hence,

$$\Phi(\mathfrak{z}) = -a(0,0)\log\frac{|(\mathfrak{z},\mathfrak{z})|}{|(\ell,\ell')|^2} + \sum_{h\in L^\sharp/L}\ \sum_{\substack{n\in\mathbb{Z}-Q(h)\\ n<0}} a(n,h)\,\varepsilon^*\left(\psi_{n,h}\right)(\mathfrak{z}).$$

Now, recall the definition of $\Psi_{n,h}$ and the location of zeros of $\Psi_{n,h}$ (see p. 53) these imply that

$$|\Psi_f(\mathfrak{z})| = C\exp\Big(-\frac{1}{4}\sum_{h\in L^\sharp/L}\sum_{\substack{n\in\mathbb{Z}-Q(h)\\ n<0}} a(h,n)\psi_{n,h}(\mathfrak{z})\Big)$$

with a real constant C. Hence,

$$|\Psi_f(\mathfrak{z})|\left(\frac{(\mathfrak{z},\mathfrak{z})}{|(\ell,\ell')|}\right)^{\frac{a(0,0)}{4}} = C'\exp\left(-\frac{1}{4}\Phi(\mathfrak{z})\right).$$

Since $\Phi(\mathfrak{z})$ is invariant under Γ_L, it follows that the left hand side of this equation is invariant, too, and hence $\Psi_f(\mathfrak{z})$ is a modular form of weight $\frac{1}{2}a(0,0)$ with some multiplier system for Γ_L. □

The infinite product expansion in part 3. of the Theorem 2.29 can be simplified considerable for suitable lattices L, for instance, we have the following Corollary [see 35, Corollary 1].

Corollary 2.30. *Assume that L is the direct sum of a unimodular lattice $\mathcal{O}_\mathbb{F}\ell + \mathcal{D}_\mathbb{F}^{-1}\ell'$ and a definite lattice D. Then, in the notation of Theorem 2.29, for every Weyl chamber $W\subset\mathcal{H}_{\ell,\ell'}$, the infinite product expansion of $\Psi_f(\mathfrak{z})$ takes the form*

$$\Psi_f(\mathfrak{z}) = Ce\left(\frac{(\rho_f(W),\mathfrak{z})}{(\ell,\ell')}\right)\prod_{\substack{\mu\in K^\sharp\\ (\mu,\varepsilon_K(W))_\mathbb{R}>0}}\left(1-e\left(\frac{(\mu,\mathfrak{z})}{(\ell,\ell')}\right)\right)^{a(\mu,-Q(\mu))}.$$

Proof. It easily seen that $N_{e_1} = N_\ell = 1$, hence $L_0 = L$. Further, the elements $K^\sharp/K$ and $L^\sharp/L$ are in one-to-one correspondence, and one can identify $\mu\in K^\sharp$ with $h\in L^\sharp$, its preimage under the projection p. Finally, the ℓ-component of μ is given by $\mu_3 e_3$ with a rational number μ_3, and it follows by (2.3.6) that

$$\operatorname{Re}\left(\frac{\kappa_\mathbb{F}(\mu,\ell')}{\delta_\mathbb{F}(\ell,\ell)}\right) = \operatorname{Re}\left(\delta_\mathbb{F}^{-1}|\kappa_\mathbb{F}|^2\mu_3\right) = 0.$$

□

Remark 2.31. The pull-back ε^* can also be used directly on the Borcherds lift of a weakly holomorphic modular form or, more generally, of a weak harmonic Maass form. For example, the pull-back of the functions $\Phi_{n,h}(Z)$ features in [14, Sec. 4].

Examples

Example 2.32. As an example, we consider the case where $p=q=1$, treated in [33, Chapter 6] and [34]. In $V=\mathbb{C}^2$, consider the self-dual lattice $L=\mathcal{O}_\mathbb{F}\oplus\mathcal{D}_\mathbb{F}^{-1}$ spanned by two isotropic vectors $\ell = 1$ and $\ell' = \delta^{-1}$. Then, $\mathcal{H}_{\ell,\ell'}$ is just the usual complex upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C};\ \Im(\tau) > 0\}$. The special unitary group $\mathrm{SU}(V)$ is isomorphic to $\mathrm{SL}_2(\mathbb{R})$ via

$$\mathrm{SL}_2(\mathbb{R}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & b\epsilon \\ c\epsilon^{-1} & d \end{pmatrix} \in \mathrm{SU}(V),$$

wherein $\epsilon := \delta(\ell', \ell)$ and and where matrices are written in terms of ℓ and ℓ'. The matrix groups operate on $\mathbb{H}$ through fractional linear transformations, as usual.

Since the lattice L is unimodular, the discriminant group $L^\sharp/L$ is trivial and the discrete Weil represenation reduces to the usual operation on $\mathbb{C}$.

A basis of $\mathrm{M}_0^!$ is given by the family of modular forms with principal part q^{-n} and constant term 0, $j_n = q^{-n} + O(q)$ for $n \geq 1$. For example, j_1 is the modular invariant less its constant term, $j_1 = j - 744$, indeed, $\mathrm{M}_0^! = \mathbb{C}[j]$. Note also that j_n can expressed through the non-holomorphic Poincaré series F_{-n} though

$$j_n(\tau) = F_{-n}(\tau, 1) - 24\sigma(n),$$

where $\sigma(n)$ denotes the divisor sum $\sum_{d|n} d$.

In determining the Weyl vector for the lift of j_n, it is necessary to take into account the contribution of the constant term $b_n(0,1)$ (see [34, Sec. 9]). The Borcherds product for the lift of this term is given by

$$\Psi_1(\tau)^{24\sigma(n)} = \left(\eta(\tau)\eta(-\bar{\kappa}_{\mathbb{F}})\right)^{24\sigma(n)}.$$

Now, the (multiplicative) Borcherds lift Ψ_{j_n} of j_n ($n \in \mathbb{N}$) is a meromophic modular form of weight 0 on $\mathbb{H}$, with zeros poles lying along a special cycle given by

$$\operatorname{div}(\Psi_{j_n}) = \frac{1}{2} \sum_{\substack{\lambda \in L \\ (\lambda,\lambda) = n}} [\tau_\lambda],$$

where the $[\tau_\lambda]$'s denotes the Γ_L-orbits of points defined by $(\mathfrak{z}(\tau_\lambda), \lambda) = 0$, with $\mathfrak{z}(\tau_\lambda) = \ell' + \tau_\lambda \delta_{\mathbb{F}}(\ell, \ell)\ell$. In fact, the τ_λ are Heegner points in the sense of CM-theory for elliptic curves.

The Weyl chambers with respect to j_n are stripe-shaped regions of $\mathbb{H}$ defined as follows: Denote by $1 = t_1 \leq t_2 \leq \ldots t_{d(n)} = n$ the positive divisors of n, arranged by their size and set $t_0 = 0$ and $t_{n+1} = \infty$. Then, the Weyl chambers are given by

$$W(t_i, t_{i+1}) = \left\{\tau \in \mathbb{C};\ |\delta_{\mathbb{F}}| t_i^2 < 2\Im\tau n < |\delta_{\mathbb{F}}| t_{i+1}^2\right\}, \quad i = 1, \ldots, n$$

and

$$W(0,1) = \left\{\tau \in \mathbb{C};\ 0 < \Im\tau < \tfrac{1}{2}|\delta_{\mathbb{F}}|\right\}, \quad W(n, \infty) = \left\{\tau \in \mathbb{C};\ \tfrac{1}{2}|\delta_{\mathbb{F}}| n < \Im\tau\right\}.$$

For each Weyl chamber $W(t_i, t_{i+1})$, we have a Borcherds product expansion, absolutely convergent for all τ in the complement of the set of poles satisfying $|\delta_{\mathbb{F}}|\Im\tau > 2n$, given by

$$\Psi_{j_n}(\tau; W(t_i, t_{i+1})) = C e\left(\rho_2 \tau - \bar{\kappa}_{\mathbb{F}} \rho_1\right) \prod_{\substack{l,k \in \mathbb{Z} \\ nl > -kt_i^2}} \left(1 - e\left(lr - \bar{\kappa}_{\mathbb{F}}\right)\right)^{a(kl)},$$

where C is a constant of absolute value one, and the components ρ_1 and ρ_2 of the Weyl vector are given by

$$\rho_1 = -\sum_{\substack{t|n \\ t \geq t_{i+1}}} t, \quad \text{and} \quad \rho_2 = -\sum_{\substack{t|n \\ 0<t\leq t_i}} \frac{n}{t}.$$

Remark 2.33. In order to determine the Weyl vector in the previous Example 2.32, in [33, 34] the additive Borcherds lift with respect to the quadratic space $K \otimes \mathbb{R} = \mathbb{Z}^2 \otimes \mathbb{R}$ was explicitly calculated, too. For the functions $j_n \in \mathrm{M}_0^!$, it is given by

$$\Phi^K(Y; j_n) = \frac{4\sqrt{2}\pi}{|Y|} \sum_{\substack{t>0 \\ t|n}} \left(\left| ty_2 + \frac{n}{t} y_1 \right| - \left| ty_2 + \frac{n}{t} y_1 \right| \right).$$

Further, the additive lift for $\mathrm{SO}(2, 2)$ and the multiplicative lift are explicitly determined in [33, Chapter 6], thus covering a case not treated in [5] and [2]. The Borcherds product expansion for the lift of j_n is absolutely convergent for those Z in $\mathcal{H}_p \simeq \mathbb{H} \times \mathbb{H}$ which are not contained in the set of poles and satisfy $\Im(y_1)\Im(y_2) > n$. It is given by

$$\Psi(Z; j_n) = e(\rho_1 z_2 + \rho_2 z_1) \prod_{\substack{m,n\in\mathbb{Z} \\ ((m,n),W)_{\mathbb{R}}>0}} (1 - e(mz_2 + nz_1))^{c(mn)},$$

with ρ_1 and ρ_2 from Example 2.32. The Weyl chambers here can be described as follows. Denote by $\mathcal{K}^-$ the connected component of the quadratic cone from Remark 2.10 with $\Im(Z) \in \mathcal{K}^-$. Then (with similar notation to Example 2.32) one has

$$W(t_i, t_{i+1}) = \{Y \in \mathcal{K}^-;\ t_i^2 y_2 < y_1 n < t_{i+1}^2 y_2\}.$$

Example 2.34. We mention that Yang and Ye in [57] use the results of [35] to construct examples of modular forms for $\mathrm{U}(2, 1)$. More precisely, in the first part of their paper, they construct a basis for each of the spaces of weakly holomorphic modular forms of negative integer weight for the congruence subgroup $\Gamma_0(4)$ having poles at only one of the three cusps $\{0, \frac{1}{2}, \infty\}$ of this group and with a quadratic character. They then use an induction process to obtain vector valued modular forms for the hermitian lattice $L = \mathbb{Z}[i] \oplus \mathbb{Z}[i] \oplus \frac{1}{2}\mathbb{Z}[i]$, equipped with the form $(x, y) = x_1\bar{y}_3 + x_3\bar{y}_1 + x_2\bar{y}_2$.

In the second part of their paper, they explicitly calculate the lift for forms F_n obtained from a basis of $M_{-1}^{!,\infty}\left(\Gamma_0(4), \chi_{-4}^k\right)$, the space of weight -1 weakly holomorphic forms for $\Gamma_0(4)$, having a pole at ∞ and with the quadratic character $\chi_{-4}(\cdot) = \left(\frac{-4}{\cdot}\right)$. The lift is an automorphic form of weight $32\sum_{d|n}\chi_{-4}(n/d)d^2 + 2\sum_{d|n}\chi_{-4}(d)d^2$ for Γ_L. They give an explicit description of the Heegner divisor, of the Weyl chambers and of the factors in the Borcherds product. They also determine the value at the boundary using [35, Theorem 5] (see Theorem 2.35 below).

2.5.2. Boundary values

The behavior of Ψ_f near the cusp $[\ell]$ can be determined either by calculating the behavior of Borcherds products on boundary components of $\mathcal{H}_p$ (cf. the examples in [2, Section

13]) and pulling back under ε, or by expanding the infinite products from Theorem 2.29 and taking the limit $\tau \to i\infty$. The first approach was used in the proof of [33, Theorem 4.3.3], while the second approach was used in the proof of [35, Theorem 5]. Assume that the width of the cusp $[\ell]$ is given by $N_\ell = 1$. One gets

Theorem 2.35. *Let W be a Weyl chamber, such that the cusp corresponding to ℓ is contained in the closure of W (viewed as a subset of $\mathcal{C}_{\mathrm{U}}$). If the cusp is neither a zero nor a pole of Ψ_f, then the limit $\lim_{r\to\infty}\Psi_f(ir,\sigma)$ is given by*

$$\lim_{r\to\infty}\Psi_f(ir,\sigma) = Ce\left(\overline{\rho_f(W)_\ell}\right)\prod_{\substack{\mu\in K^\sharp\\ \mu=a\kappa_{\mathbb{F}}\ell\\ a\in\mathbb{Q}_{<0}}}\left(1-e\left(a\bar{\kappa}_{\mathbb{F}}\right)\right)^{a(\mu,0)}.$$

Proof. We denote the ℓ- and ℓ'-components of ρ_f by ρ_ℓ and $\rho_{\ell'}$ and the definite part by $\rho_D \in D\otimes_{\mathcal{O}_{\mathbb{F}}}\mathbb{C}$. Since $\rho_f \in K\otimes_{\mathbb{Z}}\mathbb{Q}$, we have $\rho_\ell\ell = \rho_3 e_3$, $\rho_{\ell'}\ell' = \rho_4 e_4$, where ρ_3 and ρ_4 denote the e_3- and e_4-components. Similarly for a lattice vector $\lambda\in K^\sharp$, we write

$$\lambda = \lambda_\ell\ell + \lambda_{\ell'}\ell' + \lambda_D = \lambda_3 e_3 + \lambda_4 e_4 + \lambda_D.$$

With (2.3.6), which gives e_3 and e_4 in terms of ρ_ℓ and $\rho_{\ell'}$, we get

$$(\rho_f,\mathfrak{z}) = \bar{\rho}_\ell(\ell,\ell') + \bar{\rho}_{\ell'}\tau\delta_{\mathbb{F}}|(\ell',\ell)|^2 + (\rho_D,\sigma) = \bar{\kappa}_{\mathbb{F}}\rho_3(\ell,\ell') - 2\tau\rho_4(\ell,\ell') + (\rho_D,\sigma).$$

The Weyl vector in the product expansion of Ψ_f is thus given by

$$e\left(\frac{(\rho_F,\mathfrak{z})}{(\ell,\ell')}\right) = e\left(\bar{\kappa}_{\mathbb{F}}\rho_3 - 2\rho_4\tau + \frac{(\rho_D,\sigma)}{(\ell,\ell')}\right).$$

Clearly, $\Psi_f(\mathfrak{z})$ has a zero at infinity if $\rho_4<0$ and a pole if $\rho_4>0$. From now on, we assume that $\lim_{r\to\infty}\Psi_f(\tau,\sigma)$ is neither zero nor infinity.

Next, we claim that λ_4 is non-positive. To see this, consider the Weyl chamber condition $(\varepsilon_K(W),\lambda)_{\mathbb{R}}>0$ for $\mathfrak{z}$ with $\tau = ir$, $r\gg 0$ and σ of fixed norm We have:

$$(\lambda,\mathfrak{z}(ir,\sigma)) = \bar{\kappa}_{\mathbb{F}}\lambda_3(\ell,\ell') - 2ir\lambda_4(\ell,\ell') + (\lambda_D,\sigma).$$

Clearly, for large r, the Weyl chamber condition, $(\varepsilon_K(W),\lambda)_{\mathbb{R}}>0$ is satisfied only if $\lambda_4\le 0$. Since the corresponding factor in the product is trivial for $\lambda_4<0$, we can restrict to λ with $\lambda_4=\lambda_{\ell'}=0$.

Thus, in a suitable neighborhood of infinity, Ψ_f can be written in the form

$$\Psi_f(\tau,\sigma) = Ce\left(\bar{\rho}_\ell + \frac{(\rho_D,\sigma)}{(\ell,\ell')}\right)\prod_{\substack{\lambda\in K^\sharp\\ \lambda_{\ell'}=0\\ (\varepsilon_K(W),\lambda)_{\mathbb{R}}>0}}\left(1-e\left(\bar{\lambda}_\ell + \frac{(\lambda_D,\sigma)}{(\ell,\ell')}\right)\right)^{a(\lambda,-Q(\lambda_D))}. \tag{2.5.2}$$

As an automorphic form, $\Psi_f(\tau,\sigma)$ has a Fourier-Jacobi expansion of the form

$$\Psi_f(\tau,\sigma) = \sum_{n\ge 0} c_n(\sigma)e\left(\frac{n}{N}\tau\right). \tag{2.5.3}$$

Note that by the non-vanishing assumption, $n \in \mathbb{Z}$, and by regularity, $n \geq 0$. We thus have

$$c_0 = \lim_{r\to\infty} \Psi_f(ir,\sigma). \tag{2.5.4}$$

Beside the Fourier-Jacobi expansion (2.5.3), the Borcherds product can also be rewritten as a series by expanding each factor as a binomial series and taking the resulting product. Thus, with the binomial series expansion, the right hand side of (2.5.2) becomes

$$Ce\left(\bar{\rho}_\ell + \frac{(\rho_D,\sigma)}{(\ell,\ell')}\right) \cdot \prod_{\substack{\lambda\in K^\sharp \\ \lambda_{\ell'}=0 \\ (\varepsilon_K(W),\lambda)_\mathbb{R}>0}} \sum_{n\geq 0} (-1)^n \binom{a(\lambda,-Q(\lambda_D))}{n} e\left(\bar{\lambda}_\ell + \frac{(\rho_D,\sigma)}{(\ell,\ell')}\right)^n.$$

By multiplying all remaining factors, we obtain

$$e\left(\bar{\rho}_\ell + \frac{(\rho_D,\sigma)}{(\ell,\ell')}\right) \cdot \left[1 + \sum_{k>0} \sum_{\substack{\lambda_1,\dots,\lambda_k\in K^\sharp \\ (\varepsilon_K(W),\lambda_k)_\mathbb{R}>0 \\ \lambda_{k,\ell'}=0}} \sum_{\substack{n_1,\dots,n_k\in\mathbb{Z} \\ n_i\geq 0}} b\big((\lambda_i,n_i)_{i=1,\dots,k}\big) e\left(\sum_{i=1}^k n_i \frac{(\lambda_i,\mathfrak{z})}{(\ell,\ell')}\right)\right],$$

with coefficients $b\big((\lambda_i,n_i)_{i=1,\dots,k}\big)$ indexed by tuples of lattice vectors $\lambda_i \in K^\sharp$ and integers n_i. We set $\tilde{\lambda} := \sum_{i=1}^k n_i\lambda_i$. clearly, $\tilde{\lambda} \in K^\sharp$ and $\tilde{\lambda}_{\ell'} = 0$, since $K^\sharp$ is a $\mathbb{Z}$-module. Further, since the λ_i satisfy the Weyl chamber condition $(\varepsilon_K(W),\lambda)_\mathbb{R} > 0$ and the n_i are non-negative, each $\tilde{\lambda}$ also satisfies $\big(\varepsilon_K(W),\tilde{\lambda}\big)_\mathbb{R} \geq 0$. Comparing coefficients with (2.5.4) gives

$$a_0 = C\left[e\left(\bar{\rho}_\ell + \frac{(\rho_D,\sigma)}{(\ell,\ell')}\right) + e(\bar{\rho}_\ell) \sum_{\substack{\tilde{\lambda}\in K^\sharp \\ (\varepsilon_K(W),\tilde{\lambda})_\mathbb{R}\geq 0}} b(\tilde{\lambda}) e\left(\overline{\tilde{\lambda}}_\ell + \frac{(\lambda_D+\rho_D,\sigma)}{(\ell,\ell')}\right)\right].$$

As the left hand side is constant, it follows that $\rho_D = 0$ and further that $\tilde{\lambda}_D = 0$ for all $\tilde{\lambda}$. Whence $\lambda_D = 0$ for all those λ, which contribute non-trivial factors to the Borcherds product (2.5.2). Thus, re-inserting into the right-hand side of (2.5.2) we get

$$\lim_{r\to\infty} \Psi_f(ir,\sigma) = Ce(\bar{\rho}_\ell) \prod_{\substack{\lambda=\lambda_\ell\ell\in K^\sharp \\ (\varepsilon_K(W),\lambda)_\mathbb{R}>0}} \big(1 - e(\bar{\lambda}_\ell)\big)^{a(\lambda,0)}.$$

Since $\lambda = \lambda_\ell\ell$ is contained in $K^\sharp$, it follows that $\lambda = ae_3 = a\kappa_\mathbb{F}\ell$, with $a \in \mathbb{Q}$. By the Weyl chamber condition, $0 < \Im(a\bar{\kappa}_\mathbb{F}) = -\frac{1}{2}|\delta_\mathbb{F}|a$. Thus, $a < 0$,as claimed. □

2.6. Modularity of generating series

In this section, we review the result from [35, Sec. 10], which represents an analogue of Borcherds' result on the modularity of Heegner divisors in the case of modular surfaces for orthogonal groups from [3]. A result similar to our Theorem 2.37 below, has been obtained independently by Liu in [47], using rather different methods. See [48] for some newer developments in this direction.

We now introduce the residue-paring $\{\cdot,\cdot\}$, which is closely related to the pairing $\{\cdot,\cdot\}'$ introduced in Section 1.3 (see p.25). Let $\mathbb{C}[L^\sharp/L][q^{-1}]$ be the space of Fourier polynomials (including constant terms) and $\mathbb{C}[L'/L][[q]]$ the space of formal power series. Now, a non-degenerate pairing between these two spaces, called the residue-pairing, can be defined by putting

$$\{f,g\} := \sum_{\substack{n\le 0\\ h\in L^\sharp/L}} a(h,n)b(h,-n),$$

for $f=\sum_{h,n\le 0} a(h,n)q^n \in \mathbb{C}[L^\sharp/L][q^{-1}]$ and $g=\sum_{h,m\ge 0} b(h,m)q^m \in \mathbb{C}[L^\sharp/L][[q]]$. The space $\mathrm{M}^!_{L,1-p}$ can be identified with a subspace of $\mathbb{C}[L^\sharp/L][q^{-1}]$ by mapping a weakly holomorphic modular form to the non-positive part of its Fourier expansion. Likewise, the space $\mathrm{M}_{L^-,1+p}$ can be identified with a subspace of $\mathbb{C}[L^\sharp/L][[q]]$ by mapping a holomorphic modular form to its Fourier expansion.

Using Serre duality for vector-bundles on Riemann surfaces, in [3], Borcherds showed that[7] the space $\mathrm{M}^!_{L^-,1-p}$ is the orthogonal complement of $\mathrm{M}_{L,1+p}$ with respect to the residue-paring $\{\,,\,\}$. Since the pairing is non-degenerate and $\mathrm{M}_{L,1+p}$ has finite dimension, $\mathrm{M}_{L,1+p}$ is also the orthogonal complement of $\mathrm{M}_{L^-,1-p}$. In particular, the following holds [cf. 3, Theorem 10.3] (see also Proposition 1.10)

Lemma 2.36. *A formal power series $\sum_h\sum_{n>0} b(n,h)q^n\mathfrak{e}_h \in \mathbb{C}[L^\sharp/L]\otimes\mathbb{C}[[q]]$ is the Fourier expansion of a modular form $g\in\mathrm{M}_{L,1+p}$ if and only if*

$$\sum_{h\in L^\sharp/L}\sum_{\substack{n\in\mathbb{Z}-Q(h)\\ n\le 0}} a(h,n)b(h,-n)=0$$

for every $f=\sum_{n,h} a(h,n)q^n\mathfrak{e}_h\in\mathrm{M}^!_{L^-,1-p}$.

By a result of McGraw [see 50, Theorem 5.6] the spaces $\mathrm{M}^!_{L^-,1-p}$ and $\mathrm{M}_{L,1+p}$ have bases of modular forms with integer coefficients. Thus, a statement analogous to Lemma 2.36 holds for power series and modular forms over $\mathbb{Q}$. Moreover, it suffices to check the vanishing condition for every f with integral Fourier coefficients.

Consider $\mathrm{CH}^1(X)$, the first Chow group of the modular variety $X=\mathbb{D}/\Gamma_L$. Recall that $\mathrm{CH}^1(X)$ is isomorphic to the Picard group $\mathrm{Pic}(X)$.

Let $\pi:\tilde X\to X$ be a desingularization and denote by $\mathcal{B}=\mathcal{B}(\tilde X)$ the group of boundary divisors of $\tilde X$. We now consider a modified Chow group, the quotient $\mathrm{CH}^1(\tilde X)/\mathcal{B}$. Put $(\mathrm{CH}^1(\tilde X)/\mathcal{B})_{\mathbb{Q}}=(\mathrm{CH}^1(\tilde X)/\mathcal{B})\otimes_{\mathbb{Z}}\mathbb{Q}$.

Denote by $\mathcal{L}_k$ the sheaf of meromorphic automorphic forms on X. By the theory of Baily-Borel, there is a positive integer $n(\Gamma)$, such that if k is a positive integer divisible by $n(\Gamma)$, the sheaf $\mathcal{L}_k$ is an algebraic line bundle and thus defines an element in $\mathrm{Pic}(X)$.

[7] In keeping with the notation from the beginning of Section 2.4, compared to [2], we have switched ρ_L and ρ_{L^-}.

The pullback of $\mathcal{L}_k$ to $\tilde{X}$ defines a class in $\mathrm{CH}^1(\tilde{X})/\mathcal{B}$, which we denote $c_1(\mathcal{L}_k)$. More generally, if k is rational, we choose an integer n such that nk is a positive integer divisible by $n(\Gamma)$ and put $c_1(\mathcal{L}_k) = \frac{1}{n}c_1(\mathcal{L}_{nk}) \in (\mathrm{CH}^1(\tilde{X})/\mathcal{B})_{\mathbb{Q}}$.

As the Heegner divisors are $\mathbb{Q}$-Cartier on X, their pullbacks define elements in the modified Chow group $(\mathrm{CH}^1(X)/\mathcal{B})_{\mathbb{Q}}$.

Theorem 2.37. *The generating series in* $\mathbb{Q}[L'/L][[q]] \otimes (\mathrm{CH}^1(\tilde{X})/\mathcal{B})_{\mathbb{Q}}$ *given by*

$$A(\tau) = c_1(\mathcal{L}_{-1/2}) + \sum_{h \in L^\sharp/L} \sum_{\substack{n \in \mathbb{Z}+Q(h) \\ n>0}} \pi^*\big(\mathbb{D}(n,h)\big)\, q^n \mathfrak{e}_h$$

is a modular form in $\mathrm{M}_{L,1+p}$ *with values in* $(\mathrm{CH}^1(\tilde{X})/\mathcal{B})_{\mathbb{Q}}$*, i.e.* $A(\tau)$ *is contained in* $\mathrm{M}_{L,1+p} \otimes (\mathrm{CH}^1(\tilde{X})/\mathcal{B})_{\mathbb{Q}}$.

Proof. This follows from Theorem 2.29 and Lemma 2.36. Indeed, by the Lemma it suffices to show that

$$a(0,0)\cdot c_1(\mathcal{L}_{-1/2}) + \sum_{h \in L^\sharp/L} \sum_{\substack{n \in \mathbb{Z}-Q(h) \\ n<0}} a(h,n)\, \pi^*\mathbb{D}(-n,h) = 0 \quad \text{in} \quad (\mathrm{CH}^1(\tilde{X})/\mathcal{B})_{\mathbb{Q}},$$

for every $f = \sum_{h,n} a(h,n)q^n\mathfrak{e}_h$ in $\mathrm{M}^!_{L^-,1-p}$ with integral Fourier coefficients. But this follows immediately from Theorem 2.29, as the Borcherds lift Ψ_f of f is an automorphic form with divisor $\frac{1}{2}\sum_{n,h} a(h,n)\mathbb{D}(-n,h)$ of weight $\frac{1}{2}a(0,0)$, i.e. up to torsion a rational section of $\mathcal{L}_{a(0,0)/2}$. □

Theorem 2.37 is one of several modularity results for generating series, that we will encounter in the following sections. See Theorem 3.23 in Section 3.6 and Theorem 2.45 in Section 2.7 below.

2.7. Local Borcherds products

In this section, we will give a brief overview of [37], concerning local Borcherds products. *Local* here refers to a neighborhood of a cusp of the form $U_\epsilon(\ell)$, a notation that we recall from Section 2.1.2. Such local Borcherds products were introduced by Bruinier and Freitag in the context of orthogonal groups [see 7] and have since appeared in different contexts, e.g. Hilbert modular forms in [13, Chap. 2, Section 3.2] and Siegel modular threefolds in [24].

The aim of [37] is to study the local Picard group over a boundary component of X_Γ for a unitary modular group Γ. Since the construction is local in nature, it suffices to study one fixed boundary component, associated with the cusp $[\ell]$ of $\mathcal{H}_{\ell,\ell'}$, as in Section 2.1.2.

2.7.1. Local Picard group and local cohomology

With the notation of Section 2.1.2, we consider the fixed isotropic space $\mathbb{C}\ell = I_{\ell,\mathbb{C}}$ corresponding to the cusp at infinity $[\ell]$ of $\mathcal{H}_{\ell,\ell'}$. Using the Baily-Borel compactification $X^*_{\Gamma,BB}$, the local Picard group $\mathrm{Pic}(X_\Gamma, \ell)$ is defined as the direct limit of the Picard groups of the regular loci of the open neighborhoods $U_\epsilon(\ell)$ of this cup,

$$\mathrm{Pic}(X_\Gamma, \ell) = \varinjlim \mathrm{Pic}\left(U_\epsilon^{reg}(\ell)\right).$$

As the Heisenberg group $H_\ell := \mathrm{Heis}(\Gamma)$ has finite index in the stabilizer $\Gamma_\ell = \mathrm{stab}_\Gamma(\ell)$ of the cusp, the local Picard group can be described, up to torsion, through the direct system $\mathrm{Pic}\left(H_\ell \backslash U_\epsilon(\ell)\right)$. Since Γ_ℓ/H_ℓ operates on the direct limit $\varinjlim \mathrm{Pic}\left(H_\ell \backslash U_\epsilon(\ell)\right)$, for the invariant part, we have

$$\mathrm{Pic}(X_\Gamma, \ell) \otimes \mathbb{Q} = \left(\varinjlim \mathrm{Pic}\left(H_\ell \backslash U_\epsilon(\ell)\right) \otimes \mathbb{Q}\right)^{\Gamma_L}.$$

Thus, to describe the position of local divisors in the Picard group, at least up to torsion, it suffices to consider a Picard group $\mathrm{Pic}\left(H_\ell \backslash U_\epsilon(\ell)\right)$ for a fixed sufficiently small $\epsilon > 0$.

In the following, we will assume that ϵ is small enough for the map $H_\ell \backslash U_\epsilon \hookrightarrow X_\Gamma$ from p.35 to be an open immersion.

Local divisors Recall the definition of the special cycles $\mathbb{D}(\lambda)$ $(\lambda \in L^\sharp)$ and $\mathbb{D}(n, h)$ $(n \in \mathbb{Z}, h \in L^\sharp/L)$. Clearly, for $\mathbb{D}(\lambda)$ to intersect the neighborhood $U_\epsilon(\ell)$, the lattice vector λ must lie in the complement of ℓ with respect to $(\cdot,\cdot)$. Hence, $\lambda = \lambda_\ell \ell + \lambda_D$ with $\lambda_D \in W_\mathbb{F} = D \otimes \mathbb{F}$ and $\mathbb{D}(\lambda)$ is defined by an equation of the form

$$\lambda_\ell(\ell, \ell') + (\lambda_D, \sigma) = 0.$$

Now for such a λ consider its orbit under the action of the Heisenberg group H_ℓ. Clearly, since $\lambda \perp \ell$, the action of the translations $n(0, r) \in C_\ell(\Gamma)$ is trivial. The orbit under $H_\ell/C_\ell(\Gamma)$, i.e. under the elements $n(w, 0)$ with $w \in D_\Gamma$ (recall Lemma 2.5), is given by

$$\lambda - t\ell, \quad \text{with} \quad t \in \{(w, \lambda); w \in D_\Gamma\} \subseteq \mathcal{D}_\mathbb{F}^{-1}.$$

Hence, the group H_ℓ operates on the set $\lambda + \mathcal{D}_\mathbb{F}^{-1}\ell$ with only finitely many orbits and therefore, the special cyle defined by setting

$$\mathbb{D}(\lambda)_\ell := \sum_{\alpha \in \mathcal{D}_\mathbb{F}^{-1}} \mathbb{D}(\lambda + \alpha\ell)$$

is invariant under the action of the Heisenberg group H_ℓ. It thus defines an element of the divisor class group $\mathrm{Div}\left(H_\ell \backslash U_\epsilon(\ell)\right)$.

Now, to obtain a 'local' version of the Heegner divisors $\mathbb{D}(n, h)$. Consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{Div}\left(X_\Gamma\right) & \longrightarrow & \mathrm{Div}\left(H_\ell \backslash U_\epsilon(\ell)\right) \\ \downarrow & & \downarrow \\ \mathrm{Div}(\mathcal{H}_{\ell,\ell'}) & \longrightarrow & \mathrm{Div}\left(U_\epsilon(\ell)\right), \end{array} \tag{2.7.1}$$

where maps from left to right are the pre-images under the open immersion $H_\ell \backslash U_\epsilon(\ell) \hookrightarrow \Gamma\backslash\mathcal{H}_{\ell,\ell'} = X_\Gamma$ and the inclusion of the subset $U_\epsilon(\ell) \hookrightarrow \mathcal{H}_{\ell,\ell'}$, respectively, while from the top to bottom they are given by the pre-image under the projection maps.

Now we define $\mathbb{D}(n,h)_\ell$ as the image in $H_\ell\backslash U_\epsilon(\ell)$ of the divisor $\mathbb{D}(n,h) \in \mathrm{Div}(X_\Gamma)$ in the diagram (2.7.1). We will use the same notation, $\mathbb{D}(n,h)_\ell$, to denote the associated H_ℓ-invariant divisor in $\mathrm{Div}\,(U_\epsilon(\ell))$.

If ϵ is sufficiently small, the divisor $\mathbb{D}(n,h)_\ell$ is the restriction of $\mathbb{D}(n,h)$ to $U_\epsilon(\ell)$, and, indeed, the restriction of the locally finite sum of component cycles $\mathbb{D}(\lambda)$ of $\mathbb{D}(n,h)$. However, here only the λ's with $\lambda \perp \ell$ contribute.

Note that if $\mathbb{D}(n,h)_\ell$ is nonzero, then h is contained in the following subgroup of $L^\sharp/L$

$$\mathcal{L} := \left\{\mu \in L^\sharp/L; \quad 2(\ell,\mu)_\mathbb{R} \equiv 0 \mod M_1 \quad \text{and} \quad |\delta_\mathbb{F}|\Im(\ell,\mu) \equiv 0 \mod M_2\right\},$$

where M_1 and M_2 are the integers uniquely determined by $2(\ell, L)_\mathbb{R} = M_1\mathbb{Z}$ and by $|\delta_\mathbb{F}|\Im(L,\ell) = M_2\mathbb{Z}$, respectively. Then, for $h \in \mathcal{L}$ the divisor $\mathbb{D}(n,h)_\ell$ can be written in the form

$$\mathbb{D}(n,h)_\ell = \sum_{\substack{\delta\in D \\ Q\left(\delta+\dot{h}\right)=n}} \mathbb{D}(\delta+\dot{h})_\ell,$$

where we use the notation[8] that $\dot{h}$ denotes a representative of $h \in \mathcal{L}$, with $\dot{h} \in L^\sharp \cap \ell^\perp$, fixed once and for all for every such h. Note that there is a surjective homomorphism given by

$$\pi : \mathcal{L} \to D^\sharp/D, \qquad h \mapsto \dot{h}_D,$$

where $\dot{h}_D$ denotes the definite part of $\dot{h}$.

The local cohomology The Picard groups $\mathrm{Pic}\,(H_\ell\backslash U_\epsilon(\ell))$ can also be described through a local cohomology group. As usual, for a group G acting on an abelian group A, the n-th cohomology group is defined as follows

$$\mathrm{H}^n(G,A) = \frac{\ker\left(C^n(G,A) \xrightarrow{\partial} C^{n+1}(G,A)\right)}{\mathrm{im}\left(C^{n-1}(G,A) \xrightarrow{\partial} C^n(G,A)\right)},$$

with C^n the set of n-cocycles, consisting of all functions $f : G^n \to A$, and with the coboundary operator ∂. Now consider $G = H_\ell$, acting trivially on $A = \mathbb{Z}$.

Now denote by $\mathcal{O}_\epsilon = \mathcal{O}_\epsilon(U_\epsilon(\ell))$ the sheaf of holomorphic functions on $U_\epsilon(\ell)$ and by $\mathcal{O}_\epsilon^*$ the sheaf of invertible holomorphic functions. Through the action of H_ℓ on $\mathcal{O}_\epsilon$ and $\mathcal{O}_\epsilon^*$, induced from the action on $U_\epsilon(\ell)$, the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ \imath\ } \mathcal{O}_\epsilon \xrightarrow{\ e\ } \mathcal{O}_\epsilon^* \longrightarrow 0 \tag{2.7.2}$$

induces an exact sequence of cohomology groups

$$\mathrm{H}^1(H_\ell, Z) \longrightarrow \mathrm{H}^1\left(H_\ell,\mathcal{O}_\epsilon\right) \longrightarrow \mathrm{H}^1\left(H_\ell,\mathcal{O}_\epsilon^*\right) \xrightarrow{\delta} \mathrm{H}^2\left(H_\ell,\mathbb{Z}\right) \longrightarrow \mathrm{H}^2\left(H_\ell,\mathcal{O}_\epsilon\right). \tag{2.7.3}$$

[8] This notation was originally introduced in [7, Section 4].

Now, $\mathrm{Pic}(H_\ell \backslash U_\epsilon(\ell))$ is given by the cohomology group $\mathrm{H}^1\left(H_\ell \backslash U_\epsilon(\ell), \mathcal{O}_\epsilon^*\right)$. Since the open neighborhoods $U_\epsilon(\ell)$ are contractible, all analytic line bundles on $U_\epsilon(\ell)$ are trivial. Thus, in fact

$$\mathrm{Pic}(H_\ell \backslash U_\epsilon(\ell)) = \mathrm{H}^1\left(H_\ell, \mathcal{O}_\epsilon^*\right).$$

Now, denote by $\mathcal{P}_\epsilon$ the functions in $\mathcal{O}_\epsilon$ which are periodic for the action of $\mathbb{Z}N_{\Gamma,\ell} \simeq C_\ell(\Gamma)$ (see Lemma 2.5), similar for $\mathcal{P}_\epsilon^*$. Since $C_\ell(\Gamma)$, the center of the Heisenberg group, is a normal subgroup satisfying $H_\ell/\mathbb{Z}N_{\Gamma,\ell} = D_\Gamma$, and as $N_{\Gamma,\ell}\mathbb{Z}\backslash U_\epsilon(\ell)$ is contractible, we have $\mathrm{H}^p\left(H_\ell, \mathcal{O}_\epsilon\right) = \mathrm{H}^p\left(D_\Gamma, \mathcal{P}_\epsilon\right)$ for $p = 1, 2, \ldots$. Therefore, from (2.7.2) and (2.7.3), we get another exact sequence:

$$\frac{\mathrm{Hom}(D_\Gamma, \mathcal{P}_\epsilon)}{\mathrm{Hom}(D_\Gamma, \mathbb{Z})} \longrightarrow \mathrm{Pic}\left(H_\ell \backslash U_\epsilon(\ell)\right) \longrightarrow \mathrm{H}^2(H_\ell, \mathbb{Z}) \longrightarrow \mathrm{H}^2(H_\ell, \mathcal{O}_\epsilon)\,. \tag{2.7.4}$$

Moreover, since D_Γ is a free group, the following sequence is also exact

$$0 \longrightarrow \mathrm{Hom}\left(D_\Gamma, \mathbb{Z}\right) \longrightarrow \mathrm{Hom}\left(D_\Gamma, \mathcal{P}_\epsilon\right) \longrightarrow \mathrm{Hom}\left(D_\Gamma, \mathcal{P}_\epsilon^*\right) \longrightarrow 0.$$

Thus, from (2.7.4) we obtain the new exact sequence

$$\mathrm{Hom}\left(D_\Gamma, \mathcal{P}_\epsilon^*\right) \longrightarrow \mathrm{Pic}\left(H_\ell \backslash U_\epsilon(\ell)\right) \longrightarrow \mathrm{H}^2\left(H_\ell, \mathbb{Z}\right) \longrightarrow \mathrm{H}^2\left(H_\ell, \mathcal{O}_\epsilon\right).$$

Thus, the local Picard group can be studied through $\mathrm{H}^2(H_\ell, \mathbb{Z})$.

2.7.2. Local Borcherds products

Now, our aim is to describe the position of the local Heegner cycles in the cohomology group. For this purpose, given $\lambda \in L^\sharp \cap \ell^\perp$, one defines an absolutely convergent infinite product with zero-divisor $\mathbb{D}(\lambda)_\ell$.

From here on, until the end of the section, we assume that $(\ell', \ell) = \delta_{\mathbb{F}}^{-1}$ and that the constant δ in the setup of the Siegel domain $\mathcal{H}_{\ell,\ell'}$ is equal to $\delta_{\mathbb{F}}$.

