Liftings and Borcherds Products

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August 11, 2017

Introduction

The present course notes are based on three lectures held by the author during a preparatory course for the conference 'L-functions and automorphic forms'. Their purpose is to give a brief introduction to theta-liftings, in which input functions (usually modular forms) are 'lifted' by integrating them against a suitable theta-function. The main focus lies on the singular theta-lift of Borcherds [4], which leads up to the construction of Borcherds products through a multiplicative lifting. This lifting yields meromorphic modular forms for an indefinite orthogonal group of signature $(2, n)$, $n \geq 2$, which take their zeros and poles along certain arithmetically defined divisors called Heegner divisors and which posses absolutely convergent infinite product expansions (called 'Borcherds product expansions').

Special cases of such infinite products were already obtained by Borcherds in an earlier paper [3], however using completely different methods. This construction was originally motivated by the theory of generalized Lie (super-)algebras (see e.g. [36] or [15]).

The singular theta-lift we will concentrate on takes weakly holomorphic modular forms (see Definition 0.1 below) for the elliptic modular group $\mathrm{SL}_2(\mathbb{Z})$ and lifts them to modular forms for an indefinite orthogonal group.

It should be mentioned that the theoretical reason, why such a lifting using a theta-function is possible, is that $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{SO}(2, n)$ form what is called a dual reductive pair in the sense of Howe [see 26].

Overview:

In the first section, we give a few examples of liftings that can be realized as theta lifts. This includes a special case of Borcherds' original construction from [3].

In Section two, we go through the construction of symmetric domains for indefinite orthogonal groups of signature $(2, n)$, $n \geq 1$. Further, we define orthogonal modular groups related to lattices (Section 2.2) and introduce Heegner divisors (Section 2.3). The section closes with a definition of orthogonal modular forms (see p. 17).

The main section, Section 3 (p. 18) covers the singular theta-lift:

First, we study the metaplectic double cover of $\mathrm{SL}_2(\mathbb{Z})$, a representation of which is used to define vector valued modular forms (see p. 19), generalizing the usual definition of scalar valued modular forms, see Definition 0.1.

Next, in Section 3.2, we introduce the Siegel theta-function which is employed in the lifting, and formulate the theta-integral. We will indicate, why in this particular case it is necessary to consider, on the one hand, weakly holomorphic forms as input functions, and, on the other hand, to use a regularized integral.

The regularization procedure is described in detail in Section 3.3. We derive one of the main properties of Borcherds' singular lifting, namely the location and type of its singularities (Theorem 3.2). Also we briefly outline some of the main steps used in the actual evaluation of the theta-lift, without going into further detail (see p. 27).

Finally, in Section 3.4 the singular theta-lift is used to define the multiplicative lifting:

Borcherds products are explained as solutions of a multiplicative Cousin problem, namely of finding a meromorphic functions with divisor supported on the singularities of the singular theta-lift. We formulate a version of Borcherds' theorem [4, Theorem 13.3], with a simplified form of the infinite product expansion.

0.1 Basic definitions and notation

Throughout these notes, as usual, the integers are denoted by $\mathbb{Z}$, and the positive integers by $\mathbb{N}$. Also, $\mathbb{Q}$ is the field of rational numbers, $\mathbb{R}$ denotes the reals, and $\mathbb{C}$ the complex numbers.

We recall some basic definitions from the theory of modular forms, details of which can be found in many places, for example in [16], [27], [2], [12, part I] or [32].

As usual, the complex upper half-plane is denoted by $\mathbb{H} = \{z \in \mathbb{C}; \Im z > 0\}$. Throughout, τ will be used to denote a point in $\mathbb{H}$, with $\tau = u + iv$, with u and v the real and the imaginary part of τ, respectively. Also, we denote by $\mathbb{H}^*$ the union of $\mathbb{H}$ with its rational boundary points, i.e. $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$.

The special linear group $\mathrm{SL}_2(\mathbb{Z}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right); a, b, c, d \in \mathbb{Z}, ad - bc = 1\}$ operates on $\mathbb{H}$ by fractional linear transformations,

$$M\tau = \frac{a\tau + b}{c\tau + c} \quad \text{if} \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A standard fundamental domain for this operation is given by

$$\mathcal{F} := \left\{\tau = u + iv;\ |z| > 1, -\tfrac{1}{2} < u < \tfrac{1}{2}\right\}.$$

Also, recall that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the two matrices $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Beside $\mathrm{SL}_2(\mathbb{Z})$, known as the full (elliptic) modular group, subgroups of finite index are also called modular groups. These include the families of *congruence subgroups*, most importantly $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$ for N a positive integer, their *level*:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; c \equiv 0 \bmod N \right\},$$
$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; c \equiv 0 \bmod N, a \equiv d \equiv 1 \bmod N \right\},$$
$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; b \equiv c \equiv 0 \bmod N, a \equiv d \equiv 1 \bmod N \right\}.$$

Note that $\mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(1) = \Gamma_1(1) = \Gamma(1)$.

Let Γ be a modular group. The Γ-equivalence classes of $\mathbb{Q} \cup \{i\infty\}$ are called the *cusps* of Γ. The equivalence class of $\{i\infty\}$ is usually referred to as *the cusp at* ∞. Note that for the full modular group $\mathrm{SL}_2(\mathbb{Z})$, this is the only cusp.

Now, we recall the definition of modular forms.

Definition 0.1. *Let Γ be a modular group, k an integer and χ a group character of Γ. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a holomorphic modular form of weight k for Γ, with character χ, denoted $f \in \mathrm{M}_k(\Gamma, \chi)$ if*

1. *$f(M\tau) = \chi(M)(c\tau + d)^k f(\tau)$ for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,*
2. *f is holomorphic at all cusps.*

If further f vanishes at all cusps, f is called a cusp form. The space of cusp forms (for Γ, with weight k and character χ) is denoted $\mathrm{S}_k(\Gamma, \chi)$.

Contrastingly, if instead of satisfying condition 2. f is only meromorphic at the cusps, f is called a weakly holomorphic modular form. The space of weakly holomorphic modular forms is denoted $\mathrm{M}^!_k(\Gamma, \chi)$.

Clearly, we have $\mathrm{S}_k(\Gamma, \chi) \subset \mathrm{M}_k(\Gamma, \chi) \subset \mathrm{M}^!_k(\Gamma, \chi)$. Similarly, the notations $\mathrm{S}_k(\Gamma)$, $\mathrm{M}_k(\Gamma)$ and $\mathrm{M}^!_k(\Gamma)$ are used, if the character is trivial.

More generally, we will also consider modular forms of half-integer weight. For this, if $k \in \frac{1}{2}\mathbb{Z}$, one has to replace condition 1. in the definition and require, in its place

$$f(M\tau) = \chi(M) j(M, \tau)^{2k} f(\tau) \qquad \text{for all} \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

with a suitable automorphy factor $j(M, \tau)$. In particular, if $\Gamma = \Gamma_0(4N)$ the automorphy factor is given by $j(M, \tau) = \theta_0(M\tau)/\theta_0(\tau)$, where $\theta_0(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2}$ is the usual Jacobi theta function [see 27, Chapter IV].

Finally, modular forms have Fourier expansions since the matrix $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, one of the two generators of $\mathrm{SL}_2(\mathbb{Z})$, acts on $\mathbb{H}$ as $\tau \mapsto \tau + 1$. As every modular group Γ, being of finite index, contains some power of T, modular forms are periodic with positive integer periods, and hence can be expanded as Fourier series around the cusp at infinity and around all other cusps. Thus, for example if Γ is modular group with $T \in \Gamma$ (e.g. one of the $\Gamma_0(N)$'s), and k an integer, the Fourier expansion of $f \in \mathrm{M}_k^!(\Gamma)$ around ∞ takes the form

$$f(\tau) = \sum_{m \gg -\infty} a(m) q^m, \qquad q = e(\tau) = e^{2\pi i \tau}.$$

There are only finite many non-zero terms with $m < 0$. Further, if $f \in \mathrm{M}_k(\Gamma)$, then $a(m) \neq 0$ only for $m \geq 0$. Finally, if f is a cusp form, $a(m) \neq 0$ implies $m > 0$.

Table 1: Some frequently used notation:

$\mathbb{H}$	The complex upper half-plane (p. 2).
$\mathcal{F}$	A standard fundamental domain $\subseteq \mathbb{H}$ (p. 2).
$\Gamma_0(N)$, $\Gamma_1(N)$, $\Gamma(N)$	Principal congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ (p. 3).
$\mathrm{S}_k(\Gamma)$, $\mathrm{M}_k(\Gamma)$, $\mathrm{M}^!_k(\Gamma)$	Spaces of modular forms for a modular group Γ
$\mathrm{S}_k(\Gamma,\chi), \mathrm{M}_k(\Gamma,\chi), \mathrm{M}^!_k(\Gamma,\chi)$	(see Definition 0.1).
$\theta_0 = \sum_{n\in\mathbb{Z}} q^{n^2}$	The Jacobi theta-function.
$\mathrm{M}^+_k(\Gamma)$	A Kohnen plus-space (see p. 7).
$X_0(N)$	A modular curve $\simeq \Gamma_0(N)\backslash\mathbb{H}$,
$X_0(N)^*$	its compactification.
$V = V(\mathbb{Q})$, $V(\mathbb{R})$	A quadratic space over $\mathbb{Q}$, with $V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R}$ and signature $(2, n)$ (see p. 11)
$q(\cdot)$, $(\cdot,\cdot)$	The quadratic and the bilinear form of V,
$x^2 = (x, x) = 2q(x)$	(see p. 11)
$\mathrm{SO}(V)$, $\mathrm{O}(V)$	The orthogonal and the special orthogonal group of V,
$\mathrm{O}^+(V)$	the spinor kernel in $\mathrm{SO}(V)$ (see p. 11).
$V(\mathbb{C}) = V(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$	The complexification of $V(\mathbb{R})$.
$\mathbb{D}$, $\mathcal{K}$, $\mathcal{H}$	Models for the symmetric domain of $\mathrm{SO}(V)(\mathbb{R})$, see Section 2.1.
ℓ, ℓ'	Isotropic lattice vectors in V with $(\ell, \ell') = 1$.
$V_0(\mathbb{R}) = V(\mathbb{R}) \cap \ell \cap \ell'$	A Lorentzian subspace.
Z, z, Z_ℓ, $w(z)$	See Section 2.1.2, p. 13
L, L', L/L'	A lattice in V, its dual and the discriminant group, (p. 15).
Γ_L	The discriminant kernel and
X_Γ	the modular variety $\Gamma_L\backslash\mathbb{D}$. (p. 15).
$\mathbb{D}_\lambda$	A primitive Heegner divisor (Def. 2.2, p. 16),
$\mathcal{Z}(\mu, m)$	a Heegner divisor of index (μ, m).
$\mathrm{Mp}_2(\mathbb{R})$	The metaplectic double cover of $\mathrm{SL}_2(\mathbb{R})$ (p. 18).
$\mathrm{Mp}_2(\mathbb{Z}) = \widetilde{\mathrm{SL}_2}(\mathbb{Z})$, $\widetilde{\Gamma_0}(N)$	The pre-images of $\mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_0(N)$ in $\mathrm{Mp}_2(\mathbb{R})$.
ρ_L, ρ^*_L	The Weil representation and its dual.
$\mathbb{C}[L'/L]$	The group algebra of L'/L.
S_{k,ρ_L}, M_{k,ρ_L}, $\mathrm{M}^!_{k,\rho_L}$	Space of vector valued modular forms (p. 19.
H_{k,ρ_L}, H^+_{k,ρ_L}	Spaces of harmonic Mass forms, see (p. 19)
$\langle\cdot,\cdot\rangle$	A hermitian pairing on $\mathbb{C}[L'/L]$
$\Theta_L(z, \tau)$	The Siegel theta function for L (p. 21)
$d\mu = \frac{dudv}{v^2}$	The left-invariant measure on $\mathbb{H}$.
$\int^{reg}$	Regularized integral (Section 3.3).
$\Phi(z, f)$	The singular theta lift of f.
$\Psi(z, f)$	The multiplicative lift of f (Section 3.4)

1 Examples of liftings

In this section we will give some examples for liftings, all of which can, in fact, be realized as theta-liftings, using the theory of Howe duality (which is beyond the scope of the present course notes [see 26]). However, this is not the only way such liftings can be formulated, and indeed, the examples in this section were originally constructed using other methods.

