Automorphic Products on Unitary Groups

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To my parents

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Zusammenfassung

Das Ziel der vorliegenden Dissertation ist es, die Konstruktion von Borcherdsprodukten für unitäre Gruppen der Signatur $(1, q)$ über imaginär-quadratischen Zahlkörpern durchzuführen.

Die Grundlage hierfür bildet die Arbeit [5] von Borcherds. In dieser wird die singuläre Theta-Korrespondenz dazu verwendet, eine multiplikative Liftung von schwach holomorphen vektorwertigen Modulformen für die elliptische Modulgruppe $\mathrm{SL}_2(\mathbb{Z})$ zu meromorphen automorphen Formen für orthogonale Gruppen der Signatur $(2, b)$ zu realisieren. Die so erhaltenen Funktionen verfügen über eine Darstellung als unendliche Produkte, wodurch sie als Verallgemeinerung klassischer Eta-Produkte angesehen werden können. Sie werden nach ihrem Entdecker als Borcherdsprodukte bezeichnet. Diese Liftung war von Borcherds bereits in einer vorherigen Arbeit [4] konstruiert worden, die hierbei verwendete Methode war jedoch deutlich weniger konzeptuell. Tatsächlich ist die Konstruktion in [5] weit allgemeiner; sie liefert auch eine additive Liftung, welche eine Reihe vorher bekannter Liftungen als Spezialfälle umfasst, welche sich durch Theta-Korrespondenzen realisieren lassen.

Von einer singulären Theta-Korrespondenz spricht man im vorliegenden Fall, da das Theta-Integral

$$\int_{\mathscr{F}} \langle f(\tau), \Theta(\tau, Z) \rangle \, y^{b/2} \frac{dx\,dy}{y^2}$$

stark divergiert und erst durch ein aus der theoretischen Physik stammendes Verfahren regularisiert werden muss, welches von Harvey und Moore [34] auf Integrale mit Theta-Kernen übertragen wurde, siehe auch [38].

Neben der unendlichen Produktentwicklung sei an dieser Stelle auch auf eine weitere charakteristische Eigenschaft der von Borcherds konstruierten automorphen Formen hingewiesen: Ihre Pol- und Nullstellengebilde werden durch die Hauptteile der Fourierentwicklung der als Eingabewerte für die Liftung dienenden Funktionen vorgegeben. Diese Eigenschaft erlaubt somit die Konstruktion von Funktionen mit vorgegeben Divisoren auf den jeweiligen orthogonalen Modulvarietäten, was als einer der Gründe für die vielseitigen Anwendungen, welche aus Borcherds' Konstruktion hervorgegangen sind, angesehen werden kann. Nach Borcherds spricht man in diesem Zusammenhang von „Heegner-Divisoren", eine Begriffsbildung, welche auf die, einen Spezialfall darstellenden, Heegner-Punkte auf elliptischen Modulkurven hinweist.

In der vorliegenden Dissertation wird nun das für die Borcherdsprodukte zentrale Resultat, Theorem 13.3 aus [5], auf unitäre Gruppen der Signatur $(1, q)$ übertragen. Die dazu gewählte Methode ist die des Rückzugs unter einer Einbettung zwischen den hermitesch symmetrischen Gebieten der Gruppen $\mathrm{SU}(1, q)$ und $\mathrm{SO}(2, 2q)$.

Sei hierzu $\mathbb{F} = \mathbb{Q}(\sqrt{d})$ mit d einer negativen ganzen Zahl und V, $\langle \cdot, \cdot \rangle$ ein hermitescher Raum über $\mathbb{F}$. Dann besitzt V die Struktur eines quadratischen Raums über $\mathbb{Q}$ mit der quadratischen Form, welche zu der symmetrischen Bilinearform $(\cdot, \cdot) := \mathrm{Tr}_{\mathbb{F}/\mathbb{Q}} \langle \cdot, \cdot \rangle$ assoziiert ist, woraus man eine Inklusion der Isometriegruppen erhält, nämlich von $\mathrm{SU}(V)(\mathbb{R})$ in $\mathrm{SO}(V)(\mathbb{R})$. Diese Beobachtung ermöglicht es, eine Einbettung zwischen den zugehörigen symmetrischen Gebieten zu konstruieren.

Diese erfolgt im dritten Kapitel der vorliegenden Arbeit. In den vorausgehenden beiden Kapiteln werden einige Grundlagen hierzu bereitgestellt.

Im ersten Kapitel wird die Theorie der symmetrischen Gebiete und der automorphen Formen zunächst für unitäre und danach für orthogonale Gruppen entwickelt. In dem Abschnitt über unitäre Gruppen werden auch einige Elemente der Theorie hermitescher Gitter bereitgestellt. Außerdem wird im Anschluss an die Konstruktion des symmetrischen Gebiets auch die Kompaktifizierung der unitären Modulvarietät nach Baily-Borel beschrieben. Unitäre Modulformen und ihre Fourier-Jacobi Entwicklungen schließen diesen Abschnitt. In dem Abschnitt über orthogonale Gruppen wird besonderer Wert auf die Konstruktion verschiedener Realisierungen des symmetrischen Gebiets gelegt, da diese für die spätere Einbettung von großer Bedeutung sind. Ebenfalls ausführlich behandelt wird die geometrische Struktur seiner Randkomponenten. Die Definition der Modulformen wird durch eine Beschreibung ihrer Fourier-Entwicklung und die Behandlung ihres Verhaltens auf Randkomponenten des symmetrischen Gebiets ergänzt.

Im zweiten Kapitel wird die Konstruktion von Borcherds referiert. Vorher werden dafür notwendige Begriffe wie die Weil-Darstellung und die Definition von Weyl-Kammern eingeführt, wozu auch die Theorie von quadratischen Gittern vertieft wird. Besonders relevant ist hier der Begriff der Heegner-Divisoren, dessen Definition ausführlich behandelt wird.

Das vierte Kapitel beinhaltet die wichtigsten Ergebnisse der Arbeit. Zunächst werden Heegner-Divisoren und Weyl-Kammern auf dem symmetrischen Gebiet der unitären Gruppe eingeführt, woraufhin dann das Hauptresultat dieser Dissertation, ein Analogon zu Borcherds' Satz 13.3 aus [5], formuliert und bewiesen werden kann. Ein Korollar gibt eine einfachere Version für den wichtigen Spezialfall unimodularer Gitter an, und allgemeiner für Gitter, die sich in einen unimodularen isotropen Teil und einen definiten Teil zerlegen lassen.

Das Kapitel schließt mit einer Untersuchung der Werte, welche die vorher konstruierten Borcherdsprodukte auf den Randpunkten der Baily-Borel Kompaktifizierung annehmen.

Im abschließenden fünften Kapitel wird als Anwendung des Hauptsatzes sowie einer auf Bruinier zurückgehenden Verallgemeinerung des Borcherds-Lifts die Situation diskutiert, in welcher das zugrunde liegende Gitter eine hermitesche hyperbolische Ebene ist und somit der hermitesche Raum V die Signatur $(1,1)$ hat. Hier lässt sich das symmetrische Gebiet der $\mathrm{SU}(1,1)$ mit der klassischen oberen Halbebene $\mathbb{H} = \{z \in \mathbb{C};\ \Im z > 0\}$ der Gaußschen Zahlenebene identifizieren. Die Borcherds-Liftung stellt in diesem speziellen Fall eine Korrespondenz zwischen skalarwertigen meromorphen Modulformen zur elliptischen Modulgruppe $\mathrm{SL}_2(\mathbb{Z})$ dar. Geliftet werden dabei schwach holomorphe Modulformen vom Gewicht 0, also Modulfunktionen. Das Bild der Liftung besteht wiederum aus meromorphen Modulformen, die durch eine unendliche Produktentwicklung gegeben sind. Wir konstruieren hier eine Familie von Beispielen, wobei neben der Ausgangsfunktion auch der Zahlkörper als Parameter in die Liftung eingeht. Ebenfalls untersucht werden die CM-Ordnungen von Heegner-Divisoren in diesem Fall.

Mein Dank gilt in erster Linie meinem Betreuer Herrn Prof. Dr. Bruinier, der es mir ermöglicht hat, in diesem faszinierenden Themengebiet zu forschen, und ohne dessen Rat und Anregungen diese Dissertation nicht hätte zustande kommen können. Weiter danke ich Herrn Prof. Dr. Funke, der sich bereit erklärt hat, als deren Zweitgutachter zu fungieren. Ferner danke ich Herrn Ehlen für das Korrekturlesen des Skriptums. Der DFG bin ich für finanzielle Unterstützung dankbar, sowohl im Rahmen des Graduiertenkollegs 1269 „Globale Strukturen" an der Universität zu Köln als auch im Rahmen des Forschungsprojekts „Schwache Maaßformen" am Fachbereich Mathematik der TU Darmstadt.

0 Introduction

In two seminal papers, [3] and [5], Borcherds constructed a lifting from weakly holomorphic modular forms on the elliptic modular group $\mathrm{SL}_2(\mathbb{Z})$ to meromorphic automorphic forms on orthogonal groups of signature $(2, b)$, which have infinite product expansions. The aim of the present thesis is to generalize these results to unitary groups.

Let V be a vector space over $\mathbb{Q}$, equipped with a non-degenerate symmetric bilinear form $(\cdot,\cdot)$ of signature $(2, b)$ and the attached quadratic form $q(\cdot) = \frac{1}{2}(\cdot,\cdot)$. Denote by $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$ the corresponding real quadratic space.

Let $L \subset V$ be an even lattice. To simplify the following discussion, assume that L is *unimodular*.

By a *weakly holomorphic modular form* f for $\mathrm{SL}_2(\mathbb{Z})$ we mean a holomorphic function on the complex upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C};\ \Im\tau > 0\}$, with the usual transformation behavior of a modular form and a Fourier expansion

$$f(\tau) = \sum_{n \gg -\infty} c(n) q^n, \tag{*}$$

where $q = e^{2\pi i\tau}$, as usual. Thus, f is allowed to have a pole at the cusp ∞. The principal part of f is the Fourier polynomial

$$\sum_{-\infty \ll n < 0} c(n) q^n.$$

The functions serving as inputs for the Borcherds lift are weakly holomorphic modular forms of weight $1 - b/2$, having integer coefficients in their principal part.

Let f be such a modular form. The lifting constructed by Borcherds associates to f a meromorphic function $\Psi(f;Z)$ on the symmetric domain of the orthogonal group $\mathrm{O}(2,b) \simeq \mathrm{O}(V)(\mathbb{R})$. We briefly recall the construction of the hermitian symmetric domain. Let $V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C}$ be the complex quadratic space obtained from V by extension of scalars, with $(\cdot,\cdot)$ extended to a symmetric $\mathbb{C}$-bilinear form. Then, the symmetric domain for the operation of $\mathrm{O}(V)(\mathbb{R})$ on $V_{\mathbb{R}}$ can be described as one of the connected components of the following subset of the complex projective space $\mathbb{P}(V_{\mathbb{C}})$:

$$\mathcal{K}_{\mathrm{O}} = \left\{[W];\ (W,W) = 0, \left(W, \bar{W}\right) > 0\right\}.$$

We choose one connected component as a projective model for the symmetric domain and denote it by $\mathcal{K}_{\mathrm{O}}^+$. The boundary components of $\mathcal{K}_{\mathrm{O}}^+$ are given by rational isotropic subspaces. In particular, rational isotropic vectors correspond to zero-dimensional boundary components in the sense of Baily-Borel. Let $e \in L$ be primitive and isotropic and let $e' \in L$ be isotropic with $\left(e, e'\right) = 1$. Denote by K the lattice $L \cap e^{\perp} \cap e'^{\perp}$. Then, K is an even lattice of signature $(1, b-1)$. The symmetric domain can be realized as a subset of the Lorentzian space $K \otimes_{\mathbb{Z}} \mathbb{C}$ as follows. For a line $[W]$ in $\mathcal{K}_{\mathrm{O}}^+$, there is a unique representative Z_L of the form $Z_L = e' - q(Z)e + Z$, where $Z = X + iY \in K \otimes_{\mathbb{Z}} \mathbb{C}$ with imaginary part $Y \in K \otimes_{\mathbb{Z}} \mathbb{R}$ satisfying $q(Y) > 0$. Denote by $\mathcal{H}_{\mathrm{O}}$ the set of such Z. The assignment $Z \mapsto [(1, -q(Z), Z)]$ is a biholomorphic map from $\mathcal{H}_{\mathrm{O}}$ to $\mathcal{K}_{\mathrm{O}}^+$.

The set $\mathscr{H}_O \subset K \otimes_{\mathbb{Z}} \mathbb{C}$ is called the *tube domain model* for the symmetric domain. Indeed $\mathscr{H}_O$ is a tube domain as it can be written in the form $K \otimes \mathbb{R} + i\mathscr{C}$ with $\mathscr{C}$ an open connected subset, more precisely a cone, in $K \otimes_{\mathbb{Z}} \mathbb{R}$. The connected component of the identity in $\mathrm{O}(V)(\mathbb{R})$, denoted $\mathrm{O}^+(V)(\mathbb{R})$, acts on $\mathscr{H}_O$ through fractional linear transformations.

The orthogonal group $\mathrm{O}(L)$ of the lattice L, is an arithmetic subgroup of $\mathrm{O}(V)$. Its subgroup $\Gamma_L^O = \mathrm{O}(L) \cap \mathrm{O}^+(V)$ acts on $\mathscr{H}_O$. Denote by $\mathfrak{X}_\Gamma$ the quotient $\Gamma_L^O \backslash \mathscr{H}_O$. By the theory of Baily-Borel, it carries the structure of a quasi-projective algebraic variety.

The image of the Borcherds lift, $\Psi(f, Z)$, transforms as an automorphic form for Γ_L^O. Its weight is given by $c(0)/2$ with $c(0)$ the constant term in the Fourier expansion of the input f.

The zeros and poles of $\Psi(f, Z)$ lie along special divisors, so called *Heegner divisors*. These can be described as follows: Given a vector $\lambda \in L$ with $q(\lambda) < 0$, the orthogonal complement $\lambda^\perp$ is a quadratic subspace of codimension one and signature $(2, b-1)$.

For a negative integer n, the locally finite sum

$$H(n) = \sum_{\substack{\lambda \in L \\ q(\lambda)=n}} \lambda^\perp \subset \mathscr{K}_O^+$$

defines a Γ-invariant divisor on $\mathscr{H}_O$ called the *Heegner divisor* of index m. It is the inverse image of an arithmetic divisor on $\mathfrak{X}_\Gamma$.

The divisor of $\Psi(f, Z)$ is given by a finite linear combination of such divisors $H(n)$:

$$\mathrm{div}(\Psi) = \sum_{\substack{n<0 \\ c(n)\neq 0}} c(n) H(n).$$

The lift $\Psi(f, Z)$ can be expanded as an infinite product, a so called *Borcherds product*. The product expansion is absolutely convergent around the cusps and takes the form

$$\Psi(f, Z) = C e\Big(\big(\rho_f(W), Z\big)\Big) \prod_{\substack{\lambda \in K \\ (\lambda, W)>0}} \Big(1 - e\big((Z, \lambda)\big)\Big)^{c(q(\lambda))},$$

here, as usual $e(z) := \exp(2\pi i z)$, C is a constant of absolute value 1, ρ_f denotes a so-called Weyl vector and $(\lambda, W) > 0$ is a positivity condition.

Finally, the lifting is multiplicative in the sense that $\Psi(f+g, Z) = \Psi(f, Z) \cdot \Psi(g, Z)$.

In [3], Borcherds gives a first construction of such infinite products. He proves that they transform correctly under suitable generators of Γ_L^O and shows meromorphic continuation with the correct sets of zeros and poles using Hardy-Ramanujan-Rademacher asymptotics. The construction in [5] is both more general and more conceptual, with the lifting realized as a singular theta lift between the two groups $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{O}(2, b)$, which form a dual reductive pair in the sense of Howe. The roots of Borcherds' original construction in [3] lie grounded in the representation theory of generalized Kac-Moody algebras, where infinite products appear as denominator identities, compare [4]. The relationship between Borcherds products and both generalized Kac-Moody and Super Lie Algebras, see [55] for an introduction, is still a very active field of research. For examples related to the theory of Bosonic strings, see Scheithauer, [56] and [57].

Besides having expansions as infinite products, the property of taking their zeros and poles along prescribed divisors is most characteristic for Borcherds products and is crucial for many applications they have found. Indeed, there is a duality theory giving precisely the obstructions to realizing a given divisor through a Borcherds product, see [6] and [9], it comes by way of a pairing between the spaces of weakly holomorphic modular forms of weight $1-b/2$ and non-zero holomorphic cusp forms of weight $1+b/2$.

The theory of geometric moduli spaces is an area of research, where this feature has proven particularly fruitful, since moduli spaces can often be realized as quotient varieties of a similar type as $\mathfrak{X}_\Gamma$. For example, for the moduli space of complex polarized $K3$-surfaces, in [33], Gritsenko, Hulek and Sankaran use such a realization combined with Borcherds' construction to show that the moduli space is of general type in many instances where this was not previously established. Another example is provided by Allcock and Freitag, [1]. They exhibit the moduli space of marked cubic surfaces as an intersection of cubic hypersurfaces in nine-dimensional complex projective space. Automorphic forms with prescribed divisors are a key ingredient for this geometric construction.

The close link between the geometry of arithmetic cycles on modular surfaces and the Borcherds correspondence has prompted numerous developments in the arithmetic geometry of these surfaces. A generalization of Borcherds' construction towards a lifting into the cohomology was carried out by Bruinier in [9]. The lifting he constructs takes weak Maass forms as inputs to square-integrable harmonic 1, 1-forms which represent Chern classes in the second cohomology of $\mathfrak{X}_\Gamma$.

Later, Bruinier and Funke showed, see [14], that this map is in a sense adjoint to a lifting from the cohomology with compact support of locally symmetric spaces to modular forms originally constructed by Kudla and Millson, see [42], [43], [44], and extended by Funke [30].

Further, such generalized Borcherds lifts can be linked to Green currents in the sense of an extended Arakelov theory due to Burgos, Kramer and Kühn, [21], a relationship examined by Bruinier, Burgos and Kühn in [10].

Through the moduli space properties of the quotients $\mathfrak{X}_\Gamma$ the Heegner divisors can often be interpreted as special cycles in the sense of [41], see also [39]. Through this interpretation, they can often be linked to integral models. The close relationship between the (generalized) Borcherds lift, intersection theory, CM-theory and L-functions has been a focus of very active research, for example by Bruinier and Yang, see [17], [18] and, as joint work with Kudla, [19].

The aim of the present dissertation is to make Borcherds products available in the context of unitary groups of signature $(1,q)$. We describe briefly the set-up and the construction of the symmetric domain.

Let d be a negative square-free integer and $\mathbb{F}$ the imaginary quadratic number field $\mathbb{Q}(\sqrt{d})$. Denote by $\mathcal{O}_\mathbb{F}$ the ring of integers in $\mathbb{F}$ and by $D_\mathbb{F}$ the discriminant of $\mathbb{F}$ and set $\delta=\sqrt{D_\mathbb{F}}$, taking the principal branch of the square root. If $d\equiv 2,3 \bmod 4$, we have $D_\mathbb{F}=4d$ and $\mathcal{O}_\mathbb{F}$ is given by $\mathbb{Z}+\sqrt{d}\mathbb{Z}$, whereas if $d\equiv 1 \bmod 4$ the discriminant $D_\mathbb{F}$ is equal to d and $\mathcal{O}_\mathbb{F}=\mathbb{Z}+\frac{1}{2}(1+\sqrt{d})\mathbb{Z}$. Denote by $\mathscr{D}_\mathbb{F}^{-1}$ the inverse different ideal.

Let V, $\langle\cdot,\cdot\rangle$ an indefinite non-degenerate hermitian space over $\mathbb{F}$ with $\langle\cdot,\cdot\rangle$ a hermitian form of signature $(1,q)$. Assume that $\langle\cdot,\cdot\rangle$ is linear in the first and conjugate-linear in the second argument. Further, let L be an even hermitian lattice, that is, an $\mathcal{O}_\mathbb{F}$-module with $L\otimes_{\mathcal{O}_\mathbb{F}}\mathbb{F}=V$ and $\langle\cdot,\cdot\rangle\,|_L$ the hermitian form on L, which, since L is even, takes values in $\mathbb{Z}$.

As before, in the context of quadratic spaces, we assume for the purpose of this discussion that L is unimodular, thus for any $v \in V$ with $\langle v, L\rangle \subseteq \mathscr{D}_{\mathbb{F}}^{-1}$ we have $v \in L$. Then, L is also unimodular as a $\mathbb{Z}$-module with respect to the bilinear form given by $(\cdot,\cdot) := \mathrm{Tr}_{\mathbb{F}/\mathbb{Q}}\langle\cdot,\cdot\rangle$.

Recall how the hermitian symmetric space for the unitary group $\mathrm{SU}(1,q)$ of the hermitian space V, $\langle\cdot,\cdot\rangle$ can be realized as an unbounded domain. Let ℓ be a primitive isotropic vector in L and $\ell' \in L$ a second vector with $\langle \ell, \ell' \rangle \neq 0$.

The symmetric domain of $\mathrm{SU}(1,q)$ is isomorphic to the projective cone of positive definite one-dimensional subspaces

$$\mathscr{K}_{\mathrm{U}} = \{[v] \in \mathbb{P}(V_{\mathbb{R}});\ \langle v, v\rangle > 0\}.$$

The isotropic line $[\ell]$ is a boundary component of $\mathscr{K}_{\mathrm{U}}$. To each subspace in $\mathscr{K}_{\mathrm{U}}$ we can attach a unique representative z of the form $z = \ell' - \tau\langle \ell', \ell\rangle\,\delta\ell + \sigma$, with $\tau \in \mathbb{C}$ and $\sigma \in \mathbb{C}^{q-1}$ with σ negative definite. The positivity condition $\langle z, z\rangle > 0$ then allows us to consider the following *Siegel domain* as an affine model of the symmetric domain

$$\mathscr{H}_{\mathrm{U}} := \{(\tau,\sigma) \in \mathbb{C} \times \mathbb{C}^{q-1};\ 2\mathfrak{J}\tau|\langle \ell, \ell'\rangle|^2 > -\langle \sigma, \sigma\rangle\}.$$

In this model, ℓ corresponds to the cusp ∞.

Denote by $\mathrm{SU}(L)$ the unitary group of L. It is an arithmetic subgroup of $\mathrm{SU}(V)$, operating on $\mathscr{H}_{\mathrm{U}}$. Let Γ be a subgroup of finite index in $\mathrm{SU}(L)$. The modular variety for the unitary group Γ is given by the quotient

$$X_\Gamma = \Gamma \backslash \mathscr{H}_{\mathrm{U}}.$$

The geometry of such quotients has been the object of intensive study, in particular when V has signature $(1,2)$ and X_Γ is a Picard modular surface, see for example Cogdell [22] or Holzapfel [36].

Let k be an integer. A unitary automorphic form f of weight k for a subgroup of finite index in $\mathrm{SU}(L)$ is a holomorphic function f on $\mathscr{H}_{\mathrm{U}}$ which transforms according to

$$f\left(\gamma(\tau,\sigma)\right) = j(\gamma;\tau,\sigma)^k f(\tau,\sigma) \quad \text{for every} \quad \gamma \in \mathrm{SU}(L),$$

where $j : \Gamma \times \mathscr{H}_{\mathrm{U}} \to \mathbb{C}^\times$ is the factor of automorphy induced by the action of $\mathrm{SU}(L)$.

The arithmetic of unitary modular forms is particularly rich. We mention only a few examples for work in this field. Shimura has made many contributions, starting with the seminal [62], which develops the theory for unitary groups $\mathrm{U}(p,q)$ of general signature, and their relationship to Siegel modular forms. The theory of arithmetic theta functions and of L-values is pursued in [63]. For Picard modular surfaces, an overview of the arithmetic of L-series is given by [46]. Recent work by Murase and Sugano ([50] and [52]) combines a theory of arithmetic theta functions due to Shintani, see [51], with a lifting developed by Kudla in [40].

We now proceed to describe the main results of this thesis. We need a suitable definition of Heegner divisors in the unitary context. Given a lattice vector $\lambda \in L$ with $\langle \lambda, \lambda\rangle = n$ a negative integer, the complement with respect to $\langle\cdot,\cdot\rangle$ is a codimension one hermitian subspace of $V_{\mathbb{R}}$ and defines a codimension one subset of the projective cone $\mathscr{K}_{\mathrm{U}}$, which we denote by $\mathbf{H}_\lambda$. This defines a subset of $\mathscr{H}_{\mathrm{U}}$ supporting a primitive divisor on $\mathscr{H}_{\mathrm{U}}$, which we also denote by $\mathbf{H}_\lambda$.

For a negative integer n, we define the *Heegner divisor* $\mathbf{H}(n)$ of index n as the locally finite sum

$$\mathbf{H}(n) := \sum_{\substack{\lambda \in L \\ \langle \lambda, \lambda \rangle = n}} \mathbf{H}_\lambda .$$

This is a $\mathrm{SU}(L)$-invariant divisor on $\mathscr{H}_{\mathrm{U}}$.

Without loss of generality, set $\langle \ell, \ell' \rangle = -\delta^{-1}$. Write $\mathscr{O}_{\mathbb{F}}$ in the form $\mathbb{Z} + \zeta\mathbb{Z}$, with $\zeta = \sqrt{d}$ or $\zeta = \frac{1}{2}(1 + \sqrt{d})$ depending on the whether the discriminant is odd or even.

Then, for L a unimodular lattice containing an isotropic vector ℓ, our main result can be stated as follows:

Theorem. *Given a weakly holomorphic modular form f for $\mathrm{SL}_2(\mathbb{Z})$ of weight $1-q$, with Fourier expansion of the form (*) Assume that f has integer coefficients in its principle part and a constant coefficient $c(0) \in 2\mathbb{Z}$. Then, there is a meromorphic function $\Xi_f(\tau, \sigma)$ on $\mathscr{H}_{\mathrm{U}}$ with the following properties*

1. *$\Xi_f(\tau, \sigma)$ is an automorphic form of weight $c(0)/2$ for $\mathrm{SU}(L)$.*
2. *The zeros and poles of Ξ_f lie on Heegner divisors. We have*

$$\mathrm{div}(\Xi_f) = \sum_{\substack{n<0 \\ c(n) \neq 0}} c(n)\,\mathbf{H}(n),$$

   with the Heegner divisors $\mathbf{H}(n)$ as introduced above.

3. *Near the cusp corresponding to ℓ, the function $\Xi_f(\tau, \sigma)$ has an absolutely converging infinite product expansion of the form*

$$\Xi_f(\tau, \sigma) = C e\left(\delta \langle z, \rho_f(W) \rangle\right) \prod_{\substack{\lambda \in K \\ (\lambda, W) > 0}} \Big(1 - e\left(\delta \langle z, \lambda \rangle\right)\Big),$$

   here, as above $z = \ell' + \tau\ell + \sigma$, while K denotes a $\mathbb{Z}$-submodule of L given by the complement of ℓ and $\zeta\ell'$ in L with respect to the bilinear form $\mathrm{Tr}_{\mathbb{F}/\mathbb{Q}} \langle \cdot, \cdot \rangle$. Further, $\rho_f(W)$ is a Weyl vector and $(\lambda, W) > 0$ denotes a positivity condition analogous to that in Borcherds' theorem.

4. *The lifting is multiplicative. We have $\Xi_{f+g}(\tau, \sigma) = \Xi_f(\tau, \sigma) \cdot \Xi_g(\tau, \sigma)$.*

This simplified version of the product expansion can be found in corollary 4.2.4, the main theorem 4.2.1 is phrased in a form suitable for non-unimodular lattices and arbitrary non-zero value of $\langle \ell, \ell' \rangle$.

Our approach to proving this result can be summarized as follows:

We identify the hermitian space V, $\langle \cdot, \cdot \rangle$ over $\mathbb{F}$ with the underlying quadratic space over $\mathbb{Q}$, where the bilinear form is given by $(\cdot, \cdot) := \mathrm{Tr}_{\mathbb{F}/\mathbb{Q}} \langle \cdot, \cdot \rangle$. This rational quadratic space of signature $(2, 2q)$ we denote by V'. Then, the (special) unitary group can be identified with a subgroup of the (special) orthogonal group. The inclusion $\mathrm{SU}(V) \hookrightarrow \mathrm{SO}(V)$ induces an embedding of the attached symmetric domains. A suitable realization of this embedding makes it possible to transfer most of the results leading up to theorem 13.3 in [5] to the setting of unitary groups.

As sketched above, we introduce Heegner divisors on $\mathscr{H}_U$ and show that these can be considered as the pull back of the Heegner divisors $H(n)$ as occur in Borcherds' theorem. With this and an appropriate definition of Weyl chambers and the positivity condition, we can complete the proof of our main theorem, along the lines of Borcherds' proof from [5].

It should be noted that Freitag, in [28], uses an embedding of a somewhat similar type to construct modular forms a particular unitary group of signature $(1,4)$, related to the moduli space of marked cubics occuring in [1], both by Borcherds' additive and multiplicative lifts.

The content of the chapters

We give a brief synopsis of each individual chapter.

The first chapter covers the basic theory of unitary and orthogonal modular forms. In the first section, we first recall some facts on imaginary quadratic number fields and introduce hermitian lattices. A main focus is the construction of the Siegel domain model and an examination of the parabolic subgroups of $\mathrm{SU}(L)$. We then describe the Baily-Borel compactification of the modular variety. After the definition of unitary modular forms, the section closes with a description of Fourier-Jacobi expansions and a proof of the Koecher principle in this case. The second section of this chapter covers the theory for orthogonal groups of signature $(2,b)$. We assume as known the theory of lattices and quadratic modules over $\mathbb{Z}$ and focus mainly on the construction of the different models of the symmetric domain leading up the tube domain model $\mathscr{H}_O$. Under the assumption that the lattice L splits two hyperbolic planes over $\mathbb{Z}$, we give a refined system of coordinates for $\mathscr{H}_O$ which will be used in the construction of the embedding later on, in chapter 3. Further, we describe the boundary components of $\mathscr{K}_O$ and the parabolic subgroups of $\mathrm{O}(V)$. Then, we introduce orthogonal automorphic forms and describe some properties of their Fourier expansions.

Chapter two describes the theory of Borcherds and some necessary prerequisites. For one thing, these are the weakly holomorphic modular forms. In general, if the lattice is not unimodular, vector valued modular forms transforming under the Weil representation ρ_L of $\mathrm{SL}_2(\mathbb{Z})$ take the place of the modular forms described above. The next main prerequisite are the Heegner divisors, already mentioned above. Weyl chambers are another subject treated in this section. They can be seen as connected components of the symmetric domain separated by hyperplanes corresponding to Heegner divisors. Finally, we briefly describe how the Borcherds lift as given in [5] is implemented as a singular theta lift, and how the wildly divergent theta integral

$$\Phi_L(Z,f) = \int_{\mathscr{F}} \langle f(\tau), \Theta(\tau, Z) \rangle \, y^{b/2} \frac{dx\,dy}{y^2}$$

can be regularized, using a recipe due to Harvey and More [34]. We then reproduce Borcherds' theorem 13.3 in sufficient generality for our purposes.

In chapter three we construct the embedding between the symmetric domains $\mathscr{H}_U$ and $\mathscr{H}_O$ mentioned above. The chapter has two main parts. In the first part, we describe the embedding in terms of $\mathrm{SU}(V)$ and $\mathrm{SO}(V)$, and of their arithmetic and their parabolic subgroups. We then deal with the issue of choosing rational isotropic lattice vectors corresponding to the cusps and extend these to a basis for the hyperbolic part of V', $(\cdot,\cdot)$ compatible with the choice of ℓ and $\ell' \in V$, $\langle \cdot, \cdot \rangle$. In the second part, we examine the requirements the embedding has to meet: It has to be compatible, on the one hand, with the natural complex structure of the hermitian space $V_{\mathbb{R}}$, $\langle\cdot,\cdot\rangle$ and, on the other hand, with the complex structure of the complexified space $V_{\mathbb{C}}$ obtained from the real quadratic space $V_{\mathbb{R}}$, $(\cdot,\cdot)$. Further, the image of (τ,σ) under the embedding has to be contained in the tube domain $\mathscr{H}_{\mathrm{O}}$, so it has to conform with the normalization and positivity condition for the attached representative Z_L in $\mathscr{K}_{\mathrm{O}}^{+}$.

With the basis for the hyperbolic part of $V_{\mathbb{R}}$, $(\cdot,\cdot)$ determined in section 3.1.2 we then given an explicit description of the embedding in the coordinates of the Siegel domain $\mathscr{H}_{\mathrm{U}}$ and the tube domain $\mathscr{H}_{\mathrm{O}}$. As a final point we show that the embedding is well behaved on the boundary of the Baily-Borel compactification, through an explicit description for the basis of the topology around a boundary component of $\mathscr{H}_{\mathrm{O}}$ corresponding to a cusp of $\mathscr{H}_{\mathrm{U}}$.

Chapter four contains the main results of the present thesis. In this chapter, as sketched above, we introduce Heegner divisors for $\mathscr{H}_{\mathrm{U}}$, showing that they can be interpreted as the restriction of Heegner divisors on $\mathscr{H}_{\mathrm{O}}$ to the image of $\mathscr{H}_{\mathrm{U}}$ under the embedding. We also demonstrate how the concept of Weyl chambers can be transferred to the unitary setting. The main part of the chapter is, of course, the formulation and proof of the main theorem 4.2.1 and a corollary thereof, corollary 4.2.4, which can be used for example, when the lattice L is unimodular to give the version of the theorem presented above. Additionally, in the final section of the chapter, we derive a result on the behaviour of Borcherds products on the boundary of the symmetric domain, theorem 4.3.3. For a unimodular lattice, it reads:

$$\lim_{\tau\to i\infty}\Xi_f(\tau,\sigma)=e\big(-\bar{\zeta}\rho_{f,1}^{W}\big)\prod_{\substack{\lambda=\lambda_1\zeta\ell\in K,\\ \lambda_1\in\mathbb{Z}}}\Big(1-e\big(-\lambda_1\bar{\zeta}\big)\Big)^{c(0)},$$

where $\rho_{f,1}^{w}$ is a component of the Weyl vector. This expression can be interpreted as a CM-value of an eta-product.

The last chapter is devoted to the construction of examples for Borcherds products on $\mathrm{SU}(1,1)$. Here, the unimodular lattice L is given by the hyperbolic plane $\mathscr{O}_{\mathbb{F}}\oplus\mathscr{D}_{\mathbb{F}}^{-1}$ and the hermitian space V by $L\otimes_{\mathscr{O}_{\mathbb{F}}}\mathbb{F}$.

In this case, $\mathscr{H}_{\mathrm{U}}$ can be identified with the classical complex upper half-plane $\mathbb{H}=\{z\in\mathbb{C}\,;\,\Im z>0\}$. The inputs in this case are modular functions, which can be described as polynomials with integer coefficients in the modular invariant $j(\tau)$. The image also consists of meromorphic modular forms for the elliptic modular group $\mathrm{SL}_2(\mathbb{Z})$, with infinite product expansions similar to that of the eta-function. In fact, the eta-function is found to be the lift of a constant function.

To recover the Weyl vectors in explicit form, we use results of Bruinier from [9] and [16] in combination with corollary 4.2.4 of the main theorem from chapter 4.

As an example for the infinite products constructed this way, let n be a square-free integer with $n>0$. Then, there is a unique modular function f_n in $\mathbb{Z}[j]$ of the form $f_n=q^{-n}+O(q)$. A Borcherds product expansion for the lift of f_n is given by

$$\Xi_{f_n}(\tau)=e\big(-\sigma(n)\tau\big)\prod_{\substack{k,l\in\mathbb{Z}\\ ((l,k),W)>0}}\Big(1-e\big(k\tau-l\bar{\zeta}\big)\Big)^{c(kl)},$$

converging absolutely for τ with $\Im\tau>\frac{n}{|\delta|}$. Here, $\sigma(n)$ denotes the divisor sum $\sum_{d|n}d$, and W a Weyl chamber, which in this particular example can be described as the half-plane $\Im\tau>n$ in $\mathbb{C}$.

Outlook

It would be interesting to study further example cases for the unitary Borcherds lift. For instance in signature $(1,2)$ the modular varieties are Picard modular surfaces, and the Heegner divisors have applications to their CM-theory, see for example Kudla and Rapoport [39]. To furnish suitable input functions, which in this case would be vector-valued, constructions such as in [13] or [58] can be used.

It appears that the method used to derive the main theorem of this dissertation is also suitable to extend Bruinier's generalization, from [9], of the Borcherds lift to unitary groups. It will be interesting to work on this and to see how the approach can be modified to recover as much as possible of the geometric information encoded in the Bruinier-Borcherds lift.

A unitary version of the Bruinier-Borcherds lift is expected to give Green objects in the sense of Arakelov theory for unitary modular varieties. In particular, examples of this for Picard modular surfaces should contribute further to their already rich arithmetic theory.

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1 Lattices, groups and symmetric domains

The present chapter provides the basic concepts and notation used throughout the remainder of the thesis. It covers the basic theory of unitary groups of signature $(1,q)$ and of orthogonal groups of signature $(2,b)$ as isometries of hermitian and quadratic spaces, respectively. It deals with the construction of models for symmetric domains of these groups and describes the basic theory of automorphic forms. Since much of the material covered can be considered as well known, proofs are often omitted or only sketched. This applies particularly to section 1.2, dealing with the theory for orthogonal groups, at the beginning of which, on p. 34, we have listed some references.

1.1 Unitary groups

Automorphic forms for unitary groups of signature (q,q) were first studied by Hel Braun in the 30s. However, Shimura was the first to consider the case of general signature (p,q) in [62].

In this section, we cover the basic theory of modular forms for unitary groups of signature $(1,q)$. We start with the basic setup of a hermitian space V, $\langle\cdot,\cdot\rangle$ over an imaginary quadratic number field $\mathbb{F}=\mathbb{Q}(\sqrt{d})$, with $d<0$. At first we assume only that $\langle\cdot,\cdot\rangle$ has signature (p,q) with $q \geq p$. We treat hermitian lattices in V as $\mathcal{O}_\mathbb{F}$-modules, where $\mathcal{O}_\mathbb{F}$ denotes the ring of integers in $\mathbb{F}$, and introduce unitary and special unitary groups $\mathrm{U}(p,q)$ and $\mathrm{SU}(p,q)$ and unitary modular groups related to a lattice L. We give a model Gr_U of the hermitian symmetric domain of $\mathrm{SU}(p,q)$ as a Grassmannian. From this point on, we specialize to signature $(1,q)$ and describe the construction of a further model, the Siegel domain $\mathscr{H}_\mathrm{U}$. For a unitary modular group Γ, we then study the stabilizer of the cusps of $\mathscr{H}_\mathrm{U}$ and carry out the compactification of the modular variety $X_\Gamma=\Gamma\backslash\mathscr{H}_\mathrm{U}$. Finally, we define automorphic forms on $\mathscr{H}_\mathrm{U}$ and study their Fourier-Jacobi expansion. A proof of the Koecher principle and a discussion of the value taken by modular forms at the cusp ∞ closes the section.

1.1.1 The subjacent space

The starting point for all our considerations is a vector space over an imaginary quadratic number field, equipped with non-degenerate indefinite hermitian form.

Imaginary quadratic number fields

Denote by d a square-free negative integer. Consider the imaginary quadratic number field $\mathbb{F}=\mathbb{Q}(\sqrt{d})$. Denote by $D_\mathbb{F}$ the discriminant of $\mathbb{F}$, by δ the square root[1] of $D_\mathbb{F}$, and by $\mathcal{O}_\mathbb{F}$ the ring of integers of $\mathbb{F}$. We have

$$D_\mathbb{F}=\begin{cases} d & \text{if } d\equiv 1 \mod 4,\\ 4d & \text{if } d\equiv 2,3 \mod 4,\end{cases}$$

[1] By the square root of a complex number we always mean the principle branch of the square root, unless stated otherwise.

and $\mathcal{O}_{\mathbb{F}} = \mathbb{Z} + \zeta\mathbb{Z}$, where

$$\zeta = \begin{cases} \delta/2 & \text{if } D_F \text{ is even,} \\ \frac{1}{2}(1+\delta) & \text{if } D_F \text{ is odd.} \end{cases}$$

The *inverse different* ideal $\mathcal{D}_{\mathbb{F}}^{-1}$ (also called *complementary ideal*) is the $\mathbb{Z}$-dual of $\mathcal{O}_{\mathbb{F}}$ with respect to the trace $\mathrm{Tr}_{\mathbb{F}/\mathbb{Q}}$.

The inverse of $\mathcal{D}_{\mathbb{F}}^{-1}$, is the $\mathcal{O}_{\mathbb{F}}$-ideal generated by δ, $\mathcal{D}_{\mathbb{F}} = \delta\mathcal{O}_{\mathbb{F}}$. And thus, as a fractional ideal,

$$\mathcal{D}_{\mathbb{F}}^{-1} = \frac{1}{\delta}\mathcal{O}_{\mathbb{F}}.$$

We fix an embedding of $\mathbb{F}$ into $\mathbb{C}$ and thus view $\mathbb{F}$ as a subfield of $\mathbb{C}$. Then, for an element $\alpha = a + b\sqrt{d}$ in $\mathbb{F}$, the Galois conjugate is the same as the complex conjugate $\bar{\alpha} = a - b\sqrt{d}$. The trace $\mathrm{Tr}_{\mathbb{F}/\mathbb{Q}}$ and the norm $\mathrm{N}_{\mathbb{F}/\mathbb{Q}}$ are given by

$$\mathrm{Tr}_{\mathbb{F}/\mathbb{Q}}(\alpha) = \alpha + \bar{\alpha} = 2a = 2\Re\alpha,$$
$$\mathrm{N}_{\mathbb{F}/\mathbb{Q}}(\alpha) = \alpha\,\bar{\alpha} = a^2 + |d|b^2 = (\Re\alpha)^2 + (\Im\alpha)^2.$$

The hermitian space V

Let V be a vector space of dimension $p+q$ over $\mathbb{F}$, equipped with a non-degenerate hermitian form $\langle\cdot,\cdot\rangle$, indefinite of signature (p,q), with $q \geq p$, $p \neq 0$. We define $\langle\cdot,\cdot\rangle$ to be linear in the first and conjugate-linear in the second argument,

$$\langle \alpha x, \beta y\rangle = \alpha\bar{\beta}\,\langle x, y\rangle, \qquad \text{for all} \quad \alpha, \beta \in \mathbb{F}.$$

By extension of scalars, we extend $\langle\cdot,\cdot\rangle$ to the complex space $V_{\mathbb{R}} = V \otimes_{\mathbb{F}} \mathbb{C}$. As a hermitian space, $V_{\mathbb{R}}$ is isometric to the standard hermitian the pseudo-euclidean space $\mathbb{C}^{p,q}$, with form $x_1\bar{y}_1 + \ldots + x_p\bar{y}_p - x_{p+1}\bar{y}_{p+1} - \cdots - x_{p+q}\bar{y}_{p+q}$.