Definition 2.38. Let $\lambda \in L^\sharp$ be a positive norm lattice in the orthogonal complement of ℓ. The local Borcherds product $\Psi_\lambda(\mathfrak{z})$ is defined as

$$\Psi_\lambda(\mathfrak{z}) := \prod_{\alpha \in \mathcal{O}_{\mathbb{F}}} \left(1 - e\left(\sigma\left(\Im\alpha\right)\left[(\lambda, \mathfrak{z}) + \frac{\alpha}{|\delta_{\mathbb{F}}|^2}\right]\right)\right),$$

where $\sigma(\Im\alpha) \in \{\pm 1\}$ is $+1$ if $\Im\alpha \geq 0$ and -1 otherwise.

Clearly, $\Psi_\lambda(\mathfrak{z})$ is an absolutely convergent infinite product with divisor $\mathbb{D}(\lambda)_\ell$. Note that $\Psi_\lambda(\mathfrak{z})$ can also be written in the form

$$\Psi_\lambda(\mathfrak{z}) = \prod_{\substack{p \pmod{D_{\mathbb{F}}} \\ q \in \mathbb{Z}}} \left(1 - e\left(\sigma(q)\left[(\lambda, \mathfrak{z}) + \frac{1}{|D_{\mathbb{F}}|}(p + \kappa q)\right]\right)\right),$$

where $\sigma(q) = \mathrm{sign}(q)$ if $q \neq 0$ and $\sigma(0) = +1$. Further, we remark that without the sign σ, which of course assures the convergence of the infinite product Ψ_λ, the (formal) product would be invariant under the operation of H_ℓ.

As it is defined however, $\Psi_\lambda(\mathfrak{z})$ is invariant only under the translations $n(0, r) \in C_\ell(\Gamma)$, while the operation of the Eichler elements $n(w, 0)$ with $w \in D_\Gamma$ gives rise to a non-trivial automorphy factor, given by

$$J_\lambda(n(w, r), \mathfrak{z}) = \frac{\Psi_\lambda\left(n(w, 0)\mathfrak{z}\right)}{\Psi_\lambda(\mathfrak{z})} \qquad (n(w, r) \in H_\ell)\,.$$

We will use this automorphy factor, viewed as a one-cocycle attached to $\mathbb{D}(\lambda)_\ell$ to define a two-cocycle, with which to find the Chern class of this special cycle. After a brief calculation, one finds [see 37, Prop. 4.1]

Proposition 2.39. *The automorphy factor J_λ associated associated with $\Psi_\lambda(\mathfrak{z})$ takes the form*

$$J_\lambda(n(w, r), \mathfrak{z}) = e\left(-2|D_\mathbb{F}|(\lambda, \mathfrak{z})(\lambda, w)_\mathbb{R} - 2(\lambda, w)^2_\mathbb{R}\kappa_\mathbb{F} + (\lambda, w)_\mathbb{R}\left(\kappa_\mathbb{F} + 1\right)\right),$$

where $\kappa_\mathbb{F}$ is element of $\mathcal{O}_\mathbb{F}$ with $\Im\kappa_\mathbb{F} = \frac{1}{2}\delta_\mathbb{F}$ and $\mathcal{O}_\mathbb{F} = \mathbb{Z} + \mathbb{Z}\kappa_\mathbb{F}$.

Now, one can define a two-cocycle a follows. One chooses a holomorphic function $A(g, \mathfrak{z})$ which satisfies $J_\lambda(g, \mathfrak{z}) = e(A(g, \mathfrak{z}))$, and sets

$$c(g, g') = A(gg', \mathfrak{z}) - A(g, g'\mathfrak{z}) - A(g', \mathfrak{z}) \quad (\forall g, g' \in H_\ell)\,.$$

Thus, the map $(g, g') \mapsto c(g, g')$ defines a two-cocyle. Note that $c(g, g')$ is independent of the choice of $A(g, \mathfrak{z})$ and changes only by a co-boundary after multiplying J_λ with a trivial automorphy factor. The two-cocycle obtained in this manner is a representative for the Chern class of $\mathbb{D}(\lambda)_\ell$ in the second cohomology group $\mathrm{H}^2(H_\ell, \mathbb{Z})$. Directly by calculating $c(n(w, 0), n(w', 0))$ with $w, w' \in D_\Gamma$ (clearly, it suffices to consider Eichler transformations from H_ℓ) one finds the following Proposition [37, Prop. 4.2].

Proposition 2.40. *The Chern class $\delta\left(\mathbb{D}(\lambda)_\ell\right)$ of the local divisor $\mathbb{D}(\lambda)_\ell$ in $\mathrm{H}^2(H_\ell, \mathbb{Z})$ is determined by the two-cocycle*

$$[c_\lambda]:\ (n(w, 0), n(w', 0)) \longmapsto -2|\delta_\mathbb{F}|(\lambda, w)_\mathbb{R}\Im(\lambda, w') = \Im\left(-|\delta_\mathbb{F}|(\lambda, w)_\mathbb{R}(\lambda, w')\right).$$

We now define a bilinear form $F_\lambda(\cdot, \cdot)$ by setting for $a, b \in W_\mathbb{F}$

$$F_\lambda(a, b) = (\lambda, a)_\mathbb{R}(\lambda, b) = (a, \lambda)(\lambda, b) + (\lambda, a)(\lambda, b). \tag{2.7.5}$$

Note that the value of $F_\lambda(\cdot, \cdot)$ depends only on the definite part λ_D of λ.

Then, by Proposition 2.40 the Chern class $\delta\left(\mathbf{D}\right)$ of a finite linear combination

$$\mathbf{D} := \sum_{\substack{\lambda \in L^\sharp \cap \ell^\perp \\ Q(\lambda) > 0}} a(\lambda)\mathbb{D}(\lambda)_\ell \tag{2.7.6}$$

with integer coefficients $a(\lambda)$ is determined by the two-cocycle

$$(n(w,0),n(w',0)) \longmapsto \sum_{\substack{\lambda\in L^\sharp\cap\ell^\perp\\ Q(\lambda)>0}} \Im\left(|\delta_{\mathbb{F}}|F_\lambda(w,w')\right).$$

In particular, we can thus describe the Chern class $\delta\left(\mathbb{D}(n,h)_\ell\right)$ for a local Heegner divisor of the form $\mathbb{D}(n,h)_\ell$. Now, we want to find out when this Chern class is a torsion element in $\mathrm{H}^2(H_\ell,\mathbb{Z})$.

2.7.3. Torsion critera

Bilinear forms in the cohomology Consider the set of real-valued bilinear forms $B : W\otimes W \to \mathbb{R}$ and denote this set by BIL. Naturally, BIL has the structure of a real vector space. Let $\mathrm{BIL}_{\mathbb{Z}} \subset \mathrm{BIL}$ the subset of forms which are $\mathbb{Z}$-valued on the lattice D_Γ.

Associate to every element $B\in\mathrm{BIL}$ a two-cocycle in $C^2(H_\ell,\mathcal{O}_\epsilon)$ by setting

$$B(n(w,r),n(w',r')) := B(w,w') \qquad (n(w,r),n(w',r')\in H_\ell),$$

and denote by $[B]$ its class in $\mathrm{H}^2(H_\ell,\mathcal{O}_\epsilon)$. Similarly, we attach to every element of $\mathrm{BIL}_{\mathbb{Z}}$ a class in $\mathrm{H}^2(H_\ell,\mathbb{Z})$. By composition with the natural map from (2.7.3), $\mathrm{H}^2(H_\ell,\mathbb{Z})\to \mathrm{H}^2(H_\ell,\mathcal{O}_\epsilon)$, we get a sequence

$$\mathrm{BIL}_{\mathbb{Z}} \longrightarrow \mathrm{H}^2(H_\ell,\mathbb{Z}) \longrightarrow \mathrm{H}^2(H_\ell,\mathcal{O}_\epsilon).$$

This sequence turns out to be exact. Indeed, one can show the following [see 37, Prop. 3.1]:

Proposition 2.41. *The image of* BIL *in* $\mathrm{H}^2(H_\ell,\mathcal{O}_\epsilon)$ *vanishes.*

Proof. We give a brief sketch of the proof, see loc. cit. for details. It suffices to consider two cases, either that B is the imaginary part of a symmetric complex-valued bilinear form or that B is the real part of a complex hermitan form.

We consider the first case. Thus let $G : W\times W\to\mathbb{C}$ be a symmetric bilinear form with $\Im G = B$ and consider the following $\mathcal{O}_\epsilon$-valued one-cocycle:

$$u(n(w,r),\mathfrak{z}) = \frac{i}{2}\left(\delta_{\mathbb{F}}^{-1}G(w,\sigma)+\frac{1}{2}\overline{G(w,w)}\right).$$

The image under the coboundary map is given by

$$\begin{aligned}\partial u(n(w,r),n(w',r'),\mathfrak{z}) &= \frac{1}{2i}\left(G(w',w)-\frac{1}{2}\left(\overline{G(w,w')}+\overline{G(w',w)}\right)\right)\\ &= \frac{1}{2i}\left(G(w,w')-\overline{G(w',w)}\right) = B(w',w).\end{aligned}$$

Thus, we see that B is trivialized by a cochain and thus $[B]=0$ in $\mathrm{H}^2(H_\ell,\mathcal{O}_\epsilon)$.

The second case, where $B=\operatorname{Re} H$ for a hermitian form H is quite similar. (Clearly, it suffices to consider these two cases.) $\square$

The following Lemma[9] is crucial, since it gives a straight forward criterion for bilinear forms to be torsion elements in the cohomology group $\mathrm{H}^2(H_\ell, \mathbb{Z})$, which we can apply to linear combinations of two-cocycles of the form $[c_\lambda]$ from Proposition 2.40. For the proof see [37].

Lemma 2.42 ([37, Lemma 3.1])**.** *The kernel of the map $\mathrm{BIL}_{\mathbb{Z}} \mapsto \mathrm{H}^2(H_\ell, \mathbb{Z})$ is the cyclic group generated by the antisymmetric bilinear form*

$$\frac{1}{N_{\Gamma,\ell}|\delta_{\mathbb{F}}|}\Im(\cdot,\cdot).$$

In particular, the image of an element $B \in \mathrm{BIL}_{\mathbb{Z}}$ is a torsion element in the cohomology group $\mathrm{H}^2(H_\ell, \mathbb{Z})$ if an only if B and $|\delta_{\mathbb{F}}|^{-1}\Im(\cdot,\cdot)$ are linear dependent over $\mathbb{Z}$.

Torsion criteria for local Heegner divisors Using the criterion from Lemma 2.42 and Proposition 2.40, one can now give the following criterion for a linear combination of local Heegner divisors to be torsion.

Lemma 2.43 ([37, Lemma 4.1, Corollary 4.1])**.** *Let $\mathbf{D}$ be a finite linear combination of local Heegner divisors of the form (2.7.6), i.e.*

$$\mathbf{D} = \sum_{\substack{\lambda \in L^\sharp \cap \ell^\perp \\ Q(\lambda) > 0}} a(\lambda)\mathbb{D}(\lambda)_\ell.$$

Then, the following statements hold

1. *The Chern class $\delta(\mathbf{D})$ of $\mathbf{D}$ is a torsion element in $\mathrm{H}^2(H_\ell, \mathbb{Z})$ if and only if the following equation holds*

$$\sum_{\substack{\lambda \in L^\sharp \cap \ell^\perp \\ Q(\lambda) > 0}} a(\lambda)\left[F_\lambda(w, w') - \frac{Q(\lambda)}{p-1}(w, w')\right] = 0$$

   for all $w, w' \in D_\Gamma$.

2. *Let B_λ be the complex-valued bilinear form defined by $B_\lambda(a, b) := (\lambda, a)(\lambda, b)$ for $a, b \in W$. If $\delta(\mathbf{D})$, the Chern class of $\mathbf{D}$, is a torsion element in $\mathrm{H}^2(H_\ell, \mathbb{Z})$, then*

$$\sum_{\substack{\lambda \in L^\sharp \cap \ell^\perp \\ Q(\lambda) > 0}} a(\lambda)\,\mathrm{trace}\, B_\lambda = 0.$$

   (Where the trace is taken with respect to the standard orthogonal basis $v_1, \ldots, v_m$ of V, $(\cdot,\cdot)$.)

[9] This result is originally due to Freitag.

Proof. By Proposition 2.40 and the subsequent remarks, the Chern class of $\mathbf{D}$ is given by a linear combination of cocycles $[c_\lambda]$ in $\mathrm{H}^2(H_\ell, \mathbb{Z})$. By Proposition 2.41 the images of the $[c_\lambda]$'s and hence that of $\delta(\mathbf{D})$ in $\mathrm{H}^2(H_\ell, \mathcal{O}_\epsilon)$ vanish.

Further, by Lemma 2.42, $\delta(\mathbf{D})$ is a torsion element in $\mathrm{H}^2\left(H_\ell, \mathbb{Z}\right)$ precisely if there is a rational number R for which the equation

$$\sum_{\substack{\lambda \in L^\sharp \cap \ell^\perp \\ Q(\lambda)>0}} a(\lambda)|\delta_{\mathbb{F}}| \cdot \Im\left(F_\lambda(w, w')\right) = R\frac{\Im(w, w')}{|\delta_{\mathbb{F}}|} \tag{2.7.7}$$

holds for all $w, w' \in D_\Gamma$. We can now extend this equation linearly to $W = D_\Gamma \otimes \mathbb{C}$, noting that D_Γ has full rank in W. We thus get

$$\sum_{\substack{\lambda \in L^\sharp \cap \ell^\perp \\ Q(\lambda)>0}} a(\lambda)|\delta_{\mathbb{F}}| \cdot \left(F_\lambda(w, w')\right) = R\frac{(w, w')}{|\delta_{\mathbb{F}}|}. \tag{2.7.8}$$

Indeed, by multiplying any pair of vectors $(w, w') \in D_\Gamma \times D_\Gamma$ by a purely imaginary multiple, from (2.7.7), one obtains a similar equation with real parts of $F_\lambda(\cdot, \cdot)$ and $(\cdot, \cdot)$, respectively. Adding the two equations yields (2.7.8).

Now, to determine R, we take the trace of both sides of (2.7.8), using the standard orthogonal basis $v_2, \ldots, v_{m-1}$ for W. Of course, $\operatorname{trace}(\cdot, \cdot) = m - 2 = p - 1$. Recall from (2.7.5) that we can write F_λ in the form $(\lambda, b)(\lambda, a) + (\lambda, b)(\lambda, a)$. Thus, defining a hermitian form by $H_\lambda(a, b) = (a, \lambda)(\lambda, b)$ $(a, b \in W)$, we can write $F_\lambda = B_\lambda + H_\lambda$. An easy calculation yields $\operatorname{trace} H_\lambda = (\lambda, \lambda)$ and we get

$$R \cdot (p-1) = \sum_{\substack{\lambda \in L^\sharp \cap \ell^\perp \\ Q(\lambda)>0}} a(\lambda)|D_{\mathbb{F}}|\left(Q\left(\lambda\right) + \operatorname{trace} B_\lambda\right).$$

Repeating the same calculation using the trace for the basis iv_j, $j = 2, \ldots, m-1$, which, of course is a standard orthogonal basis for W, too, quite naturally, we get the same result for the traces of the two hermitian forms $(\cdot, \cdot)$ and H_λ. However, the new trace of B_λ is given by $-B_\lambda$. Thus, we conclude that the contribution of the B_λ terms must necessarily vanish, yielding the condition in the second part of the Lemma. Hence R is given by

$$R = \sum_{\substack{\lambda \in L^\sharp \cap \ell^\perp \\ Q(\lambda)>0}} a(\lambda) \cdot |D_{\mathbb{F}}|\frac{Q\left(\lambda\right)}{p-1}.$$

Reintroducing this into (2.7.8) completes the proof of the first part and thus of the Lemma. □

We can now formulate one of the main results of [37]. It describes the position of local Heegner divisors of the form $\mathbb{D}(n, h)_\ell$ in the local Picard group. Note that such cycles are given by a finite linear combination of local divisors $\mathbb{D}(\lambda)_\ell$ and, as we have seen, the Chern class $\delta\left(\mathbb{D}(\lambda)_\ell\right)$ depends only on the projection λ_D.

Theorem 2.44 ([37, Theorem 4.1]). *Let $\mathbf{D}$ be finite linear combination of local Heegner divisors of the form*

$$\mathbf{D} = \frac{1}{2}\sum_{h\in\mathcal{L}}\sum_{\substack{m\in\mathbb{Z}+Q(h)\\ n<0}} a(h,n)\mathbb{D}(-n,h)_\ell,$$

with integer coefficients $a(h,n)$ satisfying $a(-h,n) = a(h,n)$. Then $\mathbf{D}$ is a torsion element in the Picard group $\operatorname{Pic}(H_\ell\backslash U_\epsilon(\ell))$ if and only if for all $w, w' \in D_\Gamma$ the equation

$$\sum_{h\in\mathcal{L}}\sum_{\substack{n\in\mathbb{Z}+Q(h)\\ n<0}} a(h,n) \sum_{\substack{\lambda\in D^\sharp\\ \lambda+D\equiv\pi(h)\\ Q(\lambda)=-n}} \left[F_\lambda(w,w') - \frac{Q(\lambda)}{p-1}(w,w')\right] = 0. \tag{2.7.9}$$

holds. Further, as a necessary condition for this to be the case, the following equation must hold for the bilinear form $B_\lambda(\cdot,\cdot) = (\lambda,\cdot)(\lambda,\cdot)$:

$$\sum_{h\in\mathcal{L}}\sum_{\substack{n\in\mathbb{Z}+Q(h)\\ n<0}} a(h,n) \sum_{\substack{\lambda\in D^\sharp\\ \lambda+D\equiv\pi(h)\\ Q(\lambda)=-n}} \operatorname{trace} B_\lambda = 0.$$

Proof. We give an abridged version of the proof from [37], see there for more details. First note that if $\mathbf{D}$ is a torsion element, the first equation (2.7.9) follows from the first part Lemma 2.43 and second equation follows from the necessary condition in the second part of that Lemma.

For the converse, assume that (2.7.9) holds for all $w, w' \in D_\Gamma$. One now uses Proposition 2.39 to explicitly construct an automorphy factor $J_{\mathbf{D}}(g, \mathfrak{z})$ for $g \in H_\ell$ and $\mathfrak{z} \in U_\epsilon(\ell)$, describing $\mathbf{D}$ in $\operatorname{Pic}(H_\ell\backslash U_\epsilon(\ell))$. One then has to show that all factors of $J_{\mathbf{D}}$ can be expressed through (suitable) powers of trivial automorphy factors and factors of finite order, and hence that $\mathbf{D}$ is a torsion element in the local Picard group.

An automorphy factor for $\mathbf{D}$ is given as follows

$$\begin{aligned} J_{\mathbf{D}}(g,\mathfrak{z}) &= \prod_{\substack{h\ in\mathcal{L}\\ n\in\mathbb{Z}+Q(h)\\ n<0}} \prod_{\substack{\mu\in D\\ Q(\mu+\dot{h})=-n}} J_{\mu+\dot{h}}(g,\mathfrak{z})^{a(h,n)/2} \\ &= \prod_{\substack{h\ in\mathcal{L}\\ n\in\mathbb{Z}+Q(h)\\ n<0}} \prod_{\substack{\mu\in D\\ Q(\mu+\dot{h})=-n}} e\Big(-2|D_{\mathbb{F}}|\Big(\mu+\dot{h},\mathfrak{z}\Big)\Big(\mu+\dot{h}_D,w\Big)_{\mathbb{R}} \\ &\qquad - 2\Big(\mu+\dot{h}_D,w\Big)_{\mathbb{R}}^2\kappa_{\mathbb{F}} + \Big(\mu+\dot{h}_D,w\Big)_{\mathbb{R}}(\kappa_{\mathbb{F}}+1)\Big)^{a(h,n)/2}. \end{aligned}$$

Since $a(-h,n) = a(h,n)$, the last terms, being linear in $\mu + \dot{h}_D$ cancel. Now, write $\big(\mu+\dot{h},\mathfrak{z}\big) = \big(\mu+\dot{h}_D,\mathfrak{z}\big) + \big(\dot{h}-\dot{h}_D,\mathfrak{z}\big)$. Since $\dot{h} \perp \ell$, the second term is given by $\overline{\dot{h}_\ell}(\ell,\ell')$ and one can write the factors in $J_{\mathbf{D}}$ in the form

$$\left[e\left(-2\delta_{\mathbb{F}}\overline{\dot{h}_\ell}\Big(\mu+\dot{h}_D,w\Big)_{\mathbb{R}}\right)e\Big(-2|D_{\mathbb{F}}|F_{\mu+\dot{h}_D}(w,w) - 2\kappa_{\mathbb{F}}\operatorname{Re}F_{\mu+\dot{h}_D}(w,w)\Big)\right]^{\frac{a(h,n)}{2}},$$

Applying (2.7.9) to the second factor, and writing $\lambda_D = \mu + \dot{h}_D$ (and omitting the $\frac{1}{2}a(h,n)$ power for the time being), we get

$$e\left(-2\delta_{\mathbb{F}}\overline{\dot{h}_\ell}(\lambda_D, w)_{\mathbb{R}}\right) e\left(\frac{2|D_{\mathbb{F}}|}{p-1}Q(\lambda_D)\left[(w,\sigma) + \frac{Q(w)}{2\delta_{\mathbb{F}}}\right]\right) e\left(-\frac{2\operatorname{Re}\kappa_{\mathbb{F}}}{p-1}Q(\lambda)Q(w)\right) \tag{2.7.10}$$

Clearly, the last factor has finite order, and is thus a torsion element in the Picard group. To deal with the first and second term consider the following trivial automorphy factors

$$j_1(n(w,0),\mathfrak{z}) = e\left(\delta_{\mathbb{F}}^{-1}(\lambda_D, w)\right) \qquad j_2(n(w,0),\mathfrak{z}) = e\left(c\left(-(w,\sigma) + (2\delta_{\mathbb{F}})^{-1}(w,w)\right)\right)$$

which arise from the action of the Eichler elements of the form $n(w,0) \in H_\ell$ on the two invertible functions $f_1(\mathfrak{z}) = e((\lambda_D,\sigma))$ and $f_2 = e(c\tau)$ with $c \in \mathbb{Q}^\times$. Now, the second factor in (2.7.10) is a rational power of j_2. For the first factor in (2.7.10), bear in mind that $|\delta_{\mathbb{F}}|\Im\dot{h}_\ell$ is rational (in fact, half-integer). Further, after multiplying with suitable powers of j_1, since $|\delta_{\mathbb{F}}|\Im(\lambda_D, w)$ and $\operatorname{Re}\dot{h}_\ell$ are rational numbers, only a torsion element remains.

Thus, each of the finitely many factors in $J_{\mathbf{D}}$ is either of finite order or a rational power of a trivial automorphy factor. Hence, $\mathbf{D}$ is a torsion element in $\operatorname{Pic}(H_\ell\backslash U_\epsilon(\ell))$. □

2.7.4. Application to modular forms and an obstruction result

Now, let $l = p+1$, and consider the space of cusp forms S_{l,L^-} of weight l transforming under the Weil representation for L^-.

We will introduce a subspace $\mathrm{S}^{\Theta}_{l,L^-}$ of S_{l,L^-} spanned by certain theta series. For this, define polynomials $p_1(u,v,w)$ and $p_2(u,v,w)$ by setting

$$\begin{aligned} p_1(u,v,w) &:= \operatorname{Re} F_u(v,w) - \frac{Q(u)}{p-1}(v,w)_{\mathbb{R}}, \\ p_2(u,v,w) &:= \Im F_u(v,w) - \frac{Q(u)}{p-1}\Im(w,v). \end{aligned}$$

Note that, using these polynomials, we can write (2.7.9) in the form

$$\sum_{h\in\mathcal{L}}\sum_{\substack{m\in\mathbb{Z}+Q(h)\\ n<0}} a(h,n) \sum_{\substack{\lambda\in D^\sharp\\ \lambda+D\equiv\pi(h)\\ Q(\lambda)=-n}} [p_1(\lambda,w,w') + ip_2(\lambda,w,w')] = 0.$$

Now set $P(u,v) := p_1(u,v,v)$. Note that P is homogeneous with degree two in u and harmonic in both indeterminates. Then, by a well-known results from the theory of theta series [cf. 2, Theorem 4.1], for every $v \in W$, $v \neq 0$, the definite theta series defined as

$$\begin{aligned} \Theta_P(\tau; v) &:= \sum_{\lambda\in D^\sharp} P(\lambda, v) e\left(Q(\lambda)\tau\right)\mathfrak{e}_\lambda \\ &= \sum_{\mu\in D^\sharp/D}\sum_{\substack{n\in\mathbb{Z}-Q(\mu)\\ n>0}}\Bigg(\sum_{\substack{\lambda\in D^\sharp\\ \lambda+D\equiv\mu \pmod L\\ Q(\lambda)=n}} P(\lambda,v)\Bigg)\cdot e(n\tau)\mathfrak{e}_h \end{aligned} \tag{2.7.11}$$

is a cusp form of weight $p+1$ transforming under ρ_{L^-}. The second line in (2.7.11) gives the Fourier expansion, and we note that the Fourier coefficients are exactly the inner sum in (2.7.9) with the pair (w,w') restricted to the diagonal of $D_\Gamma \times D_\Gamma$. Now letting the parameter v vary over W, we define $\mathrm{S}^{\Theta}_{l,L^-}$ as the subspace of cusp forms spanned by the theta series $\Theta_P(\tau;v)$ $(v \in W_{\mathbb{C}})$.

With these considerations the following is merely a restatement of Theorem 2.44.

Theorem 2.45. *A finite linear combination of local Heegner divisors of the form*

$$\mathbf{D} = \frac{1}{2}\sum_{h\in\mathcal{L}} \sum_{\substack{n\in\mathbb{Z}+Q(h)\\ n<0}} a(h,n)\mathbb{D}(-n,h)_\ell,$$

with integer coefficients $a(h,n)$ satisfying $a(-h,n)=a(h,n)$ is a torsion element in the local Picard group $\operatorname{Pic}(H_\ell\backslash U_\epsilon(\ell))$ *if any only if*

$$\sum_{h\in\mathcal{L}} \sum_{\substack{n\in\mathbb{Z}+Q(h)\\ n<0}} a(h,n)b(\pi(h),-n) = 0 \tag{2.7.12}$$

for every cusp form $g = \sum_{\mu\in D^\sharp/D}\sum_{n\in\mathbb{Z}-Q(\mu)} b(\mu,n)e(n\tau)\mathfrak{e}_\mu$ in $\mathrm{S}^{\Theta}_{l,L^-}$.

Remark 2.46. While the construction of local Borcherds products is quite unrelated to that of Borcherds products through a singular theta-lift, the two are related through the Heegner divisors and such obstruction statements as Theorem 2.45. Indeed, the argument in the proof of Theorem 2.37, obtained by combining Theorem 2.29 and Lemma 2.36 can be stated as follows: A given linear combination of special cycles $\mathbf{H} = \sum_h\sum_{n<0} a(h,n)\mathbb{D}(-n,h)$ is the divisor a Borcherds product if and only if the sum $\sum_h\sum_{n<0} a(h,n)b(h,-n)$ vanishes for every cusp form $g \in \mathrm{S}_{1+p,L^-}$ with Fourier coefficients $b(h,n)$. In particular, since $\mathrm{S}^{\Theta}_{1+p,L^-}$ is contained in S_{1+p,L^-}, in this case the restriction of $\mathbf{H}_\ell$ also fulfills (2.7.12) (note that only $h\in\mathcal{L}$ can occur).

We remark further, that if $\mathbf{H}$ the divisor of a Borcherds product $\Psi_f(\mathfrak{z})$, there is a linear combination $\mathbf{H}_{\mathrm{O}}$ of Heegner divisors of the form $\mathbb{D}_{\mathrm{O}}(-n,h) \subset \mathcal{H}_p$ which restricts to $\mathbf{H}$ on the image $\varepsilon(\mathcal{H}_{\ell,\ell'})$. Since by a result of Borcherds in [3], the obstruction space for Borcherds products for the orthogonal groups is $\mathrm{S}_{L^-,1+p}$, the same as in the case of Borcherds products for unitary groups. Thus, $\mathbf{H}_{\mathrm{O}}$ is the divisor of a Borcherds product on $\mathcal{H}_p$ (the pull-back of which under ε is just $\Psi_f(\mathfrak{z})$).

By the results of Bruinier and Freitag [7], the divisors of Borcherds products on $\mathcal{H}_p$ restrict to torsion elements in local Picard groups for boundary components of $\mathcal{H}_p$, and these, moreover are the divisors of local Borcherds products there.

Finally, since the embedding ε is well-behaved when it comes to the neighborhoods of the cusp, the restriction of a special cycle $\mathbf{H}$ to $\operatorname{Pic}(H_\ell\backslash U_\epsilon(\ell))$ can be interpreted as the preimage of a local Heegner divisor (like in [7]) in a neighborhood of the boundary of $\mathcal{H}_p$, which is just the restriction of $\mathbf{H}_{\mathrm{O}}$.

Thus, if $\mathbf{H}$ is the divisor of a Borcherds product on $\mathrm{U}(p,1)$, not only is $\mathbf{H}_{\mathrm{O}}$ the divisor of a Borcherds product on $\mathrm{O}(2p,2)$, but both divisors also restrict to divisors of local Borcherds products, unitary or orthogonal, respectively.

3. The singular theta lift in arbitrary signature (p, q)

In this chapter, we present the construction of a Schwartz-form ψ, which coincides with φ_0 in signature $(p, 1)$ and takes its role in arbitrary signature. We use it to construct two types Green functions, one through a 'singular' following [43] and another using a singular theta-lift of Borcherds type associated to ψ, which generalizes important properties of the Borcherds lift.

For the most part, we follow the joint paper of the author and Jens Funke [25] fairly closely. Some introductory remarks are in order first.

3.1. Introduction

Besides the infinite products expansions, the Borcherds lift in signature $(p, 2)$ for orthogonal groups has a number of remarkable properties, a few of which we have already seen in Chapter 2. It takes its zeros and poles along arithmetically defined special cycles prescribed by the Fourier expansion of the input functions. By the work of Bruinier [5] these geometric properties extend to a lifting into the cohomology, which can be utilized in the construction of Green objects related to special cycles, see for example [8, 11]. And, hence further can be employed to show the modularity of generating series, two key aspects of the Kudla program [41], see [43] for an overview.

Similar properties are shared by the unitary version in signature $(p, 1)$. While the product expansion in [33, 34, 35] and the modularity result from [35] (see Section 2.6) represent merely first steps in this direction, the deep arithmetic-geometric results from [14], [19] and [15, 16] demonstrate the usefulness of the geometric lifting in the unitary setting, similar to the orthogonal case.

Of course, beside the Gaussian φ_0 there are many other Schwartz forms which can be used to construct a singular theta lift of Borcherds type. In fact, Borcherds' fairly general construction of the singular theta lift in [2] encompasses many previously known examples for theta lifts, such as the Shintani lift [55], Niwa's realization of the Shimura lift [52] or the Doi-Naganuma lift [18, 51, 58].

Generalizing to arbitrary signature (p, q), the theta lift constructed using the Gaussian $\varphi_0^{p,q}$ retains some useful properties. For example, the eigenvalue equation proven by Bruinier in [5, Chapter 4] generalizes to any signature (see Hufler's thesis [39] for a proof of this for unitary groups). However, other features are less salient.

In the seminal paper [10] Bruinier and Funke found a suitable generalization of the geometric Borcherds lift (for orthogonal groups), which works across all signatures. For this, they turned to the work of Kudla and Millson, who in a series of papers, [44], [45] and [46] constructed a geometric lifting using a Schwartz form φ_{KM} valued in the differential forms. In fact, Kudla and Millson constructed two versions of this form, one for orthogonal and the other for unitary groups. We will review the properties of the unitary version of φ_{KM} in Section 3.2 below.

Somewhat more implicitly, Kudla and Millson also constructed a second Schwartz form ψ' related to φ_{KM} via $d\psi' = L\varphi_{KM}$, where L denotes the Maass lowering operator. Denote by ψ the Schwartz form with $d^c\psi = \psi'$. This form coincides with φ_0 if the signature is $(p, 2)$ or $(p, 1)$ for orthogonal or unitary groups, respectively.

Now, Bruinier and Funke introduced a lift of Borcherds type using the Schwartz form[1] ψ' and studied the relationship between this lift on the one hand and the Kudla-Millson lift with the Schwartz form φ_{KM} on the other hand.

In particular, they established an adjointness result between the two geometric theta lifts, and, further, under a more geometric point of view, they showed a current equation for the theta lift they had introduced.

In the present chapter, in Section 3.3 we will give an explicit construction of a Schwartz form ψ with $dd^c\psi = \varphi_{KM}$ for unitary groups $\mathrm{U}(p, q)$. We will then proceed to construct Green forms, first through a 'singular' Schwartz form Ψ associated to ψ following Kudla [43]. Second, following Bruinier [5], using the geometric singular theta lift of Borcherds type for the Schwartz form ψ. Also, we examine the properties of this singular theta lift and establish analogous results to those in [10], concerning current equations and adjointness to (the unitary version of) the Kudla-Millson lift.

Furthermore, we show a modularity result for a generating series of the differences between the Green forms of Kudla type and the Green forms of Bruinier type, along the lines of [19] (see Section 3.6). Finally, in Section 3.7, we introduce a Green form depending on a complex parameter s and identify it with a Green form constructed by Oda and Tsusuki [53].

3.2. The Kudla-Millson form φ_{KM}

Recall the notation from Sections 1.1 and 1.2. Consider the complex $[\mathcal{S}(V) \otimes \mathcal{A}^\bullet(\mathbb{D})]^G$ of G-invariant Schwartz functions on V with values in the differential forms on $\mathbb{D}$. Note that evaluation at the base point z_0 yields an isomorphism

$$[\mathcal{S}(V) \otimes \mathcal{A}^\bullet(\mathbb{D})]^G \simeq \left[\mathcal{S}(V) \otimes \bigwedge\nolimits^{\bullet}(\mathfrak{p}^*)\right]^K.$$

We use the same symbol for corresponding objects. Note also

$$\varphi_0(x, z) \in \left[\mathcal{S}(V) \otimes \mathcal{A}^0(\mathbb{D})\right]^G,$$

and evaluation at the base point gives $\varphi_0(x) = \varphi_0(x, z_0) = e^{-\pi \sum_{i=1}^m |z_i|^2} \in \mathcal{S}(V)^K$.

[1]Actually, in signature $(b^+, 2)$ they considered the 'original' version with the Gaussian form φ_0.

Following [44, Proposition 5.2] and [46, Section 5], we define the two differential operators

$$\mathcal{D} = \frac{1}{2^{2q}} \prod_{\mu=p+1}^{m} \left\{ \sum_{\alpha=1}^{p} \left(\bar{z}_\alpha - \frac{1}{\pi} \frac{\partial}{\partial z_\alpha} \right) \otimes A'_{\alpha\mu} \right\} = \frac{1}{2^{2q}} \prod_{\mu=p+1}^{m} \left\{ \sum_{\alpha=1}^{p} \mathcal{D}_\alpha \otimes A'_{\alpha\mu} \right\}$$
$$\text{and} \quad \overline{\mathcal{D}} = \frac{1}{2^{2q}} \prod_{\mu=p+1}^{m} \left\{ \sum_{\alpha=1}^{p} \left(z_\alpha - \frac{1}{\pi} \frac{\partial}{\partial \bar{z}_\alpha} \right) \otimes A''_{\alpha\mu} \right\} = \frac{1}{2^{2q}} \prod_{\mu=p+1}^{m} \left\{ \sum_{\alpha=1}^{p} \overline{\mathcal{D}}_\alpha \otimes A''_{\alpha\mu} \right\},$$

where $\frac{\partial}{\partial z_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} - i\frac{\partial}{\partial y_\alpha}\right)$ and $\frac{\partial}{\partial \bar{z}_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} + i\frac{\partial}{\partial y_\alpha}\right)$. Further, $A'_{\alpha\mu}$ and $A''_{\alpha\mu}$ denote the left multiplication by $\xi'_{\alpha\mu}$, $\xi''_{\alpha\mu}$, respectively. Also, we have set

$$\mathcal{D}_\alpha := \left(\bar{z}_\alpha - \frac{1}{\pi} \frac{\partial}{\partial z_\alpha} \right) \quad \text{and} \quad \overline{\mathcal{D}}_\alpha := \left(z_\alpha - \tfrac{1}{\pi} \tfrac{\partial}{\partial \bar{z}_\alpha} \right).$$

Now, the Kudla-Millson Schwartz form from [44] is defined as

$$\varphi_{KM} := \mathcal{D}\overline{\mathcal{D}}\varphi_0 \in \left[\mathcal{S}(V) \otimes \bigwedge\nolimits^{q,q} (\mathfrak{p}^*) \right]^K \simeq \left[\mathcal{S}(V) \otimes \mathcal{A}^{q,q}(\mathbb{D}) \right]^G.$$

Thus, using multi-index notation with $\underline{\alpha} = \{\alpha_1, \ldots, \alpha_q\}$ and $\underline{\beta} = \{\beta_1, \ldots, \beta_q\}$, it takes the form

$$\varphi_{KM} = \frac{1}{2^{2q}} \sum_{\underline{\alpha}, \underline{\beta}} \mathcal{D}_{\underline{\alpha}} \overline{\mathcal{D}}_{\underline{\beta}} \varphi_0 \otimes \Omega_q(\underline{\alpha}; \underline{\beta}),$$

where $\mathcal{D}_{\underline{\alpha}} = \prod_{j=1}^{q} \mathcal{D}_{\alpha_j}$ and

$$\begin{aligned} \Omega_q(\underline{\alpha}; \underline{\beta}) &= \xi'_{\alpha_1 p+1} \wedge \cdots \wedge \xi'_{\alpha_q p+q} \wedge \xi''_{\beta_1 p+1} \cdots \wedge \xi''_{\beta_q p+q} \\ &= (-1)^{q(q-1)/2} \xi'_{\alpha_1 p+1} \wedge \xi''_{\beta_1 p+1} \wedge \cdots \wedge \xi'_{\alpha_q p+q} \wedge \xi''_{\beta_q p+q}. \end{aligned}$$

The properties of the Schwartz form φ_{KM} are summarized in the following theorem.

Theorem 3.1 (Kudla-Millson)**.** *The Schwarz form φ_{KM} has the following properties:*

i) φ_{KM} is an eigenfunction of weight $p+q$ under the operation of K' [see 44].

ii) As a differential form, $\varphi_{KM}(x,z)$ is closed for every $x \in V$ [see 44, Section 4].

iii) The Thom Lemma holds for φ_{KM} [see 45, Theorem 4.1], i.e.,

$$\int_{\Gamma_x \backslash \mathbb{D}} \eta \wedge \varphi_{KM}(x) = i^{-q} \left(\int_{\Gamma_x \backslash \mathbb{D}(x)} \eta \right) e^{-\pi(x,x)}$$

for any compactly supported closed differential $2(p-1)q$ form η on $\Gamma_x \backslash \mathbb{D}$.

3.3. The Schwartz form ψ

We define another Schwartz form ψ by setting

$$\psi := \frac{2i(-1)^{q-1}}{2^{2(q-1)}} \sum_{\substack{\underline{\alpha}=\{\alpha_1,\dots,\alpha_{q-1}\} \\ \underline{\beta}=\{\beta_1,\dots,\beta_{q-1}\}}} \mathcal{D}_{\underline{\alpha}} \overline{D}_{\underline{\beta}}\, \varphi_0 \otimes \Omega_{q-1}(\underline{\alpha}; \underline{\beta})$$

where

$$\begin{aligned} &\Omega_{q-1}(\underline{\alpha}; \underline{\beta}) \\ &= (-1)^{q(q-1)/2} \sum_{j=1}^{q} \xi'_{\alpha_1 p+1} \wedge \xi''_{\beta_1 p+1} \wedge \cdots \wedge \widehat{\xi'_{\cdot p+j} \wedge \xi''_{\cdot p+j}} \cdots \wedge \xi'_{\alpha_{q-1} p+q} \wedge \xi''_{\beta_{q-1} p+q}. \end{aligned}$$

The notation here indicates that in the j-th term of the sum, ξ' and ξ'' with second index $p+j$ are omitted.

In Section 3.3.2 we will employ the Fock model of the Weil representation to show

Proposition 3.2. *The Schwartz form ψ has the following properties.*

(i) It is invariant under the operation of K, that is,

$$\psi \in \left[\mathcal{S}(V) \otimes \bigwedge\nolimits^{q-1,q-1} (\mathfrak{p}^*)\right]^K \simeq \left[\mathcal{S}(V) \otimes \mathcal{A}^{q-1,q-1}(\mathbb{D})\right]^G.$$

(ii) Under the operation of K', ψ is an eigenfunction of weight $p+q-2$.

The main property linking φ_{KM} and ψ is the following.

Theorem 3.3. *Let $d = \frac{1}{2}\left(\partial + \bar\partial\right)$ and $d^c = \frac{1}{4\pi i}\left(\partial - \bar\partial\right)$ be the standard exterior derivatives acting on $\mathcal{A}^\bullet(\mathbb{D})$, and let $L_\kappa = -2iv^2 \frac{\partial}{\partial \bar\tau}$ be the Maass lowering operator of weight κ acting on functions on the upper half plane. Then*

$$L_{p+q}\, \varphi_{KM}(x, \tau, z) = dd^c\, \psi(x, \tau, z).$$

This implies

$$v \frac{\partial}{\partial v} \varphi^0_{KM}(\sqrt{v}x, z) = dd^c \psi^0(\sqrt{v}x, z).$$

Proof. The proof is carried out in Section 3.3.2, again using the Fock model. □

In order to derive a more explicit description of the Schwartz form ψ, when evaluated at the base point z_0, we examine the properties of the differential operators $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_\alpha$ for $\alpha \in \{1, \dots, p\}$. First, we note that all the differential operators commute, i.e. $\mathcal{D}_\alpha \mathcal{D}_\beta = \mathcal{D}_\beta \mathcal{D}_\alpha$, $\bar{\mathcal{D}}_\alpha \bar{\mathcal{D}}_\beta = \bar{\mathcal{D}}_\beta \bar{\mathcal{D}}_\alpha$ and $\mathcal{D}_\alpha \bar{\mathcal{D}}_\beta = \bar{\mathcal{D}}_\alpha \mathcal{D}_\beta$ for all $\alpha, \beta \in \{1, \dots, p\}$.