1.1 Convolution of L-series

Our first two examples were discovered in the 1970s using convolution of L-series:

1. The *Shimura lift*, discovered by Goro Shimura [37] which takes certain half-integer weight cusp form of level $4N$ ($N \geq 1$) to integral weight modular forms of level $2N$.

2. The *Doi-Naganuma correspondence*, between modular forms for the elliptic modular group $\mathrm{SL}_2(\mathbb{Z})$ and modular forms for the Hilbert modular group, constructed by Koji Doi and Hidehisa Naganuma [see 18].

1.1.1 Shimura's lifting

Let us turn to the Shimura lift first, an overview of which can be found e.g. in [35, Chapt. 3].

Suppose that N and κ are positive integers, with N square-free, and that χ is a character modulo N. Further, assume that g is a cusp form of half-integer weight contained in $\mathrm{S}_{\kappa+\frac{1}{2}}(\Gamma_0(4N), \chi)$, with Fourier expansion given by

$$g(\tau) = \sum_{n=1}^{\infty} b(n) q^n.$$

Let t be a positive square-free integer and define a Dirichlet character Ψ_t by setting

$$\Psi_t(n) := \chi(n) \cdot \left(\frac{-1}{n}\right)^{\kappa} \left(\frac{t}{n}\right) \qquad (n \in \mathbb{N}).$$

Denote by $L(s, \Psi_t) = \sum_{n>0} \Psi_t(n) n^{-s}$ the Dirichlet L-series attached to Ψ_t.

Further, let $\{a_t(n)\}_{n=1,2,\ldots}$ be a sequence of complex numbers given by

$$\sum_{n=1}^{\infty} \frac{a_t(n)}{n^s} = L(s - \kappa + 1, \Psi_t) \cdot \sum_{n=1}^{\infty} \frac{b\big(tn^2\big)}{n^s}.$$

Then, the q-expansion with coefficients $a_t(n)$ defines a modular form, called the *Shimura lift* of g:

$$\sum_{n=1}^{\infty} a_t(n) q^n =: S_{t,\kappa}(g)(\tau),$$

contained in $\mathrm{M}_{2\kappa}(\Gamma_0(2N), \chi^2)$. Further, if $\kappa \geq 2$, the lift $S_{t,\kappa}(g)$ is a cusp form, whereas for $\kappa = 1$, $S_{t,\kappa}(g)$ is cuspidal only for certain g. (More precisely, for g contained in the orthogonal complement of the subspace spanned by unary theta series [see 35, p. 53].)

In 1975, Shinji Niwa [see 34] refined Shimura's lifting and realized it as a theta-lift.

Kohnen's theory We introduce the Kohnen plus space $\mathrm{M}^+_{\kappa+\frac{1}{2}}(\Gamma_0(4N))$. It consists of modular forms with Fourier expansions of the form

$$g(z) = \sum_{(-1)^\kappa n \equiv 0,1 \bmod 4} b(n)\, q^n, \tag{1}$$

with coefficients $b(n) \neq 0$ only for n which satisfy $(-1)^\kappa n \equiv 0, 1 \pmod 4$. The plus space was introduced by Winfred Kohnen as he studied the properties of the Shimura lift with respect to Hecke operations [see 28].

Furthermore, extending Shimura's results in [29, 30], he showed that the two spaces of newforms $\mathrm{S}^{+,new}_{\kappa+\frac{1}{2}}(\Gamma_0(4N))$ and $\mathrm{S}^{new}_{2\lambda}(\Gamma_0(N))$ are isomorphic. The isomorphism is given by a linear combination of Shimura lifts. Some authors refer to this Hecke-invariant isomorphism as the 'Shimura correspondence'.

1.1.2 The Doi-Naganuma correspondence

Our next example is due to Doi and Naganuma [18] and was discovered at around the same time as Shimura's lifting. See [13, Sections 1.7, 1.10] and [12, II. Section 3.1] for details.

In order to formulate the correspondence, we briefly recall some facts about Hilbert modular forms [see 12, II. Sections 1.3, 1.6]: Let $d > 1$ be a square-free integer and denote by K the real quadratic field $K = \mathbb{Q}(\sqrt{d})$. We shall assume that the narrow class number of K is one.

Denote by $\mathcal{O}_K$ the ringer of integers in K and by $\mathfrak{d}^{-1}$ the inverse different ideal. Further, for $a \in K$ denote by a' the Galois conjugate of a.

The special linear group $\mathrm{SL}_2(K)$ is embedded into $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ through the two real embeddings of K. It acts on $\mathbb{H} \times \mathbb{H}$ through fractional linear transformations. For $z = (z_1, z_2) \in \mathbb{H}^2$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \left(\frac{az_1 + b}{cz_1 + d}, \frac{a'z_2 + b'}{c'z_2 + d'} \right) \quad \text{if} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K).$$

The Hilbert modular group $\Gamma_K = \mathrm{SL}_2(\mathcal{O}_K)$ acts properly discontinuously.

Let k be an integer. A holomorphic Hilbert modular form F for Γ_K of (parallel) weight k is a holomorphic function $F : \mathbb{H}^2 \to \mathbb{C}$ which transforms according to

$$F(\gamma z) = \big(cz_1 + d\big)^k \big(c'z_2 + d'\big)^k F(z) \qquad \text{for all} \quad \gamma \in \Gamma_K, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2}$$

We denote by $\mathrm{M}_{H,k}(\Gamma_K)$ the space of holomorphic Hilbert modular forms of weight k for Γ_K. Note that by the Koecher principle [see 12, II.Theorem 1.20] if a Hilbert modular form F is holomorphic on $\mathbb{H}^2$, it is automatically holomorphic at the cusp ∞, and indeed at all cusps. Here, as usual, by the cusps of $\mathbb{H}^2$, we mean the Γ_K-equivalence classes of elements in $\mathbb{P}^1(K)$.

We will describe what it means for a Hilbert modular form F to holomorphic at the cusp ∞ using the Fourier expansion. From this one can obtain the description for the other cusps through conjugation, noting that for any $\kappa \in \mathbb{P}^1(K)$, one can take $\rho \in \mathrm{SL}_2(K)$ with $\rho\infty = \kappa$.

Since the stabilizer of ∞ in Γ_K contains a finite index subgroup acting by translations [see 12, p. 113], for $F \in \mathrm{M}_{H,k}(\Gamma_L)$, with the transformation behavior (2), this implies the existence of a Fourier expansion of the following form:

$$F(z) = a(0) + \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} a(\nu) e\left(\mathrm{tr}(\nu z)\right).$$

Here, the sum ranges over totally positive ν (denoted $\nu \gg 0$), if, as implied by the Koecher principle, F is holomorphic at ∞. Then, one further sets $F(\infty) = a(0)$. Finally F is called a cusp form, if in addition to F being holomorphic, one has $F(\infty) = 0$.

Now, given $F \in \mathrm{M}_{H,k}(\Gamma_K)$ with Fourier coefficients $a(\nu)$, we introduce a Dirichlet series denoted $L(s, F)$ as follows:

$$L(s, F) := \sum_{\substack{\nu \in \mathfrak{d}^{-1}/U \\ \nu \gg 0}} a(\nu)\, \mathrm{N}(\nu\mathfrak{d})^{-s}.$$

Here, U denotes the set of squares of totally positive units in $\mathcal{O}_K$, while for an ideal $\mathfrak{a}$ the norm is denoted $\mathrm{N}(\mathfrak{a})$.

Now, we are ready to describe the Doi-Naganuma lifting: Suppose $f(\tau) = \sum_{n\geq 0} a(n)q^n$ is a Hecke eigenform in $\mathrm{M}_k(\Gamma_0(1))$, with even weight k. Let $L(s, f)$ be the attached Dirichlet series and denote by $L(s, f, \chi_d)$ a twist by the quadratic character $\chi_d = \left(\frac{d}{\cdot}\right)$:

$$L(s, f) = \sum_{n>0} a(n)n^{-s}, \quad L(s, f, \chi_D) = \sum_{n>0} \chi_d(n)a(n)n^{-s}.$$

Denote by $L_{DN}(s)$ the product of these two Dirichlet series,

$$L_{DN}(s, f) := L(s, f) \cdot L(s, f, \chi_d).$$

Then, there is a Hilbert modular form $DN(f) \in \mathrm{M}_{H,k}(\Gamma_K)$, the *Doi-Naganuma lift* of f, with precisely this Dirichlet series, so that $L(s, DN(f)) = L_{DN}(s, f)$.

Remark 1.1. *Of course this is not exactly the way Doi and Naganuma originally stated their result in [18]. In 1973, Naganuma obtained the following version [see 33]: Assume that $d = p$ is a prime and let $K = \mathbb{Q}(\sqrt{p})$. Let $f(\tau) = \sum_n a(n)q^n$ be a normalized Hecke eigenform in $\mathrm{M}_k(\Gamma_0(p), \chi_p)$, with χ_p a character of order two, and let $f^\rho(\tau) = \sum_n \overline{a(n)}q^n$. Then, we have $L(s, DN(f)) = L(s, f) \cdot L(s, f^\rho)$ and $DN(f) \in \mathrm{M}_{H,k}(\Gamma_K)$.*

1.2 Borcherds products

In [3] Richard E. Borcherds introduced his famous multiplicative lifting. The methods he used are totally unrelated to either convolutions of L-series or, indeed, theta-correspondences. In contrast to this, Borcherds' later, much more general construction in [4], which we will study in Section 3, is formulated as a theta-lift.

For now, though, we describe only a special case from [3]: Here, the input functions for the multiplicative lifting are contained in $\mathrm{M}^{+,!}_{\frac{1}{2}}(\Gamma_0(4))$, i.e they are weakly holomorphic modular forms of level 4, weight $\frac{1}{2}$ and satisfy a plus-space condition like in (1). They are lifted to meromorphic modular forms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$, which have infinite product expansions, and, in their Fourier expansion (around the cusp at infinity), integral Fourier coefficients and leading coefficient one. Further, they take their zeros and poles along linear combinations of rational divisors, called Heegner divisors:

Heegner Divisors (Classical) Heegner divisors are subsets of $\mathbb{H}$ arising as the pre-images under $\mathbb{H} \to X_0(N)$ ($N \in \mathbb{N}$) of certain rational divisors on the modular curve $X_0(N) \simeq \Gamma_0(N)\backslash\mathbb{H}$, for a precise definition [see 21, Section IV.1].

In the present setting, the level N is 1, and Heegner divisors are given as follows: Let D be a negative integer, with D a square modulo 4. Let a, b, c with $a > 0$ be integers satisfying $b^2 - 4ac = D$. Thus, a, b, c are the coefficients of an integral binary quadratic form, with D as its discriminant.

A point $\tau \in \mathbb{H}$ satisfying $a\tau^2 + b\tau + c = 0$ is then called a CM-point of discriminant D. Finally, the Heegner divisor of discriminant D consists of all CM-points of that discriminant. Often, it is useful to consider divisors supported at cusps as Heegner divisors, too.

We will encounter a generalization of this concept of Heegner divisors in Section 2 and Section 3 below.