Note that V also carries a structure as a quadratic space over $\mathbb{Q}$, since the trace of $\langle\cdot,\cdot\rangle$ defines a non-degenerate rational bilinear form

$$(\cdot,\cdot) = \mathrm{Tr}_{\mathbb{F}/\mathbb{Q}}\langle\cdot,\cdot\rangle = 2\Re\langle\cdot,\cdot\rangle.$$

This extends to $V_{\mathbb{R}}$ as a real bilinear form, of signature $(2p, 2q)$. As a quadratic space, $V_{\mathbb{R}}$ is isometric to $\mathbb{R}^{2p,2q}$. The attached quadratic form is denoted by $q(\cdot)$.

Since $\langle x, x\rangle = q(x)$, for $x \in V_{\mathbb{R}}$ we will sometimes use $q(\cdot)$ as a shorthand for the norm with respect to $\langle\cdot,\cdot\rangle$. Note that $q(\alpha x) = |\alpha|^2 q(x)$.

Finally, we write x^2 for (x,x), the norm with respect to $(\cdot,\cdot)$.

Remark. *Note that $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)$ determine each other uniquely, as can be seen from*

$$\langle x, y\rangle = \Re\langle x, y\rangle + i\Im\langle x, y\rangle = \frac{1}{2}\big((x,y) + i\,(x, iy)\big).$$

Example 1.1.1. *The most basic example for an indefinite hermitian space occurs when p and q are both 1. Let $V \simeq \mathbb{F}^2$, be such a space with the hermitian form $\langle\cdot,\cdot\rangle$ given by*

$$\langle u, v\rangle = \big\langle (u_1, u_2), (v_1, v_2)\big\rangle = u_1\bar{v}_2 + u_2\bar{v}_1.$$

Alternatively, after a change of basis, $\langle\cdot,\cdot\rangle$ takes the form

$$\langle u, v\rangle = \big\langle (x_1, x_2), (y_1, y_2)\big\rangle = x_1\bar{y}_1 - x_2\bar{y}_2.$$

and the complex hermitian space $V_{\mathbb{R}}$, $\langle\cdot,\cdot\rangle$ is identified with the pseudo-euclidean space $\mathbb{C}^{1,1}$, mentioned above.

The Gram matrices for these two bases are given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The first basis gives $V_{\mathbb{R}}$ a geometry as a hyperbolic, the second as a pseudo-euclidean complex space.

Remark 1.1.2. *If V is a hermitian space over $\mathbb{F}$, we call a decomposition of the form $V = H \oplus V'$ with H an* isotropic *$\mathbb{F}$-subspace of signature $(1,1)$ a* Witt decomposition*. For $p + q \geq 3$, such a decomposition always exists. This follows from a theorem of Hasse-Minkowski applied to V as a quadratic space over $\mathbb{Q}$ with the quadratic form $q(x) = \frac{1}{2}\langle x, x\rangle$.*

In contrast, of course, the complex hermitian space $V_{\mathbb{R}}$, $\langle\cdot,\cdot\rangle$ can always be decomposed into a hyperbolic part, consisting of p copies of a signature $(1,1)$-space and a definite part of dimension $q - p$.

1.1.2 Lattices and unitary groups

In this subsection, we introduce hermitian lattices in V, generalizing the usual definition of lattices as $\mathbb{Z}$-modules in quadratic spaces. We describe the concepts of integrality and evenness for lattices in this context and give the basic example for unimodular lattice, the hyperbolic plane of $\mathscr{O}_{\mathbb{F}}$. Also, we consider unitary groups with respect to $\langle\cdot,\cdot\rangle$ and define modular subgroups.

Hermitian lattices

Definition 1.1.3. *An* $\mathscr{O}_{\mathbb{F}}$-lattice *in V is a finitely generated $\mathscr{O}_{\mathbb{F}}$-submodule L such that $L \otimes_{\mathscr{O}_{\mathbb{F}}} \mathbb{F} = V$.*

An $\mathscr{O}_{\mathbb{F}}$-lattice L is integral*, if*

$$\langle \lambda, \mu\rangle \in \mathscr{D}_{\mathbb{F}}^{-1} \quad \textit{for all} \quad \lambda, \mu \in L.$$

Further, L is called even*, if*

$$\langle \lambda, \lambda\rangle \in \mathbb{Z} \quad \textit{for any} \quad \lambda \in L.$$

Remark 1.1.4. *For a lattice to be even implies that it is also integral.*

Proof. Assume L to be even. Consider the following identity, which holds for any $\lambda,\ \mu \in L$,

$$\mathrm{Tr}_{\mathbb{F}/\mathbb{Q}} \langle \lambda, \mu \rangle = \langle \lambda + \mu, \lambda + \mu \rangle - \langle \lambda, \lambda \rangle - \langle \mu, \mu \rangle .$$

By assumption, the right hand side is integral, thus the trace of $\langle \lambda, \mu \rangle$ is an integer, too. Now, as L is an $\mathcal{O}_{\mathbb{F}}$-module, clearly $x\lambda \in L$ for any $x \in \mathcal{O}_{\mathbb{F}}$. But then, $\mathrm{Tr}_{\mathbb{F}/\mathbb{Q}}\, x \langle \lambda, \mu \rangle \in \mathbb{Z}$ for any x. In other words, $\langle \lambda, \mu \rangle \in \mathcal{D}_{\mathbb{F}}^{-1}$ by the definition of the inverse different. This, in turn means L is integral. □

Note that an $\mathcal{O}_{\mathbb{F}}$-lattice in V is not only a hermitian lattice with the restriction of the form $\langle \cdot, \cdot \rangle$, but also carries the structure of a $\mathbb{Z}$-lattice – a finitely generated $\mathbb{Z}$-module – with the ($\mathbb{Z}$-)bilinear form induced by the trace,

$$(\lambda, \mu) = \mathrm{Tr}_{\mathbb{F}/\mathbb{Q}} \langle \lambda, \mu \rangle .$$

Remark 1.1.5. *Recall the usual definition of integrality and evenness for lattices over $\mathbb{Z}$, by which L is integral, if $(\lambda, \mu) \in \mathbb{Z}$ for all $\lambda,\ \mu \in L$, and L is even if $(\lambda, \lambda) \in 2\mathbb{Z}$ for any $\lambda \in L$.*

The hermitian lattice L with the form $\langle \cdot, \cdot \rangle$ is integral (even), precisely if L with the bilinear form $(\cdot, \cdot)$ is integral (even) as a $\mathbb{Z}$-lattice.

Proof. For L integral: If $\langle \lambda, \mu \rangle \in \mathcal{D}_{\mathbb{F}}^{-1}$, then by definition $\mathrm{Tr}_{\mathbb{F}/\mathbb{Q}} \langle \lambda, \mu \rangle \in \mathbb{Z}$. Conversely, if $\mathrm{Tr}_{\mathbb{F}/\mathbb{Q}} \langle \lambda, \mu \rangle \in \mathbb{Z}$ for any $\lambda,\ \mu$, $\langle \lambda, \mu \rangle$ must be in $\mathcal{D}_{\mathbb{F}}^{-1}$, since L is an $\mathcal{O}_{\mathbb{F}}$-lattice, as observed in the proof of the previous remark.

For L even: Since $\langle \cdot, \cdot \rangle$ is hermitian, $\langle \lambda, \lambda \rangle$ is real for any λ, thus $\langle \lambda, \lambda \rangle = \frac{1}{2}(\lambda, \lambda)$. From this the equivalence of the two definitions is immediate. □

Definition 1.1.6. *For an $\mathcal{O}_{\mathbb{F}}$-lattice $L \subset V$, the $\mathcal{O}_{\mathbb{F}}$-dual is defined as the set*

$$L' = \left\{ x \in V\ ;\ \langle x, \lambda \rangle \in \mathcal{D}_{\mathbb{F}}^{-1}, \quad \textit{for all} \quad \lambda \in L \right\}.$$

The dual L' of an $\mathcal{O}_{\mathbb{F}}$-lattice L is also an $\mathcal{O}_{\mathbb{F}}$-lattice.

Remark 1.1.7. *The dual lattice L' is also the $\mathbb{Z}$-dual of L with respect to $(\cdot, \cdot)$,*

$$L' = \{ x \in V\ ;\ (x, \lambda) \in \mathbb{Z}, \quad \textit{for all} \quad \lambda \in L \}.$$

Remark 1.1.8. *The lattice L is integral if and only if $L \subseteq L'$. For an integral lattice L, the bi-dual $(L')'$ is equal to L.*

Definition 1.1.9. *Let L be an integral $\mathcal{O}_{\mathbb{F}}$-lattice. The quotient a L'/L is a finite $\mathcal{O}_{\mathbb{F}}$ module, called the* discriminant group *of L.*

If $L = L'$, we say that L is unimodular.

Remark 1.1.10. *Somewhat more generally the term* hermitian lattice *can be defined by considering the ring of multipliers of L, which is given by*

$$\mathcal{O}_L = \{ x \in \mathbb{F}\ ;\ xL \subseteq L \}.$$

Clearly, $\mathbb{Z} \subset \mathcal{O}_L$. If the inclusion is proper, it can be shown that $\mathcal{O}_L$ is an order in $\mathbb{F}$ – and thus of finite index in $\mathcal{O}_{\mathbb{F}}$, see [45]. One then calls L hermitian and says that L has complex multiplication with $\mathcal{O}_L$.

Example 1.1.11. *Consider the $\mathcal{O}_{\mathbb{F}}$-module $H = \mathcal{O}_{\mathbb{F}} \oplus \mathcal{D}_{\mathbb{F}}^{-1}$, generated by 1 and δ^{-1}. After tensoring with $\mathbb{F}$ we have $H \otimes_{\mathcal{O}_{\mathbb{F}}} \mathbb{F} = \mathbb{F}^2$, so H is a lattice in $V = \mathbb{F}^2$. With the indefinite hermitian form $\langle \cdot, \cdot \rangle$ given by*

$$\big\langle (x, y), (x', y') \big\rangle = x\bar{y}' + y\bar{x}',$$

the lattice H has signature $(1, 1)$, is even and unimodular. A lattice isomorphic to H (as an $\mathcal{O}_{\mathbb{F}}$-module) is called a hyperbolic plane over $\mathcal{O}_{\mathbb{F}}$.

Unitary groups

The *unitary group* of V is the subgroup of $\mathrm{GL}(V)$ preserving the hermitian form,

$$\mathrm{U}(V) = \big\{ g \in \mathrm{GL}(V)\,;\ \langle gx, gy \rangle = \langle x, y \rangle \quad \text{for all} \quad x, y \in V \big\},$$

the *special unitary group* $\mathrm{SU}(V)$ is the intersection

$$\mathrm{SU}(V) = \mathrm{U}(V) \cap \mathrm{SL}(V).$$

We consider both as algebraic groups defined over $\mathbb{Q}$. The real points of these, $\mathrm{U}(V)(\mathbb{R})$ and $\mathrm{SU}(V)(\mathbb{R})$, are the unitary and special unitary group of $V_{\mathbb{R}}$, respectively.

Let L be a hermitian lattice in V. The isometries of L define arithmetic subgroups, the integer points of $\mathrm{U}(V)$ and $\mathrm{SU}(V)$,

$$\mathrm{U}(L) = \big\{ g \in \mathrm{U}(V);\ g(L) = L \big\} \quad \text{and} \quad \mathrm{SU}(L) = \mathrm{U}(L) \cap \mathrm{SU}(V).$$

Groups commensurable with $\mathrm{U}(L)$ are arithmetic subgroups of $\mathrm{U}(V)$, in the sense defined in [49]. We shall consider the subgroups of finite index in $\mathrm{SU}(L)$ as *unitary modular groups*

We will be particularly interested in modular groups of the following form:

Consider the action of $\mathrm{U}(L)$ and $\mathrm{SU}(L)$ on the discriminant group L'/L. The subgroup of $\mathrm{SU}(L)$ which acts trivially is called the *discriminant kernel* and is denoted by Γ_L^{U}. It has finite index in $\mathrm{SU}(L)$.

1.1.3 Models for the symmetric domain

The Grassmannian

As a hermitian space of signature (p, q), we can decompose V into a sum of maximal definite subspaces, $V = P \oplus Q$ with P positive definite and Q negative definite, of dimension p and q, respectively.

The direct product of the unitary groups attached to $P_{\mathbb{R}}$ and $Q_{\mathbb{R}}$,

$$\mathscr{C} = \mathrm{U}(P)(\mathbb{R}) \times \mathrm{U}(Q)(\mathbb{R}) \quad \hookrightarrow \quad \mathrm{U}(V)(\mathbb{R}),$$

is a maximal compact subgroup of $\mathrm{U}(V)(\mathbb{R})$ stabilizing $P_{\mathbb{R}}$ and $Q_{\mathbb{R}}$. The intersection with $\mathrm{SU}(V)(\mathbb{R})$, $\mathscr{C} \cap \mathrm{SU}(V)(\mathbb{R})$, is maximal compact in $\mathrm{SU}(V)(\mathbb{R})$. A symmetric domain for the operation of $\mathrm{SU}(V)(\mathbb{R})$ on $V_{\mathbb{R}}$ is given by the quotient

$$\mathrm{SU}(V)(\mathbb{R}) / \big(\mathscr{C} \cap \mathrm{SU}(V)(\mathbb{R})\big),$$

which is isomorphic to the *Grassmannian manifold* consisting of the positive definite p-dimensional subspaces of $V_{\mathbb{R}}$, denoted by Gr_{U}. This so called *Grassmannian model* is our starting point for the construction of further models of the symmetric domain.

Note that Gr_{U} can also serve as a model of the symmetric domain for the operation of $\mathrm{U}(V)(\mathbb{R})$, since $\mathrm{Gr}_{\mathrm{U}} \simeq \mathrm{U}(V)(\mathbb{R}) / \mathscr{C}$.

A positive cone

From now on, let us concentrate on the case where p is equal to 1. Thus V has signature $(1,q)$ and the points of Gr_{U} correspond to the one-dimensional positive subspaces of $V_{\mathbb{R}}$. From this, we have an interpretation of Gr_{U} as the following cone in projective space

$$\mathscr{K}_{\mathrm{U}} := \left\{[z] \in \mathbb{P}(V_{\mathbb{R}})\,;\ \langle z,z\rangle > 0\right\},$$

where z denotes some representative vector for the positive line $[z]$. We have $\mathrm{Gr}_{\mathrm{U}} \simeq \mathscr{K}_{\mathrm{U}}$ as a complex manifold and as an algebraic variety.

We will define an affine model for the symmetric domain presently, subject to a special choice of basis for a signature $(1,1)$-subspace of V.

Assumption 1.1.12. *We assume in the following that the hermitian $\mathbb{F}$-vector space V, $\langle\cdot,\cdot\rangle$ of signature $(1,q)$ contains an isotropic vector. Further we assume that we are given an $\mathscr{O}_{\mathbb{F}}$-lattice L in V, hermitian with the restriction of the form $\langle\cdot,\cdot\rangle$, such that $V = L\otimes_{\mathscr{O}_{\mathbb{F}}}\mathbb{F}$.*

This assumption is non-trivial only if V has signature $(1,1)$ or $(1,2)$, recall remark 1.1.2.

A basis for the hyperbolic part

Given the hermitian space V of signature $(1,q)$ and an even hermitian $\mathscr{O}_{\mathbb{F}}$-lattice in L in V, we choose a primitive isotropic lattice vector $\ell \in L$ and an element $\ell' \in L'$ with $\langle \ell,\ell'\rangle \neq 0$. A lattice vector λ in L is called *primitive* if $\lambda = \mu\lambda'$, for $\mu \in \mathscr{O}_{\mathbb{F}}$ implies $\mu \in \mathscr{O}_{\mathbb{F}}^{\times}$.

Definition 1.1.13. *We call two lattice vectors ℓ and ℓ' a* hyperbolic pair, *if $\ell \in L$ is primitive isotropic and $\ell' \in L'$ with $\langle \ell,\ell'\rangle \neq 0$.*

Consider the isotropic subspace of signature $(1,1)$ spanned by ℓ and ℓ' over $\mathbb{F}$, we can view this as the hyperbolic part of V. Denote by D the set of lattice vectors of L which lie in the complement (with respect to $\langle\cdot,\cdot\rangle$) of this space

$$D = \left\{\lambda \in L\,;\ \langle \lambda,\ell\rangle = \langle \lambda,\ell'\rangle = 0\right\}. \tag{1.1.1}$$

Clearly, $D\otimes_{\mathscr{O}_{\mathbb{F}}}\mathbb{F}$ is a definite hermitian subspace of signature $(0,q-1)$ as V splits into a direct sum

$$V = \left(\mathbb{F}\ell \oplus \mathbb{F}\ell'\right) \oplus \left(D\otimes_{\mathscr{O}_{\mathbb{F}}}\mathbb{F}\right) = \left(H\otimes_{\mathscr{O}_{\mathbb{F}}}\mathbb{F}\right) \oplus \left(D\otimes_{\mathscr{O}_{\mathbb{F}}}\mathbb{F}\right),$$

with the signature $(1,1)$ subspace spanned by ℓ and ℓ' isomorphic to the hyperbolic plane $H\otimes_{\mathscr{O}_{\mathbb{F}}}\mathbb{F}$ over $\mathbb{F}$.

Example 1.1.14. *Consider the hyperbolic plane $H = \mathscr{O}_{\mathbb{F}} \oplus \mathscr{D}_{\mathbb{F}}^{-1}$ as in example 1.1.11. And V the hermitian space, of signature $(1,1)$, given by $V = H\otimes_{\mathscr{O}_{\mathbb{F}}}\mathbb{F}$. Then, a hyperbolic pair can be obtained by setting $\ell = 1 \in \mathscr{O}_{\mathbb{F}}$, clearly, a primitive element, and choosing some element in $\mathscr{D}_{\mathbb{F}}$, for example $-\delta^{-1}$. Thus, we set $\ell' = -\delta^{-1}$ – in which case ℓ and ℓ' are also a basis of H as an $\mathscr{O}_{\mathbb{F}}$-module. Thus, $H = \mathscr{O}_{\mathbb{F}}\ell \oplus \mathscr{O}_{\mathbb{F}}\ell'$. Written as vectors in V, $\ell = (1,0)$, $\ell' = (0,-\delta^{-1})$. By the definition of the hermitian form $\langle\cdot,\cdot\rangle$ in the example, we have*

$$\langle \ell,\ell'\rangle = \delta^{-1},\ \langle \ell,\ell\rangle = \langle \ell',\ell'\rangle = 0.$$

Remark. *In the previous example, both members of the hyperbolic pair ℓ, ℓ' are actually isotropic vectors – in general though, given a lattice of signature $(1,q)$ satisfying assumption 1.1.12, we can not expect this to be the case-*

Lemma 1.1.15. *Let L be an even* unimodular *lattice containing a primitive isotropic vector ℓ. Then, L can be written in the form $L = \left(\mathcal{O}_{\mathbb{F}} \oplus \mathcal{D}_{\mathbb{F}}^{-1}\right) \oplus D = H \oplus D$ as the direct sum of a hyperbolic plane and a definite part, with $\langle H, D\rangle = 0$.*

Proof. Since ℓ is primitive, by the next lemma 1.1.16, $L = L'$ contains a vector ℓ', such that ℓ, ℓ' are a hyperbolic pair and $\left\langle \ell, \ell' \right\rangle = \delta^{-1}$.

Let D be the complement in L (with respect to $\langle \cdot, \cdot \rangle$) of $\mathcal{O}_{\mathbb{F}}\ell \oplus \mathcal{O}_{\mathbb{F}}\ell'$. Then, we can write any $\lambda \in L$ in the form

$$\lambda = a\ell + b\ell' + \lambda_D, \quad \text{with} \quad a, b \in \mathbb{F}, \lambda_D \in D \otimes \mathbb{F}.$$

We show a, $b \in \mathcal{O}_{\mathbb{F}}$ and $\lambda_D \in D$, whence the claim follows.

We have $\langle \lambda, \ell \rangle = b \left\langle \ell', \ell \right\rangle = -b\delta^{-1}$. Since L is integral and ℓ, $\lambda \in L$, we have $\langle \lambda, \ell \rangle \in \mathcal{D}_{\mathbb{F}}^{-1}$. The, clearly $b \in \delta \mathcal{D}_{\mathbb{F}}^{-1} = \mathcal{O}_{\mathbb{F}}$. Similarly. $\left\langle \lambda, \ell' \right\rangle = a\delta^{-1} + b \left\langle \ell', \ell' \right\rangle$, is contained in $\mathcal{D}_{\mathbb{F}}^{-1}$ as L is integral. Since $b \in \mathcal{O}_{\mathbb{F}}$ and $\left\langle \ell, \ell' \right\rangle \in \mathbb{Z}$, as L is even, it follows that $a \in \mathcal{O}_{\mathbb{F}}$, as well.

Finally, since $a, b \in \mathcal{O}_{\mathbb{F}}$, the vectors $a\ell$ and $b\ell'$ are contained in L, too. It follows that $\lambda_D = \lambda - a\ell - b\ell'$ is contained in $D \subset L$. So every $\lambda \in L$ can be written in the above form, as claimed. □

Lemma 1.1.16. *Let L be an even lattice and L' its dual. Let be v a primitive vector contained in L. Then $\left\langle v, L' \right\rangle = \mathcal{D}_{\mathbb{F}}^{-1}$.*

Proof. Clearly, $0 \neq \left\langle v, L' \right\rangle \subset \mathcal{D}_{\mathbb{F}}^{-1}$, since $\langle \cdot, \cdot \rangle$ is non-degenerate and $L' \otimes_{\mathcal{O}_{\mathbb{F}}} \mathbb{F} = V$. We assume that the inclusion is proper. So, $\left\langle v, L' \right\rangle$ is a fractional ideal $\mathfrak{a} \subsetneq \mathcal{D}_{\mathbb{F}}^{-1}$.

Then, there exists a prime ideal $\mathfrak{p} \subset \mathcal{O}_{\mathbb{F}}$ with $\mathcal{D}_{\mathbb{F}}\mathfrak{a} \subset \mathfrak{p}$. Now, take an element $\lambda \in \mathfrak{p}^{-1} \setminus \mathcal{O}_{\mathbb{F}}$ (the existence is clear). Now,

$$\left\langle \lambda v, L' \right\rangle \subset \lambda \mathfrak{a} \subset \mathfrak{p}^{-1}\mathfrak{a} \subset \mathcal{D}_{\mathbb{F}}^{-1},$$

since $\mathfrak{a} \subset \mathfrak{p}\mathcal{D}_{\mathbb{F}}^{-1}$. Thus, λv is contained in the bi-dual $(L')'$ of L. Since L is integral, $(L')' = L$. So, $\lambda v \in L$. But since $\lambda^{-1} \in \mathfrak{p}$, we get $\lambda^{-1}(\lambda v) = v$ in contradiction to the primitiveness of v. □

The Siegel domain model

Consider the following set of representatives in $V_{\mathbb{R}}$ for $\mathcal{K}_{\mathrm{U}}$,

$$\widetilde{\mathcal{K}_{\mathrm{U}}^{1}} := \left\{ z \in V_{\mathbb{R}}\ ;\ \langle z, z \rangle > 0 \quad \text{and} \quad \langle z, \ell \rangle = \left\langle \ell', \ell \right\rangle \right\}. \tag{1.1.2}$$

Under the canonical projection $\pi : V_{\mathbb{R}} \to \mathbb{P}(V_{\mathbb{R}})$ this set bijects onto $\mathcal{K}_{\mathrm{U}}$. The condition on $\langle z, \ell \rangle$ for all $z \in \widetilde{\mathcal{K}_{\mathrm{U}}^{1}}$ fixes the ℓ'-component to 1.

We shall write each representative z in the following form:

$$z = \ell' - \tau \left\langle \ell', \ell \right\rangle \delta\ell + \sigma, \quad \text{with} \quad \sigma \in D \otimes_{\mathcal{O}_{\mathbb{F}}} \mathbb{C} \text{ and } \quad \tau \in \mathbb{C}. \tag{1.1.3}$$

The positivity condition $\langle z, z\rangle > 0$ now reads

$$2|\langle \ell, \ell' \rangle|^2 |\delta| \Im\tau > -\langle \sigma, \sigma\rangle - \langle \ell', \ell' \rangle .$$

Since one component is fixed, we can identify $\widetilde{\mathscr{K}_{\mathrm{U}}^1}$ with a subset of $\mathbb{C}^q$, parameterized by the coordinates τ and σ.

Definition 1.1.17. *Given a hermitian lattice L and a fixed choice of a hyperbolic pair $\ell \in L$, $\ell' \in L'$ spanning the hyperbolic part of V over $\mathbb{F}$, the* Siegel domain model *for* Gr_{U} *is defined as*

$$\mathscr{H}_{\mathrm{U}} := \left\{ (\tau, \sigma) \in \mathbb{C} \times \left(D \otimes_{\mathscr{O}_{\mathbb{F}}} \mathbb{C} \right) ; \quad 2|\langle \ell, \ell' \rangle|^2 |\delta| \Im\tau > -\langle \sigma, \sigma\rangle - \langle \ell', \ell' \rangle \right\} . \tag{1.1.4}$$

Note that if $q = 1$ the Siegel domain $\mathscr{H}_{\mathrm{U}}$ is isomorphic to the usual upper halfplane $\mathbb{H}$ in $\mathbb{C}$. This is also the *only* case, in which $\mathscr{H}_{\mathrm{U}}$ is a *tube-domain*, i.e. a subset of $\mathbb{C}^n$, here $n = 1$, of the form $\mathbb{R}^n + iD$ with D an open subset of $\mathbb{R}^n$.

Remark 1.1.18. *Clearly, the construction of the Siegel domain depends on the choice of the hyperbolic pair ℓ and ℓ'. Another point of view is to say that the choice of ℓ and ℓ' fixes an embedding of the set $\mathscr{H}_{\mathrm{U}} \subset \mathbb{C} \times \mathbb{C}^{q-1}$ defined by (1.1.4) into $\mathscr{K}_{\mathrm{U}}$ through the assignment of $(\tau, \sigma) \mapsto z$.*

Now, let μ be another primitive isotropic lattice vector. By a theorem of Witt, there is $g \in \mathrm{SU}(V)$, with $\mu = g\ell$. Then μ and $g\ell'$ are a hyperbolic pair according to the above definition. The action of the unitary modular group Γ on the Siegel domain, defined with respect to μ and $g\ell'$ is the same as that of the conjugate group $g\Gamma g^{-1}$ on the Siegel domain defined with respect to ℓ and ℓ'.

Remark. *An other standard affine model for the symmetric domain of* $\mathrm{SU}(V)$ *is the q-ball,*

$$\mathscr{B}_q = \left\{ x \in \mathbb{C}^{0,q} ; -\langle x, x\rangle < 1 \right\} = \left\{ (x_1, \ldots, x_q) \in \mathbb{C}^q ; \textstyle\sum_i |x_i|^2 < 1 \right\}.$$

It is constructed by fixing an isometry from $V_{\mathbb{R}}$ to $\mathbb{C}^{1,q}$ with a pseudo-euclidean basis $e_1, \ldots, e_{q+1}$, so that $\langle z, z\rangle = |z_1|^2 - |z_2|^2 - \cdots - |z_{q+1}|^2$ and normalizing the one positive coordinate z_1 for a representative of $\mathscr{K}_{\mathrm{U}}$ as 1. Thus, $z = e_1 + x$ and the positivity condition gives $1 + \langle x, x\rangle > 0$.

Remark. *More generally, for unitary groups of the type* $\mathrm{U}(p, q)$*, Shimura gives a realization of the symmetric domain as an unbounded domain in the $q \times p$-matrices with complex entries, with $p \times p$ and $(q - p) \times p$ matrices playing the roles of τ and σ in the above definition, see [62].*

The unitary modular variety

Definition 1.1.19. *Let Γ be a subgroup of finite index in* $\mathrm{SU}(L)$*. The unitary* modular variety *X_Γ is the quotient of the symmetric domain under the operation of Γ,*

$$X_\Gamma \simeq \Gamma \backslash \mathrm{Gr}_{\mathrm{U}} \simeq \Gamma \backslash \mathrm{SU}(V)(\mathbb{R}) / (\mathscr{C} \cap \mathrm{SU}(V)(\mathbb{R})) .$$

The modular variety is a complex space of dimension q; it is regular if Γ is torsion free.

The modular variety X_Γ is also called a *ball-quotient*, a term referring to the model $\mathscr{B}_q$.

Remark 1.1.20. *The existence of torsion free arithmetic subgroups of* $\mathrm{SU}(L)$ *is always assured. Indeed, by a theorem of Borel, [8], there exist ideals $\mathfrak{A}$ in $\mathscr{O}_{\mathbb{F}}$ for which the subgroup of Γ_L^{U} acting trivially on $L/\mathfrak{A}L$, called the* congruence subgroup of level $\mathfrak{A}$*, is torison free.*

1.1.4 Cusps and parabolic subgroups

The boundary points of the symmetric domain correspond to non-zero isotropic vectors in V. $\mathbb{F}$-*rational boundary components* are given by primitive isotropic lattice vectors from L. If Γ is a unitary modular group, a *cusp* is a Γ-equivalence class of $\mathbb{F}$-rational boundary components.

The choice of a hyperbolic pair as above and in particular of a primitive isotropic lattice vector $\ell \in L$ defines a cusp, as the projective line $[\ell] \in \mathbb{P}(V_{\mathbb{R}})$ is an $\mathbb{F}$-rational boundary point of the symmetric domain. For the following, choose once and for all a hyperbolic pair ℓ and ℓ'.

We will now consider which elements in $\mathrm{SU}(V)$ which stabilize the cusp. More details on the parabolic subgroups of $\mathrm{U}(p,q)$ in general may be found in [60] and in [64]. For the case of $\mathrm{U}(1,2)$ a treatment closer to the one presented here can be found in [22] and in [50], [51], [52]. Also, the structure of the stabilizers of a points in $\mathscr{H}_{\mathrm{U}}$ is treated in [37].

The stabilizer of a cusp

Denote by $P(\ell)$ the stabilizer in $\mathrm{SU}(V)$ of the cusp attached to ℓ,

$$P(\ell) := \{g \in \mathrm{SU}(V);\ [g\ell] = [\ell]\}.$$

We want to describe the structure of $P(\ell)$. To this aim, we consider three types of transformations on V:

I. **Translations:** For $h \in \mathbb{Q}$ define the linear map

$$[h,0]:\ v \mapsto v - \langle v,\ell\rangle\, \delta h \ell. \tag{1.1.5}$$

In the coordinates (τ,σ) of $\mathscr{H}_{\mathrm{U}}$, this acts as

$$[h,0]:\ (\tau,\sigma) \longmapsto (\tau + h, \sigma).$$

II. **Eichler transformations:** For $t \in D \otimes_{\mathcal{O}_{\mathbb{F}}} \mathbb{F}$, define

$$[0,t]:\ v \mapsto v + \langle v,\ell\rangle\, t - \langle v,t\rangle\, \ell - \frac{1}{2}\langle v,\ell\rangle \langle t,t\rangle\, \ell. \tag{1.1.6}$$

The transformation $[0,t]$ is called an Eichler transformation or an Eichler element in $\mathrm{SU}(V)$. Its action on $\mathscr{H}_{\mathrm{U}}$ is given by

$$[0,t]:\ (\tau,\sigma) \longmapsto \left(\tau + \frac{\langle \sigma,t\rangle}{\delta \langle \ell',\ell\rangle} + \frac{1}{2}\frac{\langle t,t\rangle}{\delta}, \sigma + \langle \ell',\ell\rangle\, t\right).$$

III. **Embedded** $\mathrm{SU}(D \otimes_{\mathcal{O}_{\mathbb{F}}} \mathbb{F})$**:** The special unitary group of $D \otimes_{\mathcal{O}_{\mathbb{F}}} \mathbb{F}$ is embedded into $\mathrm{SU}(V)$, acting trivially on the hyperbolic part of V spaned by ℓ and ℓ'. On $\mathscr{H}_{\mathrm{U}}$, an element g of this subgroup acts as

$$g:\ (\tau,\sigma) \mapsto (\tau, g\sigma).$$

It is easily checked that all three types of transformations belong to $\mathrm{SU}(V)$ and stabilize $\mathbb{C}\ell$. The set of real points $P(\ell)(\mathbb{R})$ of the algebraic group $P(\ell)$ can be obtained by considering $[h,0]$ with $h\in\mathbb{R}$, $[0,t]$ with $t\in D\otimes_{\mathcal{O}_\mathbb{F}}\mathbb{C}$ and $g\in\mathrm{SU}(D\otimes_{\mathcal{O}_\mathbb{F}}\mathbb{C})\subset\mathrm{SU}(V)(\mathbb{R})$.

Observe that the translations $[h,0]$, $h\in\mathbb{Q}$ commute and form an Abelian algebraic group, isomorphic to the additive group $(\mathbb{Q},+)$. The intersection with $\mathrm{SU}(L)$ is isomorphic to $(\mathbb{Z},+)$.

Among the Eichler transformations $[0,t]$, one has the following commutation relation:

$$[0,t_1]\circ[0,t_2]=[\frac{\Im\langle t_1,t_2\rangle}{2|\delta|},0]\circ[0,t_1+t_2].$$

Also, translations commute with Eichler transformations. We set $[h,t]:=[h,0]\circ[0,t]$.

Definition 1.1.21. *The* Heisenberg group $H(\ell)$ *is the algebraic group defined on the set of pairs* $[h,t]\in\mathbb{Q}\times\big(D\otimes_{\mathcal{O}_\mathbb{F}}\mathbb{F}\big)$ *by the group law*

$$[h,t]\circ[h',t']=\big[h+h'+\frac{\Im\langle t,t'\rangle}{2|\delta|},t+t'\big].$$

The center of the Heisenberg group consists of all translations $[h,0]$.

To sum up, we have exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{R} & \to & H(\ell)(\mathbb{R}) & \to & D\otimes_{\mathcal{O}_\mathbb{F}}\mathbb{C} & \to & 0, \quad \text{and}\\ 0 & \to & \mathbb{Q} & \to & H(\ell) & \to & D\otimes_{\mathcal{O}_\mathbb{F}}\mathbb{F} & \to & 0.\\ & & h & \mapsto & [h,0] & & & & \\ & & & & [h,t] & \mapsto & t & & \end{array}$$

Finally, the full stabilizer of the cusp attached to ℓ is obtained by taking the semidirect product of $H(\ell)$ with the embedded unitary group of the definite subspace $D\otimes_{\mathcal{O}_\mathbb{F}}\mathbb{F}$,

$$P(\ell)=H(\ell)\ltimes\mathrm{SU}\big(D\otimes_{\mathcal{O}_\mathbb{F}}\mathbb{F}\big).$$

The center of $P(\ell)$ is the subgroup generated by all translations $[h,0]$, for $h\in\mathbb{Q}$.

Given a modular group Γ, we denote by $\Gamma(\ell)$ the stabilizer of the cusp in Γ,

$$\Gamma(\ell)=\Gamma\cap P(\ell).$$

For such a group, the following proposition holds, of which we sketch the proof.

Proposition 1.1.22. *Let* Γ *be a subgroup of finite index in* $\mathrm{SU}(L)$. *There exists a positive integer* N *and a lattice* $\tilde{D}$ *of finite index in* D *such that*

$$[h,t]\in\Gamma(\ell),\qquad \textit{for all}\quad h\in N\mathbb{Z},\ t\in\tilde{D}.$$

Also,

$$\frac{\Im\langle t,t'\rangle}{2|\delta|}\in N\mathbb{Z}\quad\textit{for all}\quad t,t'\in\tilde{D}.$$

Proof. Assume that L is an even $\mathscr{O}_{\mathbb{F}}$-lattice.

For the first part of the statement consider (1.1.5), with $v \in L$. Then $\langle v, \ell \rangle \in \mathscr{D}_{\mathbb{F}}^{-1}$, since L is an integral lattice, while δh is in $\mathscr{D}_{\mathbb{F}}$ for any integer h. Thus, since $\mathscr{O}_{\mathbb{F}}\ell \subset L$, the shifted vector $[h,0]v$ remains in L and $[h,0] \in \mathrm{SU}(L)$. By assumption Γ is of finite index, thus, $[h,0]^N = [Nh,0] \in \Gamma$ for some positive integer N.

For the second part of the statement, a similar consideration of (1.1.6) shows that $[0,t]v$ with $t \in D$ translates a lattice vector $v \in L$ by some element of $\mathscr{D}_{\mathbb{F}}^{-1}L = L'$. The observation that $[0,t]^k = [0,kt]$ (easily verified by calculation) and the fact that L'/L is finite show the existence of a lattice $\tilde{D}$ of finite index in D for $\Gamma = \mathrm{SU}(L)$. Finally, if Γ is of finite index in $\mathrm{SU}(L)$, as for the first part of the statement, a repeated application of the finite index argument completes the proof for this case, as well. □

1.1.5 Compactification

Let Γ be a modular group and denote by $\Gamma(\ell)$ the stabilizer in Γ of the cusp corresponding to ℓ, as in the previous subsection.

In order to compactify the modular variety $X_\Gamma = \Gamma\backslash\mathscr{H}_{\mathrm{U}}$, we introduce suitable neighborhoods of infinity in the affine model:

Definition 1.1.23. *For a real number $C > 0$, define the sets*

$$\mathscr{H}_{\mathrm{U}}^C = \left\{(\tau,\sigma);\ 2|\delta||\langle \ell',\ell\rangle|^2\,\mathfrak{I}(\tau) + \langle\sigma,\sigma\rangle + \langle \ell',\ell'\rangle > C\right\} \subset \mathbb{C}\times(D\otimes_{\mathscr{O}_{\mathbb{F}}}\mathbb{C})$$
$$\text{and}\quad \overline{\mathscr{H}_{\mathrm{U}}^C} = \mathscr{H}_{\mathrm{U}}^C \cup \{\infty\}.$$

Note that sets of the form $\mathscr{H}_{\mathrm{U}}^C$ are invariant under the action of $\Gamma(\ell)$.

We define the following topology on $\mathscr{H}_{\mathrm{U}} \cup \{\infty\}$: A subset $U \subset \mathscr{H}_{\mathrm{U}} \cup \{\infty\}$ is called *open*, if

- $U \cap \mathscr{H}_{\mathrm{U}}$ is open in the usual sense and
- if $\infty \in U$, then U contains $\overline{\mathscr{H}_{\mathrm{U}}^C}$ for some $C > 0$.

From this definition, it is clear that the action of $\Gamma(\ell)$, continuous on $\mathscr{H}_{\mathrm{U}}$, extends continuously to $\mathscr{H}_{\mathrm{U}} \cup \{\infty\}$. Further, clearly, with the quotient topology, the quotient space

$$\Gamma(\ell)\backslash\left(\mathscr{H}_{\mathrm{U}}\cup\{\infty\}\right) = \left(\Gamma(\ell)\backslash\mathscr{H}_{\mathrm{U}}\right)\cup\{\infty\},$$

is locally compact. A family of compact neighborhoods of infinity is given by the sets

$$\Gamma(\ell)\backslash\overline{\mathscr{H}_{\mathrm{U}}^C},\quad \text{for}\quad C\in\mathbb{R},\, C>0.$$

To see that $\left(\mathscr{H}_{\mathrm{U}} \cup \{\infty\}\right)$ is a complex space, proceed as follows.

Denote by $X_\Gamma^* = \Gamma(\ell)\backslash\left(\mathscr{H}_{\mathrm{U}}\cup\{\infty\}\right)$. We define the complex structure on X_Γ^* as follows. For an open set $V \subset X_\Gamma^*$, let $U \subset \mathscr{H}_{\mathrm{U}} \cup \{\infty\}$ be the inverse image of V under the canonical projection $\mathrm{pr} : \mathscr{H}_{\mathrm{U}} \cup \{\infty\} \to X_\Gamma^*$. Further we let U' be the inverse image of U in $\mathscr{H}_{\mathrm{U}}$. We have the diagram

$$\begin{array}{ccccc} \mathscr{H}_{\mathrm{U}} & \longrightarrow & \mathscr{H}_{\mathrm{U}}\cup\{\infty\} & \xrightarrow{\mathrm{pr}} & X_\Gamma^* \\ \uparrow & & \uparrow & & \uparrow \\ U' & \longrightarrow & U & \longrightarrow & V \end{array}.$$

Now, define $\mathcal{O}(V)$ to be the ring of continuous functions $f : V \to \mathbb{C}$, such that the pull back $\mathrm{pr}^*(f)$ is holomorphic on U'. By the usual methods of algebraic geometry, this defines a sheaf of rings $\mathcal{O}$ on X^*_Γ, and the pair $\left(X^*_\Gamma, \mathcal{O}\right)$ is a locally ringed space. By the theory of Baily Borel [2], the following statement holds:

Proposition 1.1.24. *The space* $\left(X^*_\Gamma, \mathcal{O}\right)$ *is a normal complex space.*

However, X^*_Γ still has singularities at the cusps and at elliptic fixed points of Γ.

Resolution of the singularity at a cusp:

The singularities at the cusps can be resolved, cf. [35] and [36]. It suffices to resolve the cusp at infinity.

Consider the quotient of $\mathcal{H}_{\mathrm{U}}$ with $N\mathbb{Z}$, where N is the positive number defined in proposition 1.1.22; in other words, we quotient out the action of the translations in $\Gamma(\ell)$.