Further, by direct calculation, we get

$$\mathcal{D}_\alpha \varphi_0 = 2\bar z_\alpha \varphi_0, \quad \bar{\mathcal{D}}_\alpha \varphi_0 = 2 z_\alpha \varphi_0 \quad \text{and} \quad \mathcal{D}_\alpha \bar{\mathcal{D}}_\alpha \varphi_0 = \left(4|z_\alpha|^2 - \tfrac{2}{\pi}\right) \varphi_0.$$

In fact (see e.g., [45, p. 303 (6.41)]),

$$\mathcal{D}_\alpha^k \bar{\mathcal{D}}_\alpha^k \varphi_0 = \left(\mathcal{D}_\alpha \bar{\mathcal{D}}_\alpha\right)^k \varphi_o = \left(\frac{1}{\pi}\right)^k 2^k k! L_k \left(2\pi |z_\alpha|^2\right) \varphi_0, \tag{3.3.1}$$

where $L_k(t) = \frac{e^t}{k!}\left(\frac{d}{dt}\right)^k \left(e^{-t} t^k\right)$ is the k-the Laguerre polynomial. More generally, we get

$$\mathcal{D}_\alpha^l \bar{\mathcal{D}}_\alpha^k \varphi_0 = 2^k \sum_{m=0}^{l} \binom{l}{m} \sum_{n=0}^{\min(m,k)} \bar{z}_\alpha^{l-n} z_\alpha^{k-n} \binom{m}{n} \frac{k!}{(k-n)!} \left(\frac{-1}{\pi}\right)^n \varphi_0. \tag{3.3.2}$$

Hence the Schwartz form ψ can be expressed using (in general non-homogeneous) polynomials $P^{2q-2}_{\underline{\alpha},\underline{\beta}} \in \mathcal{P}(V)$ as follows:

$$\psi(x, z_0) = \frac{2i(-1)^{q-1}}{2^{2(q-1)}} \sum_{\underline{\alpha},\underline{\beta}} P^{2q-2}_{\underline{\alpha},\underline{\beta}}(x) \varphi_0(x) \otimes \Omega_{q-1}(\underline{\alpha}; \underline{\beta}), \tag{3.3.3}$$

$$\psi^0(x, z_0) = \frac{2i(-1)^{q-1}}{2^{2(q-1)}} \sum_{\underline{\alpha},\underline{\beta}} P^{2q-2}_{\underline{\alpha},\underline{\beta}}(x)\, e^{-2\pi R(x,z_0)} \otimes \Omega_{q-1}(\underline{\alpha}; \underline{\beta}). \tag{3.3.4}$$

The following lemma is easily obtained.

Lemma 3.4. *For any pair of multi-indices $\underline{\alpha}$, $\underline{\beta} \in \{1, \dots, p\}^{q-1}$, the attached polynomial $P^{2q-2}_{\underline{\alpha},\underline{\beta}}(x)$ has the following properties:*

1. *It has degree $2q-2$ and depends only on V_+.*
2. *The leading term is given by*

$$2^{2(q-1)} \prod_{l=1}^{q-1} \bar{z}_{\alpha_l} \prod_{k=1}^{q-1} z_{\beta_k}.$$

3. *All monomials occurring in $P^{2q-2}_{\underline{\alpha},\underline{\beta}}(x)$ have even degree.*
4. *The constant term is non-zero if and only if for every $\alpha \in \{1, \dots, p\}$ the multiplicity of α in the multi-indices $\underline{\alpha}$ and $\underline{\beta}$ is the same. In which case, $P^{2q-2}_{\underline{\alpha},\underline{\beta}}(x)$ is a product of Laguerre functions, and the constant term is given by*

$$2^{q-1} \left(\frac{-1}{\pi}\right)^{q-1} \prod_{\alpha \in \underline{\alpha}} m(\alpha)!,$$

where $m(\alpha)$ is the multiplicity of α.

In particular, the situation in part 4 of the lemma occurs when $x = z_\alpha v_\alpha$, and only the terms with $\underline{\alpha} = \underline{\beta} = (\alpha, \alpha, \dots, \alpha)$ are non-zero.

Example 3.5. Consider the special case of signature $(1,q)$. Then, there is only one multi-index $\underline{\alpha} = \underline{\beta} = \underline{1}$. Hence, by (3.3.1)

$$\begin{aligned}\psi_{1,q} &= \frac{2i(-1)^{q-1}}{2^{2(q-1)}} \left(\mathcal{D}_1\bar{\mathcal{D}}_1\right)^{q-1} \varphi_0 \otimes \Omega_{q-1}(\underline{1},\underline{1}) \\ &= \frac{2i(-1)^{q-1}}{2^{(q-1)}} \frac{(q-1)!}{\pi^{q-1}} L_{q-1}\left(2\pi|z_1|^2\right) \varphi_0 \otimes \Omega_{q-1}(\underline{1},\underline{1}).\end{aligned}$$

We write

$$P_\psi(x,z_0) = \frac{2i(-1)^{q-1}}{2^{2(q-1)}} \sum_{\underline{\alpha},\underline{\beta}} P^{2q-2}_{\underline{\alpha},\underline{\beta}}(x) \otimes \Omega_{q-1}(\underline{\alpha};\underline{\beta}) \tag{3.3.5}$$

for the polynomial part of ψ. Furthermore, it will be convenient to write $P^{2q-2}_{\underline{\alpha},\underline{\beta}}$ as a sum of its homogeneous components,

$$P^{2q-2}_{\underline{\alpha},\underline{\beta}}(x) = \sum_{\ell=0}^{q-1} P^{2q-2}_{\underline{\alpha},\underline{\beta};2\ell}(x),$$

with 2ℓ the respective weight. Note that $P^{2q-2}_{\underline{\alpha},\underline{\beta};2\ell}(wx) = |w|^{2\ell} P^{2q-2}_{\underline{\alpha},\underline{\beta};2\ell}(x)$ for any $w \in \mathbb{C}$.

Remark 3.6. Note that besides (3.3.2) the polynomials $P^{2q-2}_{\underline{\alpha},\underline{\beta}}(x)$ can also be expressed using derivatives of Laguerre functions by (3.3.1) or, alternatively through Hermite functions in the real and imaginary parts of the z_α's as indeterminates.

For this, recall the definition of the Hermite polynomials H_k for $k \geq 0$. They are given by

$$H_k(t) = (-1)^k e^{t^2} \left(\frac{d}{dt}\right)^k e^{-t^2} = e^{t^2/2}\left(t - \frac{d}{dt}\right)^k e^{-t^2/2}.$$

Since the operators $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_\alpha$ commute for $\alpha \in \{1,\dots,p\}$, with (3.3.1) we get

$$\begin{aligned}\mathcal{D}^k_\alpha \bar{\mathcal{D}}^k_\alpha \varphi_0 &= \left(\frac{-1}{\pi}\right)^k 2^k k!\, L_k\left(2\pi|z_\alpha|^2\right)\varphi_0 = \left(\mathcal{D}_\alpha\bar{\mathcal{D}}_\alpha\right)^k \varphi_0 \\ &= \left[\left(x_{\alpha,1} - \frac{1}{2\pi}\frac{d}{dx_{\alpha,1}}\right)^2 + \left(x_{\alpha,2} - \frac{1}{2\pi}\frac{d}{dx_{\alpha,2}}\right)^2\right]^k \varphi_0 \\ &= (2\pi)^{-k} \sum_{l=0}^{k} \binom{k}{l} H_{2(k-l)}(\sqrt{2\pi}x_{\alpha,1}) H_{2l}(\sqrt{2\pi}x_{\alpha,2})\varphi_0. \end{aligned} \tag{3.3.6}$$

Hence, for all $\alpha \in \{1,\dots,p\}$ one has the identity

$$\begin{aligned}\mathcal{D}^{k+1}_\alpha \bar{\mathcal{D}}^k_\alpha \varphi_0 = (2\pi)^{-k-\frac{1}{2}} \sum_{l=0}^{k} \binom{k}{l} \Big[& H_{2(k-l)+1}\left(\sqrt{2\pi}x_{\alpha,1}\right) H_{2l}\left(\sqrt{2\pi}x_{\alpha,2}\right) \\ & - iH_{2(k-l)}\left(\sqrt{2\pi}x_{\alpha,1}\right) H_{2l+1}\left(\sqrt{2\pi}x_{\alpha,2}\right)\Big]\varphi_0.\end{aligned}$$

Proceeding inductively, one can use this to work out further factors of $P^{2q-2}_{\underline{\alpha},\underline{\beta}}$.

Remark 3.7. We note that $\mathcal{D}_\alpha\varphi_0 = \overline{\bar{\mathcal{D}}_\alpha\varphi_0}$. Indeed, by Remark 3.6

$$\begin{aligned}\mathcal{D}_\alpha\varphi_0 &= \left[\left(x_{\alpha,1} - \frac{1}{2\pi}\frac{d}{dx_{\alpha,1}}\right) - i\left(x_{\alpha,2} - \frac{1}{2\pi}\frac{d}{dx_{\alpha,2}}\right)\right]\varphi_0 \\ &= \frac{1}{\sqrt{2\pi}}\left(H_1(\sqrt{2\pi}x_{\alpha 1}) - iH_1(\sqrt{2\pi}x_{\alpha,2})\right)\varphi_0 = (x_\alpha - iy_\alpha)\varphi_0 = \overline{\bar{\mathcal{D}}_\alpha\varphi_0}.\end{aligned}$$

Remark 3.8. We remark that ψ can be also be defined using a so called 'homotopy' operator h, somewhat like in [46]. Set

$$h := \prod_{\mu=p+1}^{m}\prod_{\alpha,\alpha'=1}^{p}\left(\bar{z}_{\alpha'} + \frac{1}{\pi}\frac{\partial}{\partial z_{\alpha'}}\right)\left(z_\alpha + \frac{1}{\pi}\frac{\partial}{\partial \bar{z}_\alpha}\right)\otimes A'^*_{\alpha\mu}A''^*_{\alpha'\mu},$$

where $A^*_{\alpha\mu}$ denotes the left multiplication with the dual of $\xi'_{\alpha\mu}$. Then, one has the following relation between ψ as φ_{KM}, which can serve as an alternative definition for ψ

$$\psi = \frac{q^2}{4\pi}h\,\varphi_{KM}.$$

We sketch a proof of this using the Fock model of the Weil representation in Remark 3.11 below.

3.3.1. Calculations in the Fock model

In this section, we prove the main properties of the Schwartz functions introduced above. We use the polynomial Fock model for the Weil representation, see Appendix A.2.2. We use the intertwining map $\iota : \mathcal{S}(V) \longrightarrow \mathcal{P}(\mathbb{C}^{2(p+q)})$ between the Schrödinger model and the space of complex polynomials in $2(p+q)$ variables, on which the action of the Weil representation ω is given by the Fock model. Note that $\iota(\varphi_0) = 1$. Further main properties of the intertwining operator are summarized in Lemma A.3.

We abbreviate the variables in the Fock model for $\mathrm{U}(p,q)\times\mathrm{U}(1,1)$ by $z''_\alpha = z''_{\alpha 1}$, $z'_\alpha = z'_{\alpha 2}$, $z'_\mu = z'_{\mu 1}$ and $z''_\mu = z''_{\mu 2}$. We then have (see Lemma A.3):

$$\mathcal{D} = \frac{1}{2^{2q}}\left(\frac{-i}{\sqrt{2\pi}}\right)^q\prod_\mu\sum_{\alpha=1}^{p} z''_\alpha\otimes A'_{\alpha\mu} \quad\text{and}\quad \bar{\mathcal{D}} = \frac{1}{2^{2q}}\left(\frac{-i}{\sqrt{2\pi}}\right)^q\prod_\mu\sum_{\beta=1}^{p} z'_\beta\otimes A''_{\beta\mu}.$$

By applying this to $1\otimes 1 = \iota(\varphi_0\otimes 1)$, we see that φ_{KM} is given by

$$\varphi_{KM} = \frac{(-1)^q}{2^{3q}\pi^{2q}}\sum_{\substack{\alpha_1,\dots,\alpha_q\\ \beta_1,\dots,\beta_q}} z''_{\alpha_1}\cdots z''_{\alpha_q}z'_{\beta_1}\cdots z'_{\beta_q}\otimes\Omega_q(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_q),$$

while the form ψ is given by

$$\psi = \frac{2i}{2^{3(q-1)}\pi^{2(q-1)}}\sum_{\substack{\alpha_1,\dots,\alpha_{q-1}\\ \beta_1,\dots,\beta_{q-1}}} z''_{\alpha_1}\cdots z''_{\alpha_{q-1}}z'_{\beta_1}\cdots z'_{\beta_{q-1}}\otimes\Omega_{q-1}(\alpha_1,\dots,\alpha_{q-1};\beta_1,\dots,\beta_{q-1}).$$

3.3.2. Proof of Proposition 3.2

We first verify that ψ has the correct transformation behavior under the operation of $\mathfrak{k}' \simeq \mathfrak{so}_2(\mathbb{R})$.

Lemma 3.9. *Under the operation of $\mathfrak{k}'$, the form ψ has weight $p+q-2$. That is,*

$$\omega\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \psi = i(p+q-2)\psi.$$

Proof. We use the formula for the operation of the generators of $\mathfrak{k}'$ through the Weil representation from Lemma A.5 on p. 140. As $\mathfrak{su}(W) \simeq \mathfrak{sl}_2(\mathbb{R})$, we are mainly interested in the behavior of ψ under the operation of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ (while of course, $\left(\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right)$ generates the center). We have

$$\omega\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = i\left[\sum_{\alpha=1}^{p} z''_\alpha \frac{\partial}{\partial z''_\alpha} + \sum_{\alpha'=1}^{p} z'_{\alpha'} \frac{\partial}{\partial z'_{\alpha'}} - \sum_{\mu'=p+1}^{p+q} z'_{\mu'} \frac{\partial}{\partial z'_{\mu'}} - \sum_{\mu=p+1}^{p+q} z''_\mu \frac{\partial}{\partial z''_\mu}\right] + i(p-q).$$

Bearing in mind that ψ doesn't depend on $z'_{\mu'}$ and z''_μ the claim now follows from

$$\sum_{\alpha=1}^{p} z''_\alpha \frac{\partial}{\partial z''_\alpha}\psi = \sum_{\alpha'=1}^{p} z'_{\alpha'} \frac{\partial}{\partial z'_{\alpha'}}\psi = (q-1)\psi,$$

which is easily checked. $\square$

Lemma 3.10. *The Schwartz form ψ is invariant under the operation of $\mathfrak{k}$.*

Proof. We need to show $Z(\psi) = 0$ for all $Z \in \mathfrak{k}$. Using the explicit formula for ψ given above (and ignoring constants), this means, using that Z acts as a derivation,

$$\begin{aligned} 0 = &\sum_{\substack{\alpha_1,\dots,\alpha_{q-1} \\ \beta_1,\dots,\beta_{q-1}}} \omega(Z)\left(z''_{\alpha_1}\cdots z''_{\alpha_{q-1}} z'_{\beta_1}\cdots z'_{\beta_{q-1}}\right) \otimes \Omega_{q-1}(\alpha_1,\dots,\alpha_{q-1};\beta_1,\dots,\beta_{q-1}) \\ &+ \sum_{\substack{\alpha_1,\dots,\alpha_{q-1} \\ \beta 1,\dots,\beta_{q-1}}} z''_{\alpha_1}\cdots z''_{\alpha_{q-1}} z'_{\beta_1}\cdots z'_{\beta_{q-1}} \otimes Z.\left(\Omega_{q-1}(\alpha_1,\dots,\alpha_{q-1};\beta_1,\dots,\beta_{q-1})\right). \end{aligned}$$

Now let $Z = Z'_{\alpha\beta} \in \mathrm{Hom}(V'_+, V'_+)$. Then the Weil representation action gives

$$\begin{aligned} &\omega(Z'_{\alpha\beta})\left(z''_{\alpha_1}\cdots z''_{\alpha_{q-1}} z'_{\beta_1}\cdots z'_{\beta_{q-1}}\right) \otimes \Omega_{q-1}(\alpha_1,\dots,\alpha_{q-1};\beta_1,\dots,\beta_{q-1}) \\ &= -\sum_{j=1}^{q-1} z''_\alpha z''_{\alpha_1}\cdots \widehat{z''_{\alpha_j}}\cdots z''_{\alpha_{q-1}} z'_{\beta_1}\cdots z'_{\beta_{q-1}} \otimes \Omega_{q-1}(\alpha_1,\dots,\beta,\dots,\alpha_{q-1};\beta_1,\dots,\beta_{q-1}) \\ &\quad + \sum_{j=1}^{q-1} z''_{\alpha_1}\cdots z''_{\alpha_{q-1}} z'_\beta z'_{\beta_1}\cdots \widehat{z'_{\beta_j}}\cdots z'_{\beta_{q-1}} \otimes \Omega_{q-1}(\alpha_1,\dots,\alpha_{q-1};\beta_1,\dots,\alpha,\dots,\beta_{q-1}). \end{aligned}$$

Now $\mathfrak{k} \simeq \operatorname{Hom}(V'_+, V'_+)$ acts on $\mathfrak{p}_+ \simeq \operatorname{Hom}(V_-, V_+)$ by composition. We obtain

$$Z'_{\alpha\beta}.Z'_{\alpha_j\mu} = -\delta_{\beta\alpha_j} Z'_{\alpha\mu},$$

and hence for the dual action we see

$$Z'_{\alpha\beta}.\xi'_{\alpha_j\mu} = \delta_{\alpha\alpha_j}\xi'_{\beta\mu}.$$

In the same way we see

$$Z'_{\alpha\beta}.\xi''_{\beta_j\mu} = -\delta_{\beta\beta_j}\xi''_{\alpha\mu}.$$

This gives

$$\begin{aligned}
&z''_{\alpha_1}\cdots z''_{\alpha_{q-1}} z'_{\beta_1}\cdots z'_{\beta_{q-1}} \otimes Z'_{\alpha\beta}.\Omega_{q-1}(\alpha_1,\dots,\alpha_{q-1};\beta_1,\dots,\beta_{q-1})\\
&= \sum_{j=1}^{q-1} z''_{\alpha_1}\cdots z''_{\alpha}\cdots z''_{\alpha_{q-1}} z'_{\beta_1}\cdots z'_{\beta_{q-1}}\Omega_{q-1}(\alpha_1,\dots,\beta,\dots\alpha_{q-1};\beta_1,\dots,\beta_{q-1})\\
&= -\sum_{j=1}^{q-1} z''_{\alpha_1}\cdots z''_{\alpha_{q-1}} z'_{\alpha'_1}\cdots z'_{\beta}\cdots z'_{\alpha'_{q-1}}\Omega_{q-1}(\alpha_1,\dots,\alpha_{q-1};\beta_1,\dots,\alpha,\dots\beta_{q-1}).
\end{aligned}$$

Combining all this shows $Z'_{\alpha\beta}\psi = 0$, as desired.

We now consider the action of $Z'_{\mu\nu} \in \operatorname{Hom}(V'_-, V'_-)$. The Weil representation action on ψ clearly vanishes. Now the action on $\mathfrak{p}^+$ is given by $Z'_{\mu\nu}.Z'_{\alpha\mu'} = \delta_{\mu\mu'}Z'_{\alpha\nu}$ and hence

$$Z'_{\mu\nu}\xi'_{\alpha_j\mu'} = -\delta_{\nu\mu'}\xi'_{\alpha_j\mu} \qquad \text{and} \qquad Z'_{\mu\nu}\xi''_{\beta_j\mu'} = \delta_{\mu\mu'}\xi''_{\beta_j\nu}.$$

From this it is easy to see that

$$Z'_{\mu\nu}\Omega_{q-1}(\alpha_1,\dots,\alpha_{q-1};\beta_1,\dots,\beta_{q-1}) = 0.$$

for all multi-indices $\underline{\alpha} = \{\alpha_1,\dots,\alpha_{q-1}\}$, $\underline{\beta} = \{\beta_1,\dots,\beta_{q-1}\}$. □

Proof of Theorem 3.3

Recall

$$d = \frac{1}{2}\left(\partial + \bar\partial\right), \qquad d^c = \frac{(\partial - \bar\partial)}{4\pi i}, \qquad dd^c = -\frac{1}{4\pi i}\partial\bar\partial. \tag{3.3.7}$$

In the Fock model, the differential operators ∂, $\bar\partial$ are given by (see Lemmas A.4, A.5)

$$\partial = \sum_{\alpha,\mu}\left[\frac{1}{4\pi}z''_\alpha z'_\mu - 4\pi\frac{\partial^2}{\partial z'_\alpha \partial z''_\mu}\right]\otimes A'_{\alpha\mu}, \qquad \bar\partial = \sum_{\beta,\nu}\left[\frac{1}{4\pi}z'_\beta z''_\nu - 4\pi\frac{\partial^2}{\partial z''_\beta \partial z'_\nu}\right]\otimes A''_{\beta\nu}.$$

For the lowering operator L, we have by Lemma A.5(ii)

$$L = -4\pi\sum_\gamma \frac{\partial^2}{\partial z''_\gamma \partial z'_\gamma} + \frac{1}{4\pi}\sum_\mu z''_\mu z'_\mu.$$

To simplify notation, we drop all constants and consider

$$\varphi'_{KM} = \sum_{\substack{\alpha_1,\dots,\alpha_q\\ \beta_1,\dots,\beta_q}} z''_{\alpha_1}\cdots z''_{\alpha_q} z'_{\beta_1}\cdots z'_{\beta_q} \otimes \xi'_{\alpha_1 p+1}\wedge\cdots\xi'_{\alpha_q p+q}\wedge\xi''_{\beta_1 p+1}\wedge\cdots\wedge\xi''_{\beta_q p+q},$$

$$\psi' = \sum_{\substack{\alpha_1,\dots,\alpha_{q-1}\\ \beta_1,\dots,\beta_{q-1}}} z''_{\alpha_1}\cdots z''_{\alpha_{q-1}} z'_{\beta_1}\cdots z'_{\beta_{q-1}}$$

$$\otimes\sum_{j=1}^{q}\xi'_{\alpha_1 p+1}\wedge\cdots\wedge\widehat{\xi'_{\cdot p+j}}\cdots\wedge\xi'_{\alpha_{q-1}p+q}\wedge\xi''_{\beta_1 p+1}\wedge\cdots\wedge\widehat{\xi''_{\cdot p+j}}\wedge\cdots\wedge\xi''_{\beta_{q-1}p+q}.$$

Then the claim is equivalent to

$$L\varphi'_{KM} = (-1)^{q-1}4\pi\partial\bar\partial\psi',$$

which we show by a direct calculation of both sides. We have

$$L\varphi'_{KM} = \frac{1}{4\pi}\left(\sum_{\mu} z''_\mu z'_\mu\right)\varphi'_{KM}$$

$$-4\pi\sum_{\underline{\alpha},\underline{\beta}}\sum_{j,k=1}^{q}\delta_{\alpha_j\beta_k} z''_{\alpha_1}\cdots\widehat{z''_{\alpha_j}}\cdots z''_{\alpha_{q-1}} z'_{\beta_1}\cdots\widehat{z'_{\beta_k}}\cdots z'_{\beta_{q-1}}$$

$$\otimes\xi'_{\alpha_1 p+1}\wedge\cdots\wedge\xi'_{\alpha_j p+j}\wedge\cdots\xi'_{\alpha_q p+q}\wedge\xi''_{\beta_1 p+1}\wedge\cdots\wedge\xi''_{\beta_k p+k}\wedge\cdots\wedge\xi''_{\beta_q p+q}.$$

On the other hand,

$$\partial\bar\partial\psi' = \frac{1}{16\pi^2}\sum_{\alpha,\beta,\mu,\nu}\left(z''_\alpha z'_\beta z'_\nu z''_\mu\otimes\xi'_{\alpha\nu}\wedge\xi''_{\beta\mu}\right)\ \psi'$$

$$-\sum_{\substack{\alpha_1,\dots,\alpha_{q-1}\\ \beta_1,\dots,\beta_{q-1}\\ \alpha,\beta,\mu}} z''_{\alpha_1}\cdots z''_{\alpha_{q-1}}\frac{\partial}{\partial z'_\alpha}\left(z'_\beta z'_{\beta_1}\cdots z'_{\beta_{q-1}}\right)$$

$$\otimes\xi'_{\alpha\mu}\wedge\xi''_{\beta\mu}\wedge\sum_{j=1}^{q}\xi'_{\alpha_1 p+1}\wedge\cdots\widehat{\xi'_{\cdot p+j}}\cdots\wedge\xi'_{\alpha_{q-1}p+q}\wedge\xi''_{\beta_1 p+1}\wedge\cdots\widehat{\xi''_{\cdot p+j}}\cdots\wedge\xi''_{\beta_{q-1}p+q}.$$

For the first term, it is easy to see that only the terms with $\mu=\nu$ contribute and one obtains

$$(-1)^{q-1}\frac{1}{16\pi^2}\left(\sum_\mu z''_\mu z'_\mu\right)\varphi'_{KM}.$$

For the second, only terms with $\mu = p+j$ contribute and one obtains

$$(-1)^q\sum_{\substack{\alpha_1,\dots,\alpha_{q-1}\\ \beta_1,\dots,\beta_{q-1}\\ \alpha_0,\beta_0}} z''_{\alpha_1}\cdots z''_{\alpha_{q-1}}\sum_{k=0}^{q-1}\delta_{\alpha_0\beta_k} z'_{\beta_0} z'_{\beta_1}\cdots\widehat{z_{\beta_k}}\cdots z'_{\beta_{q-1}}$$

$$\otimes\sum_{j=1}^{q}\xi'_{\alpha_1 p+1}\wedge\cdots\wedge\xi'_{\alpha_0 p+j}\cdots\wedge\xi'_{\alpha_{q-1}p+q}\wedge\xi''_{\beta_1 p+1}\wedge\cdots\wedge\xi''_{\beta_0 p+j}\wedge\cdots\wedge\xi''_{\beta_{q-1}p+q}.$$

Now comparing the formulas for $L\varphi'_{KM}$ and $\partial\bar{\partial}\psi'$ gives the claim.

The auxiliary form $d^c\psi$

Beside ψ itself, we also use the form $d^c\psi$, see Section 3.4.1. Here, we give a more explicit description of this auxiliary form. Consider $(4\pi)^{-1}\partial\psi$. It is given by

$$\begin{aligned}
\frac{1}{4\pi}\partial\psi &= \frac{i}{2^{3(q-1)}\pi^{2(q-1)}}\frac{1}{2\pi}\sum_{\substack{\underline{\alpha},\underline{\beta}\\ \gamma,\mu}} z''_\gamma z'_\mu z''_{\underline{\alpha}} z'_{\underline{\beta}} \otimes \xi'_{\gamma\mu} \wedge \Omega_{q-1}(\underline{\alpha};\underline{\beta})\\
&= \frac{i}{2^{3q-2}\pi^{2q-1}}\sum_{\substack{\gamma,\underline{\alpha}\\ \underline{\beta}}} z''_\gamma z''_{\alpha_1}\cdots z''_{\alpha_{q-1}} z'_{\underline{\beta}} \sum_{j=1}^{q}(-1)^{j-1} z'_{p+j}\\
&\qquad \otimes \xi'_{\alpha_1 p+1}\wedge\cdots\xi'_{\gamma p+j}\cdots\wedge\xi'_{\alpha_{q-1}p+q}\wedge\xi''_{\beta_1 p+1}\wedge\cdots\widehat{\xi''_{\cdot p+j}}\cdots\wedge\xi''_{\beta_{q-1}p+q}.
\end{aligned}$$

Similarly, $(4\pi)^{-1}\bar{\partial}\psi$ is given by

$$\begin{aligned}
\frac{1}{4\pi}\bar{\partial}\psi &= \frac{i}{2^{3q-2}\pi^{2q-1}}\sum_{\substack{\underline{\alpha}\\ \gamma,\underline{\beta}}} z''_{\underline{\alpha}} z'_\gamma z'_{\beta_1}\cdots z'_{\beta_{q-1}} \sum_{j=1}^{q}(-1)^{q+j} z''_{p+j}\\
&\qquad \otimes \xi'_{\alpha_1 p+1}\wedge\cdots\widehat{\xi'_{\cdot p+j}}\cdots\wedge\xi'_{\alpha_{q-1}p+q}\wedge\xi''_{\beta_1 p+1}\wedge\cdots\xi''_{\gamma p+j}\cdots\wedge\xi''_{\beta_{q-1}p+q}.
\end{aligned}$$

Now by (3.3.7), $d^c\psi$ is the difference of these two terms.

Finally, in the Schrödinger model, $d^c\psi$ takes the following explicit form (note that $\mathcal{D}_\mu\varphi_0 = 2\bar{z}_\mu\varphi_0$.): We have

$$d^c\psi(x) = \frac{1}{2^{3q-1}\pi^{2q-1}}\left[\sum_{\substack{\underline{\alpha},\gamma\\ \underline{\beta}}}\mathcal{D}_{\underline{\alpha}}\mathcal{D}_\gamma\bar{\mathcal{D}}_{\underline{\beta}}\varphi_0(x)\otimes Q'_{\underline{\alpha},\gamma;\underline{\beta}}(x) - \sum_{\substack{\underline{\alpha}\\ \gamma,\underline{\beta}}}\mathcal{D}_{\underline{\alpha}}\bar{\mathcal{D}}_\gamma\bar{\mathcal{D}}_{\underline{\beta}}\varphi_0(x)\otimes Q''_{\underline{\alpha};\underline{\beta},\gamma}(x)\right]. \tag{3.3.8}$$

Her $Q'_{\underline{\alpha},\gamma;\underline{\beta}}(x)$ and $Q''_{\underline{\alpha};\underline{\beta},\gamma}(x)$ are given by

$$\begin{aligned}
&Q'_{\underline{\alpha},\gamma;\underline{\beta}}(x)\\
&= \sum_{j=1}^{q}(-1)^{j-1} z_{p+j}\otimes\xi'_{\alpha_1 p+1}\wedge\cdots\xi'_{\gamma p+j}\cdots\wedge\xi'_{\alpha_{q-1}p+q}\wedge\xi''_{\beta_1 p+1}\wedge\cdots\widehat{\xi''_{\cdot p+j}}\cdots\wedge\xi''_{\beta_{q-1}p+q}\\
&Q''_{\underline{\alpha};\underline{\beta},\gamma}(x)\\
&= \sum_{j=1}^{q}(-1)^{q+j}\bar{z}_{p+j}\otimes\xi'_{\alpha_1 p+1}\wedge\cdots\widehat{\xi'_{\cdot p+j}}\cdots\wedge\xi'_{\alpha_{q-1}p+q}\wedge\xi''_{\beta_1 p+1}\wedge\cdots\xi''_{\gamma p+j}\cdots\wedge\xi''_{\beta_{q-1}p+q}.
\end{aligned}$$

Remark 3.11. Taking up Remark 3.8, we note that in the Fock model, the explicit form of ψ can be easily recovered from the alternative definition $\psi = \frac{q^2}{4\pi}h\varphi_{KM}$, and we give a brief sketch of how this can be used to prove of Theorem 3.3.

In the Fock model, the homotopy operator h takes the form

$$h = \sum_{\mu=p+1}^{m} \sum_{\alpha,\alpha'=1}^{p} \frac{\partial^2}{\partial z''_\alpha \partial z'_{\alpha'}} \otimes A'^*_{\alpha\mu} A''^*_{\alpha'\mu}.$$

By a straightforward calculation (the omitted constants can be easily checked), one has

$$\begin{aligned} h\varphi'_{KM} &= \sum_{\substack{\alpha,\alpha' \\ \mu}} \frac{\partial^2}{\partial z''_\alpha \partial z'_{\alpha'}} \otimes A'^*_{\alpha\mu} A''^*_{\alpha'\mu} \sum_{\substack{\alpha_1,\dots,\alpha_q \\ \beta_1,\dots,\beta_q}} z''_{\alpha_1} \cdots z''_{\alpha_q} z'_{\beta_1} \cdots z'_{\beta_q} \otimes \Omega_q\left(\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_q\right) \\ &= \sum_{\substack{\alpha_1,\dots,\alpha_{q-1} \\ \beta_1 m\dots,\beta_{q-1}}} z''_{\alpha_1} \cdots z''_{\alpha_{q-1}} z'_{\beta_1} \cdots z'_{\beta_{q-1}} \otimes \Omega_{q-1}\left(\alpha_1, \dots, \alpha_{q-1}, \beta_1, \dots, \beta_{q-1}\right) = \psi'. \end{aligned}$$

Now for the proof of Theorem 3.3. Since φ_{KM} is closed, one can use anticommutators to calculate $\partial\bar{\partial} h\varphi_{KM}$. A short calculation using Clifford identities [see 46, p.158f] shows that

$$\left\{\bar{\partial}, h\right\} \varphi_{KM} = q^2 \left(\sum_{\alpha,\nu} z''_\nu \frac{\partial}{\partial z''_\alpha} \otimes A'^*_{\alpha\mu} \right) \varphi_{KM}.$$

Quite similarly,

$$\left\{\partial, h\right\} \varphi_{KM} = q^2 \left(\sum_{\beta',\nu'} z'_{\nu'} \frac{\partial}{\partial z'_{\beta'}} \otimes A''^*_{\beta'\nu'} \right) \varphi_{KM}.$$

Now, set $\partial_+ = \frac{1}{4\pi} \sum_{\alpha,\mu} z''_\alpha z'_\mu$ and $\partial_- = 4\pi \sum_{\alpha,\mu} \frac{\partial^2}{\partial z'_\alpha \partial z''_\mu}$, the the first and second term of ∂ in the Fock model (see p. 3.3.2). Similarly, define $\bar{\partial}_+$ and $\bar{\partial}_-$ as the first and second term of $\bar{\partial}$. Then

$$\partial\bar{\partial}\left(h\varphi_{KM}\right) = \left\{\partial\bar{\partial}, h\right\} \varphi_{KM} = \left\{\partial_+\bar{\partial}_+ + \partial_-\bar{\partial}_-, h\right\} \varphi_{KM},$$

since $\partial z''_\mu$ and $\partial z'_\nu$ both kill φ_{KM} and $h\varphi_{KM}$. Further, $\partial_-\varphi_{KM} = 0$. Now, again by a calculation involving Clifford identities, and somewhat similar to [46], one gets

$$\partial\partial h\varphi_{KM} = \frac{q^2}{4\pi} L\varphi_{KM}.$$

We remark that the auxiliary form $d^c\psi$ can be calculated directly from the anticommutators via

$$d^c\psi = \frac{(\partial - \bar{\partial})}{4\pi i} \left(h\varphi_{KM}\right) = \frac{1}{4\pi i} \left(\{\partial, h\} - \{\bar{\partial}, h\}\right) \varphi_{KM} = -\frac{4\pi i}{q^2} \left(\{\partial, h\} - \{\bar{\partial}, h\}\right) \varphi_{KM}.$$

Moreover, one may define two new homotopy operators

$$h_\partial := \frac{1}{4\pi i} \{\partial, h\}, \qquad h_{\bar{\partial}} := \frac{1}{4\pi i} \{\bar{\partial}, h\}$$

with $d^c\psi = \left(h_\partial - h_{\bar{\partial}}\right) \varphi_{KM}$.

3.4. A singular Schwartz form

Analogously to Kudla [43] for $\mathrm{O}(p,2)$, we define for $x \neq 0$ the singular Schwartz form

$$\Psi^0(x,z) := -\int_1^\infty \psi^0(\sqrt{t}x,z)\frac{dt}{t}. \tag{3.4.1}$$

The form Ψ^0 has its singularities where $R(x,z)=0$, i.e., precisely along the cycles $\mathbb{D}(x)$. Thus, in particular, $\Psi^0(x,z)$ is smooth for $(x,x)\leq 0$. We also set

$$\Psi(x,z) = \Psi^0(x,z)e^{-\pi(x,x)}.$$

Recall the definition of the incomplete Γ-function, $\Gamma(s,a) = \int_a^\infty t^{s-1}e^{-t}dt$ (see Appendix B). The following lemma is obtained by a straightforward calculation.

Lemma 3.12. *At the base point $z=z_0$, the singular Schwartz form Ψ^0 is given by*

$$\Psi^0(x,z_0) = \frac{2i(-1)^{q-1}}{2^{2(q-1)}}\sum_{\underline{\alpha},\underline{\beta}}\left[\sum_{\ell=0}^{q-1} P^{2q-2}_{\underline{\alpha},\underline{\beta};2\ell}(x)\,(2\pi R(x,z_0))^{-\ell}\,\Gamma\left(\ell,2\pi R(x,z_0)\right)\right] \otimes \Omega_{q-1}(\underline{\alpha};\underline{\beta}).$$

We conclude that $R(x,z)^{q-1}\Psi^0(x,z)$ extends to a smooth differential $(q-1,q-1)$-form on $\mathbb{D}$.

While it should be emphasized that Ψ is not a Schwartz function on V, we nonetheless define (as if Ψ had weight $p+q$)

$$\Psi(x,\tau,z) = \Psi^0(\sqrt{v}x,z)e^{\pi i(x,x)\tau} \quad (\tau\in\mathbb{H}).$$

This is motivated by the second statement in the Proposition below. Note

$$\Psi(x,\tau,z) = -\left(\int_v^\infty \psi^0(\sqrt{t}x,z)\frac{dt}{t}\right)e^{\pi i(x,x)\tau}. \tag{3.4.2}$$

From the definition of Ψ and the properties of ψ, we get

Proposition 3.13. *Outside the singularities, $\Psi(x,\tau,z)$ has the following properties:*

1. *For d and d^c the standard exterior differentials on $\mathcal{A}^\bullet(\mathbb{D})$, we have outside $\mathbb{D}(x)$*

$$dd^c\,\Psi(x,\tau,z) = \varphi_{KM}(x,\tau,z).$$

2. *We have*

$$L_{p+q}\Psi(x,\tau,z) = \psi(x,\tau,z),$$

with the Maass lowering operator L_{p+q} as before.

Proof. 1. This follows from Theorem 3.3 and the rapid decay of the Schwartz form φ_{KM}:

$$\begin{aligned} dd^c\Psi(x,\tau,z) &= -\left(\int_v^\infty dd^c\psi(\sqrt{t}x,z)\frac{dt}{t}\right) e^{\pi i(x,x)\tau} \\ &= -\left(\int_v^\infty \frac{\partial}{\partial t}\varphi^0_{KM}(\sqrt{t}x,z)\,dt\right) e^{\pi i(x,x)\tau} = \varphi_{KM}(x,\tau,z). \end{aligned}$$

2. Immediately from the definition,

$$\begin{aligned} L_{p+q}\Psi(x,\tau,z) &= 2iv\frac{\partial}{\partial\bar\tau}\left(\int_v^\infty \psi^0(\sqrt{t}x,z)\frac{dt}{t}\right) e^{\pi i(x,x)\tau} \\ &= -v\left(\frac{\partial}{\partial v}\int_v^\infty \psi^0(\sqrt{t}x,z)\frac{dt}{t}\right) e^{\pi i(x,x)\tau} = \psi^0(\sqrt{v}x,z)e^{\pi i(x,x)\tau} = \psi(x,\tau,z), \end{aligned}$$

again by rapid decay. □

3.4.1. The current equation

We denote by $\mathcal{A}_c^k(\mathbb{D})$ the space of compactly supported differential forms on $\mathbb{D}$ of degree k. Recall that a locally integrable degree k-form ω on $\mathbb{D}$ defines a current, i.e., a (continuous) linear functional on the compactly supported forms of complementary degree, via

$$[\omega](\eta) := \int_{\mathbb{D}} \eta\wedge\omega \qquad \left(\eta\in\mathcal{A}_c^{2pq-k}(\mathbb{D})\right).$$

Furthermore, for the exterior derivatives of a current $[\omega]$ we have

$$dd^c[\omega](\eta) := [\omega](dd^c\eta).$$

The goal of this section is to prove the following generalization of the Thom Lemma, see Theorem 3.1 iii), the proof will be carried out in the next two subsections, using the same method as employed in [10] and [26].

Theorem 3.14. *Let $x\in V$ and let $\delta_{Z(x)}$ denote the delta current for the special cycle $Z(x)$. Then*

$$dd^c[\Psi^0(x)] + (-i)^q\delta_{Z(x)} = [\varphi^0_{KM}(x)]$$

as currents on $\Gamma_x\backslash\mathbb{D}$. In other words, we have

$$\int_{\Gamma_x\backslash\mathbb{D}} dd^c\eta\wedge\Psi^0(x) + (-i)^q\int_{Z(x)}\eta = \int_{\Gamma_x\backslash\mathbb{D}}\eta\wedge\varphi^0_{KM}(x)$$

for any $\eta\in\mathcal{A}_c^{2(p-1)q}(\Gamma_x\backslash\mathbb{D})$.