The multiplicative lifting Let $\tilde{H}(\tau)$ denote the following generating series

$$\tilde{H}(\tau) := \sum_{\substack{n \equiv 0,3 \bmod 4 \\ n \geq 0}} H(n) q^n,$$

where $H(n)$ are the usual Hurwitz class numbers. They are modified class numbers given as follows [see 14, Section 5.3.2]: For $n = 0$ one sets $H(0) = -\frac{1}{12}$. Otherwise, for $n > 0$, if $h(-n)$ is the usual class number of primitive positive definite quadratic forms with discriminant $-n$, then

$$H(n) = \sum_{d^2 | n} w\left(\frac{n}{d^2}\right) \cdot h\left(-\frac{n}{d^2}\right) \qquad \text{where} \quad w(n) = \begin{cases} \frac{1}{3} & n = 3, \\ \frac{1}{2} & n = 4, \\ 1 & n > 4. \end{cases}$$

In particular, if $-n < -4$ is a fundamental discriminant, $H(n) = h(-n)$.

Now, let $f(\tau)$ be a weakly holomorphic modular form contained in $\mathrm{M}^{+,!}_{\frac{1}{2}}(\Gamma_0(4))$, and assume that the Fourier expansion of f around the cusp at ∞ is given by $\sum_{n>n_0} a(n)q^n$ with integer coefficients $a(n)$, with $a(n) = 0$ unless $n \equiv 0, 1 \pmod 4$. Then, the Borcherds lift $\Psi(\tau, f)$ of f is a meromorphic modular form of weight $a(0)$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ which has an absolutely converging infinite product expansion (a 'Borcherds product') as follows [see 3, Theorem 14.1]:

$$\Psi(\tau, f) = q^{-h} \prod_{n=1}^{\infty} (1 - q^n)^{a(n^2)} . \tag{3}$$

Here, h denotes the constant coefficient of the product $f(\tau)\tilde{H}(\tau)$.

Further $\Psi(\tau, f)$ has integer coefficients in its Fourier expansion around infinity, and leading coefficient one. Also, its divisor is supported on a linear combination of Heegner divisors or possibly the cusp. More precisely, if $\tau \in \mathbb{H}$ is a CM-point of discriminant $D < 0$, its multiplicity in $\mathrm{div}(\Psi(\tau, f))$ is given by $\sum_{n>0} a(Dn^2)$.

We note two further important properties:

1. The map $\Psi : f \mapsto \Psi(\tau, f)$ is *multiplicative*, with $\Psi(f + g) = \Psi(f)\Psi(g)$.
2. Any meromorphic modular form for the modular group $\mathrm{SL}_2(\mathbb{Z})$, the divisor of which is a linear combination of Heegner divisors (possibly including the cusp), can be realized as a Borcherds product $\Psi(f)$ for some $f \in \mathrm{M}^{+,!}_{\frac{1}{2}}(\Gamma_0(4))$.

By these two properties, the map Ψ becomes an *isomorphism* between the additive group $\mathrm{M}^{+,!}_{\frac{1}{2}}(\Gamma_0(4))$ and the multiplicative group of meromorphic modular forms satisfying the conditions given above for $\Psi(\tau, f)$.

Examples We present some examples following [3, Section 14] and [35, Section 4.2]. A basis for the space $\mathrm{M}^{+,!}_{\frac{1}{2}}(\Gamma_0(4))$ consists of functions $\{f_d\}_{d\equiv 0,3(4)}$ given by

$$f_0(\tau) = 1 + \sum_{n>0} 2q^{n^2}, \qquad f_d(z) = q^{-d} + \sum_{D>0} a(D, d)\, q^D, \quad d = 3, 4, 7, \ldots . \tag{4}$$

Note that $f_0(\tau)$ is simply the Jacobi theta-function $\theta_0(\tau)$. Given f_0 and f_3, further f_d's can be obtained inductively by observing that $f_{d-4}(\tau)j(4\tau)$ has the leading term q^{-d}. (For an explicit formula defining $f_3(\tau) = q^{-3} - 248q + \ldots$, see [35, (4.4), p. 70] or [3, Example 2, p. 202]).

From (3) and (4), one has:

$$\Psi(\tau, f_d) = q^{-H(d)} \prod_{n=1}^{\infty} (1 - q^n)^{a(n^2,d)} .$$

with $H(d)$ a Hurwitz class number as defined above. For applications of this formula see [35, Chapter 4].

Now, for two examples:

1. Let $f(z) = 12f_0(\tau) = 12\theta_0(\tau)$. Then, $f(z) = 12 + 24q + 24q^4 + \ldots$ and for $\Psi(\tau, f)$, we have
$$\Psi(\tau, f) = q \prod_{n>0} (1 - q^n)^{24} = \Delta(\tau),$$
which is just the usual modular discriminant function, with divisor supported at the cusp.

2. Consider $g(\tau) = 4f_0(\tau) + f_3(\tau)$. Then, one finds that $\Psi(\tau, g) = E_4(\tau)$, the Eisenstein series of weight 4, since this is the only holomorphic modular form of weight 4 with leading coefficient one. Modulo the action of $\mathrm{SL}_2(\mathbb{Z})$, the divisor $\mathrm{div}(\Psi(g))$ is, of course, given by $\zeta = \frac{1}{2}(1 + \sqrt{-3})$.

2 Orthogonal groups

We give a brief introduction to the theory of symmetric domains for indefinite orthogonal groups and of orthogonal modular forms. Further details on these topics can be found in a number of places, for instance [6], [24] or [19].

In this section, let $V = V(\mathbb{Q})$ be a quadratic space over $\mathbb{Q}$ of signature $(2, n)$, $n \geq 1$, endowed with a non-degenerate indefinite bilinear form, denoted $(\cdot, \cdot)$. Let $q(x) = \frac{1}{2}(x, x)$ be the attached quadratic form. Further, we will often the notation $x^2 = (x, x)$. Denote by $V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R}$, the real quadratic space obtained from $V(\mathbb{Q})$ by extension of scalars, with $(\cdot, \cdot)$ likewise extended to a real-valued form. For later use, we also introduce the notation $V(\mathbb{C}) = V \otimes_{\mathbb{Q}} \mathbb{C}$ for the complexified space with $(\cdot, \cdot)$ extended to a complex bilinear form.

The orthogonal group of V is denoted $\mathrm{O}(V)$. Considered as an algebraic group defined over $\mathbb{Q}$, its set of real points is given by $\mathrm{O}(V)(\mathbb{R})$, the orthogonal group of $V(\mathbb{R})$. Similarly, the special orthogonal groups of $V(\mathbb{Q})$ and $V(\mathbb{R})$ are denoted by $\mathrm{SO}(V)$ and $\mathrm{SO}(V)(\mathbb{R})$, respectively.

Now, there is an exact sequence with the spin group Spin_V, wherein θ denotes the spinor norm:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_V(\mathbb{Q}) \longrightarrow \mathrm{SO}(V) \xrightarrow{\ \theta\ } \mathbb{Q}^\times/\mathbb{Q}^{\times 2}. \tag{5}$$

Looking at the sets of real points, the image of $\mathrm{Spin}_V(\mathbb{R})$ (which, of course is the kernel of θ) is the connected component of the identity in $\mathrm{SO}(V)(\mathbb{R})$. It is referred to as the *spinor kernel* and denoted $\mathrm{O}^+(V)(\mathbb{R})$.

2.1 Models for the symmetric domain of $\mathrm{SO}(V)$

Let K_{SO} be a maximal compact (path-connected) subgroup of $\mathrm{SO}(V)(\mathbb{R})$. A symmetric domain for the operation of $\mathrm{SO}(V)(\mathbb{R})$ on $V(\mathbb{R})$ is given by the quotient

$$\mathrm{SO}(V)(\mathbb{R})/K_{SO}.$$

It is isomorphic to the *Grassmannian* of two-dimensional positive definite oriented subspaces, called the Grassmannian model:

$$\mathbb{D} := \{v \subset V(\mathbb{R});\, \dim v = 2, q\,|_v \geq 0, v \text{ oriented}\}.$$

Note that $\mathbb{D}$ has two connected components, they correspond to the two choices of orientation and are stabilized by the spinor-kernel $\mathrm{O}^+(V)(\mathbb{R})$.

Also, each $v \in \mathbb{D}$, through the decomposition $V(\mathbb{R}) = v \oplus v^\perp$, fixes an isometry between $V(\mathbb{R})$ and the standard pseudo-euclidean space $\mathbb{R}^{2,n}$, with quadratic form $q(x) = \frac{1}{2}\left(x_1^2 + x_2^2 - x_3^2 - \cdots - x_{n+2}^2\right)$. Denoting the special orthogonal groups of $\mathbb{R}^{2,n}$, $\mathbb{R}^{2,0}$ and $\mathbb{R}^{0,n}$ by $\mathrm{SO}(2,n)$, $\mathrm{SO}(2)$ and $\mathrm{SO}(n)$, respectively, we obtain an isomorphism

$$\mathrm{SO}(V)(\mathbb{R})/K_{SO} \simeq \mathrm{SO}(2,n)/\left(\mathrm{SO}(2) \times \mathrm{SO}(n)\right).$$

Remark. *For the orthogonal group* $\mathrm{O}(V)(\mathbb{R})$ *a symmetric domain is given by*

$$\mathrm{O}(V)(\mathbb{R})/K_O \simeq \mathrm{O}(2,n)/\left(\mathrm{O}(2) \times \mathrm{O}(n)\right),$$

with K_O *a maximal compact subgroup. In this case, the Grassmannian model consists simply of the two-dimensional positive-definite subspaces of* $V(\mathbb{R})$ *(without orientation), and there is only one connected component.*

2.1.1 The projective cone model

Let $V(\mathbb{C})$ be the complexified space $V(\mathbb{C}) = V \otimes_{\mathbb{Q}} \mathbb{C}$, as above. Further, denote by $\mathbb{P}V(\mathbb{C})$ the projective space

$$\mathbb{P}V(\mathbb{C}) = \left(V(\mathbb{C}) \setminus \{0\}\right)/\mathbb{C}^\times,$$

and by $\pi : V(\mathbb{C}) \setminus \{0\} \longrightarrow \mathbb{P}V(\mathbb{C})$ the canonical projection.

The *positive cone model* $\mathcal{K}$ is defined as the following subset of $\mathbb{P}V(\mathbb{C})$:

$$\mathcal{K} := \left\{[Z] \in \mathbb{P}V(\mathbb{C});\, (Z,Z) = 0, \left(Z,\bar{Z}\right) > 0\right\},$$

a complex projective manifold of dimension n with two connected components.

Given $Z \in V(\mathbb{C})$ with $\pi(Z) \in \mathcal{K}$, write Z in the form $Z = X + iY$ with $X, Y \in V(\mathbb{R})$. From the definition of $\mathcal{K}$, we have

$$(X,Y) = 0 \qquad \text{and} \quad X^2 = Y^2 > 0.$$

In other words, if $\pi(Z) \in \mathcal{K}$, the real and the imaginary part of Z constitute an orthogonal, normalized and *oriented* basis for a two-dimensional positive subspace of $V(\mathbb{R})$.

Thus, immediately, we have an isomorphism between the models $\mathbb{D}$ and $\mathcal{K}$ given by a real-analytic map:

$$\begin{array}{ccc} \mathcal{K} & \longrightarrow & \mathbb{D} \\ [Z] & \longmapsto & \mathbb{R}X + \mathbb{R}Y. \end{array}$$

We take note of the following properties of $\mathcal{K}$:

1. The special orthogonal group acts on $\mathcal{K}$, with $g[Z] = [gZ]$ for $g \in \mathrm{SO}(V)(\mathbb{R})$.

2. There is an element of order two which interchanges the two connected components of $\mathcal{K}$, thus acting by complex conjugation. In contrast to this, the action of the spinor kernel $\mathrm{O}^+(V)(\mathbb{R})$ stabilizes the connected components.