Using the holomorphic map

$$(\tau, \sigma) \mapsto \left(e\left(\frac{\tau}{N}\right), \sigma\right), \tag{1.1.7}$$

we identify the quotient with a subset of $\mathbb{C}^q$,

$$N\mathbb{Z}\backslash\mathcal{H}_{\mathrm{U}} = \left\{(q, \sigma);\ 0 < |q| < \exp\left(\frac{-\pi\left(\langle\sigma,\sigma\rangle + \langle\ell',\ell'\rangle\right)}{N|\delta||\langle\ell',\ell\rangle|^2}\right)\right\}.$$

which can be considered as a bundle of punctured discs over $\mathbb{C}^{q-1}$. Then, $N\mathbb{Z}\backslash\mathcal{H}_{\mathrm{U}}$ can be viewed as the total space of a complex analytic bundle of punctured discs over the Abelian variety $\tilde{D}\backslash\mathbb{C}^q$.

Adding the midpoint to each disc gives the bundle

$$\widetilde{N\mathbb{Z}\backslash\mathcal{H}_{\mathrm{U}}} = \left\{(q, \sigma);\ |q| < e\left(\frac{\left(\langle\sigma,\sigma\rangle + \langle\ell',\ell'\rangle\right)}{2\delta N|\langle\ell',\ell\rangle|^2}\right)\right\}.$$

Now, the action of the Eichler elements in $\Gamma(\ell)$, i.e. of $\Gamma(\ell)/N\mathbb{Z} \simeq \tilde{D}$, on $\widetilde{N\mathbb{Z}\backslash\mathcal{H}_{\mathrm{U}}}$, given by

$$(q, \sigma) \mapsto \left(q \cdot e\left(\frac{1}{N}\left(\frac{\langle\sigma, t\rangle}{\delta\,\langle\ell',\ell\rangle} + \frac{\langle t, t\rangle}{2\delta}\right)\right), \sigma + \langle\ell',\ell\rangle\, t\right)$$

is well defined at $q = 0$. This action is free and hence $\widetilde{N\mathbb{Z}\backslash\mathcal{H}_{\mathrm{U}}}$ is an analytic manifold for which there is a holomorphic and proper map

$$\tilde{D}\backslash\left(\widetilde{N\mathbb{Z}\backslash\mathcal{H}_{\mathrm{U}}}\right) \longrightarrow \Gamma(\ell)\backslash\mathcal{H}_{\mathrm{U}} \cup \{\infty\} = X^*_\Gamma.$$

1.1.6 Modular forms

Definition and automorphy factor

Consider the preimage of $\mathcal{K}_{\mathrm{U}}$ under the canonical projection $\pi : V_{\mathbb{R}} \to \mathbb{P}(V_{\mathbb{R}})$,

$$\widetilde{\mathcal{K}_{\mathrm{U}}} = \left\{v \in V_{\mathbb{R}} \setminus \{0\};\ \langle v, v\rangle > 0\right\}.$$

$\widetilde{\mathcal{K}_{\mathrm{U}}}$ is a principal $\mathbb{C}^\times$ bundle over $\mathcal{K}_{\mathrm{U}}$, which can be viewed as a restriction of the tautological bundle of $\mathbb{P}(V_{\mathbb{R}})$.

Definition 1.1.25. *Let* Γ *be a subgroup of finite index in* $\mathrm{SU}(L)$*,* k *a rational integer and* $\chi : \Gamma \to S^1$ *a unitary character. A* holomorphic (meromorphic) automorphic form *with respect to* Γ *of weight* k *and character* χ *is a function* $f : \widetilde{\mathscr{K}_{\mathrm{U}}} \to \mathbb{C}$ *(or* $\bar{\mathbb{C}}$ *if meromorphic) with the following properties*

- f *is holomorphic (meromorphic) on* $\widetilde{\mathscr{K}_{\mathrm{U}}}$*,*
- $f(\gamma z) = \chi(\gamma) f(z)$ *for all* $\gamma \in \Gamma$*,*
- $f(tz) = t^{-k} f(z)$ *for all* $t \in \mathbb{C}^\times$*.*

Such functions uniquely correspond to the Γ-invariant holomorphic sections of $\mathscr{K}_{\mathrm{U}}^{\otimes k}$.

For a fixed choice of the cusp ℓ, the subset $\widetilde{\mathscr{K}_{\mathrm{U}}^1}$ of $\widetilde{\mathscr{K}_{\mathrm{U}}}$ is defined through the normalization $\langle z, \ell \rangle = \langle \ell', \ell \rangle$ in (1.1.2). This uniquely defines a holomorphic and nowhere vanishing section $\mathfrak{s}_{\ell,\ell'}$ of $\mathscr{K}_{\mathrm{U}}$,

$$\mathfrak{s}_{\ell,\ell'} : \mathscr{K}_{\mathrm{U}} \longrightarrow \widetilde{\mathscr{K}_{\mathrm{U}}^1},$$

and a projection $\mathfrak{s}'$ from $\widetilde{\mathscr{K}_{\mathrm{U}}}$ to $\widetilde{\mathscr{K}_{\mathrm{U}}^1}$, given by $\mathfrak{s}' = \mathfrak{s}_{\ell,\ell'} \circ \pi$.

The action of $\mathrm{SU}(V)(\mathbb{R})$ gives rise to a holomorphic automorphy factor

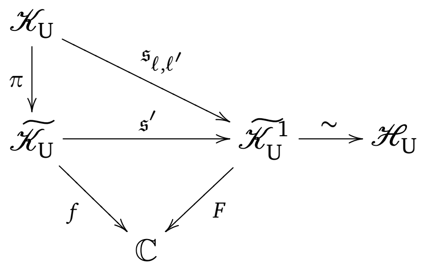
$$j : \begin{cases} \mathrm{SU}(V) \times \widetilde{\mathscr{K}_{\mathrm{U}}^1} & \to & \mathbb{C}^\times \\ (\gamma, z) & \mapsto & j(\gamma, z) \end{cases} \tag{1.1.8}$$

given by $j(\gamma, z) = \langle \gamma z, \ell \rangle / \langle \ell', \ell \rangle$.

Now, $\mathfrak{s}_{\ell,\ell'}$ induces an isomorphism $f \mapsto f \circ \mathfrak{s}'$ between holomorphic automorphic forms for Γ and holomorphic functions $F : \widetilde{\mathscr{K}_{\mathrm{U}}^1} \to \mathbb{C}$, which transform according to

$$F\big(\mathfrak{s}'(\gamma z)\big) = \chi(\gamma)\, j\big(\gamma, z\big)^k F(z), \quad \text{for all } \gamma \in \Gamma. \tag{1.1.9}$$

The relationship between f and F, the section $\mathfrak{s}_{\ell,\ell'}$ and the projections $\mathfrak{s}$, π is illustrated by the following diagram:



The corresponding section of $\widetilde{\mathscr{K}_{\mathrm{U}}^{\otimes k}}$ then takes the form

$$[z] \longmapsto F(\mathfrak{s}_{\ell,\ell'}[z]) \cdot \mathfrak{s}_{\ell,\ell'}([z])^{\otimes k}.$$

Using the unique association of the function F on $\widetilde{\mathscr{K}_{\mathrm{U}}^1}$ to the automorphic form f we can give another definition of automorphic form, alternative to definition 1.1.25, as follows.

Definition 1.1.26. *Use the same notation as in definition 1.1.25. A holomorphic automorphic form of weight* k *and character* χ *on the group* Γ *is a holomorphic function* $F : \widetilde{\mathscr{K}_{\mathrm{U}}^1} \to \mathbb{C}$ *which exhibits the transformation behavior as in* (1.1.9)*. Meromorphic automorphic forms are defined similarly.*

Now, consider holomorphic automorphic forms with trivial character. Since Γ-invariance is required, such functions pull back to the modular variety $X_\Gamma = \Gamma\backslash\mathscr{H}_{\mathrm{U}}$ on which they correspond to sections of the bundle $\mathscr{L}_\Gamma^{\otimes k}$, where

$$\mathscr{L}_\Gamma = \Gamma\backslash\widetilde{\mathscr{K}_{\mathrm{U}}} \longrightarrow X_\Gamma$$

denotes the holomorphic line bundle induced by the action of Γ on $\widetilde{\mathscr{K}_{\mathrm{U}}}$. A similar consideration, for a cover of X_Γ applies to automorphic forms with character.

Now, a priori not all sections of $\mathscr{L}_\Gamma^{\otimes k}$ can be continued to holomorphic sections on the full modular variety obtained by compactifying X_Γ. For this reason, a condition of 'regularity at the cusp' will be required for *modular forms*, see definition 1.1.29 below. However, it will turn out that in almost all cases this condition is automatically met for holomorphic automorphic forms by the *Koecher principle*, see theorem 1.1.30 below.

We postpone these considerations until we have introduced the Fourier-Jacobi expansion of an automorphic form around a cusp.

Fourier-Jacobi expansion and Koecher principle

In the following, let f denote a holomorphic automorphic form of weight k for a modular group Γ. We consider f as a function in the variables τ and σ, defined on $\mathscr{H}_{\mathrm{U}}$. To make this clear, we denote

$$\begin{aligned} f_\ell(\tau,\sigma) &:= f(z), \\ \text{where} \quad z &= \ell' - \delta\,\langle \ell',\ell\rangle\,\tau\ell + \sigma \;\in \widetilde{\mathscr{K}_{\mathrm{U}}^1}. \end{aligned}$$

It is easily checked that the automorphy factor $j(\gamma, z)$ is identically 1 for all $\gamma \in P(\infty)$. Hence, except for a possible character, f_ℓ is invariant under $\Gamma(\infty)$. As a consequence, f_ℓ can be expanded in a Fourier-Jacobi series around $(0,\sigma)$.

Proposition 1.1.27. *A holomorphic automorphic form f with character χ has a Fourier-Jacobi expansion of the form*

$$f_\ell(\tau,\sigma) = \sum_{n\in\mathbb{Z}+r} a_n(\sigma) e\left(\frac{n}{N}\tau\right), \tag{1.1.10}$$

where N is the positive rational number defined in proposition 1.1.22 and r is a constant rational number, with $r \geq 0$. If χ is trivial for all $[h,0]$, $h \in N\mathbb{Z}$, then $r = 0$. The Fourier-Jacobi coefficients $a_n(\sigma)$ are holomorphic functions with the following transformation behavior

$$a_n([0,t]\sigma) = \chi([0,t]) \cdot a_n(\sigma)\, e\left(-\frac{n}{N}\left(\frac{\langle \sigma, t\rangle}{\delta\,\langle \ell',\ell\rangle} + \frac{\langle t,t\rangle}{2\delta}\right)\right) \quad \textit{for all} \quad t \in \tilde{D}, \tag{1.1.11}$$

$$a_n(\gamma\sigma) = \chi(\gamma)\cdot a_n(\sigma) \quad \textit{for all} \quad \gamma \in \mathrm{SU}(D \otimes_{\mathscr{O}_{\mathbb{F}}} \mathbb{C}) \cap \Gamma, \tag{1.1.12}$$

where $\tilde{D}$ is the lattice from proposition 1.1.22.

As an immediate consequence of (1.1.11), if $r \equiv 0 \mod 1$, the coefficient $a_0(\sigma)$ is constant, hence denoted a_0.

Proof. The existence of a (partial) Fourier expansion is immediate from the invariance of f_ℓ under all $[h,0] \in \Gamma(\infty)$, since h is then from $N\mathbb{Z}$, so the expansion must be of the form (1.1.10), at least if f has trivial character. Otherwise, $[h,0]f_\ell(\tau,\sigma)$ must be equal both to $f_\ell(\tau+h,\sigma)$ and to $\chi([h,0])f_\ell(\tau,\sigma)$, by automorphy. Write $\chi([h,0])$ in the form $e(hr/N)$ for some positive rational number r. Here too, we recover the Fourier expansion given above.

From $f_\ell\big([0,t](\tau,\sigma)\big) = \chi\big([0,t]\big)f_\ell(\tau,\sigma)$, for $[0,t] \in \Gamma(\infty)$, we get (1.1.11) and (1.1.12) by expanding both sides via (1.1.10) and comparing terms. □

Remark 1.1.28. *The Fourier-Jacobi coefficients a_n exhibit the transformation behavior of theta functions. Note however that, in contrast to the coefficients occuring in the Fourier-Jacobi expansion of an orthogonal modular form, see proposition 1.2.29, the $a_n(\sigma)$ are not Jacobi-forms, see remark 1.2.30.*

Definition 1.1.29. *A holomorphic automorphic form f is called a* modular form *if only terms with $n \geq 0$ occur in its Fourier-Jacobi expansion, so*

$$f_\ell(\tau,\sigma) = \sum_{\substack{n\in\mathbb{Z}+r\\ n\geq 0}} a_n(\sigma)\, e\left(\frac{n}{N}\tau\right).$$

We then say that f is regular at the cusp (corresponding to ℓ).

The meaning of 'regularity at the cusps' is clarified by the following consideration. Assume for simplicity $b=0$. Under the map $\tau \to q = e(\tau/N)$ in (1.1.7) for z to approach the cusp ℓ means

$$(q,\sigma) \to (0,\sigma) \quad \text{in} \quad \widetilde{\mathscr{H}_{\mathrm{U}}/N\mathbb{Z}},$$

corresponding to the limit $\Im\tau \to \infty$. The Fourier series expansion is replaced by a Laurent series expansion around $q=0$. If negative powers q^n, $n<0$ occur, then f has a pole at the cusp, whereas if only non-negative n occur, the q-series is a power series and f has a well defined value at $q=0$. So f is regular at the cusp, if the Fourier expansion around $(0,\sigma)$ contains no terms a_n with $n<0$.

Theorem 1.1.30 (Koecher principle)**.** *Let f be a holomorphic automorphic form on Γ of weight k and character χ. Then, under the condition that V has signature $(1,q)$ with $q>1$, the Fourier-Jacobi expansion of f takes the form*

$$f_\ell(\tau,\sigma) = \sum_{\substack{n\in\mathbb{Z}+r\\ n\geq 0}} a_n(\sigma)e\big(\frac{n}{N}\tau\big),$$

i.e. $a_n(\sigma) \equiv 0$ for $n<0$. In other words, f is actually a modular form.

Proof. Assume $n<0$. Taking norms on both sides of (1.1.11) – recall that $|\chi|=1$ – we get

$$\begin{aligned}
\left|a_n([0,t]\sigma)\right| &= |a_n(\sigma)| \cdot \exp\left(\pi i \frac{-n}{N}(2i)\left[\Im\left[\frac{\langle\sigma,t\rangle}{\delta\,\langle\ell',\ell\rangle}\right] + \Im\frac{\langle t,t\rangle}{2\delta}\right]\right)\\
&= |a_n(\sigma)| \cdot \exp\left(\frac{+2\pi n}{N}\left[-\Re\left[\frac{\langle\sigma,t\rangle}{|\delta|\,\langle\ell',\ell\rangle}\right] - \frac{\langle t,t\rangle}{2|\delta|}\right]\right)\\
&= |a_n(\sigma)| \cdot \exp\left(-\frac{\pi n}{|\delta|N}\left[\operatorname{Tr}\frac{\langle\sigma,t\rangle}{\langle\ell',\ell\rangle} + \langle t,t\rangle\right]\right)\\
&= |a_n(\sigma)| \cdot \exp\left(\frac{\pi n}{|\delta|N}\left[-\left\langle \frac{\sigma}{\langle\ell',\ell\rangle}+t, \frac{\sigma}{\langle\ell',\ell\rangle}+t\right\rangle + \left\langle \frac{\sigma}{\langle\ell',\ell\rangle}, \frac{\sigma}{\langle\ell',\ell\rangle}\right\rangle\right]\right).
\end{aligned}$$

If we define a real analytic function of σ by

$$g(\sigma) := \left|a_n(\sigma)\right| \cdot \exp\left(\frac{\pi n}{|\delta|N} \frac{\langle\sigma,\sigma\rangle}{|\langle\ell,\ell'\rangle|^2}\right),$$

it follows that $g([0,t]\sigma) = g(\sigma)$ for all $t \in \tilde{D}$, in other words, $g(\sigma)$ is periodic with period lattice $\tilde{D}$. Hence, there is a constant $M > 0$ with

$$g(\sigma) \leq M \quad \text{and thus} \quad |a_n(\sigma)| \leq M \cdot \exp\left(-\frac{\pi n \langle\sigma,\sigma\rangle}{N|\delta||\langle\ell,\ell'\rangle|^2}\right), \quad \text{for all} \quad \sigma \in D \otimes_{\mathcal{O}_{\mathbb{F}}} \mathbb{C}.$$

Since by assumption $n < 0$ and $\langle\cdot,\cdot\rangle$ is negative definite on $D \otimes_{\mathcal{O}_{\mathbb{F}}} \mathbb{C}$, the norm $|a_n(\sigma)|$ tends to zero for increasing $|\sigma|$. Since $a_n(\sigma)$ is a holomorphic function of σ on all of $D \otimes_{\mathcal{O}_{\mathbb{F}}} \mathbb{C}$, it follows that $a_n(\sigma)$ is the constant function 0. □

Let us now take the limit $\tau \to i\infty$, for fixed σ. Doing this term by term using the Koecher principle we get

$$\lim_{\substack{\Im\tau\to\infty \\ \Re\tau \text{ bounded}}} f_\ell(\tau,\sigma) = \lim_{\Im\tau\to\infty} \sum_{\substack{n\in\mathbb{Z}+r \\ n\geq 0}} a_n(\sigma) e\left(\frac{n}{N}\tau\right) = \begin{cases} 0 & \text{if } r \not\equiv 0 \mod 1, \\ a_0 & \text{if } r \equiv 0 \mod 1. \end{cases}$$

Remark 1.1.31. *The value of the limit depends only on the* Γ*-equivalence class of* ℓ*. However, it does depend on the particular choice of* ℓ *as a representative of the cusp* $[\ell]$*, insofar as replacing* ℓ *by* $C\ell$*,* $C \neq 0$*, replaces* a_0 *by* $C^k a_0$*, when* k *is the weight of* f*.*

It should now be clear that the following definition makes sense.

Definition 1.1.32. *A* cusp form f *is a modular form which vanishes at the cusps, for any choice of* Γ*-equivalence class of cusps. Equivalently, for each cusp, there is no constant term* a_0 *in the Fourier expansion of* f *around this cusp.*

1.2 Orthogonal Groups

In this section, we explore the analogous terms and objects to those considered in the previous section, however in the context of orthogonal groups acting on rational quadratic spaces of signature $(2,b)$, with $b \geq 2$. The rational space underlying the hermitian space V of the previous section is just such a space, with the bilinear form induced by the trace. So, later on, in chapter 3 we will fix $b = 2q$.

References for the material in this section include [9] ch. 3, also ch. 1 of [20] which covers much of the basic theory, [29], which is a study of modular varieties but also gives a thorow overview of much the theory of orthogonal modular forms, and, finally the unpublished set of course notes [26].

1.2.1 Quadratic spaces and orthogonal groups

Let V be a vector space over $\mathbb{Q}$, equipped with a non-degenerate symmetric bilinear form $(\cdot,\cdot)$ of signature $(2,b)$ and the attached quadratic form $q(\cdot)$, where $q(x)=\frac{1}{2}(x,x)$. Denote by $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$ the corresponding real quadratic space. By extension of scalars, $(\cdot,\cdot)$ is extended to a real bilinear form on $V_{\mathbb{R}}$. We denote the orthogonal group of V by $\mathrm{O}(V)$ and the special orthogonal group by $\mathrm{SO}(V)$. The real points of these algebraic groups are $\mathrm{O}(V)(\mathbb{R})$ and $\mathrm{SO}(V)(\mathbb{R})$. Further, we denote by $\mathrm{O}^+(V)(\mathbb{R})$ the *spinor kernel*, the image of the homomorphism from the spin group $\mathrm{Spin}(V)(\mathbb{R})$ of $V_{\mathbb{R}}$ to $\mathrm{SO}(V)(\mathbb{R})$ and the kernel of the map θ sending an element of $\mathrm{SO}(V)(\mathbb{R})$ to its spinor norm in the following exact sequence (see [15], p. 135)

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}(V)(\mathbb{R}) \longrightarrow \mathrm{SO}(V)(\mathbb{R}) \xrightarrow{\theta} \mathbb{R}^\times/(\mathbb{R}^\times)^2.$$

The spinor kernel is the connected component of the identity in $\mathrm{SO}(V)(\mathbb{R})$, consisting of the orientation preserving transformations.

Lattices and the discriminant kernel

A lattice L in V is a $\mathbb{Z}$-module with $L\otimes_{\mathbb{Z}}\mathbb{Q}=V$. As before, we assume all lattices to be even, i.e. (μ,μ) is an even number for all μ in L. We denote by $\mathrm{O}(L)$, $\mathrm{SO}(L)$ and $\mathrm{O}^+(L)$ the integral orthogonal groups

$$\mathrm{O}(L)=\{g\in \mathrm{O}(V)\,;\ gL\subset L\},$$
$$\mathrm{SO}(L)=\mathrm{O}(L)\cap \mathrm{SO}(V), \qquad \mathrm{O}^+(L)=\mathrm{O}(L)\cap \mathrm{O}^+(V).$$

As usual, the dual lattice of L is denoted by L'. The quotient L'/L is the discriminant group of L. The action of the integral orthogonal groups on L' induces an action L'/L. The subgroup acting trivially is called the *discriminant kernel*. We denote by $\mathrm{SO}_d(L)$ the discriminant kernel in $\mathrm{SO}(L)$ and by Γ_L^{O} the discriminant kernel in $\mathrm{O}^+(L)$. Both are subgroups of finite index in $\mathrm{SO}(L)$ and thus arithmetic subgroups, as in [49], of $\mathrm{SO}(V)$.

Besides Γ_L^{O}, we consider subgroups of finite index in $\mathrm{SO}(L)$ as *orthogonal modular groups*.

The Grassmannian model

The symmetric domain attached to $\mathrm{SO}(V)$ is given by the quotient

$$\mathrm{SO}(V)(\mathbb{R})/\mathscr{C},$$

where $\mathscr{C}$ is a maximal path-connected, compact subgroup of $\mathrm{SO}(V)(\mathbb{R})$. Under an isometry of $V_{\mathbb{R}}$ to the standard pseudo-euclidean real quadratic space $\mathbb{R}^{2,b}$, with the quadratic form

$$q(x)=\frac{1}{2}(x_1^2+x_2^2-x_3^2-\ldots-x_{2+b}^2),$$

the subgroup $\mathscr{C}$ is isomorphic to $\mathrm{SO}(2)\times\mathrm{SO}(b)$. The symmetric domain can most easily be described through its isomorphism to the *Grassmannian manifold* Gr_{O}, whose points correspond to oriented two-dimensional positive definite subspaces of $V_{\mathbb{R}}$. We treat Gr_{O} as the standard model for the symmetric domain, from which further models will be derived.

Each subspace $v \in \mathrm{Gr}_O$ fixes an isometry between $V_{\mathbb{R}}$ and the pseudo-euclidean quadratic space $\mathbb{R}^{2,b}$ mentioned above. This isometry introduces an isomorphism between the stabilizer of v in $\mathrm{SO}(V)(\mathbb{R})$, which is isomorphic to $\mathscr{C}$, and $\mathrm{SO}(2) \times \mathrm{SO}(b) \subset \mathrm{SO}(2, b)$.

A priori, from the definition, it is not clear that Gr_O carries a complex structure. It can however be fitted with one, as we will describe below.

Remark 1.2.1. *A symmetric domain for the orthogonal group* $\mathrm{O}(V)(\mathbb{R})$ *can be obtained quite similarly as the group quotient*

$$\mathrm{O}(V)(\mathbb{R})/\mathscr{C}' \simeq \mathrm{O}(2, b)/\big(\mathrm{O}(2) \times \mathrm{O}(b)\big),$$

with a maximal compact subgroup $\mathscr{C}' \subset \mathrm{O}(V)(\mathbb{R})$*, isomorphic to* $\mathrm{O}(2) \times \mathrm{O}(b)$ *under an isometry between* $V_{\mathbb{R}}$ *and* $\mathbb{R}^{2,b}$*.*

Here too, there is a Grassmannian model, which we denote Gr'_O*, the points of which correspond to positive two-dimensional subspaces which, in this case however, are not oriented.*

Remark 1.2.2. *The Grassmannian model* Gr_O *is not connected, since its points consist of subspaces of both opposing orientations. In contrast, the Grassmannian* Gr'_O *introduced in remark 1.2.1 is connected.*

1.2.2 Coordinates for the Grassmannian

Next, we fix basis vectors for an isotropic subspace of $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$, corresponding to a decomposition of the lattice L. These will play a pivotal role in the construction of further models for the symmetric domain in section 1.2.3 below. Also, more directly, they can be used to introduce coordinates on the points of the Grassmannian itsself.

Witt decomposition and basis for the hyperbolic part

If the quadratic space V of signature $(2, b)$ contains an isotropic vector with rational coordinates, it is possible to carry out a *Witt decomposition* of V into a sum of a hyperbolic plane and a Lorentzian space W of signature $(1, b)$,

$$V = H_1 \oplus W. \tag{1.2.1}$$

If W also contains a rational isotropic vector, we can further decompose W into a hyperbolic plane and a definite subspace V_0. Then V can be written in the form

$$V = H_1 \oplus H_2 \oplus V_0. \tag{1.2.2}$$

Remark 1.2.3. *In general, by a theorem of Hasse-Minkowski, an indefinite quadratic space over* $\mathbb{Q}$ *always contains an isotropic vector, if it has dimension* ≥ 5*. So the assumption that* V *splits two hyperbolic planes is trivial for* $b \geq 5$*.*

If $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$ contains an isotropic vector, we can assume the existence of a primitive isotropic lattice vector in L. Thus, we make the following definition.

Definition 1.2.4. *Let e be a primitive isotropic vector in L and e' a lattice vector from L' with $(e, e') = 1$. We call two such vectors e, e' a* hyperbolic pair.

By the notation e and e' we will always mean a hyperbolic pair.

Given a hyperbolic pair e, e', we can decompose V as in (1.2.1) with the rational isotropic subspace H_1 spanned by e and e'. Then, the Lorentzian space W is given by $(L \otimes_{\mathbb{Z}} \mathbb{Q}) \cap e^{\perp} \cap e'^{\perp}$. We denote by K the lattice

$$K = L \cap e^{\perp} \cap e'^{\perp} \tag{1.2.3}$$

of signature $(1, b-1)$. Clearly, $W = K \otimes_{\mathbb{Z}} \mathbb{Q}$.

Since $V \cap e^{\perp} = W \oplus \mathbb{Q}e$, and e is isotropic, the projection

$$p_K : \begin{cases} V_{\mathbb{R}} & \longrightarrow & W_{\mathbb{R}} \\ x & \longmapsto & x_K \end{cases} \tag{1.2.4}$$

is an isometry for all $x \in V_{\mathbb{R}} \cap e^{\perp}$.

Assuming that a Witt decomposition of the form (1.2.2) exists, a particular choice for such a decomposition may be described by giving a basis $e_1, e_2, e_3, e_4 \in V$ for the hyperbolic part, with e_1, e_2 spanning the first hyperbolic plane H_1 and e_3, e_4 spanning the second hyperbolic plane H_2.

Definition 1.2.5. *We fix the following notation: By $e_1, \ldots, e_4$ denote a basis of the hyperbolic part $H_1 \oplus H_2$ of V as a rational space, satisfying*

$$(e_1, e_2) = 1, \ (e_3, e_4) = 1 \quad \textit{and} \quad (e_i, e_j) = 0 \quad \textit{for} \quad i \le j, \ (i, j) \notin \{(1,2), (3,4)\}.$$

Further, it will be assumed that e_1, e_2 corresponds to a hyperbolic pair e, e' as in definition 1.2.4 above – with the additional property that e_2 is isotropic. In particular, we assume e_1 and e_2 to be lattice vectors with $e_1 \in L$ and $e_2 \in L$.

With $e = e_1$ and $e' = e_2$, the lattice K introduced above is given by $K = L \cap e_1^{\perp} \cap e_2^{\perp}$. We denote by D the definite lattice $K \cap e_3^{\perp} \cap e_4^{\perp}$. Then, $V_0 = D \otimes_{\mathbb{Z}} \mathbb{Q}$.

Assumption 1.2.6. *Often, we will use the notation $e_1, \ldots, e_4$ and additionally require e_3 and e_4 to be lattice vectors, with $e_3 \in K \subset L$ and $e_4 \in K'$.*

Doing this requires the assumption that L contains two independent, perpendicular isotropic vectors, so that $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$ has a decomposition of the form (1.2.2).

Further, if the hyperbolic part of L is unimodular, we can choose all the e_i from L.

Remark. *From chapter 3 onward, we consider only rational quadratic spaces which can be identified with the underlying space of a hermitian space over a number field $\mathbb{F}$, as in section 1.1. Given such a hermitian $\mathbb{F}$-space V, $\langle \cdot, \cdot \rangle$ of signature $(1, q)$ and assuming, as in assumption 1.1.12, that V contains an isotropic vector ℓ, the $\mathbb{F}$-span of ℓ contains two linearly independent rational isotropic vectors. Thus, as a rational quadratic space, V, $\mathrm{Tr}_{\mathbb{F}/\mathbb{Q}} \langle \cdot, \cdot \rangle$ splits two hyperbolic planes as in* (1.2.2).

Grassmannian coordinates

In the following, let e and e' be a hyperbolic pair of lattice vectors.

Now, given a subspace $v \in \mathrm{Gr}_{\mathrm{O}}$, we decompose $V_{\mathbb{R}} = v \oplus v^{\perp}$. We denote by subscripts $_v$ and $_{v^\perp}$ the orthogonal projections onto v and $v^{\perp}$. Thus, e_v is the projection of e to v. We have

$$v = w \oplus \mathbb{R}e_v, \quad \text{and} \quad v^{\perp} = w^{\perp} \oplus \mathbb{R}e_{v^\perp},$$

with w a one-dimensional, positive definite subspace perpendicular to e and $w^{\perp}$ its complement in $V_{\mathbb{R}} \cap e^{\perp}$. Further,

$$V_{\mathbb{R}} = \left(w \oplus \mathbb{R}e_v\right) \oplus \left(w^{\perp} \oplus \mathbb{R}e_{v^\perp}\right).$$

For projections onto w and $w^{\perp}$ we use the same notation as for v and $v^{\perp}$.

Consider the vector

$$\mu = -e' + \frac{e_v}{2e_v^2} + \frac{e_{v^\perp}}{2e_{v^\perp}^2} = -e' + \frac{e_v}{e_v^2} - \frac{e}{2e_v^2}, \tag{1.2.5}$$

the last equality resulting from $0 = e^2 = e_v^2 + e_{v^\perp}^2$. Now μ_v, given by

$$\mu_v = -e'_v + \frac{e_v}{2e_v^2},$$

is not contained in w, thus $v = w + \mathbb{R}\mu_v$. Similarly, $v^{\perp} = w^{\perp} + \mathbb{R}\mu_{v^\perp}$. Note however that μ_v is not perpendicular to w. Also, since $\mu \perp e$, we have $\mu_K^2 = \mu^2$.

We will use μ and w as coordinates on the Grassmannian manifold.

1.2.3 The tube domain model

We now proceed to introduce a complex structure on the Grassmannian Gr_{O}. This leads to a model of the symmetric domain of $\mathrm{SO}(V)$ as a tube domain, i.e. a subset of a complex space $\mathbb{C}^n$ of the form $\mathbb{R}^n + iC$ with an open connected subset $C \subset \mathbb{R}^n$. In the present case, n is $b-1$.

Complexification

Let $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $V_{\mathbb{R}}$. Extend the bilinear form $(\cdot,\cdot)$ to a symmetric, $\mathbb{C}$-bilinear form on $V_{\mathbb{C}}$. Denote by $\mathbb{P}(V_{\mathbb{C}})$ the associated projective space and by

$$\pi : Z_L \longmapsto [Z_L],$$

the canonical projection, where $Z_L \in V_{\mathbb{C}}$. The subscript $_L$ is due to $V_{\mathbb{C}} = L \otimes_{\mathbb{Z}} \mathbb{C}$.

In $P(V_{\mathbb{C}})$, consider the *zero-quadric*, $\mathcal{N}$ given by

$$\mathcal{N} = \left\{[Z_L] \in P(V_{\mathbb{C}})\,;\ \left(Z_L, Z_L\right) = 0\right\}, \tag{1.2.6}$$

and its open subset, the positive cone

$$\mathscr{K}_{\mathrm{O}} = \{[Z_L] \in \mathscr{N} \,;\, (Z_L, \bar{Z}_L) > 0\}\,. \tag{1.2.7}$$

Note that both $\mathscr{N}$ and $\mathscr{K}_{\mathrm{O}}$ are well defined, as either of the conditions $(Z_L, Z_L) = 0$ and $(Z_L, Z_L) > 0$ holds for every element of the projective line $\mathbb{C}^{\times} Z_L$ iff it holds for one element.

As a complex projective manifold $\mathscr{K}_{\mathrm{O}}$ consists of two connected components, which are preserved under the operation of $O^{+}(V)(\mathbb{R})$. We choose one of these and denote it by $\mathscr{K}_{\mathrm{O}}^{+}$.

Also, we denote by $\widetilde{\mathscr{N}}$, $\widetilde{\mathscr{K}_{\mathrm{O}}}$ and $\widetilde{\mathscr{K}_{\mathrm{O}}^{+}}$ the preimages under the canonical projection π.

Consider a vector $Z_L \in V_{\mathbb{C}}$, written in the form $Z_L = X_L + iY_L$ with X_L, $Y_L \in V_{\mathbb{R}}$. Then, $[Z_L]$ is an element of $\mathscr{K}_{\mathrm{O}}$, exactly if

$$X_L \perp Y_L \quad \text{and} \quad X_L^2 = Y_L^2 > 0. \tag{1.2.8}$$

Note that $[\bar{Z}_L]$ lies in $\mathscr{K}_{\mathrm{O}}$, too. However, since $Y_L \neq 0$ for any Z_L in $\widetilde{\mathscr{K}_{\mathrm{O}}}$, we see that $[Z_L]$ and $[\bar{Z}_L]$ lie in different connected components.

From (1.2.8) it is clear that X_L, Y_L span a two-dimensional positive subspace of $V_{\mathbb{R}}$. If we consider X_L and Y_L as an oriented basis, this defines a point in the Grassmannian Gr_{O}. The same subspace, but with inverse orientation is defined by X_L, $-Y_L$, corresponding to the conjugate $\bar{Z}_L$.

On the other hand, for an oriented subspace v corresponding to a point of Gr_{O} we can always choose a basis X_L, Y_L satisfying (1.2.8). Thus, we have a bijection between Gr_{O} and $\mathscr{K}_{\mathrm{O}}$, and this induces a complex structure on Gr_{O}.

The operation of complex conjugation corresponds to switching connected components of $\mathscr{K}_{\mathrm{O}}$ and to passing from one element in Gr_{O} to the unique element defining the same subspace but with the inverse orientation.

Remark 1.2.7. *Consider* $\mathrm{Gr}'_{\mathrm{O}}$, *the Grassmannian consisting of non-oriented two dimensional positive subspaces – which we have introduced as a model for the symmetric domain of* $\mathrm{O}(V)$, *see remark 1.2.1. Either component of* $\mathscr{K}_{\mathrm{O}}$ *bijects to* $\mathrm{Gr}'_{\mathrm{O}}$ *and the choice of one of these components corresponds to choosing a* 'spin-orientation' *on* $\mathrm{Gr}'_{\mathrm{O}}$, *that is, a continuously varying choice of orientation for each* $v \in \mathrm{Gr}'_{\mathrm{O}}$.

Next, we will realize $\mathscr{K}_{\mathrm{O}}^{+}$ as a tube domain model, denoted $\mathscr{H}_{\mathrm{O}}$. We will consider this as a model for the symmetric domain of $\mathrm{SO}(V)$.

The tube domain

Assume we have fixed a hyperbolic pair e and e' as in the previous subsection. Recall the decomposition $V_{\mathbb{R}} = W_{\mathbb{R}} \oplus \mathbb{R}e' \oplus \mathbb{R}e$, where $W_{\mathbb{R}} = K \otimes \mathbb{R}$.

Given $Z_L \in V_{\mathbb{C}}$, we may write $Z_L = Z + ae' + be$, with $Z \in K \otimes \mathbb{C}$, $a, b \in \mathbb{C}$. For brevity, we denote this as $Z_L = (b, a, Z)$.

Now, $[Z_L] \in \mathscr{K}_{\mathrm{O}}$ has a unique representative $Z_L = (b, 1, Z)$ with $a = 1$. With this normalization, the conditions in (1.2.6) imply $b = -q(Z) - q(e')$. Thus

$$Z_L = (-q(Z) - q(e'), 1, Z). \tag{1.2.9}$$

Now, Z suffices to uniquely determine a (normalized) Z_L.

Writing $Z = X + iY$ with $X, Y \in K \otimes_{\mathbb{Z}} \mathbb{R} = W_{\mathbb{R}}$, we have $X = p_K(X_L)$ and $Y = p_K(Y_L)$, with the orthogonal projection p_K introduced in 1.2.4. Then, $Y_L \perp e$ and thus $Y_L^2 = Y^2$.

On the other hand, to $Z \in K \otimes_{\mathbb{Z}} \mathbb{C}$ with $\Im Z \in K \otimes_{\mathbb{Z}} \mathbb{R}$ of positive norm, associate Z_L through (1.2.9); this lies in $\mathscr{K}_{\mathrm{O}}$. Denote by $\mathscr{H}_{\mathrm{O}}^{\pm}$ the set of such Z,

$$\mathscr{H}_{\mathrm{O}}^{\pm} := \left\{ Z = X + iY \in K \otimes \mathbb{C} \,;\, Y^2 > 0 \right\}.$$

Thus, we have a bijection

$$\begin{aligned} \mathscr{H}_{\mathrm{O}}^{\pm} &\longrightarrow \quad \mathscr{K}_{\mathrm{O}} \\ Z &\longmapsto \quad [Z_L], \end{aligned} \tag{1.2.10}$$

with Z_L given by (1.2.9). Note that, like $\mathscr{K}_{\mathrm{O}}$, the set $\mathscr{H}_{\mathrm{O}}^{\pm}$ has two connected components,

$$\mathscr{H}_{\mathrm{O}}^{\pm} = \left(K \otimes_{\mathbb{Z}} \mathbb{R} + i\mathscr{C}^{+}\right) \cup \left(K \otimes_{\mathbb{Z}} \mathbb{R} + i\mathscr{C}^{-}\right),$$

where $\mathscr{C}^{\pm}$ is either of the two components of the cone $\{Y \in K \otimes_{\mathbb{Z}} \mathbb{R} \,;\, Y^2 > 0\}$.

Definition 1.2.8. *The* tube domain model $\mathscr{H}_{\mathrm{O}}$ *of* Gr_{O} *is the component of* $\mathscr{H}_{\mathrm{O}}^{\pm}$ *which is sent to* $\mathscr{K}_{\mathrm{O}}^{+}$ *under this mapping.*

We denote by $\widetilde{\mathscr{K}_{\mathrm{O}}^{+}} \subset V_{\mathbb{C}}$ *the preimage of* $\mathscr{K}_{\mathrm{O}}^{+}$ *under the canonical projection and by* $\widetilde{\mathscr{K}_{\mathrm{O},1}^{+}}$ *the subset consisting of representatives* Z_L *of the form* (1.2.9) *for* $Z \in \mathscr{H}_{\mathrm{O}}$*,*

$$\widetilde{\mathscr{K}_{\mathrm{O},1}^{+}} := \left\{ Z_L = (-q(Z), 1, Z) \,;\, Z \in \mathscr{H}_{\mathrm{O}} \right\}.$$

How do $Z \in \mathscr{H}_{\mathrm{O}}$ and the corresponding vectors $Z_L \in V_{\mathbb{C}}$, X_L and Y_L in $V_{\mathbb{R}}$, relate to the previously defined coordinates μ and w?

We have

$$\begin{aligned} Z_L &= \left(-q(Z) - q\left(e'\right), 1, Z\right), \\ X_L &= \left(q(Y) - q(X) - q\left(e'\right), 1, X\right), \\ Y_L &= \left(-(X, Y), 0, Y\right). \end{aligned}$$

With X_L and Y_L as a basis for $v \in \mathrm{Gr}_{\mathrm{O}}$,

$$e_v = \frac{1}{Y^2}\left[\left(e, X_L\right) X_L + \left(e, Y_L\right) Y_L\right] = \frac{1}{Y^2} X_L.$$

From this, by an easy calculation, we get

$$\begin{gathered} \mu = \left(-q(X) - q\left(e'\right), 0, X\right) \quad \text{and} \quad w = \mathbb{R} Y_L, \\ \text{thus} \quad \mu_K = X,\ w_K = \mathbb{R} Y. \end{gathered} \tag{1.2.11}$$

Remark 1.2.9. *The real analytic maps between* Gr_{O}*,* $\mathscr{K}_{\mathrm{O}}$ *and* $\mathscr{H}_{\mathrm{O}}$ *permit us to consider functions defined on any one of these models for the symmetric domain as functions on the other models. Real analytic functions on one model correspond to real analytic functions on any other model.*

Further. while the isomorphism between Gr_{O} *and* $\mathscr{K}_{\mathrm{O}}$ *is real analytic, clearly, that between* $\mathscr{K}_{\mathrm{O}}^{+}$ *and* $\mathscr{H}_{\mathrm{O}}$ *is holomorphic. With the induced complex structure on* Gr_{O} *we can also consider a holomorphic (or meromorphic) function on* $\mathscr{H}_{\mathrm{O}}$ *as (pulling back to) a holomorphic (or meromorphic) function on* Gr_{O}*.*

Refined coordinates

Given a basis e_1, e_2, e_3, e_4 for the hyperbolic part of $L \otimes_{\mathbb{Z}} \mathbb{Q}$, with e_1, e_2 a hyperbolic pair and all e_i, $i = 1, \ldots, 4$ isotropic, we can introduce somewhat refined coordinates on the tube domain. These can be used to described the components of $\mathscr{H}_{\mathrm{O}}^{\pm}$ more explicitly.