With this we can now define a Green current for the special cycles $Z(n,h) \subset X$. Namely, for $m \in \mathbb{Q}$, $h \in L^\sharp/L$ satisfying $n \equiv (h,h) \mod \mathbb{Z}$ and a real parameter $w > 0$ we introduce the Green form of Kudla type on X by setting

$$\Xi^K(n,w,h)(z) := \sum_{\substack{\lambda \in L+h \\ (\lambda,\lambda)=n \\ \lambda \neq 0}} \Psi^0(\sqrt{2w}\lambda, z). \tag{3.4.3}$$

Then by Theorem 3.14 we immediately obtain the following

Corollary 3.15. *The singular differential $(q-1,q-1)$-form $\Xi^K(n,w,h)$ defines a Green current for the cycle $Z(n,h)$ on X.*

Local integrability

Proposition 3.16. *Let $x \in V$. Then $\Psi^0(x)$ and $d^c\Psi^0(x)$ are locally integrable differential forms on $\mathbb{D}$.*

Proof. We view a top-degree differential form $\phi \in \mathcal{A}^{2pq}(\mathbb{D})$ via the Hodge $*$-operator as a (K-invariant) function on G. We pick suitable coordinates on $\mathbb{D}$, using the decomposition $G = HAK$, where H is the stabilizer of the first basis vector v_1 of V, A is a one parameter subgroup $A = \{a_t = \exp(tX_{1p+q});\ t \in \mathbb{R}\}$. Set $A_0 = \{a_t : t \geq 0\}$. Now, using Flensted-Jensen theory with the quantity $\delta(H)$ determined in Appendix A.1, see (A.1.1), we get

$$\begin{aligned} \int_{\mathbb{D}} \phi = \int_G \phi(g)\,dg = C \int_{A_0} \int_H \phi(ha_t)\delta(H) dh dt \\ = C \int_{A_0} \int_H \phi(ha_t) \sinh(t)^{2q-1} \cosh(t)^{2p-1}\,dh\,dt, \end{aligned} \tag{3.4.4}$$

with C a positive constant, depending on the normalization of the invariant measures, see [21, Sec. 2] or [53, Section 2] for further details.

Now $\Psi^0(x)$ is smooth unless $(x,x) > 0$. In that case we may assume that $x = \sqrt{n}v_1$, for some $n > 0$. Then for $\eta \in \mathcal{A}_c^{2(pq-(q-1))}(\mathbb{D})$. We set $\phi = \eta \wedge \Psi(x)$ and see

$$\phi(ha_t) = \eta(ha_t) \wedge \Psi^0(a_t^{-1}h^{-1}\sqrt{n}v_1),$$

wherein

$$a_t^{-1}h^{-1}\sqrt{n}v_1 = \cosh(t)\sqrt{n}v_1 - \sinh(t)\sqrt{n}v_{p+q}.$$

Hence,

$$\left(a_t^{-1}h^{-1}\sqrt{n}v_1\right)_{z_0} = -\sinh(t)\sqrt{n}v_{p+q} \quad \text{and} \quad \left(a_t^{-1}h^{-1}\sqrt{n}v_1\right)_{z_0^\perp} = \cosh(t)\sqrt{n}v_1. \tag{3.4.5}$$

Thus, we have (see Lemma 3.12),

$$\begin{aligned} \Psi^0(a_t^{-1}h^{-1}\sqrt{n}v_1) = \frac{2i(-1)^{q-1}}{2^{2(q-1)}} \Bigg[\sum_{\ell=0}^{q-1} \left(2\pi n \sinh^2(t)\right)^{-\ell} \Gamma\left(\ell, 2\pi n \sinh^2(t)\right) \\ \cdot \sum_{\underline{\alpha},\underline{\beta}} P^{2q-2}_{\underline{\alpha},\underline{\beta};2\ell}\left(\kappa\sqrt{n}\cosh(t)v_1\right)\Bigg] \otimes \Omega_{q-1}(\underline{\alpha};\underline{\beta}). \end{aligned}$$

We conclude that the integrand of (3.4.4), i.e.,

$$\eta(ha_t)\wedge\Psi^0(a_t^{-1}h^{-1}\sqrt{n}v_1)\sinh(t)^{2q-1}\cosh(t)^{2p-1}$$

is bounded, in fact, vanishes, as $t\to 0$. Further, as η has compact support, the integral is convergent.

For the local integrability of $d^c\Psi(x)$ the reasoning is similar, but a bit more tedious. Again, we may assume that $x=\sqrt{n}v_1$, with $n>0$. Further, note that we only need to consider highest-degree terms.

Note $d^c\Psi^0(x)=-\int_1^\infty d^c\psi^0(\sqrt{s}x)\frac{ds}{s}$, which can be evaluated similarly to Lemma 3.12. By (3.3.8), $d^c\psi$ consists of two parts. Both involve polynomials of degree $2q-1$ which depend on the positive coordinates of x (note that there is no constant part). If by (3.4.5), we set $x=\cosh(t)\sqrt{n}v_1$, only the polynomials which depend exclusively on the first vector can contribute to $d^c\Psi^0(x)$. From their highest-degree terms, we get

$$2^{-(2q-1)}\cosh(t)^{2q-1}n^{q-\frac{1}{2}}\sqrt{s}^{2q-1}.$$

Also, in (3.3.8) there are linear homogeneous polynomials in the negative coordinates, $Q_{\alpha'_q,\underline{\alpha}_{(q-1)}}$ and $Q'_{\alpha_q,\underline{\alpha}'_{(q-1)}}$. From them, again by (3.4.5) we have contributions of

$$-\sqrt{s}\,\sqrt{n}\sinh(t)$$

Hence, gathering the contributions of the non-vanishing highest-degree terms, we still have the integral

$$\begin{aligned}\int_1^\infty s^{q-1}e^{2\pi R(x,z_0)}ds &= (2\pi R(x,z_0))^{-q}\,\Gamma(q,2\pi R(x,z_0))\\ &= (2\pi\sinh^2(t))^{-q}\,\Gamma(q,2\pi\sinh^2(t)).\end{aligned}$$

Thus, up to sign, for $t\to 0$ the behavior of $d^c\Psi(a_t^{-1}h^{-1}\sqrt{n}v_1)$ is dominated by terms of the form

$$\frac{(-1)^{q-1}\pi}{2^{2q-1}}\sinh(t)\cosh(t)^{2q-1}\left(\sinh^2(t)\right)^{-q}\Gamma\left(q,2\pi\sinh^2(t)\right). \tag{3.4.6}$$

In particular, it follows that the integrand in

$$\int_{A_0}\int_H\eta(ha_t)\wedge\left(d^c\Psi(a_t^{-1}h^{-1}\sqrt{n}v_1)\right)\sinh(t)^{2q-1}\cosh(t)^{2p-1}\,dh\,dt,$$

remains bounded as $t\to 0$, and hence the integral converges. $\square$

Proof of the current equation

Proof of Theorem 3.14. Let $\eta\in\mathcal{A}_c^{2(p-1)q}(\Gamma_x\backslash\mathbb{D})$, not necessarily closed. First note using $(dd^c\eta)\wedge\Psi^0(x)=(d\eta)\wedge d^c\Psi^0(x)-d^c\left(d\eta\wedge\Psi^0(x)\right)$ and Stokes' theorem

$$\int_{\Gamma_x\backslash\mathbb{D}}(dd^c\eta)\wedge\Psi^0(x)=-\int_{\Gamma_x\backslash\mathbb{D}}(d\eta)\wedge d^c\Psi^0(x)+\lim_{\epsilon\to 0}\int_{\Gamma_x\backslash\partial(\mathbb{D}-U_\epsilon(x))}(d\eta)\wedge d^c\Psi^0(x),$$

where U_ϵ, ($\epsilon > 0$) denotes an open neighborhood of the cycle $\mathbb{D}(x)$. Next we show that the limit on the right hand side vanishes. We may again assume $x = \sqrt{n}v_1$, with $n > 0$ and use the HAK coordinates introduced in the proof of Proposition 3.16 and Appendix A.1. Then for $\epsilon > 0$, an open neighborhood of $\mathbb{D}(v_1)$ is defined by

$$U_\epsilon = \mathbb{D} - (H \times A_\epsilon)\,, \tag{3.4.7}$$

with $A_\epsilon = \{a_t : t \geq \epsilon\}$. With the analog of the integral formula from (3.4.4) and (A.1.1), the limit can be written as

$$C \lim_{\epsilon\to 0} \int_{\Gamma_{v_1}\backslash H} \eta(ha_\epsilon) \wedge \Psi^0(a_\epsilon^{-1}h^{-1}\sqrt{n}v_1) \sinh(\epsilon)^{2q-1} \cosh(\epsilon)^{2p-1}\, dh$$

for some constant C. Only the highest degree term of $\Psi(a_t^{-1}h^{-1}\sqrt{n}v_1)$ (see Lemma 3.12) can contribute. Further, note that, since $\left(a_t^{-1}h^{-1}\sqrt{n}v_1\right)_{z_0^\perp} = \cosh(t)\sqrt{n}v_1$ by (3.4.5), we have $P^{2q-2}_{\underline{\alpha},\underline{\beta};2q-2}(\sqrt{n}\cosh(t)v_1) \neq 0$ only for $\underline{\alpha} = \underline{\beta} = (1,\dots,1)$, thus, up to constants, the highest degree term is given by

$$\left(n\sinh^2(t)\right)^{-(q-1)} \Gamma\left(q-1, 2\pi n \sinh^2(t)\right) (2\sqrt{n}\cosh(t))^{2q-1}.$$

Hence, comparing powers of $\sinh(t)$ we see that the integrand goes to zero for $t = \epsilon \to 0$, and the limit vanishes as claimed.

Now, since $dd^c\Psi(x) = \varphi_{KM}(x)$, we have

$$\begin{aligned} -\int_{\Gamma_x\backslash\mathbb{D}} (d\eta)\wedge d^c\Psi^0(x) &= \int_{\Gamma_x\backslash\mathbb{D}} \eta\wedge dd^c\Psi^0(x) - \int_{\Gamma_x\backslash\mathbb{D}} d\left(\eta\wedge d^c\Psi^0(x)\right) \\ &= \int_{\Gamma_x\backslash\mathbb{D}} \eta\wedge\varphi^0_{KM}(x) + \lim_{\epsilon\to 0}\int_{\Gamma_x\backslash\partial(\mathbb{D}-U_\epsilon(x))} \eta\wedge d^c\Psi^0(x), \end{aligned}$$

again by applying Stokes' theorem. Thus it remains to show that

$$\lim_{\epsilon\to 0}\int_{\Gamma_x\backslash\partial(\mathbb{D}-U_\epsilon(x))} \eta\wedge d^c\Psi^0(x) = (-i)^q \int_{Z(x)} \eta.$$

We have to consider the limit of the same integral as in the proof of second part of Proposition 3.16:

$$C\lim_{\epsilon\to 0}\int_{\Gamma_{v_1}\backslash H} \eta(ha_\epsilon)\wedge d^c\Psi^0(a_{-\epsilon}h^{-1}\sqrt{n}v_1)\cosh(\epsilon)^{2p-1}\sinh(\epsilon)^{2q-1}\,dh, \tag{3.4.8}$$

with a non-zero constant C, independent of η. With (3.4.6) we see that for both parts of $d^c\Psi^0(x)$, the integral is bounded as $t = \epsilon \to 0$. We have

$$\begin{aligned} C\lim_{\epsilon\to 0}\int_H \eta(ha_\epsilon)\wedge\left(d^c\Psi^0(a_{-\epsilon}h^{-1}\sqrt{n}v_1)\right)\cosh(\epsilon)^{2p-1}\sinh(\epsilon)^{2q-1}\,dh \\ = \tilde{C}\int_H \eta(h)\,dh, \end{aligned}$$

with a constant $\tilde{C}$ independent of η. By Kudla-Millson theory [see 45, Theorem 6.4], we see that $\tilde{C} = (-i)^q$ for η closed, see Theorem 3.1 iii)..

To summarize, we have showed that for all $\eta \in \mathcal{A}_c^{2(p-1)q}(\Gamma_x \backslash \mathbb{D})$,

$$\int_{\Gamma_x \backslash \mathbb{D}} (dd^c \eta) \wedge \Psi^0(x) = \int_{\Gamma_x \backslash \mathbb{D}} \eta \wedge \varphi_{KM}^0(x) - (-i)^q \int_{Z(x)} \eta,$$

as claimed. ☐

3.5. The regularized theta integral

In the following, we set $\kappa = p+q-2$ and $k = -\kappa = -(p+q)+2$. We define a vector-valued theta function for the Schwartz form ψ introduced in Section 3.3, following the general recipe from Section 1.2.1. Hence, we set

$$\Theta(\tau, z) := \Theta(\tau, z; \psi)_L := \big(\theta(h, z, \psi)_h\big)_{h \in L^\sharp/L} = \sum_{h \in L^\sharp/L} \theta(\tau, h, \psi_h)\mathfrak{e}_h \quad (\tau \in \mathbb{H}, z \in \mathbb{D}),$$

with

$$\theta(\tau, z, \psi)_h = \sum_{\lambda \in L+h} \psi\left(\sqrt{2}\lambda, \tau, z\right) \quad (h \in L^\sharp/L).$$

(Recall Remark 1.2 concerning the factor of $\sqrt{2}$ in the exponential.) Explicitly,

$$\Theta(\tau, z) = \sum_{h \in L^\sharp/L} \sum_{\lambda \in L+h} P_\psi\left(\sqrt{2}v\lambda, z\right) e^{4\pi v(\lambda_z, \lambda_z) + 2\pi i(\lambda,\lambda)} \mathfrak{e}_h, \tag{3.5.1}$$

where $P_\psi(x,z) \in \left[\mathcal{P}(V) \otimes \mathcal{A}^\bullet(\mathbb{D})\right]^G$ is the polynomial part of ψ, see (3.3.5).

Following [2, 5, 10], for a weak harmonic Maass form $f \in \mathrm{H}^+_{k,L^-}$, we consider the regularized theta integral (evaluated using the regularization procedure from Section 1.4)

$$\Phi(z, f, \psi) := \int^{reg}_{\mathrm{SL}_2(\mathbb{Z})\mathbb{H}} \left\langle f(\tau), \overline{\Theta(\tau, z)} \right\rangle_L d\mu. \tag{3.5.2}$$

We call $\Phi(z, f, \psi)$ the regularised lift of f.

3.5.1. Singularities and current equation

Let f be a harmonic weak Maass form with holomorphic Fourier coefficients $a^+(h,n)$, $h \in L^\sharp/L$, $n \in \mathbb{Q}_{<0}$. We define a locally finite cycle $\mathbb{D}(f)$ on $\mathbb{D}$ by

$$\mathbb{D}(f) := \sum_{h \in L^\sharp/L} \sum_{n \in \mathbb{Q}_{<0}} a^+(h,n)\mathbb{D}(n,h)$$

and denote by $Z(f)$ the image of $\mathbb{D}(f)$ on X.

Proposition 3.17. *The regularized lift $\Phi(z, f, \psi)$ converges to a smooth differential form on $\mathbb{D}$ with singularities along the cycle $\mathbb{D}(f)$. In a small neighborhood of $w \in \mathbb{D}$, the singularities are of type*

$$-\sum_{h\in L^\sharp/L}\sum_{\substack{n\in\mathbb{Q}\\ n<0}} a^+(h,n) \sum_{\substack{\lambda\in L+h\\ (\lambda,\lambda)=-n\\ \lambda\in w^\perp}} \Psi^0(\sqrt{2}\lambda,\tau,z),$$

i.e., the difference of $\Phi(z, f, \psi)$ and this sum extends to a smooth form.

Proof. The argument closely follows [10, Sec. 5]. It suffices to consider the integral up to smooth functions. Due to the rapid decay of the non-holomorphic part of f, the integral converges for f^- to a smooth form, and we only need to consider

$$\sum_h \lim_{t\to\infty}\int_{\mathcal{F}_t}^{reg} f_h^+(\tau)\theta(\tau,z,\psi)_h v^{-s}\,d\mu.$$

Also, since the integral over $\mathcal{F}_1$ is smooth, it suffices to consider the integral over $v > 1$:

$$\sum_h \lim_{t\to\infty}\int_1^t\int_{-\frac{1}{2}}^{\frac{1}{2}} f_h^+(\tau)\theta(\tau,z,\psi)_h v^{-s-2}du\,dv. \tag{3.5.3}$$

Now, the integration over u picks out the constant term in the Fourier expansion of the integrand, which in the notation of (3.5.1) is given by

$$v\sum_h\sum_{\lambda\in L+h} a^+(h,-(\lambda,\lambda))P_\psi(\sqrt{2v}\lambda,z)e^{4\pi v(\lambda_z,\lambda_z)}.$$

For (3.5.3) we therefore obtain

$$\sum_{\lambda\in L^\sharp} a^+(\lambda,-(\lambda,\lambda))\int_1^\infty P_\psi(\sqrt{2v}\lambda,z)e^{4\pi v(\lambda_z,\lambda_z)}v^{-s-1}dv. \tag{3.5.4}$$

For a relatively compact open neighborhood $U \subset \mathbb{D}$, define the set

$$S_f(U,\epsilon) = \left\{\lambda\in L^\sharp;\, a^+\big(\lambda,-(\lambda,\lambda)\big)\neq 0 \quad \text{and } |(\lambda_z,\lambda_z)|<\epsilon \quad \text{for some } z\in U\right\}.$$

By reduction theory, this set is finite, as f^+ has only finitely many non-vanishing Fourier coefficients in its principal part.

Using standard arguments, like in [10], one finds that in (3.5.4) the sum of all terms with $\lambda \in L^\sharp - S_f(U,\epsilon)$ is majorized by a convergent sum, $\sum_{\lambda\in L^\sharp}\exp\left(-C(\lambda,\lambda)_z\right)$ for some $C > 0$, and hence converges. Further, in (3.5.4), the term with $\lambda = 0$ is given by $a^+(0,0)P_\psi(0,z)\int_1^\infty \frac{1}{v^{s+1}}dv$, which falls out after regularization.

Finally, all that remains of (3.5.4) is the following finite sum, which dictates the singularities in U:

$$\sum_{0\neq\lambda\in S_f(U,\epsilon)} a^+(\lambda,-(\lambda,\lambda))\int_1^\infty P_\psi(\sqrt{2v}\lambda,z)e^{4\pi v(\lambda_z,\lambda_z)}v^{-s-1}dv.$$

Clearly, the integral has meromorphic continuation to the entire s-plane, and for $s = 0$ is equal to $-\Psi^0(\sqrt{2}\lambda, \tau, z)$, cf. (3.4.1). Hence, the singularity for $z \in U$ is dictated by

$$-\sum_{\substack{\lambda \in S_f(U,\epsilon) \\ \lambda \neq 0}} a^+(\lambda, -(\lambda, \lambda))\Psi^0(\sqrt{2}\lambda, \tau, z).$$

In particular, z is a singular point precisely if $R(\lambda, z) = -(\lambda_z, \lambda_z) = 0$ for some $\lambda \in S_f(U, \epsilon) - \{0\}$. □

The singular theta lift as a current Using the relationship between the singular theta lift and the singular Schwartz form Ψ, already seen in the proof of Proposition 3.17, we derive a current equation for $\Phi(f, \psi)$. The role of φ_{KM} in Theorem 3.14 is now played by

$$\Lambda_\psi(f) := dd^c\Phi(z, f, \psi), \tag{3.5.5}$$

where $f \in \mathrm{H}^+_{k,L^-}$.

Theorem 3.18. *The singular theta lift $\Phi(z, f, \psi)$ and the lifting $\Lambda_\psi(f)$ satisfy the following current equation on X:*

$$dd^c[\Phi(f, \psi)] + (-i)^q\delta_{Z(f)} = [\Lambda_\psi(f)].$$

Proof. This follows directly from Theorem 3.14. For $x \in V$, we have

$$dd^c[\Psi^0(x)] + (-i)^q\delta_{\Gamma(x)\backslash\mathbb{D}(x)} = [\varphi^0_{KM}(x)]. \tag{3.5.6}$$

As usual, denote the Fourier coefficients of f^+ by $a^+(\lambda, n)$ for $\lambda \in L^\sharp$, $n \in \mathbb{Q}$. For any relatively compact open neighbourhood $U \subset \mathbb{D}$ and any $\epsilon > 0$, we consider the set $S_f(U, \epsilon)$ from p. 95. Then, from the left hand side of (3.5.6), we get

$$dd^c \sum_{\substack{\lambda \in S_f(U,\epsilon) \\ \lambda \neq 0}} a^+(\lambda, -(\lambda, \lambda)) \left[\Psi^0(\sqrt{2}\lambda)\right] + (-i)^q \sum_{\substack{\lambda \in S_f(U,\epsilon) \\ \lambda \neq 0}} a^+(\lambda, -(\lambda, \lambda))\delta_{Z(\lambda)}.$$

Now, by Proposition 3.17, and after taking the (locally finite) union over neighborhoods U containing singular points, we get the current associated to (the singular part of) $\Phi(z, f, \psi)$ plus the delta current for the cycle $Z(f)$:

$$dd^c[\Phi(f, \psi)] + (-i)^q\delta_{Z(f)}.$$

(Note that, through Stokes' theorem, the current is determined by the singular part.)

Repeating the same steps on the right hand side of (3.5.6), by using the identity $dd^c\Psi(x, \tau, z) = \varphi_{KM}(x, \tau, z)$ (see Proposition 3.13), we recover the current

$$[dd^c\Phi(f, \psi)] = [\Lambda_\psi(f)],$$

as claimed. □

3.5.2. Adjointness to the Kudla-Millson lift

We now show an adjointness result analogous to [10, Theorem 6.1, Theorem 6.2]. Denote by $\Theta(\tau, z, \varphi_{KM})$ the theta function for the Schwartz form φ_{KM} from Section 3.3 (see, [44, 45, 46]). By Theorem 3.3 it is a closed differential (q, q)-form (in z), which has weight $p+q$ as a modular form (in τ). The Kudla-Millson lift Λ_{KM} is now defined for any rapidly decreasing $2(p-1)q$-form η through the assignment

$$\eta \longmapsto \Lambda_{KM}(\eta) := \int_X \eta \wedge \Theta(\tau, z, \varphi_{KM}).$$

This map factors through the de Rham cohomology with compact supports on X. By [46, Theorem 2] if η is closed, $\Lambda(\tau, \eta)$ is a holomorphic modular form.

From Section 1.3 recall the definition of the pairing $\{\cdot,\cdot\}'$ between the spaces M_{k,L^-} and $\mathrm{H}^+_{k,L}$: For $f \in \mathrm{H}^+_{k,L}$ with $f^+ = \sum_{h,n} a^+(h,n)e(n\tau)\mathfrak{e}_h$ and $g \in \mathrm{M}_{k,L^-}$ with q-expansion $g = \sum_{h,n} b(h,n)e(n\tau)\mathfrak{e}_h$, the pairing is given by

$$\{g, f\}' = (g, \xi_k(f))_{2-k,L} - \sum_{h \in L^\sharp/L} a^+(h,0)b(h,0) = \sum_{h \in L^\sharp/L} \sum_{\substack{n \in \mathbb{Q} \\ n<0}} a^+(h,n)b(h,-n).$$

The following theorem is analogue of results by Bruinier and Funke in the setting of orthogonal groups [see 10, Theorem 6.1 – 6.3].

Theorem 3.19. *The lift Λ_ψ has the following properties:*

1. *Let $f \in \mathrm{H}^+_{k,L^-}$. Then*

$$(\Theta(\cdot, z, \varphi_{KM}), \xi_k(f))_{2-k,L} + a^+(0,0)\varphi_{KM}(0) = \Lambda_\psi(f)$$

as differential forms on X. In particular, $\Lambda_\psi(f)$ extends to a smooth closed (q,q)-form of moderate growth.

2. *The Kudla-Millson lift Λ_{KM} and Λ_ψ are adjoint in the sense that*

$$(\eta, \Lambda_\psi(f))_X = \{\Lambda_{KM}(\eta), f\}'$$

for any $f \in \mathrm{H}^+_{k,L^-}$ and any rapidly decreasing closed $2(p-1)q$-form η.

We note that, in particular, if $f \in \mathrm{M}^!_{k,L^-}$, we have $\Lambda_\psi(f) = a^+(0,0)\varphi_{KM}(0)$.

Corollary 3.20. *For any rapidly decreasing closed $2(p-1)q$-form η and any $f \in \mathrm{H}_{k,L}$, we have*

$$(\eta, \Lambda_\psi(f))_X = \int_{Z(f)} \eta.$$

Proof of the Theorem. 1. We have

$$L_{2-k}\Theta(\tau, z, \varphi_{KM}) = \Theta(\tau, z, dd^c\psi),$$

since $L\varphi_{KM}(0) = dd^c\psi(0)$, Hence, we have

$$\begin{aligned}\lim_{t\to\infty}\int_{\mathcal{F}_t}\left\langle L_{2-k}\Theta(\tau,z,\varphi_{KM}),\bar{f}\right\rangle d\mu &= \int_{\mathcal{F}}^{reg}\left\langle L_{2-k}\Theta(\tau,z,\varphi_{KM}),\bar{f}\right\rangle d\mu\\ &= \int_{\mathcal{F}}^{reg}\left\langle\Theta(\tau,z,dd^c\psi),\bar{f}\right\rangle d\mu,\end{aligned}$$

and this quantity defines a smooth form on $\mathbb{D}-\mathbb{D}(f)$, which extends smoothly to $\mathbb{D}$. With [10, Lemmas 6.6, 6.7] we get the following identity, valid outside $\mathbb{D}(f)$:

$$(\Theta(z,\varphi_{KM}),\xi_k(f))_{2-k,L} = \int_{\mathcal{F}}^{reg}\left\langle\Theta(\tau,z,dd^c\psi),\bar{f}\right\rangle d\mu + a^+(0,0)\varphi_{KM}(0).$$

Now, the statement follows by showing that

$$\int_{\mathcal{F}}^{reg}\left\langle\Theta(\tau,z,dd^c\psi),\bar{f}\right\rangle d\mu = dd^c\int_{\mathcal{F}}^{reg}\left\langle\Theta(\tau,z,\psi),\bar{f}\right\rangle d\mu. \tag{3.5.7}$$

First, note that

$$\int_{\mathcal{F}}^{reg}\left\langle\Theta(\tau,z,\psi),\bar{f}\right\rangle d\mu = \lim_{t\to\infty}\int_{\mathcal{F}_t}\left(\left\langle\Theta(\tau,z,\psi),\bar{f}\right\rangle - a^+(0,0)v\right)d\mu + Ca^+(0,0), \tag{3.5.8}$$

with a constant C, coming from the regularization of the constant term. Arguing along the same lines as in the proof of Proposition 3.17, we see that in the integrand, the sum over $\lambda\in L^\sharp - S_f(U,\epsilon)$ (see p. 95) converges uniformly for any relatively compact open neighborhood $U\subset\mathbb{D}$ and any $\epsilon>0$. For the remaining terms, with $\lambda\in S_f(U,\epsilon)$ the integrand decays exponentially.

Thus, switching the order of differentiation from the right hand side of (3.5.7) and the limit from (3.5.8) is justified, which completes the proof.

2. The second statement follows from the first, the proof is exactly like the one of [10, Theorem 6.3], which we briefly reproduce here. Denote by $(\cdot,\cdot)_X$ the natural pairing between closed forms of complementary degree (where one is rapidly decreasing and the other of moderate growth). We have

$$\begin{aligned}(\eta,\Lambda_\psi(f))_X &= \left(\eta,(\Theta(\cdot,z,\varphi_{KM}),\xi_k(f))_{k,L}\right)_X\\ &= \left((\eta,\Theta(\cdot,z,\varphi_{KM}))_X,\xi_k(f)\right)_{k,L} = \{\Lambda_{KM}(\eta),f\}.\end{aligned}$$

Note only that the order of integration can be switched by absolute convergence. □

3.6. Comparison of the two Green forms

In this section, we compare the Green forms of Kudla type $\mathcal{G}^K(m,w,h)$, for $m\in\mathbb{Q}$, $h\in L^\sharp/L$ and $w\in\mathbb{R}_{>0}$, and those of Bruinier type $\mathcal{G}^B(m,h)$ (see below). The aim is to transfer some of the results of Ehlen and Sankaran from [19] to the present setting. Thus, we obtain a modularity result for the difference of the generating series of the two Green forms.

3.6.1. Green form of Bruinier type

We first introduce the Green form of Bruinier type.

Recall Definition 1.13 for Hejhal Poincaré series of weight k and index (n,h), $h \in L^\sharp/L$, $n \in \mathbb{Z}$ for $\tau \in \mathbb{H}$, $s \in \mathbb{C}$ with $\sigma = \mathrm{Re}(s) > 1$:

$$F_{n,h}(\tau,s) = \frac{1}{4\Gamma(2s)} \sum_{A \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \mathcal{M}_s(4\pi|n|v)e^{2\pi i n u}\mathfrak{e}_h \mid_{k,L^-} A, \tag{3.6.1}$$

where $\mathcal{M}_s(t) = t^{-\frac{k}{2}} M_{-\frac{k}{2},s-\frac{1}{2}}(t)$, with the M-Whittaker function $M_{\kappa,\mu}(t)$.

Set $s_0 = 1 - \frac{k}{2}$. For fixed $s = s_0$, the Poincaré series $F_{n,h}(\tau, s_0)$ have principal part $q^n\mathfrak{e}_h$ and form a basis of H^+_{k,L^-}, [see 5, Proposition 1.12]. Note further that by [10, Remark 3.10] the image under the ξ-operator, $\xi_k(F_{n,h}(\tau,s_0))$ is a holomorphic, cuspidal Poincaré series of index $(-n,h)$.

We now introduce two Green forms through the regularized pairing (see p. 28) of the Hejhal Poincaré series with $\Theta(\tau,z)$. First, we define the Bruinier type Green form $\mathcal{G}^B(n,h)$ by setting

$$\mathcal{G}^B(n,h)(z) := \left(F_{n,h}(\tau,s_0), \Theta(\cdot,z)\right)^{reg}_{L^-}, \tag{3.6.2}$$

i.e., the regularized theta lift of the weak Maass form $F_{n,h}(\tau,s_0)$. By Theorem 3.18 $\mathcal{G}^B(n,h)$ is thus a Green current for the cycle $Z(n,h)$.

3.6.2. The Kudla type Green form as a theta lift

Following [19, Section 2.4], we introduce truncated Poincaré series $P_{n,w,h}$ with $n \in \mathbb{Z}$, $w \in \mathbb{R}_{>0}$ and $h \in L^\sharp/L$, of weight $k = 2-(p+q)$:

$$P_{n,w,h}(\tau) = \frac{1}{2} \sum_{A \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \left[\sigma_w(\tau) q^{-n}\mathfrak{e}_h\right] \mid_{k,L^-} A,$$

$$\text{where} \quad \sigma_w(\tau) = \begin{cases} 1 & \text{if} \quad v \geq w \\ 0 & \text{if} \quad v < w. \end{cases}$$

Further, if $n \notin \frac{1}{2}(h,h) + \mathbb{Z}$ we set $P_{n,w,h} = 0$.

Proposition 3.21. *The regularized pairing $(P_{n,w,h}, \Theta(\cdot,z))^{reg}_{L^-}$ exists. On $\mathbb{D} \setminus \mathbb{D}(n,h)$, it satisfies the identity*

$$\left(P_{n,w,h}, \Theta(\cdot,z)\right)^{reg}_{L^-} = -\Xi^K(n,w,h) - \delta_{n,0}\delta_{h,0}\psi(0)\log(w).$$

The Kudla type Green form $\Xi^K(n,w,h)$ can thus be expressed as a regularized theta lifting. This also affords an (albeit discontinuous) extension of $\Xi^K(n,w,h)$ to all $\mathbb{D}$.

Proof. Assume that $z \notin \mathbb{D}(n,h)$. We evaluate the regularized pairing by unfolding using the modularity of Θ and see

$$\begin{aligned}(P_{n,w,h}, \Theta(\cdot,z))^{reg}_{L^-} &= \underset{s=0}{\mathrm{CT}} \lim_{t\to\infty} \int_{\mathcal{F}_t - \mathcal{F}_w} \sum_{\substack{\lambda\in L+h\\(\lambda,\lambda)=n}} q^{-n}\psi(\sqrt{2v}\lambda)v^{-s}d\mu \\ &= \underset{s=0}{\mathrm{CT}} \int_w^\infty \sum_{\substack{\lambda\in L+h\\(\lambda,\lambda)=n}} \psi^0(\sqrt{2v}\lambda, z)v^{-s-1}dv.\end{aligned}$$

Now, for $n \neq 0$ this extends smoothly to the entire s-plane and for $s=0$, we obtain

$$-\sum_{\substack{\lambda\in L+h\\(\lambda,\lambda)=n}} \Psi^0(\sqrt{2v}\lambda, z) = -\Xi^K(n,w,h).$$

Similarly, for $n=0$ we obtain $-\Xi^K(n,w,h)$ from the sum over $\lambda\neq 0$. The term for $\lambda = 0$ contributes

$$\psi(0)\underset{s=0}{\mathrm{CT}}\lim_{t\to\infty}\int_w^t v^{-s-1}dv = -\psi(0)\underset{s=0}{\mathrm{CT}}\lim_{t\to\infty}\tfrac{1}{s}\left(t^{-s}-w^{-s}\right) = -\psi(0)\log(w). \qquad \square$$

3.6.3. The difference of the two Green forms as a modular form

Now, with the results of [19], we can show that the difference of $\mathcal{G}^K(n,v)$ and $\mathcal{G}^B(n)$ is, essentially a modular form.

Lemma 3.22. *The difference*

$$(P_{n,w,h}, \Theta(\cdot,z))^{reg}_{L^-} - (F_{n,h}, \Theta(\cdot,z))^{reg}_{L^-}$$

extends to a smooth differential $(q-1,q-1)$-form on $\mathbb{D}$.

Proof. Since the principal part of $F_{n,h}$ is given by $q^{-n}\mathfrak{e}_h$ this is immediate from Proposition 3.17 and Proposition 3.21. $\square$

We now assume $p+q>2$. Using [19, Theorem 1.1], we show the following:

Theorem 3.23. *Assume $p+q>2$, and fix $z\in\mathbb{D}$. The generating series*

$$F(\tau,z) = -\log(v)\psi(0)\mathfrak{e}_0 - \sum_{n\in\mathbb{Q}}\left(\Xi^K(n,v) - \mathcal{G}^B(n)\right)(z)\,q^n$$

is an element of $\mathrm{A}^!_{p+q,L}$. Furthermore, F satisfies $L_{p+q}(F)(\tau,z) = -\Theta(\tau,z)$ and is orthogonal to cusp forms.

Proof. We observe that $\Theta(\tau, z; \psi)$, as a function on $\mathbb{H}$ is contained in the space $\mathrm{A}^{mod}_{(p+q-2),L}$, see Definition 1.11. Clearly by Proposition 3.21 the generating series above can be written as

$$\sum_{n\in\mathbb{Q}} \sum_{h\in L^\sharp/L} \left(P_{n,v,h} - F_{n,h}, \Theta_h(\cdot, z)\right)^{reg}_{L^-} q^n \mathfrak{e}_h.$$

Since κ is an integer and satisfies $\kappa = p+q-2 > 0$, by [19, Theorem 1.1], this generating series, as a function on $\mathbb{H}$, is the q-expansion of a modular form F in $\mathrm{A}^!_{p+q,L}$, which satisfies $L_{p+q}(F) = -\Theta$, has trivial principal part and trivial cuspidal holomorphic projection, i.e. for every cusp form G in $S_{\kappa,L}$, the (regularized) Petersson product $\langle F, G\rangle^{reg}$ vanishes. □

Remark 3.24. We note that Theorem 3.23 also gives a different approach to the duality statement Theorem 3.19. Namely, consider $dd^cF(\tau)$ and take the Petersson inner product with the holomorphic Poincare series $\xi_k(F_{n,h}(\tau, s_0))$ of index $(-n, h)$. This vanishes and computing the inner product explicitly (using the formulas for holomorphic projection) one obtains Theorem 3.19. We leave the details to the reader.

We thank Stephan Ehlen for this comment.

Remark 3.25. As Stephan Ehlen also pointed out, it should be possible to show that the generating series from Theorem 3.23 also satisfies a current equation.

3.7. Poincaré series

In this section we introduce and study the form $\mathcal{G}^B_s(n, h)$ depending on a complex parameter s and identify it with the Green form constructed by Oda-Tsuzuki [53].

Namely, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) = \sigma > 1$, we define

$$\mathcal{G}^B_s(n, h)(z) := \lim_{t\to\infty} \int_{\mathcal{F}_t} \langle F_{n,h}(\tau, s), \Theta(\tau, z)\rangle_L \, d\mu$$

Similar to Section 2.2 in [5] it can be seen that the (regularized) integral converges for σ sufficiently large and can be analytically continued to the region $\sigma > 1$ with $s \neq s_0$.

Remark 3.26. We can also define $\mathcal{G}^B_{s_0}(n, h)(z)$ for $s = s_0$ as the constant term of the Laurent expansion of $\mathcal{G}^B_s(n, h)(z)$ at $s = s_0$. We note that $\mathcal{G}^B(n, h)$, see (3.6.2) and $\mathcal{G}^B_{s_0}(n, h)$ are not quite identical; due to the different regularization procedures, they differ by a smooth term. See [5, Proposition 2.11] for further details in the orthogonal case.

To ease the comparison with the work of Oda-Tsuzuki, we use the identification of differential forms on $\mathbb{D}$ with K-invariant functions on G with values in $\bigwedge^\bullet \mathfrak{p}^*$. In our situation, this means to consider $\mathcal{G}^B_s(n, h)$ as a function on G with values in $\bigwedge^{q-1,q-1} \mathfrak{p}^*$ by first setting $\psi(x, g) := \psi(g^{-1}x, z_0)$ for $g \in G$ and then defining

$$\mathcal{G}^B_s(n, h)(g) := \lim_{t\to\infty} \int_{\mathcal{F}_t} \langle F_{n,h}(\tau, s), \Theta(\tau, g)\rangle_L \, d\mu.$$

It is then clear that $\mathcal{G}^B_s(n, h)$ is holomorphic in s in the convergent range.

3.7.1. An eigenvalue equation

Now, we show that the Green form $\mathcal{G}_s^B(m,h)$ satisfies an eigenvalue equation under the action of the Casimir element for $\mathrm{U}(p,q)$ as the one in [53], Theorem 18 (iii) (with a different normalization of the holomorphic parameter s). The overall strategy follows of [5, Chapter 4.1] using results of Shintani [55] and additionally Hufler [39]. We denote by $\mathrm{C}_{\mathrm{SL}_2}$, $\mathrm{C}_{\mathrm{U}(p,q)}$ and $\mathrm{C}_{\mathrm{O}(2p,2q)}$ the respective Casimir elements of $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{U}(p,q)$ and $\mathrm{O}(2p,2q)$ in the universal enveloping algebra.

Let $\phi = \phi(x,\tau,z_0)$ be a Schwartz form and κ the weight of $\phi(\tau)$ under the Weil representation. As ϕ satisfies condition (1.19) of [55] with $m = 2\kappa$, by [55, Lemma 1.4] we have

$$\begin{aligned}\omega(g'_\tau)\mathrm{C}_{\mathrm{SL}_2}\,\phi(x) &= 4\left[v^2\left(\frac{\partial^2}{\partial^2 u}+\frac{\partial^2}{\partial^2 v}\right)-\kappa i v\frac{\partial}{\partial u}\right]\omega(g_\tau)\phi(x)\\ &= -4\left[\Delta_\kappa - v\kappa\frac{\partial}{\partial v}\right]\omega(g_\tau)\phi(x),\end{aligned}$$

wherein $g'_\tau = \left(\begin{smallmatrix}\sqrt{v} & u\sqrt{v}^{-1}\\ & \sqrt{v}^{-1}\end{smallmatrix}\right)$. By a brief calculation we thus have

$$4\Delta_\kappa\phi(x,\tau) = \kappa(\kappa-2)\phi(x,\tau) - v^{-\frac{\kappa}{2}}\omega(g'_\tau)\mathrm{C}_{\mathrm{SL}_2}\,\phi(x).$$

Now, by [55, Lemma 1.5] we have with $m = \dim_{\mathbb{C}}(V) = p+q$

$$\mathrm{C}_{\mathrm{SL}_2}\,\phi(x) = \left[\mathrm{C}_{\mathrm{O}(2p,2q)} + m(m-2)\right]\phi(x).$$

We note that the operation of $\mathrm{SL}_2(\mathbb{R})$ by the Weil representation commutes with $\mathrm{C}_{\mathrm{O}(2p,2q)}$. Hence, we get

$$4\Delta_\kappa\phi(x,\tau) = [\kappa(\kappa-2)-m(m-2)]\,\phi(x,\tau) - \mathrm{C}_{\mathrm{O}(2p,2q)}\,\phi(x,\tau).$$

Now, by a result of Hufler [see 39, Satz 6.10], who carries out the analogous computations for the Schwartz form φ_0,

$$\mathrm{C}_{\mathrm{U}(p,q)}\,\phi(x) = \mathrm{C}_{\mathrm{O}(2p,2q)}\phi(x) - 2\left(\Im\left(\sum_{j=1}^m z_j\frac{\partial}{\partial z_j}\right)\right)^2\phi(x). \tag{3.7.1}$$

Now set $\phi = \psi$. The second term on the right hand side of (3.7.1) vanishes for ψ and with $\kappa = p+q-2 = m-2$, we get

$$4\Delta_\kappa\psi = -4\kappa\psi - \mathrm{C}_{\mathrm{U}(p,q)}\psi.$$

The following Lemma is an immediate consequence.