2.1.2 The tube domain model

Suppose there are two isotropic vectors ℓ, $\ell' \in V(\mathbb{Q})$, with $(\ell, \ell') = 1$. Later on, we will further require there to be an integral lattice $L \subset V$ with $\ell \in L$ and that ℓ' is in contained in the dual lattice L' (see Section 2.2).

Consider the subspace $V_0(\mathbb{R}) = V(\mathbb{R}) \cap \ell^\perp \cap \ell'^\perp$. This is a Lorentzian space, as the restriction $(\cdot,\cdot)\,|_{V_0}$ is a quadratic form with signature $(1, n-1)$. The complexification $V_0(\mathbb{C})$ is a complex quadratic space with the extension of $(\cdot,\cdot)\,|_{V_0}$, as usual. Now, the *tube domain model* is defined as the set

$$\mathcal{H} := \{z = x + iy \in V_0(\mathbb{C});\, q(y) > 0\}. \tag{6}$$

There is an isomorphism between $\mathcal{H}$ and $\mathcal{K}$ given by

$$\mathcal{H} \xrightarrow{\sim} \mathcal{K} : z \longmapsto \big[Z_\ell(z) := z + \ell' - q(z)\ell\big].$$

Whence further,

$$\mathcal{H} \xrightarrow{\sim} \mathbb{D} :\; z \longmapsto w(z) := \mathbb{R}\Re Z_\ell(z) + \mathbb{R}\Im Z_\ell(z).$$

A first non-trivial example for this construction is the following:

Example 2.1. *Let $n = 1$. Then, $V_0(\mathbb{C}) = \mathbb{C}$ and we have*

$$\mathcal{H} = \Big\{z = x + iy \in \mathbb{C}; (\Im z)^2 > 0\Big\} \simeq \mathbb{H} \cup \overline{\mathbb{H}}.$$

We remark at this point, that it may sometimes be useful to restrict to one connected component, as the example shows.

The action of $G = \mathrm{SO}(V)(\mathbb{R})$ on $\mathcal{H}$ is described by the following diagram (with $g \in G$):

$$\begin{array}{ccc}
\mathcal{K} & \xrightarrow{\;[Z] \mapsto g[Z]\;} & \mathcal{K} \\
{\scriptstyle z \mapsto [Z_\ell(z)]}\Big\uparrow & & \Big\uparrow{\scriptstyle gz \mapsto [Z_\ell(gz)]} \\
\mathcal{H} & \xrightarrow[\;z \mapsto gz\;]{} & \mathcal{H}
\end{array}$$

In order for this diagram to commute, we must have

$$\big[gZ_\ell(z)\big] = \big[Z_\ell(gz)\big] \quad (\forall g \in G, \forall z \in \mathcal{H}).$$

Thus, an automorphy factor $j(g,z) : G \times \mathcal{H} \to \mathbb{C}$ is defined by setting

$$gZ_\ell(z) = j(g,z) Z_\ell(gz) \quad (g \in G, z \in \mathcal{H}).$$

Note that if g is actually contained in $g \in \mathrm{SO}(V_0)(\mathbb{R})$, this automorphy factor is trivial.

Example 2.2. *Again, let $n = 1$. Further, let the* level N *be an integer, $N \geq 1$. We consider the space*

$$V = \{x \in \mathrm{Mat}(2 \times 2, \mathbb{Q})\,;\ \mathrm{tr}(x) = 0\}\,,$$

with the quadratic form $q(x) = -N\det(x)$ and the bilinear form $(x, y) = +N\,\mathrm{tr}(xy)$. Setting

$$\ell = \begin{pmatrix} 0 & 1/N \\ 0 & 0 \end{pmatrix}, \quad \ell' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{we get} \quad V_0 = \mathbb{Q}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Also, clearly, $\ell^2 = \ell'^2 = 0$ and $(\ell, \ell') = 1$. The isomorphisms between the tube domain, the projective cone and the Grassmannian model are given by

$$\begin{array}{ccccc} \mathcal{H} = \mathbb{H} \cup \bar{\mathbb{H}} & \longrightarrow & \mathcal{K} & \longrightarrow & \mathbb{D} \\ z = x + iy & \longmapsto & \left[\begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix}\right] & \longmapsto & \mathbb{R}\Re\left(\begin{smallmatrix} z & -z^2 \\ 1 & -z \end{smallmatrix}\right) + \mathbb{R}\Im\left(\begin{smallmatrix} z & -z^2 \\ 1 & -z \end{smallmatrix}\right). \end{array}$$

Now, consider the subgroup of $\mathrm{GL}_2(\mathbb{R})$ consisting of matrices A with $\det(A) = \pm 1$. One can define an isometric action on $V(\mathbb{R})$ by setting

$$(A, X) \mapsto AXA^{adj},$$

where A^{adj} denotes the usual adjoint matrix of A, i.e. with $AA^{adj} = \det(A)E_2$. Thus, there is a homomorphism $\{A \in \mathrm{GL}_2(\mathbb{R}); \det(A) = \pm 1\} \longrightarrow \mathrm{O}(V)(\mathbb{R})$. Its kernel is a subgroup of order two, as clearly A and $-A$ have the same image. We note that $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{O}^+(V)(\mathbb{R})$.

On $\mathcal{K}$, the action is given as follows: Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$ with $\det A = \pm 1$. Then,

$$A\left[\begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix}\right] = \left[A\begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix}A^{adj}\right] = \left[\begin{pmatrix} (az+b)(cz+d) & -(az+b)^2 \\ (cz+d)^2 & -(az+b)(cz+d) \end{pmatrix}\right].$$

The automorphy factor thus is given by $j(g, z) = (cz + d)^2$. Also, we see that the action on $\mathcal{H}$ is compatible with the usual action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H} \cup \bar{\mathbb{H}}$ through Möbius transformations $z \mapsto \frac{az+b}{cz+d}$.

Example 2.3. *Let $n = 2$. A commonly used model for this case is the following*

$$V = \mathrm{Mat}\big(2 \times 2, \mathbb{Q}\big), \qquad q(X) = -\det(X).$$

After setting

$$\ell = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \ell' = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

the subspace V_0 is given by

$$V_0 = \left\{\begin{pmatrix} 0 & x_1 \\ x_2 & 0 \end{pmatrix};\ x_1, x_2 \in \mathbb{Q}\right\}.$$

Now, a subset of $\widetilde{\mathcal{K}}$ *of* $V(\mathbb{C})$ *with* $\pi\big(\widetilde{\mathcal{K}}\big) = \mathcal{K}$ *is given by*

$$\widetilde{\mathcal{K}} = \left\{ \begin{pmatrix} z_1 z_2 & z_1 \\ z_2 & 1 \end{pmatrix} ; z_1, z_2 \in \mathbb{C} \right\}.$$

Hence, for the tube domain, we have

$$\mathcal{H} = \left\{ \begin{pmatrix} z_1 z_2 & z_1 \\ z_2 & 1 \end{pmatrix} \in \widetilde{\mathcal{K}} ; \Im z_1 \cdot \Im z_2 > 0 \right\} \simeq \big(\mathbb{H} \times \mathbb{H}\big) \cup \big(\overline{\mathbb{H}} \times \overline{\mathbb{H}}\big).$$

We can define an isometric action of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ *on* $V(\mathbb{R})$ *by setting*

$$(A, B)X = AXB^{adj} \qquad (A, B \in \mathrm{SL}_2(\mathbb{R}), X \in V(\mathbb{R}))\,,$$

with B^{adj} *the adjoint matrix of* B*. From this we get a homomorphism* $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \to \mathrm{O}(V)(\mathbb{R})$*, which can be shown to be an isogeny. Its image is the connected component* $\mathrm{O}(V)(\mathbb{R})^+$ *and the kernel is a subgroup of order* 4 *[see 19, p. 15]. The action on* $\mathcal{H}$ *is compatible with the usual action of* $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ *on* $\mathbb{C} \times \mathbb{C}$*:*

$$\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) : \quad (z_1, z_2) \longmapsto \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2} \right).$$

We remark that through $\mathrm{SL}_2(K) \hookrightarrow \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ *(see p. 7), one has a homomorphism from* $\mathrm{SL}_2(K)$ *to* $\mathrm{O}(V)(\mathbb{R})^+$*. Hence, the symmetric domain of the Hilbert modular group can be considered as a connected component of* $\mathcal{H}$*.*

2.2 Lattices and modular groups

In the following, let L be an even integral lattice in V, meaning that $\lambda^2 \in 2\mathbb{Z}$ for all $\lambda \in L$ (i.e. $q(\lambda) \in \mathbb{Z}$ for all λ). Let L' be the dual lattice of L, defined as

$$L' = \{v \in V(\mathbb{R});\ (\lambda, v) \in \mathbb{Z} \quad \text{for all } \lambda \in L\} \supseteq L.$$

The quotient L'/L is called the *discriminant group* of L. Let $\mathrm{SO}(L)$ be the group of isometries of L in $\mathrm{SO}(V)$. By $\Gamma_L \subset \mathrm{SO}(L)$, we denote the *discriminant kernel* of L, the subgroup acting trivially on the discriminant group. By a *modular group* we shall understand a subgroup $\Gamma \subset \mathrm{SO}(L)$ which is commensurable with the discriminant kernel. In particular, a modular group has finite index in $\mathrm{SO}(L)$.

Let us introduce one further notation. As in section 2.1.2, let ℓ, ℓ' be isotropic vectors with $(\ell, \ell') = 1$ and, further, assume that $\ell \in L$ and $\ell' \in L'$. Then, we denote by L_0 the Lorentzian lattice given by $L \cap \ell^\perp \cap \ell'^\perp$. Note that $V_0(\mathbb{Q}) = L_0 \otimes \mathbb{Q}$, where V_0 is the Lorentzian space used in the construction of the tube domain.

Definition 2.1. *Let* $\Gamma \subseteq \Gamma_L$ *be a modular group. The quotient* $X_\Gamma = \Gamma \backslash \mathbb{D}$ *is called the (non-compact)* modular variety *associated to* Γ*. By the theory of Baily-Borel, there is a compactification, which we denote by* X_Γ^**. See [19, Chapter II]. For a more general background [see 5, Sections I.4, I.5].*

Remark 2.1. *The compactified modular variety X_Γ^* gives rise to a Shimura variety [see 7, Section 1.5] (of course, one has to take the non-archimedian places into account for this, too).*

Example 2.4. *In the setup of Examples 2.1 and 2.2, and using the same notation, the following set L is an even integral lattice and L' its dual:*

$$L = \left\{ \begin{pmatrix} b & -a/N \\ c & -b \end{pmatrix};\ a,b,c \in \mathbb{Z} \right\}, \qquad L' = \left\{ \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix};\ a,b,c \in \mathbb{Z} \right\}.$$

The discriminant group L'/L is isomorphic to $\mathbb{Z}/2N\mathbb{Z}$. It is easily verified that $\Gamma_0(N)$ acts trivially on the discriminant group and, in fact, $\Gamma_0(N) = \Gamma_L \cap \mathrm{O}^+(L)$.

In classical language, the modular variety corresponding to the quotient $\Gamma\backslash\mathcal{H}$ is given by the modular curve $X_0(N)$. In particular, for $N = 1$, we have $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}^* \simeq X_0(1)^*$, [see 16, Section 2.4, Section 7]. Its points correspond to isogeny classes of elliptic curves (more generally, the points of $X_0(N)$ describe cyclic N-isogenies of elliptic curves).