Writing out Z as $z_1 e_3 + z_2 e_4 + \mathfrak{z}$, with $\mathfrak{z} \in D \otimes \mathbb{C}$, we have

$$Z_L = \left(-q(Z), 1, z_1, z_2, \mathfrak{z}\right).$$

Write x_i, y_i for $i = 1, 2$ and $\mathfrak{x}$, $\mathfrak{y}$ for the real and imaginary parts of z_i and $\mathfrak{z}$, respectively. In this setting – recall all e_i are assumed to be isotropic – the condition $q(Y) > 0$ reads

$$y_1 y_2 + q(\mathfrak{y}) > 0.$$

Since $D \otimes_{\mathbb{Z}} \mathbb{R}$ is negative definite, this implies that either y_1 and y_2 are both positive or both negative. Thus, the hyperplane $y_1 = 0$ separates the set of vectors $Y \in K \otimes_{\mathbb{Z}} \mathbb{R}$ with positive norm into the two cones $\mathscr{C}^+$ and $\mathscr{C}^-$ introduced above. We denote by $\mathscr{C}^+$ the *positive cone* consisting of Y with $y_1, y_2 > 0$. We may assume that $\mathscr{H}_{\mathrm{O}}$ is given by $K \otimes \mathbb{R} + i\mathscr{C}^+$.

Note that the orientation reversing transformations in $\mathrm{SO}(V)(\mathbb{R}) \setminus \mathrm{O}^+(\mathbb{R})$ switch the two cones, sending $\mathscr{H}_{\mathrm{O}}$ to its conjugate $\overline{\mathscr{H}_{\mathrm{O}}}$, which thus may be considered as a generalized 'lower half-plane'.

By construction the disjoint union $\mathscr{H}_{\mathrm{O}}^{\pm} = \mathscr{H}_{\mathrm{O}} \cup \overline{\mathscr{H}_{\mathrm{O}}}$ bijects to $\mathscr{K}_{\mathrm{O}}$ under the map (1.2.10), associating $[Z_L]$ to Z.

So the symmetric domain for $\mathrm{SO}(V)(\mathbb{R})$ introduced on page 1.2.1 above corresponds more precisely to the union of $\mathscr{H}_{\mathrm{U}}$ and its conjugate. However, we lose nothing in restricting to one connected component, corresponding to $\mathscr{K}_{\mathrm{O}}^{+}$ and $\mathscr{H}_{\mathrm{O}}$.

The modular variety

Let Γ be an orthogonal modular group. The *orthogonal modular variety* $\mathfrak{X}_\Gamma$ is the quotient of the symmetric domain under the action of Γ. We define this as

$$\mathfrak{X}_\Gamma := \Gamma \backslash \mathscr{K}_{\mathrm{O}}^{+}.$$

Due to the isomorphism between $\mathscr{K}_{\mathrm{O}}^{+}$ and $\mathscr{H}_{\mathrm{O}}$, we can also write this as $\mathfrak{X}_\Gamma = \Gamma \backslash \mathscr{H}_{\mathrm{O}}$. By the theory of Baily-Borel [2], $\mathfrak{X}_\Gamma$ is a quasi-projective algebraic variety, see below. There is some abuse of language in this definition, since we originally considered the symmetric domain of $\mathrm{SO}(V)$ as Gr_{O}, while here, we form the quotient of only one connected component. In fact, we have $\mathfrak{X}_\Gamma \simeq \Gamma \backslash \mathrm{Gr}'_{\mathrm{O}}$ with the Grassmannian $\mathrm{Gr}'_{\mathrm{O}}$ of non-oriented two-dimensional spaces $\mathrm{Gr}'_{\mathrm{O}}$.

1.2.4 Cusps and boundary components

By a *cusp* of $\mathscr{H}_{\mathrm{O}}$ we mean an equivalence class of non-zero rational isotropic subspaces of V under the operation of Γ_L^{O}. Thus, the choice of a hyperbolic pair e, e' with e a primitive isotropic lattice vector serves to define a cusp of $\mathscr{H}_{\mathrm{O}}$.

If $\tilde{e}$ is an isotropic lattice vector in V, which is $\mathrm{O}^+(L)$-inequivalent to e, by a theorem of Witt, there is a $g \in \mathrm{O}^+(V)$ with $\tilde{e} = ge$. Then $\left(ge', \tilde{e}\right) = \left(e', e\right)$ and we can define the tube domain (or more precisely its identification with a subset of $K \otimes_{\mathbb{Z}} \mathbb{C}$) with respect to $\tilde{e}$ and ge'. The operation of $\gamma \in \mathrm{O}^+(L)$ on the tube domain thus defined is the same as that of $g\gamma g^{-1}$ on the tube domain with respect to e and e'.

Since, as we will see, isotropic subspaces define boundary components of $\mathscr{K}_\mathrm{O}$ and thus of $\mathscr{H}_\mathrm{O}$ we also use the term 'cusp' to mean a *rational* boundary component.

We will examine isotropic subspaces and boundary components on the following pages, and also study parabolic elements in the orthogonal group $\mathrm{O}^+(V)$.

Boundary components

As in the unitary case, the boundary components of the symmetric domain for the orthogonal group correspond to non-zero isotropic subspaces of $V_\mathbb{R}$. However, since the signature of $V_\mathbb{R}$ is $(2, b)$, both one- and two-dimensional isotropic subspaces can occur. For the following, see [11], section 2, as well as [26] and [29].

The boundary of the symmetric domain is most easily described in terms of $V_\mathbb{R}$ and the projective model $\mathscr{K}_\mathrm{O}^+ \subset \mathbb{P}(V_\mathbb{C})$.

- Let $F \subset V_\mathbb{R}$ be an isotropic line. Then, F represents a boundary point of $\mathscr{K}_\mathrm{O}^+$. Such a boundary point is called *special*. A zero-dimensional boundary component is a set consisting of one such point.

- Let $F \subset V_\mathbb{R}$ be a two-dimensional totally isotropic subspace. Boundary points which are described by elements of $F_\mathbb{C} = F \otimes_\mathbb{R} \mathbb{C}$ and are not special are called *generic*. The set of all generic boundary points attached to F is called a one-dimensional boundary component of $\mathscr{K}_\mathrm{O}^+$ in $\mathscr{N}$.

Lemma 1.2.10. *There is a bijective correspondence between (zero- and one-dimensional) boundary components of $\mathscr{K}_\mathrm{O}$ in $\mathscr{N}$ and non-zero isotropic subspaces $F \subset V_\mathbb{R}$ (of dimension one and two, respectively). The boundary of $\mathscr{K}_\mathrm{O}^+$ consists of the disjoint union of all boundary components.*

A boundary component attached to a two-dimensional isotropic subspace F can be described as follows: In $V_\mathbb{R}$, there is a second (totally) isotropic subspace F' such that $F + F'$ is the hyperbolic part of $V_\mathbb{R}$ and the sum of two hyperbolic planes (over the reals). Fix a basis of $F + F'$ of the form $f_1, f_3 \in F$, $f_2, f_4 \in F'$ with $(f_1, f_2) = (f_3, f_4) = 1$ and $(f_i, f_j) = 0$ in all other cases where $i \leq j$.

Write (z_1, z_2, z_3, z_4) for $z_1 f_1 + \cdots + z_4 f_4$. Then F consists of elements of the form $(z_1, 0, z_3, 0)$. We consider the boundary point of $\mathscr{K}_\mathrm{O}^+$ attached to such a vector in F. We may assume that is not a multiple of a real point, in particular that z_1 and z_2 are both non-zero. We normalize $z_1 = 1$ and write the boundary point in the form $(1, 0, \tau, 0)$, with $\tau \in \mathbb{C} \setminus \{0\}$. This is a boundary point of $\mathscr{K}_\mathrm{O}$ in $\mathscr{N}$. We may assume that the point $(1, 1, i, i)$ is contained in $\mathscr{K}_\mathrm{O}^+$. Otherwise, we need only replace f_3 and f_4 by $-f_3$, $-f_4$.

Then, the following defines an embedding of two copies of the usual complex upper half-planes $\mathbb{H}$ into $\mathscr{K}_\mathrm{O}$:

$$\begin{array}{ccc} \mathbb{H} \times \mathbb{H} & \longrightarrow & \widetilde{\mathscr{K}_\mathrm{O}^+} \\ (\tau_1, \tau_2) & \longmapsto & (-\tau_1\tau_2, 1, \tau_1, \tau_2). \end{array}$$

Now, sending τ_1 to $i\infty$ corresponds to heading towards the boundary of $\mathcal{K}_0$. By taking the limit in the projective space $\mathbb{P}(V_{\mathbb{C}})$, we get

$$\lim_{t\to\infty}\left[-it\tau_2,1,it,\tau_2\right]=\lim_{t\to\infty}\left[-\tau_2,\frac{1}{it},1,\frac{\tau_2}{it}\right]=\left[-\tau_2,0,1,0\right].$$

Thus, the point $(-\tau,0,1,0)$ is a boundary point of $\mathcal{K}_0^+$ exactly if τ has positive imaginary part. The one-dimensional boundary component of $\mathcal{K}_0^+$ attached to F can therefore be considered as a copy of an upper half-plane $\mathbb{H}$.

Besides these generic boundary points, there are also special boundary points represented by $F_{\mathbb{C}}$. It can be shown that the set of all boundary points represented by $F_{\mathbb{C}}$ can be identified with $\mathbb{H}\cup\mathbb{R}\cup\{\infty\}$.

Rational boundary components

A boundary component is called *rational* if the corresponding isotropic subspace F is defined over $\mathbb{Q}$.

Remark. *Clearly, a choice of basis $e_1,\dots,e_4$ for the hyperbolic part of V fixes two complementary totally isotropic subspaces and thus defines a rational boundary component. In particular, this is true if the basis consists of lattice vectors, with e_1, e_2 a hyperbolic pair and e_3 in K and e_4 in K', though of course, the definition of a rational boundary component does not immediately depend on the lattice L but on the $\mathbb{Q}$-form $V=L\otimes_{\mathbb{Z}}\mathbb{Q}$.*

The union of $\mathcal{K}_0^+$ with all rational boundary components is denoted by $\mathcal{K}_0^{+,*}$.

The rational orthogonal group $\mathrm{O}^+(V)$ acts on $\mathcal{H}_0^*$. Consider a subgroup Γ of finite index in $\mathrm{O}^+(L)$. Whereas the quotient $\mathfrak{X}_\Gamma=\Gamma\backslash\mathcal{H}_0$ is not compact, by the theory of Baily-Borel [2], the quotient

$$\mathfrak{X}_\Gamma^*=\Gamma\backslash\mathcal{H}_0^*,$$

carries the structure of a quasi-projective algebraic variety.

Locally, for any point $s\in\mathcal{H}_0^*$, the stabilizer of s in Γ defines an open embedding

$$\mathrm{Stab}_\Gamma(s)\backslash\mathcal{H}_0^*\longrightarrow X_\Gamma^*$$

of a neighborhood of the image of s.

As detailed in [11], the stabilizer of a point s in a boundary component of $\mathcal{H}_0^*/\Gamma$ is contained in the normalizer of this boundary component.

The normalizer of a boundary component

For an non-zero isotropic two-dimensional subspace F of V denote by N_F the *normalizer* of F in $\mathrm{O}^+(V)$ and by C_F its *centralizer* in $\mathrm{O}^+(V)$,

$$N_F=\left\{g\in\mathrm{O}^+(V);\ g(F)=F\right\},\quad C_F=\left\{g\in\mathrm{O}^+(V);\ g\,|_F=\mathrm{Id}\right\}.$$

Arithmetic subgroups denoted $C_F(\Gamma)$ and $N_F(\Gamma)$ are given by the intersection of C_F and N_F, respectively, with an arithmetic group $\Gamma\subset\mathrm{O}^+(L)$.

We will consider four types of transformations in N_F, three of which arise from Eichler elements of $\mathrm{O}^+(V)$.

Definition 1.2.11. *An* Eichler transformation *or* Eichler element (of $\mathrm{O}^+(V)(\mathbb{R})$) *is defined for an isotropic vector* $u \in V_{\mathbb{R}}$ *and a* $v \in V_{\mathbb{R}}$*, with* $u \perp v$ *as a map of the following form*

$$E(u,v) : \begin{cases} V_{\mathbb{R}} & \longrightarrow & V_{\mathbb{R}} \\ a & \longmapsto & a - (a,u)\,v + (a,v)\,u - q(v)\,(a,u)\,u. \end{cases} \tag{1.2.12}$$

Then we have $E(u,v) \in \mathrm{O}^+(V)(\mathbb{R})$.

We note some useful properties of Eichler elements, which can easily be verified by calculation:

- For fixed u, Eichler transformations form an additive group, isomorphic to the complement $u^\perp$ of u in V, since

$$E(u,v_1) \circ E(u,v_2) = E(u, v_1 + v_2) \qquad \text{for any } v_1,\ v_2 \perp u.$$

- If both u and v are from an even lattice L then $E(u,v)$ acts trivially on the discriminant group, thus, $E(u,v) \in \Gamma_L^{\mathrm{O}}$.
- Let u, u' be isotropic vectors perpendicular to each other and v, v' perpendicular to both u and u'. Then, the following identities hold

$$\begin{aligned} E(u,v) \circ E(u',v') &= E\big(u', v' + (v,v')\,u\big) \circ E(u,v) \quad \text{and} \\ E(u,u') &= E(u',u)^{-1}. \end{aligned} \tag{1.2.13}$$

Since by assumption V contains at least two independent rational isotopic vectors, there is a decomposition of V into subspaces of the form (1.2.2). Thus,

$$V = H_1 \oplus H_2 \oplus V_0,$$

with H_i, $i = 1,2$ hyperbolic planes and V_0 a definite subspace. Considering how u and v can be chosen from these subspaces results in three different types of Eichler transformations, see below. Further elements of N_F are given by the embedded $\mathrm{O}^+(V_0)$.

As usual, we denote by L an even lattice with $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$. We assume that the hyperbolic part has been equipped with a basis $e_1, \ldots, e_4$ of the form introduced in 1.2.2, with e_1 primitive isotropic from L, $e_2 \in L'$. Further assume that $e_3 \in K \subset L$, with $K = L \cap e_1^\perp \cap e_2^\perp$, and $e_4 \in K'$.

We assume F to be the two-dimensional isotropic subspace of V defined by e_1 and e_3.

For a vector $X \in V_{\mathbb{R}}$, we abbreviate

$$X = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + \mathfrak{x} \quad \text{as} \quad X = (x_1, x_2, x_3, x_4, \mathfrak{x}),$$

while on the tube domain for $Z = z_1 e_3 + z_2 e_4 + \mathfrak{z}$ we write $(z_1, z_2, \mathfrak{z})$, as usual.

We list four types of transformations. Note that all of them actually lie in the centralizer of F, cf. [11].

I. Eichler transformation of the type $E(e_3, te_1)$ with $t \in \mathbb{Q}$:

$$X \mapsto (x_1 + tx_4, x_2, x_3 - tx_2, x_4, \mathfrak{x}).$$

In (refined) tube-domain coordinates, we have

$$Z \mapsto (z_1 - t, z_2, \mathfrak{z}).$$

II. Transformations of the form $E(e_1, \lambda)$ with $\lambda \in D \otimes \mathbb{Q}$:

$$X \mapsto (x_1 - (\lambda, \mathfrak{x}) - q(\lambda)\, x_2, x_2, x_3, x_4, \mathfrak{x} + x_2\lambda).$$

On the tube domain, these act as translations of the definite part,

$$Z \mapsto (z_1, z_2, \mathfrak{z} + \lambda).$$

III. Eichler elements of the form $E(e_3, \mu)$ with $\mu \in D \otimes \mathbb{Q}$:

$$X \mapsto (x_1, x_2, x_3 - \big(\mu, \mathfrak{x}\big) - q\big(\mu\big)\, x_4, x_4, \mathfrak{x} + x_4\mu).$$

On the tube domain we have

$$Z \mapsto (z_1 - \big(\mu, \mathfrak{z}\big) - q\big(\mu\big)\, z_2, z_2, \mathfrak{z} + z_2\mu).$$

IV. The orthogonal group $\mathrm{O}^+(V_0)$ embedded into $\mathrm{O}^+(V)$, and thus into $\mathrm{SO}(V)$, acts as usual on V_0. On the tube domain, we simply have

$$Z \mapsto (z_1, z_2, \gamma(\mathfrak{z})), \quad \text{for} \quad \gamma \in \mathrm{O}^+(V_0).$$

Obviously, transformations of this type can not be represented by Eichler elements of the types given in I through III and vice versa, unless they are the identity.

Transformations of the types I, II and III form additive groups, isomorphic to $\mathbb{Q}$ for type I, and V_0 for types II and III. Further, types I and II commute and thus form a group isomorphic to $\mathbb{Q} \times V_0$. Denote by

$$[\lambda, 0, t] := E(e_1, te_3) \circ E(e_1, \lambda) \qquad \text{and}$$
$$[0, \mu, 0] := E(e_3, \mu).$$

Then, from (1.2.13), we obtain the following commutation relation

$$[0, \mu, 0] \circ [\lambda, 0, t] = [\lambda, 0, t + \big(\lambda, \mu\big)] \circ [0, \mu, 0].$$

From this, it is clear that transformations of types I, II and III above form a group, as a semidirect product. The direct factor is the additive group of $[0, \mu, 0]$, $\mu \in V_0$, formed by the transformations of type III.

Definition 1.2.12. *The set of triples $[\lambda, \mu, t]$ with λ, $\mu \in V_0$ and $t \in \mathbb{Q}$ is a group, the* rational Heisenberg group of V_0 *denoted* $\mathrm{H}(V_0)$. *The group law is given by*

$$[\lambda, \mu, t] \circ [\lambda', \mu', t'] = [\lambda + \lambda', \mu + \mu', t + t' + \big(\mu, \lambda'\big)]$$

The Heisenberg group is an algebraic group. Its real points are given by triples with $\lambda, \mu \in D \otimes \mathbb{R}$ and $t \in \mathbb{R}$. For elements of the integral Heisenberg group $\mathrm{H}(D)$, *in turn, λ, μ are lattice vectors from D and t is an integer.*

Clearly, the Heisenberg group can be embedded into $O^+(V)$ by the map

$$[\lambda,\mu,t] \longmapsto E(e_1, te_3+\lambda)\circ E(e_3,\mu),$$

under which $H(D)$ is sent to a subgroup of the discriminant kernel Γ_L^O. (For this, it is necessary that both $e_1 \in L$ and $e_3 \in L$, as assumed.)

Remark 1.2.13. *The elements of the type $[0,0,t]$ form a normal subgroup of* $H(V_0)$. *In conjunction with the projection $[\lambda,\mu,0]\mapsto(\lambda,\mu)$ this gives rise to an exact sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Q} & \longrightarrow & H(V_0) & \longrightarrow & V_0\times V_0 & \longrightarrow 0 \\ & & t & \longmapsto & [0,0,t] & & & \\ & & & & [\lambda,\mu,0] & \longmapsto & (\lambda,\mu). & \end{array}$$

The centralizer C_F of F in $O^+(V)$ is given by the semidirect product $H(V_0)\ltimes O^+(V_0)$. Finally, the normalizer N_F of F can be described through the following exact sequence

$$1 \longrightarrow C_F \longrightarrow N_F \longrightarrow GL^+(F) \longrightarrow 1,$$

where $GL^+(F)$ is the image of the natural homomorphism $N_F \to GL(F)\simeq GL(2,\mathbb{Q})$, bearing in mind that N_F is a subset of $O^+(V)$.

Remark. *Since $u,\ v\in L$ implies $E(u,v)\in\Gamma_L^O$ for an even lattice L, in particular, for a subgroup Γ of finite index in $\Gamma(L)$ there is a lattice in $K\otimes_{\mathbb{Z}}\mathbb{Q}$ such that $E(e_1,\kappa)\in\Gamma$ for any κ in the lattice. A similar statement holds for Eichler elements of the form $E(e_3,\nu)$, so that in fact there is some lattice in $D\otimes_{\mathbb{Z}}\mathbb{Q}$ and an integer M such $[\kappa,\nu,m]\in\Gamma$ for all κ, ν from this lattice and all $m\in M\mathbb{Z}$.*

The action of $SL_2(\mathbb{R})$ on $\mathscr{H}_O$ and the Jacobi group

The following may be found in greater detail in [20], Chapter 3 and also in [26]. We have seen how the centralizer of the rational boundary component attached to the isotropic subspace spanned by e_1 and e_3 acts on the tube domain $\mathscr{H}_O$. We want to get a more explicit description of the normalizer, as well. All coordinates are given with respect to the basis $e_1,\dots,e_4$ for the hyperbolic part of V consisting of lattice vectors. To simplify the following discussion, we assume that the hyperbolic part of L is unimodular and that all e_i are contained in L.

First, we consider a homomorphism from $SL_2(\mathbb{R})\times SL_2(\mathbb{R})$ to $O^+(V)(\mathbb{R})$, defined as follows: For a vector X in $V_{\mathbb{R}}$ write

$$X=(x_1,x_2,x_3,x_4,\mathfrak{x}) \quad \text{as} \quad X=\left(\begin{pmatrix} x_4 & x_2\\ -x_1 & x_3\end{pmatrix},\mathfrak{x}\right).$$

On the hyperbolic part of $V_{\mathbb{R}}$, the quadratic form is now given by the determinant. Thus,

$$q(X)=\det\begin{pmatrix} x_4 & x_2\\ -x_1 & x_3\end{pmatrix}+q(\mathfrak{x}).$$

Next, assign to $(A,B)\in SL_2(\mathbb{R})\times SL_2(\mathbb{R})$ the linear map

$$X\mapsto \left(A\begin{pmatrix} x_4 & x_2\\ -x_1 & x_3\end{pmatrix}B^t,\mathfrak{x}\right). \tag{1.2.14}$$

Note that the matrix B is transposed. Since the quadratic form (as well as the orientation) is preserved, this defines a homomorphism to $\mathrm{O}^+(V)$, which, conveniently, restricts to a homomorphism into $\Gamma(L)$ for $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$, since the hyperbolic part of L is unimodular.

The e_1–e_3–plane is preserved by the operation of the subgroup $\{E_2\} \times \mathrm{SL}_2(\mathbb{Q})$, where E_2 denotes the identity matrix. Consider $\mathrm{SL}_2(\mathbb{Q})$ as embedded into $\mathrm{O}^+(V)$ via the composition of the inclusion into $\{E_2\} \times \mathrm{SL}_2(\mathbb{Q})$ and the above homomorphism (1.2.14).

Note that the embedded $\mathrm{SL}_2(\mathbb{Q})$ acts on $\mathrm{H}(V_0)$ by conjugation.

Definition 1.2.14. *The* (rational) Jacobi group $\mathrm{J}(V_0)$ *is defined as the semidirect product of the embedded* $\mathrm{SL}_2(\mathbb{Q})$ *with* $\mathrm{H}(V_0)$*,* $\mathrm{J}(V_0) = \mathrm{SL}_2(\mathbb{Q}) \ltimes \mathrm{H}(V_0)$*. An* integral Jacobi-group $\mathrm{J}(D)$ *is obtained as the product* $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathrm{H}(D)$*.*

The elements of $\mathrm{J}(V_0)$ *are denoted as* $[M][\lambda, \mu, t]$ *with* $M \in \mathrm{SL}_2(\mathbb{Q})$*,* λ*,* $\mu \in D \otimes_{\mathbb{Z}} \mathbb{Q}$ *and* $t \in \mathbb{Q}$*.*

Of course, this embedding also extends to the real groups $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{O}^+(V)(\mathbb{R})$ giving the real points $\mathrm{J}(V_0)(\mathbb{R})$ of the algebraic group.

Now, the action of $\mathrm{J}(V_0)(\mathbb{R})$ on the tube domain can be given in explicit terms, see (1.2.17), below. However, in contrast to the Heisenberg group, the action of the Jacobi-group gives rise to a non-trivial automorphy factor, which we shall determine in example 1.2.17 below.

We shall first explore this concept in a somewhat more general setting.

1.2.5 Automorphy factor

Denote by e, e' lattice vectors spanning a hyperbolic plane in $L \otimes_{\mathbb{Z}} \mathbb{Q}$, with e primitive isotropic and $(e, e') = 1$. By $\widetilde{\mathscr{K}_{\mathrm{O}}^+}$ denote the preimage in $V_{\mathbb{C}}$ of $\mathscr{K}_{\mathrm{O}}^+$ under the canonical projection $\pi : V_{\mathbb{C}} \to \mathbb{P}(V_{\mathbb{C}})$.

Recall how, the association in (1.2.9) of a unique $Z_L \in \widetilde{\mathscr{K}_{\mathrm{O},1}^+} \subset \widetilde{\mathscr{K}_{\mathrm{O}}^+}$ to $Z \in \mathscr{H}_{\mathrm{O}}$ combined with the canonical projection was used to introduce the bijection between $\mathscr{H}_{\mathrm{O}}$ and $\mathscr{K}_{\mathrm{O}}^+$, see (1.2.10),

$$Z \quad \longmapsto Z_L = \big(Z, 1, -q(Z) - q(e')\big) \quad \longmapsto [Z_L].$$

As $\gamma \in \mathrm{O}^+(V)(\mathbb{R})$ acts on $\mathscr{H}_{\mathrm{O}}$ and on $\mathscr{K}_{\mathrm{O}}^+$, the following diagram is commutative

$$\begin{array}{ccc} \mathscr{K}_{\mathrm{O}}^+ & \xrightarrow{[Z_L] \mapsto [\gamma Z_L]} & \mathscr{K}_{\mathrm{O}}^+ \\ {\scriptstyle Z \mapsto [Z_L]} \uparrow & & \uparrow {\scriptstyle \gamma Z \mapsto [(\gamma Z)_L]} \\ \mathscr{H}_{\mathrm{O}} & \xrightarrow{Z \mapsto \gamma Z} & \mathscr{H}_{\mathrm{O}}. \end{array} \tag{1.2.15}$$

Indeed, given $Z_L \in \widetilde{\mathscr{K}_{\mathrm{O},1}^+}$, it is clear that γZ_L lies in the zero-quadric $\widetilde{\mathscr{N}}$ and further in $\widetilde{\mathscr{K}_{\mathrm{O}}^+}$, with $q(\gamma Z_L) = 0$ and $\big(\gamma Z_L, \overline{\gamma Z_L}\big) > 0$, because $\gamma \in \mathrm{O}^+(V)(\mathbb{R})$. However, we can not expect the e'-component of γZ_L to be equal to 1, so $\gamma Z_L \notin \widetilde{\mathscr{K}_{\mathrm{O},1}^+}$, in general. Now, $(\gamma Z)_L$ denotes the (normalized) element of $\widetilde{\mathscr{K}_{\mathrm{O},1}^+}$ associated to $\gamma Z \in \mathscr{H}_{\mathrm{O}}$. By commutativity of the diagram (1.2.15), we have $[\gamma Z_L] = [(\gamma Z)_L]$ as an equality in $\mathbb{P}(V_{\mathbb{C}})$. The factor of proportionality between γZ_L and $(\gamma Z)_L$ is given by the e'-component of γZ_L. Since $(e, e') = 1$, we have

$$(\gamma Z)_L = \frac{1}{(\gamma Z_L, e)} \gamma Z_L.$$

Thus, the action of $\mathrm{O}^+(V)(\mathbb{R})$ on $\mathscr{H}_{\mathrm{O}}$ induces a non-trivial *factor of automorphy*,

$$J : \begin{cases} \mathrm{O}^+(V) \times \widetilde{\mathscr{K}^+_{\mathrm{O},1}} & \rightarrow & \mathbb{C}^\times \\ (\gamma, Z_L) & \mapsto & J(\gamma, Z_L), \end{cases}$$

given by

$$J(\gamma, Z_L) = (\gamma Z_L, e)\,.$$

(Compare this to (1.1.8) in the unitary case.)

Remark 1.2.15. *From the definition of Eichler transformations* (1.2.12)*, it is easily seen that the e' component of Z_L is left unchanged by the Eichler transformation of the form $E(e,\kappa)$ with $\kappa \in K \otimes_{\mathbb{Z}} \mathbb{R}$. See p. 44, where the Eichler elements listed under II are of this type, with $\kappa \in K \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus*

$$J\big(E(e,\kappa), Z_L\big) \equiv 1, \quad \textit{identically, for all} \quad \kappa \in K \otimes_{\mathbb{Z}} \mathbb{R}.$$

The process by which Z_L is assigned to Z can be described from a slightly different point of view, too:

Consider the restriction of the tautological bundle of $\mathbb{P}(V_{\mathbb{C}})$ to $\mathscr{K}^+_{\mathrm{O}}$ which we denote by $\mathscr{L}_{\mathscr{K}_{\mathrm{O}}}$. Note that this is isomorphic to the preimage $\widetilde{\mathscr{K}^+_{\mathrm{O}}}$ under the canonical projection of $V_{\mathbb{C}}$ to $\mathbb{P}(V_{\mathbb{C}})$, interpreted as a $\mathbb{C}^\times$-bundle over $\mathscr{K}^+_{\mathrm{O}}$. From the action of $\mathrm{O}^+(V)(\mathbb{R})$ on $\mathscr{L}_{\mathscr{K}_{\mathrm{O}}}$ there is a bundle

$$\mathscr{L}_O = \mathrm{O}^+(V)(\mathbb{R})\backslash\mathscr{L}_{\mathscr{K}_{\mathrm{O}}} \longrightarrow \mathrm{O}^+(V)(\mathbb{R})\backslash\mathscr{H}_{\mathrm{O}}. \tag{1.2.16}$$

The assignment $Z \mapsto [Z_L]$ defines a holomorphic, nowhere vanishing section of $\mathscr{L}_O$.

We close these considerations by giving some examples for the calculation of the automorphy factor.

Example 1.2.16. *Remark 1.2.15 can be generalized: Consider the Heisenberg group* $\mathrm{H}(V_0)(\mathbb{R})$ *attached to a one-dimensional boundary component, defined on p. 45. With a matching basis for the hyperbolic part of $V_{\mathbb{R}}$, $e_1, \ldots, e_4$, the group* $\mathrm{H}(V_0)(\mathbb{R})$*, can be embedded into* $\mathrm{O}^+(V)(\mathbb{R})$ *using Eichler elements of the forms $E(e_3, te_1)$, $E(e_1,\kappa)$ and $E(e_3,\eta)$ with $t \in \mathbb{R}^\times$, $\kappa, \eta \in K \otimes_{\mathbb{Z}} \mathbb{R}$.*

Inspecting the description on p. 44 for the action Eichler elements in $\mathrm{H}(V_0)$ *of these types it is clear that* $\mathrm{H}(V_0)$ *acts trivially on the e_2-component of any vector in $V_{\mathbb{C}}$. It follows that for $\gamma \in \mathrm{H}(V_0)(\mathbb{R})$, the automorphy factor $J(\gamma, Z) = 1$, on all of $\mathscr{H}_{\mathrm{O}}$. Also, clearly, transformations from* $\mathrm{SO}(V_0)$ *as embedded into* $\mathrm{SO}(V)$ *have trivial automorphy factor. Thus, for any γ in the centralizer of the boundary component attached to e_1 and e_3, $J(\gamma, Z_L) \equiv 1$, identically.*

Now, to give the automorphy factor for the real Jacobi-group $\mathrm{J}(V_0)(\mathbb{R})$ it suffices to consider the action of elements $[M]$ with $M \in \mathrm{SL}_2(\mathbb{R})$. We use the same notation as we did for the definition of $\mathrm{J}(V_0)(\mathbb{R})$, on 49f, in particular $e_1, \ldots, e_4$ denotes a basis for the hyperbolic part of $V_{\mathbb{R}}$ consisting of lattice vectors.

Example 1.2.17. *Consider the action of* $\mathrm{SL}_2(\mathbb{R})$ *as embedded into the Jacobi group* $\mathrm{J}(V_0)(\mathbb{R})$*. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. Write $Z_L = (Z_1, Z_2, Z_3, Z_4, \mathfrak{z})$, where Z_3 and Z_4 correspond to the (refined) tube domain coordinates z_1 and z_2, respectively. By* (1.2.14)*, we have*

$$[M]Z_L = \left(\begin{pmatrix} Z_4 & Z_2 \\ -Z_1 & Z_3 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^t, \mathfrak{z}\right) = \left(\begin{pmatrix} aZ_4 + bZ_2 & cZ_4 + dZ_2 \\ -aZ_1 + bZ_3 & -cZ_1 + dZ_3 \end{pmatrix}, \mathfrak{z}\right).$$

The e_2–component of $[M]Z_L$ is given by $cZ_4 + dZ_2$. Since $Z_2 = 1$ for $Z_L \in \widetilde{\mathcal{K}}^+_{O,1}$, the automorphy factor $J([M], Z)$ equals $cZ_4 + d = cz_2 + d$, with the tube domain coordinate z_2. The action of the embedded $\mathrm{SL}_2(\mathbb{R})$ on the tube domain now comes out as

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (z_1, z_2, \mathfrak{z}) \longmapsto \left(z_1 + \frac{cq(\mathfrak{z})}{cz_2 + d}, \frac{az_2 + b}{cz_2 + d}, \frac{\mathfrak{z}}{cz_2 + d} \right). \tag{1.2.17}$$

1.2.6 Automorphic forms

With the definition of the automorphy factor, we have all that is needed to introduce automorphic forms on arithmetic subgroups of $\mathrm{O}^+(L)$.

Two definitions

Definition 1.2.18. *Let Γ be a subgroup of finite index in $\mathrm{O}^+(L)$, k an integer and χ a unitary character on Γ. A holomorphic* automorphic form *on Γ of weight k and with character χ is a holomorphic function $F : \mathcal{H}_O \to \mathbb{C}$, satisfying*

$$F(\gamma Z) = \chi(\gamma)\, J(\gamma, Z)^k\, F(Z) \quad \text{for all} \quad \gamma \in \Gamma.$$

Meromorphic *automorphic forms are defined similarly.*

A holomorphic automorphic form is called a modular form *if it is regular on the boundary of $\mathcal{H}_O$.*

However, as it turns out, by the Koecher principle 1.2.24 this is always the case if the signature is $(2, b)$ with $b > 2$.

Such modular forms correspond uniquely to sections of $\mathcal{L}_O^{\otimes k}$, where $\mathcal{L}_O$ is the bundle defined in (1.2.16) .

From this, we can also draw an alternative definition which resembles more closely definition 1.1.25 in the unitary case. This also sheds some light on the requirement of regularity.

Definition 1.2.19. *For a subgroup Γ of finite index in $\mathrm{O}^+(L)$, an integer k and a unitary character χ, a holomorphic* automorphic form *of weight k, with character χ, is a function $f : \widetilde{\mathcal{K}}^+_O \to \mathbb{C}$ with the following properties:*

1. *f is holomorphic on $\widetilde{\mathcal{K}}^+_O$,*
2. *f is homogeneous of degree $-k$, i.e. $f(ta) = t^{-k} f(a)$, for all $t \in \mathbb{C} \setminus \{0\}$,*
3. *f is Γ-invariant, i.e. $f(\gamma a) = \chi(\gamma) f(a)$, for all $\gamma \in \Gamma$.*

(Meromorphic automorphic forms are defined similarly.) Further, f is called a modular form *if it can be continued to a holomorphic function on the boundary of $\mathcal{K}^+_O$.*

To an automorphic form according to this definition we can uniquely assign an automorphic form according to the earlier definition 1.2.18, and vice versa: Denote by $\jmath_e$ the map $Z \mapsto Z_L$ with $Z_L \in \widetilde{\mathcal{K}}^+_{O,1}$. Given an automorphic form $f : \widetilde{\mathcal{K}}^+_O \to \mathbb{C}$ of weight k for the modular group Γ, with character χ, put

$$f_e(Z) := f(\jmath_e(Z)).$$

It is easily verified that f_e thus defined is an automorphic form by definition 1.2.18:

$$\begin{aligned} f_e(\gamma Z) &= f(\jmath(\gamma Z)) = f\left(J(\gamma,Z)^{-1}\cdot \gamma Z_L\right) \\ &=^2 J(\gamma,Z)^k f(\gamma Z_L) =^3 \chi(\gamma)\, J(\gamma,Z)^k f(Z_L) \\ &= \chi(\gamma)\, J(\gamma,Z)^k f_e(Z). \end{aligned}$$

Given an automorphic form $F : \mathscr{H}_{\mathrm{O}} \to \mathbb{C}$ of weight k for the modular group Γ, write each $a \in \widetilde{\mathscr{K}_{\mathrm{O}}^{+}}$ in the form $a = sZ_L$ with $Z_L \in \widetilde{\mathscr{K}_{\mathrm{O},1}^{+}}$ and $s \in \mathbb{C}^{\times}$. Put

$$f(a) := s^{-k} F\left(\jmath_e^{-1} Z_L\right) = s^{-k} F(Z).$$

If F is holomorphic, so is f, thus, we only need to verify properties 2 and 3 from definition 1.2.19. That $f(a)$ satisfies 2 is immediate from the definition:

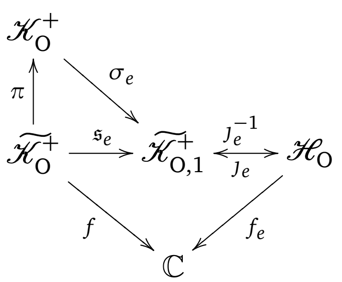
$$f(ta) = (st)^{-k} F\left(\jmath_e^{-1} Z_L\right) = t^{-k} f(a).$$

Now, we show that $f(a)$ satisfies property 3 as well:

$$\begin{aligned} f(\gamma a) &= f(\gamma(sZ_L)) = f\left(s J(\gamma,Z)\, \jmath(\gamma Z)\right) \\ &=^2 s^{-k} J(\gamma,Z)^{-k} f\left(\jmath(\gamma Z)\right) = s^{-k} J(\gamma,Z)^{-k} F(\gamma Z) \\ &= s^{-k} \chi(\gamma)\, J(\gamma,Z)^{-k} J(\gamma,Z)^k F(Z) \\ &= \chi(\gamma) s^{-k} F(Z) = \chi(\gamma) f(a). \end{aligned}$$

The notation f_e used in (1.2.6) is to make clear that the assignment of an automorphic form on $\mathscr{H}_{\mathrm{O}}$ to an automorphic form f on $\widetilde{\mathscr{K}_{\mathrm{O}}^{+}}$ depends on the choice of the cusp e and of e'. Recall that a similar notation was introduced in the unitary case.

Remark 1.2.20. *The following diagram may serve to illustrate the relationship between $\mathscr{K}_{\mathrm{O}}^{+}$, its affine preimages $\widetilde{\mathscr{K}_{\mathrm{O}}^{+}}$ and $\widetilde{\mathscr{K}_{\mathrm{O},1}^{+}}$ and the tube-domain. And also that between an automorphic form f according to definition 1.2.19 and the corresponding f_e, automorphic according to definition 1.2.18.*



Here, π is the canonical projection, $\jmath_e$ the association $Z \mapsto Z_L$, σ_e the unique section of π with $\sigma_e : [Z_L] \mapsto Z_L$ and $\mathfrak{s}_e$ the projection of $a = sZ_L \in \widetilde{\mathscr{K}_{\mathrm{O}}^{+}}$ to $Z_L \in \widetilde{\mathscr{K}_{\mathrm{O},1}^{+}}$. With the exception of π all these maps depend on the choice of e and e'.

Remark 1.2.21. *The action of $\mathrm{O}^+(L)$ on $\widetilde{\mathscr{K}_{\mathrm{O}}^{+}}$, induces an action on functions $\widetilde{\mathscr{K}_{\mathrm{O}}^{+}} \to \mathbb{C}$. Clearly, the first and second property in definition 1.2.19 are invariant under this action, while a Γ-invariant function f, satisfying $f(\gamma a) = f(a)$, satisfies $f(\gamma' g a) = f(ga)$ for $\gamma' \in g\Gamma g^{-1}$ for fixed $g \in \mathrm{O}^+(L)$. Thus, $f \circ g$ is an automorphic form for a group conjugate to Γ. The corresponding automorphic form on $\mathscr{H}_{\mathrm{O}}$ is given by $J(g,Z)^{-k} f_e(gZ)$. (A similar consideration applies if f has a non-trivial character, bearing in mind that Γ has finite index in $\mathrm{O}^+(L)$.)*

Fourier series

We want to expand orthogonal modular forms as Fourier series. This can be achieved with Eichler transformations of the form considered above, $E(e,\kappa)$ with $\kappa \in K \otimes_{\mathbb{Z}} \mathbb{Q}$. As noted earlier, such transformations leave the e'–component of Z_L untouched and thus act on $\mathscr{H}_O$ with trivial automorphy factor.

Let f be an automorphic form on some orthogonal modular group Γ and with character χ. Assume for a moment that Γ is the full discriminant kernel Γ_L^O and χ is trivial on Eichler elements of the form $E(e,\kappa)$. We then have

$$f_e\big(E(e,-\kappa)Z\big) = f_e\big(Z+\kappa\big) = f_e(Z)$$

for all $\kappa \in K$, since $E(e,\kappa) \in \Gamma(L)$. Thus, f_e is periodic with period lattice K and can be expanded as a Fourier series

$$f_e(Z) = \sum_{\lambda \in K'} a(\lambda) e\big((\lambda, Z)\big).$$

More generally, for χ any non-trivial unitary character, there is vector $\varrho \in K \otimes_{\mathbb{Z}} \mathbb{Q}$, unique modulo K', such that

$$f_e(Z+\kappa) = e\big((\varrho,\kappa)\big) \cdot f_e(Z) \tag{1.2.18}$$

holds. From this relation, one gets a Fourier expansion of the form

$$f_e(Z) = \sum_{\lambda \in \varrho + K'} a(\lambda) e\big((\lambda, Z)\big).$$

Of course, if Γ is an arbitrary arithmetic subgroup of $\mathrm{O}^+(L)$, we must replace K with some period lattice in $K \otimes_{\mathbb{Z}} \mathbb{Q}$. We summarize the most important considerations in the following proposition

Proposition 1.2.22. *Let f be an automorphic form on a subgroup Γ of finite index in Γ_L^O, of weight k and with character χ. Then, on $\mathscr{H}_O$, f can be expanded as a Fourier series of the form*

$$f_e(Z) = \sum_{\lambda \in H' + \varrho} a(\lambda) e\big((Z, \lambda)\big),$$

where H' is a sublattice of K' and ϱ is the vector from $K \otimes_{\mathbb{Z}} \mathbb{Q}$ defined by the relation (1.2.18). *If χ is trivial, then $\varrho = 0$ (modulo K'). Further, if Γ contains* $\mathrm{H}(D)$*, we have $H' = K'$. In particular, this is the case if $\Gamma = \Gamma_L^O$.*

Of course, the Fourier coefficients can be calculated by the integrals

$$a(\lambda) = \frac{1}{|H'/H|} \int_{H'/H} f_e(Z) e\big(-(\lambda, Z)\big)\, dZ.$$

Remark 1.2.23. *The Fourier coefficients of f satisfy the relationship*

$$a(\gamma\lambda) = \chi(\gamma)\, a(\lambda), \quad \textit{for} \quad \gamma \in \Gamma \cap \mathrm{O}^+(K \otimes_{\mathbb{Z}} \mathbb{Q}),$$

as an immediate consequence of the transformation behavior of f_e.