Lemma 3.27. *The theta function $\Theta(\tau,z)$, considered as a function on $\mathbb{H}$, satisfies the following differential equation:*

$$4\Delta_\kappa\Theta(\tau,z_0) = \left[-4\kappa - \mathrm{C}_{\mathrm{U}(p,q)}\right]\Theta(\tau,z_0).$$

Noting that the Poincaré series $F_{m,h}$ is an eigenfunction of Δ_k with eigenvalue $\frac{\kappa^2}{4} + \frac{\kappa}{2} + s(1-s)$ [see 5, p. 29], we have the following analogue of [5, Lemma 4.4], the proof of which is quite similar:

Lemma 3.28. *For the regularized pairing of $\Theta(\tau, z)$ and the Maass Poincaré series $F_{n,h}$ of weight $-\kappa$, we have*

$$\begin{aligned}(F_{n,h}, \Delta_\kappa \Theta(\cdot, z))^{reg}_{L^-} &= (\Delta_{-\kappa} F_{n,h}, \Theta(\cdot, z))^{reg}_{L^-} - \kappa\,(F_{n,h}, \Theta(\cdot, z)))^{reg}_{L^-} \\ &= \left(\frac{\kappa^2}{4} - \frac{\kappa}{2} + s(1-s)\right)(F_{n,h}, \Theta(\cdot, z)))^{reg}_{L^-}.\end{aligned}$$

By combining the two Lemmas we get

Theorem 3.29. *Recall $\kappa = p + q - 2$. The Green form $\mathcal{G}^B_s(n,h)$ is an eigenfunction of the Casimir operator $\mathrm{C}_{\mathrm{U}(p,q)}$, with*

$$\mathrm{C}_{\mathrm{U}(p,q)}\, \mathcal{G}^B_s(n,h) = \left((2s-1)^2 - (\kappa+1)^2\right) \mathcal{G}^B_s(n,h). \tag{3.7.2}$$

Proof. Due to locally uniform convergence of the regularized lift and all partial derivatives, we have

$$\begin{aligned}\mathrm{C}_{\mathrm{U}(p,q)}\,(F_{n,h}(\cdot,s), \Theta(\cdot, z))^{reg}_{L^-} &= \left(F_{n,h}(\cdot,s), \mathrm{C}_{\mathrm{U}(p,q)}\Theta(\cdot, z)\right)^{reg}_{L^-} \\ &= -4\,(F_{n,h}(\cdot,s), (\Delta_\kappa\Theta)(\cdot, z))^{reg}_{L^-} - 4k\,(F_{n,h}(\cdot,s), \Theta(\cdot, z))^{reg}_{L^-},\end{aligned}$$

by Lemma 3.27. The statement then follows by Lemma 3.28. □

3.7.2. Unfolding against the Poincaré series

In this section, we calculate $\mathcal{G}^B(n,h)(z_0)$ by unfolding the theta integral against the Poincaré series $F_{n,h}(\tau, s)$. To facilitate notation we write

$$\mathbf{P}^\psi_{2\ell}(\lambda) := \frac{2i(-1)^{q-1}}{2^{2(q-1)}} \sum_{\underline{\alpha},\underline{\beta}} P^{2q-2}_{\underline{\alpha},\underline{\beta};2\ell}(\lambda) \otimes \Omega_{q-1}(\underline{\alpha}; \underline{\beta}) \tag{3.7.3}$$

for the homogeneous component of degree 2ℓ of the polynomial part $P_\psi(\lambda)$ of the Schwartz form ψ.

Theorem 3.30. *We have*

$$\begin{aligned}\mathcal{G}^B_s(n,h) &= \frac{(2\pi|n|)^{s-\frac{k}{2}}}{2\Gamma(2s)} \\ &\times \sum_{\substack{\lambda\in h+L \\ (\lambda,\lambda)=n}} \sum_{\ell=0}^{q-1} \mathbf{P}^\psi_{2\ell}(\lambda) \frac{\Gamma(s-\frac{k}{2}+\ell)}{\left(2\pi\left(\lambda_{z_0^\perp}, \lambda_{z_0^\perp}\right)\right)^{s-\frac{k}{2}+\ell}}\, {}_2F_1\left(s-\frac{k}{2}+\ell, s+\frac{k}{2}; 2s; \frac{|n|}{\left(\lambda_{z_0^\perp}, \lambda_{z_0^\perp}\right)}\right).\end{aligned}$$

Here ${}_2F_1$ denotes the standard Gaussian hypergeometric function.

Proof. From the definition of $F_{n,h}$ (3.6.1), and using the unitarity of ρ_L and the transformation property of $\Theta(\tau)$ we have

$$
\begin{aligned}
&\mathcal{G}_s^B(n,h)\\
&= \frac{1}{4\Gamma(2s)} \int_{\mathcal{F}}^{reg} \langle \sum_{A \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \mathcal{M}_s(4\pi|n|\Im(A\tau)) e^{2\pi i n \operatorname{Re}(A\tau)} j(A,\tau)^{-k} \mathfrak{e}_h, \rho_L(A)\Theta(\tau,z_0)\rangle_{L^-}\, d\mu\\
&= \frac{1}{4\Gamma(2s)} \int_{\mathcal{F}}^{reg} \sum_{A \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \mathcal{M}_s(4\pi|n|\Im(A\tau)) e^{2\pi i n \operatorname{Re}(A\tau)} \theta_h(A\tau, z_0)\, d\mu.
\end{aligned}
$$

Now, arguing exactly as in [5, p.55f], the unfolding (justified by absolute convergence for $\sigma > 1 + \frac{p}{2} + \frac{q}{2}$) is allowed, and we obtain

$$\mathcal{G}_s^B(n,h) = \frac{2}{4\Gamma(2s)} \int_{v=0}^{\infty} \int_{u=0}^{1} \mathcal{M}_s(4\pi|n|v) e^{2\pi i n u} \theta_h(\tau,z_0) v^{-2}\, du\, dv.$$

Inserting the Fourier expansion of $\theta_h(\tau,z)$ and integrating over u one sees

$$
\begin{aligned}
&\frac{(4\pi|n|)^{-\frac{k}{2}}}{2\Gamma(2s)} \int_{v=0}^{\infty} \sum_{\substack{\lambda \in h+L\\ (\lambda,\lambda)=-n}} M_{-\frac{k}{2}, s-\frac{1}{2}}(4\pi|n|v) e^{4\pi\left(\lambda_{z_0},\lambda_{z_0}\right)v - 2\pi(\lambda,\lambda)v} v^{-\frac{k}{2}-1} \sum_{\ell=0}^{q-1} v^{\ell} \mathbf{P}_{2\ell}^{\psi}(\sqrt{2}\lambda)\\
&\quad = \frac{(4\pi|n|)^{-\frac{k}{2}}}{2\Gamma(2s)} \sum_{\substack{\lambda \in h+L\\ (\lambda,\lambda)=-n}} \sum_{\ell=0}^{q-1} 2^{\ell} \mathbf{P}_{2\ell}^{\psi}(\lambda) \int_{v=0}^{\infty} v^{-\frac{k}{2}+\ell-1} M_{-\frac{k}{2}, s-\frac{1}{2}}(4\pi|n|v) e^{-2\pi v(\lambda,\lambda)_{z_0}}\, dv.
\end{aligned}
$$

The integrals are Laplace transforms, which can be evaluated as usual [see 20, p. 215]. We get for each integral

$$\frac{(4\pi|n|)^{s}}{\left(4\pi\left(\lambda_{z_0^\perp},\lambda_{z_0^\perp}\right)\right)^{s-\frac{k}{2}+\ell}} \Gamma\left(s-\tfrac{k}{2}+\ell\right)\, {}_2F_1\left(s-\tfrac{k}{2}+\ell, s+\tfrac{k}{2}; 2s; \frac{|n|}{\left(\lambda_{z_0^\perp},\lambda_{z_0^\perp}\right)}\right),$$

and the result follows. $\square$

We denote the individual summands for $\mathcal{G}_s^B(n,h)$ in Theorem 3.30 by $\phi_s(\lambda)$, that is,

$$
\begin{aligned}
&\phi_s(\lambda) :=\\
&\frac{(2\pi|n|)^{s-\frac{k}{2}}}{2\Gamma(2s)} \sum_{\ell=0}^{q-1} \mathbf{P}_{2\ell}^{\psi}(\lambda) \frac{\Gamma(s-\frac{k}{2}+\ell)}{\left(2\pi\left(\lambda_{z_0^\perp},\lambda_{z_0^\perp}\right)\right)^{s-\frac{k}{2}+\ell}}\, {}_2F_1\left(s-\tfrac{k}{2}+\ell, s+\tfrac{k}{2}; 2s; \tfrac{|n|}{\left(\lambda_{z_0^\perp},\lambda_{z_0^\perp}\right)}\right).
\end{aligned}
$$

Proposition 3.31. *Assume $m > 0$. Let H be the stabilizer of λ in G. Then*

(i)

$$\phi_s(\lambda) \in C^{\infty}\left((G - HK)/K; \bigwedge^{(q-1),(q-1)} \mathfrak{p}^*\right)$$

(ii) $\phi_s(\lambda)$ *is holomorphic in* s.

(iii) *Let* $\lambda = \sqrt{n}v_1$ *and consider* $g = a_t = \exp(tX_{1p+q})$ *as in the proof of Proposition 3.16. Then there exists a non-zero constant* C *such that*

$$\lim_{t\to 0} t^{2(q-1)}\phi_s(\lambda, a_t) = C\Omega_{q-1}(\underline{1}, \underline{1}).$$

(iv) *With the hypothesis as in (iv) we have*

$$\phi_s(\lambda, a_t) = O(e^{-(2Re(s)+p+q)t)})$$

as $t \to \infty$.

Proof. (i) and (ii) are clear. Now assume $\lambda = \sqrt{n}v_1$ and take $g = a_t = \exp(tX_{1p+q})$. Then $a_t^{-1}\lambda_{z_0^\perp} = \cosh(t)\sqrt{n}v_1$, and we calculate

$$\begin{aligned}
\phi_s(\lambda, a_t) = \frac{1}{2\Gamma(2s)} \sum_{\ell=0}^{q-1} \mathbf{P}^{\psi}_{2\ell}(\sqrt{n}v_1) & \\
\times \quad \frac{\Gamma(s-\frac{k}{2}+\ell)}{(2\pi n)^\ell (\cosh t)^{2s-k+2\ell}} & {}_2F_1\Big(s-\tfrac{k}{2}+\ell, s+\tfrac{k}{2}; 2s; \frac{1}{\cosh^2 t}\Big) \\
= \frac{1}{2\Gamma(2s)} \sum_{\ell=0}^{q-1} \mathbf{P}^{\psi}_{2\ell}(v_1) \frac{\Gamma(s-\frac{k}{2}+\ell)}{(2\pi)^\ell (\cosh t)^{2s-k}} & {}_2F_1\Big(s-\tfrac{k}{2}+\ell, s+\tfrac{k}{2}; 2s; \frac{1}{\cosh^2 t}\Big) \\
= \frac{1}{2\Gamma(2s)} \sum_{\ell=0}^{q-1} \mathbf{P}^{\psi}_{2\ell}(v_1) \frac{\Gamma(s-\frac{k}{2}+\ell)}{(2\pi)^\ell (\cosh t)^{2s-k}} \left(\frac{\sinh t}{\cosh t}\right)^{-2\ell} & {}_2F_1\Big(s+\tfrac{k}{2}-\ell, s-\tfrac{k}{2}; 2s; \frac{1}{\cosh^2 t}\Big).
\end{aligned}$$

Here we used ${}_2F_1(a, b; c, z) = (1-z)^{c-a-b}{}_2F_1(c-a, b-a; c, z)$. Then (iii) follows from the second line of the previous equation, while (iv) from the third line, properties of $\mathbf{P}^{\psi}_{2\ell}(v_1)$ and ${}_2F_1(s+\frac{k}{2}-(q-1), s-\frac{k}{2}; 2s; 1) = \Gamma(2s)\Gamma(q-1)/\Gamma(s-\frac{k}{2}+q-1)\Gamma(s+\frac{k}{2})$. $\square$

Oda and Tsuzuki in [53], Theorem 18, show that the properties (i)-(iv) in Theorem 3.31 together with the Casimir equation uniquely determine the function. Using Theorem 3.29 we conclude

Corollary 3.32. *The Green forms* $\mathcal{G}^B_s(n, h)$ *agree (up to a constant) with the (global) Green forms constructed by Oda and Tsuzuki in [53].*

Remark 3.33. Similarly one can evaluate the regularized pairing of $\Theta(\tau, z)$ with the non-holomorphic Eisenstein series

$$E_h(\tau, s) = \sum_{A\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} v^s \mathfrak{e}_h \mid_{k,L^-} A,$$

corresponding to $\mathcal{G}_s^B(0,h)$. After unfolding, and integration one has

$$(E_h(\cdot,s), \Theta(\cdot,z))_{L^-}^{reg} \mid_{z=z_0} = 2\sum_{\ell=0}^{q-1} \frac{\Gamma(s+\ell)}{(2\pi)^{s+\ell}} \sum_{\substack{\lambda\in L+h \\ (\lambda,\lambda)=0}} (\lambda_{z_0}, \lambda_{z_0})^{-s-\ell} \mathbf{P}_{2\ell}^{\psi}(\lambda).$$

This expression can be written in terms of Eisenstein series for the discriminant kernel Γ_L in $G = \mathrm{U}(V)$. After setting

$$\zeta_{h,\lambda}(s) := \sum_{\substack{a\in\mathcal{O}_{\mathbb{F}}^{\times} \\ a\lambda\in L+h}} \mathrm{N}_{\mathbb{F}/\mathbb{Q}}(a)^{-s}, \qquad P(L) = \{\lambda \in L^{\sharp};\, \lambda \text{ primitive}, (\lambda,\lambda) = 0\}.$$

where, as usual, $\mathbb{F}$ is the underlying imaginary quadratic field, one obtains

$$2\sum_{\ell=0}^{q-1} \frac{\Gamma(s+\ell)}{\left|\mathcal{O}_{\mathbb{F}}^{\times}\right|(2\pi)^{s+\ell}} \sum_{\lambda\in\Gamma_L\backslash P(L)} \zeta_{h,\lambda}(s)\, \mathbf{P}_{2\ell}^{\psi}(\lambda) \sum_{\gamma\in\Gamma_{L,\lambda}\backslash\Gamma_L} (\lambda_{\gamma z_0}, \lambda)^{-s-\ell}.$$

Here, $\Gamma_{L,\lambda}$ denotes the stabilizer of λ in Γ_L.

The properties of these Eisenstein series should be of interest for future investigation.

4. The Fourier-Jacobi expansion of the singular theta lift

Our intent in the present chapter is to evaluate the singular theta lift $\Phi(z, f, \psi)$ with f a weak harmonic Maass form from H^+_{k,L^-} and to explicitly determine its Fourier-Jacobi expansion. For this purpose, we adapt a method introduced by Kudla in [40]. We will explain this procedure in Section 4.2, and evaluate the actual unfolding integrals in Section 4.5. From this, we recover an explicit form of the Fourier-Jacobi expansion in Section 4.4. For our approach it is advantageous to use the mixed model of the Weil representation, thus, as a first step we will translate the Schwartz form ψ into the mixed model in Section 4.1, using the intertwining operators from Appendix A.2.1.

Some problems remain open for future work. For example, it should be interesting to study the behavior of the lift on the boundary of $\mathbb{D}$ in terms of the Fourier-Jacobi expansion. In particular, this seems quite feasible in signature $(1, q)$ where the compactification theory is essentially the same (via complementary spaces) as that for $(p, 1)$ described in Section 2.1.2. Also, we should mention that for $p, q > 1$ we determine the Fourier-Jacobi expansion only up to a constant (see Remark 4.7). To explicitly calculate this final contribution is thus also left open for the time being.

4.1. Passage to the mixed model

As usual, let (p, q) be the signature of the complex hermitian space V. In the following, to emphasize the dependence on the signature we will denote Schwartz functions ϕ in $[\mathcal{S}(V) \otimes \mathcal{A}^\bullet(\mathbb{D})]^G$ by $\phi_{p,q}$. Recall that evaluation at the base point $z_0 \in \mathbb{D}$ yields an isomorphism

$$\left[\mathcal{S}(V) \otimes \mathcal{A}^\bullet(\mathbb{D})\right]^G \simeq \left[\mathcal{S}(V) \otimes \bigwedge\nolimits^\bullet(\mathfrak{p}^*)\right]^K.$$

Further, recall the definition of the Schwartz form $\psi = \psi_{p,q}$ contained in $[\mathcal{S}(V) \otimes \mathcal{A}^{q-1,q-1}(\mathbb{D})]^G$ from Section 3.3,

$$\psi_{p,q} = \frac{2i(-1)^{q-1}}{2^{2(q-1)}} \sum_{\substack{\underline{\gamma}=\{\gamma_1,\ldots,\gamma_{q-1}\} \\ \underline{\delta}=\{\delta_1,\ldots,\delta_{q-1}\}}} \mathcal{D}_{\underline{\gamma}} \overline{\mathcal{D}}_{\underline{\delta}} \otimes \Omega_{q-1}(\underline{\gamma}, \underline{\delta}) \in \left[\mathcal{S}(V) \otimes \bigwedge\nolimits^\bullet(\mathfrak{p}^*)\right]^K,$$

with $\mathcal{D}_{\underline{\gamma}}$ and $\overline{\mathcal{D}}_{\underline{\delta}}$ as defined in Section 3.2, and finally, recall that $\psi_{p,q}$ has weight $r = p+q-2$ under the operation of $\mathrm{SL}_2(\mathbb{R}) \simeq \mathrm{U}(1,1)$.

We now use the notation from appendix A.2.1. Thus, let ℓ and ℓ' be two isotropic vectors, defined as $\ell = \frac{1}{\sqrt{2}}(v_1 + v_m)$ and $\ell' = \frac{1}{\sqrt{2}}(v_1 - v_m)$. The complement $V \cap \ell^\perp \cap \ell'^\perp$ is denoted by W, and for $x \in V$, we write $x = (\alpha, x_0, \beta) = \alpha\ell + x_0 + \beta\ell'$, with $x_0 \in W$. The real and imaginary parts of the coordinates are denoted by subscripts, e.g. $\alpha = \alpha_1 + i\alpha_2$.

Now, as in Section A.2.1, passage to the mixed model is obtained by a partial Fourier transform in α, the coordinate attached to ℓ. We denote the new ℓ-coordinate by β'.

The partial Fourier-transform of $\psi_{p,q}$, evaluated at the base-point is given as follows (with the partial Fourier-transform of $\varphi_0^{p,q}$ from (A.2.1)):

Proposition 4.1. *For a multi-index $\underline{\gamma}$ denote by n_γ the multiplicity of 1 in $\underline{\gamma}$ and denote by $\tilde{\gamma}$ the multi-index obtained from $\underline{\gamma}$ by removing all occurrences of* 1.

The partial Fourier transform of $\psi_{p,q}$ is given by

$$\widehat{\psi_{p,q}}(\beta', x_0, \beta) = \frac{2i(-1)^{q-1}}{2^{2(q-1)}} \sum_{\substack{\underline{\gamma}=\{\gamma_1,\ldots,\gamma_{q-1}\} \\ \underline{\delta}=\{\delta_1,\ldots,\delta_{q-1}\}}} \left(\frac{-i\sqrt{\pi}}{\sqrt{2}}\right)^{n_\gamma+n_\delta} (\beta'-i\beta)^{n_\delta} \left(\bar{\beta}'-i\bar{\beta}\right)^{n_\gamma} P_{\tilde{\gamma},\tilde{\delta}}(x_{0,+})\hat{\varphi}_0^{p,q} \otimes \Omega_{q-1}(\underline{\gamma};\underline{\delta}).$$

Here, $P_{\tilde{\gamma},\tilde{\delta}}(x_{0,+})$ denotes the polynomial in the positive components of x_0, i.e. $z_2, \ldots z_p$, obtained by applying $\mathcal{D}_{\tilde{\gamma}}\bar{\mathcal{D}}_{\tilde{\delta}}$ to φ_0. Its degree is $2q-2-n_\gamma-n_\delta$. Further, $\hat{\varphi}_0^{p,q}$ is given by

$$\hat{\varphi}_0^{p,q} = \exp\left(-\pi\left(|\beta'-i\beta|^2 + 2\Im\left(\beta'\bar{\beta}\right) + 2i(x_0,x_0)_{z_0}\right)\right).$$

In particular, for $(p,q) \in \{(p,1),(q,1)\}$. we have

$$\widehat{\psi}_{p,1} = 2i\,\hat{\varphi}_0^{p,1} \otimes 1 \quad \textit{and} \tag{4.1.1}$$

$$\widehat{\psi}_{1,q} = \frac{2i\pi^{q-1}}{2^{3(q-1)}} (\beta'-i\beta)^{q-1} \left(\bar{\beta}'-i\bar{\beta}\right)^{q-1} \hat{\varphi}_0^{1,q} \otimes \Omega_{q-1}\left(\underline{1};\underline{1}\right). \tag{4.1.2}$$

Proof. Follows immediately from Lemma 4.2. □

Lemma 4.2. *Denote by φ_{k_1,k_2} the Schwartz function given by $\mathcal{D}_1^{k_1}\bar{\mathcal{D}}_1^{k_2}\varphi_0$. Then, the partial Fourier transform of φ_{k_1,k_2} with respect to α is given by*

$$\hat{\varphi}_{k_1,k_2}(\beta',\beta,x_0) = \left(\frac{-i\sqrt{\pi}}{\sqrt{2}}\right)^{k_1+k_2} (\beta'-i\beta)^{k_2}\left(\bar{\beta}'-i\bar{\beta}\right)^{k_1} \hat{\varphi}_0^{p,q}.$$

Proof. If $k_1 = k_2$, $\varphi_{k,k}(x)$ takes the form

$$\left(\frac{-1}{\pi}\right)^k 2^k k!\, L_k\left(2\pi|z_1|^2\right)\varphi_0(x) = \left(\frac{-1}{\pi}\right)^k 2^k k!\, L_k\left(2\pi|\alpha+\beta|^2\right)\varphi_0(x),$$

and the statement follows from Lemma B.4 and the conclusions immediately following that Lemma, see p. 145. Indeed,

$$\begin{aligned}\hat{\varphi}_k(\beta',\beta,x_0) &= \left(\frac{-\pi}{2}\right)^k \left((\beta_1'-i\beta_1)^2 + (\beta_2'-i\beta_2)^2\right)^k \hat{\varphi}_0^{p,q} \\ &= \left(\frac{-\pi}{2}\right)^k (\beta'-i\beta)^k\left(\bar{\beta}'-i\bar{\beta}\right)^k \hat{\varphi}_0^{p,q}.\end{aligned}$$

For the general case, we can assume $k_1 > k_2$. Further, it suffices to consider $\varphi_{k+1,k}$, as the rest follows through symmetry and by induction. Set $x_i = \alpha_i + \beta_i$, $i = 1, 2$. By Remark 3.6 we have

$$(2\pi)^{k+\frac{1}{2}} \mathcal{D}_1^{k+1} \bar{\mathcal{D}}_1^k \varphi_0 = (2\pi)^{k+\frac{1}{2}} \mathcal{D}_1 \left(\mathcal{D}_1^k \bar{\mathcal{D}}_1^k \varphi_0 \right) =$$

$$\sum_{l=0}^{k} \binom{k}{l} \left[H_{2(k-l)+1}(\sqrt{2\pi} x_1) H_{2l}(\sqrt{2\pi} x_2) - i H_{2(k-l)}(\sqrt{2\pi} x_1) H_{2l+1}(\sqrt{2\pi} x_2) \right] \varphi_0.$$

Thus, with Lemma B.3, and arguing as before, the Fourier transform of $\varphi_{k+1,k}$ with respect to α is given by

$$\frac{(-i\sqrt{\pi})^{2k+1}}{2^{\frac{2k+1}{2}}} \sum_{l=0}^{k} \binom{k}{l} \left[(\beta_1' - i\beta_1)^{2(k-l)+1} (\beta_2' - i\beta_2)^{2l} - i(\beta_1' - i\beta_1)^{2(k-l)} (\beta_2' - i\beta_2)^{2l+1} \right] \hat{\varphi}_0^{p,q}$$

$$= i\sqrt{\pi}(-1)^{k+1} \left(\frac{\pi}{2}\right)^k \left((\beta_1' - i\beta_1) - i(\beta_2' - i\beta_2) \right) \left((\beta_1' - i\beta_1)^2 + (\beta_2' - i\beta_2)^2 \right)^k \hat{\varphi}_0^{p,q}$$

$$= i\sqrt{\pi}(-1)^{k+1} \left(\beta' - i\beta\right)^k \left(\bar{\beta}' - i\bar{\beta}\right)^{k+1} \hat{\varphi}_0^{p,q}.$$

This proves the Lemma. □

4.1.1. Intertwining for G' and G.

Up to here, through Proposition 4.1 one has $\psi_{p,q}$ in the mixed model only at the base-point z_0 of $\mathbb{D}$, and with τ fixed at the base point i of the complex upper-half plane $\mathbb{H}$. Moving away from the respective base points is accomplished by applying the intertwining operators for $G' = \mathrm{SL}_2(\mathbb{R})$ from Lemma A.1 and for G from Lemma A.2.

Intertwining for G'. First the operation of G'. With the notation from Proposition 4.1, let $P_{\tilde{\gamma},\tilde{\delta};\ell}$ denote the homogeneous component of weight ℓ of the polynomial $P_{\tilde{\gamma},\tilde{\delta}}$:

$$P_{\tilde{\gamma},\tilde{\delta}}(x_{0,+}) = \sum_{\ell=0}^{2q-2-n_\gamma-n_\delta} P_{\tilde{\gamma},\tilde{\delta};\ell}(x_{0,+}). \tag{4.1.3}$$

Using Lemma A.1, which gives the intertwining for the operation of $g_\tau = \begin{pmatrix} \sqrt{v} & u\sqrt{v}^{-1} \\ 0 & \sqrt{v}^{-1} \end{pmatrix}$, we get $\psi_{p,q}(x,\tau)$ in the mixed model, given by $\widehat{\psi_{p,q}}(x,\tau) = \mathcal{F}(\omega(g_\tau)\psi_{p,q}(x))(\beta',\beta)$.

Proposition 4.3. *In the mixed model the Schwartz form $\psi_{p,q}(x,\tau)$ takes the following form:*

$$\widehat{\psi_{p,q}}((\eta,x_0),\tau) = \frac{2i(-1)^{q-1}}{2^{2(q-1)}} \sum_{\substack{\underline{\gamma}=\{\gamma_1,\dots,\gamma_{q-1}\} \\ \underline{\delta}=\{\delta_1,\dots,\delta_{q-1}\}}} \left(-i\pi^{\frac{1}{2}}\right)^{n_\gamma+n_\delta} (2v)^{-\frac{n_\gamma+n_\delta}{2}} (\beta' + \bar{\tau}\beta)^{n_\delta} \left(\bar{\beta}' + \bar{\tau}\beta\right)^{n_\gamma}$$
$$\cdot \sum_{\ell=0}^{2q-2-n_\gamma-n_\delta} v^{\frac{\ell}{2}} P_{\tilde{\gamma},\tilde{\delta};\ell}(x_{0,+}) \cdot \widehat{\varphi_0}^{p,q} \otimes \Omega_{q-1}(\underline{\gamma};\underline{\delta}), \tag{4.1.4}$$

with the polynomials $P_{\tilde{\gamma},\tilde{\delta};\ell}$ from (4.1.3).

Proof. The Proposition follows directly from Lemma A.1 with (A.2.1). Recall only that $\psi_{p,q}$ has weight $r = p + q - 2$. □

A special case of the proposition, when the signature of V is $(1, q)$, is the following:

Corollary 4.4. *In the mixed model, $\psi_{1,q}(x, \tau)$ is given by*

$$\begin{aligned}\widehat{\psi_{1,q}}((\eta, x_0), \tau) &= \frac{2i\pi^{q-1}}{2^{3(q-1)}} v^{-q+1} \Big([\beta, \beta'] g'_\tau, [1, i]\Big)^k \Big(\overline{[\beta, \beta'] g'_\tau}, [1, i]\Big)^k \widehat{\varphi_0}^{1,q} \otimes \Omega_{q-1}(\underline{1}; \underline{1}) \\ &= \frac{2i\pi^{q-1}}{2^{3(q-1)}} v^{-q+1} \left(\beta' + \bar{\tau}\beta\right)^{q-1} \left(\bar{\beta}' + \bar{\tau}\bar{\beta}\right)^{q-1} \cdot \widehat{\varphi_0}^{1,q} \otimes \Omega_{q-1}(\underline{1}; \underline{1}).\end{aligned}$$

Note that for the special case of signature $(p, 1)$, the Schwartz form $\psi(x, \tau)$ is given by the Gaußian $2i\varphi_0^{p,1}(x, \tau)$ and hence, by (A.2.1) the partial Fourier transform at the base point is given by

$$\begin{aligned}\widehat{\psi_{p,1}}((\eta, x_0), \tau) &= 2i \exp\left(-\frac{\pi}{v}\left(|\beta'|^2 + |\bar{\tau}\beta|^2 + 2u \operatorname{Re}\left(\beta'\bar{\beta}\right)\right) + 2\pi\tau(x_0, x_0)\right) \\ &= 2i \exp\left(-\frac{\pi}{v}\left(|\beta' + \bar{\tau}\beta|^2 + 2v \operatorname{Re}\left(\beta'\bar{\beta}\right)\right) + 2\pi\tau(x_0, x_0)\right).\end{aligned}$$

Intertwining for G Now, the intertwining operators for the operation of G, with the Levi decomposition $G = NAM$ are given by Lemma A.2 (see p. 138). From Proposition 4.3, we thus get

Proposition 4.5. *Let $g = m(w, 0) \circ m(0, r) \circ a(t)$. Then, $\widehat{\psi_{p,q}}(gz_0)$ is given by*

$$\begin{aligned}\widehat{\psi_{p,q}}((\eta, x_0), \tau)(gz_0) = \frac{2i(-1)^{q-1}}{2^{2(q-1)}} &\sum_{\substack{\underline{\gamma}=\{\gamma_1,\ldots,\gamma_{q-1}\} \\ \underline{\delta}=\{\delta_1,\ldots,\delta_{q-1}\}}} \left(-i\pi^{\frac{1}{2}}\right)^{n_\gamma + n_\delta} (2v)^{-\frac{n_\gamma+n_\delta}{2}} t^{2+\frac{n_\gamma+n_\delta}{2}} \\ \cdot \left(\beta' + \bar{\tau}\beta\right)^{n_\delta} \left(\bar{\beta}' + \bar{\tau}\beta\right)^{n_\gamma} &\sum_{\ell=0}^{2q-2-n_\gamma-n_\delta} v^{\frac{\ell}{2}} P_{\tilde{\gamma},\tilde{\delta};\ell}(x_{0,+} - \beta w_+) \cdot \widehat{\varphi}_0^{p,q}((t\eta, x_{0,+} - \beta w_+), \tau) \\ &\cdot e\left(r\Im(\beta'\bar{\beta}) + \frac{1}{2}\operatorname{Re}(\beta'\bar{\beta})(w, w) - \operatorname{Re}\left(\beta'(x_0, w)\right)\right) \otimes \Omega_{q-1}(\underline{\gamma}; \underline{\delta}).\end{aligned}$$

We note that the intertwining operation for the action of $\mu \in M \simeq \mathrm{SU}(p-1, q-1)$ is the identity.

4.1.2. Fourier expansion of a weak harmonic Maass form in the mixed model

Let $f \in \mathrm{H}^+_{k,L^-}$ be a weak harmonic Maass with weight $k = 2 - (p + q)$ under the discrete Weil representation of ρ_{L^-}. Let $a^+(h, n)$ and $a^-(h, n)$ denote the Fourier coefficients of its holomorphic part and non-holomorphic part, respectively. The Fourier expansion of f in the mixed model can be described as follows, see [40, p. 23]:

$$f(\tau) = \sum_{n \gg -\infty} \hat{a}^+(n) q^n + \hat{a}^-(0) v^{1-k} + \sum_{\substack{n \gg -\infty \\ n \neq 0}} \hat{a}^-(n) \Gamma\left(1-k, 4\pi |n| v\right) e^{2\pi i n u}$$

where for the Fourier coefficients we have set

$$\hat{a}^\pm(n) := \sum_{\substack{\lambda = \lambda_\ell \ell + \lambda_W + \lambda_{\ell'} \ell \\ \lambda \in L^\sharp / L}} a^\pm(\lambda, n) \hat{\mathfrak{e}}_\lambda = \sum_{\substack{\lambda = \lambda_\ell \ell + \lambda_W + \lambda_{\ell'} \ell \\ \lambda \in L^\sharp / L}} a^\pm(\lambda, n) e\left(-\lambda_2 \cdot \beta'\right) \mathfrak{e}_\lambda.$$

4.2. Evaluating the singular theta lift based on Kudla's approach

To evaluate the regularized theta integral and to calculate the Fourier-Jacobi expansion of $\Phi(z, f, \psi)$ from Chapter 3.5 we employ a method recently introduced by Kudla in [40]. The key observation is that, due to invariance under the action[1] of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, the theta function can be decomposed along Γ-orbits of $\eta = [\beta, \beta']$:

$$\begin{aligned}
\Theta(\tau, z; \psi_{p,q}) &= \sum_{h \in L^\sharp / L} \sum_{\substack{\lambda \in L + h \\ \lambda = (\alpha, x_0, \beta)}} \mathcal{F}\left(\omega\left(g_\tau, \psi_{\sqrt{2}}\right) \psi_{p,q}(\lambda, z)\right) \mathfrak{e}_h \\
&= \sum_{h \in L^\sharp / L} \sum_{\substack{(\eta, x_0) \in L + h \\ \eta = [\beta, \beta'],\, x_0 \in W}} \widehat{\psi_{p,q}}\left(\sqrt{2}(\eta, x_0), \tau, z\right) \mathfrak{e}_h \\
&= \sum_{\eta / \sim} \sum_{\gamma \in \Gamma_\eta \backslash \Gamma} \theta_{\gamma\eta}(\tau, z),
\end{aligned}$$

with

$$\theta_{\gamma\eta}(\tau, z) = \sum_{h \in L^\sharp / L} \sum_{\substack{x_0 \in W \\ (\eta, x_0) \in L + h}} \widehat{\psi_{p,q}}\left(\sqrt{2}(\eta g, x_0), \tau, z\right) \mathfrak{e}_h$$

Here, Γ_η denotes the stabilizer of η in Γ, and $\eta / \sim$ denotes the orbit of η under the action of Γ. A set of orbit representatives is given by the following lemma.

Lemma 4.6. *([see 40, p. 20]) The orbits of matrices in* $\mathrm{M}_2(\mathbb{Q})$ *under the operation of* $\mathrm{SL}_2(\mathbb{Z})$ *with their respective sets of representatives and stabilizers in* $\mathrm{SL}_2(\mathbb{Z})$ *are the following:*

1. *The zero orbit, stabilized by the whole of* $\mathrm{SL}_2(\mathbb{Z})$.

[1] While Γ' would be more in keeping with our convention that $G' = \mathrm{SL}_2(\mathbb{R})$, in this chapter, to lighten notation, we denote the elliptic modular group by Γ.

2. *The orbits of rank 1 matrices. A set of representatives is given by*

$$\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \quad \text{with } a > 0 \text{ or } a = 0 \text{ and } b > 0.$$

   They are stabilized by $\mathrm{SL}_2(\mathbb{Z})_\infty = \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right);\, n \in \mathbb{Z}\}$.

3. *The orbits of rank 2 matrices. A set of representatives is given by all matrices of the form*

$$\begin{pmatrix} a & b \\ 0 & \alpha \end{pmatrix} \quad \text{with } a, \alpha \in \mathbb{Q}^\times, a > 0 \quad \text{and} \quad b \in \mathbb{Q} \mod a\mathbb{Z}. \tag{4.2.1}$$

   The stabilizer of any such orbit is trivial.

Consequently, the hermitian product in the integrand of the regularized integral

$$\int_{\Gamma\backslash\mathbb{H}}^{reg} \left\langle f(\tau), \overline{\Theta(\tau, z)} \right\rangle_L d\mu$$

decomposes along Γ-orbits, which can be sorted according to the rank of their representatives,

$$\langle f, \overline{\Theta} \rangle_L = \sum_{i=0}^{2} \sum_{\substack{\eta/\sim \\ \mathrm{rank}(\eta)=i}} \sum_{\gamma \in \Gamma_\eta \backslash \Gamma} \left\langle f(\gamma\tau), \overline{\theta_{\gamma\eta}(\tau, z)} \right\rangle_L.$$

Since each term is Γ-invariant, the regularized integral can be decomposed similarly. Thus, we write

$$\Phi(z, f, \psi) = \sum_{i=0}^{2} \Phi_i(z, f, \psi), \quad \text{with}$$
$$\Phi_i(z, f, \psi) := \int_{\Gamma\backslash\mathbb{H}}^{reg} \sum_{\substack{\eta=[\beta,\beta']/\sim \\ \mathrm{rank}(\eta)=i}} \sum_{\gamma \in \Gamma_\eta \backslash \Gamma} \left\langle f(\gamma\tau), \overline{\theta_\eta(\gamma\tau, z)} \right\rangle_L v^{-2} du\, dv.$$

Moreover, due to rapid decay of the integrand, the integrals can be evaluated for each term separately, with fixed η, and summed up later, as

$$\Phi_i(z, f, \psi) =: \sum_{\substack{\eta=[\beta,\beta']/\sim \\ \mathrm{rank}(\eta)=i}} \sum_{\gamma \in \Gamma_\eta \backslash \Gamma} \sum_{n \gg -\infty} \left(\hat{a}^+(n)\phi_i(n,\eta)^+ + \hat{a}^-(n)\phi_i(n,\eta)^-\right) \qquad (i = 0, 1, 2),$$

with $\hat{a}^\pm$ the mixed model Fourier coefficients introduced in Section 4.1.2, and with

$$\phi_i(n,\eta)^+ = \int_{\Gamma_\eta\backslash\mathbb{H}}^{reg} e^{2\pi i n u} \theta_\eta(\tau, z)\, d\mu(\tau)$$
$$\phi_i(n,\eta)^- = \int_{\Gamma_\eta\backslash\mathbb{H}}^{reg} \Gamma(1-k, 4\pi|n|) e^{2\pi i n u} \theta_\eta(\tau, z)\, d\mu(\tau) \qquad (n \neq 0),$$
$$\phi_i(0,\eta)^- = \int_{\Gamma_\eta\backslash\mathbb{H}}^{reg} \theta_\eta(\tau, z) v^{1-k}\, d\mu(\tau).$$

While for $i = 0$, the domain of integration is the usual fundamental domain $\Gamma\backslash\mathbb{H}$, for $i = 1$, it is given by $\Gamma_\infty\backslash\mathbb{H}$. Finally, for $i = 2$, integration ranges over the whole upper half plane.

An advantage of this approach is that the Fourier-Jacobi expansion of the theta lift is more easily calculated this way. In fact, the constant term of the Fourier expansion is obtained from the rank 1 terms of the lift and the zero-orbit, while the remaining terms can be read off from the terms for non-singular η, see Section 4.4 below.

To facilitate calculation across different signatures, we will first evaluate the integrals at the base point $z = z_0 \in \mathbb{D}$ and apply intertwining operators for the operation of G to the individual terms $\phi_i(n, \eta)^\pm$ after integration. These unfolding integrals are evaluated in Section 4.5.

Recall the NAM decomposition of G from Sections 2.1.1 and A.2.1 (see p. 138). Assume that for every z in $\mathbb{D}$ a continuous choice of $n(w, r) \in N$, $a(t) \in A$ and $\mu \in M$ has been fixed such that $z = (n(w, r)a(t)\mu)(z_0)$. We will use $w, r.a$ and μ as coordinates to describe the Fourier-Jacobi expansion, see Theorem 4.14. Since the Fourier-Jacobi expansion is closely linked to the operation of the Heisenberg group N, this appears as a natural choice for our purpose.

Remark 4.7. As mentioned in the introduction to this chapter, we should point out that we will not evaluate the rank 0 term $\Phi_0(z, f, \psi)$, in general, at least not if $p, q > 1$. This term is given by a convolution integral of the harmonic Maass form f with an indefinite theta series for the lattice $L \cap W$. Of course, if $q = 1$, there is no contribution of the rank 0 term, so $\Phi_0 \equiv 0$. While if $p = 1$, the theta series is definite, and this kind of integral, at least for $f \in \mathrm{M}^!_{k,L^-}$, has already been treated by Borcherds [see 2, Sec. 9]. Hence, in this special case, the value of $\Phi_0(z_0, f)$ can be deduced from Borcherds' results, see cite [40].

4.3. The lift at the base point

In this section, we state our first main result, an explicit expression for the singular theta lift of a weak harmonic Maass form f evaluated at the base point. Our Theorem 4.9 covers the general case of signature (p, q) with $p, q \geq 1$ and $p + q > 2$. The calculations for this are carried out in Section 4.5. Two special cases, where either $q = 1$ or $p = 1$ are treated in Examples 4.11 and 4.12.

Recall the kernel function from Proposition 4.3. We have

$$\begin{aligned}\widehat{\psi_{p,q}}(\sqrt{2}(\eta, x_0), \tau) = \frac{2i(-1)^{q-1}}{2^{2(q-1)}} &\sum_{\substack{\underline{\gamma}=\{\gamma_1,\dots,\gamma_{q-1}\} \\ \underline{\delta}=\{\delta_1,\dots,\delta_{q-1}\}}} \left(-i\sqrt{\pi}\right)^{n_\gamma+n_\delta} \left(\beta' + \bar{\tau}\beta\right)^{n_\delta} \left(\bar{\beta}' + \bar{\tau}\beta\right)^{n_\gamma} \\ \cdot \sum_{\ell=0}^{2q-n_\gamma-n_\delta-2} 2^{\frac{\ell}{2}} v^{\frac{(\ell-n_\gamma-n_\delta)}{2}} &\quad P_{\tilde{\gamma},\tilde{\delta};\ell}(x_{0,+}) \exp\left(-\frac{2\pi}{v}\left(\left|\beta' + \bar{\tau}\beta\right|^2 + 2v\Im\left(\beta'\bar{\beta}\right)\right)\right) \\ &\cdot e\big(\bar{\tau}(x_{0,-}, x_{0,-}) + \tau(x_{0,+}, x_{0,+})\big) \otimes \Omega_{q-1}(\underline{\gamma}; \underline{\delta}),\end{aligned} \tag{4.3.1}$$

where n_γ and n_δ denote the number of 1's occurring in the respective multi-index, and $\tilde{\gamma}$ and $\tilde{\delta}$ denote the remaining multi-indices after striking out all occurences of 1. The ranks of $\tilde{\gamma}$ and $\tilde{\delta}$ are $q-1-n_\gamma$ and $q-1-n_\delta$, respectively.