2.3 Special cycles

For the following, [cf. 24, Section 2.1.2] or [cf. 6, p. 119]. Let $W \subset V(\mathbb{R})$ be a negative definite one-dimensional subspace. Then, a codimension-one sub-Grassmannian is given by

$$\mathbb{D}_W := \{v \in \mathbb{D};\ v \perp W\} \ \subset \mathbb{D}.$$

It defines a codimension-one submanifold of the projective cone $\mathcal{K}$, also denoted by $\mathbb{D}_W$ which, in turn, corresponds to a subset of the tube domain. In the following, if w is a negative definite vector, we further simplify notation by setting $\mathbb{D}_w := \mathbb{D}_{\mathbb{R}w}$.

Example 2.5. *Taking up the $n = 1$ examples 2.1, 2.2 and 2.4 set $N = 2$ and consider*

$$w = \begin{pmatrix} b/4 & c/2 \\ -a/2 & -b/4 \end{pmatrix}, \quad \textit{with} \quad a,b,c \in \mathbb{Z} \quad \textit{and} \quad b^2 - 4ac < 0.$$

Then,

$$\begin{aligned} \mathbb{D}_w &= \left\{ z \in \mathbb{H} \cup \bar{\mathbb{H}};\ 2\,\mathrm{tr}\left(w \cdot \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \right) = 0 \right\} \\ &= \left\{ z \in \mathbb{H} \cup \bar{\mathbb{H}};\ az^2 + bz + c = 0 \right\}. \end{aligned}$$

Then, $\mathbb{D}_w$ consists of CM-points in $\mathbb{H} \cup \bar{\mathbb{H}}$. (By a common abuse of notation, $\mathbb{D}_w$ is also used to denote the subset of the tube domain.)

The case where W is defined by a lattice vector is particularly important. As before, let L be an even integral lattice, and L' it dual. We define:

Definition 2.2.

1. *Assume that λ is a lattice vector with $\lambda \in L'$ and with $q(\lambda) = m$, $m \in \mathbb{Z}_{<0}$. Then, $\mathbb{D}_\lambda$ is called the* primitive Heegner divisor *attached to λ.*

2. *Let $\gamma \in L'/L$ be an element of the discriminant group and m a negative integer. The* Heegner divisor of index (γ, m) *is defined as*

$$\mathcal{Z}(\gamma, m) := \sum_{\substack{\lambda \in \gamma + L \\ q(\lambda) = m}} \mathbb{D}_\lambda. \tag{7}$$

The sum runs over a system of representatives for $\gamma \in L'/L$.

Note that the sum in (7) is Γ_L-invariant. Thus, $\mathcal{Z}(\gamma, m)$ is, in fact, the pre-image under the canonical projection of a divisor on the modular variety $X_{\Gamma_L} = \Gamma_L \backslash \mathcal{H}$. Usually, the term Heegner divisor is used both for the divisor on X_{Γ_L} and for its pre-image. Also by abuse of notation, both are denoted $\mathcal{Z}(\gamma, m)$.

2.4 Modular Forms

We use the notation established before. Hence, let L be an even integral lattice, and $\Gamma_L \subset \mathrm{SO}(L)$ the discriminant kernel of L. Also assume that the isotropic vectors from Section 2.1.2 are lattice vectors, with $\ell \in L$, $\ell' \in L'$. Then, the tube domain is contained in $V_0(\mathbb{C}) = L_0 \otimes \mathbb{C}$ with $L_0 = L \cap \ell^\perp \cap \ell'^\perp$.

Definition 2.3. *Let k be an integer and Γ an orthogonal modular group. A function $f : \mathcal{H} \to \mathbb{C}$ is called a holomorphic modular form of weight k on Γ, if the following conditions are satisfied:*

1. *$f(\gamma z) = j(\gamma, z)^k f(z)$ for all $\gamma \in \Gamma$.*

2. *f is holomorphic on $\mathcal{H}$.*

3. *f is holomorphic on the boundary of $\mathcal{H}$.*

Note that by the Koecher principle ([see 19, Theorem IV.3.6]), for holomorphic modular forms, the third condition can be omitted if $n > 2$. More generally, the Koecher principle is valid, if the Witt-rank of $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$, i.e. the dimension of a maximal totally isotropic subspace, is less than n. (For example, this is the case for Hilbert modular forms, cf. p. 7.)

Meromorphic (etc.) modular forms are defined similarly, with 2. and 3. replaced by suitable conditions on $\mathcal{H}$ and on the boundary components. Also, the definition can easily be extended to accommodate for half-integral weights and multiplier systems.

We will not say much about the properties of modular forms for orthogonal groups, but let us at least mention that they admit Fourier expansions:

If f is a modular form for a modular group Γ, as in Definition 2.3, there is a lattice M in V_0 such that $f(z + \mu) = f(z)$ for all $\mu \in M$. For example, if $\Gamma = \Gamma_L$, then $M = L_0$. Thus, f has a Fourier expansion of the form

$$f(z) = \sum_{\mu \in M'} a(\mu) e\left((\mu, z)\right).$$

Due to the Koecher principle or, if necessary, by condition 3., μ's with $a(\mu) \neq 0$ satisfy a positivity condition [see 19, Section IV.3].

3 The singular theta lift

For this section, recall our convention that $\tau = u + iv$ denote a point in the complex upper half-plane $\mathbb{H}$. In the following, $\sqrt{\tau} = \tau^{1/2}$ is the principal branch of the complex square root, with $\arg(\sqrt{\tau}) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Further, z shall denote a point in $\mathcal{H}$ and $w(z)$ the attached positive definite subspace in $\mathbb{D}$.

We would like to mention some general references, which, among them, cover most of this section: Beside the original works of Borcherds [4] and of Bruinier [6], these are [36] and the lecture notes [7].

3.1 The Weil representation

Consider the metaplectic group $\mathrm{Mp}_2(\mathbb{R})$, the double cover of $\mathrm{SL}_2(\mathbb{R})$. It can be written as the set of pairs $(M, \phi(\tau))$, with $M \in \mathrm{SL}_2(\mathbb{R})$ and $\phi(\tau)$ a holomorphic square root of $c\tau + d$. In particular, $\mathrm{Mp}_2(\mathbb{Z})$ is generated by the elements

$$S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right), \quad \text{and} \quad T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right).$$

The center of $\mathrm{Mp}_2(\mathbb{Z})$ is generated by

$$Z = S^2 = (TS)^3 = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i \right).$$

If Γ is an elliptic modular group, we denote the pre-image under $\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$ by $\widetilde{\Gamma}$, i.e. $\widetilde{\Gamma_1} = \mathrm{Mp}_2(\mathbb{Z})$, $\widetilde{\Gamma_0}(N)$ etc.

Now, there is a representation ρ_L of $\mathrm{Mp}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[L'/L]$, defined through the action of the above generators on the basis elements $\mathfrak{e}_\mu$:

$$\begin{aligned} \rho_L(T)\mathfrak{e}_\mu &= e\left(q(\mu)\right)\mathfrak{e}_\mu, \\ \rho_L(S)\mathfrak{e}_\mu &= \frac{\sqrt{i}^{n-2}}{\sqrt{|L'/L|}} \sum_{\nu \in L'/L} e\left(-(\mu, \nu)\right)\mathfrak{e}_\nu. \end{aligned}$$

Also, the action of Z is given by $\rho_L(Z)\mathfrak{e}_\mu = i^{n-2}\mathfrak{e}_\mu$.

Essentially, ρ_L is the Weil representation, for more details we refer to Shintani [see 38] and, for a description using the language of adeles, to [7, Sections 3.1, A].

Remark 3.1. *If n is even, the representation ρ_L of $\mathrm{Mp}_2(\mathbb{Z})$ factors through a representation of $\mathrm{SL}_2(\mathbb{Z})$. Also, the representation factors over the finite group $\mathrm{Mp}_2(\mathbb{Z}/N_L\mathbb{Z})$, where N_L is the level L, defined as the the smallest positive integer N satisfying $Nq(\gamma) \in \mathbb{Z}$ for all $\gamma \in L'$; if n is even, ρ_L factors over $\mathrm{SL}_2(\mathbb{Z}/N_L\mathbb{Z})$.*

We denote the standard hermitian scalar product on $\mathbb{C}[L'/L]$ by $\langle \cdot, \cdot \rangle$, i.e.

$$\Big\langle \sum_{\mu \in L'/L} a_\mu \mathfrak{e}_\mu, \sum b_\mu \mathfrak{e}_\mu \Big\rangle = \sum_{\mu \in L'/L} a_\mu \overline{b_\mu}. \tag{8}$$

With this, for $\mu, \nu \in L'/L$ and $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$, the matrix coefficient $\rho_{\mu\nu}(M, \phi)$ of the representation ρ_L is given by

$$\rho_{\mu\nu}(M, \phi) = \langle \rho_L(M, \phi)\mathfrak{e}_\mu, \mathfrak{e}_\nu \rangle.$$

Finally, the dual representation ρ_L^* for $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$ given in terms of its matrix coefficients is the complex conjugate of the matrix $\big(\rho_{\mu\nu}(M, \phi)\big)_{\mu,\nu \in L'/L}$.

We briefly recall the definitions of vector-valued modular forms for the representation ρ_L, more details can be found in the course notes of Claudia Alfes-Neumann [1].

Definition 3.1. *Let $k \in \frac{1}{2}\mathbb{Z}$ be a half-integer. A smooth function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ which transforms under ρ_L according to*

$$(M, \phi)f(\tau) = \phi(\tau)^{2k}\rho_L(M, \phi)f(M\tau), \quad ((M, \phi) \in \mathrm{Mp}_2(\mathbb{Z}))$$

is called

1. *a* weakly holomorphic modular form*, if f is holomorphic on $\mathbb{H}$ and meromorphic at the cusp ∞,*
2. *a* holomorphic modular form*, if f is holomorphic on $\mathbb{H}$ and at the cusp, Further, f is called a* cusp form *if f is holomorphic and vanishing at the cusp.*

We denote the by $\mathrm{S}_{k,\rho_L} \subset \mathrm{M}_{k,\rho_L} \subset \mathrm{M}^!_{k,\rho_L}$ the spaces of cusp forms, holomorphic modular forms and weakly holomorphic modular forms transforming under the Weil representation, respectively.

We remark that, similarly, vector valued modular forms can be defined for the dual representation ρ_L^*, i.e S_{k,ρ_L^*}, M_{k,ρ_L^*} and $\mathrm{M}^!_{k,\rho_L^*}$.

Next, following [10, Section 3] we introduce harmonic Maass forms.

Definition 3.2.

Let $k \in \frac{1}{2}\mathbb{Z}$ A twice continuously differentiable function $f : \mathbb{H} \to \mathbb{C}$ is called a harmonic Maass form (or harmonic weak Maass form) with representation ρ_L for $\mathrm{Mp}_2(\mathbb{Z})$ if

1. *$(M, \phi)f(\tau) = \phi(\tau)^{2k}\rho_L(M, \phi)f(M\tau)$ for all $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$,*
2. *There is a $C > 0$ such that $f(\tau) = \mathrm{O}\big(e^{Cv}\big)$ as $v \to \infty$ (uniformly in u),*
3. *f is annihilated by the weight-k Laplace operator, $\Delta_k f(\tau) = 0$, with*

$$\Delta_k = -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + iku \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v} \right).$$

We denote the space of harmonic Maass forms by H_{k,ρ_L}.

Further we denote by H^+_{k,ρ_L} the subspace of harmonic Maass forms f, which additionally to 1.–3. satisfy the following condition: The image of f under the ξ-operator

$$\xi_k := 2iv^k \frac{\partial}{\partial \bar{z}}$$

is a cusp form for the dual representation ρ_L^* with $\xi_k(f)(\tau) \in \mathrm{S}_{2-k,\rho_L^*}$.