It is worth noting that an automorphic form can also be expanded as a Fourier series in terms of the real part X of Z. We can employ this to write the Fourier expansion of f in terms of the Grassmannian coordinates μ and w. Assume for the following that $\Gamma = \Gamma_L^{\mathrm{O}}$ and χ is trivial – this is merely to simplify notation, the general result is similar.

Since $K \subset V$, a translation by $\kappa \in K$ only acts on the real part of Z. Thus, if we consider f_e as a function $f_e(X,Y)$ of the real and imaginary parts of Z, we have $f_e(X+\kappa, Y) = f_e(X,Y)$ and consequently a Fourier expansion of the form

$$f_e(X,Y) = \sum_{\lambda \in K'} a'(\lambda, Y)\, e\big((X,\lambda)\big).$$

Comparing terms, we see that the coefficient $a'(\lambda)$ in this expansion is related to the coefficient $a(\lambda)$ in Proposition 1.2.22 by

$$a'(\lambda, Y) = a(\lambda)\, e\big((iY,\lambda)\big) = a(\lambda)\, \exp\big(-2\pi\,(Y,\lambda)\big).$$

Since by (1.2.11), $X = \mu_K$ and Y is a representative of the subspace w_K, we can also consider this as a Fourier series in the Grassmannian coordinates μ_K and w_K.

The more customary Fourier series in the complex variable Z can also be reformulated in terms of the Grassmannian coordinates. Since $w_K = \mathbb{R}Y$, the projection λ_w of λ onto w, equals $(\lambda, Y)\, Y/Y^2$. Taking norms, we have

$$|(\lambda, Y)| = |\lambda_w| \quad \text{and} \quad \frac{|Y^2|}{|Y|} = \frac{|\lambda_w|}{|e_v|},$$

whence

$$(Z,\lambda) = (X,\lambda) + i\,(Y,\lambda) = (\mu,\lambda) + i\,\frac{\operatorname{sgn}\,(\lambda, w_k)\, |\lambda_w|}{|e_v|}.$$

The Fourier expansion now takes the form

$$f_e(Z) = f_e(\mu, w) = \sum_{\lambda \in K'} a(\lambda) e\left((\mu,\lambda) + i \operatorname{sgn}\,(\lambda, w_K) \cdot \frac{|\lambda_w|}{|e_v|} \right). \tag{1.2.19}$$

Koecher principle

Let $e_1, \ldots, e_4$ be a basis for the hyperbolic part of V as in 1.2.2, with $e_1 \in L$ corresponding to the cusp of $\mathscr{H}_{\mathrm{O}}$. Denote by $\mathscr{C}_+$ the positive cone in $K \otimes \mathbb{R}$ given by

$$X = (x_1, x_2, \mathfrak{x}) \in K \otimes \mathbb{R} \quad \text{with} \quad x_1 > 0,\, x_1 x_2 + q(\mathfrak{x}) > 0, \tag{1.2.20}$$

and by $\overline{\mathscr{C}}_+$ its closure, on which $q(X) \geq 0$ and $x_1, x_2 \geq 0$.

Proposition 1.2.24 (Koecher principle). *If V has signature $(2,b)$ with $b \geq 3$, the following holds: A holomorphic function $f_e : \mathscr{H}_O \to \mathbb{C}$ satisfying*

$$f_e(Z+k) = f_e(Z) \quad \textit{for} \quad k \in K',$$
$$f_e(\gamma Z) = f_e(Z) \quad \textit{for} \quad \gamma \in \Gamma(L) \cap O^+(K),$$

has a Fourier series expansion of the form

$$f_e(Z) = \sum_{\substack{\lambda \in K' \\ \lambda \in \overline{\mathscr{C}}_+}} a(\lambda) e\big((Z,\lambda)\big),$$

where only $\lambda \in K'$ occur in the sum, which satisfy the semi-positivity condition $\lambda \in \overline{\mathscr{C}}_+$.

For a proof, see [26]. The proof is of a quite similar vain as that in the case of Siegel modular forms, see [27].

Remark 1.2.25. *Using the theory of* Siegel domains *it can be shown that this implies the regularity of f_e on the boundary of the Satake compactification of $\Gamma\backslash\mathscr{H}_O$, see [26].*

Cusp forms

Definition 1.2.26. *A modular form f is called a* cusp form *if its Fourier expansion is supported only on the (open) positive cone,*

$$f_e(Z) = \sum_{\substack{\lambda \in K \\ \lambda \in \mathscr{C}^+}} a(\lambda) e\big((Z,\lambda)\big).$$

Fourier-Jacobi expansion and induced Jacobi forms

A Jacobi form is a function on $\mathbb{H} \times V_{0,\mathbb{C}}$ defined through its transformation behavior under the Jacobi-group introduced in 1.2.14.

For the following, we use [20] as a reference. For simplicity, we assume that L is the orthogonal sum of a definite part and two hyperbolic planes spanned by e_1, e_2 and e_3, e_4. In particular, $e_i \in L$ for all i.

Note that by the definition of the positive cone $\mathscr{C}^+$, for the e_4-component z_2 of $Z \in \mathscr{H}_O$ we have $\Im z_2 > 0$, so z_2 can be identified with an element of the upper half plane $\mathbb{H}$ in the complex numbers.

Definition 1.2.27. *Let G be an arithmetic subgroup of $\mathrm{J}(D)$, k an integer, χ a unitary character of order e, and $m \in 1/e\mathbb{Z}$. A* Jacobi form *on G of character χ, weight k and index m is a holomorphic function $\phi : \mathbb{H} \times V_{0,\mathbb{C}} \to \mathbb{C}$, with the following transformation behavior:*

$$\phi(\tau, \mathfrak{z} + \lambda + \tau\mu) = \chi([\lambda,\mu,t]) \exp\big(2\pi i m\big((\mathfrak{z},\mu) + q(\mu)\,\tau + t\big)\big)\, \Phi(\tau,\mathfrak{z}), \tag{1.2.21}$$

$$\phi([M]\tau, [M]\mathfrak{z}) = \chi([M])\,(c\tau+d)^k \exp\left(-2\pi i m \frac{cq(\mathfrak{z})}{c\tau+d}\right) \phi(\tau,\mathfrak{z}), \tag{1.2.22}$$

for all $[\lambda,\mu,t] \in \mathrm{H}(D)$ and $M \in \mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{J}(D)$, where the action on $\mathbb{H}$ and V_0 is explained through the action on the tube domain, see p. 44 for $\mathrm{H}(D)$ *and* (1.2.17) *on p. 48 for* $\mathrm{SL}_2(\mathbb{Z})$.

Remark. *This definition generalizes the original definition by Eichler and Zagier in [24], which is the classical reference on Jacobi forms for the elliptic modular group. A reference on the more general concept of 'mixed automorphic forms' is [47].*

Without proof, we quote the following from [20], Lemma 3.1.11.

Lemma 1.2.28. *A holomorphic function ϕ on $\mathbb{H} \times V_{0,\mathbb{C}}$ is a Jacobi form on an arithmetic subgroup $G \subset \mathrm{J}(D)$ of weight k, character χ and index m exactly if the function on $\mathscr{H}_{\mathrm{O}}$ given by*

$$F(z_1, z_2, \mathfrak{z}) = \phi(z_2, \mathfrak{z}) \cdot e(mz_1)$$

transforms like an orthogonal modular form for G with the same weight k and the same character.

Proposition 1.2.29 (Fourier-Jacobi expansion)**.** *Let Γ be a subgroup of finite index in $\Gamma(L)$ containing $\mathrm{H}(D)$. Let f be a modular form on Γ of weight k and with character χ of order e. Then, for every non-negative $m \in 1/e\mathbb{Z}$ there exists a Jacobi-form ϕ_m of weight k and index m on $\Gamma \cap \mathrm{J}(D)$ with character χ, such that*

$$f(z_1, z_2, \mathfrak{z}) = \sum_{\substack{m \in 1/e\mathbb{Z} \\ m \geq 0}} \phi_m(z_2, \mathfrak{z}) e(mz_1).$$

We refer to the Jacobi forms ϕ_m occurring as coefficients in this expansion as being *induced* from the orthogonal modular form f.

Remark 1.2.30. *That the coefficients in the Fourier-Jacobi expansion of an orthogonal modular form are Jacobi-forms, stands out in contrast to the unitary modular forms in section 1.1. The reason for this is, of course, that the boundary components of the symmetric domain $\mathscr{H}_{\mathrm{U}}$ for the unitary group are only zero-dimensional. Thus, there is no copy of $\mathbb{H}$ contained in the boundary of $\mathscr{H}_{\mathrm{U}}$ and no subgroup isomorphic to $\mathrm{SL}_2(\mathbb{Z})$ in the normalizer of a boundary component. The transformation behavior in* (1.1.11) *of the Fourier-Jacobi coefficients $a_n(\sigma)$ under the unitary Heisenberg group is, however, analogous to that of the Jacobi-forms in* (1.2.21).

The Siegel operator

We wish to clarify somewhat the relationship between the Fourier expansion of a modular form and its behavior on the boundary of $\mathscr{K}_{\mathrm{O}}$.

Assume that we have a basis $e_1, \ldots, e_4$ for the hyperbolic part of V consisting of isotropic lattice vectors. This fixes the cusp as the zero-dimensional rational boundary component attached to e_1, which, in turn, is contained in the one-dimensional boundary component attached to the rational isotropic space spanned by e_1 and e_3.

Given Z_L in $\widetilde{\mathscr{K}_{0,1}^{+}}$, the intersection in projective space of $[Z_L]$ with the boundary of $\mathscr{K}_{\mathrm{O}}$, corresponds to the limit

$$\lim_{it \to i\infty} \left[-itz_2, 1, it, z_2, \mathfrak{z}\right] = \lim_{t \to \infty} \left[-z_2, \frac{1}{iz}, 1, \frac{z_2}{it}, \frac{\mathfrak{z}}{it}\right] = \left[-z_2, 0, 1, 0\right].$$

Recall that $z_2 \in \mathbb{H}$ parameterizes the points in the boundary component.

Next, we consider the value of a modular form f_{e_1} on the boundary component. For this, we write $\lambda \in K'$ as $\lambda = (\lambda_1, \lambda_2, \lambda_D)$. Using the Koecher principle, we have

$$\begin{aligned}\lim_{t\to\infty} f_{e_1}(it, z_2, \mathfrak{z}) &= \lim_{t\to\infty} \sum_{\substack{\lambda\in K' \\ \lambda_1,\lambda_2\geq 0 \\ q(\lambda)\geq 0}} a(\lambda) e\left(it\lambda_2 + z_2\lambda_1 + (\mathfrak{z}, \lambda_D)\right) \\ &= \lim_{t\to\infty} \sum_{\substack{\lambda\in K' \\ \lambda_1\geq 0,\, \lambda_2=0 \\ q(\lambda)\geq 0}} a(\lambda)\, e\left(z_2\lambda_1 + (\mathfrak{z}, \lambda_D)\right),\end{aligned}$$

since all terms with $\lambda_2 > 0$ vanish in the limit. But then, in the remaining terms, we must have $\lambda_D = 0$, as well, because $q(\lambda) = q(\lambda_D) \geq 0$ and λ_D is from the negative definite lattice D'. Thus, the value of f on the boundary is given by

$$\lim_{t\to\infty} f_{e_1}(it, z_2, \mathfrak{z}) = \sum_{\substack{(\lambda_1,0,0)\in K' \\ \lambda_1\geq 0}} a(\lambda_1, 0, 0)\, e\left(z_2\lambda_1\right). \tag{1.2.23}$$

Since z_2 is in $\mathbb{H}$, this can be seen as a Fourier expansion in one variable on the upper half-plane.

Remark. *Since the Fourier expansion in* (1.2.23) *is regular, we can say that the Koecher principle implies regularity on the boundary. More generally, it can be shown that the semi-positivity condition $\lambda \in \overline{\mathscr{C}}_+$ is equivalent to regularity on the boundary.*

Remark 1.2.31. *If f is a cusp form, then $\lim_{t\to\infty} f_{e_1} = 0$. This is implied by* (1.2.23) *and the (strict) positivity condition $\lambda \in \mathscr{C}^+$ in definition 1.2.26. In other words, cusp forms vanish at the cusp.*

The *Siegel operator* is an operator taking modular forms on $\mathscr{H}_O$ to functions on $\mathbb{H}$, defined as

$$(f \mid \Phi)(\tau) := \sum_{\lambda_1\geq 0} a(\lambda_1, 0, 0) e^{2\pi i \lambda_1 \tau}.$$

The resulting function is in fact an elliptic modular form for some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Remark. *The value of a modular form on a zero-dimensional boundary component, corresponding to a one-dimensional isotropic space $\mathbb{Q}e$, can be calculated as follows. For $Z_L \in \widetilde{\mathscr{K}}^+_{0,1}$, the intersection of $[Z_L]$ with the boundary of $\mathscr{K}_O$ is given by*

$$\lim_{t\to\infty} [Z_L] = \lim_{t\to\infty} [-t^2 q(Z), 1, tZ] = [1, 0, 1].$$

The value of a modular form f on the boundary is simply

$$\lim_{t\to\infty} f_e(Z) = \lim_{t\to\infty} \sum_{\lambda\in K'} a(\lambda) e\big((tZ, \lambda)\big) = a(0).$$

Cusp forms, in this context, can be characterized as having a Fourier expansion in which only λ with $q(\lambda) > 0$ occur.

2 Theta lifts and Borcherds theory for $\mathrm{O}(2,b)$

This chapter describes the lifting constructed by Borcherds, and some of the concepts needed to formulate it.

In the form given in [5], this lifting is an example for a, in this case, singular theta correspondence. It generalizes early known theta correspondences or theta lifts, such as those constructed by Shimura, Doi-Naganuma, Maass, Gritsenko and others, see in this order, [61], [23], [48], [32].

Many of these liftings, notably the Shimura and the Doi-Naganuma lift, were first constructed through correspondences between L-series attached to automorphic forms. A more general approach is based on the concept of a dual reductive pair due to Howe. Two reductive algebraic groups are dual in this sense if, as subgroups of a larger group, they are commutants of each other. In such a situation, intertwining operators between the representations of these groups exist, which may be used to 'lift' automorphic forms from one group to automorphic forms on the other group. In particular, for a dual pair of subgroups in a metaplectic group (a double cover of the symplectic group), this lifting takes the form of an integral transformation, with a theta function as integration kernel.

A first example of such a lifting was constructed by Shintani, see [65]. Using his setup as a model, in [67], Zagier reimplemented the Doi-Naganuma lift, which takes elliptic modular forms to modular forms for a Hilbert modular group, as a theta lift, while Niwa similarly reformulated the Shimura lift, see [53]. A theta lift which generalizes the liftings of Maass and Gritsenko was constructed by Oda in [54], for the dual reductive pair $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{O}(2,b-2)$.

The original construction of automorphic products by Borcherds in [3] was not based on a theta lift. However, the physicists Harvey and More discovered in [34] that Borcherds' result could be obtained from a 'singular theta correspondence', in which the input functions for the lift are allowed to have poles at the cusps. The term 'singular' is due to the fact the theta integral, in this case, is widely divergent. However, Harvey and More also showed how a regularization recipe from theoretical physics could be applied to this integral.

Borcherds then used this singular theta correspondence to implement his lifting in [5], generalizing many of the previous examples. His general construction is an additive lifting, from which a multiplicative lifting is then derived. We will describe this construction in more detail in section 2.2 below.

Before we explore some of the concepts needed to formulate the Borcherds lift, we would like to mention one more example for a theta lift, which is not included as a special case in Borcherds' construction. This is a lifting due to Kudla, who in [40] implemented a setup analogous to that of Sintani for the dual reductive pair $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{U}(1,q)$, providing a lifting to modular forms for unitary groups.

2.1 Prerequisites for Borcherds theory

In this section we discuss some of the basics needed to formulate Borcherds' result. First we recall the definition of modular forms for the elliptic modular group $\mathrm{SL}_2(\mathbb{Z})$; the usual definition must be extended to accomodate weakly holomorphic modular forms, which are allowed to have a pole at the cusp $i\infty$ and, in general, for vector valued weakly holomorphic forms of rational weight.

Also, the image of the lift consists of meromorphic functions having their zeros and poles along so called Heegner-divisors, which we discuss, too. Finally, to formulate the infinite product expansions we need to somewhat reexamine the theory of quadratic lattices and also to introduce the concept of Weyl chambers as subsets of the Grassmannian $\mathcal{G}(K)$ of the Lorentzian space $W = K \otimes_{\mathbb{Z}} \mathbb{Q}$.

2.1.1 The Weil representation and vector valued modular forms

The functions serving as inputs for the Borcherds lift are vector valued elliptic modular forms, that is holomorphic functions on the complex upper half-plane $\mathbb{H} = \{z \in \mathbb{C}\,;\ \Im z > 0\}$ transforming under a representation of the elliptic modular group $\mathrm{SL}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[L/L']$, the so called Weil representation. Such vector valued modular forms are a generalization of classical – i.e. scalar valued – elliptic modular forms on the upper half plane $\mathbb{H}$. Further, the modular forms we consider need only be weakly holomorphic and are thus allowed to have poles at the cusp $i\infty$.

Scalar valued modular forms

Recall that $\mathrm{GL}_2(\mathbb{R})^+$ acts on the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C}\,; \Im\tau > 0\}$ of $\mathbb{C}$, through Möbius transformations,

$$\mathrm{GL}_2(\mathbb{R})^+ \ni \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \,:\, \tau \mapsto \frac{a\tau + b}{c\tau + d}.$$

For an integer k, the Petersson slash operator on functions $f : \mathbb{H} \to \mathbb{C}$ is defined as

$$M\,|_k\, f = \det(M)^{k/2}(c\tau + d)^{-k} f(M\tau), \quad \text{for} \quad M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R}).$$

Now consider the elliptic modular group $\mathrm{SL}_2(\mathbb{Z})$. Denote by $\Gamma(N)$ the principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$,

$$\Gamma(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod N \right\}. \tag{2.1.1}$$

And by $\Gamma_0(N)$ and $\Gamma_1(N)$ the congruence subgroups

$$\Gamma_0(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),\ c \equiv 0 \bmod N \right\}, \tag{2.1.2}$$

$$\Gamma_1(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),\ c \equiv 0 \bmod N,\ a, d \equiv 1 \bmod N \right\}. \tag{2.1.3}$$

More generally, a *congruence subgroup* of $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ is a subgroup of finite index containing the principal congruence subgroup $\Gamma(N)$ for some $N \geq 1$.

Definition 2.1.1. *Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, χ a character of Γ and k an integer. A (scalar valued)* weakly holomorphic *(elliptic) modular form on Γ of weight k with character χ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ which is*

1. *invariant under the weight k slash operator,* $M\,|_k\, f = f$, *for any* $M \in \Gamma$,

2. *at most meromorphic at the cusps.*

If f is holomorphic at the cusps, it is called a holomorphic modular form *(or simply a modular form), a modular form vanishing at the cusps is called a* cusp form. *The space of weight k weakly holomorphic modular forms on* Γ *with character* χ *is denoted by* $\mathscr{M}_k^!(\Gamma,\chi)$, *the subspace of holomorphic modular forms by* $\mathscr{M}_k(\Gamma,\chi)$ *and that of cusp forms by* $\mathscr{S}_k(\Gamma,\chi)$.

Example 2.1.2. *The Eisenstein series* G_k *with k an integer,* $k \geq 3$,

$$G_k(\tau) = \sum_{\substack{a,b\in\mathbb{Z}\\(a,b)\neq(0,0)}} \frac{1}{(a\tau+b)^k},$$

is a holomorphic modular form of weight k for the full elliptic modular group $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. *Note that* G_k *vanishes identically for odd k. For even k, its q-expansion is given by*

$$G_k = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!}\sum_{n\geq 1}\sigma_{k-1}(n)\,q^n, \quad \textit{with} \quad \sigma_k(n) = \sum_{d|n} d^k.$$

Example 2.1.3. *The usual elliptic j-function, can be defined as*

$$j(\tau) = 1728\frac{20G_4^3}{20G_4^3 - 49G_3^2}.$$

The j-function is holomorphic on $\mathbb{H}$ *and invariant under the Petersson slash operator of weight* 0 *for the full elliptic modular group, thus,* $j(\tau)$ *is contained in* $\mathscr{M}_0^!(\Gamma(1))$.

Necessarily, the j-function has a pole at the cusp $i\infty$, *which is also apparent from the q-expansion*

$$j(\tau) = q^{-1} + 744 + 196884\,q + 21493760\,q^2 + 86429970\,q^3 + \cdots.$$

In fact, the space of weight 0 *weakly holomorphic forms* $\mathscr{M}_0^!(\Gamma(1))$ *is isomorphic to* $\mathbb{C}[j(\tau)]$.

The Weil representation

For $z \in \mathbb{C}$ denote by $\sqrt{z}$ the principal branch of the square root, which has $\arg\sqrt{z} \in (-\pi/2, \pi/2]$. For any integer k set $z^{k/2} = \sqrt{z}^k$.

Denote by $\mathrm{Mp}_2(\mathbb{R})$ the *metaplectic group*, the double cover of $\mathrm{SL}_2(\mathbb{R})$. The metaplectic group can be realized by the two choices of holomorphic square roots for $\tau \mapsto c\tau + d$, where $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. The elements of $\mathrm{Mp}_2(\mathbb{R})$ can be written as

$$\big(M, \phi(\tau)\big), \quad \text{where} \quad M = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$$

and ϕ is a holomorphic function

$$\phi : \mathbb{H} \to \mathbb{C}, \quad \text{with} \quad \phi(\tau)^2 = c\tau + d.$$

The product of two elements (M_1, ϕ_1) and (M_2, ϕ_2) in $\mathrm{Mp}_2(\mathbb{R})$ is given by

$$\big(M_1, \phi_1(\tau)\big)\,\big(M_1, \phi_1(\tau)\big) = \big(M_1 M_2, \phi_1\big(M_2\tau\big)\,\phi_2(\tau)\big),$$

where $M \in \mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ as usual, via $M\tau = \frac{a\tau+b}{c\tau+d}$. Clearly, there is an embedding of $\mathrm{SL}_2(\mathbb{R}) \hookrightarrow \mathrm{Mp}_2(\mathbb{R})$. This is a local isomorphism.

The group $\mathrm{Mp}_2(\mathbb{Z})$ is the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map $\mathrm{Mp}_2(\mathbb{R}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{R})$. This group is generated by the elements

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right) \quad \text{and} \quad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right).$$

A standard generator for the center of $\mathrm{Mp}_2(\mathbb{Z})$ is

$$Z = S^2 = (TS)^3 = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i\right).$$

In the following let L be an even lattice over $\mathbb{Z}$ with a symmetric, non-degenerate $\mathbb{Z}$-bilinear form $(\cdot,\cdot)$ of signature $(2, b)$, as considered in section 1.2. The $\mathbb{Z}$-dual in $L \otimes_{\mathbb{Z}} \mathbb{Q}$ with respect to $(\cdot,\cdot)$ is the dual lattice L'. Then, the discriminant group L'/L is a finite Abelian group and the discriminant form is the modulo 1 reduction of $(\cdot,\cdot)$ on L'. Denote by $q(\cdot)$ the quadratic form associated to $(\cdot,\cdot)$ as well as its modulo 1 reduction, which is a $\mathbb{Q}/\mathbb{Z}$ valued quadratic form on L'/L.

The *Weil representation* ρ_L is a unitary representation of $\mathrm{Mp}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[L'/L]$. We denote by $\mathfrak{e}_\gamma$ for $\gamma \in L'/L$ the standard basis elements of $\mathbb{C}[L'/L]$. Then, ρ_L can be defined through the action of the generators T and S introduced above, which is given by

$$\begin{aligned} \rho_L(T)\,\mathfrak{e}_\gamma &= e\big(-q(\gamma)\big)\mathfrak{e}_\gamma, \\ \rho_L(S)\,\mathfrak{e}_\gamma &= \frac{\sqrt{i}^{\,b-2}}{\sqrt{|L'/L|}} \sum_{\delta \in L'/L} e\big(-(\gamma,\delta)\big)\mathfrak{e}_\delta. \end{aligned} \tag{2.1.4}$$

For details, see [5]. Further, the action of Z is given by

$$\rho_L(Z)\,\mathfrak{e}_\gamma = i^{b-2}\mathfrak{e}_{-\gamma}. \tag{2.1.5}$$

Proposition 2.1.4. *If b is even, the representation ρ_L of* $\mathrm{Mp}_2(\mathbb{Z})$ *factors through a representation of* $\mathrm{SL}_2(\mathbb{Z})$*. Further, in this case, ρ_L factors though the finite group* $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$*. where N is the* level *of L, the smallest integer with $Nq(\gamma) \in \mathbb{Z}$ for all $\gamma \in L'$. If b is odd, ρ_L only factors through a double cover of* $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

Remark. *More generally, for a lattice of signature (b^+, b^-) the conditions on $b \bmod 2$ in the proposition and on the following pages are replaced by conditions on $b^+ + b^- \bmod 2$. In the present case, $b^+ + b^- = 2 + b \equiv b \bmod 2$, of course.*

If b is even, the negative identity matrix $-E_2 \in \mathrm{SL}_2(\mathbb{Z})$ acts as $\rho_L\big(-E_2\big)\mathfrak{e}_\gamma = (-1)^{(b-2)/2}\mathfrak{e}_{-\gamma}$.

The standard hermitian scalar product on the group algebra $\mathbb{C}[L'/L]$ is defined as

$$\Big\langle \sum_\gamma a_\gamma \mathfrak{e}_\gamma, \sum_\gamma b_\gamma \mathfrak{e}_\gamma \Big\rangle = \sum_\gamma a_\gamma \bar{b}_\gamma \mathfrak{e}_\gamma.$$

The matrix coefficients $\rho_{\gamma,\delta}\left(M, \phi\right)$, for $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$ and γ, $\delta \in L'/L$, of the representation ρ_L are defined as

$$\rho_{\gamma,\delta}\left(M, \phi\right) = \left\langle \rho_L(M, \phi)\mathfrak{e}_\delta, \mathfrak{e}_\gamma \right\rangle,$$

see [9]. There is also dual representation for ρ_L, denoted ρ_L^*. Its matrix coefficients can be obtained by conjugating those of ρ_L.

Let $\kappa \in \frac{1}{2}\mathbb{Z}$ and f be a $\mathbb{C}[L'/L]$ valued function on $\mathbb{H}$. For (M,ϕ) we define the Petersson slash operator as

$$\left(f\mid_\kappa (M,\phi)\right)(\tau) = \phi(\tau)^{-2\kappa}\rho_L(M,\phi)^{-1}f(M\tau),$$

where $M\tau = \frac{a\tau+b}{c\tau+d}$, as usual. Similarly, for the dual representation ρ_L^*, the slash operator

$$\left(f\mid_\kappa^* (M,\phi)\right)(\tau) = \phi(\tau)^{-2\kappa}\rho_L^*(M,\phi)^{-1}f(M\tau)$$

defines a 'dual' operation on the $\mathbb{C}[L'/L]$ valued function on $\mathbb{H}$.

Proposition 2.1.5. *Let f be a holomorphic function $f : \mathbb{H} \to \mathbb{C}[L'/L]$. Write f in the form $f = \sum_{\gamma\in L'/L} f_\gamma \mathfrak{e}_\gamma$ with holomorphic component functions f_γ. If for some $\kappa \in \frac{1}{2}\mathbb{Z}$, f is invariant under the $\mid_\kappa$ operation of $T \in \mathrm{Mp}_2(\mathbb{Z})$, f has a Fourier expansion of the form*

$$f(\tau) = \sum_{\gamma\in L'/L}\sum_{n\in\mathbb{Z}+q(\gamma)} c(n,\gamma)e(n\tau)\mathfrak{e}_\gamma.$$

If on the other hand, f is invariant under the dual operation, $\mid_\kappa^*$ for some half-integer κ, then f has a Fourier expansion of the form

$$f(\tau) = \sum_{\gamma\in L'/L}\sum_{n\in\mathbb{Z}-q(\gamma)} c(n,\gamma)e(n\tau)\mathfrak{e}_\gamma.$$

Proof. From the definition of the slash operator, using (2.1.4), for f under $\mid_\kappa T$ it is clear that each $\gamma \in L'/L$, the function $f_\gamma(\tau)\,e(q(\gamma)\tau)$ is periodic with period 1 and thus can be expanded into a Fourier series, as claimed.

Similarly, $f\mid_\kappa^* T = f$ implies that $f_\gamma(\tau)\,e(-q(\gamma)\tau)$ is periodic for each γ. □

Definition 2.1.6. *Let $\kappa \in \frac{1}{2}\mathbb{Z}$, A function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is a* weakly holomorphic vector valued modular form *of weight κ with respect to the Weil representation ρ_L on* $\mathrm{Mp}_2(\mathbb{Z})$ *(or on* $\mathrm{SL}_2(\mathbb{Z})$*, when b is even), if f satisfies*

1. *$f\mid_\kappa (M,\phi) = f$ for all $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$,*

2. *f is holomorphic on $\mathbb{H}$,*

3. *f is meromorphic at the cusp $i\infty$.*

The space of weakly holomorphic vector valued modular forms with respect to ρ_L is denoted by $\mathscr{M}_\kappa^!(\rho_L)$, the subspace of holomorphic modular forms*, those which are holomorphic at the cusp, is denoted by $\mathscr{M}_\kappa(\rho_L)$. The space of* cusp forms*, vanishing at the cusp, by $\mathscr{S}_\kappa(\rho_L)$.*

For the dual representation ρ_L^*, vector valued modular forms are defined similarly and denoted by $\mathscr{M}_\kappa^!(\rho_L^*)$, $\mathscr{M}_\kappa(\rho_L^*)$ and $\mathscr{S}_\kappa(\rho_L^*)$, respectively.

From the definition, it follows that a weakly holomorphic vector valued modular form f is invariant under the $|_\kappa$ operation of $T \in \mathrm{Mp}_2(\mathbb{Z})$, and for even $b+2$, under the operation of the matrix $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. By the preceding proposition 2.1.5, f has a Fourier expansion of the form

$$f(\tau) = \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n \gg -\infty}} c(n,\gamma) e(n\tau) \mathfrak{e}_\gamma,$$

in which only finitely many coefficients $c(n,\gamma)$ with $n < 0$ occur, since f has at most a pole at the cusp $i\infty$. If f is holomorphic, n runs over non-negative values only.

Remark 2.1.7. *Assume b to be even. Let f be a vector valued modular form in $\mathcal{M}^!_\kappa(\rho_L)$ with component functions f_γ, $\gamma \in L'/L$. The transformation behavior under the negative identity matrix $-E_2 \in \mathrm{SL}_2(\mathbb{Z})$ implies that $f_\gamma = f_{-\gamma}$ if $k \equiv 1 + b/2 \bmod 2$ and $f_\gamma = -f_{-\gamma}$ if $k \not\equiv 1 + b/2 \bmod 2$.*

Borcherds inputs

In Borcherds theory, weakly holomorphic modular forms from $\mathcal{M}^!_{1-b/2}(\rho_L)$ play an important role, as they serve as inputs for the multiplicative Borcherds lift, which we discuss below.

Remark 2.1.8. *If $f \in \mathcal{M}^!_{1-b/2}(\rho_L)$ as a consequence of* (2.1.5) *and $|_{1-b/2}$ invariance, the following relation holds between Fourier coefficients of f:*

$$c(n,\gamma) = c(n,-\gamma).$$

As the orthogonal group $\mathrm{O}^+(L)$ operates on the dual lattice L' there is an induced operation of $\mathrm{O}^+(L)$ on the group algebra $\mathbb{C}[L'/L]$ given by

$$g : \sum_{\gamma \in L'/L} c_\gamma \mathfrak{e}_\gamma \longmapsto \sum_{\gamma \in L'/L} c_\gamma \mathfrak{e}_{g(\gamma)}.$$

Definition 2.1.9. *Let f be a weakly holomorphic modular form contained in $\mathcal{M}^!_{1-b/2}(\rho_L)$ with component functions f_γ, $\gamma \in L'/L$. We define the* automorphism group *of f in $\mathrm{O}^+(L)$ as*

$$\mathrm{O}^+_f(L) = \left\{ g \in \mathrm{O}^+(L)\,;\; f_\gamma = f_{g(\gamma)} \quad \text{for every} \quad \gamma \in L'/L \right\}.$$

The automorphism group $\mathrm{O}^+_f(L)$ is a subgroup of finite index in $\mathrm{O}^+(L)$, which contains the discriminant kernel Γ^{O}_L as a subgroup.

2.1.2 Heegner divisors and Weyl chambers

Besides elliptic modular forms and the Weil representation, which can be considered as prerequisites for most theta lifts, several further concepts still need to be described, which are more specific to the theory surrounding Borcherds' work. In this, we largely follow Bruinier, [9].

To describe the theory of Borcherds in a sufficiently general setting, we need some facts on lattice as quadratic modules, which have not yet been covered in section 1.2. For the following, see [9], further details can be found in [5] and [29], for example.

Let L be an even lattice of signature $(2,b)$ and e, e' a hyperbolic pair as per definition 1.2.4, with $e\in L$ primitive and isotropic, $e'\in L'$ with $\left(e,e'\right)=1$. As usual, we denote by K the sublattice $L\cap e^{\perp}\cap e'^{\perp}$.

Consider the orthogonal projection p_K from $L\otimes_{\mathbb{Z}}\mathbb{Q}$ to $K\otimes_{\mathbb{Z}}\mathbb{Q}$ introduced in (1.2.4). Denote by a subscript $\cdot_K$ the image of p_K. For $\lambda\in L'$, $\lambda_K:=p_K(\lambda)$ is contained in K', the dual lattice of K in $K\otimes_{\mathbb{Z}}\mathbb{Q}$. However, $\lambda\in L$ does not imply $\lambda_K\in K$. In fact, K' is not necessarily contained in L'.

We now introduce a projection p with the property that $p(L)=K$. Clearly, there is a unique positive integer N such that $(e,L)=N\mathbb{Z}$, called the *level of the cusp e*. Let $f\in L$ be a lattice vector with $\left(f,e\right)=N$.

Proposition 2.1.10. *([9], prop. 2.2) L can be written as a direct sum $K\oplus\mathbb{Z}e\oplus Zf$.*

Proof. Let $\lambda\in L$. The vector $\tilde{\lambda}-(\lambda,e/N)f-\left(\lambda,e'\right)e+(\lambda,e/N)\left(f,e'\right)e$ is contained in L. Now, as is easily verified, $\tilde{\lambda}\perp e$ and $\tilde{\lambda}\perp e'$. Therefore $\tilde{\lambda}$ is contained in K. Hence, $\lambda\in K+\mathbb{Z}f+\mathbb{Z}e$. The directness of the sum is obvious. □

As a consequence, there is an isometric embedding $K'\to L'$ defined by

$$\gamma\mapsto\gamma-\left(\gamma,f\right)\frac{e}{N}$$

and an induced map $K'\to L'/L$. The kernel of this map is given by the set $\left\{\gamma\in K;\,\left(\gamma,f\right)\in N\mathbb{Z}\right\}$.

Consider the following sublattice of L'

$$L_0'=\left\{\lambda\in L';\,(\lambda,e)\equiv 0\mod N\right\}. \tag{2.1.6}$$

By the definition of N, L is contained in L_0'. The quotient L_0'/L is a subgroup of the discriminant group L'/L, given by

$$L_0'/L=\left\{\lambda\in L'/L\,;\,(\lambda,e)\equiv 0\mod N\right\}.$$

Proposition 2.1.11. *The projection $p:\,L_0'\to K'$ defined by*

$$p(\lambda)=\lambda_K-\frac{(\lambda,e)}{N}f_K \tag{2.1.7}$$

has the property $p(L)=K$. It induces a surjective map from $L_0'/L\to K'/K$.

Proof. For an arbitrary $\lambda\in L$ write $\lambda=\gamma+af+be$, with $\gamma\in K$. Then, the map from $L_0'\to L'\cap e^{\perp}$, $\lambda\mapsto(\lambda,e)/N$, combined with the orthogonal projection p_K from $L'\cap e^{\perp}\to K'$, sends λ to γ. □

Heegner divisors

The functions constructed in [5] by means of the Borcherds lift have their poles and zeros along certain Γ_L^{O}-invariant divisors, called *Heegner divisors*, generalizing classical Heegner points on modular curves and Hirzebruch-Zagier divisors on *Hilbert modular varieties*. We recall some basic facts on divisors first.

Divisors

For the following, compare [9] and [31]. Let X be a normal complex space, with every point in X locally irreducible. A divisor is a formal linear combination

$$D = \sum n_Y Y, \qquad \text{with} \quad n_Y \in \mathbb{Z},$$

of irreducible closed analytic subsets Y of codimension 1, such that the support, $\operatorname{supp}(D) = \bigcup_{n_Y \neq 0} Y$ is a closed analytic subset of everywhere pure codimension 1.

Let Γ be group of biholomorphic transformations acting properly discontinuously on X. Consider a divisor D on the quotient X/Γ. The pullback $\pi^*(D)$ of D under the canonical projection $\pi : X \to X/\Gamma$, is a Γ-invariant divisor on X. For every irreducible component Y of the inverse image of supp(D), the multiplicity of Y in $\pi^*(D)$ equals the multiplicity of $\pi(Y)$ in D.

The divisors occurring in connection with Borcherds' singular theta lift are defined through the complement of negative definite rational subspaces as follows.

Heegner divisors

Given a vector of negative norm $\lambda \in L'$, the orthogonal complement $\lambda^\perp$ in $V_\mathbb{R}$ is a quadratic space of signature $(2, b-1)$. All positive definite two-dimensional subspaces contained in $\lambda^\perp$ form a subset of the Grassmannian Gr_O, which is also denoted by $\lambda^\perp$,

$$\lambda^\perp = \{v \in \mathrm{Gr}_\mathrm{O}\,;\ (a, \lambda) = 0, \quad \text{for all} \quad a \in v\}.$$

Each point v in the Grassmannian is an oriented subspace of $V_\mathbb{R}$. For each $v \in \mathrm{Gr}_\mathrm{O}$, there is a subspace $v' \in \mathrm{Gr}_\mathrm{O}$ with inverse orientation. Clearly, v is contained in the sub-Grassmannian $\lambda^\perp$ exactly if $v' \in \lambda^\perp$, too. Thus $\lambda^\perp$ has two connected components, one for each connected component of Gr_O. It suffices to consider one of these connected component, which corresponds to the component $\mathscr{K}_\mathrm{O}^+$ of the positive projective cone $\mathscr{K}_\mathrm{O}$.

Then, $\lambda^\perp$ defines a closed analytic subset of the cone $\mathscr{K}_\mathrm{O}^+$ in the complex projective space $\mathbb{P}(V_\mathbb{C})$, consisting of all positive, norm-zero lines $[Z_L] \in \mathscr{K}_\mathrm{O}^+$ with $\left(Z_L, \lambda\right) = 0$.

Through the normalized representatives of $Z_L \in \widetilde{\mathscr{K}_{\mathrm{O},1}^+}$, the subset of $\mathscr{K}_\mathrm{O}^+$ given by $\lambda^\perp$ defines a closed, analytic subset of the tube domain, which we also denote by $\lambda^\perp$.

Definition 2.1.12. *Let $\lambda \in L'$ be a vector of negative norm and write λ in the form $\lambda = \lambda_K + ae' + be$, with $\lambda_K \in K'$, $a \in \mathbb{Z}$, $b \in \mathbb{Q}$. We denote by $\lambda^\perp$ the subset*

$$\lambda^\perp = \left\{Z \in \mathscr{H}_\mathrm{O};\quad aq(Z) - \left(Z, \lambda_K\right) + aq\left(e'\right) - b = 0\right\}.$$

Then, $\lambda^\perp$ defines a prime divisor on $\mathscr{H}_\mathrm{O}$.

Now, given $\beta \in L'/L$ and $m \in \mathbb{Z}$ with $m < 0$, we consider the following subset of Gr_{O}:

$$\bigcup_{\substack{\lambda\in\beta+L\\ q(\lambda)=m}} \lambda^{\perp} \tag{2.1.8}$$

Since for any subset $U \subset \mathrm{Gr}_{\mathrm{O}}$ with compact closure, the set

$$S(m,\beta,U) = \{\lambda = \beta + L; \quad q(\lambda) = m,\ \exists\, v \in U \quad \text{with} \quad v \perp \lambda\}$$

is finite, cf. [9], p. 48. The union over $\lambda^{\perp}$ in (2.1.8) is a locally finite union of codimension one sub-Grassmannians, which naturally remains finite when restricted to either connected component. It may thus be interpreted as the support of a divisor on the symmetric domain. From this observation, it should also be clear that the following is well-defined.