First, we introduce some notation which will be used in this and the following sections:

Notation 4.8. If $\eta = [\beta, \beta']$ is non-singular, define $\mathbf{A}_\eta$, A_η, $\mathbb{A}_\eta$, $\mathbf{B}_\eta$ and $\mathbf{C}_\eta$ by setting

$$\begin{aligned} \mathbf{A}_\eta(n,x_0) &:= n + 2|\beta|^2 + (x_0,x_0), \quad \mathrm{A}_\eta(n,x_0) := n - 2|\beta|^2 + (x_0,x_0) \\ \mathbb{A}_\eta(n,x_0) &:= \frac{1}{2}\left(n + (x_0,x_0)\right)^2 + 2|\beta|^4 + 2|\beta|^2 (x_0,x_0)_{z_0} + 2|\beta|^2 n, \\ &= \frac{1}{2}\mathbf{A}_\eta(n,x_0)^2 - 4(x_{0,-},x_{0,-}) \end{aligned} \tag{4.3.2}$$

and

$$\mathbf{B}_\eta := 2\left(|\beta'|^2|\beta|^2 - \mathrm{Re}\left(\beta'\bar{\beta}\right)^2\right) = 2\Im\left(\beta'\bar{\beta}\right)^2 \quad \text{and} \quad \mathbf{C}_\eta := \frac{\mathrm{Re}(\beta'\bar{\beta})}{|\beta|^2}. \tag{4.3.3}$$

When n or x_0 are fixed, we will drop either (or both) and just write, e.g. $\mathbb{A}_\eta$ if both x_0 and n are fixed. We note that both $\mathbf{B}_\eta$ and $\mathbb{A}_\eta$ are non-negative real numbers, and if $\beta \neq 0$ they are both positive.

Further, generalizing [5, (3.25)], we introduce the following special function

$$\begin{aligned} \mathcal{V}_{N,\mu}(A,B,c) &:= \int_0^\infty \Gamma(N-1,cv) v^{-\mu} e^{-Av-Bv^{-1}} dv \qquad \left(N \geq 2, \mathrm{Re}(A+c), \mathrm{Re}(B) > 0\right) \\ &= 2(N-2)! \sum_{r=0}^{N-2} \frac{c^r}{r!} \left(\frac{A+c}{B}\right)^{\frac{\mu-r-1}{2}} K_{r+1-\mu}\left(2\sqrt{(A+c)B}\right), \end{aligned} \tag{4.3.4}$$

see Lemma B.1 concerning the evaluation of the integral. Also note that for $N = p+q$ one has $N-1 = 1-k$ and $N-2 = -k = \kappa$.

Now, with the notation from Section 4.2, consider the terms $\phi_i(n,\eta)^\pm$, $i = 1,2$. Since the differential form part $\Omega_{q-1}(\underline{\gamma},\underline{\delta})$ of the Schwartz form $\psi_{p,q}$ depends on the multi-indices $\underline{\gamma}$, $\underline{\delta} \in \{1,\dots,p\}^{q-1}$, each term $\phi_i(n,\eta)^\pm$ can be decomposed as

$$\phi_i(n,\eta)^\pm = \frac{2i(-1)^{q-1}}{2^{2(q-1)}} \sum_{\underline{\gamma},\underline{\delta}} \phi_i^{\underline{\gamma},\underline{\delta}}(n,\eta)^\pm \otimes \Omega_{q-1}\left(\underline{\gamma},\underline{\delta}\right).$$

Hence, for $f \in \mathrm{H}^+_{k,L^-}$ the regularized integral $\Phi(z_0,f,\psi)$ can be written in the form

$$\begin{aligned} \Phi(z_0,f,\psi) &= \Phi_{12}(z_0,f,\psi) + \Phi_0(z_0,f,\psi) \\ &= \sum_{i=1}^{2}\left(\Phi_i(z_0,f^+,\psi) + \Phi_i(z_0,f^-,\psi)\right) + \Phi_0(z_0,f,\psi) \end{aligned}$$

with

$$\Phi_i(z_0,f^\pm,\psi)=\frac{2i(-1)^{q-1}}{2^{2(q-1)}}\sum_{\underline{\gamma},\underline{\delta}}\Big(\sum_{\substack{\eta=[\beta,\beta']/\sim\\ \mathrm{rank}(\eta)=i}}\sum_{\gamma\in\Gamma_\eta\backslash\Gamma}\sum_{n\gg-\infty}\hat{a}^\pm(n)\phi_i^{\gamma,\delta}(n,\eta)^\pm\Big)\otimes\Omega_{q-1}\big(\underline{\gamma},\underline{\delta}\big).$$

The individual terms $\phi_i^{\gamma,\delta}(n,\eta)^\pm$ $(i=1,2)$ are calculated in Section 4.5. The Lemmas there are the basis for the proof the main result of this section, Theorem 4.9.

We fix some more notation. Given n_γ,n_δ as defined above, with $0\le n_\gamma,n_\delta\le q-1$, and $0\le M\le n_\gamma+n_\delta$, set

$$R_{n_\delta,n_\gamma}(\eta,M):=\sum_{\substack{0\le\mu_1\le n_\delta\\0\le\mu_2\le n_\gamma\\ \mu_1+\mu_2=M}}\beta^{\mu_1}\bar{\beta}^{\mu_2}\beta'^{n_\delta-\mu_1}\bar{\beta}'^{n_\gamma-\mu_2}\binom{n_\delta}{\mu_1}\binom{n_\gamma}{\mu_2}.$$

Now, we can state the Theorem.

Theorem 4.9. *For a weak harmonic Maass form $f\in\mathrm{H}^+_{k,L^-}$ with Fourier coefficients $a^\pm(h,n)$ $(h\in L^\sharp/L, n\in\mathbb{Q},-\infty\ll n)$ the regularized theta integral $\Phi_{12}(z_0,f,\psi)$ is given by*

$$\Phi_{12}(z_0,f,\psi)=\frac{2i(-1)^{q-1}}{2^{2(q-1)}}\sum_{\underline{\gamma},\underline{\delta}}\Big\{\sum_{i=1}^{2}\sum_{\substack{\eta=[\beta,\beta']/\sim\\ \mathrm{rank}(\eta)=i}}$$
$$\cdot\sum_{\gamma\in\Gamma_\eta\backslash\Gamma'}\sum_{n\gg-\infty}\Big(\hat{a}^+(n)\phi_i^{\gamma,\delta}(n,\eta)^+(z_0)+\hat{a}^-(n)\phi_i^{\gamma,\delta}(n,\eta)^-(z_0)\Big)\Big\}\otimes\Omega_{q-1}\big(\underline{\gamma},\underline{\delta}\big),$$

where for fixed m and η the contributions to the inner sum are the following:

1. *The non-singular terms $\phi_2^{\gamma,\delta}(n,\eta)^+(z_0)$ and $\phi_2^{\gamma,\delta}(n,\eta)^-(z_0)$ are given by the sum*

$$\begin{aligned}&\big(-i\sqrt{\pi}\big)^{n_\gamma+n_\delta}\sum_{\ell=0}^{2q-2-n_\gamma-n_\delta}2^{\frac{\ell+1}{2}}P_{\tilde\gamma,\tilde\delta,\ell}(x_{0,+})\sum_{M=0}^{n_\gamma+n_\delta}R_{n_\delta,n_\gamma}(\eta,M)\\&\cdot\sum_{j=0}^{\lfloor\frac{M}{2}\rfloor}\frac{1}{\pi^j}\sum_{h=0}^{M-2j}\frac{i^h}{2^{h+3j}}\mathrm{A}_\eta^h\left(-\mathbf{C}_\eta\right)^{M-2j-h}|\beta|^{-2h-2j}\binom{M-2j}{h}\frac{M!}{j!(M-2j)!}\\&\qquad\cdot\mathfrak{a}_\nu^\pm(\eta;n)\exp\Big(-2\pi i\mathbf{C}_\eta\left((x_0,x_0)+n\right)\Big).\end{aligned}\tag{4.3.5}$$

   The factor $\mathfrak{a}_\nu^\pm(\eta;n)$ in the inner sum, with index $\nu=h+j+\frac{1}{2}(\ell-n_\gamma-n_\delta-1)$ takes the form

$$\mathfrak{a}_\nu^+(\eta;n)=\left(\frac{\mathbb{A}_\eta}{\mathbf{B}_\eta}\right)^{-\frac{\nu}{2}}K_\nu\left(\frac{2\pi}{|\beta|^2}\left(\mathbb{A}_\eta\mathbf{B}_\eta\right)^{\frac{1}{2}}\right),\tag{4.3.6}$$

for $\phi_2^{\gamma,\delta}(n,\eta)^+(z_0)$, while while for $\phi_2^{\gamma,\delta}(n,\eta)^-(z_0)$ if $n \neq 0$, it is given by

$$\mathfrak{a}_\nu^-(\eta;n) = \tfrac{1}{2}\mathcal{V}_{p+q,1-\nu}\left(\pi\left(\frac{\mathbb{A}_\eta}{|\beta|^2} - 2n\right), \frac{\pi \mathbf{B}_\eta}{|\beta|^2}, 4\pi|n|\right). \tag{4.3.7}$$

Finally for $n=0$, the non-holomorphic part $\phi_{\underline{2}}^{\gamma,\delta}(0,\eta)^-$ takes the same form as $\phi_{\underline{2}}^{\gamma,\delta}(0,\eta)^+$ but with a shifted index, as ν is replaced by $\nu-k+1$.

2. *The terms $\phi_1^{\gamma,\delta}(n,\eta)^+(z_0)$ and $\phi_1^{\gamma,\delta}(n,\eta)^-(z_0)$ consist of a sum*

$$\left(-i\sqrt{\pi}\right)^{n_\gamma+n_\delta}\beta'^{n_\gamma}\bar{\beta}'^{n_\delta}\sum_{\ell=0}^{2q-2-n_\gamma-n_\delta} 2^{\frac{\ell}{2}}P_{\tilde{\gamma},\tilde{\delta},\ell}(x_{0,+})\mathfrak{b}_\nu^\pm(\eta;n),$$

wherein the factor $\mathfrak{b}_\nu^\pm(\eta;n)$ with index $\nu = \frac{1}{2}(\ell - n_\gamma - n_\delta) - 1$ is given by

$$\mathfrak{b}_\nu^+(\eta;n) = 2|\beta'|^\nu\left(2|(x_{0,-},x_{0,-})|\right)^{-\frac{\nu}{2}}K_\nu\left(2\sqrt{2}\pi|\beta'||(x_{0,-},x_{0,-})|\right),$$

for the holomorphic term and by

$$\mathfrak{b}_\nu^-(\eta;n) = \mathcal{V}_{p+q,1-\nu}\left(2\pi(x_{0,-},x_{0,-}), \pi|\beta'|^2, 4\pi|n|\right)$$

for the non holomorphic term if $n \neq 0$.

For $n=0$, the contribution for the non-holomorphic part is the same as for the holomorphic part, but with index shifted by $-k+1$, i.e. with $\mathfrak{b}_{\nu'}^-(\eta;0) = \mathfrak{b}_{\nu-k+1}^+(\eta;0)$.

Proof. Follows directly from the calculations in Section 4.5. The results for the non-singular terms can be found in Lemmas 4.22 and 4.23, for the results on rank 1 terms see Lemmas 4.24 and 4.25. □

Remark 4.10. In Theorem 4.9, whenever the index ν is a half-integer, i.e. $\nu \equiv \frac{1}{2} \pmod 1$, the Bessel functions can be expressed through Bessel polynomials.

For example, for the non-singular terms, if $\frac{1}{2}(\ell - n_\gamma - n_\delta - 1) \equiv \frac{1}{2} \pmod 1$, the Bessel function in (4.3.6) can be replaced by an expression of the form

$$\frac{1}{2}|\beta|\left(\mathbb{A}_\eta\mathbf{B}_\eta\right)^{-\frac{1}{4}}h_{\nu'}\left(\left(2\pi\sqrt{\mathbb{A}_\eta\mathbf{B}_\eta}\right)^{-1}\right)\exp\left(-\frac{2\pi}{|\beta|^2}\left(\mathbb{A}_\eta\mathbf{B}_\eta\right)^{\frac{1}{2}}\right), \tag{4.3.8}$$

with $\nu' = |\nu| - \frac{1}{2}$. Likewise, in this case, the Bessel functions occuring in the expansion of $\mathcal{V}_{p+q,1-\nu}$ in (4.3.7) can be replaced by a similar expression, with $\nu' = |\nu + r| - \frac{1}{2}$ and $\mathbb{A}_\eta$ replaced by $\mathbb{A}_\eta + 4|n||\beta|^2 - 2n|\beta|^2$.

For the rank 1 terms, if $\frac{1}{2}(\ell - n_\gamma - n_\delta) \equiv \frac{1}{2} \pmod 1$, the Bessel functions in the holomorphic term $\phi_1^{\gamma,\delta}(n,\eta)^+$ can be replaced by expressions of the form

$$\frac{1}{2}|\beta'|^{-\frac{1}{2}}\left(2(x_{0,-},x_{0,-})\right)^{-\frac{1}{4}}h_{\nu'}\left(\left(2\sqrt{2}\pi|\beta'||(x_{0,-},x_{0,-})|^{\frac{1}{2}}\right)^{-1}\right),$$

with $\nu' = |\nu| - \frac{1}{2}$. Similarly for the Bessel function in the expansion of $\mathcal{V}_{p+q,1-\nu}$ of the non-holomorphic terms, but with $\nu' = |\nu + r| - \frac{1}{2}$ and $2(x_0,x_0)$ shifted by $4|n|$.

Example 4.11. The lift for signature $(p,1)$ presents an interesting example. Recall that in this signature the Schwartz form is given by

$$\widehat{\psi}_{p,1}(\sqrt{2}(\eta, x_0), \tau) = 2i \exp\left(-\frac{2\pi}{v}\left[|\beta' + \bar{\tau}\beta|^2 + 2v\Im\left(\beta'\bar{\beta}\right)\right]\right) e^{2\pi i\tau(x_0,x_0)} \otimes 1,$$

so the polynomials and the differential form are trivial. It is thus not surprising that, that the expressions from Theorem 4.9 simplify considerably.

For the non-singular terms, the contribution to the lift of of $f \in \mathrm{H}^+_{1-p,L^-}$ due to the holomorphic part f^+ is given by (for fixed η and n)

$$\begin{aligned}\phi_2(n,\eta)^+(z_0) &= \sqrt{2}\left(\frac{\mathbb{A}_\eta}{\mathbf{B}_\eta}\right)^{+\frac{1}{4}} K_{-\frac{1}{2}}\left(\frac{2\pi}{|\beta|^2}\left(\mathbb{A}_\eta\mathbf{B}_\eta\right)^{\frac{1}{2}}\right) e^{-2\pi i\mathbf{C}_\eta((x_0,x_0)+n)} \\ &= \frac{\sqrt{2}}{2}|\beta|\mathbf{B}_\eta{}^{-\frac{1}{2}} \exp\left(-\frac{2\pi}{|\beta|^2}|\mathbf{A}_\eta|\left(\tfrac{1}{2}\mathbf{B}_\eta\right)^{\frac{1}{2}}\right) e^{-2\pi i\mathbf{C}_\eta((x_0,x_0)+n)},\end{aligned}$$

using (B.1.4). Note that here $\mathbb{A}_\eta = \frac{1}{2}\mathbf{A}_\eta^2$. The contribution of the non-holomorphic part f^- (for fixed η and m) is given by

$$\phi_2(n,\eta)^-(z_0) = \frac{\sqrt{2}}{2}\mathcal{V}_{\kappa+2,\frac{3}{2}}\left(\frac{\pi\mathbb{A}_\eta}{|\beta|^2} - 2\pi n, \frac{\pi\mathbf{B}_\eta}{|\beta|^2}, 4|n|\pi\right) e^{-2\pi i\mathbf{C}_\eta((x_0,x_0)+n)}.$$

Since the index $\nu = \frac{1}{2}$ is a half-integer, after expressing $\mathcal{V}_{\kappa+2,3/2}$ as a sum, we can use Bessel polynomials to write $\phi_2(n,\eta)^-$ in the form

$$\begin{aligned}\phi_2(n,\eta)^-(z_0) = \sqrt{2}(p-1)! &\sum_{r=0}^{p-1}\frac{(4\pi|n|)^r}{r!}\left(\mathbb{A}_\eta + 2|\beta|^2(2|n|-n)\right)^{-\frac{r}{2}}\mathbf{B}_\eta^{\frac{r-1}{2}} \\ &\cdot h_{\max(0,r-1)}\left(\left[\frac{2\pi}{|\beta|^2}\left(\left(\mathbb{A}_\eta + 2|\beta|^2(2|n|-n)\right)\mathbf{B}_\eta\right)^{\frac{1}{2}}\right]^{-1}\right) \\ &\cdot \exp\left(-\frac{2\pi}{|\beta|^2}\left(\left(\mathbb{A}_\eta + 2|\beta|^2(2|n|-n)\right)\mathbf{B}_\eta\right)^{\frac{1}{2}}\right) e^{-2\pi i\mathbf{C}_\eta((x_0,x_0)+n)}.\end{aligned}$$

The rank 1 terms $\phi_1(n,\eta)^+$ are easily calculated directly. For a 'one-line' version of the calculations in Section 4.5.2, note that the domain of integration is given by $0 \le u \le 1$ and $0 \le v < \infty$. Thus, the integral over u just picks out the constant term of the Fourier expansion and terms are non-zero only if $(x_0, x_0) = -n$ (note that x_0 is positive definite). Then, recalling that $\eta = [0, (a,b)^t]$, the integral over v is given by

$$\begin{aligned}\phi_1(n,\eta)^+(z_0) &= \operatorname*{CT}_{s=0}\left[\int_0^\infty \exp\left(-\frac{2\pi}{v}|\beta'|^2\right) v^{-s-2}dv\right] \\ &= \left(2\pi|\beta'|^2\right)^{-s-1}\Gamma(s+1)\Big|_{s=0} = \frac{1}{2\pi}\frac{1}{a^2+b^2}\Gamma(1).\end{aligned}$$

For the contribution of the non-holomorphic part, one has (for $n \neq 0$) :

$$\phi_1(n,\eta)^-(z_0) = \quad (p-1)! \sum_{r=0}^{(p-1)} \frac{\pi^r}{r!} (4|n|)^{\frac{r}{2}-\frac{1}{4}} |\beta'|^{r-\frac{1}{2}} h_r \left(\frac{1}{4\pi|\beta'||n|^{\frac{1}{2}}} \right) e^{-4\pi|\beta'||n|^{\frac{1}{2}}}.$$

For $n = 0$, one has $\phi_1(0,\eta)^-(z_0) = (2\pi(a^2+b^2))^{-(p+1)}\Gamma(p+1)$, from (B.1.1) with $1-k = \kappa+1 = p$.

Example 4.12. Another important example is the case of signature $(1,q)$. As we have seen in Proposition 4.1 the Schwartz form $\widehat{\psi}_{1,q}$ takes a fairly simple form, as there is only one pair of multi-indices, given by $\underline{\gamma} = \underline{\delta} = (1,\dots,1)$. Thus $n_\delta = n_\gamma = q-1$ and there are no polynomials $P_{\tilde{\gamma},\tilde{\delta}}(x_{0,+})$. Hence,

$$\Phi(z,f,\psi) = \frac{2i(-1)^{q-1}}{2^{2(q-1)}} \Big(\sum_{i=1}^{2} \sum_{\substack{\eta=[\beta,\beta']/\sim \\ \mathrm{rank}(\eta)=i}} \sum_{\gamma\in\Gamma_\eta\backslash\Gamma} \sum_{n\gg-\infty} \hat{a}^{\pm}(n)\phi_i(n,\eta)^{\pm} \Big) \otimes \Omega_{q-1}\big(\underline{1},\underline{1}\big).$$

We note that index from Theorem 4.9 is half-integer and Remark 4.10 applies. Thus, the non-singular terms – excluding the case $n=0$ for the non-holomorphic part – are given by

$$\begin{aligned}
\phi_2(n,\eta)^{\pm}(z_0) = (-\pi)^{q-1}\frac{\sqrt{2}}{2} \sum_{M=0}^{2(q-1)} R_{q-1,q-1}(\eta;M) \sum_{j=0}^{\lfloor\frac{M}{2}\rfloor} \frac{1}{\pi^j}|\beta|^{-2(M-j)} \\
\cdot \sum_{h=0}^{M-2j} \frac{i^h}{2^{h+3j}} \mathrm{A}_\eta^h(-\mathbf{C}_\eta)^{M-2j-h}|\beta|^{2h-2j+1}\binom{M-2j}{h}\frac{M!}{j!(M-2j)!} \\
\cdot \exp\left(-\frac{2\pi}{|\beta|^2}(\mathbb{A}_\eta\mathbf{B}_\eta)^{\frac{1}{2}}\right) e\left(-\mathbf{C}_\eta\left((x_0,x_0)+n\right)\right) \\
\cdot \begin{cases} \mathbb{A}_\eta^{-\frac{\nu}{2}-\frac{1}{4}}\mathbf{B}_\eta^{\frac{\nu}{2}-\frac{1}{4}} h_{\nu'}\left(\dfrac{|\beta|^2}{2\pi\left(\mathbb{A}_\eta\mathbf{B}_\eta\right)^{\frac{1}{2}}}\right) & \text{for } \phi^+, \\ (q-1)!\displaystyle\sum_{r=0}^{q-1}\frac{(4\pi|n|)^r}{r!}\mathbb{A}_\eta^{-\frac{\nu+r}{2}-\frac{1}{4}}\mathbf{B}_\eta^{\frac{\nu+r}{2}-\frac{1}{4}} h_{\nu''}\left(\dfrac{|\beta|^2}{2\pi\left(\mathbb{A}_\eta\mathbf{B}_\eta\right)^{\frac{1}{2}}}\right) & \text{for } \phi^-, \end{cases}
\end{aligned}$$

wherein

$$R_{q-1,q-1}(\eta;M) = \sum_{\substack{0\le\mu_1,\mu_2\le q-1 \\ \mu_1+\mu_2=M}} |\beta|^M|\beta'|^{q-1}\,\mathrm{Re}\left(\beta'^{-\mu_1}\bar{\beta}'^{-\mu_2}\right)\binom{q-1}{\mu_1}\binom{q-1}{\mu_2},$$

whilst the indices ν, ν' and ν'' are given by

$$\nu = h+j-q-\frac{1}{2}, \qquad \nu' = |\nu|-\frac{1}{2}, \qquad \nu'' = |\nu+r|-\frac{1}{2}.$$

$$\text{So, for example} \qquad \nu'' = \begin{cases} q-r-h-j-1 & \text{if } \quad q > h+j+r, \\ r+h+j-q & \text{otherwise.} \end{cases}$$

As before, for $n=0$, the terms for the non-holomorphic part (both rank 1 and rank 2) can be obtained from the respective holomorphic term after an index shift by $-k+1$.

For the rank 1 terms one has for $n \neq 0$, by part 2. of Theorem 4.9, the holomorphic term

$$\phi_1(n,\eta)^+(z_0) = (-\pi)^{q-1}|\beta'|^{q-2}\left(2|n|\right)^{\frac{q}{2}} K_q\left(2\sqrt{2}\pi|\beta'||n|^{\frac{1}{2}}\right).$$

Note that the index $\nu = -q$ and that $(x_{0,-},x_{0,-}) = -n$. For $n \neq 0$ the non-holomorphic term is given by

$$\begin{aligned}\phi_1(n,\eta)^-(z_0) =& 2(-\pi)^{q-1}|\beta'|^{q-2}(q-1)!\sum_{r=0}^{q-1} 2^{\frac{q-r}{2}}|\beta'|^r\frac{(4\pi|n|)^r}{r!}\\ &\cdot\left(Q\left(x_{0,-}\right)+2|n|\right)^{\frac{q-1}{2}} K_{q-r}\left(2\pi|\beta'|\sqrt{2}\sqrt{Q\left(x_{0,-}\right)+2|n|}\right),\end{aligned}$$

while for $n=0$, one has

$$\begin{aligned}\phi_1(0,\eta)^+(z_0) &= (-\pi)^{q-1}|\beta'|^{q-2}\left(2|Q\left(x_{0,-}\right)|\right)^{\frac{q}{2}} K_q\left(2\sqrt{2}\pi|\beta'||Q\left(x_{0,-}\right)|^{\frac{1}{2}}\right)\\ \text{and}\quad \phi_1(0,\eta)^-(z_0) &= (-\pi)^{q-1}|\beta'|^{2(q-1)}K_0\left(2\sqrt{2}\pi|\beta'||Q\left(x_{0,-}\right)|^{\frac{1}{2}}\right).\end{aligned}$$

4.4. Determining the Fourier-Jacobi expansion

In this section, we determine the Fourier-Jacobi expansion of the lift $\Phi(z_0,f,\psi)$ for $f\in \mathrm{H}^+_{k,L^-}$ by applying the action of the parabolic subgroup to the lift at the base-point from Section 4.3.

4.4.1. Operation of the parabolic subgroup in Γ

To begin, we study the operation of P_ℓ on $\Phi(z_0;f)$ through the intertwining operators from Lemma A.2. For this, keep in mind that the theta function $\Theta(\tau,z)$ is formed using a factor of $\sqrt{2}$ in η and x_0, see Sections 1.2.1 and 3.5, which has to be taken into account in all exponential factors from Lemma A.2.

On rank 2 terms: Let us first consider the action on the non-singular terms. Recall the definitions of $\mathbf{B}_\eta$, $\mathbf{C}_\eta$ adn $\mathbb{A}_\eta$ from Notation 4.8 above.

Clearly, $\mathbf{B}_\eta$ and $\mathbf{C}_\eta$ are invariant under the operation of $n(w,r)\in N$ and $\mu\in M$, as they do not depend on x_0. Further, under the operation of $a(t)\in A$, the expression $\mathbf{C}_\eta$ is invariant, while $a(t)\mathbf{B}_\eta = \mathbf{B}_{t\eta} = t^4\mathbf{B}_\eta$. The quotient $\mathbf{B}_\eta^{\frac{1}{2}}|\beta|^{-2}$ is again invariant. Quite contrastingly, we have

$$\begin{gathered} n(w,0)\mathbb{A}_\eta(m,x_0) = \mathbb{A}_\eta(n,x_0-\beta w) \quad (n(w,0)\in N),\\ a(t)\mathbb{A}_\eta(n,x_0) = \mathbb{A}_{t\eta}(m,x_0)\ (a(t)\in A), \qquad \mu\mathbb{A}_\eta(n,x_0) = \mathbb{A}_\eta(n,\mu^{-1}x_0)\ (\mu\in M),\end{gathered}$$

and similarly for $\mathbf{A}_\eta(n, x_0)$ and $\mathrm{A}_\eta(n, x_0)$. Also recall that by part 3. of Lemma A.2 the entire expression has to multiplied with t^2.

The operation of the translations $n(0, r) \in N$ is most easily described: The terms $\phi_2^{\gamma,\delta}(n, \eta)^\pm$ are just multiplied with a factor (see Lemma A.2) of

$$\exp\left((2\pi ir\Im\left(2\beta'\bar{\beta}\right)\right) = \exp\left(-4\pi ira\alpha\right).$$

The operation of $n(0, w)$ is more complicated. It affects the polynomials $P_{\tilde{\gamma},\tilde{\delta},\ell}(x_{0,+})$. and all factors containing either $\mathbb{A}_\eta$ and A_η. From the last factor in (4.5.2) and (4.5.3), one has

$$\begin{aligned} &e\left(-\mathbf{C}_\eta\left(n + (x_0 - \beta w, x_0 - \beta w)\right)\right) = \\ &\quad e\left(-\mathbf{C}_\eta\left(n + (x_0, x_0)\right)\right) \cdot e\left(-\mathbf{C}_\eta\left[|\beta|^2(w, w) - 2\operatorname{Re}\left(\beta(x_0, w)\right)\right]\right). \end{aligned}$$

Further, again by Lemma A.2, the term $\phi_2^{\gamma,\delta}(n, \eta)^\pm$ gets a factor of

$$e\left(\operatorname{Re}\left(2\beta'\bar{\beta}\right)\tfrac{1}{2}(w, w) - 2\operatorname{Re}\left(\beta'(x_0, w)\right)\right).$$

Multiplying the two factors, we have

$$\begin{aligned} e&\left(\left[-\mathbf{C}_\eta|\beta|^2 + \operatorname{Re}\left(\beta'\bar{\beta}\right)\right](w, w) - 2\mathbf{C}_\eta\operatorname{Re}\left(\beta(x_0, w)\right) - 2\operatorname{Re}\left(\beta'(x_0, w)\right)\right) \\ &= e\left(\operatorname{Re}\left[\left(2\operatorname{Re}\left(\beta'\bar{\beta}\right)|\beta|^{-1} - 2\beta'\right)(x_0, w)\right]\right) \\ &= e\left(\operatorname{Re}(x_0, w)\left(2\tfrac{ab}{a} - 2b\right) + 2\alpha\Im(x_0, w)\right) = e\left(2\alpha\Im(x_0, w)\right). \end{aligned} \tag{4.4.1}$$

since $\beta = a > 0$ and $\beta' = b + i\alpha$.

Remark 4.13. For the case of signature $(p, 1)$, studied in Example 4.11 on p. 117, we get a somewhat simpler expression for the operation of the elements $n(w, 0) \in N$. Recall that in this signature $\mathbb{A}_\eta = \frac{1}{2}\mathbf{A}_\eta^2$. Thus, in the exponential, from (4.4.1) we have (with $\beta = a$, $\beta' = b + i\alpha$)

$$\exp\left(-\frac{2\pi}{|\beta|^2}|\mathbf{A}_\eta(x_0 - \beta w)|\left(\tfrac{1}{2}\mathbf{B}_\eta\right)^{\frac{1}{2}}\right)e\left(-\mathbf{C}_\eta\left((x_0, x_0) + n\right) + 2\alpha\Im(x_0, w)\right).$$

Considering the first term, since $\mathbf{B}_\eta = 2\alpha^2a^2$ and $a > 0$ one has

$$\begin{aligned} &\exp\left(\mp\frac{2\pi}{|\beta|^2}\left(\tfrac{1}{2}\mathbf{B}_\eta\right)^{\frac{1}{2}}\left|\mathbf{A}_\eta(x_0) - 2\operatorname{Re}\left(\beta(x_0, w)\right) + |\beta|^2(w, w)\right|\right) = \\ &\exp\left(-\frac{2\pi}{|\beta|^2}|\mathbf{A}_\eta(x_0)|\left(\tfrac{1}{2}\mathbf{B}_\eta\right)^{\frac{1}{2}}\right) \cdot \exp\left(-2\pi\epsilon'\frac{\alpha}{a}\left[a^2(w, w) - 2a\operatorname{Re}(x_0, w)\right]\right), \end{aligned} \tag{4.4.2}$$

where we have set $\epsilon' := \operatorname{sign}(\alpha\mathbf{A}_\eta(x_0 - aw))$ (recall that $\alpha \neq 0$). Collecting the second factor in (4.4.2) and the factor $e(-\mathbf{C}_\eta(n + (x_0, x_0)) + 2\alpha\Im(x_0, w))$ we get

$$\begin{aligned} &e\Big(i\epsilon'\alpha\left(a(w, w) - 2\operatorname{Re}(x_0, w)\right) + 2\alpha\Im(x_0, w) - \frac{b}{a}\left(n + (x_0, x_0)\right)\Big) \\ &= \begin{cases} e\Big(-\frac{b}{a}\left(n + (x_0, x_0)\right) + i\alpha\left(a(w, w) - 2(x_0, w)\right)\Big) & (\alpha\mathbf{A}_\eta(x_0 - aw) > 0), \\ e\Big(-\frac{b}{a}\left(n + (x_0, x_0)\right) + i\alpha\left(-a(w, w) + 2(w, x_0)\right)\Big) & (\alpha\mathbf{A}_\eta(x_0 - aw) < 0). \end{cases} \end{aligned}$$

On rank 1 terms: Now, for the operation on the rank 1 terms. Now, $\eta = [0, \beta']$ with $\beta' = a + ib$. The elements $a(t) \in A$ and $\mu \in M$ operate as usual, so $a(t)\beta' = t\beta'$ and $\mu : x_0 \mapsto \mu^{-1}x_0$. The operation of the Heisenberg group is much simpler in this case: $n(0,r)$ operates trivially, while $n(w,0)$ only operates through the multiplicative factor from Lemma A.2, given by

$$e\left(-\operatorname{Re}\left(\sqrt{2}(a+ib)\left(\sqrt{2}x_0, w\right)\right)\right) = e\big(-2\operatorname{Re}\left(\beta'(x_0,w)\right)\big).$$

4.4.2. The Fourier-Jacobi expansion

Now, we want to determine the Fourier-Jacobi expansion of $\Phi(z,f,\psi)$, i.e. an expansion of the form

$$\Phi(z,f,\psi) = \sum_{\kappa\in\mathbb{Q}} a_\kappa(\sigma)e^{2\pi i\kappa \operatorname{Re}\tau_\ell},$$

where τ_ℓ is coordinate attached to the ℓ-component. Let us assume that for the base point $\operatorname{Re}\tau_\ell = 0$. This can easily be realized through a suitable choice of coordinate in $\mathbb{D}$. For example, in the Siegel domain model of the symmetric domain (see (2.1.1)) in signature $(p,1)$, we have $\tau_\ell = \tau$, and the base point is given by i.

Consider $z = gz_0$ with $g = n(w,0)n(0,r)a(t)\mu \in G$, and write $\Phi(z,f,\psi)$ in the form

$$\Phi(gz_0,f,\psi) = c_0(t,w,\mu) + \sum_{\kappa\in\mathbb{Q}^\times} c_\kappa(w,t,r,\mu)e^{2\pi i\kappa r}.$$

Now, only the non-singular part of the lift, $\Phi_2(z,f,\psi)$ transforms under the action of the center of N, while the rank 0 and rank 1 contributions are invariant. Hence, the constant term of the Fourier-Jacobi expansion is given by

$$c_0(t,w,\mu) = \Phi_0(z,f,\psi) + \Phi_1(z,f,\psi),$$

and all other terms, for $\kappa > 0$, come from $\Phi_2(z,f,\psi)$. In this case, $\kappa \neq 0$ is given by (possibly a constant multiple of) $a\alpha$. Hence,

$$\begin{aligned} &c_\kappa(t,w,r,\mu) = \\ &\sum_{\underline{\gamma},\underline{\delta}}\sum_b\sum_n\sum_\lambda e\left(-\bar{\lambda}_2\beta'\right)\left[a^+(\lambda,n)\mathcal{F}(\hat{g})\circ\phi_2^{\underline{\gamma},\underline{\delta}}(n,\eta)^+ + a^-(\lambda,n)\mathcal{F}(\hat{g})\circ\phi_2^{\underline{\gamma},\underline{\delta}}(n,\eta)^-\right]\end{aligned} \tag{4.4.3}$$

These terms, as well as the rank 1 contribution to $c_0(t,w,\mu)$, can be obtained by applying the group operation to the results from Theorem 4.9.

Theorem 4.14. *For $z \in \mathbb{D}$, denote by $g_z \in G$ an element with $g_z z_0 = z$, and let t,w,r,μ be the parameters of its NAM decomposition, i.e. $g_z = n(w,0)n(0,r)a(t)\mu$. Then, the Fourier-Jacobi expansion of the singular theta lift $\Phi(z,f,\psi)$ for a weak harmonic Maass form $f \in \mathrm{H}^+_{k,L^-}$ is given by*

$$\frac{2^{2(q-1)}}{2i(-1)^{q-1}}\Phi(z,f,\psi) = \frac{2^{2(q-1)}}{2i(-1)^{q-1}}\Phi(g_z z_0,f,\psi) = c_0(t,w,\mu) + \sum_{\kappa\in\mathbb{Q}^\times} c_\kappa(t,w,\mu)e^{2\pi i\kappa r},$$

where for $\kappa \neq 0$ the Fourier-Jacobi coefficients can be written in the form

$$c_\kappa(t,w,\mu) = \sum_{\substack{a,\alpha \\ a\alpha=\kappa}} \sum_{\underline{\gamma},\underline{\delta}} \Big(\sum_b \sum_n \sum_\lambda A_\kappa^{\underline{\gamma},\underline{\delta}}\left(n,\lambda,\left(\begin{smallmatrix} a & b \\ 0 & \alpha \end{smallmatrix}\right)\right)(t,w,\mu) e(-\bar{\lambda}_2\beta') \Big) \otimes \Omega_{q-1}(\underline{\gamma},\underline{\delta}), \tag{4.4.4}$$

while the constant coefficient $c_0(t,w,\mu)$ consists of a contribution of rank 1 terms, which can be written in the form

$$\sum_{\underline{\gamma},\underline{\delta}} \Big(\sum_{a,b} \sum_n \sum_\lambda B^{\underline{\gamma},\underline{\delta}}\left(n,\lambda,\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\right)(t,w,\mu) e(-\bar{\lambda}_2\beta') \Big) \otimes \Omega_{q-1}(\underline{\gamma},\underline{\delta}) \tag{4.4.5}$$

and, if $p > 1$, a contribution of the 0-orbit, which we omit. (However, see Corollary 4.15 for the case of signature $(p,1)$.)