Note that the component functions of an elliptic modular form $f = \sum_\mu f_\mu \mathfrak{e}_\mu$ are scalar valued elliptic modular forms of the appropriate type (i.e. weakly holomorphic, holomorphic or cuspidal) and the same weight for (at least) the principal congruence subgroup $\Gamma(N_L)$, where the level N_L is determined as in Remark 3.1. The same applies for harmonic Maass forms.

Due to invariance under $T \in \mathrm{Mp}_2(\mathbb{Z})$, a weakly holomorphic modular form f with representation ρ_L, admits a Fourier expansion around the cusp ∞ of the following form:

$$f(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\mu) \\ m \gg -\infty}} c(\mu, m) q^m \mathfrak{e}_\mu, \tag{9}$$

with only finitely many $m < 0$ for which $c(\mu, m) \neq 0$. If f is a holomorphic modular form, then $c(\mu, m) \neq 0$ only for $m \geq 0$, and for a cusp form, $c(\mu, m) \neq 0$ only for $m > 0$.

The Fourier expansion of a harmonic Maass form $f \in \mathrm{H}^+_{k,\rho_L}$ $(k \neq 1)$ consists of a holomorphic part f_+ similar to (9) and a non-holomorphic part f_- involving certain special functions, see for example [1, Section 3]. We will need the Fourier expansion only in the case where $k < 1$, for which it takes the following form:

$$\begin{aligned} f(\tau) &= f^+(\tau) + f^-(\tau) \\ &= \sum_{\mu \in L'/L} \sum_{m \gg -\infty} c^+(m, \mu) q^m \mathfrak{e}_\mu + \sum_{\mu \in L'/L} \sum_{m<0} c^-(m, \mu) \Gamma\left(1-k, 4\pi|m|v\right) q^m \mathfrak{e}_\mu, \end{aligned} \tag{10}$$

with the incomplete Gamma function $\Gamma(a, x) = \int_x^\infty e^{-r} r^{a-1} dr$ [cf. 17, 8.2.2].

Remark 3.2. *As Bruinier and Funke have shown [see 10] the condition $\xi(f)(\tau) \in \mathrm{S}_{2-k,\rho_L^*}$ for $f \in \mathrm{H}^+_{k,\rho_L}$ has immediate consequences for the growth behavior of f: Denote by $P(f)$ the principal part of f, i.e. the Fourier polynomial given by*

$$P(f)(\tau) := \sum_{\mu \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\mu) \\ 0 > m \gg -\infty}} c(\mu, m) q^m \mathfrak{e}_\mu.$$

Then, for $f \in \mathrm{H}^+_{k,\rho_L}$, $f - P(f)$ decays exponentially as $v \to \infty$. For the Fourier expansion given in (10) *(for $k \neq 1$), this is can also be seen from the asymptotic behavior of the incomplete Gamma function.*

3.2 Siegel theta functions

In this section we want to introduce the Siegel theta-function attached to the lattice L, integrating against which will yield the theta-lift. For a concise yet very readable treatment in the language of representation theory see [31].

Definite theta functions To begin, we start with a simple example for a theta-function attached to a lattice. For this, let M be a positive definite even lattice, of rank $l \geq 1$

and endowed with a quadratic form $q(\cdot)$. Then, generalizing the well known Jacobi theta-function $\theta_0(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}$, one sets, if M is unimodular

$$\Theta_M(\tau) = \sum_{\lambda\in M} q^{\frac{1}{2}\lambda^2} = \sum_{\lambda\in M} e\big(q(\lambda)\,\tau\big).$$

Otherwise, if M'/M is non-trivial, one sets

$$\Theta_M(\tau) = \sum_{\mu\in M'/M}\sum_{\lambda\in\mu+M} e\big(q(\lambda)\,\tau\big)\mathfrak{e}_\mu.$$

Clearly, in both cases the series converges absolutely and uniformly and hence defines a holomorphic function on $\mathbb{H}$. Using Poisson summation, it is fairly straight-forward to show that $\Theta_M(\tau)$ transforms as a modular form of weight $l/2$,

If, contrastingly, the lattice is indefinite, to assure absolute convergence of the theta-series, we have to replace $q(\lambda)$ by a majorant.

The Siegel theta function Thus, let L be an indefinite even lattice, as in section 2, with $L \subset V$ and with $V = L \otimes \mathbb{Q}$ an indefinite quadratic space of signature $(2, n)$. The quadratic form is again denoted $q(\cdot)$. We will now attach an absolutely convergent theta-series to L and at the same time obtain a function on $\mathbb{H} \times \mathbb{D}$.

Recall that $\mathbb{D}$ consists of maximal positive definite (oriented) subspaces. Given a maximal positive definite subspace $w \subset V(\mathbb{R})$, we decompose $V = w \oplus w^\perp$. Naturally, $w^\perp$ is negative definite. Writing $a \in V(\mathbb{R})$ as $a_w + a_{w^\perp}$, the majorant $q_w(a)$ is given by $q(a_w) - q(a_{w^\perp})$.

Further, recall that to every $z \in \mathcal{H}$, we can associate a positive definite subspace $w(z) \in \mathbb{D}$. To simplify notation, we write a_z and $a_{z^\perp}$ for the projections $a_{w(z)}$ and $a_{w(z)^\perp}$, respectively. Now, for $\tau = u + iv \in \mathbb{H}$, we define

$$\frac{1}{2}(x,x)_{z,\tau} := q(x)\,u + q_{w(z)}(x)v = q(x_z)\,\tau + q(x_{z^\perp})\,\bar{\tau}. \qquad (x \in V(\mathbb{R}))\,.$$

Then, for every $z \in \mathcal{H}$, the following function, called the *Gaussian*, is rapidly decreasing,

$$\phi(x,z,\tau) := e\left(\tfrac{1}{2}(x,x)_{z,\tau}\right), \tag{11}$$

in other words, ϕ is a Schwartz function on $V(\mathbb{R})$.

This leads to the following definition of a theta-function attached to L:

Definition 3.3. *The Siegel theta-function $\Theta_L(\tau,z) : \mathbb{H}\times\mathbb{D} \to \mathbb{C}[L'/L]$ is given by*

$$\Theta_L(\tau,z) = \sum_{\mu\in L'/L} \theta_\mu(\tau,z)\mathfrak{e}_\mu, \tag{12}$$

with component functions

$$\theta_\mu(\tau,z) = \sum_{\lambda\in\mu+L} \phi(\lambda,z,\tau) = \sum_{\lambda\in\mu+L} e\left(\tau q(\lambda_z) + \bar{\tau} q(\lambda_{z^\perp})\right). \tag{13}$$

Due to the rapid decay of the Gaussian, the series defining $\Theta_L(\tau, z)$ is absolutely convergent. Its transformation behavior is given by the following theorem, which can be proved using Poisson summation [see 4, Theorem 4.1].

Theorem 3.1. *For $\gamma = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \phi(\tau)\right) \in \mathrm{Mp}_2(\mathbb{Z})$, we have*

$$\Theta_L(\gamma\tau, z) = \phi(\tau)^2 \overline{\phi(\tau)}^n \rho_L(\gamma) \Theta_L(\tau, z).$$

Also, $\Theta_L(\tau, z)$ is invariant under $\mathrm{SO}(V)(\mathbb{R})$.

It is worth mentioning that the theorem in [4] is much more general. For example, Borcherds allows for arbitrary signature (p, n) and also covers Siegel theta-functions with a fairly general harmonic polynomial as an additional factor together with $\phi(x, z, \tau)$.

The theta integral Now, let $f \in \mathrm{M}_{k,\rho_L}$ be a modular form transforming under the Weil representation ρ_L, and consider the theta-integral given by

$$\int_{\mathcal{F}} \langle f(\tau), \Theta_L(\tau, z)\rangle v \, d\mu \qquad (\text{with} \quad d\mu = \frac{du\, dv}{v^2}). \tag{14}$$

Here, $\mathcal{F}$ denotes a fundamental domain for the operation of $\mathrm{Mp}_2(\mathbb{Z})$, while $\langle \cdot, \cdot \rangle$ is the hermitian scalar product on $\mathbb{C}[L'/L]$ from (8). Note that $d\mu$ is the left-invariant Haar measure for the operation of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{H}$.

By theorem 3.1, if f has weight $k = 1 - n/2$, the expression under the integral is invariant under $\mathrm{Mp}_2(\mathbb{Z})$. Thus, we may expect to evaluate the integral by using unfolding.

However, there are two problems:

1. The space $\mathrm{M}_{1-\frac{n}{2},\rho_L}$ is often trivial. Indeed, if $n > 2$, then $\mathrm{M}_{1-\frac{n}{2},\rho_L} = \{0\}$.
2. A possible solution is to extend to $\mathrm{M}^!_{1-\frac{n}{2},\rho_L}$, allowing f to be weakly holomorphic. However this entails a new difficulty: The integral in (14) no longer converges. (Hence the name 'singular' theta-lift.)

Thus, if we admit weakly holomorphic modular form contained in $\mathrm{M}^!_{1-\frac{n}{2},\rho_L}$ as input functions, which is desirable, we have to replace the theta-integral in (14) by a suitably regularized integral. This is what we will do in the next section.

Remark 3.3. *To avoid these difficulties, one can also use a more refined kernel function instead of the Gaussian. Most commonly, one introduces a homogeneous polynomial as a further factor, the degree of which then enters into the transformation behavior of the theta-function.*

An example for this is the following kernel function, φ_r $(r \in \mathbb{N})$ defined as

$$\varphi_r(\lambda, z, \tau) := \frac{(\lambda, \overline{w(z)})}{(y, y)^r} \phi(\lambda, z, \tau) \quad (z \in \mathcal{H}).$$

With this kernel function, the theta-integral is $\mathrm{Mp}_2(\mathbb{Z})$-invariant for input functions of weight $k = 1 - \frac{n}{2} + r$. Indeed, for suitable $r > 1$, the space M_{k,ρ_L} is non-trivial. Also, in this case the theta-integral converges without need for any regularization. The kernel function φ_r leads to the Shintani-Oda-Gritsenko lifting, see [38]

3.3 The regularized theta lift

We set $k = 1 - \frac{n}{2}$. Somewhat more generally, following Bruinier-Funke [10], we extend $\mathrm{M}^!_{k,\rho_L}$ to H^+_{k,ρ_L}, the space of harmonic Maass forms introduced in section 3.1. Recall from (10) the Fourier expansion for a harmonic Maass form $f \in \mathrm{H}^+_{k,\rho_L}$ (note that $k < 1$):

$$\begin{aligned} f(\tau) &= \sum_{\mu \in L'/L} \left(f^+_\mu(\tau) + f^-_\mu(\tau)\right) \mathfrak{e}_\mu \\ &= \Big[\sum_{m \gg -\infty} c^+(m,\mu) q^m + \sum_{m<0} c^-(m,\mu)\Gamma\left(1-k, 4\pi|m|v\right) q^m\Big] \mathfrak{e}_\mu, \end{aligned} \tag{15}$$

where we denote by f^+_μ and f^-_μ the components of the holomorphic part f^+ and the non-holomorphic part f^- of f, respectively. Recall that each component function $f_\mu = f^+_\mu + f^-_\mu$ is a scalar valued harmonic Maass form.

Note the asymptotic behavior of the non-holomorphic part for $v \to \infty$: Since the incomplete Γ-functions (or, more generally the M-Whittaker functions they are related to) are of rapid decay [see 17, Sections 8.11, 8.12 and 13.21], f^- decays rapidly, too. (Also, see Remark 3.2.)

Thus, for the question of convergence or non-convergence of the theta-integral in (14), only the f^+ part plays a role. So, to formulate the necessary regularization recipe, we look at the integral $\int_{\mathcal{F}} \langle f^+, \Theta_L \rangle v \, d\mu$.