Definition 2.1.13. *Given $\beta \in L'/L$ and $m \in \mathbb{Z}$, $m < 0$. We define a Γ_L^{O}-invariant divisor on $\mathscr{H}_{\mathrm{O}}$, called a* Heegner divisor of discriminant (m,β) *and denoted $H(m,\beta)$, as follows*

$$H(m,\beta) := \sum_{\substack{\lambda\in\beta+L\\ q(\lambda)=m}} \lambda^{\perp}. \tag{2.1.9}$$

The support of $H(m,\beta)$ is given by the (locally finite) union

$$\bigcup_{\substack{\lambda\in\beta+L\\ q(\lambda)=m}} \lambda^{\perp}.$$

The Γ_L^{O} invariance of $H(m,\beta)$ follows directly from the definition as the sum is carried over a system of representatives of L'/L. Through this, in turn, we can consider $H(m,\beta)$ as the inverse image of a divisor on $\mathfrak{X}_\Gamma = \Gamma\backslash\mathscr{H}_{\mathrm{O}}$.

Definition 2.1.14. *Since $H(m,\beta)$ is Γ_L^{O} invariant, it is the inverse image under the canonical projection $\mathscr{H}_{\mathrm{O}} \to \mathfrak{X}_\Gamma$ of a divisor on the modular variety $\mathfrak{X}_\Gamma = \Gamma\backslash\mathscr{H}_{\mathrm{O}}$, which is also referred to as a* Heegner divisor *of discriminant (m,β) and, for simplicity, also denoted by $H(m,\beta)$.*

Weyl chambers

In Borcherds theory, *Weyl chambers* occur as disjoint subsets of the symmetric domain. The multiplicative Borcherds lift is holomorphic on these subsets and has an infinite product expansion on each of them.

For the following, we need to assume that the Lorentzian lattice $K = L \cap e^{\perp} \cap e'^{\perp}$ contains an isotropic vector. Consider the Lorentzian space $W_{\mathbb{R}} = K \otimes_{\mathbb{Z}} \mathbb{R}$ and the attached Grassmannian of one-dimensional positive definite subspaces, denoted $\mathscr{G}(K)$. This Grassmannian is a model for the symmetric space of $\mathrm{O}(W)(\mathbb{R})$. For $\gamma \in K'$ with $q(\gamma) = n < 0$, a divisor $H(m,\gamma)$ on $\mathscr{G}(K)$ can be defined similarly as in definition 2.1.13. In particular, its support, a subset of $\mathscr{G}(K)$ can be defined as in 2.1.8, replacing L by K. Then, $\operatorname{supp} H(m,\gamma)$ is a union of codimension 1 sub-Grassmannians. Hence, the set $\mathcal{G}(K) - H(m,\gamma)$ is disjoint. Its components are called *Weyl chambers of index* (m,γ).

The Grassmannian can be realized as a upper half-space in a hyperbolic space of dimension $b-1$,

$$\mathcal{G}(K) \simeq \mathcal{H} = \left\{(x_0, x_1, \dots, x_{b-2}) \in \mathbb{R}^{b-1}\,;\; x_0 > 0\right\},$$

with $\mathbb{R}^{b-1} = \mathbb{R} \times (D \otimes_{\mathbb{Z}} \mathbb{R})$, where D is a negative definite lattice of rank $b-2$ contained in K. The half-space $\mathcal{H}$ is the called the *upper half-space model* of $\mathcal{G}(K)$.

Another realization of $\mathcal{G}(K)$ can be obtained as follows: Let $z \in K$ be primitive and isotropic. Then, the set

$$\mathcal{C}_K = \left\{v \in K \otimes_{\mathbb{Z}} \mathbb{R}\,;\; v^2 = 1, (v,z) > 0\right\}$$

is called the *hyperboloid model* of $\mathcal{G}(K)$. The identification between $\mathcal{C}_K$ and $\mathcal{G}(K)$ is carried out via $v \mapsto \mathbb{R}v$.

Recall that the tube domain $\mathcal{H}_O$ can be written as $K \otimes_{\mathbb{Z}} \mathbb{R} + i\mathcal{C}^+$ with $\mathcal{C}^+$ a positive cone, see p. 40. For $Z = X + iY \in \mathcal{H}_O$, the imaginary part Y is contained in $\mathcal{C}^+ \subset K \otimes \mathbb{R}$ and satisfies $Y^2 > 0$. Then, the line $\mathbb{R}Y$ defines an element of the Grassmannian $\mathcal{G}(K)$ and we may can identify $Y/|Y|$ with a point in the hyperboloid model.

Definition 2.1.15. *Let m be a negative integer, $\beta \in L'$ be a vector with $q(\beta) = m$. Denote by N the level of the cusp e, as on p. 63, and denote by $p(\beta) \in K$ the image of β under the projection* (2.1.7).

A Weyl chamber *W of index (m,β) is a subset of $\mathcal{H}_O$ defined as follows:*

If $(\beta, e) \equiv 0 \mod N$, put

$$W := \left\{Z = X + iY \in \mathcal{H}_O\,;\quad \frac{Y}{|Y|} \in U\right\} \subset \mathcal{H}_O,$$

where U is a Weyl chamber of index $(m, p(\beta))$ in $\mathcal{G}(K)$.

If, otherwise, $(\beta, e) \not\equiv 0 \mod N$, define W to be the whole of $\mathcal{H}_O$.

Since this definition associates Weyl chambers in $\mathcal{H}_O$ to Weyl chambers in $\mathcal{G}(K)$, given a Weyl chamber W in $\mathcal{G}(K)$, we will usually denote the corresponding Weyl chamber in $\mathcal{H}_O$ by W, too.

For a Weyl chamber W of $\mathcal{G}(K)$ and an element $\kappa \in K'$, we write $(W,\kappa) > 0$ if $(v,\kappa) > 0$ for every v in the interior of W – for this, consider W as a subset of the hyperboloid model.

Lemma 2.1.16 (cf. [9], lemma 3.2). *Let $W \subset \mathcal{G}(K)$ be a Weyl chamber of index (m,γ) and assume that $\kappa \in K'$ with $q(\kappa) \geq 0$ or $q(\kappa) = m$, $\kappa + L = \pm\gamma$. Suppose that $(v_0, \kappa) > 0$ for a $v_0 \in W$. Then $(W,\kappa) > 0$.*

Proof. Obviously, $\kappa \neq 0$. Assume the existence of a $v_1 \in W$ with $(\kappa, v_1) \leq 0$. Since W is connected and (κ, v) is continuous, there exists a $v_2 \in W$ with $(\kappa, v_2) = 0$. Since $v_2 \in \mathcal{C}^+$, clearly $v_2^\perp \subset K \otimes \mathbb{R}$ is a negative definite subspace, thus $q(\kappa) < 0$. We may assume $q(\kappa) = m$ but then, $v_2 \in \lambda^\perp$ implies $v_2 \in H(m,\kappa)$ so v_2 can not be contained in W. Contradiction. □

Weyl chambers associated with modular forms

The Weyl chambers and Heegner divisors which arise in the context of the Borcherds lift and its generalizations, are related to the Fourier expansions of the vector valued modular forms serving as inputs for the lift.

Just as for $f \in \mathscr{M}^!_{1-b/2}(\rho_L)$ we have defined an automorphism group $\mathrm{O}^+_f(L)$, we define Weyl chambers with respect to f.

Definition 2.1.17. *Let $f : \mathbb{H} \to \mathbb{C}[L'/L]$ be a weakly holomorphic modular form for* $\mathrm{Mp}_2(\mathbb{Z})$ *of weight $1-b/2$ with principal part*

$$\sum_{\beta \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\beta) \\ n<0}} c(n,\beta) e(n\tau) \mathfrak{e}_\gamma .$$

Then, the components of

$$\mathscr{G}(K) - \bigcup_{\beta \in L'_0/L} \bigcup_{\substack{n \in \mathbb{Z}+q(\beta) \\ n<0 \\ c(n,\beta)\neq 0}} H\left(n, p(\beta)\right) \tag{2.1.10}$$

are called Weyl chambers *of $\mathscr{G}(K)$* with respect to f.

Remark 2.1.18. *We extend the notation $(W,\kappa)>0$ to Weyl chambers of this form, as well. By repeated application of lemma 2.1.16 to all indices (n,γ) with $\gamma = p(\beta)$ for β with $c(n,\beta)\neq 0$, we see that $(W,\kappa)>0$ holds precisely if $(v_0,\kappa)>0$ for one $v_0 \in W$.*

2.2 The Borcherds lift

Given a weakly holomorphic modular form $f : \mathbb{H} \to \mathbb{C}[L'/L]$ on $\mathrm{Mp}_2(\mathbb{Z})$, the theta lift constructed by Borcherds is given by the regularized theta integral

$$\Phi_L(Z,f) = \int_{\mathscr{F}}^{\mathrm{reg}} \langle f(\tau), \Theta_L(\tau,Z)\rangle \, y^{b/2} \frac{dx\,dy}{y^2}, \quad \text{with } \tau \in \mathbb{H},\ Z \in \mathscr{H}_{\mathrm{O}}.$$

Here, $\mathscr{F} \simeq \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ denotes a fundamental domain for the operation of the elliptic modular group $\mathrm{SL}_2(\mathbb{Z})$ and $\langle\,,\rangle$ the usual Petersson scalar product.

The *Siegel theta function* Θ_L attached to the lattice L is defined as follows.

$$\Theta_L(\tau,Z) = \sum_{\gamma \in L'/L} \theta_\gamma(\tau,Z)\mathfrak{e}_\gamma \quad \tau \in \mathbb{H},\ Z \in \mathscr{H}_{\mathrm{O}}, \tag{2.2.1}$$

with component functions

$$\theta_\gamma(\tau,Z) = \sum_{\lambda \in \gamma + L} e\left(\tau\, q(\lambda_Z) + \bar{\tau}\, q(\lambda_{Z^\perp})\right). \tag{2.2.2}$$

The notation λ_Z means projection onto the two-dimensional positive subspace $v = \mathbb{R}X_L + \mathbb{R}Y_L$ of $V_\mathbb{R}$ associated to Z, while $\lambda_{Z^\perp}$ is the projection to the complement, $v^\perp$. Note that with $Z_L = X_L + iY_L = (-q(Z) - q(e'), 1, Z)$ we have

$$\lambda_Z = \frac{1}{Y^2}\left(\lambda, X_L\right) X_L + \frac{1}{Y^2}\left(\lambda, Y_L\right) Y_L.$$

The divergent integral is regularized as follows, using a recipe due to Harvey and Moore [34]. For $u \in \mathbb{R}_{>0} \cup \{\infty\}$ denote by $\mathcal{F}_u$ the region

$$\mathcal{F}_u = \left\{\tau\,;\ |\tau| \geq 1, |\Re\tau| \leq \tfrac{1}{2}, 0 < \Im\tau \leq u\right\} \subset \mathbb{H},$$

and by $\mathcal{F}_\infty$ the usual fundamental domain $\mathcal{F}$. Now, for $s \in \mathbb{C}$ consider the limit

$$\lim_{u\to\infty} \int_{\mathcal{F}_u} \langle f(\tau), \Theta_L(\tau, Z)\rangle\, y^{b/2-s} \frac{dx\,dy}{y^2}.$$

Assuming the limit exists for $\Re(s)$ sufficiently large, and can be continued to a meromorphic function defined for all $s \in \mathbb{C}$, the regularized integral is defined as the constant term of its Laurent expansion around $s = 0$,

$$\int_{\mathcal{F}}^{\text{reg}} \langle f(\tau), \Theta(\tau, Z)\rangle\, y^{b/2} \frac{dx\,dy}{y^2} := \mathcal{C}_{s=0} \left[\lim_{u\to\infty} \int_{\mathcal{F}_u} \langle f(\tau), \Theta_L(\tau, Z)\rangle\, y^{b/2-s} \frac{dx\,dy}{y^2}\right].$$

If there is a pole at $s = 0$, a slightly different definition is needed, see [5], p. 22f. Yet another variation of this, based on a different regularization procedure, is introduced in [9], p. 47.

Remark 2.2.1. *Actually, the theta functions Borcherds introduces in [5] are of a somewhat more general type then* (2.2.1)*, which makes it possible to admit a much larger space of input functions for the theta lift. For the multiplicative lift under consideration here, however, a theta function of the above type suffices.*

The regularized theta integral defines an additive lifting for input functions $f \in \mathcal{M}_k^!(\rho_L)$ with k a half-integer $\geq 1 - b/2$. In the special case where $f \in \mathcal{M}_{1-b/2}^!(\rho_L)$, the additive lift $\Phi_L(Z, f)$ can be used to define a *multiplicative lifting*, $\Psi_L(Z, f)$, the properties of which are given in theorem 2.2.3 below. To summarize, Ψ_L is a meromorphic function in $\mathcal{H}_O$ having as divisor a special cycle attached to f, see (2.2.3) below, and defined through the property that $-2\log\|\Psi_L(Z, f)\|_{Pet}^2 = \Phi_L(Z, f)$ (up to some constants, see remark 2.2.6).

Remark 2.2.2. *For the following, cf. [12], lemma 6.6. Denote by D the divisor given in theorem 2.2.3. The additive lift $\Phi_L(Z, f)$ has logarithmic singularities along D, see [5], §6. Furthermore, it can be shown that Φ_L is pluriharmonic, i.e. all mixed second derivatives $\partial\bar{\partial}\Phi_L$ vanish. From this, the existence of the meromorphic function Ψ_L satisfying $\Phi_L = \log|\Psi_L|$ and having divisor D follows. Ψ_L is then uniquely determined up to a constant of modulus* 1.

The multiplicative Cousin problem, see [31], is universally solvable for $\mathcal{H}_O$, since $\mathcal{H}_O$ is convex. Hence, there exists a meromorphic function g with divisor D. Then, $\Phi_L - \log|g|$ extends to a pluriharmonic real analytic function on $\mathcal{H}_O$. Since further $\mathcal{H}_O$ is simply connected, there exists a holomorphic function h such that $\Re h = \Phi_L - \log|g|$. Then, $\Phi_L = \log\left|e^h \cdot g\right|$ and we may put $\Psi_L = e^h \cdot g$.

Besides having its poles and zeros on Heegner divisors the function Ψ_L also has the distinctive property of having an infinite product expansion as a Borcherds product.

The term 'multiplicative lift' results from the fact that $\Psi_L(Z, f + g) = \Psi_L(Z, f) \cdot \Psi_L(Z, g)$.

Theorem 2.2.3 (Borcherds, [5], th. 13.3)**.** *Let L be an even lattice of signature $(2,b)$, $e \in L$ primitive isotropic, and $e' \in L'$ with $(e,e') = 1$. Assume that $K = L \cap e^{\perp} \cap e'^{\perp}$ also contains an isotropic vector. Let $f \in \mathscr{M}^{!}_{1-b/2}(\rho_L)$ be a weakly holomorphic modular form, with principal part having integral Fourier coefficients, i.e. $c(n,\gamma) \in \mathbb{Z}$ for $n < 0$. Assume further that the constant coefficient $c(0,0)$ is twice an integer.*

Then there is a meromorphic function $\Psi_L(Z,f)$ on $\mathscr{H}_O$ with the following properties:

i) $\Psi_L(Z,f)$ is an automorphic form of weight $c(0,0)/2$ on the orthogonal group $O^+_f(L)$ with a unitary character χ of finite order.

ii) The divisor of $\Psi_L(Z,f)$ on $\mathscr{H}_O$ is given by

$$\operatorname{div}(\Psi_L) = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\beta) \\ n<0}} c(n,\beta) H(n,\beta). \tag{2.2.3}$$

The multiplicities of the $H(n,\beta)$ are 2, if $2\beta = 0$ in L'/L, and 1 otherwise. Note that $c(n,\beta) = c(n,-\beta)$ and $H(n,\beta) = H(n,-\beta)$. (Recall (2.1.9) for the definition of $H(n,\beta)$.)

iii) Under the assumptions of the Koecher principle, $\Psi(Z,f)$ is a holomorphic modular form on $O^+_f(L)$ if the orders $c(n,\beta)$ in the above divisor are all positive.

iv) The functions Φ_L and Ψ_L are related by

$$\log|\Psi_L(Z,f)| = -\frac{\Phi_L(Z,f)}{4} - \frac{c(0,0)}{2}\left(\log|Y_L| + \frac{1}{2}\Gamma'(1) + \log\sqrt{2\pi}\right). \tag{2.2.4}$$

v) Let W be a Weyl chamber with respect to f. Let $m_0 = \min\{n \in \mathbb{Q};\ c(n,\gamma) \neq 0\}$. On the set of $Z \in \mathscr{H}_O$, for which $Y^2 > 2|m_0|$ and which belong to the complement of the set of poles of $\Psi_L(Z,f)$, the function $\Psi_L(Z,f)$ has a normally convergent infinite product expansion

$$\Psi_L(Z,f) = C e\left((\rho_f(W), Z)\right) \prod_{\substack{\lambda \in K' \\ (\lambda,W)>0}} \prod_{\substack{\delta \in L_0'/L \\ p(\delta)=\lambda+K}} \left(1 - e\left((\delta,e') + (\lambda,Z)\right)\right)^{c(q(\lambda),\delta)},$$

where C is a constant of absolute value 1.

Remark 2.2.4. *The vector $\rho_f(W) \in K \otimes_{\mathbb{Z}} \mathbb{R}$ is the* Weyl vector *attached to f and W. Sometimes, it can be computed explicitly by means of the results in [5], section 10. Also, a more direct way to determine ρ_f is often afforded by a result of Bruinier, cf. [9], theorem 3.4.*

We employ this later on, in chapter 5 when calculating Borcherds products for weakly holomorphic scalar valued modular forms of weight 0.

Remark 2.2.5. *The theorem remains true, without the additional assumption $c(0,0) \in 2\mathbb{Z}$. In this case however, we must allow for $\Psi_L(Z,f)$ to have fractional weight. Automorphic forms of weight r, with $r \in \mathbb{Q}$, and with a suitable multiplier system of this weight can be defined somewhat similarly to definition 1.2.18. For details see [9] p. 86 or [20], p. 31f.*

Thus, in particular, for $c(0,0) \notin 2\mathbb{Z}$, the multiplier system χ from the theorem is, in general no longer a character. Its order however remains finite. If $b \geq 3$, where $(2,b)$ is the signature of V, this follows by a result of Margullis, see [9] p. 87. Whereas, for $b = 1$, 2, it can be shown using the embedding trick, see [7].

Remark 2.2.6. *From* (2.2.4) *we have* $-4\log|\Psi| = \Phi + c(0,0)\log|Y|^2 + A$*, with some constant* A*. Hence,*

$$\Phi + A = -2\log\left(|\Psi|^2|Y|^{c(0,0)}\right) = -2\log||\Psi||^2_{Pet,c(0,0)/2},$$

with the weight $c(0,0)/2$ *Petersson norm of* Ψ*.*

3 Embedding from the unitary to the orthogonal world

In this chapter, we bring together the objects studied in sections 1.1 and 1.2. We consider a hermitian space V over an imaginary quadratic number field $\mathbb{F} = \mathbb{Q}(\sqrt{d})$ as in section 1.1, with an indefinite non-degenerate hermitian form $\langle\cdot,\cdot\rangle$ of signature $(1,q)$. The ringer of integers of $\mathbb{F}$ is denoted by $\mathcal{O}_{\mathbb{F}}$, the discriminant $D_{\mathbb{F}}$ and the generator of the different ideal by δ. Considered as a vector space over $\mathbb{Q}$, the vector space V comes equipped with the bilinear form $(\cdot,\cdot)$ given by the trace $\mathrm{Tr}_{\mathbb{F}/\mathbb{Q}}\langle\cdot,\cdot\rangle$. Thus, $V_{\mathbb{R}} = V \otimes \mathbb{R}$ is both a complex hermitian space as in section 1.1 and a real quadratic space of signature $(2,2q)$, as in section 1.2. To avoid confusion, we introduce the following notation. By V' we denote the $\mathbb{Q}$-vector space[1] underlying V and by $V'_{\mathbb{R}} = V' \otimes_{\mathbb{Q}} \mathbb{R}$ the real space underlying $V_{\mathbb{R}}$.

Let L be an even $\mathcal{O}_{\mathbb{F}}$-lattice (of maximal rank) in the hermitian space V. We have $L \otimes_{\mathcal{O}_{\mathbb{F}}} \mathbb{F} = V$ as an identity of $\mathcal{O}_{\mathbb{F}}$-modules but also $L \otimes_{\mathbb{Z}} \mathbb{Q} = V'$ as an identity of the underlying $\mathbb{Z}$-modules. As a quadratic $\mathbb{Z}$-module with the bilinear form $(\cdot,\cdot)$ and the attached quadratic form $q(\cdot)$, L is even in the usual sense, see remark 1.1.5.

The identification of the hermitian space V, $\langle\cdot,\cdot\rangle$ with the quadratic space V', $(\cdot,\cdot)$ gives rise to an inclusion of groups, sending $\mathrm{U}(1,q)$ into $\mathrm{O}(2,2q)$, since, forcibly, an endomorphism fixing $\langle\cdot,\cdot\rangle$ also fixes its trace, $(\cdot,\cdot)$. We can identify $\mathrm{U}(1,q)$ with a subgroup of $\mathrm{O}(2,2q)$. This embedding of groups, in turn, gives rise to an embedding of symmetric domains. We first describe this at the level of the Grassmannian models Gr_{U} and Gr_{O}. Further, we derive embeddings of the positive cone $\mathcal{K}_{\mathrm{U}}$ into the positive cone $\mathcal{K}_{\mathrm{O}}$, and of the Siegel domain model $\mathcal{H}_{\mathrm{U}}$ into the tube domain $\mathcal{H}_{\mathrm{O}}$.

At this point, however, the issue of complex structures comes into play: On the one hand, $\mathcal{K}_{\mathrm{U}}$ and $\mathcal{H}_{\mathrm{U}}$ come naturally with the complex structure of the hermitian space $V_{\mathbb{R}}$, or more precisely of the attached projective space $\mathbb{P}(V_{\mathbb{R}})$ in case of $\mathcal{K}_{\mathrm{U}}$. On the other hand, $\mathcal{K}_{\mathrm{O}}$ and the tube domain $\mathcal{H}_{\mathrm{O}}$ are equipped with the complex structure of the space $V_{\mathbb{C}}$, which is the complexification of the underlying *real space* $V'_{\mathbb{R}}$; this complex structure is derived from the choice of orientation on the points of the Grassmannian Gr_{O}, recall section 1.2.3. We have to take care so as to realize the embedding in a manner compatible with both complex structures.

A further important question is the choice of the cusp for either $\mathcal{H}_{\mathrm{U}}$ or for the tube domain $\mathcal{H}_{\mathrm{O}}$. This corresponds to choosing an $\mathbb{F}$-rational isotropic subspace of V for a cusp of $\mathcal{H}_{\mathrm{U}}$ and of a rational isotropic subspace for a cusp of $\mathcal{H}_{\mathrm{O}}$, compare for this the construction in 1.1 and 1.2. In either case, the isotropic subspace corresponding to the cusp at infinity is fixed by choosing a primitive isotropic lattice vector from L, denoted either ℓ or e, respectively.

We will fix a choice of ℓ, primitive in L as a $\mathcal{O}_{\mathbb{F}}$-module and isotropic with respect to $\langle\cdot,\cdot\rangle$. We then set $e = \ell$. Clearly ℓ is also primitive in L as a $\mathbb{Z}$-module and isotropic for $(\cdot,\cdot)$. This, way, the point at infinity of $\mathcal{H}_{\mathrm{U}}$, corresponding to ℓ, will come to lie on the boundary of $\mathcal{H}_{\mathrm{O}}$ under the embedding. Since $\mathcal{O}_{\mathbb{F}}\ell \subset L$ is a rank two $\mathbb{Z}$-submodule of L, clearly, it contains a second isotropic vector, linearly independent over $\mathbb{Q}$. Since L contains two rational isotropic vectors, $V' = L \otimes_{\mathbb{Z}} \mathbb{Q}$ splits two hyperbolic planes. With the notation introduced in 1.2.5, $\ell = e_1$, and the second isotropic vector is denoted e_3. Then, two further vectors e_2 and e_4, with e_1, e_2 and

[1] Confusion with the notation for the dual used in the context of lattices is unlikely.

e_3, e_4 each spanning a hyperbolic plane over $\mathbb{Q}$, are needed to describe the hyperbolic part of V' and to introduce coordinates on $\mathscr{H}_{\mathrm{O}}$, see p. 41.

Similarly, over $\mathbb{F}$, the hyperbolic part of V is spanned by ℓ and a second vector ℓ', with $\ell' \in L'$ and $\langle \ell, \ell' \rangle \neq 0$. These two vectors, a hyperbolic pair as in definition 1.1.13, are both needed for the construction of the Siegel domain $\mathscr{H}_{\mathrm{U}}$. The $\mathbb{F}$-span of ℓ and ℓ' viewed as a quadratic space over $\mathbb{Q}$ is, of course, the hyperbolic part of V' and isometric to the two hyperbolic planes over $\mathbb{Q}$ spanned by $e_1, \dots, e_4$.

Thus, after setting $e_1 = \ell$ and choosing e_3 from $\mathscr{O}_{\mathbb{F}}\ell$, we can determine suitable vectors e_2 and e_4 from the $\mathbb{F}$-span of ℓ and ℓ'. Using these vectors, we can then completely describe the embedding of $\mathscr{H}_{\mathrm{U}}$ into $\mathscr{H}_{\mathrm{O}}$, with the Siegel domain coordinates τ and σ used for the former and the (refined) tube domain coordinates Z_1, Z_2 and $\mathfrak{z}$ used for the latter.

With the *additional assumption* that ℓ' is isotropic, we can take e_2 and e_4 from $\mathbb{F}\ell'$. Without this assumption we can still choose e_2 and e_4 from $\mathbb{F}\ell'$, if the discriminant $D_{\mathbb{F}}$ is even, see remark 3.2.10 below, but not if $D_{\mathbb{F}}$ is odd. (The main point here is that the hyperbolic planes $\mathbb{Z}e_1 + \mathbb{Z}e_2$ and $\mathbb{Z}e_3 + \mathbb{Z}e_4$ have to be perpendicular.)

Since not making this assumption offers only a slight gain in generality, offset by a considerable complication of notation and a lengthening of all further calculations, we *require here that ℓ' be isotropic*. We will be actively using this assumption from section 3.1.2 onward.

3.1 The Embedding of $\mathrm{SU}(1,q)$ into $\mathrm{SO}(2,2q)$

In this section, we first consider how the identification of the real space $V'_{\mathbb{R}}$ underlying the hermitian space $V_{\mathbb{R}}$, $\langle\cdot,\cdot\rangle$ with the quadratic space $V'_{\mathbb{R}}$, $(\cdot,\cdot) = 2\mathfrak{R}\langle\cdot,\cdot\rangle$, induces an embedding of the attached isometry groups and how these carry through to their arithmetic and their parabolic subgroups.

Further, in section 3.1.2 we deal with the issue of determining rational basis vectors for the hyperbolic part of the quadratic space $V'_{\mathbb{R}}$, $(\cdot,\cdot)$, as mentioned above in the introduction to this chapter. We determine these vectors $e_1, \dots, e_4$ from a hyperbolic pair ℓ and ℓ' in the hermitian space V,$\langle\cdot,\cdot\rangle$. The results is given in (3.1.1) below.

3.1.1 Setup and general considerations

Consider the hermitian space V, $\langle\cdot,\cdot\rangle$ as a vector space over the number field $\mathbb{F}$ and the complex space $V_{\mathbb{R}} = V \otimes_{\mathbb{F}} \mathbb{C}$. Also consider the underlying spaces V' and $V'_{\mathbb{R}}$ as vector spaces over $\mathbb{Q}$ and $\mathbb{R}$, respectively, equipped with the bilinear form $(\cdot,\cdot)$ and the attached quadratic form $q(\cdot)$.

As algebraic groups, $\mathrm{U}(V)$, the isometry group of the hermitian space V, $\langle\cdot,\cdot\rangle$, and $\mathrm{O}(V)$, the isometry group of the quadratic space[2] V', $(\cdot,\cdot)$ are defined over $\mathbb{Q}$ as subgroups of the algebraic group $\mathrm{GL}(V)$. The corresponding sets of real points are $\mathrm{U}(V)(\mathbb{R})$, $\mathrm{O}(V)(\mathbb{R})$ and $\mathrm{GL}(V)(\mathbb{R})$.

Clearly, $\mathrm{U}(V)(\mathbb{R})$ is a subgroup of $\mathrm{O}(V)(\mathbb{R})$, since for $\gamma \in \mathrm{GL}(V)(\mathbb{R})$

$$\langle \gamma v, \gamma v \rangle = \langle v, v \rangle \quad \text{implies} \quad (\gamma v, \gamma v) = (v, v).$$

[2] We will not denote $\mathrm{O}(V)$ or any of its subgroups by $\mathrm{O}(V')$, as the mention of the respective group should suffice to make clear that we are working over a quadratic space.

Also, $\mathrm{SU}(V)(\mathbb{R})$ is a subgroup of $\mathrm{SO}(V)(\mathbb{R})$. In fact, as $\mathrm{SU}(V)(\mathbb{R})$ is connected, we have $\mathrm{SU}(V)(\mathbb{R}) \subset \mathrm{O}^+(V)(\mathbb{R})$ as an inclusion of groups.

Remark 3.1.1. *Given a maximal compact subgroup $\mathscr{C}$ of $\mathrm{U}(V)(\mathbb{R})$, under the inclusion $\mathrm{U}(V)(\mathbb{R}) \hookrightarrow \mathrm{O}(V)(\mathbb{R})$, as a subgroup of $\mathrm{O}(V)(\mathbb{R})$, $\mathscr{C}$ remains compact, and thus a compact subgroup of $\mathrm{O}(V)(\mathbb{R})$. Similarly for maximal compact subgroups of $\mathrm{SU}(V)(\mathbb{R})$.*

Let L be an even hermitian lattice in V, with $V = L \otimes_{\mathscr{O}_{\mathbb{F}}} \mathbb{F}$, equipped with the hermitian form $\langle \cdot, \cdot \rangle$. Interpreted as a $\mathbb{Z}$-module, L is a $\mathbb{Z}$-submodule of the rational space V', with $L \otimes_{\mathbb{Z}} \mathbb{Q} = V'$, and comes equipped with the quadratic form $q(\cdot)$. The definitions of integral and of even $\mathscr{O}_{\mathbb{F}}$-lattices given in section 1.1.2, are equivalent to the usual definitions of integral and of even lattices, when these lattices are interpreted as $\mathbb{Z}$-modules with the bilinear form given by the trace, $(\cdot, \cdot) = \mathrm{Tr}_{\mathbb{F}/\mathbb{Q}} \langle \cdot, \cdot \rangle$, see remark 1.1.5. Similarly, L', the $\mathscr{O}_{\mathbb{F}}$-dual of L is also the $\mathbb{Z}$-dual, see remark 1.1.7.

Now, consider the isometry groups of L for the hermitian form $\langle \cdot, \cdot \rangle$ and for the bilinear form $(\cdot, \cdot)$; these are the arithmetic subgroups $\mathrm{SU}(L) \subset \mathrm{SU}(V)$ and $\mathrm{SO}(L) \subset \mathrm{SO}(V)$. Under the embedding $\mathrm{SU}(V) \hookrightarrow \mathrm{O}^+(V)$, $\mathrm{SU}(L)$ is isomorphic to a subgroup of $\mathrm{O}^+(L)$. What is more, the discriminant kernel $\Gamma_L^{\mathrm{U}} \subset \mathrm{SU}(L)$ is isomorphic to a subgroup of the discriminant kernel $\Gamma_L^{\mathrm{O}} \subset \mathrm{O}^+(L)$, as both by definition consist of the isometries which act trivially on L'/L.

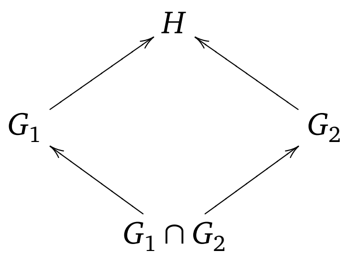
In the following, we will usually identify $\mathrm{SU}(L)$ with its embedded image in $\mathrm{SO}(L)$ and make no notational distinction between the two.

Lemma 3.1.2. *Let Γ be a subgroup of finite index in $\mathrm{O}^+(L)$. Then, the intersection $\Gamma \cap \mathrm{SU}(L)$ has finite index in $\mathrm{SU}(L)$. Also if Γ is an orthogonal modular group, i.e. of finite index in the discriminant kernel Γ_L^{O}, the intersection with the discriminant kernel Γ_L^{U} is a unitary modular group.*

Proof. Apply the group theoretic lemma 3.1.3 below to $H = \mathrm{O}^+(L)$, $G_2 = \Gamma$ and $G_1 = \mathrm{SU}(L)$ for the first part. Similarly, for the second part of statement, set $H = \Gamma_L^{\mathrm{O}}$, $G_1 = \Gamma_L^{\mathrm{U}}$ and let G_2 be a modular group, i.e. a subgroup of finite index in Γ_L^{O}. □

Lemma 3.1.3. *Let H be a group and G_1 and G_2 subgroups of H, with G_2 of finite index. Then, $G_1 \cap G_2$ is of finite index in G_1.*

Proof. Consider the following diagram of groups, in which all arrows are inclusion and G_2 is of finite index in H.



We want to show that the coset decomposition of G_1 as the disjoint union $\uplus (G_1 \cap G_2)g$ with $g \in (G_1 \cap G_2)\backslash G_1$ is finite. For this, consider cosets of the form $G_2 g \subset H$, with $g \in G_1$. If two such cosets coincide, i.e. $G_2 g_1 = G_2 g_2$ for $g_1, g_2 \in G_1$, then $g_1 g_2^{-1} \in G_2$ and thus also in $G_2 \cap G_1$. Thus, $(G_1 \cap G_2)g_1 = (G_1 \cap G_2)g_2$ and it follows that the map given by

$$\begin{cases} (G_1 \cap G_2)\backslash G_1 & \to & G_2\backslash H \\ (G_1 \cap G_2)g & \mapsto & G_2 g \end{cases}$$

is injective. However, since $G_2\backslash H$ is a finite set, this implies that $(G_1 \cap G_2)\backslash G_1$ also is finite, which completes the proof. $\square$

Complex scalars as endomorphisms of $V'_{\mathbb{R}}$

When comparing the complex hermitian space $V_{\mathbb{R}}$, $\langle\cdot,\cdot\rangle$ and the real quadratic space $V'_{\mathbb{R}}$, $(\cdot,\cdot)$, besides the operation of the isometry groups $\mathrm{U}(V)(\mathbb{R})$ and $\mathrm{O}(V)(\mathbb{R})$, we must also consider the operation of $\mathbb{C}^\times$, as scalars of the former space act on the latter space. This is essential when, explicitly given a vector in the complex space $V_{\mathbb{R}}$, we want to rewrite it as a vector in the real space $V'_{\mathbb{R}}$, in terms of some basis of $V'_{\mathbb{R}}$.

The group $\mathbb{C}^\times$ operates on $V_{\mathbb{R}}$, $\langle\cdot,\cdot\rangle$ by multiplication, and scalar factors behave in the usual manner with respect to the hermitian product.

Naturally, complex scalars also operate as endomorphisms on the underlying real space $V'_{\mathbb{R}}$ and on the quadratic space $V'_{\mathbb{R}}$, $(\cdot,\cdot)$. For $\mu \in \mathbb{C}$, the identity $\big(\mu v, w\big) = \mu\,(v,w)$ holds only if $\mu \in \mathbb{R}$. The following identity, however, holds for any $\mu \in \mathbb{C}$:

$$\big(\mu v, \mu w\big) = 2\Re\,\langle \mu v, \mu w\rangle = |\mu|^2\,(v,w), \quad \text{for} \quad v \in V'_{\mathbb{R}},\ w \in V'_{\mathbb{R}}.$$

Purely imaginary numbers act on the quadratic space $V'_{\mathbb{R}}$ as endomorphism sending vectors in $V'_{\mathbb{R}}$ to their orthogonal complement with respect $(\cdot,\cdot)$: Given $\mu \in i\mathbb{R}$, with $\mu \neq 0$, and $v \in V'_{\mathbb{R}}$, we have

$$\big(\mu v, v\big) = 2\Re\big(\mu\,\langle v,v\rangle\big) = 2\Re\big(i\Im\mu \cdot \langle v,v\rangle\big) = 0,$$

hence $\mu v \in v^\perp$. In particular, $\pm i$ is an element of $\mathrm{O}(V)(\mathbb{R})$, since $|i| = 1$, sending v to $\pm i v \in v^\perp$. Thus, given a complex line $\mathbb{C}v \subset V_{\mathbb{R}}$, the corresponding two dimensional real subspace of $V'_{\mathbb{R}}$ has v and iv as an *orthogonal* basis.

To clarify whether a complex number is acting as a scalar of $V_{\mathbb{R}}$ or a (non-scalar) endomorphism of $V'_{\mathbb{R}}$ we define the following notation, which we will be using where confusion seems likely, in particular, when dealing with two different complex structures later on.

Definition 3.1.4. *For an element $w \in \mathbb{C}\setminus\mathbb{R}$, we denote by $\hat{w}$, the endomorphism of the real space $V'_{\mathbb{R}}$ induced by the action of w as a scalar of the complex space $V_{\mathbb{R}}$.*

For typographic reasons, the endomorphism given by the operation of i on $V'_{\mathbb{R}}$ is denoted $\hat{\imath}$. Note that $\hat{\imath}$ can be identified with an element in $\mathrm{SO}(V)(\mathbb{R})$.

Proof. We show that $\hat{\imath} \in \mathrm{SO}(V)(\mathbb{R})$. Since $|i| = 1$, we have $\hat{\imath} \in \mathrm{O}(V)(\mathbb{R})$. Consider a basis $e_1,\dots,e_{q+1}$ of $V_{\mathbb{R}}$, $\langle\cdot,\cdot\rangle$. Then e_1, $\hat{\imath}e_1$, $\dots$, e_{q+1}, $\hat{\imath}e_{q+1}$ form basis of $V'_{\mathbb{R}}$, $(\cdot,\cdot)$, with $\big(e_j, \hat{\imath}e_j\big) = 0$, for $j = 1,\dots,q+1$. In this basis, the matrix representing the endomorphism $\hat{\imath}$ has block diagonal form with $q+1$ blocks of the form $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ each of which has determinant 1. $\square$

Parabolic subgroups

Next, we want to find out how parabolic subgroups behave under the embedding $\mathrm{SU}(V)(\mathbb{R}) \hookrightarrow \mathrm{SO}(V)(\mathbb{R})$. Somewhat more generally than in section 1.1.4, where on p. 27 we introduced the unitary Heisenberg group $H(\ell)$ for a fixed choice of cusp ℓ, we consider transformations

$\mathscr{T}(e,h)$ and $\mathscr{E}(e,t) \in \mathrm{SU}(V)(\mathbb{R})$. These will be interpreted in terms of parabolic elements in $\mathrm{SO}(V)(\mathbb{R})$.

Let u be a vector in $V_{\mathbb{R}}$ with $\langle u,u\rangle = 0$ and κ be a real number. Then, it is easily seen that the transformation

$$\mathscr{T}(u,\kappa) : v \mapsto v - \kappa \langle v,u\rangle\, iu$$

is contained in $\mathrm{SU}(V)(\mathbb{R})$. Note that the translation $[h,0]$ in the unitary Heisenberg group $H(\ell)$ is just such a transformation, for $u = \ell$ and $\kappa = h|\delta|$.

Now, recall definition 1.2.11 for the orthogonal Eichler elements in $\mathrm{O}^+(V)(\mathbb{R})$.

Proposition 3.1.5. *Given an isotropic vector $e \in V_{\mathbb{R}}$ and a real number $\kappa \in \mathbb{R}$, the transformation $\mathscr{T}(u,h) \in \mathrm{SU}(V)(\mathbb{R})$ is mapped to the Eichler element $E(u,\frac{h}{2}\hat{\imath}u) \in \mathrm{O}^+(V)(\mathbb{R})$ under the embedding $\mathrm{SU}(V)(\mathbb{R}) \hookrightarrow \mathrm{O}^+(V)(\mathbb{R})$.*

Proof. Since $u \perp \hat{\imath}u$, $E(u,\frac{h}{2}\hat{\imath}u)$ is an Eichler element as per definition 1.2.11. Because $\hat{\imath}u$ is isotropic, we have

$$\begin{aligned} E\big(u,\tfrac{h}{2}\hat{\imath}u\big)(v) &= v - \frac{1}{2}h(v,u)\hat{\imath}u + \frac{1}{2}(v,h\hat{\imath}u)u - 0 \\ &= v - h\Re\langle v,u\rangle \imath u + h\Im\langle v,u\rangle u = v - h\langle v,u\rangle\, iu = \mathscr{T}(u,h)(v). \end{aligned}$$

□

Given a vector $t \in V_{\mathbb{R}}$ with $\langle t,u\rangle = 0$, denote by $\mathscr{E}(u,t)$ the element of $\mathrm{SU}(V)(\mathbb{R})$ given by

$$\mathscr{E}(u,t) :\ v \mapsto v + \langle v,u\rangle\, t - \langle v,t\rangle\, u - \frac{1}{2}\langle v,u\rangle\langle t,t\rangle\, u.$$

Note that the unitary Eichler element $[0,t]$ in $H(\ell)$ can be written as $\mathscr{E}(\ell,t)$, with ℓ as usual a primitive isotropic lattice vector corresponding to a cusp of $\mathscr{H}_{\mathrm{U}}$ and $t \in D \otimes_{\mathscr{O}_{\mathbb{F}}} \mathbb{F}$, see p. 27.