The coefficients in (4.4.5) and (4.4.4) are given as follows[2]*: For the rank 1 contributions to the constant term one has (as usual $\beta' = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$)*

$$\begin{aligned} &B^{\underline{\gamma},\underline{\delta}}(n,\lambda,\beta')(t,w,\mu) = \\ &\quad \left(-i\sqrt{\pi}\right)^{n_\gamma+n_\delta} t^{n_\gamma+n_\delta+2} \beta'^{n_\gamma} \bar{\beta}'^{n_\delta} \sum_{\ell=0}^{2q-2-n_\gamma-n_\delta} 2^{\frac{\ell}{2}+1} P_{\tilde{\gamma},\tilde{\delta},\ell}(x_{0,+}) e\left(-\operatorname{Re}\left(\beta'(x_0,w)\right)\right) \cdot \\ &\quad \cdot \Big[a^+(\lambda,n) B_n^+(\beta';t) + a^-(\lambda,n) B_n^-(\beta';t) \Big], \end{aligned} \tag{4.4.6}$$

wher $B_n^+(\beta';t)$ and $B_n^-(\beta',t)$ denote contributions of the holomorphic and the non-holomorphic terms, respectively. Setting $\nu = \frac{1}{2}(\ell - n_\gamma + n_\delta)$, they are given by

$$\begin{aligned} B_n^+(\beta';t) &= t^\nu |\beta'|^\nu \left(2|(x_{0,-},x_{0,-})|\right)^{-\frac{\nu}{2}} K_\nu\left(2\sqrt{2}\pi t|\beta'||(x_{0,-},x_{0,-})|^{\frac{1}{2}}\right), \\ B_n^-(\beta';t) &= \tfrac{1}{2} a(\lambda,n)^- \mathcal{V}_{p+q,1-\nu}\left(2\pi(x_{0,-},x_{0,-}), \pi t^2|\beta'|^2, 4\pi|n|\right) \quad (n \neq 0), \\ B_0^-(\beta';t) &= t^{\nu-k+1}|\beta'|^{\nu-k+1} \left(2|(x_{0,-},x_{0,-})|\right)^{\frac{k-\nu-1}{2}} \cdot K_{\nu-k+1}\left(2\sqrt{2}\pi t|\beta'||(x_{0,-},x_{0,-})|^{\frac{1}{2}}\right). \end{aligned}$$

The coefficients for $\kappa \in \mathbb{Q}^\times$, coming from the contributions of the rank 2 terms are given as follows (with $\eta = \left(\begin{smallmatrix} a & b \\ 0 & \alpha \end{smallmatrix}\right)$):

$$\begin{aligned} &A_\kappa^{\underline{\gamma},\underline{\delta}}(n,\lambda,\eta) = \\ &\quad \left(-i\sqrt{\pi}\right)^{n_\gamma+n_\delta} t^{n_\gamma+n_\delta+2} \sum_{\ell=0}^{2q-2-n_\gamma-n_\delta} 2^{\frac{\ell+1}{2}} P_{\tilde{\gamma},\tilde{\delta},\ell}(x_{0,+}-\beta w) \sum_{M=0}^{n_\gamma+n_\delta} R_{n_\delta,n_\gamma}(\eta,M) \sum_{j=0}^{\lfloor \frac{M}{2} \rfloor} \frac{1}{\pi^j} \\ &\quad \cdot \sum_{h=0}^{M-2j} \frac{i^h}{2^{h+3j}} \mathrm{A}_{t\eta}^h(x_0-\beta w) \left(-\mathbf{C}_\eta\right)^{M-2j-h} t^{-2h-2j} |\beta|^{-2h-2j} \binom{M-2j}{h} \frac{M!}{j!(M-2j)!} \\ &\quad \cdot e\Big(-\mathbf{C}_\eta\left((x_0,x_0)+n\right) + 2\alpha\Im(x_0,w)\Big) \cdot \Big[a^+(\lambda,n) A_n^+(\eta;t,w) + a^-(\lambda,n) A_n^-(\eta;t,w)\Big], \end{aligned} \tag{4.4.7}$$

[2]Since the notation is already quite heavy, we have suppressed the μ-dependence here, writing x_0 instead of $\mu'^{-1}x_0$.

wherein $A_n^+(\eta;t,w)$ *and* $A_n^-(\eta;t,w)$ *denote contribution, which come from the holomorphic and non-holomorphic terms, and are given by*

$$A_n^+(\eta;t,w) = t^{2\nu}\left(\frac{\mathbb{A}_{t\eta}(x_0-\beta w)}{\mathbf{B}_\eta}\right)^{-\frac{\nu}{2}} K_\nu\left(\frac{2\pi}{|\beta|^2}\left(\mathbb{A}_{t\eta}(x_0-\beta w)\mathbf{B}_\eta\right)^{\frac{1}{2}}\right),$$

$$A_n^+(\eta;t,w) = \tfrac{1}{2}\mathcal{V}_{p+q,1-\nu}\left(\pi\left(\frac{\mathbb{A}_{t\eta}(x_0-\beta w)}{t^2|\beta|^2}-2n\right),\frac{\pi}{t^2|\beta|^2}\mathbf{B}_{t\eta},4|n|\pi\right) \quad (n\neq 0),$$

$$A_0^+(\eta;t,w) = t^{2(\nu-k+1)}\left(\frac{\mathbb{A}_{t\eta}(x_0-\beta w)}{\mathbf{B}_\eta}\right)^{-\frac{\nu-k+1}{2}} K_{\nu-k+1}\left(\frac{2\pi}{|\beta|^2}\left(\mathbb{A}_{t\eta}(x_0-\beta w)\mathbf{B}_\eta\right)^{\frac{1}{2}}\right).$$

Proof. After setting

$$A_k^{\gamma,\delta}(n,\lambda,\eta)(t,w,r,\mu) := a^+(\lambda,n)\mathcal{F}(\hat{g})\circ\phi_2^{\gamma,\delta}(n,\eta)^+ + a^-(\lambda,n)\mathcal{F}(\hat{g})\circ\phi_2^{\gamma,\delta}(n,\eta)^-,$$

$$B^{\gamma,\delta}_{-}(n,\lambda,\eta)(t,w,\mu) := a^+(\lambda,n)\mathcal{F}(\hat{g})\circ\phi_1^{\gamma,\delta}(n,\eta)^+ + a^-(\lambda,n)\mathcal{F}(\hat{g})\circ\phi_1^{\gamma,\delta}(n,\eta)^-,$$

the results follow directly from Theorem 4.9 and the consideration concerning the group operation on the terms $\phi_i(n,\eta)^\pm$ from the beginning of this Section (see pp. 119), including the factor of t^2 from the intertwining operation of $a(t)$ from Lemma A.2.3. □

Corollary 4.15. *In signature* $(p,1)$ *the Fourier-Jacobi expansion of* $\frac{1}{2i}\Phi(z,f,\psi)$ *takes the form*

$$c_0(t,w) + \sum_{\kappa\in\mathbb{Q}^\times} c_\kappa(t,w)e^{2\pi i\kappa\operatorname{Re}\tau_\ell},$$

where the constant term $c_0(t,w)$ *is given by*

$$c_0(t,w) = 4\pi I_0 + t^2\sum_{\beta'=(a,b)}\sum_\lambda\Bigg[a^-(\lambda,0)\frac{1}{(2\pi t^2|\beta'|)^{p+1}}\Gamma(p+1) +$$

$$\sum_{n\neq 0}\Bigg(a^+(\lambda,n)\frac{1}{2\pi t^2|\beta'|^2} + a^-(\lambda,n)(p-1)!\sum_{r=0}^{p-1}\frac{\pi^r}{r!}(4|n|)^{\frac{r}{2}-\frac{1}{4}}t^{r-\frac{1}{2}}|\beta'|^{r-\frac{1}{2}}$$

$$\cdot\, h_r\left(\frac{1}{4\pi t|\beta'||n|^{\frac{1}{2}}}\right)e^{-4\pi t|\beta'||n|^{\frac{1}{2}}}\Bigg)\Bigg]\cdot e\Big(-\lambda_2\beta' - \operatorname{Re}\left(\beta'(x_0,w)\right)\Big),$$

with a rational[3] *constant* I_0*, which can be evaluated using the methods of [2], see [40]. The coefficients* $c_\kappa(t,w)$ $(\kappa>0)$ *take the form*

$$c_\kappa(t,w) = \sum_{a,b}\sum_m\sum_\lambda A_\kappa(n,\lambda,[\beta,\beta'])(t,w)e^{-2\pi i\lambda_2\beta'}.$$

wherein

$$A_\kappa(n,\lambda,[\beta,\beta'])(t,w) = \left(a^+(\lambda,n)\frac{\sqrt{2}}{2}\frac{t|\beta|}{\mathbf{B}_\eta^{\frac{1}{2}}} + a^-(\lambda,n)A_n^-(\eta;t,w)\right)$$

$$\cdot\exp\left(-2\frac{\pi}{|\beta|^2}|\mathbf{A}_{t\eta}(x_0-\beta w)|\left(\tfrac{1}{2}\mathbf{B}_\eta\right)^{\frac{1}{2}} - 2\pi i\big[\mathbf{C}_\eta\left((x_0,x_0)+n\right)+2\alpha\Im(x_0,w)\big]\right),$$

[3] In fact, I_0 is an integer for $f\in \mathrm{M}^!_{k,L^-}$.

with a non-holomorphic term $A_n^-(\eta; t, w)$, given by

$$\sqrt{2}(p-1)! \sum_{r=0}^{p-1} \frac{(4|n|\pi)^r}{r!} t^{2r+1+2} |\beta| \mathbf{B}_\eta^{\frac{r-1}{2}} \left(\tfrac{1}{2}\mathbf{A}_{t\eta}^2(x_0 - \beta w) - 2t^2|\beta|^2 (2|n| - n)\right)^{-\frac{r}{2}}$$
$$\cdot\, h_{\max\{0,r-1\}} \left(\frac{|\beta|^2}{2\pi} \left(\left(\tfrac{1}{2}\mathbf{A}_{t\eta}(x_0 - \beta w) + 2t^2|\beta|^2(2|n| - n)\right)\mathbf{B}_\eta\right)^{-\frac{1}{2}} \right).$$

Proof. Follows directly from Example 4.11 and Remark 4.13 after taking into account the factors of t^2 from the intertwining operation of $a(t)$. □

The case of signature $(1, q)$**.** In this case where $p = 1$, there is no contribution to from the 0 orbit, since $\widehat{\psi}_{1,q}((0, x_0), \tau) = 0$. In other words, $\Phi_0(z, f, \psi) \equiv 0$. Hence, Theorem 4.14 gives the complete Fourier-Jacobi expansion in this case. Thus, with Example 4.12 we get the following.

Corollary 4.16. *In signature $(1, q)$, the singular theta lift of a weak harmonic Maass form $f \in \mathrm{H}_{k,L^-}$ with Fourier coefficients $a^+(\lambda, m)$ and $a^-(\lambda, m)$ has the following Fourier-Jacobi expansion*

$$\frac{2^{2(q-1)}}{2i(-1)^{q-1}} \Phi(z, f, \psi) = \left(c_0(t, w) + \sum_{\kappa \in \mathbb{Q}^\times} c_\kappa(t, w) e^{2\pi i \kappa r} \right) \otimes \Omega_{q-1,q-1}(\underline{1}, \underline{1}),$$

with

$$c_0(t, w) = \sum_{ab} \sum_{n} \sum_{\lambda} B^{\underline{1},\underline{1}}\left(m, \lambda, \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\right)(t, w) e\left(-\lambda_2(a + ib)\right) \quad \textit{and}$$
$$c_\kappa(t, w) = \sum_{\substack{a,\alpha \\ a\alpha = \kappa}} \sum_{b} \sum_{m} \sum_{\lambda} A_\kappa^{\underline{1},\underline{1}}\left(m, \lambda, \left(\begin{smallmatrix} a & b \\ & \alpha \end{smallmatrix}\right)\right)(t, w) e\left(-\lambda_2(a + ib)\right) \quad (\kappa \neq 0).$$

Herein, the coefficients $B^{\underline{1},\underline{1}}$ are given by

$$B^{\underline{1},\underline{1}}(n, \lambda, \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right))(t, w, \mu) = (-\pi)^{q-1} \Big[a(\lambda, n)^+ t^q |\beta'|^{q-2} (2|n|)^{\frac{q}{2}} K_q \left(2\sqrt{2}\pi|\beta'||n|^{\frac{1}{2}}\right)$$
$$+\, a(\lambda, n)^- t^{2(q-1)} |\beta|^{2(q-1)} \mathcal{V}_{q+1,1-q}\left(2\pi Q\left(x_{0,-}\right), \pi t^2 |\beta'|^2, 4\pi|n|\right) \Big] \left(-\operatorname{Re}\left(\beta'(x_0, w)\right)\right)$$

for $n \neq 0$ and by

$$B^{\underline{1},\underline{1}}(0, \lambda, \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right))(t, w, \mu) = (-\pi)^{q-1} 2t^q |\beta'|^{q-2} e\left(-\operatorname{Re}\left(\beta'(x_0, w)\right)\right)$$
$$\left[a(\lambda, 0)^+ |Q\left(x_{0,-}\right)|^{\frac{q}{2}} K_q \left(2^{\frac{3}{2}}\pi|\beta'||Q\left(x_{0,-}\right)|^{\frac{1}{2}}\right) + a(\lambda, 0)^- t^q |\beta|^q K_0 \left(2^{\frac{3}{2}}\pi|\beta'||Q\left(x_{0,-}\right)|^{\frac{1}{2}}\right) \right]$$

for $n = 0$.

The coefficients $A^{1,1}_{\kappa}$ are given as follows (with $\eta = \left(\begin{smallmatrix} a & b \\ & \alpha \end{smallmatrix}\right)$):

$$A^{1,1}_{\kappa}(n,\lambda,\eta) = (-\pi)^{q-1} t^{2(q-1)} \frac{\sqrt{2}}{2} \sum_{M=0}^{2(q-1)} R_{q-1,q-1}(\eta, M)$$
$$\cdot \sum_{j=0}^{\lfloor \frac{M}{2} \rfloor} \frac{1}{\pi^j} \sum_{h=0}^{M-2j} \frac{i^h}{2^{h+3j}} \mathbb{A}^h_{t\eta}(x_0 - \beta w) \left(-\mathbf{C}_\eta\right)^{M-2j-h} t^{2h-2j} |\beta|^{-2h-2j} \binom{M-2j}{h} \frac{M!}{j!(M-2j)!}$$
$$\cdot \left[a^+(\lambda,n) A^+_n(\eta;t,w) + a^-(\lambda,n) A^-_n(\eta;t,w)\right] \cdot e\Big(-\mathbf{C}_\eta\left((x_0,x_0)+n\right) + 2\alpha\Im(x_0,w)\Big),$$

where A^+_n, the contribution from the holomorphic part, is given by

$$A^+_n(\eta;t,w) = t^{2\nu-1}\left(\mathbb{A}_{t\eta}(x_0-\beta w)\right)^{-\frac{\nu}{2}-\frac{1}{4}} \mathbf{B}_\eta{}^{\frac{\nu}{2}-\frac{1}{4}} \exp\left(-\frac{2\pi}{|\beta|^2}\left(\mathbb{A}_{t\eta}(x_0-\beta w)\mathbf{B}_\eta\right)^{\frac{1}{2}}\right)$$
$$\cdot h_{\nu'}\left(\frac{|\beta|^2}{2\pi\left(\mathbb{A}_{t\eta}(x_0-\beta w)\mathbf{B}_\eta\right)^{\frac{1}{2}}}\right)$$

with the index ν given by $h+j+1-q$ and $\nu' = |\nu| - \frac{1}{2} = \max(q-1-h-j, h+j-q)$. The contribution of the non-holomorphic part A^-_n is given by

$$A^-_n(\eta;t,w) = (q-1)! \sum_{r=0}^{q-1} \frac{(4\pi|n|)^r}{r!} \left(\mathbb{A}_{t\eta}(x_0-\beta w) + 2t^2|\beta|^2(2|n|-n)\right)^{-\frac{\nu}{2}-\frac{r}{2}-\frac{1}{4}}$$
$$\cdot \left(t^4 \mathbf{B}_\eta\right)^{\frac{\nu}{2}+\frac{r}{2}-\frac{1}{4}} \exp\left(-\frac{2\pi}{|\beta|^2}\left(\mathbb{A}_{t\eta}(x_0-\beta w) + 2t^2|\beta|^2(2|n|-n)\right)^{\frac{1}{2}} \mathbf{B}_\eta^{\frac{1}{2}}\right)$$
$$\cdot h_{\nu''}\left(\frac{|\beta|^2}{2\pi}\left(\mathbb{A}_{t\eta}(x_0-\beta w) + 2t^2|\beta|^2(2|n|-n)\right)^{-\frac{1}{2}} \mathbf{B}_\eta^{-\frac{1}{2}}\right),$$

with $\nu'' = |\nu + r| - \frac{1}{2}$ for $n \neq 0$. For $n = 0$, we have

$$A^-_0(\eta;t,w) =$$
$$t^{2\nu-k+1}\left(\mathbb{A}_{t\eta}(x_0-\beta w)\right)^{-\frac{\nu}{2}+\frac{k}{2}-\frac{1}{4}} \mathbf{B}_\eta{}^{\frac{\nu}{2}-\frac{k}{2}-\frac{1}{4}} \cdot h_{|\nu-k|-\frac{1}{2}}\left(\frac{|\beta|^2}{2\pi\left(\mathbb{A}_{t\eta}(x_0-\beta w)\mathbf{B}_\eta\right)^{\frac{1}{2}}}\right).$$

Proof. Follows immediately from Theorem 4.14 and Example 4.12. □

4.5. Calculation of the unfolding integrals

In this section, we will evaluate the unfolding integrals for the theta lift, providing all the Lemmas used in the proof of Theorem 4.9 above. We use the notation introduced in Section 4.3 (in particular, see Notation 4.8).

4.5.1. Non-singular terms

First, we calculate the contribution of the terms where the matrix $\eta = [\beta, \beta']$ is non-singular. By Lemma 4.6 a set of representatives for these η under the operation of $\mathrm{SL}_2(\mathbb{Z})$ is given by the rational matrices

$$\begin{pmatrix} a & b \\ 0 & \alpha \end{pmatrix} \in \mathrm{M}_2(\mathbb{Q}) \quad \text{with} \quad a > 0,\ \alpha \neq 0 \text{ and } b \pmod{a\mathbb{Z}}.$$

Since the stabilizer is trivial, for fixed η, the unfolding integrals take the form

$$\begin{aligned}
&\sum_n \left(\hat{a}^-(n)\phi_2(n,\eta)^-(z_0) + \hat{a}^+(n)\phi_2(n,\eta)^+(z_0)\right) = \\
&= \sum_n \hat{a}^+(n) \int_{\mathbb{H}}^{reg} \widehat{\psi_{p,q}}\left(\sqrt{2}(\eta, x_0), \tau, z_0\right) e^{2\pi i n u} e^{-2\pi n v} v^{-s-2} du\, dv \\
&+ \sum_n \hat{a}^-(n) \int_{\mathbb{H}}^{reg} \sum_{\gamma \in \Gamma_\eta \backslash \Gamma} \widehat{\psi_{p,q}}(\sqrt{2}(\eta, x_0), \tau, z_0) e^{2\pi i n u} \Gamma(1-k, 4\pi|n|v) v^{-s-2} du\, dv.
\end{aligned}$$

Now, the inner integral, over u ranging through $\mathbb{R}$, is simply a Fourier-transform, which we shall calculate first. For the outer integral over v, ranging over $\mathbb{R}_{>0}$, we will make use of the integral representations of Bessel functions (cf. Appendix B.1).

Preliminary lemmas

Let us first gather some lemmas. The first three will help us to calculate the Fourier transform.

Lemma 4.17. *The Fourier transform*

$$\int_{-\infty}^{\infty} e^{-\frac{2\pi}{v}\left(|\beta'|^2 + |\tau|^2|\beta|^2 + 2u\,\mathrm{Re}\left(\beta'\bar{\beta}\right)\right) + 2\pi i u (x_0, x_0) + 2\pi v((x_{0,-}, x_{0,-}) - (x_{0,+}, x_{0,+}))} e^{2\pi i n u} du,$$

with $n \neq 0$ is given by

$$e^{+2\pi n v} \frac{v^{\frac{1}{2}}}{\sqrt{2}|\beta|} \exp\left(-\frac{v\pi}{|\beta|^2}\mathbb{A}_\eta - \frac{\pi}{v|\beta|^2}\mathbf{B}_\eta\right) e\big(-\mathbf{C}_\eta\left((x_0, x_0) + n\right)\big),$$

with $\mathbb{A}_\eta = \frac{1}{2}\left(n + |\beta|^2 + (x_0, x_0)\right)^2 - 4(x_{0,+}, x_{0,+})|\beta|^2$, $\frac{1}{2}\mathbf{B}_\eta = |\beta'|^2|\beta|^2 - \mathrm{Re}\left(\beta'\bar{\beta}\right)^2$ and $\mathbf{C}_\eta = \mathrm{Re}\left(\beta'\bar{\beta}\right)|\beta|^{-2}$ from (4.3.2) and (4.3.3).

Proof. With (B.2.1) setting

$$\begin{aligned}
A &= \frac{i|\beta|^2}{v}, \quad B = \frac{2i}{v}\mathrm{Re}(\beta'\bar{\beta}) + (x_0, x_0) \quad \text{and} \\
C &= i\Big[\frac{|\beta'|^2}{v} + v|\beta|^2 + v\left((x_{0,+}, x_{0,+}) - (x_{0,-}, x_{0,-})\right)\Big],
\end{aligned} \tag{4.5.1}$$

one obtains the Fourier transform

$$\frac{\sqrt{v}}{\sqrt{2}|\beta|}\exp\left(2\pi i\left(-\frac{n^2}{4A}-\frac{nB}{2A}-\frac{B^2}{4A}+C\right)\right)=$$
$$\frac{\sqrt{v}}{\sqrt{2}|\beta|}\exp\Bigg(-\pi v\left[\frac{n^2}{2|\beta|^2}+\frac{n(x_0,x_0)}{|\beta|^2}+\frac{(x_0,x_0)^2}{2|\beta|^2}+2\left((x_{0,+},x_{0,+})-(x_{0,-},x_{0,-})\right)+2|\beta|^2\right]$$
$$-\frac{\pi}{v}\left[2|\beta'|^2-\frac{2\operatorname{Re}\left(\beta'\bar{\beta}\right)^2}{|\beta|^2}\right]-2\pi i\frac{\operatorname{Re}(\beta'\beta)}{|\beta|^2}\left((x_0,x_0)+n\right)\Bigg).$$

After multiplying with $e^{-2\pi nv}$, one has

$$\frac{\sqrt{v}}{\sqrt{2}|\beta|}\exp\Bigg(-\frac{v\pi}{|\beta|^2}\left[\frac{1}{2}\left(n+2|\beta|^2+(x_0,x_0)\right)^2-4(x_{0,-},x_{0,-})|\beta|^2-2n|\beta|^2\right]-\pi v2n$$
$$-\frac{2\pi}{v|\beta|^2}\left[|\beta'|^2|\beta|^2-\operatorname{Re}\left(\beta'\bar{\beta}\right)^2\right]\Bigg)\cdot e^{-2\pi i\frac{\operatorname{Re}(\beta'\bar{\beta})}{|\beta|^2}((x_0,x_0)+n)}$$
$$=\frac{\sqrt{v}}{|\beta|}e^{-\frac{v\pi}{|\beta|^2}\mathbb{A}_\eta-\frac{\pi}{v|\beta|^2}\mathbf{B}_\eta}e^{-2\pi i\mathbf{C}_\eta((x_0,x_0)+n)},$$

as claimed. □

Lemma 4.18. *The Fourier transform*

$$\int_{-\infty}^{\infty}(\beta'+\bar{\tau}\beta)^{k_1}(\bar{\beta}'+\tau\bar{\beta})^{k_2}e^{-\frac{2\pi}{v}\left(|\beta'|^2+|\beta|^2|\tau|^2+2u\operatorname{Re}\left(\beta'\bar{\beta}\right)\right)+2\pi i(\tau(x_{0,+},x_{0,+})+\bar{\tau}(x_{0,-},x_{0,-}))}e^{2\pi inu}du,$$

is given by

$$e^{+2\pi nv}\frac{\sqrt{v}}{\sqrt{2}|\beta|}\tilde{p}_\eta(v,n)\exp\left(-\frac{v\pi}{|\beta|^2}\mathbb{A}_\eta-\frac{\pi}{v|\beta|^2}\mathbf{B}_\eta\right)e^{-2\pi i\mathbf{C}_\eta((x_0,x_0)+n)},$$

with a polynomial $\tilde{p}_\eta(v,n)$ of the following form

$$\sum_{M=0}^{k_1+k_2}R_{k_1,k_2}(\eta,M)\sum_{j=0}^{\left\lfloor\frac{M}{2}\right\rfloor}\sum_{\kappa=0}^{M-2j}\frac{i^\kappa v^{\kappa+j}\mathrm{A}_\eta(n)^\kappa}{|\beta|^{2\kappa+2j}}\frac{(-\mathbf{C}_\eta)^{M-2j-\kappa}}{2^{\kappa+3j}\pi^j}\frac{M!}{j!(M-2j)!}\binom{M-2j}{\kappa}.$$

wherein

$$R_{k_1,k_2}(\eta,M):=\sum_{\substack{0\le\mu_1<k_1\\0\le\mu_2\le k_2\\\mu_1+\mu_2=M}}\beta^{\mu_1}\bar{\beta}^{\mu_2}\beta'^{k_1-\mu_1}\bar{\beta}'^{k_2-\mu_2}\binom{k_1}{\mu_1}\binom{k_2}{\mu_2}.$$

Note that if $k_1=k_2=k$, the coefficient $R_{k,k}(\eta,M)$ is real and can be written in the form

$$\sum_{\substack{0\le\mu_1,\mu_2<k\\\mu_1+\mu_2=M}}\operatorname{Re}\left(\beta^{\mu_1}\bar{\beta}^{\mu_2}\beta'^{k-\mu_1}\bar{\beta}'^{k-\mu_2}\right)\binom{k}{\mu_1}\binom{k}{\mu_2}.$$

Proof. We need only calculate the contribution of the polynomial, the rest follows from Lemma 4.17. Rewrite the polynomial in the form

$$\begin{aligned}(\beta' + \bar\tau\beta)^{k_1}\left(\bar\beta' + \bar\tau\bar\beta\right)^{k_2} &= \sum_{\mu_1=0}^{k_1}\sum_{\mu_2=0}^{k_2} \bar\tau^{\mu_1+\mu_2}\beta^{\mu_1}\bar\beta^{\mu_2}\beta'^{k-\mu_1}\bar\beta'^{k-\mu_2}\binom{k_1}{\mu_1}\binom{k_2}{\mu_2}\\ &= \sum_{M=0}^{k_1+k_2}\bar\tau^M R_{k_1,k_2}([\beta,\beta'],M),\end{aligned}$$

with $R_{k_1,k_2}([\beta,\beta'],M)$ as above. Setting A, B and C as in (4.5.1) from the proof of Lemma 4.17, by Lemma B.2 one has to apply $\exp\left(\frac{v}{8\pi|\beta|^2}\frac{d^2}{du^2}\right)$ and obtains

$$\sum_{M=0}^{k_1+k_2}\sum_{j=0}^{\lfloor\frac{M}{2}\rfloor}\left[\frac{v}{8\pi|\beta|^2}\right]^j\bar\tau^{M-2j}\frac{M!}{j!(M-2j)!}R_{k_1,k_2}([\beta,\beta'],M),$$

where, further $\bar\tau$ is to be replaced by

$$-\frac{n+B}{2A} - iv = \frac{iv}{2|\beta|^2}\left(n - |\beta|^2 + (x_0,x_0)\right) - \frac{\mathrm{Re}(\beta'\bar\beta)}{|\beta|^2} = \frac{iv}{2|\beta|^2}\mathrm{A}_\eta - \mathbf{C}_\eta.$$

Thus, the polynomial part of the Fourier transform is given by

$$\begin{aligned}\tilde p_\eta(v,n) = \sum_{M=0}^{k_1+k_2} R_{k_1,k_2}(\eta;M)\sum_{j=0}^{\lfloor\frac{M}{2}\rfloor}\frac{1}{\pi^j}\frac{1}{|\beta|^{2(M-j)}}\\ \cdot\sum_{\kappa=0}^{M-2j}\frac{i^\kappa v^{\kappa+j}}{2^{\kappa+3j}}\mathrm{A}_\eta^\kappa\left(-\mathrm{Re}\left(\beta'\bar\beta\right)\right)^{M-2j-\kappa}\binom{M-2j}{\kappa}\frac{M!}{j!(M-2j)!}.\end{aligned}$$

With the definition of $\mathbf{C}_\eta$ this gives the claimed form. ☐

Lemma 4.19. *The integral*

$$\begin{aligned}&\int_{-\infty}^{\infty}(\beta'+\bar\tau\beta)^{k_1}(\bar\beta'+\bar\tau\bar\beta)^{k_2}e\big(u(x_0,x_0)\big)\\ &\cdot\exp\left(-\frac{2\pi}{v}\left(|\beta'|^2 + |\tau|^2|\beta|^2 + 2u\,\mathrm{Re}\left(\beta'\bar\beta\right)\right) + 2\pi v\left((x_{0,-},x_{0,-}) - (x_{0,+},x_{0,+})\right)\right)du,\end{aligned}$$

is given by

$$\frac{v^{\frac{1}{2}}}{\sqrt{2}|\beta|}\tilde p_\eta(v,0)\exp\left(-v\frac{\pi}{|\beta|^2}\mathbb{A}_\eta(0) - \frac{1}{v}\frac{\pi}{|\beta|^2}\mathbf{B}_\eta\right)e^{-2\pi i\mathbf{C}_\eta(x_0,x_0)},$$

where $\tilde p_\eta$ is the polynomial from Lemma 4.18, and $\mathbb{A}_\eta(0)$ is given by

$$\frac{1}{2}\left(2|\beta|^2 + (x_0,x_0)\right)^2 - 4(x_{0,+},x_{0,+})|\beta|^2.$$

Proof. The integral is a Fourier transform. Arguing similarly to the proof of Lemma 4.17, with

$$A = \frac{i|\beta|^2}{v}, \qquad B = \frac{2i}{v}\operatorname{Re}\left(\beta'\bar{\beta}\right)$$

we get the exponential factor

$$\exp\left(-\pi v\left[\frac{(x_0,x_0)^2}{2|\beta|^2} - |\beta|^2 + ((x_{0,-},x_{0,-}) - (x_{0,+},x_{0,+}))\right] - \frac{\pi}{v}\left[\frac{\operatorname{Re}\left(\beta'\bar{\beta}\right)^2}{|\beta|^2} - |\beta'|^2\right]\right) \cdot e\left(-\frac{(x_0,x_0)}{|\beta|^2}\operatorname{Re}\left(\beta'\bar{\beta}\right)\right).$$

The polynomial is calculated as in Lemma 4.18, up to the last step where, now, $\bar{\tau}$ is replaced by

$$-\frac{(x_0,x_0)-B}{2A} - iv = \frac{iv}{2|\beta'|}\left((x_0,x_0) - |\beta|^2\right) - \frac{\operatorname{Re}(\beta'\bar{\beta})}{|\beta|^2} = \frac{iv}{2|\beta|^2}\mathrm{A}_\eta(n=0) - \mathbf{C}_\eta.$$

□

Once the inner integrals are evaluated using the previous lemmas, the following two lemmas allow us to evaluate the outer integrals.

Lemma 4.20. *Let ℓ be an integer. We have the following identity*

$$\begin{aligned}\frac{1}{|\beta|}\int_0^\infty v^{-s-\frac{1}{2}-\ell}\exp\left(-\frac{v\pi}{|\beta|^2}\mathbb{A}_\eta - \frac{\pi}{v|\beta|^2}\mathbf{B}_\eta\right)dv \\ = \frac{2}{|\beta|}\left(\frac{\mathbb{A}_\eta}{\mathbf{B}_\eta}\right)^{\frac{1}{2}\left(s+\ell-\frac{1}{2}\right)} K_{-s+\frac{1}{2}-\ell}\left(\frac{2\pi}{|\beta|^2}\mathbb{A}_\eta^{\frac{1}{2}}\mathbf{B}_\eta^{\frac{1}{2}}\right).\end{aligned}$$

Now, for an integer $k > 0$ denote by h'_k the Bessel polynomial of index k and set $h'_0 = 1$. Then, for $s = 0$, we have if $\ell \leq 0$:

$$\frac{\mathbb{A}_\eta^{\frac{\ell-1}{2}}}{\mathbf{B}_\eta^{\ell/2}} h'_{-\ell}\left(\frac{|\beta|^2}{2\pi\mathbb{A}_\eta^{\frac{1}{2}}\mathbf{B}_\eta^{\frac{1}{2}}}\right)\exp\left(-\frac{2\pi}{|\beta|^2}\mathbb{A}_\eta^{\frac{1}{2}}\mathbf{B}_\eta^{\frac{1}{2}}\right),$$

whereas, if $\ell > 0$, we have

$$\frac{\mathbb{A}_\eta^{\frac{\ell-1}{2}}}{\mathbf{B}_\eta^{\ell/2}} h'_{\ell-1}\left(\frac{|\beta|^2}{2\pi\mathbb{A}_\eta^{\frac{1}{2}}\mathbf{B}_\eta^{\frac{1}{2}}}\right)\exp\left(-\frac{2\pi}{|\beta|^2}\mathbb{A}_\eta^{\frac{1}{2}}\mathbf{B}_\eta^{\frac{1}{2}}\right),$$

Proof. Recall that $\mathbb{A}_\eta$ and $\mathbf{B}_\eta$ are both positive (as β and β' are both non-zero). Thus, the first equality is immediate from (B.1.3), while the second, for $s = 0$ follows by (B.1.5). For the third, use $K_{-\nu} = K_\nu$ from (B.1.4) and argue similarly. □

Lemma 4.21. *Let ℓ be an integer. Then, the value at $s=0$ of the integral*

$$\frac{1}{|\beta|}\int_0^\infty v^{-s+\ell-\frac{1}{2}}\Gamma\left(\kappa-1,4\pi|n|v\right)e^{+2\pi nv}\exp\left(-\frac{\pi}{|\beta|^2}\left\{v\mathbb{A}_\eta+\frac{1}{v}\mathbf{B}_\eta\right\}\right)dv,$$

where $\kappa=p+q-2$, is given by

$$\begin{aligned}&\frac{2}{|\beta|}\kappa!\left(\frac{\mathbb{A}_\eta-2n|\beta|^2+4n|\beta|^2}{\mathbf{B}_\eta}\right)^{\frac{1}{2}\left(\ell-\frac{1}{2}\right)}\sum_{r=0}^{\kappa}\frac{(4\pi|n|)^r}{r!}\\&\cdot\left(\frac{\mathbb{A}_\eta-2n|\beta|^2+4|n||\beta|^2}{\mathbf{B}_\eta}\right)^{-\frac{r}{2}}K_{\frac{1}{2}-\ell+r}\left(\frac{2\pi}{|\beta|^2}\left(\mathbb{A}_\eta-2n|\beta|^2+4|n||\beta|^2\right)^{\frac{1}{2}}\mathbf{B}_\eta^{\frac{1}{2}}\right)\\&\qquad=\frac{1}{|\beta|}\mathcal{V}_{\kappa+2,\frac{1}{2}-\ell}\left(\pi\left(\frac{\mathbb{A}_\eta}{|\beta|^2}-2n\right),\frac{\pi\mathbf{B}_\eta}{|\beta|^2},4\pi|n|\right)\end{aligned}$$

Proof. Follows from Lemma 4.20 with $\mathbb{A}_\eta$ shifted by $-2n|\beta|^2$ and Lemma B.1, which in turn is a consequence of the finite series expansion of the incomplete Gamma function in (B.1.2). ☐

The contribution of the holomorphic terms

Let us now calculate $\phi_{\frac{\gamma,\delta}{2}}(n,\eta)^+$, the contribution due to the lift of the holomorphic part of the weak harmonic Maass form f, with Fourier expansion $f^+=\sum_{n\gg-\infty}\hat{a}^+(n)q^n$. We have

$$\begin{aligned}\phi_{\frac{\gamma,\delta}{2}}(m,\eta)^+&=\left(-i\sqrt{\pi}\right)^{n_\gamma+n_\delta}\sum_{\ell=0}^{r_\gamma+r_\delta}2^{\frac{\ell}{2}}P_{\tilde{\gamma},\tilde{\delta},\ell}(x_{0,+})\\&\cdot\underset{s=0}{\mathrm{CT}}\int_0^\infty v^{\frac{\ell-n_\gamma-n_\delta}{2}-s-2}e^{-2\pi(v(x_{0,+},x_{0,+})+(x_{0,-},x_{0,-}))}e^{2\pi mv}\int_{-\infty}^\infty\left(\beta'+\bar{\tau}\beta\right)^{n_\delta}\left(\bar{\beta}'+\bar{\tau}\bar{\beta}\right)^{n_\gamma}\\&\qquad\cdot\exp\left(-\frac{2\pi}{v}\left(|\beta'|^2+|\bar{\tau}|^2|\beta|^2\right)-2\pi\Im\left(\beta'\bar{\beta}\right)+2\pi iu(x_0,x_0)\right)e^{2\pi inu}du\,dv,\end{aligned}$$

After using Lemmas 4.17 and 4.18 to evaluate the inner integral, the integrand of the outer integral is given by

$$\frac{2^{-\frac{1}{2}}}{|\beta|}v^{\frac{1}{2}(\ell-n_\gamma-n_\delta)-s-\frac{3}{2}}\tilde{p}_\eta(v,n)\exp\left(-\frac{v\pi}{|\beta|^2}\mathbb{A}_\eta-\frac{\pi}{v|\beta|^2}\mathbf{B}_\eta\right)e^{-2\pi i\mathbf{C}_\eta((x_0,x_0)+n)},$$

with $\tilde{p}_\eta(v,n)$ the polynomial from Lemma 4.18, with $k_1=n_\delta$ and $k_2=n_\gamma$. Evaluating with Lemma 4.20, and using (B.1.5), we get the following result:

Lemma 4.22. *For fixed n and η and at the base point z_0, the rank two term $\phi_2^{\gamma,\delta}(n,\eta)^+$ is given by*

$$\begin{aligned}\phi_2^{\gamma,\delta}(n,\eta)^+ = \left(-i\sqrt{\pi}\right)^{n_\gamma+n_\delta} \sum_{\ell=0}^{2q-2-n_\gamma-n_\delta} 2^{\frac{\ell+1}{2}} P_{\tilde{\gamma},\tilde{\delta},\ell}(x_{0,+}) \sum_{M=0}^{n_\gamma+n_\delta} R_{n_\delta,n_\gamma}(\eta,M) \\ \cdot \sum_{j=0}^{\left\lfloor\frac{M}{2}\right\rfloor} \frac{1}{\pi^j} \sum_{h=0}^{M-2j} \frac{i^h}{2^{h+3j}} \mathbb{A}_\eta^h \left(-\mathbf{C}_\eta\right)^{M-2j-h} |\beta|^{-2h-2j} \binom{M-2j}{h} \frac{M!}{j!(M-2j)!} \\ \cdot \left(\frac{\mathbb{A}_\eta}{\mathbf{B}_\eta}\right)^{-\frac{\nu}{2}} K_\nu\left(\frac{2\pi}{|\beta|^2}\left(\mathbb{A}_\eta\mathbf{B}_\eta\right)^{\frac{1}{2}}\right) \exp\left(-2\pi i\mathbf{C}_\eta\left((x_0,x_0)+n\right)\right),\end{aligned} \tag{4.5.2}$$

where $\nu = h+j+\frac{1}{2}(\ell-n_\gamma-n_\delta)-\frac{1}{2}$. Furthermore, if $\nu \equiv \frac{1}{2} \pmod 1$, the K-Bessel functions in the last line can be replaced by

$$\frac{1}{2}|\beta|\left(\mathbb{A}_\eta\mathbf{B}_\eta\right)^{-\frac{1}{4}} h_{|\nu|-\frac{1}{2}}\left(\left(2\pi\left(\mathbb{A}_\eta\mathbf{B}_\eta\right)^{\frac{1}{2}}\right)^{-1}\right) \exp\left(-\frac{2\pi}{|\beta|^2}\left(\mathbb{A}_\eta\mathbf{B}_\eta\right)^{\frac{1}{2}}\right),$$

where h_n denotes the nth Bessel polynomial.

The contribution of the non-holomorphic part

Now, we calculate $\phi_2^{\gamma,\delta}(n,\eta)^-$, the contribution due to the non-holomorphic part f^- of a weak harmonic Maass form, with a Fourier expansion of the form

$$f^-(\tau) = \hat{a}^-(0)v^{1-k} + \sum_{\substack{n\in\mathbb{Q}\\ n\neq 0}} \hat{a}^-(n)\Gamma\left(1-k, 4\pi|n|v\right) e^{2\pi inu}.$$

Let us briefly examine the contribution due to the constant term. Using Lemma 4.19 to evaluate the inner integral over u, we get

$$\frac{2^{-\frac{1}{2}}}{|\beta|} v^{-\frac{1}{2}-k+\frac{1}{2}(\ell-n_\gamma-n_\delta)-s} \tilde{p}_\eta(v,0) \exp\left(-\frac{\pi}{|\beta^2|}\left(v\mathbb{A}_\eta(0)-\frac{1}{v}\mathbf{B}_\eta\right)\right) e^{-2\pi i\mathbf{C}_\eta(x_0,x_0)}$$

as the integrand of the integral over v, which can be evaluated exactly like for the holomorphic terms, with $n=0$ throughout and index shifted by $-(k-1)$.

Terms with $n \neq 0$: The argumentation is similar, as previously for the holomorphic part. The inner integral, over u, is evaluated exactly as before. The integrand for the integral over v takes the form

$$\frac{2^{-\frac{1}{2}}}{|\beta|} v^{\frac{1}{2}(\ell-n_\gamma-n_\delta)-s-\frac{3}{2}} \tilde{p}_\eta(v,n)\Gamma\left(k+1, 4\pi|n|v\right) e^{2\pi nv} e^{-\frac{v\pi}{|\beta|^2}\mathbb{A}_\eta - \frac{\pi}{v|\beta|^2}\mathbf{B}_\eta} e^{-2\pi i\mathbf{C}_\eta((x_0,x_0)+n)},$$

with $\tilde{p}_\eta(v,n)$ from Lemma 4.18. The integral is now evaluated using Lemma 4.21, yielding

Lemma 4.23. *Let $\kappa = p+q-2$. For fixed n and η and at the base point z_0, the contribution of the rank two orbit to the lift of the non-holomorphic part f^- of f is given by*

$$\begin{aligned}\phi_2^{\gamma,\delta}(m,\eta)^- = \left(-i\sqrt{\pi}\right)^{n_\gamma+n_\delta} &\sum_{\ell=0}^{2q-2-n_\gamma-n_\delta} 2^{\frac{\ell+1}{2}} P_{\tilde{\gamma},\tilde{\delta},\ell}(x_{0,+}) \sum_{M=0}^{n_\gamma+n_\delta} R_{n_\delta,n_\gamma}(\eta,M)\\ \cdot \sum_{j=0}^{\lfloor \frac{M}{2}\rfloor} \frac{1}{\pi^j} \sum_{h=0}^{M-2j} &\frac{i^h}{2^{h+3j}} \mathrm{A}_\eta^h \left(-\mathbf{C}_\eta\right)^{M-2j-h} |\beta|^{-2h-2j} \binom{M-2j}{h} \frac{M!}{j!(M-2j)!}\\ \cdot \mathcal{V}_{\kappa+2,\frac{3}{2}-\nu} &\left(\pi\left(\frac{\mathbb{A}_\eta}{|\beta|^2}-2n\right), \frac{\pi\mathbf{B}_\eta}{|\beta|^2}, 4\pi|n|\right) \exp\left(-2\pi i \mathbf{C}_\eta\left((x_0,x_0)+n\right)\right).\end{aligned} \tag{4.5.3}$$

where $\nu = h+j+\frac{1}{2}(\ell-n_\gamma-n_\delta)-\frac{1}{2}$.