The regularization we describe is due to Harvey, Moore [23] and Borcherds [4], [see also 6, Section 2.2]. For $t \in \mathbb{R}_{>0}$, define the truncated fundamental domain $\mathcal{F}_t$ as follows

$$\mathcal{F}_t := \mathcal{F} \cap \{\tau \in \mathbb{H}; \Im\tau \le t\} = \left\{\tau = u + iv; |\tau| > 1, -\tfrac{1}{2} < u < \tfrac{1}{2}, 0 < v \le t\right\}.$$

Clearly $\mathcal{F}_t$ is compact. Hence, since Θ_L and f^+ are holomorphic as functions of τ on $\mathbb{H}$, the definite integral

$$\int_{\mathcal{F}_t} \langle f^+, \Theta_L \rangle v \, d\mu$$

is well-defined. One can take the limit $t \to \infty$ and, providing it exists, define the regularized integral accordingly.

Actually, the constant coefficient $c^+(0,0)$ still poses a problem, as we will see presently. But, excluding this coefficient, the following regularization can be used.

Definition 3.4 (Regularization 1)**.** *If the constant term $c^+(0,0)$ in the Fourier expansion of f vanishes, the regularized integral is defined as*

$$\int_{\mathcal{F}}^{reg} \langle f, \Theta_L \rangle v \, d\mu := \lim_{t \to \infty} \int_{\mathcal{F}_t} \langle f, \Theta_L \rangle v \, d\mu.$$

We note that since the integral is definite, we are allowed to interchange the order of integration.

To see why it is necessary to require $c^+(0,0) = 0$, consider the Fourier expansions of $f^+(\tau)$ and of $\overline{\Theta}_L(\tau)$. (Note that the expression under the integral is periodic with period length 1):

$$\begin{aligned} f^+(\tau) &= \sum_{\mu \in L'/L} \sum_m c^+(\mu, m) e(m\tau) \mathfrak{e}_\mu, \\ \overline{\Theta}_L(\tau) &= \sum_{\mu \in L'/L} \sum_{\lambda \in \mu + L} e^{-4\pi v q(\lambda_z)}\, e(-q(\lambda)\,\tau) \mathfrak{e}_\mu. \end{aligned} \tag{16}$$

Due to absolute convergence, we many integrate term by term. Thus,

$$\begin{aligned} \int_{-\frac{1}{2}}^{\frac{1}{2}} \langle f^+, \Theta_L \rangle(\tau) du &= \sum_{\mu \in L'/L} \sum_m c^+(\mu, m) \sum_{\lambda \in \mu + L} e^{-4\pi q(\lambda_z) v} \int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(u(m - q(\lambda))\right) du \\ &= \sum_{\mu \in L'/L} \sum_{\lambda \in \mu + L} e^{-4\pi q(\lambda_z) v}\, c^+(\mu, q(\lambda)). \end{aligned} \tag{17}$$

Hence, the contribution of the constant term to the integral over $\mathcal{F}_t$ is given by

$$c^+(0,0) \lim_{t \to \infty} \int_{\mathcal{F}_t} v \frac{dv du}{v^2} = c^+(0,0) \lim_{t \to \infty} \int_{v=0}^{t} \frac{dv}{v} = c^+(0,0) \left[\int_{v=0}^{1} \frac{dv}{v} + \lim_{t \to \infty} \int_1^t \frac{dv}{v} \right].$$

Clearly, on the right hand side, the first integral is divergent, as is the limit of the second integral.

Thus, a slightly more elaborate regularization recipe is needed here, which of course also works if $c^+(0,0) = 0$:

Definition 3.5 (Regularization 2)**.** *If for $s \in \mathbb{C}$ with $\Re(s) \gg 0$ the limit*

$$g(s) = \lim_{t \to \infty} \int_{\mathcal{F}_t} \langle f, \Theta_L \rangle v^{1-s} d\mu$$

exists and has a meromorphic continuation on $\mathbb{C}$, then the regularized integral is defined as the constant term of the Laurent expansion of $g(s)$ at [1] $s = 0$, denoted $\mathcal{C}_{s=0}[g(s)]$:

$$\int_{\mathcal{F}}^{reg} \langle f, \Theta_L \rangle v \, d\mu := \mathcal{C}_{s=0} \left[\lim_{t \to \infty} \int_{\mathcal{F}_t} \langle f, \Theta_L \rangle v^{1-s} \, d\mu \right].$$

The regularized lift will give us a smooth function $\Phi(z, f)$ which still has some singularities. Beside determining their location, we also want to describe the behavior of $\Phi(z, f)$ around these singularities. For the following it is somewhat more natural to consider the regularized integral as a function on $\mathbb{D}$, rather than on $\mathcal{H}$.

We define the *type of a singularity* as follows:

Definition 3.6. *Let $U \subset \mathbb{D}$ be an open subset and f, g functions on a dense open subset of U. We say that f has a singularity of type g, if $f - g$ can be continued to a real analytic function on U. In this case, we write $f \simeq_U g$.*

[1] If 0 happens to be a pole, yet another, slight variation of this recipe is needed, see [6].

Let $f \in \mathrm{H}^+_{k,\rho_L}$ be a harmonic Maass form with Fourier expansion as in (15). Further, assume that $c^+(\mu, m) \in \mathbb{Z}$ for all $m < 0$. We define a Heegner divisor associated to f by setting

$$\mathcal{Z}(f) := \sum_{\mu \in L'/L} \sum_{m<0} c^+(\mu, m)\mathcal{Z}(\mu, m), \tag{18}$$

where the $\mathcal{Z}(\mu, m)$ are the Heegner divisors of index (μ, m) from Definition 2.2.

Theorem 3.2 (Borcherds-Bruinier, cf. [4, Theorem 6.2], [6, Theorem 2.12])**.** *The function $\Phi(z, f)$ given by the regularized integral*

$$\Phi(z, f) = \int_{\mathcal{F}}^{reg} \langle f, \Theta_L \rangle \, v \, d\mu, \tag{19}$$

considered as a function on $\mathbb{D}$, is real-analytic on $\mathbb{D} \setminus \sup(-2\mathcal{Z}(f))$ and takes singularities of logarithmic type along the divisor $-2\mathcal{Z}(f)$ (i.e. for every $w \in \mathbb{D}$, there is a neighborhood $w \in U \subset \mathbb{D}$ and a local equation $\operatorname{div}(g) = -2\mathcal{Z}(f) \mid_U$ with a meromorphic function g, such that $\Phi \simeq_U \log|g|$).

We give a brief sketch of the calculations involved in the proof, following [7]:

Proof. To determine the divisor of $\Phi(z, f)$, we need to work out the integral up to smooth functions. First, split up the integral into two parts, one over $z \in \mathcal{F}$ with $\Im z \leq 1$ and one over z with $\Im z > 1$.

$$\Phi(z, f) = \int_{\mathcal{F}_1}^{reg} \langle f, \Theta_L \rangle v \, d\mu + \int_{\mathcal{F}_{>1}}^{reg} \langle f, \Theta_L \rangle v \, d\mu.$$

Clearly, the first integral is smooth, and it suffices to consider the second integral. Further, due to the rapid decay of the non holomorphic part, only the contribution of f^+ matters here. Thus, consider

$$\lim_{t \to \infty} \int_{v=1}^{t} \int_{u=-\frac{1}{2}}^{\frac{1}{2}} \left\langle f^+, \Theta_L \right\rangle v^{1-s} d\mu. \tag{20}$$

Since the expression under the integral is periodic in the indeterminate τ, we can insert the Fourier expansion of f^+ and $\overline{\Theta}_L$ and carry out integration over u, as above. With (17) we get:

$$\lim_{t \to \infty} \int_{v=1}^{t} \sum_{\lambda \in L'} e^{-4\pi q(\lambda_z) v} c^+(\lambda, q(\lambda)) \frac{dv}{v^{s+1}}.$$

We now split the sum into three parts: First, the sum over $\lambda \neq 0$ with $q(\lambda) \geq 0$, second the term for $\lambda = 0$ and third the sum over λ with $q(\lambda) < 0$. Also, since the integral is definite, we can interchange the order of integration. Absolute convergence allows the limit to be taken term-wise.

So, first consider

$$\int_{v=1}^{t} \sum_{\substack{0 \neq \lambda \in L' \\ q(\lambda) \geq 0}} e^{-4\pi q(\lambda_z) v} c^+(\lambda, q(\lambda)) \frac{dv}{v^{s+1}}. \tag{21}$$

We will estimate the growth of the sum under the integral: Applying the Hecke estimate [see e.g. 12, I. Proposition 8] to the Fourier coefficients $c^+(\lambda, q(\lambda))$, we see that their asymptotic behavior as $q(\lambda)$ increases is $\mathrm{O}(e^{c\sqrt{q(\lambda)}})$ with some constant $c > 0$. We rewrite the argument of the exponential as follows:

$$-4\pi q(\lambda_z)\, v = 2\pi \left[q(\lambda_{z^\perp}) - q(\lambda_z)\right] v - 2\pi q(\lambda)\, v.$$

Note that the first term is a negative define quadratic form. It follows that the asymptotic behavior of $c^+(\lambda, q(\lambda))e^{-4\pi q(\lambda_z)v}$ is given by $\mathrm{O}(e^{-q(\lambda)})$. Hence, the integral (21) contributes only a smooth function.

Now, for the term with $\lambda = 0$: We get the integral expression

$$c^+(0,0)\int_{v=1}^{t} \frac{dv}{v^s},$$

of which, after regularization, only a constant remains.

Finally, from the third sum, with $q(\lambda) < 0$, we get the following contribution to the regularized integral

$$\sum_{\substack{\lambda\in L'\\ q(\lambda)\leq 0}} c^+(\lambda, q(\lambda))\, \mathcal{C}_{s=0}\left[\int_{v=1}^{\infty} e^{-4\pi q(\lambda_z)v}\frac{dv}{v^{1+s}}\right].$$

We can express this in terms of the incomplete Γ-function, $\Gamma(a,x) = \int_x^\infty e^{-r}r^{a-1}dr$ [cf. 17, 8.2.2], with $a = 0$ and $x = 4\pi|q(\lambda_z)|$. Thus, after regularization and *up to smooth functions*, $\Phi(z,f)$ is given by

$$\Phi(z,f) \simeq \sum_{\substack{\lambda\in L'\\ q(\lambda)\leq 0}} c^+(\lambda, q(\lambda))\Gamma\left(0, 4\pi|q(\lambda_z)|\right).$$

Now, we study the behavior of $\Phi(z,f)$ locally around a given point $w(z_0) \in \mathbb{D}$. From the definition of $\Gamma(a,x)$, by partial integration,

$$\Gamma(0,x) = -\left[e^{-r}\log(r)\right]_x^\infty + \int_x^\infty e^{-r}\log(r)\, dr.$$

one can see that near $x = 0$, the function $\Gamma(0,x)$ behaves like $-\log(x)$ and is otherwise smooth. Thus, we write the above sum as follows:

$$\sum_{\mu\in L'/L}\sum_{m<0} c^+(\mu,m)\Bigg[\sum_{\substack{\lambda\in\mu+L\\ q(\lambda)=m\\ \lambda\not\perp z_0}}\Gamma\left(0,4\pi|\lambda_z^2|\right) + \sum_{\substack{\lambda\in\mu+L\\ q(\lambda)=m\\ \lambda\perp z_0}}\Gamma\left(0,4\pi|\lambda_z^2|\right)\Bigg].$$

The first, sum over all λ with $\lambda \not\perp w(z_0)$ contributes a function which is smooth on a small neighborhood of $w(z_0)$. This can be shown using reduction theory. The remaining λ with $\lambda \perp w(z_0)$ generate a positive definite sublattice, and thus the second sum is finite. Hence, locally near $w(z_0)$ and up to smooth functions $\Phi(z, f)$ is given by the finite sum

$$-\sum_{\mu\in L'/L}\sum_{m<0} c^+(\mu, m) \sum_{\substack{\lambda\in\mu+L\\ q(\lambda)=m\\ \lambda\perp z_0}} \log|\lambda_z^2|.$$

We conclude that the divisor of $\Phi(z, f)$ is given by a *locally finite* sum of the primitive Heegner divisors $\mathbb{D}_\lambda$, and get $\operatorname{div}(\Phi) = -2\mathcal{Z}(f)$. Also, clearly, the singularities are of logarithmic type, as claimed. This completes the proof. □

Remark 3.4. *Beside its singularities, the function $\Phi(z, f)$ has a number of further remarkable properties. Just to mention a few:*

1. *Bruinier showed that $\Phi(z, f)$ is an eigenfunction of the $\mathrm{SO}(V)(\mathbb{R})$-invariant Laplacian [6, Theorem 4.6, 4.7]. He further used this result to construct a lifting into the cohomology [see 6, Chapter 5].*

2. *Also, $\Phi(z, f)$ can be used to define a smooth $(1,1)$-form on the modular variety X_Γ, which satisfies a current equation. Naturally, this leads to various geometric applications for example in Arakelov theory [see 9].*

   In particular, in the special case where f is a weakly holomorphic modular form, this current equation implies that

$$\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\left[\Phi(z, f) + c^+(0,0)\log|y|^2\right] = 0 \quad (0 \le i, j \le n)\,.$$

   This means that $\Phi(z, f)$ is pluriharmonic on $\mathcal{H} \setminus \mathcal{Z}(f)$.