Proposition 3.1.6. *Given a norm zero vector $u \in V_{\mathbb{R}}$ and $t \in V_{\mathbb{R}}$ with $\langle u,t\rangle = 0$, the transformation $\mathscr{E}(u,t) \in \mathrm{SU}(V)(\mathbb{R})$ is mapped to $E(u,\frac{1}{2}t) \circ E(\hat{\imath}u,\frac{1}{2}\hat{\imath}t) \in \mathrm{O}^+(V)(\mathbb{R})$.*

Proof. We need only show $\mathscr{E}(u,t)(v) = E(u,\frac{1}{2}t) \circ E(\hat{\imath}u,\frac{1}{2}\hat{\imath}t)(v)$ for any $v \in V_{\mathbb{R}}$. We have

$$\begin{aligned} \mathscr{E}(u,t)v &= v + \langle v,u\rangle\, t - \langle v,t\rangle\, u - \frac{1}{2}\langle v,u\rangle\langle t,t\rangle\, u \\ &= v + \big(\Re\langle v,u\rangle\, i\Im\langle v,u\rangle\big)t - \big(\Re\langle v,t\rangle + i\Im\langle v,t\rangle\big)u - \frac{1}{2}\big(\Re\langle v,u\rangle + i\Im\langle v,u\rangle\big)\langle t,t\rangle\, u. \end{aligned}$$

Now consider v as a vector in $V'_{\mathbb{R}}$ and interpret i as $\hat{\imath}$ operating on $V'_{\mathbb{R}}$, $(\cdot,\cdot)$. Then,

$$\begin{aligned} \mathscr{E}(u,t)v = v &+ \frac{1}{2}(v,u)t + \frac{1}{2}(v,\hat{\imath}u)\hat{\imath}t - \frac{1}{2}(v,t)u - \frac{1}{2}(v,\hat{\imath}t)\hat{\imath}u \\ &- \frac{1}{8}(v,u)(t,t)u - \frac{1}{8}(v,\hat{\imath}u)(\hat{\imath}t,\hat{\imath}t)\hat{\imath}u, \end{aligned}$$

since $i\Im\langle v_1,v_2\rangle = \Re\langle v_1,iv_2\rangle$ and both (t,t) and $(\hat{\imath}t,\hat{\imath}t)$ are equal to $\frac{1}{2}\langle t,t\rangle$. Thus,

$$\begin{aligned} \mathscr{E}(u,t)(v) &= v + \big(E(u,\tfrac{1}{2}t)(v) - v\big) + \big(E(\hat{\imath}u,\tfrac{\hat{\imath}}{2}t)(v) - v\big) \\ &= \Big(E(u,\tfrac{1}{2}t) \circ E(\hat{\imath}u,\tfrac{\hat{\imath}}{2}t)\Big)(v) \ = \Big(E(\hat{\imath}u,\tfrac{\hat{\imath}}{2}t) \circ E(u,\tfrac{1}{2}t)\Big)(v). \end{aligned}$$

Note that the Eichler elements occurring here commute, since t, u and $\hat{\imath}t$, $\hat{\imath}u$ span mutually perpendicular subspaces of $V'_{\mathbb{R}}$, recall (1.2.13) on p. 44. □

3.1.2 Choice of cusp and basis for the hyperbolic part

The construction of the symmetric domains $\mathscr{H}_{\mathrm{U}}$ and $\mathscr{H}_{\mathrm{O}}$ is made relative to a fixed choice of cusp. As sketched in the introduction to this chapter on p. 71, in order to explicitly describe an embedding between the two symmetric domains we must fix a choice of cusp for each. This is done by choosing primitive isotropic lattice vectors: ℓ, primitive and isotropic in L as an $\mathcal{O}_{\mathbb{F}}$-lattice, for $\mathscr{H}_{\mathrm{U}}$ and $e = e_1$, isotropic and primitive as an element of L as a quadratic $\mathbb{Z}$-module, for $\mathscr{H}_{\mathrm{O}}$.

As in section 1.1.3 we now choose ℓ, once and for all, and then, *after having chosen ℓ, we set $e_1 = \ell$.*

Let us recall the relationship between isotropic lattice vectors and boundary components. Since ℓ is *primitive* in the $\mathcal{O}_{\mathbb{F}}$-lattice L, it is not a multiple of any other vector in L by a non-unit in $\mathcal{O}_{\mathbb{F}}$. Then, $[\ell] \in P(V_{\mathbb{R}})$ represents a boundary component of $\mathrm{Gr}_{\mathrm{U}} \simeq \mathscr{K}_{\mathrm{U}}$ defined over $\mathbb{F}$, in other words, a cusp for $\mathrm{SU}(L)$.

But ℓ also defines a cusp of Gr_{O}: The fact that ℓ is primitive in L as an $\mathcal{O}_{\mathbb{F}}$ lattice it is, in particular, also primitive in L as a $\mathbb{Z}$-module, since $\mathbb{Z} \subset \mathcal{O}_{\mathbb{F}}$ (and $\mathbb{Z} \cap \mathcal{O}_{\mathbb{F}}^{\times} = \{\pm 1\}$). Since ℓ is isotropic, it thus defines a (rational) boundary point of the symmetric domain Gr_{O}.

In fact, ℓ defines a entire one-dimensional boundary component of Gr_{O}, in which this boundary point is contained: As the isotropic subspace $\mathbb{F}\ell$ of V is a two-dimensional and thus maximal totally isotropic $\mathbb{Q}$-subspace of V', $(\cdot,\cdot)$. It contains $\mathcal{O}_{\mathbb{F}}\ell$, which is a rank two $\mathbb{Z}$-submodule of L as a $\mathbb{Z}$-module. This defines a rational one-dimensional boundary component. We will examine the structure of this boundary component in more detail in section 3.3 below.

Recall from section 1.1 how the construction of the Siegel domain depends on ℓ and a further lattice vector ℓ' with $\langle \ell, \ell' \rangle \neq 0$, also, recall from section 1.2 how the construction of the tube domain $\mathscr{H}_{\mathrm{O}}$ besides e also requires a second vector e' with satisfying $(e, e') = 1$. Finally, how, for a lattice containing two isotropic vectors $e = e_1$ and e_3, refined coordinates can be introduced for $\mathscr{H}_{\mathrm{O}}$, with further vectors $e' = e_2$ and e_4 spanning a subspace complementary to e_1 and e_3.

As indicated in the introduction to this chapter, below, we will choose $e_3 \in \mathbb{F}\ell$ and then determine e_2 and e_4 from $\mathbb{F}\ell'$, in order to get a $\mathbb{Q}$-basis $e_1, \ldots, e_4$ for the hyperbolic part of V' as in definition 1.2.5, which is needed to give an explicit description of the embedding $\mathscr{H}_{\mathrm{U}} \hookrightarrow \mathscr{H}_{\mathrm{O}}$.

Stabilizer of the cusp

Having fixed the cusp, we can specialize the above propositions 3.1.5 and 3.1.6 to this particular choice and describe the image in $\mathrm{O}^+(V)$ of the parabolic elements from the unitary Heisenberg group $H(\ell)$.

Proposition 3.1.7. *Under the embedding $\mathrm{SU}(V)(\mathbb{R}) \hookrightarrow \mathrm{SO}(V)(\mathbb{R})$, the elements $[h, 0]$ and $[0, t]$ of the Heisenberg group $H(\ell) \subset \mathrm{SU}(V)(\mathbb{R})$ are mapped to transformations in $\mathrm{O}^+(V)(\mathbb{R})$, which can be expressed through Eichler elements in the following manner:*

$$\begin{aligned} [h, 0] &\longmapsto E\left(\ell, \tfrac{h}{2}\hat{\imath}\ell\right), \\ [0, t] &\longmapsto E\left(\ell, \tfrac{1}{2}t\right) \circ E\left(\hat{\imath}\ell, \tfrac{1}{2}\hat{\imath}t\right). \end{aligned}$$

Basis for the span of ℓ and ℓ'

For the following, as indicated in the introduction to this chapter, we assume ℓ' to be isotropic. The $\mathcal{O}_{\mathbb{F}}$-span of ℓ and ℓ' is a $\mathbb{Z}$-module of rank 4 and signature $(2,2)$, which, since ℓ' is isotropic, can be split into two perpendicular hyperbolic planes over $\mathbb{Z}$. We want to equip the corresponding subspace of V' with a basis $e_1 = \ell$, e_2, e_3, e_4, as in definition 1.2.5, where e_1, e_3 and e_2, e_4 span complementary maximal isotropic subspaces and further satisfy $(e_1, e_2) = (e_3, e_4) = 1$ and $(e_j, e_k) = 0$ for $j \le k$ and $(j,k) \notin \{(1,2),(3,4)\}$.

The two vectors e_1 and e_3 define the one-dimensional boundary component of the (refined) tube domain, into which we want to embed $\mathcal{H}_{\mathrm{U}}$. We set $e_1 = \ell$ and fix $e_3 \in \mathcal{O}_{\mathbb{F}}\ell$, then both e_1 and e_3 are contained in L. Next, we determine e_2 and e_4 from $\mathbb{F}\ell'$. Whether these two are contained in L' or not depends, of course, on L and on the choice of ℓ and ℓ'.

Remark 3.1.8. *The basis for the hyperbolic part of $V'_{\mathbb{R}}$ we thus determine consists of complex multiples of the basis vectors ℓ and ℓ' for the hyperbolic part of $V_{\mathbb{R}}$. This facilitates considerably rewriting $z \in V_{\mathbb{R}}$ as a vector with only real coordinates.*

The reason why we fix $\ell = e_1$ and e_3 and then determine the other vectors accordingly lies in the fact the Siegel domain coordinate τ is the ℓ-component of z, while the ℓ' component is fixed through the normalization $\langle z, \ell\rangle = \langle \ell', \ell\rangle$, recall (1.1.2).

Since we are considering the $\mathcal{O}_{\mathbb{F}}$-module L as a module over $\mathbb{Z}$, it becomes necessary to differentiate whether the number field $\mathbb{F}$ has even or odd discriminant. We can avoid some of the resulting inconveniences through the following notational convention: Write $\mathcal{O}_{\mathbb{F}}$ in the form $\mathbb{Z} + \zeta\mathbb{Z}$, where if $\mathbb{F}$ has even discriminant, ζ is given by $\sqrt{d} = \frac{1}{2}\delta$, whereas if $\mathbb{F}$ has odd discriminant, we have $\zeta = \frac{1}{2}(1+\delta)$. Note that $\Im\zeta = \frac{1}{2}|\delta|$ holds, either way, irrespective of whether $D_{\mathbb{F}}$ is odd or even. The line $\mathbb{Z}\hat{\zeta}\ell$ is isotropic and perpendicular to $\mathbb{Z}\ell$. Thus, with $e_1 = \ell$ we set $e_3 = -\hat{\zeta}\ell$; together, these two vectors span a maximal totally isotropic $\mathbb{Q}$-subspace of V', as required. See remark 3.2.6 below, concerning the choice of sign.

Now, to determine e_2 and e_4, as $\mathbb{F}$-multiples of ℓ', put $e_2 = \hat{\gamma}\ell'$ and $e_4 = \hat{\xi}\ell'$. The factors γ and ξ can be recovered from the conditions $(e_1, e_2) = (e_3, e_4) = 1$ and $(e_1, e_4) = (e_3, e_2) = 0$. Thus, for γ, we have

$$\begin{aligned}1 &= (\ell, \hat{\gamma}\ell') = 2\Re\left(\langle \ell, \ell'\rangle \bar{\gamma}\right),\\ 0 &= (-\zeta\ell, \gamma\ell') = -2\Re\left(\langle \ell, \ell'\rangle \zeta\bar{\gamma}\right) = -\Re\zeta \cdot 2\Re\left(\langle \ell, \ell'\rangle \bar{\gamma}\right) + |\delta| \cdot \Im\left(\langle \ell, \ell'\rangle \bar{\gamma}\right).\end{aligned}$$

Hence,

$$\gamma \langle \ell', \ell\rangle = \frac{1}{2} - i\frac{\Re\zeta}{2|\delta|} = \frac{\zeta}{\delta} = \begin{cases} \frac{1}{2} & \text{if } D_{\mathbb{F}} \equiv 0 \mod 2 \\ \frac{1}{2}(1+\delta^{-1}) & \text{if } D_{\mathbb{F}} \equiv 1 \mod 2. \end{cases}$$

Thus,

$$\gamma = \frac{\zeta}{\delta \langle \ell, \ell'\rangle}.$$

Similarly, ξ is determined from the conditions

$$0 = \Re\left(\langle \ell, \ell'\rangle \bar{\xi}\right) \quad \text{and} \quad 1 = \frac{1}{2}\Re\zeta \cdot \Re\left(\langle \ell, \ell'\rangle \bar{\xi}\right) + |\delta| \cdot \Im\left(\langle \ell, \ell'\rangle \bar{\xi}\right),$$

which gives $\xi = \langle \ell', \ell \rangle^{-1} \delta^{-1}$.

We can now write out $e_1, \ldots, e_4$ and summarize our considerations in the following proposition.

Proposition 3.1.9. *Let $e_1, \ldots, e_4 \in V$ be the vectors given by*

$$e_1 = \ell,\ e_2 = \frac{\zeta}{\delta \langle \ell', \ell \rangle}\ell',\ e_3 = -\zeta\ell,\ e_4 = \frac{1}{\delta \langle \ell', \ell \rangle}\ell'. \tag{3.1.1}$$

Then, the following statements hold:

1. *The vectors $e_1, \ldots, e_4$ form a basis of a signature $(2,2)$ quadratic subspace of the rational quadratic space V', $(\cdot,\cdot)$. The orthogonal complement of this space in V' is the definite subspace $D \otimes_{\mathbb{Z}} \mathbb{Q}$, with D the definite lattice introduced in* (1.1.1) *on p. 24, in the complement of ℓ and ℓ' with respect to $\langle \cdot, \cdot \rangle$.*

2. *e_1 and e_3 are contained in L, while e_2 and e_4 are contained in L'.*

3. *The pairs of vectors e_1, e_2 and e_3, e_4 span perpendicular hyperbolic planes.*

   Thus, $(e_1, e_2) = (e_3, e_4) = 1$, $q(e_i) = 0$ for $i = 1, \ldots, 4$ and $(e_j, e_k) = 0$ for $1 \leq j \leq k \leq 4$, with $(j,k) \neq (1,2), (3,4)$.

   (Without the assumption that ℓ' is isotropic, $q(e_2)$ and $q(e_4)$ may be non-zero and also, we may have $(e_2, e_4) \neq 0$, compare remark 3.2.10 below.)

Proof. That e_i, $i = 1, \ldots, 4$ form a basis for the hyperbolic part of V' is clear. The second property follows by construction, also clearly $e_1 = \ell$ and $e_3 = -\zeta\ell$ are contained in L. Further, e_2 and $e_4 \in L'$, since we have

$$\left\langle \frac{\zeta}{\delta \langle \ell', \ell \rangle}\ell', \ell \right\rangle = \frac{\zeta}{\delta} \in \mathscr{D}_{\mathbb{F}}^{-1} \qquad \text{and} \qquad \left\langle \frac{1}{\delta \langle \ell', \ell \rangle}\ell', \ell \right\rangle = \frac{1}{\delta} \in \mathscr{D}_{\mathbb{F}}^{-1}.$$

Finally, as ℓ' is isotropic and in the complement of the definite sublattice D, i.e. $\langle \ell', D \rangle = 0$, it follows that $\langle e_2, L \rangle \in \mathscr{D}_{\mathbb{F}}^{-1}$ and $\langle e_4, L \rangle \in \mathscr{D}_{\mathbb{F}}^{-1}$. Thus, e_2 and e_4 are contained in L' by definition 1.1.6. □

Example 3.1.10. *Consider a lattice of the form $L = \mathscr{O}_{\mathbb{F}} \oplus \mathscr{D}_{\mathbb{F}}^{-1} \oplus D$ with a definite $\mathscr{O}_{\mathbb{F}}$-lattice D of rank $q-1$. Let $\langle \cdot, \cdot \rangle$, when restricted to the hyperbolic plane $\mathscr{O}_{\mathbb{F}} \oplus \mathscr{D}_{\mathbb{F}}^{-1}$, be given by $\langle x, y \rangle = x_1\bar{y}_2 + x_2\bar{y}_1$, as usual. Choose ℓ and ℓ' with $\langle \ell, \ell' \rangle = \delta^{-1}$. For the basis vectors $e_1, \ldots, e_4$, we get*

$$e_1 = \ell, e_2 = -\zeta\ell', e_3 = -\zeta\ell, \quad \textit{and} \quad e_4 = -\ell',$$

with $\zeta = \frac{1}{2}\delta$ or $\zeta = \frac{1}{2}(1+\delta)$ depending on whether $\mathbb{F}$ has even or odd discriminant. All these vectors are isotropic and, since L is unimodular, contained in L.

3.2 Complex structures and symmetric domains

The embedding $\mathrm{SU}(V)(\mathbb{R}) \hookrightarrow \mathrm{SO}(V)(\mathbb{R})$ naturally induces an embedding of the respective symmetric domains, given by

$$\begin{array}{ccc} \mathrm{SU}(V)(\mathbb{R})/\mathscr{C}_{\mathrm{SU}} & & \mathrm{SO}(V)(\mathbb{R})/\mathscr{C}_{\mathrm{SO}} \\ \simeq \mathrm{SU}(1,q)/\mathrm{S}\big(\mathrm{U}(1)\times\mathrm{U}(q)\big) & \hookrightarrow & \simeq \mathrm{SO}(2,2q)/\big(\mathrm{SO}(2)\times\mathrm{SO}(2q)\big), \end{array}$$

where $\mathscr{C}_{\mathrm{SU}}$ denotes a maximal compact subgroup of $\mathrm{SU}(V)(\mathbb{R})$ and $\mathscr{C}_{\mathrm{SO}}$ a maximal path-connected compact subgroup of $\mathrm{SO}(V)(\mathbb{R})$. Clearly, $\mathscr{C}_{\mathrm{SU}}$ is a compact subgroup of $\mathscr{C}_{\mathrm{SO}}$, recall remark 3.1.1.

The embedding of the symmetric domains induces an embedding of the Grassmannian models, $\mathrm{Gr}_{\mathrm{U}} \hookrightarrow \mathrm{Gr}_{\mathrm{O}}$. This embedding can be described as follows: A point in Gr_{U} is a one-dimensional positive subspace of the complex hermitian space $V_{\mathbb{R}}$, which, as a real subspace v of $V'_{\mathbb{R}}$ is two-dimensional and also positive with respect to the bilinear form $(\cdot,\cdot)$. Write the positive one-dimensional subspace of $V_{\mathbb{R}}$ as $\mathbb{C}z$, with $z \in V_{\mathbb{R}}$, $\langle z,z\rangle > 0$. Then, the underlying real subspace $v \in V'_{\mathbb{R}}$ is spanned by z and $\hat{\imath}z$, for example. Since $q(z) = q(\hat{\imath}z) = \langle z,z\rangle > 0$, it is clear that v is positive. So, indeed v can be associated with a point in Gr_{O}, denoted $\alpha(v)$. However, in the Grassmannian Gr_{O} of *oriented* positive two-dimensional subspace, $\alpha(v)$ is not uniquely determined, since the two dimensional real space underlying $v \in \mathrm{Gr}_{\mathrm{U}}$ does not come with prescribed orientation.

Thus, we can embed two copies of $\mathscr{K}_{\mathrm{U}}$ into the oriented Grassmannian Gr_{O}, by the two different possible choices of spin-orientation on the set of two-dimensional positive subspaces of $V'_{\mathbb{R}}$.

Recall how this ambiguity also occurs in the construction of the positive cone $\mathscr{K}_{\mathrm{O}}$ and the tube domain $\mathscr{H}_{\mathrm{O}}$ and is resolved by choosing one connected component $\mathscr{K}_{\mathrm{O}}^{+}$ of the positive cone. The two choices for $\alpha(v)$ correspond to two complex conjugate lines in $\mathscr{K}_{\mathrm{O}}$.

For an embedding between the 'generalized upper half-plane' models $\mathscr{H}_{\mathrm{U}}$ and $\mathscr{H}_{\mathrm{O}}$ this freedom of choice means that we can construct the embedding to be either holomorphic or antiholomorphic, see below.

The embedding between the Grassmannians induces an embedding of the positive cones $\mathscr{K}_{\mathrm{U}}$ and $\mathscr{K}_{\mathrm{O}}$, also between their affine models and further on between $\mathscr{H}_{\mathrm{U}}$ and $\mathscr{H}_{\mathrm{O}}$, running through all isomorphisms involved in the construction of the respective models. To summarize, we have induced maps at every level of the following diagram

$$\begin{array}{ccc} \mathrm{Gr}_{\mathrm{U}} & \hookrightarrow & \mathrm{Gr}_{\mathrm{O}} \ni v \\ \updownarrow & \nearrow & \updownarrow \\ [z] \in \mathscr{K}_{\mathrm{U}} & \overset{\alpha}{\hookrightarrow} & \mathscr{K}_{\mathrm{O}} \ni [Z_L] \\ \updownarrow & & \updownarrow \\ (\tau,\sigma) \in \mathscr{H}_{\mathrm{U}} & \hookrightarrow & \mathscr{H}_{\mathrm{O}} \ni Z. \end{array} \tag{3.2.1}$$

To avoid further complicating notation, we will not introduces separate designations for all of these maps; rather we will always, either, by abuse of notation, denote them α, or call them induced from α. It will always be clear from the context, which map is meant.

Our aim is to construct the embeddings induced by α explicitly. In doing so, the following points require our attention:

Complex structure: The map from $\mathrm{Gr_U}$ to $\mathrm{Gr_O}$ is only real analytic, as is the isomorphism between $\mathrm{Gr_O}$ and $\mathscr{K}_\mathrm{O}$. The complex structure on $\mathscr{K}_\mathrm{O}$ is induced from the choice of orientation on $\mathrm{Gr_O}$. In contrast, $\mathrm{Gr_U}$ and $\mathscr{K}_\mathrm{U}$ carry a complex structure coming naturally from the complex structure of the complex hermitian space $V_\mathbb{R}, \langle\cdot,\cdot\rangle$.

Our aim in constructing the embedding, is that α be holomorphic and thus compatible with both complex structures, rather than being merely real-analytic.

Normalizations: In order for the above diagram to commute, we must also make sure to conform to all the normalizations involved in the constructions of $\mathscr{K}_\mathrm{O}$, $\widetilde{\mathscr{K}_{\mathrm{O},1}^+}$ and $\mathscr{H}_\mathrm{O}$, carried out with respect to a fixed choice of cusp $e_1 = \ell$ and of e_2.

The starting point most conductive to our considerations is the affine set $\widetilde{\mathscr{K}_\mathrm{U}^1}$ of representatives for $\mathscr{K}_\mathrm{U}$. We will consider $\mathbb{R}$-linear maps from this set to $\mathrm{Gr_O}$ and to the isomorphic complex cone $\mathscr{K}_\mathrm{O}$.

Remark 3.2.1. *As the $\mathbb{R}$-bilinear form $(\cdot,\cdot)$ is extended to a $\mathbb{C}$-bilinear form on $V_\mathbb{C}$, it should be clear that it does not become bilinear for scalars of $V_\mathbb{R}$, $\langle\cdot,\cdot\rangle$. Neither, when $(\cdot,\cdot)$ is rewritten as $2\Re\langle\cdot,\cdot\rangle$, does the sesquilinearity of $\langle\cdot,\cdot\rangle$ apply to scalars of $V_\mathbb{C}$. We will use the notation introduced in definition 3.1.4 to distinguish between complex numbers acting as scalars on $V_\mathbb{C}$ and as endomorphisms of $V'_\mathbb{R}$ and $V_\mathbb{C}$, particularly, in situations, where both occur in one expression.*

We give some examples showing how to deal with such cases. Let $a, b, c \in \mathbb{R}$, v_i, v_j, v_k vectors in $V'_\mathbb{R}$ and μ, ξ, ρ complex numbers from $\mathbb{C}^\times$:

$$\begin{aligned}\left(\xi e_i, \xi e_j\right) &= |\xi|^2(e_i, e_j),\\ \left(a\hat{\xi}e_i, b\hat{\mu}e_j + i\rho e_k\right) &= \left(a\hat{\xi}e_i, b\hat{\mu}e_j\right) + i\left(a\hat{\xi}e_i, \hat{\rho}e_k\right)\\ &= 2ab\Re\left(\xi\bar{\mu}\langle v_i, v_j\rangle\right) + i2a\Re\left(\xi\bar{\rho}\langle e_i, e_k\rangle\right),\\ \left((a+\hat{\imath}b)e_i, e_j\right) &= 2a\Re\langle e_i, e_j\rangle - 2a\Im\langle e_i, e_j\rangle.\end{aligned}$$

3.2.1 Constructing the embedding

Recall from section 1.2.3, how in the definition of the tube domain, each two-dimensional positive subspace v corresponding to a point of $\mathrm{Gr_O}$ is fitted with a basis X_L, Y_L satisfying $X_L \perp Y_L$, $X_L^2 = Y_L^2$, and normalized so as to have $\left(X_L, e\right) = 1$ and $\left(Y_L, e\right) = 0$. Then, the point $v = \mathbb{R}X_L + \mathbb{R}Y_L$ in $\mathrm{Gr_O}$ corresponds to the line in $V_\mathbb{C}$ spanned by $Z_L = X_L + iY_L \in \widetilde{\mathscr{K}_{\mathrm{O},1}^+}$.

Now, consider $z = \ell' - \delta\langle\ell', \ell\rangle\tau\ell + \sigma \in \widetilde{\mathscr{K}_\mathrm{U}^1}$. We want to map z to an element $Z_L \in \widetilde{\mathscr{K}_{\mathrm{O},1}^+}$, or equivalently, to two vectors X_L and Y_L in $V'_\mathbb{R}$, with $X_L + iY_L \in \widetilde{\mathscr{K}_{\mathrm{O},1}^+}$.

If we consider the line $\mathbb{C}z = [z] \subset V_\mathbb{R}$ as a point in the Grassmannian $\mathrm{Gr_U}$, we can associate to $[z]$ a positive two-dimensional subspace v of the quadratic space $V'_\mathbb{R}$, the two possible orientations of which give two points in the Grassmannian. For this real subspace v underlying $\mathbb{C}z$, we determine two basis vectors, denoted X_L and Y_L, conforming to the necessary normalizations, so that either $X_L + iY_L$ or $X_L - iY_L$ lies in $\widetilde{\mathscr{K}_{\mathrm{O},1}^+}$, giving $Z_L(z)$. Since X_L and Y_L are multiples of z by complex scalars of $V_\mathbb{R}$, $\langle\cdot,\cdot\rangle$, we consider z as a vector in $V'_\mathbb{R}$, with only real coordinates, and set

$$X_L = \hat{\psi}z, \quad Y_L = \hat{\xi}z, \quad \text{with} \quad \psi, \xi \in \mathbb{C}^\times, \quad \text{considered as endomorphisms of } V'_\mathbb{R}.$$

These complex factors, ψ and ξ, can be determined from the conditions which X_L and Y_L have to meet. First of all X_L and Y_L have to be perpendicular and of equal (positive) norm. Now,

$$X_L^2 = Y_L^2 \quad \text{implies} \quad |\psi|^2 = |\xi|^2, \tag{3.2.2}$$
$$X_L \perp Y_L \quad \text{implies} \quad \psi\bar{\xi} \in i\mathbb{R}. \tag{3.2.3}$$

The second implication follows from $\big(X_L, Y_L\big) = 2\,\langle z, z\rangle\, \Re(\psi\bar{\xi})$, since the norm of z is real and positive.

This condition assures that $[Z_L]$, as the image of $[z] \in \mathscr{K}_{\mathrm{U}}$ is contained in the positive cone $\mathscr{K}_{\mathrm{O}}$, defined by $\big(Z_L, \bar{Z}_L\big) > 0$, in the zero quadric $\mathscr{N}$ given by $\big(Z_L, Z_L\big) = 0$.

Further, see p. 39, X_L and Y_L are normalized with respect to the cusp e so as to satisfy

$$\big(X_L, e\big) = 1, \quad \big(Y_L, e\big) = 0. \tag{3.2.4}$$

For Z_L contained in the connected component $\widetilde{\mathscr{K}_{\mathrm{O}}^{+}}$, these condition assure that Z_L lies in the set $\widetilde{\mathscr{K}_{\mathrm{O},1}^{+}}$, of normalized representatives from $\mathscr{K}_{\mathrm{O}}^{+}$, uniquely corresponding to points in the tube domain $\mathscr{H}_{\mathrm{O}}$. Then, $\bar{Z}_L$ is contained in $\widetilde{\mathscr{K}_{\mathrm{O}}} \setminus \widetilde{\mathscr{K}_{\mathrm{O}}^{+}}$, and lies over the generalized 'lower half-plane' $\overline{\mathscr{H}}_{\mathrm{O}}$,

From (3.2.4), we get

$$0 = (\xi z, \ell) = 2\Re\big(\xi\,\langle \ell', \ell\rangle\big) \quad \text{so,} \quad \xi = \frac{ia}{2\,\langle \ell', \ell\rangle}, \quad \text{with } a \in \mathbb{R},$$
$$1 = \big(\psi z, \ell\big) = 2\Re\big(\psi\,\langle \ell', \ell\rangle\big) \quad \text{so,} \quad \psi = \frac{1 + ib}{2\,\langle \ell', \ell\rangle}, \quad \text{with } b \in \mathbb{R}.$$

Since $\psi\bar{\xi} = (-ia + ab)\big(4|\langle \ell', \ell\rangle|^2\big)^{-1}$ by (3.2.3) it follows that $b = 0$. With (3.2.2), $|a|^2 = 1$ and one obtains

$$X_L = \widehat{\left(\frac{1}{2\,\langle \ell', \ell\rangle}\right)} z \quad \text{and} \quad Y_L = \widehat{\left(\frac{\pm i}{2\,\langle \ell', \ell\rangle}\right)} z. \tag{3.2.5}$$

The image of z is now determined up to the sign of Y_L, the choice of which determines the orientation for the subspace $\mathbb{R}X_L + \mathbb{R}Y_L$ in Gr_{O} or equivalently whether we map z to a point $Z_L \in \widetilde{\mathscr{K}_{\mathrm{O}}^{+}}$ or to its conjugate $\bar{Z}_L$.

Complex structure

Of the two possible maps $z \mapsto (X_L, \pm Y_L)$ we want to choose that which is compatible with both the complex structure of the hermitian space $V_{\mathbb{R}}$, $\langle\cdot,\cdot\rangle$ and that of the complexified quadratic space $V_{\mathbb{C}}$, $(\cdot,\cdot)$.

This can be assured if the complex unit with respect to either complex structure commutes with the embedding in the sense that the image of iz equals i times the image of z, in other words $\alpha(\hat{\imath}z) = i\alpha(z)$. Then, by $\mathbb{R}$-linearity the same follows for any complex scalar, $\alpha(\hat{\mu}z) = \hat{\mu}\alpha(z)$ for any $\mu \in \mathbb{C}$.

This condition can be expressed through the commutativity of the following diagram

$$\begin{array}{ccc} z & \stackrel{\alpha}{\longmapsto} & Z_L = \quad X_L + iY_L \\ \downarrow i & & \downarrow i \\ iz & \stackrel{\alpha}{\longmapsto} & iZ_L = -Y_L + iX_L, \end{array} \tag{3.2.6}$$

where on the left side i acts as a complex scalar on the hermitian space $V_{\mathbb{R}}$ while on the right side i is the complex unit of the complexified quadratic space $V_{\mathbb{C}}$.

The diagram commutes if $Y_L(iz) = X_L(z)$ and $X_L(iz) = -Y_L(z)$. Thus, if $X_L = \hat{\psi}z$ and $Y_L = \hat{\xi}z$, we must have

$$i\psi = -\xi.$$

With (3.2.5) above, we conclude that $\xi = -i\big(2\left\langle \ell', \ell \right\rangle\big)^{-1}$ and get

$$X_L = \widehat{\left(\frac{1}{2\left\langle \ell', \ell \right\rangle}\right)} z \quad \text{and} \quad Y_L = (-\hat{\imath}) \widehat{\left(\frac{1}{2\left\langle \ell', \ell \right\rangle}\right)} z. \tag{3.2.7}$$

The image of z in $\widetilde{\mathscr{K}^+_{\mathrm{O},1}}$ is then given by the following expression

$$Z_L = X_L + iY_L = \widehat{\left(\frac{1}{2\left\langle \ell', \ell \right\rangle}\right)} z + i \widehat{\left(\frac{-\imath}{2\left\langle \ell', \ell \right\rangle}\right)} z, \tag{3.2.8}$$

in which the complex factors in parenthesis must be interpreted as elements of $\mathrm{End}_{\mathbb{R}}(V_{\mathbb{R}})$, as denoted by the use of $\widehat{\ }$, while the complex unit i preceding Y_L is a scalar of the complexified quadratic space $V_{\mathbb{C}}$.

Proposition 3.2.2. *The map $\tilde{\alpha}$ from $\mathbb{C}^{1+q}$ to $\mathbb{C}^{2+2q}$ given by $z \mapsto Z_L$, with Z_L given by* (3.2.8) *is holomorphic. In particular, the embedding $\alpha : \widetilde{\mathscr{K}^1_{\mathrm{U}}} \to \widetilde{\mathscr{K}^+_{\mathrm{O},1}}$ is holomorphic.*

Proof. We will show that the coordinates of $\mathbb{C}^{2+2q} \simeq V_{\mathbb{C}}$ depend holomorphically on those of $\mathbb{C}^{1+q}$, isomorphic to the complex space $V_{\mathbb{R}}$, $\langle\cdot,\cdot\rangle$. We can write $\tilde{\alpha}$ as the composition of multiplication with the complex scalar $2\left\langle \ell, \ell' \right\rangle^{-1}$ on $\mathbb{C}^{1+q}$ and a map β sending $z \in \mathbb{C}^{1+q}$ to $z - i(\hat{\imath}z) \in \mathbb{C}^{2+2n}$. It suffices to consider the latter map.

Equip $\mathbb{C}^{1+q}$ with a basis $e_1, \ldots, e_{1+q}$ and the underlying real space $\mathbb{R}^{2+2q}$ with the basis

$$b_1 = e_1, \ldots, b_{q+1} = e_{1+q}, b_{q+2} = \hat{\imath}e_1, \ldots, b_{2+2q} = \hat{\imath}e_{1+q}.$$

We write $z \in \mathbb{C}^{1+q}$ in the form $z = \sum_{i=1}^{1+q} z_i e_i = \sum_{i=1}^{1+q}(u_i + iv_i)e_i$, with $u_i,\ v_i \in \mathbb{R}$ for $i = 1, \ldots, q+1$.

The image of β is a vector $Z \in \mathbb{C}^{2+2q}$ with real and imaginary part X and $Y \in \mathbb{R}^{2+2q}$, given by

$$X = \sum_{j=1}^{2+2q} X_j b_j = \sum_{j=1}^{1+q} u_j b_j + \sum_{j=q+2}^{2q+2} v_{j-q-1} b_j,$$

$$Y = \sum_{j=1}^{2+2q} Y_j b_j = \sum_{j=1}^{1+q} v_j b_j + \sum_{j=q+2}^{2q+2} (-u_{j-q-1}) b_j.$$

We now verify that the X_j and Y_j, $j = 1, \dots, 2+2q$, fulfill the Cauchy-Riemann equations in their functional dependence on the u_i and v_i, $i = 1, \dots, 1+q$. We have

$$\frac{\partial X_j}{\partial u_i} = \begin{cases} \delta_{i,j} & \text{if} \quad j \leq q+1 \\ 0 & \text{if} \quad j > q+1, \end{cases} \qquad \frac{\partial X_j}{\partial v_i} = \begin{cases} 0 & \text{if} \quad j \leq q+1 \\ \delta_{i,j-q-1} & \text{if} \quad j > q+1, \end{cases}$$
$$\frac{\partial Y_j}{\partial u_i} = \begin{cases} 0 & \text{if} \quad j \leq q+1 \\ -\delta_{i,j-q-1} & \text{if} \quad j > q+1, \end{cases} \qquad \frac{\partial Y_j}{\partial v_i} = \begin{cases} \delta_{i,j} & \text{if} \quad j \leq q+1 \\ 0 & \text{if} \quad j > q+1, \end{cases}$$

where $\delta_{k,l}$ is the usual Kroencker-delta, defined as $\delta_{k,l} = 1$ if $k = l$ and 0 otherwise. Thus,

$$\frac{\partial Y_j}{\partial v_i} = \frac{\partial X_j}{\partial u_i}, \quad \text{and} \quad \frac{\partial Y_j}{\partial u_i} = -\frac{\partial X_j}{\partial v_i}, \quad \forall i, j,$$

as claimed. This completes the proof. □

From this proposition we immediately get the following.

Proposition 3.2.3. *The pullback $\alpha^* F$ of a holomorphic function $F : V_{\mathbb{C}} \to \mathbb{C}$ is holomorphic on the complex space $V_{\mathbb{R}}$, $\langle \cdot, \cdot \rangle$.*

Proof. The claim is a direct consequence of the previous proposition 3.2.2, as the pullback is just the composite of α with F restricted to the image of α. □

Some further properties of functions, besides that of being holomorphic are preserved, when pulling back under α, as well.

Remark 3.2.4. *Let $F : V_{\mathbb{C}} \to \mathbb{C}$ be homogeneous of degree d, such that $F(\lambda Z) = \lambda^d F(Z)$ for any $\lambda \in \mathbb{C}^\times$. Then, the pullback $\alpha^* F$ is also homogeneous of the same degree. Similarly, a $\mathbb{C}$-linear function has $\mathbb{C}$-linear pull-back under α.*

Proof. Both statements follow from proposition 3.2.2 combined with the $\mathbb{R}$-linearity of α. They can also be verified directly using $\mathbb{R}$-linearity and the commutativity of diagram 3.2.6. □

Remark 3.2.5. *By choosing $i\psi = -\xi$, we have made α into a $\mathbb{C}$-linear embedding of complex spaces from $V_{\mathbb{R}} \to V_{\mathbb{C}}$. Reversing the sign, by setting $i\psi = +\xi$ sends iz to $(X_L, -Y_L) = \bar{Z}_L$ giving a conjugate linear (and antiholomorphic) embedding.*

Embedding of symmetric domains and choice of cone

Recall the definition of the tube domain in section 1.2.3 and in particular the refined coordinates introduced depending on a basis as in definition 1.2.5 for the hyperbolic part of $L \otimes_{\mathbb{Z}} \mathbb{Q}$.

$$Z = (Z_1, Z_2, \mathfrak{z}) = Z_1 e_3 + Z_2 e_4 + \mathfrak{z},$$

with $\mathfrak{z}$ is from the negative definite space $D \otimes_{\mathbb{Z}} \mathbb{C}$. One can write $\mathfrak{z}$ as $\mathfrak{x} + i\mathfrak{y}$, where $\mathfrak{x}$ and $\mathfrak{y}$ are the projections of X_L and Y_L to $D \otimes_{\mathbb{Z}} \mathbb{R}$. By equation (3.2.8), we have

$$\mathfrak{z} = \widehat{\left(\frac{1}{2 \langle \ell', \ell \rangle} \right)} \sigma + i \widehat{\left(\frac{-i}{2 \langle \ell', \ell \rangle} \right)} \sigma. \tag{3.2.9}$$

Of course, if σ is given explicitly with respect to some basis of $D \otimes_{\mathcal{O}_{\mathbb{F}}} \mathbb{C}$, then $\mathfrak{z}$ can easily be calculated in explicit form, as well, assuming a suitable choice of basis for $D \otimes_{\mathbb{Z}} \mathbb{R}$, see example 3.2.7 below.

We proceed by calculating the image of the hyperbolic part of z, that is, the hyperbolic part of X_L and Y_L in $V'_{\mathbb{R}}$ and of Z_L in $V_{\mathbb{C}}$.

For this, we use the basis vectors $e_1, \dots, e_4$ for the hyperbolic part from the $\mathbb{F}$-span of ℓ and ℓ' given in (3.1.1). As before, we write the ring of integers $\mathcal{O}_{\mathbb{F}}$ of $\mathbb{F}$ in the form $\mathbb{Z} + \zeta\mathbb{Z}$, in order to deal with both the case when $D_{\mathbb{F}}$ is odd and when $D_{\mathbb{F}}$ is even.

In the following calculation, all complex scalars come from the complex structure of $V_{\mathbb{R}}$, $\langle\cdot,\cdot\rangle$. Since there is no possibility of confusion, no special notation is used for this. We use the basis elements given by (3.1.1). Bear in mind that $\Im\zeta = \frac{1}{2}|\delta|$, while $\Re\zeta$ is either 0 or $\frac{1}{2}$ depending on the discriminant $D_{\mathbb{F}}$. For X_L, we have

$$X_L(z) = \frac{1}{2\langle \ell', \ell\rangle} z = -\frac{\tau\delta}{2}\ell + \frac{1}{2\langle \ell', \ell\rangle}\ell' + \mathfrak{x}(\sigma).$$

Writing this as a vector in $V'_{\mathbb{R}}$, we get, if $D_{\mathbb{F}}$ is odd

$$\frac{1}{2\langle \ell', \ell\rangle}(1 + \delta^{-1} - \delta^{-1})\ell' - \Re\tau\left(\frac{\delta}{2} + \frac{1}{2}\right)\ell + \Re\tau\frac{1}{2}\ell + \Im\tau\frac{|\delta|}{2}\ell + \mathfrak{x}(\sigma).$$

Similarly, if $D_{\mathbb{F}}$ is even, we write

$$\frac{1}{2\langle \ell', \ell\rangle}\ell' + 0 - \Re\tau\left(\frac{\delta}{2} + 0\right)\ell + 0 + \Im\tau\frac{|\delta|}{2}\ell + \mathfrak{x}(\sigma).$$

Either way, we get

$$\begin{aligned} X_L(z) &= \frac{\zeta}{\delta\langle \ell', \ell\rangle}\ell' - \Re\zeta\frac{1}{\delta\langle \ell', \ell\rangle}\ell' + \Re\tau(-\zeta\ell) + \left(\Re\zeta\cdot\Re\tau + \frac{1}{2}\Im\tau|\delta|\right)\ell + \mathfrak{x}(\sigma) \\ &= e_2 - \Re\zeta e_4 + \Re\tau e_3 + \left(\Re\zeta\cdot\Re\tau + \Im\zeta\cdot\Im\tau\right)e_1 + \mathfrak{x}(\sigma). \end{aligned} \tag{3.2.10}$$

While for Y_L we have

$$\begin{aligned} Y_L(z) &= \frac{-i}{2\langle \ell', \ell\rangle} z = \frac{|\delta|}{2\delta\langle \ell', \ell\rangle}\ell' - \Re\tau\frac{|\delta|}{2}\ell - \Im\tau\frac{\delta}{2}\ell + \mathfrak{y}(\sigma) \\ &= \frac{|\delta|}{2}\frac{1}{\delta\langle \ell', \ell\rangle}\ell' - \Re\tau\frac{|\delta|}{2}\ell - \Im\tau\left(\frac{\delta}{2} + \Re\zeta\right)\ell + \Im\tau\cdot\Re\zeta\ell + \mathfrak{y}(\sigma) \\ &= \frac{|\delta|}{2}e_4 - \Im\tau\left[\frac{1}{2}\delta + \Re\zeta\right]\ell - \left(\Re\zeta\cdot\Im\tau - \Re\tau\frac{|\delta|}{2}\right)\ell + \mathfrak{y}(\sigma) \\ &= \Im\zeta e_4 + \Im\tau e_3 - \left(\Re\zeta\cdot\Im\tau - \Im\zeta\cdot\Re\tau\right)e_1 + \mathfrak{y}(\sigma). \end{aligned} \tag{3.2.11}$$

In the end results for X_L and Y_L, only real coordinates occur, since all complex scalars acting as endomorphisms of the underlying real space have absorbed into the basis vectors $e_1, \dots, e_4$ given by (3.1.1). The image of z in $\widetilde{\mathcal{K}^{+}_{0,1}}$ can now be calculated easily, too. Note that here, all complex coordinates occurring can be considered as scalars of $V_{\mathbb{C}}$:

$$Z_L(z) = \bar{\zeta}\tau e_1 + e_2 + \tau e_3 - \bar{\zeta} e_4 + \mathfrak{z}(\sigma). \tag{3.2.12}$$

The corresponding element Z in the tube domain is given by

$$Z = \tau e_3 - \bar{\zeta} e_4 + \mathfrak{z}(\sigma) = \begin{cases} \tau e_3 + \frac{1}{2}\delta e_4 + \mathfrak{z}(\sigma) & \text{if} \quad D_{\mathbb{F}} \equiv 0 \mod 2 \\ \tau e_3 - \bar{\omega} e_4 + \mathfrak{z}(\sigma) & \text{if} \quad D_{\mathbb{F}} \equiv 1 \mod 2, \end{cases} \tag{3.2.13}$$

where ω denotes $\frac{1}{2}(1+\delta)$.