We recall that by the definition of the special function $\mathcal{V}_{n,\nu}(A,B,c)$ we have

$$\begin{aligned}\mathcal{V}_{\kappa+2,\frac{3}{2}-\nu}\left(\pi\left(\frac{\mathbb{A}_\eta}{|\beta|^2}-2n\right), \frac{\pi\mathbf{B}_\eta}{|\beta|^2}, 4\pi|n|\right) &= \kappa!\sum_{r=0}^{\kappa}\frac{(4\pi|n|)^r}{r!}\\ \cdot\left(\frac{\mathbb{A}_\eta+4|n||\beta|^2-2n|\beta|^2}{\mathbf{B}_\eta}\right)^{-\frac{\nu}{2}-\frac{r}{2}} &K_{r+\nu}\left(\frac{2\pi}{|\beta|^2}\left(\left(\mathbb{A}_\eta+4|n||\beta|^2-2n|\beta|^2\right)\mathbf{B}_\eta\right)^{\frac{1}{2}}\right).\end{aligned}$$

Further, in every term where $\nu+r$ is a half-integer, one can set $\nu' = |r+\nu|-\frac{1}{2}$ and replace $K_{r+\nu}$ with

$$\begin{aligned}\frac{1}{2}|\beta|\left(\left(\mathbb{A}_\eta+4|n||\beta|^2-2n|\beta|^2\right)\mathbf{B}_\eta\right)^{-\frac{1}{4}} h_{\nu'}\left(\left(2\pi\left(\left(\mathbb{A}_\eta+4|n||\beta|^2-2n|\beta|^2\right)\mathbf{B}_\eta\right)^{\frac{1}{2}}\right)^{-1}\right)\\ \cdot\exp\left(-\frac{2\pi}{|\beta|^2}\left(\left(\mathbb{A}_\eta+4|n||\beta|^2-2n|\beta|^2\right)\mathbf{B}_\eta\right)^{\frac{1}{2}}\right).\end{aligned}$$

4.5.2. Rank one terms

Now, let consider the case where $\eta = [\beta,\beta']$ is of rank 1. Recall from Lemma 4.6 that a set of representatives for the orbit under $\mathrm{SL}_2(\mathbb{Z})$ operation is given by

$$[0,\beta'] = \begin{pmatrix} 0 & a\\ 0 & b\end{pmatrix}, \quad \text{with either } \alpha>0 \text{ or } a=0 \text{ and } b>0.$$

Further, the stabilizer in this case is $\mathrm{SL}_2(\mathbb{Z})_\infty$. So the domain of integration is given by

$$\Gamma'_\infty\backslash\Gamma' = \{u+iv\,;\ 0<u<1, 0<v<\infty\}.$$

Again, we begin with the contribution of the holomorphic part.

Holomorphic part

For a fixed pair of multi-indices $\underline{\gamma}$, $\underline{\delta}$, and fixed $\eta = [0, \beta']$, the rank one term $\phi_1^{\underline{\gamma},\underline{\delta}}(n,\eta)^+$ is given by the integral

$$\phi_1^{\underline{\gamma},\underline{\delta}}(n,\eta)^+ = (-i)^{n_\gamma+n_\delta}\pi^{\frac{n_\gamma+n_\delta}{2}}\beta'^{n_\delta}\bar{\beta}'^{n_\gamma}\sum_{\ell=0}^{2q-2-n_\delta-n_\gamma} 2^{\frac{\ell}{2}}P_{\tilde{\gamma},\tilde{\delta},\ell}(x_{0,+})$$
$$\cdot \operatorname*{CT}_{s=0}\int_{\mathbb{R}_{>0}}\int_0^1 v^{\frac{1}{2}(\ell-n_\gamma-n_\delta)-2-s}e^{-\frac{2\pi}{v}|\beta'|^2}e^{2\pi iu(x_0,x_0)-2\pi v(Q(x_{0,+})-Q(x_{0,-}))}e^{2\pi in\tau}du\,dv.$$

The integral over u just picks out the constant term. Hence, $n = -(x_0, x_0)$ for all non-vanishing contributions. Now, $(x_{0,-}, x_{0,-}) - (x_{0,+}, x_{0,+}) - n = 2(x_{0,-}, x_{0,-})$. Since the norm of $x_{0,-}$ is negative, the integrals over v take the form

$$\operatorname*{CT}_{s=0}\int_0^\infty v^{\frac{1}{2}(\ell-n_\gamma-n_\delta)-2-s}\exp\left(-4\pi v|(x_{0,-},x_{0,-})|-\frac{2\pi}{v}|\beta'|^2\right)dv. \tag{4.5.4}$$

If $Q(x_{0,-}) \neq 0$, by the integral representation of the Bessel-functions (B.1.3), setting $\nu = \frac{1}{2}(\ell - n_\gamma - n_\delta) - 1$ and evaluating at $s = 0$, one obtains

$$2|\beta'|^\nu\left(2|(x_{0,-},x_{0,-})|\right)^{-\frac{\nu}{2}}K_\nu\left(2\pi|\beta'|\left|\sqrt{2(x_{0,-},x_{0,-})}\right|\right).$$

Further, if ν is a half-integer, by (B.1.5), this equals

$$|\beta'|^{\nu-\frac{1}{2}}\left(2|(x_{0,-},x_{0,-})|\right)^{-\frac{\nu}{2}-\frac{1}{4}}h_{|\nu|-\frac{1}{2}}\left(\left(2\sqrt{2}\pi|\beta'||(x_{0,-},x_{0,-})|^{\frac{1}{2}}\right)^{-1}\right)e^{-2\sqrt{2}\pi|\beta'||(x_{0,-},x_{0,-})|^{\frac{1}{2}}},$$

with the Bessel polynomial $h_{\nu'}$, $\nu' = |\nu| - \frac{1}{2}$. If $Q(x_{0,-}) = 0$ the integral (4.5.4) can be evaluated using the integral representation of Γ-functions, see Example 4.11.

Lemma 4.24. *For fixed n and fixed $\eta = [0, \beta']$, at the base point z_0, and assuming that $Q(x_{0,-}) \neq 0$, the rank one term is given by*

$$\phi_1^{\underline{\gamma},\underline{\delta}}(n,\eta)^+(z_0) = \left(-i\sqrt{\pi}\right)^{n_\gamma+n_\delta}\beta'^{n_\gamma}\bar{\beta}'^{n_\delta}\sum_{\ell=0}^{2q-2-n_\gamma-n_\delta}2^{\frac{\ell}{2}}P_{\tilde{\gamma},\tilde{\delta},\ell}(x_{0,+})|\beta'|^\nu 2^{\frac{\nu}{2}}|(x_{0,-},x_{0,-})|^{\frac{\nu}{2}}$$
$$\cdot\begin{cases} 2K_\nu\left(2\sqrt{2}\pi|\beta'||(x_{0,-},x_{0,-})|^{\frac{1}{2}}\right), \\ \qquad \text{if } \nu = \frac{1}{2}(\ell-n_\gamma-n_\delta)-1 \equiv 0 \pmod 1 \\ \left(2(x_{0,-},x_{0,-})|\beta'|^2\right)^{-\frac{1}{4}}h_{\nu'}\left(\frac{1}{2\pi|\beta'|}|2(x_{0,-},x_{0,-})|^{-\frac{1}{2}}\right)\exp\left(-2\sqrt{2}|\beta'|\pi|(x_{0,-},x_{0,-})|^{\frac{1}{2}}\right) \\ \qquad \text{with } \nu' = |\nu|-\frac{1}{2},\ \text{if } \nu \equiv \frac{1}{2} \pmod 1. \end{cases}$$

Note that ν ranges from $-\frac{n_\gamma+n_\delta}{2} - 1$ to $q - 2 - \frac{n_\gamma+n_\delta}{2}$.

Non-holomorphic part

As for the lift of the holomorphic part, when $n \neq 0$ the inner integral over u picks out the constant term. The outer integral now is of the form

$$\operatorname*{CT}_{s=0} \int_0^\infty v^{\frac{1}{2}(\ell - n_\gamma - n_\delta) - 2 - s} \Gamma(\kappa + 1, 4\pi|n|) e^{-\frac{\pi}{v}|\beta'|^2} e^{2\pi v((x_{0,-}, x_{0,-}) - (x_{0,+}, x_{0,+}))} dv,$$

with $\kappa = (p+q) - 2$. Hence, using Lemma 4.21, we get the following.

Lemma 4.25. *For fixed $n \neq 0$ and fixed $\eta = [0, \beta']$, at the base point z_0, the rank one term is given by*

$$\begin{aligned}
\phi_1^{\gamma,\delta}(n,\eta)^-(z_0) &= \left(-i\sqrt{\pi}\right)^{n_\gamma + n_\delta} \beta'^{n_\gamma} \bar{\beta}'^{n_\delta} \sum_{\ell=0}^{2q-2-n_\gamma-n_\delta} 2^{\frac{\ell}{2}+1} P_{\tilde{\gamma},\tilde{\delta},\ell}(x_{0,+}) \\
&\quad \cdot \kappa! \sum_{r=0}^{\kappa} \frac{(4\pi|n|)^r}{r!} \left(\frac{2(x_{0,-}, x_{0,-}) + 4|n|}{|\beta'|^2} \right)^{-\frac{\nu+r}{2}} K_{\nu+r}\left(2\pi|\beta'| \left(2(x_{0,-}, x_{0,-}) + 4|n|\right)^{\frac{1}{2}}\right) \\
&= \left(-i\pi^{\frac{1}{2}}\right)^{n_\gamma+n_\delta} \beta'^{n_\gamma} \bar{\beta}'^{n_\delta} \sum_{\ell=0}^{\substack{2q-2\\-n_\gamma-n_\delta}} 2^{\frac{\ell}{2}} P_{\tilde{\gamma},\tilde{\delta},\ell}(x_{0,+}) \cdot \mathcal{V}_{\kappa+2,1-\nu}\left(2\pi(x_{0,-}, x_{0,-}), \pi|\beta'|^2, 4\pi|n|\right).
\end{aligned}$$

where, as usual $\kappa = p + q - 2$ and $\nu = \frac{1}{2}(\ell - n_\gamma - n_\delta) - 1$. Note that if $\nu + r$ is a half-integer, the Bessel functions in the second line can be replaced by

$$\frac{1}{2^{\frac{3}{2}} (|\beta'|)^{\frac{1}{2}}} \left(2(x_{0,-}, x_{0,-}) + 4|n|\right)^{-\frac{1}{4}} h_{\nu'} \left(\left[2^{\frac{3}{2}} \pi |\beta'| \sqrt{(x_{0,-}, x_{0,-}) + 2|n|} \right]^{-1} \right)$$
$$\cdot \exp\left(-2^{\frac{3}{2}} \pi |\beta| \sqrt{(x_{0,-}, x_{0,-}) + 2|n|} \right)$$

wherein $h_{\nu'}$ is the Bessel polynomial of index $\nu' = |\nu + r| - \frac{1}{2}$.

For $n = 0$ the contribution of the non-holomorphic part $\phi_1^{\gamma,\delta}(0,\eta)^-$ is similar to the holomorphic part $\phi_1^{\gamma,\delta}(0,\eta)^+$ (see Lemma 4.24), but with index shifted by $-k+1$ due to the power of v in the constant term of f^-.

Appendix A.

Tools from representation theory

In the present chapter, we will gather some results form representation theory used later on. Recall the notation from Chapter 1. The Lie algebra of $G = \mathrm{U}(V)$ is denoted by $\mathfrak{g}_0$, whilst $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ is its complexifcation, viewed as a right $\mathbb{C}$ vector space. We have the Cartan decomposition of $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{k}_0$ and the Harish-Chandra decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$.

A.1. Flensted-Jensen theory

In [21, Sec. 2] Flensted-Jensen derived an integral formula for connected, non-compact semi-simple Lie groups. In the following, we establish the multiplicities that go into this formula for the Lie group $G = \mathrm{SU}(p, q)$. Following [21] and [45, Sec. 4], we consider the following sub-algebras of $\mathfrak{g}_0$:

First, for the sub-algebra $\mathfrak{k}_0$, associated to the maximal compact subgroup $K \subset G$. We have the decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$$

into ± 1 eigenspaces of the Cartan-involution θ. Second, the sub-algebra $\mathfrak{h}$ associated to the stabilizer in G of the fist vector v_1, $H = \mathrm{stab}_G(v_1)$. Note that H is the fixed point set of the involution $\tau = \left(\begin{smallmatrix} -1 & \\ & \mathbf{1}_{m-1} \end{smallmatrix}\right)$ in G. Note also that H is connected. Again, we get a decomposition into ± 1 eigenspaces of, in the case, τ:

$$\mathfrak{g}_0 = \mathfrak{h}_0 + \mathfrak{q}_0.$$

Hence there is a decomposition of $\mathfrak{g}_0$

$$\mathfrak{g}_0 = \mathfrak{k}_0 \cap \mathfrak{h}_0 + \mathfrak{p}_0 \cap \mathfrak{q}_0 + \mathfrak{p}_0 \cap \mathfrak{h}_0 + \mathfrak{k}_0 \cap \mathfrak{q}_0.$$

Using the euclidean basis $v_1, v_2, \ldots$ of V, we will describe the subalgebras in this decomposition through their matrices. For $\mathfrak{k}_0 \cap \mathfrak{h} \simeq \mathfrak{s}(\mathfrak{u}(1) \times \mathfrak{u}(p-1) \times \mathfrak{u}(q))$, the matrices have vanishing trace and block diagonal form (here and in the following all omitted matrix entries are zero)

$$\begin{pmatrix} a & & \\ & A & \\ & & B \end{pmatrix} \quad \text{with} \quad a \in i\mathbb{R}, A = -\bar{A}^t, B = -\bar{B}^t.$$

While for $\mathfrak{p}_0 \cap \mathfrak{h}_0$ we have

$$\begin{pmatrix} & X \\ \bar{X}^t & \end{pmatrix}, \quad \text{with} \quad X \in \mathrm{M}_{p-1,q}(\mathbb{C}), \bar{X}^t = X.$$

Finally, the spaces $\mathfrak{p}_0 \cap \mathfrak{q}_0$ and $\mathfrak{k}_0 \cap \mathfrak{q}_0$ are given by matrices of the forms

$$\begin{pmatrix} & & \bar{x}^t \\ & & \\ x & & \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -\bar{y}^t & \\ y & & \\ & & \end{pmatrix} \qquad \text{with} \quad x \in \mathbb{C}^q, y \in \mathbb{C}^{p-1}.$$

By direct calculation, we see that the maximal abelian subspace $\mathfrak{b}_0$ of $\mathfrak{p}_0 \cap \mathfrak{q}_0$ is given by the following set of matrices:

$$\mathfrak{b}_0 = \left\{ \begin{pmatrix} & & \bar{t} \\ & & \\ t & & \end{pmatrix}; \quad t \in \mathbb{R} \cup i\mathbb{R} \right\}.$$

We study the action of $\mathfrak{b}_0$ on its complement in $\mathfrak{p}_0 \cap \mathfrak{q}_0$. For this purpose, we set

$$H = \begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix}, \qquad \tilde{H} = \begin{pmatrix} & & -i \\ & & \\ i & & \end{pmatrix}.$$

Clearly, $\mathfrak{b} = \mathbb{R}H + \mathbb{R}\tilde{H}$. Further, for a column vector $x \in \mathbb{C}^{q-1}$ define

$$p(x) = \begin{pmatrix} & & \bar{x}^t & 0 \\ & & & \\ x & & & \\ 0 & & & \end{pmatrix} \in \mathfrak{p}_0 \cap \mathfrak{q}_0, \quad k(x) = \begin{pmatrix} & & & \\ & & & \\ & & & -x \\ & & \bar{x}^t & 0 \end{pmatrix} \in \mathfrak{k}_0 \cap \mathfrak{h}_0.$$

We have

$$\begin{aligned} [H, p(x)] &= \begin{pmatrix} & & & \\ & & & \\ & & \bar{x}^t & 0 \end{pmatrix} - \begin{pmatrix} & & \\ & & x \\ & & 0 \end{pmatrix} = k(x), \\ [H, k(x)] &= \begin{pmatrix} & \bar{x}^t & 0 \\ & & \\ & & \end{pmatrix} - \begin{pmatrix} & & \\ -x & & \\ 0 & & \end{pmatrix} = p(x). \end{aligned}$$

Thus, $\beta = 1$ is a positive root of $\mathfrak{b}_0$, the intersection of its root space with $\mathfrak{k}_0 \cap \mathfrak{h}_0 + \mathfrak{p}_0 \cap \mathfrak{q}_0$ is given by $\{p(x) + k(x);\ x \in \mathbb{C}^{q-1}\}$ and has (real) dimension $p_\beta = 2q - 2$. Also, since

$$[H, \tilde{H}] = \begin{pmatrix} 2i & & \\ & & \\ & & -2i \end{pmatrix}, \qquad \left[H, \left(\begin{smallmatrix} i & \\ & -i \end{smallmatrix}\right)\right] = 2\tilde{H},$$

we conclude that $\beta = 2$ is a positive root of $\mathfrak{b}_0$, too. The attached root space is one-dimensional and contained in $\mathfrak{k}_0 \cap \mathfrak{h}_0 + \mathfrak{p}_0 \cap \mathfrak{q}_0$.

Now, for the operation of $\mathfrak{b}_0$ on $\mathfrak{k}_0 \cap \mathfrak{q}_0 + \mathfrak{p}_0 \cap \mathfrak{h}_0$. For a column vector $x \in \mathbb{C}^{p-1}$, we set

$$p'(x) = \begin{pmatrix} & & 0 \\ & & x \\ 0 & \bar{x}^t & \end{pmatrix} \in \mathfrak{p}_0 \cap \mathfrak{h}_0, \quad k'(x) = \begin{pmatrix} 0 & \bar{x}^t & \\ -x & & \\ & & \end{pmatrix} \in \mathfrak{k}_0 \cap \mathfrak{q}_0.$$

Here, quite similarly to the above, we calculate the Lie-brackets

$$[H, p'(x)] = \begin{pmatrix} 0 & \bar{x}^t & \\ -x & & \\ & & \end{pmatrix} = k'(x), \qquad [H, k'(x)] = \begin{pmatrix} & & 0 \\ & & x \\ 0 & \bar{x}^t & \end{pmatrix} = p'(x).$$

Thus, for the root $\beta = 1$ the intersection of its root space with $\mathfrak{k}_0 \cap \mathfrak{q}_0 + \mathfrak{p}_0 \cap \mathfrak{h}_0$, given by $\{p'(x) + k'(x);\ x \in \mathbb{C}^{p-1}\}$, has real dimension $q_\beta = 2p - 2$.

Now, the quantity $\delta(H)$ occurring in [21, p. 263] is given by

$$\begin{aligned} \delta(H) &= \prod_{\beta \in \Delta^+} |\sinh(\langle \beta, H \rangle)|^{p_\beta} \cosh(\langle \beta, H \rangle)^{q_\beta} \\ &= 2 \operatorname{sgn}(\langle \beta, H \rangle) \sinh(\langle \beta, H \rangle)^{2q-1} \cosh(\langle \beta, H \rangle)^{2p-1}. \end{aligned} \tag{A.1.1}$$

A.2. Models for the Weil representation

Beside the Schrödinger model of the Weil representation, introduced in Section 1.2, we will use two further models, the mixed model and the polynomial Fock model. In this section, we describe their setup and give the intertwining operators.

First, we treat the mixed model, which is used in Chapter 4.

A.2.1. The mixed model

The passage to the mixed model of the Weil representation can be realized through a partial Fourier-transform. We use hyperbolic coordinates (cf. Section 1.1.1) by setting

$$\ell := \frac{1}{\sqrt{2}} (v_1 + v_m), \qquad \ell' := \frac{1}{\sqrt{2}} (v_1 - v_m),$$

and write x in the form $\alpha\ell + x_0 + \beta\ell' = (\alpha, x_0, \beta)$, with $x_0 \in W := V \cap \ell^\perp \cap \ell'^\perp$. We denote the real and imaginary parts of the coordinates by writing $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$.

Now, passing to the mixed model amounts to calculating the partial Fourier transform with respect to the hyperbolic coordinate α attached to ℓ . Since α is a complex variable, one has to calculate the partial Fourier transform in the two real variables α_1 and α_2. The new coordinate is denoted by $\beta' = \beta'_1 + i\beta'_2$. Hence, for a Schwartz form ϕ, we set

$$\widehat{\phi}(\beta', x_0, \beta) := \int_{\mathbb{R}^2} \phi(\alpha, x_0, \beta) e^{2\pi i \left(\alpha_1 \beta'_1 + \alpha_2 \beta'_2\right)} \, d\alpha_1 d\alpha_2.$$

Note that the integral converges since the integrand is a Schwartz function.

Intertwining for $\mathrm{SL}_2(\mathbb{R})$ Now, we determine the intertwining operators for the operation of $\mathrm{SL}_2(\mathbb{R}) \simeq \mathrm{SU}(1,1)$. To facilitate notation, set $G' = \mathrm{SU}(1,1)$. Following [40], we define

$$\eta := [\beta, \beta'] = \begin{pmatrix} \beta_1 & \beta_1' \\ \beta_2 & \beta_2' \end{pmatrix} \in \mathrm{M}_2(\mathbb{R}).$$

Lemma A.1. *Let ϕ be a Schwartz function, and r its weight under the operation of $K' = \mathrm{U}(1)$. The intertwining operators for the action of G' are given by*

1. $$\mathcal{F}\left(\omega\begin{pmatrix} \sqrt{v} & \\ & \sqrt{v}^{-1} \end{pmatrix}\varphi(\cdot)\right)(\beta', \beta) = v^{-\frac{r}{2}+\frac{p-q}{2}}\frac{1}{v}\hat{\varphi}\left(\tfrac{1}{\sqrt{v}}\beta', \sqrt{v}\beta\right),$$

2. $$\mathcal{F}\left(\omega\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}\varphi(\cdot)\right)(\beta', \beta) = \hat{\varphi}\left(\beta' + u\beta, \beta\right).$$

Thus, $g_\tau = \begin{pmatrix} \sqrt{v} & \frac{u}{\sqrt{v}} \\ 0 & \frac{1}{\sqrt{v}} \end{pmatrix}$ *operates as follows:*

$$\mathcal{F}\left(\omega(g_\tau')\varphi(\cdot)\right)(\beta', \beta) = v^{-\frac{r}{2}+\frac{p+q}{2}-1}\hat{\varphi}\left(\tfrac{1}{\sqrt{v}}\left(\beta' + u\beta\right), \sqrt{v}\beta\right).$$

Proof. Direct calculation. □

Using the Lemma, one quickly obtains the (partial) Fourier transform of the Gaussian $\varphi_0(x,\tau) = \varphi_0^{p,q}(x,\tau)$. It takes the form

$$\begin{aligned} &\widehat{\varphi}_0^{p,q}((\eta, x_0), \tau) \\ &= \exp\left(-\frac{\pi}{v}\left(|\beta'|^2 + |\bar{\tau}\beta|^2 + 2u\,\mathrm{Re}\left(\beta'\bar{\beta}\right)\right) + 2\pi\bar{\tau}(x_{0,-}, x_{0,-}) + 2\pi\tau(x_{0,+}, x_{0,+})\right) \qquad (A.2.1) \\ &= \exp\left(-\frac{\pi}{v}\left(|\beta' + \bar{\tau}\beta|^2 + 2v\Im\left(\beta'\bar{\beta}\right)\right) + 2\pi\bar{\tau}(x_{0,-}, x_{0,-}) + 2\pi\tau(x_{0,+}, x_{0,+})\right). \end{aligned}$$

Intertwining for the operation of the parabolic subgroup $P_\ell \subset G$ To determine the intertwining operators for the action of G, we recall the Levi decomposition $G = NAM$ introduced in the context of signature $(p,1)$ in Section 2.1.1. In signature (p,q), we have $M \simeq \mathrm{SU}(p-1, q-1)$ and $A \simeq \mathrm{GL}([\ell])$ the elements of A and M are written as matrices in the form

$$a(t) = \begin{pmatrix} t & & \\ & 1_{m-1} & \\ & & t^{-1} \end{pmatrix} \quad \left(t \in \mathbb{R}_{>0}\right), \qquad \mu = \begin{pmatrix} 1 & & \\ & \mu' & \\ & & 1 \end{pmatrix} \quad \left(\mu' \in \mathrm{SU}(W)\right),$$

while the elements of the Heisenberg group are given by matrices of the form

$$n(0,r) = \begin{pmatrix} 1 & 0 & ir \\ & 1_{m-1} & \\ & & 1 \end{pmatrix} \quad (r \in \mathbb{R}),$$
$$n(w,0) = \begin{pmatrix} 1 & -\bar{w}^t & -\frac{1}{2}(w,w) \\ & 1_{m-1} & w \\ & & 1 \end{pmatrix} \quad (w \in W),$$

and satisfy the group law $n(w_2, 0) \circ n(w_1, 0) = n\left(w_1 + w_2, -\Im(w_2, w_1)\right)$.

Lemma A.2. *Let φ be a Schwartz form. The intertwining operators for the operation of the subgroups N, A and M are given as follows:*

1. $\mathcal{F}\big(\widehat{n(0,r)\varphi}\big)$:
$$\hat{\varphi}\left(([\beta,\beta'],x_0),\tau,z_0\right) e\left(r\Im(\beta'\bar{\beta})\right).$$
2. $\mathcal{F}\big(\widehat{n(w,0)\varphi}\big)$:
$$\hat{\varphi}\left(([\beta,\beta'],x_0-\beta w),\tau,z_0\right) e\left(\tfrac{1}{2}\operatorname{Re}(\beta'\bar{\beta})(w,w)-\operatorname{Re}\left(\beta'(x_0,w)\right)\right).$$
3. $\mathcal{F}\big(\widehat{a(t)\varphi}\big)$:
$$t^2\hat{\varphi}\left((t[\beta,\beta'],x_0),\tau,z_0\right).$$
4. $\mathcal{F}\big(\widehat{\mu\varphi}\big)$:
$$\hat{\varphi}\left([\beta,\beta'],\mu^{-1}x_0),\tau,z_o\right).$$

(Note that if either p or q is 1, M is compact.)

Proof. Since
$$g\varphi=\varphi(x,\tau,gz_0)=\varphi(g^{-1}x,\tau,z_0),$$
the operation of N and the elements of the Levi-factor are given as follows:
$$\begin{aligned}
n(0,r)\varphi &= \varphi(x,\tau,n(0,r)z_0)=\varphi((\alpha-ir\beta,x_0,\beta),\tau,z_0),\\
n(w,0)\varphi &= \varphi((\alpha,x_0,\beta),\tau,n(w,0)z_0)=\varphi(n(-w,0)(\alpha,x_0,\beta),\tau,z_0)\\
&= \varphi((\alpha+(w,x_0)-\beta\tfrac{1}{2}(w,w),x_0-w\beta,\beta),\tau,z_0),\\
a(t)\varphi &= \varphi\left((\alpha,x_0,\beta),\tau,a(t)z_0\right)=\varphi\left((t^{-1}\alpha,x_0,t\beta),\tau,z_0\right),\\
m\varphi &= \varphi((\alpha,x_0,\beta),\tau,\mu z_0)=\varphi((\alpha,\mu^{-1}x_0,\beta),\tau,z_0).
\end{aligned}$$
The claim follows easily by calculating of the partial Fourier transform in α. □

A.2.2. The polynomial Fock model

We briefly recall the setup of the polynomial Fock model of the Weil representation for the dual pair $\mathrm{U}(p,q)\times\mathrm{U}(1,1)$ used in the construction of ψ in Chapter 3, see Sections 3.3 and 3.3.1. For a review of the Fock model (for $\mathrm{U}(p,q)\times\mathrm{U}(r,s)$) we refer to [25, Appendix B]. For more on the background, see Kudla and Millson [46, 44], Adams [1] and Funke and Millson [see 27].

Recall that the Schrödinger model is given by the space of Schwartz functions $\mathcal{S}(V)$ on V. The K'-finite vectors in $\mathcal{S}(V)$ form the polynomial Fock space $S(V)$. It consists of functions on V of the form $p(z)\varphi_0(z)$, where $p(z)$ is a polynomial function on V and $\varphi_0(z)$ is the standard Gaussian on V. We use the coordinates $z_1,\dots,z_m$ in V, relative to the basis $\{v_\alpha,v_\mu\}$.

From the action of $\mathrm{U}(1,1)\times\mathrm{U}(p,q)$ on $\mathcal{S}(V)$ one obtains an action of the Lie algebra $\mathfrak{u}(1,1)(\mathbb{C})\times\mathfrak{u}(p,q)(\mathbb{C})$ on $\mathcal{P}(\mathbb{C}^{2m})$, denoted $\omega=\omega_\lambda$ (with a central character λ, here $\lambda=2\pi i$). We then have a intertwining operator $\iota:\mathcal{S}(V)\to\mathcal{P}(\mathbb{C}^{2m})$ satisfying $\iota(\varphi_0)=1$. It is given by the following Lemma (see [25], Lemma B.3).

Lemma A.3. *The intertwining operator between the Schrödinger and the Fock model satisfies*

$$\iota\left(\bar{z}_\alpha - \frac{1}{\pi}\frac{\partial}{\partial z_\alpha}\right)\iota^{-1} = -i\frac{1}{\sqrt{2\pi}}z''_\alpha, \qquad \iota\left(\bar{z}_\alpha + \frac{1}{\pi}\frac{\partial}{\partial z_\alpha}\right)\iota^{-1} = 2\sqrt{2}i\frac{\partial}{\partial z'_\alpha},$$
$$\iota\left(z_\alpha - \frac{1}{\pi}\frac{\partial}{\partial \bar{z}_\alpha}\right)\iota^{-1} = -i\frac{1}{\sqrt{2\pi}}z'_\alpha, \qquad \iota\left(z_\alpha + \frac{1}{\pi}\frac{\partial}{\partial \bar{z}_\alpha}\right)\iota^{-1} = 2\sqrt{2}i\frac{\partial}{\partial z''_\alpha},$$
$$\iota\left(\bar{z}_\mu - \frac{1}{\pi}\frac{\partial}{\partial z_\mu}\right)\iota^{-1} = i\frac{1}{\sqrt{2\pi}}z''_\mu, \qquad \iota\left(\bar{z}_\mu + \frac{1}{\pi}\frac{\partial}{\partial z_\mu}\right)\iota^{-1} = -2\sqrt{2}i\frac{\partial}{\partial z'_\mu},$$
$$\iota\left(z_\mu - \frac{1}{\pi}\frac{\partial}{\partial \bar{z}_\mu}\right)\iota^{-1} = i\frac{1}{\sqrt{2\pi}}z'_\mu, \qquad \iota\left(z_\mu + \frac{1}{\pi}\frac{\partial}{\partial \bar{z}_\mu}\right)\iota^{-1} = -2\sqrt{2}i\frac{\partial}{\partial z''_\mu}.$$

In the Fock model, the Weil representation acts as follows (see loc. cit. Lemma B.1)

Lemma A.4. *For the action of $\mathfrak{g} \simeq \mathfrak{u}(p,q)(\mathbb{C})$ on $\mathcal{P}(\mathbb{C}^{2m})$, we have the following:*

(i) The elements $Z'_{\alpha\beta}$, $Z''_{\alpha\beta}$ and $Z'_{\mu\nu}$, $Z''_{\mu\nu}$ in $\mathfrak{k}$ act by

$$\omega\lambda(Z'_{\alpha\beta}) = -\omega(Z''_{\beta\alpha}) = -z''_\alpha\frac{\partial}{\partial z''_\beta} + z'_\beta\frac{\partial}{\partial z'_\alpha},$$
$$\omega(Z'_{\mu\nu}) = -\omega(Z''_{\nu\mu}) = -z'_\nu\frac{\partial}{\partial z'_\mu} + z''_\mu\frac{\partial}{\partial z''_\nu}.$$

(ii) The elements $Z'_{\alpha\mu}$ of $\mathfrak{p}^+$ and $Z''_{\alpha\mu}$ of $\mathfrak{p}^-$ act by

$$\omega(Z'_{\alpha\mu}) = \frac{1}{4\pi}z''_\alpha z'_\mu - 4\pi\frac{\partial^2}{\partial z'_\alpha\partial z''_\mu},$$
$$\omega(Z''_{\alpha\mu}) = -4\pi\frac{\partial^2}{\partial z''_\alpha\partial z'_\mu} + \frac{1}{4\pi}z'_\alpha z''_\mu.$$

Let $\mathfrak{g}' \simeq \mathfrak{u}(1,1)(\mathbb{C})$ and $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'^{+} \oplus \mathfrak{p}''^{-}$ be the standard decomposition. For the following see [25, Lemma B.2.].

Lemma A.5. *For the action of $\mathfrak{g}' \simeq \mathfrak{u}(1,1)(\mathbb{C})$ on $\mathcal{P}(\mathbb{C}^{2m})$, we have the following*

(i) The generators of $\mathfrak{k}'$ act by

$$\omega\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = i\left[\sum_{\alpha=1}^{p} z''_\alpha\frac{\partial}{\partial z''_\alpha} - \sum_{\mu=p+1}^{p+q} z'_\mu\frac{\partial}{\partial z'_\mu}\right] + i(p-q),$$
$$\omega\left(\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right) = i\left[\sum_{\alpha=1}^{p} z'_\alpha\frac{\partial}{\partial z'_\alpha} - \sum_{\mu=p+1}^{p+q} z''_\mu\frac{\partial}{\partial z''_\mu}\right] + i(p-q).$$

(ii) In $\mathfrak{p}'^{\pm}$*, consider the elements* L *and* R*, given by*

$$L = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \in \mathfrak{p}'^{-}, \quad R = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \in \mathfrak{p}'^{+}.$$

On the Fock model, they act by

$$\begin{aligned} \omega(L) &= -4\pi \sum_{\alpha=1}^{p} \frac{\partial^2}{\partial z''_{\alpha a} \partial z'_{\alpha u}} + \frac{1}{4\pi} \sum_{\mu=p+1}^{p+q} z'_\mu z''_\mu, \\ \omega(R) &= -\frac{1}{4\pi\lambda} \sum_{\alpha=1}^{p} z''_\alpha z'_\alpha + 4\pi \sum_{\mu=p+1}^{p+q} \frac{\partial^2}{\partial z'_\mu \partial z''_\mu}. \end{aligned}$$

Note that L *and* R *which give rise to the classical Maass lowering and raising operators of* SL_2*.*

Appendix B.

Useful formulas

B.1. Special functions

In this section we recall the integral representations of some special functions and their properties.

The incomplete Gamma function First, for the convenience of the reader, we recall the integral representations of the Gamma function and the incomplete Gamma function.

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt, \qquad \Gamma(s,a) = \int_a^\infty t^{s-1} e^{-t} dt. \tag{B.1.1}$$

For $n \in \mathbb{N}_0$, we note the following identity [cf. 5, p. 74]:

$$\Gamma(n+1,a) = n! e^{-a} e_n(a) = n! e^{-a} \sum_{r=0}^{n} \frac{a^r}{r!}. \tag{B.1.2}$$

Relations for the K-Bessel functions The following integral representation for Bessel functions is well-known to number theorists, see [20, 6.(17), p. 313]

$$\int_0^\infty v^{\nu-1} \exp\left(-av - bv^{-1}\right) dv = 2\left(\frac{a}{b}\right)^{-\frac{\nu}{2}} K_\nu\left(2\sqrt{ab}\right) \qquad (\operatorname{Re} a > 0, \operatorname{Re} b > 0). \tag{B.1.3}$$

Beside this integral representation, we also make frequent use of the following relations [cf. 17, 10.27.3, 10.33.2]:

$$K_{-\nu}(x) = K_\nu(x), \quad K_{-\frac{1}{2}}(2\pi r) = K_{\frac{1}{2}}(2\pi r) = \frac{1}{2} r^{-\frac{1}{2}} e^{-2\pi r}, \tag{B.1.4}$$

$$\text{and} \qquad K_{n+\frac{1}{2}}(2\pi r) = \frac{1}{2} r^{-\frac{1}{2}} e^{-2\pi r} h_n\left(\tfrac{1}{2\pi r}\right) \quad (n \in \mathbb{Z}, n \geq 0). \tag{B.1.5}$$

Here, h_n is the n-th Bessel polynomial, explicitly given by

$$h_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!\,k!} \left(\frac{x}{2}\right)^k.$$

A further special function, which generalises [5, (3.25) on p.74] is usful for the Fourier-Jacobi expansion of the singular theta lift $\Phi(z, f, \psi)$ (see Chapter 4).

Lemma B.1. *For $n \in \mathbb{Z}$, $n \geq 2$, $\operatorname{Re}(A+c) > 0$, $\operatorname{Re} B > 0$, the special function defined as*

$$\mathcal{V}_{n,\mu}(A, B, c) := \int_0^\infty \Gamma(n-1, cv) v^{-\mu} e^{-Av - B\frac{1}{v}} dv.$$

is given by

$$2(n-2)! \sum_{r=0}^{n-2} \frac{c^r}{r!} \left(\frac{A+c}{B}\right)^{\frac{\mu-r-1}{2}} K_{r+1-\mu}(2\sqrt{(A+c)B}).$$

Further, if $\mu \equiv \frac{1}{2} \pmod 1$, we have

$$(n-2)! \pi^{\frac{1}{2}} \sum_{r=0}^{n-2} \frac{c^r}{r!} (A+c)^{\frac{1}{2}(\mu-r)-\frac{3}{4}} B^{\frac{1}{2}(r-\mu)+\frac{1}{4}} e^{-2\sqrt{(A+c)B}} h_{r-\mu+\frac{1}{2}} \left(\frac{\pi}{2\sqrt{(A+c)B}}\right).$$

Proof. Since $n-2$ is a non-negative integer, we can use the formula (B.1.2), and by (B.1.3), obtain the following:

$$\int_0^\infty \Gamma(n-1, cv) v^{-\mu} e^{-Av - Bv^{-1}} dv =$$

$$(n-2)! \sum_{r=0}^{n-2} \frac{c^r}{r!} \int_0^\infty v^{r-\mu} e^{-cv - Av - Bv^{-1}} dv =$$

$$(n-2)! \sum_{r=0}^{n-2} \frac{c^r}{r!} \cdot 2 \left(\frac{A+c}{B}\right)^{-\frac{r+1-\mu}{2}} K_{r-\mu+1}(2\sqrt{(A+c)B}).$$

The rest follows directly from (B.1.4) and (B.1.5). $\square$

B.2. Fourier transforms

Now, we gather some formulas for Fourier transforms, which come in useful for the evaluation of theta integral and for switching between the Schrödinger model and the mixed model of the Weil representation (cf. Section A.2).

For the following Lemma, see [2, Corollary 3.3].

Lemma B.2. *Let x be a real indeterminate and $p(x) \in \mathbb{C}[x]$ a polynomial. Then, the Fourier transform of the Schwartz function*

$$p(x) e^{2\pi i \left(Ax^2 + Bx + C\right)}$$

is given by

$$(-2iA)^{-\frac{1}{2}} \exp\left(\frac{i}{8\pi A} \frac{d^2}{dt^2}\right) (p(t)) \left(-\tfrac{\xi}{2A} - \tfrac{B}{2A}\right) \exp\left(2\pi i \left[-\tfrac{\xi^2}{4A} - \tfrac{\xi B}{2A} - \tfrac{B^2}{4A} + C\right]\right), \tag{B.2.1}$$

wherein ξ denotes the transformed variable.

Now, consider a special case, the Fourier transform of $p(x)e^{-\pi x^2}$. It is given by

$$\exp\left(\frac{1}{4\pi}\frac{d^2}{dt^2}\right)\big(p(t)\big)(i\xi)e^{-\pi\xi^2} := \tilde{p}(\xi)e^{-\pi x^2}. \tag{B.2.2}$$

From this, one can immediately conclude the following statements

1. The Fourier transform of $p(x+c)e^{-\pi x^2}$ is given by $\tilde{p}(\xi - ic)e^{-\pi\xi^2}$.
2. The transform of $p(-x)e^{-\pi x^2}$ is given by $\tilde{p}(\xi)e^{-\pi x^2}$.

Now, we turn to the Fourier transform of the special polynomials introduced in Chapter 3 above. Recall the definition of Hermite polynomials and Laguerre polynomials (cf. Section 3.3). For $k \geq 0$, the Hermite polynomial H_k is given by

$$H_k(t) = (-1)^k e^{t^2}\left(\frac{d}{dt}\right)^k e^{-t^2} = e^{t^2/2}\left(t - \frac{d}{dt}\right)^k e^{-t^2/2}.$$

Its Fourier transform is given by the following Lemma [see 28, Lemma 4.1]:

Lemma B.3. *The Fourier transform of the k-th Hermite polynomial $H_k(x)$ is given by*

$$\int_0^\infty \frac{1}{\sqrt{2\pi}^k} H_k(-\sqrt{\pi}x)e^{-\pi x^2}\, e^{2\pi i\xi x}dx = \left(-\sqrt{\pi}i\xi\right)^k e^{-\pi\xi^2}. \tag{B.2.3}$$

Now, for the Laguerre polynomials. Recall the definition of the k-th Laguerre polynomial L_k ($k \geq 0$),

$$L_k(t) = \frac{e^t}{k!}\left(\frac{d}{dt}\right)^k \left(e^{-t}t^k\right).$$

Its Fourier transform can be derived from the formula for the Fourier transform of the Hermite polynomials.

Lemma B.4. *Let $z = x + iy$ be a complex variable. The Fourier transform in z of the Laguerre polynomial*

$$L_k\left(\pi|z|^2\right)e^{-\pi|z|^2} \quad \textit{is given by} \quad \frac{\pi^{2k}}{2^k k!}|w|^{2k}e^{-\pi|w|^2}, \tag{B.2.4}$$

with the transformed variable w.

Proof. Recall that by (3.3.1) (see p. 81), $(\mathcal{D}_\alpha\mathcal{D}_\alpha)^k\varphi_0 = L_k(2\pi|z_\alpha|)$. Thus, the claim follows directly from Lemma B.3 via Remark 3.6 (see (3.3.6) on p. 82). Indeed, since

$$\left(\frac{-1}{\pi}\right)^k 2^k k! L_k\left(\pi|z|^2\right)e^{-\pi|z|^2} = \left(\frac{1}{2\pi}\right)^k \sum_{j=0}^k \binom{k}{j} H_{2(k-j)}\left(\sqrt{\pi}x\right) H_{2j}\left(\sqrt{\pi}y\right) e^{-\pi\left(x^2+y^2\right)},$$

after applying (B.2.3) twice and writing $w = \xi + i\eta$, in the Fourier transform we get

$$(-1)^k\pi^k \sum_{j=0}^k \binom{k}{j}\xi^{2(k-j)}\eta^{2j} = (-1)^k\pi^k|w|^{2k},$$

as claimed. □

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