The evaluation of the integral. The calculations involved in the evaluation of the regularized theta integral are quite involved and too extensive to reproduce here. But, at least, we want to outline some of the main points. Let f be a weakly holomorphic modular form with $f \in \mathrm{M}^!_{k,\rho_L}$, $k = 1 - n/2$.

Borcherds observed that the Siegel theta-function Θ_L can be expressed through the Siegel theta-function of the smaller, Lorentzian lattice $L_0 = L \cap \ell^\perp \cap \ell'^\perp$. Likewise, using the Fourier expansion of the input function f one can define a vector valued modular form $f_{L_0} : \mathbb{C} \to \mathbb{C}[L_0'/L_0]$ transforming under the Weil representation of the lattice L_0 [see 4, Theorems 5.2 and 5.3]. Using partial Poisson summation, Borcherds then decomposed the regularized theta-integral [4, Theorem 7.1], with one part given by, essentially, the regularized theta-lift for signature $(1, n-1)$ of f_{L_0}, with Θ_{L_0} as the theta-function, and further terms which are evaluated by unfolding.

The evaluation of the Lorentzian part (actually, in there, there is again a contribution of a positive definite lattice contained in L_0, which however evaluates to a constant) gives piecewise polynomial functions. The singularities can be evaluated quite similarly to the proof of Theorem 3.2, except that in the end their type is not logarithmic. They also lie along Heegner divisors, which dissect the symmetric domain of the Lorentzian orthogonal group $\mathrm{SO}(1,n)$ into connected components. On each connected component, many terms cancel, leaving only piecewise linear functions, which Borcherds gathers into a term involving a Weyl vector [cf. 4, Section 10]. This is where the Weyl chambers – connected components of $\mathcal{H}$ with wall-crossing occurring between them – and the Weyl vector terms in Theorem 3.4 below originate from: in the contribution of the Lorentzian part.

3.4 Borcherds products

Our main references for the following are [4, Section 13] and [6, Section 3.2]. In this section, we assume the signature of V to be $(2,n)$ with $n \geq 2$. Further, let f denote a weakly holomorphic modular form with $f \in \mathrm{M}^!_{k,\rho_L}$, with $k = 1 - \frac{n}{2}$.

We define $\Psi(z,f)$ as a meromorphic function on $\mathcal{H}$ with $\mathrm{div}(\Psi) = \mathcal{Z}(f)$ by setting

$$\Phi(z,f) + c^+(0,0)\log|y|^2 = -2\log|\Psi(z,f)|.$$

To see why this works, we note that the multiplicative Cousin problem is universally solvable on $\mathcal{H}$ [see 20, Section V.2], since the components of $\mathcal{H}$ are convex. Hence there exists a meromorphic function g with divisor $\mathcal{Z}(f)$; for this, one has to show that $\Phi(z,f)$ is pluriharmonic i.e. all mixed second derivatives $\partial_i\bar{\partial}_j\Phi$ $(1 \leq i,j \leq n-1)$ vanish (see Remark 3.4).

Then, $\Phi - \log|g|$ extends to a pluriharmonic real analytic function on $\mathcal{H}$. Further, this implies that there is a holomorphic function h with $\Re(h) = \Psi - \log|g|$ [see 22, Section IX.C], and one can set $\Psi = e^h g$. (For a detailed version of this argument [see 6, p. 82ff] or [cf. also 8, Lemma 6.6]).

Since $\Phi(z,f)$ is invariant, $\Psi(z,f)$ transforms under Γ_L according to

$$\Psi(\gamma z,f) = \sigma(\gamma)\cdot j(\gamma,z)^{c^+(0,0)/2}\Psi(z,f),$$

with some multiplier system σ. It can be shown that σ has at most finite order, using a result of Margulis (for $n > 2$). (For $n = 2$ an embedding trick has to be employed first.) See [4, Lemma 13.1], [6, Section 3.4]. Thus, Ψ is a meromorphic modular form of weight $c^+(0,0)/2$.

Now we are ready to formulate Borcherds' celebrated result [4, Theorem 13.3]:

Theorem 3.3 (Borcherds)**.** *Let $f \in \mathrm{M}^!_{k,\rho_L}$ be a weakly holomorphic modular form with Fourier expansion $f = \sum_{\mu.m} c^+(\mu,m)q^n$, satisfying[2] $c^+(\mu,m) \in \mathbb{Z}$ for all $m \leq 0$. Then, there is a meromorphic function $\Psi(z,f)$ on $\mathcal{H}$ with the following properties:*

i) $\Psi(z,f)$ is a modular form of weight $c^+(0,0)/2$ with respect to Γ_L with a multiplier system of (at most) finite order.

[2]If we want to avoid a rational weight for $\Psi(z,f)$, we must further assume that $c^+(0,0) \in 2\mathbb{Z}$. In this case, the multiplier system in i) is a character [see 6, Theorem 3.22 i)].

ii) The divisor of $\Psi(z,f)$ is given by

$$\operatorname{div}(\Psi(z,f)) = \mathcal{Z}(f),$$
$$\text{where} \quad \mathcal{Z}(f) = \sum_{\mu\in L'/L}\sum_{m<0} c^+(\mu.m)\mathcal{Z}(\mu.m)$$

is the Heegner divisor associated to f, see (18).

iii) For $z \in \mathcal{H}$ with $|y|^2 \gg 0$ and z in the complement of the set of poles, $\Psi(z,f)$ has an absolutely convergent infinite product expansion.

To simplify notation, instead of the general product expansion for $\Psi(z,f)$ from [4, Theorem 13.3.5], we will give a simplified version. Consider the following setup:

Assume that $L = L_0 \oplus H$, the direct sum of a lattice L_0 of signature $(1, n-1)$ and a hyperbolic plane H, i.e. a unimodular lattice of signature (1,1). We set $V_0(\mathbb{R}) = L_0 \otimes_{\mathbb{Z}} \mathbb{R}$ (so $\mathcal{H}$ is adapted to L_0). Then, part iii) of Theorem 3.3 can be formulated as follows:

Theorem 3.4. *For $z \in \mathcal{H}$ with $|y|^2 \gg 0$ and z in the complement of the set of poles, the absolutely convergent infinite product expansion of $\Psi(z,f)$ takes the following form:*

$$\Psi(z,f) = e\big((\rho_W(f), z)\big) \prod_{\substack{\lambda\in L_0' \\ (\lambda,W)>0}} \Big[1 - e\left((\lambda, z)\right)\Big]^{c^+(\lambda,q(\lambda))}.$$

Here, $W \subset V_0(\mathbb{R})$ denotes a Weyl chamber for f and $\rho_W(f) \in V_0(\mathbb{R})$ is the Weyl vector attached to W and f.

The Weyl chambers occurring in the theorem are connected components of $\mathcal{H}$; together with the associated Weyl vectors, they can often be determined explicitly, using results of Bruinier [see 6, p. 88]. It is worth noting, that while the Weyl vector parts and the infinite product parts differ depending on the Weyl chamber, the product as a whole is actually the same for all Weyl chambers.

Remark 3.5. *Assume that the signature of $V = L \otimes \mathbb{Q}$ is $(2,n)$ with $n \geq 3$. Then, by the Koecher principle, if in the sum in ii) all coefficients $c^+(\mu,m)$ are positive, it follows that $\Psi(z,f)$ is a holomorphic orthogonal modular form.*

Contrastingly, if $n = 2$, as the Koecher principle fails in general, this line of reasoning only works for those lattices L where the Witt rank is smaller than n, see Definition 2.3.

Finally, for the case $n = 1$, excluded above, Theorem 3.3 is mostly still correct, except for one caveat: The multiplier system is not guaranteed to have finite order. Bruiner and Ono give a precise criterion for this [see 11, Section 6], which in the present setting can be stated as follows: The order is finite if for all $m < 1$ the Fourier coefficients $c^+(\mu,m)$ of the input function f are rational. As can further be shown, this is equivalent to f being perpendicular to the subspace spanned by unary theta series.

Example 3.1. *Let L be the even unimodular lattice of signature $(2,2)$. Then, L is given by the direct sum of two hyperbolic planes, and the Witt rank here is 2. The space of input functions is given by $\mathrm{M}^!_0(\Gamma(1)) = \mathbb{C}[j]$, where $j = j(\tau)$ is the modular invariant.*

For example, let $J(\tau) = j(\tau) - 744$. Then,

$$\Psi(z, J) = j(z_1) - j(z_2) = q_1^{-1} \prod_{\substack{m>0 \\ n\in\mathbb{Z}}} \left(1 - q_1^m q_2^n\right)^{c(mn)},$$

with $q_1 = e(z_1)$ and $q_2 = e(z_2)$. A complete treatment of this case is carried out in [25].

Example 3.2. *We now turn to the case $n = 1$, see Remark 3.5. Consider the following lattice of the form introduced in Example 2.4 (here, $N = 1$):*

$$L = \left\{ \begin{pmatrix} b & a \\ c & -b \end{pmatrix} ;\ a, b, c \in \mathbb{Z} \right\}.$$

Then,

$$L'/L = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \right\}.$$

As the space of input functions $\mathrm{M}^!_{\frac{1}{2},\rho_L}$ is isomorphic to $\mathrm{M}^{!,+}_{\frac{1}{2}}(\Gamma_0(4))$, we recover the examples of Borcherds products from Section 1.2. (It can also be shown that the criterion of Bruinier and Ono mentioned in Remark 3.5 is satisfied.)

We remark that as elliptic modular forms these products have double the weight that they have as orthogonal modular forms. The reason for this is that $\mathrm{SL}_2(\mathbb{R})$ is isomorphic to $\mathrm{Spin}_V(\mathbb{R})$ and the map from $\mathrm{Spin}_V(\mathbb{R})$ to $\mathrm{O}^+(V)(\mathbb{R})$ is two-to-one [cf. 11, Section 5] or [cf. 4, Example 14.4].

Remark 3.6. *In [11], Bruinier and Ono study a generalization of Borcherds' construction for signature $(2,1)$, using a* twisted *Siegel theta function and with harmonic Maass forms as input functions. One of their results [11, Theorem 6.1] is the existence of generalized Borcherds products, which, however can have multiplier systems of infinite order.*

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