Remark 3.2.6. *The reason for the particular choice of sign for the basis vectors e_3 and e_4 made above, is clarified by* (3.2.13)*, as otherwise the tube domain coordinate Z_1 is given by $-\tau$. Not only would this appear somewhat unintuitive for a generalized 'upper' half-plane, but it would also make it necessary to adjust signs when comparing expressions like Fourier series between the orthogonal and the unitary side. Setting $e_3 = -\hat{\zeta}\ell$ avoids this inconvenience. Besides, having $Z_1 = +\tau$ matches our choice of connected component of the positive cone $\mathscr{K}_{\mathrm{O}}$ made in section 1.2.4, see p. 42. There, the component $\mathscr{K}_{\mathrm{O}}^+$ was fixed by assuming that it contain the point* $(1,1,i,i,0)$*. Clearly, the image of $\mathscr{K}_{\mathrm{U}}$ as determined by* (3.2.12) *lies in this component.*

Example 3.2.7. *Let L be the lattice $\mathscr{O}_{\mathbb{F}} \oplus \mathscr{D}_{\mathbb{F}}^{-1} \oplus \mathscr{O}_{\mathbb{F}}$, of signature* $(1,2)$*, with the hermitian form* $\langle \lambda, \mu \rangle = \lambda_1 \bar{\mu}_2 + \lambda_2 \bar{\mu}_1 - \lambda_3 \bar{\mu}_3$*. As a basis of $L \otimes_{\mathscr{O}_{\mathbb{F}}} \mathbb{F} = V$, we take* $\ell = (1,0,0)$*,* $\ell' = (0, -\delta^{-1}, 0)$ *and* $f = (0,0,1)$*. Then,* $\langle \ell, \ell' \rangle = \delta^{-1}$ *and* $\langle f, f \rangle = 1$*.*

The hyperbolic part of L is the hyperbolic plane $\mathscr{O}_{\mathbb{F}} \oplus \mathscr{D}_{\mathbb{F}}^{-1}$, for which $e_1, \ldots, e_4$ have been calculated in example 3.1.10. For the definite part, i.e. the rational space underlying the $\mathbb{F}$-span of f, we can take f and any multiple $\hat{\mu} f$, with $\mu \in \mathbb{F} \setminus \mathbb{Q}$.

Thus, if the discriminant of $\mathbb{F}$ is even, with

$$e_1 = \ell,\ e_2 = \frac{\delta}{2}\ell',\ e_3 = -\frac{\delta}{2}\ell,\ e_4 = -\ell',\ f_1 = f,\ f_2 = \frac{\delta}{2} f,$$

we only need to calculate the coordinates for the definite part of Z_L, $\mathfrak{z}(\sigma)$. Using (3.2.9)*, we have* $\mathfrak{x}(\sigma) = \Re\sigma f_1 + 2|\delta|^{-1}\Im\sigma f_2$ *and* $\mathfrak{y}(\sigma) = \Im\sigma f_1 - 2|\delta|^{-1}\Re\sigma f_2$*. We get*

$$Z_L(z) = -\tau\frac{\delta}{2} e_1 + e_2 + \tau e_3 + \frac{\delta}{2} e_4 + \sigma f_1 + \frac{2}{\delta}\sigma f_2.$$

If the discriminant of $\mathbb{F}$ is odd, with $e_1, \ldots, e_4$ from example 3.1.10, we may set $f_1 = f$ and $f_2 = \delta f$, for example, and get

$$Z_L(z) = \bar{\omega}\tau e_1 + e_2 + \tau e_3 - \bar{\omega} e_4 + \sigma f_1 + \delta^{-1}\sigma f_2,$$

with $\omega = \frac{1}{2}(1+\delta)$*.*

Remark 3.2.8. *The equation for the image of σ in the subspace $D \otimes_{\mathbb{Z}} \mathbb{C}$ of $V_{\mathbb{C}}$,* (3.2.9) *has a noteworthy consequence: As on the one hand $q(\mathfrak{x}(\sigma)) = q(\mathfrak{y}(\sigma))$, while on the other hand $\mathfrak{x}(\sigma) \perp \mathfrak{y}(\sigma)$, we have*

$$\mathfrak{z}(\sigma)^2 = \mathfrak{x}(\sigma)^2 - \mathfrak{y}(\sigma)^2 = 0, \quad \textit{whereas} \quad \big(\mathfrak{z}(\sigma), \bar{\mathfrak{z}}(\sigma)\big) = \mathfrak{x}(\sigma)^2 + \mathfrak{y}(\sigma)^2 < 0.$$

Since the e_1 component of Z_L is given by $-q(Z) - q(e_2)$, see (1.2.9)*, and* $q(Z) = Z_1 Z_2 + q(\mathfrak{z})$*, this also explains how τ, which is mapped to the tube domain coordinate Z_1, reappears as a factor in the e_1-component of $Z_L(z)$.*

Remark 3.2.9. *The embedding can also be described using the Grassmannian coordinates μ and w, which corresponds to the (dotted) diagonal in diagram* (3.2.1). *Either directly through the definition of μ in* (1.2.5) *or from X_L by way of* (1.2.11)*, it can be calculated that*

$$\mu(z) = \begin{cases} \Re\tau e_3 - \frac{q(\sigma)}{4|\langle \ell,\ell'\rangle|^2} e_1 & \text{if } D_{\mathbb{F}} \text{ is even,} \\ -\frac{1}{2}e_4 + \Re\tau e_3 - \left[\frac{1}{2}\Re\tau + \frac{q(\sigma)}{4|\langle \ell,\ell'\rangle|^2}\right] e_1 & \text{if } D_{\mathbb{F}} \text{ is odd.} \end{cases}$$

The subspace w is given by $\mathbb{R}w_0 = \mathbb{R}Y_L$, with Y_L as given by (3.2.11) *above.*

Remark 3.2.10 (Concerning ℓ'). *If the discriminant of $\mathbb{F}$ is even, for the choice of $e_1, \dots, e_4$ given in* (3.1.1) *and the resulting realization of the embedding $\mathcal{H}_U \hookrightarrow \mathcal{H}_O$ given in* (3.2.12)*, the condition $\ell'^2 = 0$ can be omitted.*

Proof. Assume that ℓ' is not isotropic. Then, neither e_2 nor e_4 are isotropic. Note however that both e_3 and e_4 lie in $K \otimes_{\mathbb{Z}} \mathbb{R}$, the complement of e_1 and e_2. For e_3 this is true by construction and for e_4 it follows from (3.1.1), as $e_4 = \hat{\delta}^{-1} e_2$.

Since e_2 is not isotropic, in order to satisfy $Z_L \in \mathcal{N}$, the e_1-component of Z_L must equal $-q(Z) - q(e_2)$. A brief calculation shows that Z_L as given by (3.2.12), with $\zeta = \frac{1}{2}\delta$ is indeed correctly normalized.

$$\begin{aligned} -q(Z) - q(e_2) &= -\tau\frac{\delta}{2} - \frac{\delta^2}{4} q(e_4) - q(\mathfrak{z}(\sigma)) - q(e_2) \\ &= -\tau\frac{\delta}{2} - \frac{d}{|\delta|^2}\frac{\langle \ell',\ell'\rangle}{|\langle \ell,\ell'\rangle|^2} - \frac{\langle \ell',\ell'\rangle}{4|\langle \ell,\ell'\rangle|^2} = -\tau\frac{\delta}{2}, \end{aligned}$$

as $q(\mathfrak{z}(\sigma)) = 0$ and $d|\delta|^{-2} = d(-4d)^{-1} = -\frac{1}{4}$. This all we needed to verify. □

Unfortunately, for number fields with odd discriminant, Z_L as given in (3.2.12) does not lie in $\mathcal{N}$, as $q(Z_L)$ is non-vanishing. Also, in this case, the vectors e_2 and e_4 as in (3.1.1) are no longer perpendicular, which requires further adjustments.

3.3 Behavior on the boundary

Up to now, our results on embeddings of groups and the pull-back of functions living on $\mathcal{H}_O$ to $\mathcal{H}_U$ allow us to formulate the following proposition. Recall the definitions 1.1.25 and 1.2.19 for automorphic forms on unitary and orthogonal groups.

Proposition 3.3.1. *Let f be a meromorphic function on $\mathcal{H}_O$, $f : \mathcal{H}_O \to \mathbb{C}$, which transforms like an automorphic form of weight k under the action of a subgroup Γ of finite index in $O^+(L)$. Then, the pullback $\alpha^* f$ is holomorphic on $\mathcal{H}_U$ and exhibits the transformation behavior of an automorphic form of weight k for a subgroup of finite index $\Gamma' = \Gamma \cap SU(L)$, where $SU(L)$ is considered as a subgroup of $O^+(L)$.*

Proof. Lemma 3.1.2 shows that $\Gamma' = \Gamma \cap SU(L)$ has finite index in $SU(L)$ (and similarly, if $\Gamma \in \Gamma_L^O$, we get $\Gamma' \in \Gamma_L^U$).

Denote by F the homogeneous Γ-invariant holomorphic function on $\widetilde{\mathscr{K}^{+}_{\mathrm{O},1}}$ which is associated to f. The pullback α^*F is holomorphic on $\widetilde{\mathscr{K}^1_{\mathrm{U}}}$ and homogeneous of the same degree, $-k$, as F, by propositions 3.2.3. As F is invariant under all of Γ, in particular, the restriction of F to the image $\alpha(\widetilde{\mathscr{K}^1_{\mathrm{U}}})$ is invariant under Γ' as it operates on $\widetilde{\mathscr{K}^1_{\mathrm{U}}}$, so the pullback α^*F is Γ'-invariant. Clearly, also a character χ of Γ restricts to a character of Γ' as a subgroup.

So $\alpha^*F : \widetilde{\mathscr{K}^1_{\mathrm{U}}} \to \mathbb{C}$ transforms like an automorphic form according to definition 1.1.25. As a function $(\alpha^*F)_\ell$ living on $\mathscr{H}_{\mathrm{U}}$ it is the pullback of f. □

Besides being holomorphic on the symmetric domain and displaying the appropriate transformation behavior, the definition of holomorphic modular forms includes the requirement of regularity at the cusps. When V, $\langle\cdot,\cdot\rangle$ has signature $(1,q)$ with $q > 2$, the Koecher principle can be used to show that a holomorphic automorphic form obtained by pulling back a modular form is regular at the cusp.

More generally, we want to see in how far regularity at the cusp is preserved under pullbacks, we first consider how the boundary $\partial\mathscr{H}_{\mathrm{U}}$ of $\mathscr{H}_{\mathrm{U}}$ is mapped into a component of the boundary $\partial\mathscr{H}_{\mathrm{O}}$ of $\mathscr{H}_{\mathrm{O}}$:

In the setup for the tube domain used in this section, with refined coordinates and the basis vectors $e_1,\dots,e_4$, the boundary of $\mathscr{H}_{\mathrm{O}}$ is defined through a rational maximal isotropic subspace $F = \mathbb{Q}e_1 \oplus \mathbb{Q}e_3$ of V. By construction of the embedding, we have $e_1 = \ell$ and $e_3 = \hat{\zeta}\ell$, with $\zeta \in \mathbb{F}$ linearly independent to 1 over $\mathbb{Q}$. Thus, as a rational vector space $F \simeq \mathbb{F}\ell$. Whereas $F_{\mathbb{R}} = F \otimes \mathbb{R} \simeq \mathbb{C}\ell$.

The complexified isotropic space $F_{\mathbb{C}} = F \otimes_{\mathbb{Q}} \mathbb{C}$ corresponds to a one dimensional boundary component of $\mathscr{K}_{\mathrm{O}} \in \mathbb{P}(V_{\mathbb{C}})$, while the projective line $\mathbb{C}\ell$ defines a boundary point of $\mathscr{K}_{\mathrm{U}}$. It is embedded diagonally into $F_{\mathbb{C}}$ via

$$\mathbb{C}\ell \simeq \mathbb{R}\ell \oplus \mathbb{R}\hat{\imath}\ell \to \mathbb{R}\ell \oplus i\mathbb{R}\hat{\imath}\ell \hookrightarrow \mathbb{R}\ell \oplus \mathbb{R}\hat{\imath}\ell \oplus i\left(\mathbb{R}\ell \oplus \mathbb{R}\hat{\imath}\ell\right) \simeq F \oplus iF = F_{\mathbb{C}}.$$

Thus, we have

Proposition 3.3.2. *Boundary points of $\mathscr{H}_{\mathrm{U}}$ are mapped to one-dimensional boundary components of $\mathscr{H}_{\mathrm{O}}$. The boundary point attached to a primitive isotropic lattice vector ℓ is mapped to the boundary component attached to the rational isotropic subspace $F = \mathbb{Q}\ell \oplus \hat{\zeta}\mathbb{Q}\ell$ of V'.*

In particular, since the boundary $\partial\alpha(\mathscr{H}_{\mathrm{U}})$ is contained in the boundary $\partial\mathscr{H}_{\mathrm{O}}$, if f is a function on $\mathscr{H}^*_{\mathrm{O}}$, the behavior of the pullback α^*f on the boundary of $\mathscr{H}_{\mathrm{U}}$ is dictated by that of f on the boundary of $\mathscr{H}_{\mathrm{O}}$. We now consider how the neighborhood of a cusp behaves under the embedding α. As usual, it suffices to consider the cusp ∞ of $\mathscr{H}_{\mathrm{U}}$.

Lemma 3.3.3. *Consider the cusp at infinity of $\mathscr{H}_{\mathrm{U}}$. Then the inverse image of every open neighborhood of the boundary point $\alpha(\infty)$ in the closure of $\mathscr{K}_{\mathrm{O}}$ contains an open neighborhood of infinity in $\mathscr{H}_{\mathrm{U}} \cup \{\infty\}$.*

Proof. A neighborhood in the zero quadric $\mathscr{N}$ of a point x in a one-dimensional boundary component of $\mathscr{K}_{\mathrm{O}}$ consists of a subset of the boundary component open with respect to the subspace topology and an open subset in $\mathscr{K}_{\mathrm{O}}$, a neighborhood of infinity there. More precisely, see [11], a fundamental system of neighborhoods for x is given by

$$V_\epsilon(x) = \left\{z_2 \in \mathbb{H};\ |z_2 - x| < \epsilon\right\},$$
$$U_\epsilon(x) = \left\{(z_1, z_2, \mathfrak{z}) \in \mathscr{H}_{\mathrm{O}};\ y_1 y_2 + q(\mathfrak{y}) > \epsilon^{-1}\right\},$$

where the definition of V_ϵ employs the usual identification of the one-dimensional boundary component with a classical upper half plane, recall p. 42 from section 1.2.4.

The boundary of $\mathscr{H}_{\mathrm{U}}$ consists only of one boundary point, ∞. Let x be the image of this point in a one-dimensional boundary component of $\mathscr{K}_{\mathrm{O}}$ and consider a neighborhood of infinity $U_\epsilon(x) \subset \mathscr{K}_{\mathrm{O}}$. Now, if $Z \in U_\epsilon(x) \cap \alpha(\mathscr{H}_{\mathrm{U}})$, its imaginary part Y is given by

$$Y = \Im\tau e_3 + \frac{|\delta|}{2} e_4 + \Big(\frac{-\imath}{2\,\langle \ell', \ell\rangle}\Big)\widehat{\ }\sigma.$$

And, since $Z \in U_\epsilon(x)$, the following inequality holds

$$\frac{1}{2}\Im\tau|\delta| + \frac{\langle\sigma,\sigma\rangle}{4|\langle \ell',\ell\rangle|^2} > \frac{1}{\epsilon}.$$

In other words, (τ,σ) is contained in one the neighborhoods of infinity $\mathscr{H}_{\mathrm{U}}^C$, as introduced in definition 1.1.23,

$$\mathscr{H}_{\mathrm{U}}^C = \{(\tau,\sigma) \in \mathscr{H}_{\mathrm{U}};\ 2\Im\tau|\delta||\langle \ell',\ell\rangle|^2 + \langle\sigma,\sigma\rangle > C\}.$$

Here, $C = 4|\langle \ell',\ell\rangle|^2/\epsilon$.

□

It thus follows that for a function f which is regular in a neighborhood in $\mathscr{N}$ of $\alpha(\infty)$ or more generally of the image of some cusp of $\mathscr{H}_{\mathrm{U}}$, the pullback $\alpha^* f$ is regular around that cusp.

Proposition 3.3.4. *A holomorphic function f which is regular at the cusps of $\mathscr{H}_{\mathrm{O}}$ has a pullback $\alpha^* f$ which is regular at the cusps of $\mathscr{H}_{\mathrm{U}}$. In particular, if f is a modular form, i.e. a holomorphic automorphic form which is regular at the cusps, of weight k for an orthogonal modular group Γ, then, the pullback $\alpha^* f$ is a modular form of weight k for the unitary modular group $\Gamma' = \Gamma \cap \mathrm{O}^+(L)$.*

The value of $\alpha^* f$ at the cusp at infinity is given by

$$\lim_{\tau\to i\infty} \alpha^* f = \alpha^*(\Phi \mid f),$$

where Φ is the Siegel operator introduced in section 1.2.6 on page 54.

Proof. The first part, concerning the regularity of the pullback of holomorphic functions follows from the preceding lemma. Together with proposition 3.3.1 the statement on the pullback of holomorphic functions follows as well. Finally, $\Phi \mid f$ is the function on the boundary component obtained from f by taking the limit. By regularity, its pullback, in other words its value at $\alpha(\infty)$, is the same as limit of $\alpha^*(f)$. □

4 Borchers products for SU$(1,q)$

This chapter presents the main result of the present dissertation, theorem 4.2.1. Here, the results of chapter 3 are applied to the Borcherds lift, as introduced in chapter 2, through the pull-back under the embedding of the symmetric domain for the unitary into the symmetric domain for the orthogonal group. In this manner, the theory of Borcherds products can be made available for unitary groups. The lifting we construct takes weakly holomorphic vector valued modular forms for the Weil representation of the elliptic modular group $\mathrm{SL}_2(\mathbb{Z})$ to meromorphic automorphic forms on the unitary modular group Γ_L^{U}, which have their zeros and poles along prescribed divisors and can be expanded as Borcherds products.

In section 2.1, we have discussed most of the prerequisites for Borcherds' result, theorem 13.3 in [5], prior to reproducing it as theorem 2.2.3. The concepts introduced there, need to be reconsidered in the new context of unitary groups and hermitian spaces.

4.1 Some prerequisites

As in chapters 1 and 3 let L be an even $\mathcal{O}_{\mathbb{F}}$-lattice, hermitian of signature $(1,q)$, containing a primitive isotropic vector ℓ. Let ℓ' be a vector in the dual lattice L', with $\langle \ell, \ell' \rangle \neq 0$. In the present chapter we *assume that ℓ' is also isotropic*, an assumption we have used while constructing the embedding in section 3.1.2 from p. 77 onward. Compare remark 3.2.10 on when this assumption may be omitted. Denote by D the definite sublattice of $\mathcal{O}_{\mathbb{F}}$-rank $q-1$ given by the complement, taken with respect to $\langle \cdot, \cdot \rangle$, in L of ℓ and ℓ'.

As a $\mathbb{Z}$-module with the bilinear form $(\cdot,\cdot) = \mathrm{Tr}_{\mathbb{F}/\mathbb{Q}} \langle \cdot, \cdot \rangle$, L has signature $(2, 2q)$ and rank $2+2q$. Then, the Weil representation ρ_L factors through a representation of the elliptic modular group $\mathrm{SL}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[L'/L]$, recall proposition 2.1.4. The inputs for the Borcherds lift are contained in the space $\mathcal{M}^{!}_{1-q}(\rho_L)$ of weakly holomorphic vector valued modular form of weight $1-q$ transforming under the representation ρ_L. By remark 2.1.7 their components satisfy $f_\gamma = f_\gamma$ for $\gamma \in L'/L$.

We denote by N_ℓ the *level* of the cusp associated to ℓ, the unique positive integer defined by

$$\mathrm{Tr}_{\mathbb{F}/\mathbb{Q}} \langle L, \ell \rangle = N_\ell \mathbb{Z},$$

compare the definition in chapter 2 on p. 63. Using N_ℓ, as in (2.1.6), a $\mathbb{Z}$-submodule L'_0 of the dual lattice L' is defined through

$$L'_0 := \left\{ \lambda \in L' ;\ \mathrm{Tr}_{\mathbb{F}/\mathbb{Q}} \langle \lambda, \ell \rangle \equiv 0 \mod N_\ell \right\} = \left\{ \lambda \in L' ;\ \langle \lambda, \ell \rangle \in \delta^{-1} \big(\mathbb{Z} + \zeta N_\ell \mathbb{Z} \big) \right\}. \quad (4.1.1)$$

Note that in general L'_0 is not an $\mathcal{O}_{\mathbb{F}}$-lattice. It is however a hermitian lattice in the sense of remark 1.1.10 as the ring of multipliers of L'_0 is an order $\mathcal{O}_L$ in $\mathbb{F}$ with $\mathcal{O}_{\mathbb{F}} \supseteq \mathcal{O}_L \supseteq N_\ell \mathcal{O}_{\mathbb{F}}$.

Recall from section 1.2 the Lorentzian $\mathbb{Z}$-lattice K given by $L \cap e_1^\perp \cap e_2^\perp$, where the complement is taken with respect to $(\cdot,\cdot)$. Here, $e_1 = \ell$ and e_2 is as in (3.1.1),

$$e_2 = \frac{\zeta}{\delta} \langle \ell', \ell \rangle^{-1} \ell'.$$

The lattice K contains an isotropic vector, $e_3 = -\zeta\ell$. The dual, with respect to $(\cdot,\cdot)$, is denoted K'. The order of the discriminant group K'/K is given by $\sharp L'/L = N_\ell^2 \cdot \sharp K'/K$. We have $D = K \cap e_3^\perp \cap e_4^\perp$ and $K \otimes_\mathbb{Z} \mathbb{Q} = (\mathbb{Q}e_3 \oplus \mathbb{Q}e_4) \oplus D \otimes_\mathbb{Z} \mathbb{Q}$ with the vector e_4 from (3.1.1). Note that e_4 is contained in K' and isotropic (since ℓ' is assumed to be isotropic).

Finally, recall the projection $p(\cdot) : L_0 \to K'$ introduced in proposition 2.1.11, cf. (2.1.7),

$$p(\lambda) = \lambda_K - \frac{(\lambda, e)}{N} f_K,$$

where λ_K and f_K denote the images of λ and f under the orthogonal projection p_k from V to $K \otimes_\mathbb{Z} \mathbb{Q}$, see (1.2.4). We have $p(L) = K$. Further $p(\cdot)$ induces a surjective map from L_0'/L onto K'/K, see proposition 2.1.11. Denote by f a lattice vector with $\big(f, \ell\big) = N_\ell$. Then, for $\lambda \in K'$, a set of representatives for $\beta \in L_0'/L$ with $p(\beta) = \lambda + K$ is given by $\beta - \big(\beta, f\big)\,\ell/N_\ell + be_1/N_\ell$, where b runs modulo N_ℓ, see [9], p. 45.

4.1.1 Heegner divisors and Weyl chambers

The definition of Heegner divisors from section 2.1.2 can be cleanly reformulated in the context of hermitian lattices and symmetric domains for unitary groups. The definition of Weyl chambers can also be restated in terms of the Siegel domain coordinates τ and σ.

Heegner divisors

Denote by $\lambda \in L'$ a lattice vector of negative norm. Consider the complement of λ with respect to $\langle\cdot,\cdot\rangle$ in $V_\mathbb{R}$,

$$\big\{u \in V_\mathbb{R}\,;\ \langle \lambda, u\rangle = 0\big\}.$$

This is a subspace of codimension one in $V_\mathbb{R}$. The complement of λ in $\mathbb{P}(V_\mathbb{R})$ consists of all one dimensional subspaces v of $V_\mathbb{R}$ with $\langle \lambda, v\rangle = 0$. Considering only those subspaces, on which $\langle\cdot,\cdot\rangle$ is positive definite, we get the complement of λ in the positive cone $\mathscr{K}_\mathrm{U} \simeq \mathrm{Gr}_\mathrm{U}$. To λ we can thus associate a sub-Grassmannian,

$$\mathbf{H}_\lambda := \big\{v \in \mathrm{Gr}_\mathrm{U}\,;\ \langle z, \lambda\rangle = 0, \quad \text{for all } z \in v\big\}. \tag{4.1.2}$$

Now, $\mathbf{H}_\lambda \subset \mathrm{Gr}_\mathrm{U}$ is a closed analytic subset of codimension one. It defines a prime divisor on Gr_U. Considering those representatives z for lines $[z]$ in the positive cone $\mathscr{K}_\mathrm{U}$ which lie in $\widetilde{\mathscr{K}_\mathrm{U}^1}$, we obtain a closed analytic subset of the Siegel domain model, which we also denote by $\mathbf{H}_\lambda$,

$$\begin{aligned}\mathbf{H}_\lambda &:= \big\{(\tau,\sigma) \in \mathscr{H}_\mathrm{U}\,;\ \langle z(\tau,\sigma), \lambda\rangle = 0\big\}\\ &= \Big\{(\tau,\sigma) \in \mathscr{H}_\mathrm{U}\,;\ -\tau\delta\,\langle \ell', \ell\rangle\,\bar\lambda_{\ell'} + \bar\lambda_\ell + \langle \sigma, \lambda_D\rangle = 0\Big\},\end{aligned} \tag{4.1.3}$$

where we have written λ in the form $\lambda_\ell\ell + \lambda_{\ell'}\ell' + \lambda_D$ and, as usual, $z(\tau,\sigma) = \ell' - \tau\,\langle \ell', \ell\rangle\,\delta\ell + \sigma$. Clearly, $\mathbf{H}_\lambda$ defines a prime divisor on $\mathscr{H}_\mathrm{U}$.

With a view towards Borcherds theory, we now define Heegner divisors as follows.

Definition 4.1.1. *Let m be a negative integer and $\beta \in L'/L$. We denote by $\mathbf{H}(m,\beta)$ the following divisor on $\mathscr{H}_{\mathrm{U}}$*

$$\mathbf{H}(m,\beta) = \sum_{\substack{\lambda\in\beta+L\\ \langle\lambda,\lambda\rangle=m}} \mathbf{H}_\lambda, \qquad \textit{with support} \qquad \bigcup_{\substack{\lambda\in\beta+L\\ \langle\lambda,\lambda\rangle=m}} \mathbf{H}_\lambda.$$

The divisor $\mathbf{H}(m,\beta)$ is Γ_L^{U} invariant.

It is the inverse image under the canonical projection $\mathscr{H}_{\mathrm{U}} \to X_\Gamma = \Gamma\backslash\mathscr{H}_{\mathrm{U}}$ of an algebraic divisor, called Heegner divisor *of discriminant (m,β) on X_Γ, which we also denote by $\mathbf{H}(m,\beta)$. We also refer to the divisor $\mathbf{H}(m,\beta)$ on $\mathscr{H}_{\mathrm{U}}$ as a Heegner divisor.*

Note that the sum over irreducible components $\mathbf{H}_\lambda$ in the definition of $\mathbf{H}(m,\beta)$ is locally finite. The multiplicities of the connected components of $\mathbf{H}(m,\beta)$ are 1 if $\beta \neq 0 \in L'/L$ and 2 if $\beta = 0 \in L'/L$.

The definition of $\mathbf{H}_\lambda$ is analogous to that of the prime divisor $\lambda^\perp$ in the context of quadratic spaces, recall definition 2.1.12, which was also introduced as a sub-Grassmannian and as the corresponding subset of the tube-domain.

Lemma 4.1.2. *Let λ be a lattice vector λ with $q(\lambda) < 0$, the image of $\mathbf{H}_\lambda$ under the embedding α is given by the intersection of the support of $\lambda^\perp$ on $\mathscr{H}_{\mathrm{O}}$ with the image of $\mathscr{H}_{\mathrm{U}}$,*

$$\mathbf{H}_\lambda \xrightarrow{\alpha} \lambda^\perp \cap \alpha(\mathscr{H}_{\mathrm{U}}). \tag{4.1.4}$$

Proof. With the bijection between the Grassmannian models and the projective cone models, we consider $\lambda^\perp$ and $\mathbf{H}_\lambda$ as subsets of the projective cones $\mathscr{K}_{\mathrm{O}}^+$ and $\mathscr{K}_{\mathrm{U}}$. A set of representatives for lines in $\mathbf{H}_\lambda$ is given by the vectors $z \in \widetilde{\mathscr{K}_{\mathrm{U}}^1}$ with $\langle z,\lambda\rangle = 0$, while a set of representatives for $\lambda^\perp$ is given all Z_L in $\widetilde{\mathscr{K}_{\mathrm{O},1}^+}$, with $(Z_L,\lambda) = 0$.

Let z be the preimage of $Z_L \in \widetilde{\mathscr{K}_{\mathrm{O},1}^+} \cap \alpha\big(\widetilde{\mathscr{K}_{\mathrm{U}}^1}\big)$. Then, Z is given by (3.2.8) and we have

$$\begin{aligned}
(Z_L,\lambda) = 0 \Leftrightarrow \quad & 0 = \left(\frac{z}{2\langle \ell',\ell\rangle^{\frown}},\lambda\right) + i\left(\frac{-\hat{\imath}z}{2\langle \ell',\ell\rangle^{\frown}},\lambda\right)\\
\Leftrightarrow \quad & 0 = \Re\frac{\langle z,\lambda\rangle}{\langle \ell,\ell'\rangle} + i\Im\frac{\langle z,\lambda\rangle}{\langle \ell,\ell'\rangle}\\
\Leftrightarrow \quad & 0 = \langle z,\lambda\rangle.
\end{aligned}$$

□

Thus, every point in the intersection of $\operatorname{supp}(\lambda^\perp) \subset \mathscr{H}_{\mathrm{O}}$ with $\alpha(\mathscr{H}_{\mathrm{U}})$ lies in the image of $\operatorname{supp}(\mathbf{H}_\lambda)$ and vice versa, every $z \in \operatorname{supp}(\mathbf{H}_\lambda)$ is sent to a point in $\operatorname{supp}(\lambda^\perp)$ under the embedding α. Thus, we may write

$$\alpha^*(\lambda^\perp) = \mathbf{H}_\lambda.$$

Further, the locally finite sum

$$H(m,\beta) = \sum_{\substack{\lambda\in\beta+L\\ q(\lambda)=m}} \lambda^\perp, \quad \text{pulls back to} \quad \sum_{\substack{\lambda\in\beta+L\\ \langle\lambda,\lambda\rangle=m}} \mathbf{H}_\lambda = \mathbf{H}(m,\beta).$$

Thus, we have the following corollary.

Corollary 4.1.3. *The divisor $\lambda^\perp$ on $\mathscr{H}_{\mathrm{O}}$ pulls back to the divisor $\mathbf{H}_\lambda$.*

The pullback of the Heegner divisor $H(m,\beta)$ on $\mathscr{H}_{\mathrm{O}}$ is the Heegner divisor $\mathbf{H}(m,\beta)$ on $\mathscr{H}_{\mathrm{U}}$.

4.1.2 Weyl chambers

Recall definition 2.1.15 for a Weyl chamber of $\mathscr{H}_{\mathrm{O}}$. Given $Z = X + iY \in \mathscr{H}_{\mathrm{O}}$, the imaginary part Y defines a positive line contained in the Grassmannian $\mathscr{G}(K)$, associated with the Lorentzian lattice K. Then, Z is contained in a Weyl chamber of $\mathscr{H}_{\mathrm{O}}$ if $Y/|Y|$ is contained in a Weyl chamber of $\mathscr{G}(K)$. Our definition of Weyl chambers in the symmetric domain $\mathscr{H}_{\mathrm{U}}$ is based on this definition and on the embedding α.

Definition 4.1.4. *Given a Weyl chamber W in $\mathscr{G}(K)$, we say that $(\tau,\sigma) \in \mathscr{H}_{\mathrm{U}}$ is* contained in *W, if the image $Z = \alpha(\tau,\sigma)$ is contained in W.*

The preimage of $W \cap \alpha(\mathscr{H}_{\mathrm{U}})$ is then also called a Weyl chamber *of $\mathscr{H}_{\mathrm{U}}$. Equivalently, $(\tau,\sigma) \in W$, if the imaginary part Y of $Z = \alpha(\tau,\sigma)$ defines a positive line contained in W as a subset of $\mathscr{G}(K)$.*

Since the points of $\mathscr{H}_{\mathrm{U}}$ correspond $1:1$ to vectors in $\widetilde{\mathscr{K}_{\mathrm{U}}^{1}} \subset \widetilde{\mathscr{K}_{\mathrm{U}}}$, we also consider Weyl chambers in $\mathscr{H}_{\mathrm{U}}$ as Weyl chambers in $\widetilde{\mathscr{K}_{\mathrm{U}}^{1}}$ and say that $z \in \widetilde{\mathscr{K}_{\mathrm{U}}^{1}}$ is contained in W if the corresponding element $(\tau,\sigma) \in \mathscr{H}_{\mathrm{U}}$ is contained in W.

A Weyl chamber of index (n,γ) *is a subset W of $\mathscr{H}_{\mathrm{U}}$ the image under α of which defines a Weyl chamber of $\mathscr{H}_{\mathrm{O}}$.*

We want to be able to describe the Weyl chambers of $\mathscr{H}_{\mathrm{U}}$ more explicitly in terms of τ and σ. For this we first need a more explicit description of Weyl chambers in the orthogonal setting.

Inequalities determining Weyl chambers

The Weyl chambers in $\mathscr{H}_{\mathrm{O}}$ were defined through the complement of Heegner divisors in $\mathscr{G}(K)$.

Thus, $Z = X + iY \in \mathscr{H}_{\mathrm{O}}$ is *not* contained in any Weyl chamber of index (n,γ) if for some $\lambda \in p(\gamma) + K$, with $q(\lambda) = n$,

$$\mathbb{R}\frac{Y}{|Y|} \in H(n, p(\gamma)), \quad \text{in other words, if} \quad \frac{Y}{|Y|} \in \lambda^{\perp}.$$

On the other hand, there is a Weyl chamber W of index (n,γ) containing Z, precisely if $(\lambda, Y) \neq 0$ for all $\lambda \in p(\gamma) + K$, with $q(\lambda) = n$. In this case, Y satisfies a set of at most $\sharp K'/K$ inequalities of the form

$$\frac{(Y,\lambda)}{|Y|} \gtrless 0, \qquad \text{for all } \lambda \in p(\gamma) + K,$$

which determine the Weyl chamber W in question. Now, $|Y| > 0$ and, also, λ is contained in K' and is thus perpendicular to $\ell = e_1$ and e_2. So we can rewrite each of these inequalities in the form $\left(Y_L, \lambda\right) \gtrless 0$.

Similarly, the Weyl chambers of $\mathscr{H}_{\mathrm{O}}$ attached to a vector valued modular form are determined by inequalities of the form $\left(Y_L, \lambda\right) \gtrless 0$, for λs obtained by first iterating over all pairs (n,γ) with $\gamma \in L_0'/L$ and $n \in \mathbb{Z} + q(\gamma)$, $n < 0$, for which $c(n,\gamma) \neq 0$ and finally over representatives for $p(\gamma) \in K'/K$ with norm n.

Now, let W be a Weyl chamber and assume that $Z = X + iY \in W$ is the image under α of the point $(\tau,\sigma) \in \mathscr{H}_{\mathrm{U}}$. Then, the vector $Y_L \in V_{\mathbb{R}}'$ associated with the imaginary part of Z is given by

$$Y_L = \frac{1}{2}\widehat{\left(\frac{-i}{\langle \ell', \ell\rangle}\right)} z, \quad \text{with, as usual,} \quad z = \ell' - \tau\delta \left\langle \ell', \ell \right\rangle \ell + \sigma.$$

The inequalities determining W satisfied by Y_L take the form

$$(Y_L,\lambda) = \frac{1}{2}\left(\widehat{\left(\frac{-i}{\langle \ell',\ell\rangle}\right)} z,\lambda\right) \gtrless 0$$

for an appropriate set of $\lambda \in K'$ and can thus be reformulated directly in terms of $z \in \widetilde{\mathcal{K}_{\mathrm{U}}^1}$.

These considerations are summarized in the following remark.

Remark 4.1.5. *If a Weyl chamber W of index $(n, p(\gamma))$ is explicitly defined through a set of inequalities satisfied by the representative v_0 in the hyperboloid model of $v \in \mathcal{G}(K)$, of the form*

$$s_\lambda \cdot (v_0,\lambda) > 0, \quad \text{with} \quad s_\lambda \in \{-1,1\},$$

indexed over representatives $\lambda \in p(\gamma)+K$ with $q(\lambda) = n$, then, $z(\tau,\sigma)$ is contained in W iff

$$s_\lambda \cdot \Im \frac{\langle z,\lambda\rangle}{\langle \ell',\ell\rangle} > 0 \quad \text{holds for all } \lambda.$$

The Weyl chambers defined with respect to (the principal part of) the Fourier expansion of a weakly holomorphic vector valued modular form can be treated similarly.

Positivity with respect to Weyl chambers

In Borcherds' theorem 2.2.3 lattice vectors satisfying a positivity condition with respect to a given Weyl chamber W occur. In our main theorem 4.2.1 below, a similar condition is needed.

Definition 4.1.6. *We introduce the following notation: Given a Weyl chamber W, and a vector $\lambda \in K'$, we denote $(\lambda, W) > 0$ iff*

$$\Im \frac{\langle z,\lambda\rangle}{\langle \ell',\ell\rangle} > 0 \qquad \text{for all } z \text{ in } W.$$

Note that this condition holds for every z in W if it holds for one $z_0 \in W$. So to verify whether $(\lambda, W) > 0$, it suffices to check against an arbitrary fixed $z_0 \in W$. This follows from the analogous statement for Weyl chambers in the hyperboloid model of $\mathcal{G}(K)$, see lemma 2.1.16 and remark 2.1.18.

4.2 The main theorem

For the statement of the main theorem, we briefly recall the definitions and notation from the beginning of this chapter. Let L be an even $\mathcal{O}_{\mathbb{F}}$-lattice of signature $(1,q)$ containing a primitive isotropic vector ℓ. Let $\ell' \in L'$ with $\langle \ell, \ell'\rangle \neq 0$ and, further, ℓ' isotropic. Denote by $D \subset L$ the definite lattice with $\langle D,\ell\rangle, \langle D,\ell'\rangle = 0$.

Denote by N_ℓ the level of the cusp associated to ℓ, i.e. the unique positive integer with $(L,\ell) = N_\ell\mathbb{Z}$, where as usual $(\cdot,\cdot) = \mathrm{Tr}_{\mathbb{F}/\mathbb{Q}} \langle \cdot,\cdot\rangle$; and let L_0' be the $\mathbb{Z}$-lattice associated by (4.1.1) to N_ℓ and L'. We have the inclusion $L_0' \subset L'$ and, if $N_\ell = 1$, the identity $L_0' = L'$.

Denote by K the $\mathbb{Z}$-lattice $L \cap \ell^\perp \cap e_2^\perp$, with the complement taken with respect to $(\cdot,\cdot)$, and e_2 given by $\zeta\delta^{-1}\langle \ell',\ell\rangle^{-1}\ell'$. Then, K contains the isotropic vector $e_3 = -\zeta\ell$ and $K \otimes_{\mathbb{Z}} \mathbb{Q}$ is the direct sum of the hyperbolic plane $(\mathbb{Q}e_3 \oplus \mathbb{Q}e_4)$ and the negative definite space $D \otimes_{\mathbb{Z}} \mathbb{Q}$, with e_4 as given in (3.1.1). Further, $p : L_0' \to K$ is the projection defined in (2.1.7). As usual, write z for the point $z = \ell' - \delta\langle \ell',\ell\rangle \tau\ell + \sigma \in \widetilde{\mathcal{K}_{\mathrm{U}}^1}$ attached to $(\tau,\sigma) \in \mathcal{H}_{\mathrm{U}}$.

Theorem 4.2.1. *Let L be an even hermitian lattice of signature $(1,q)$, with $q \geq 1$, and $\ell \in L$ a primitive isotropic vector. Let $\ell' \in L'$ with $\langle \ell, \ell' \rangle \neq 0$. Further assume that ℓ' is isotropic, as well.*

Given a weakly holomorphic modular form $f \in \mathscr{M}^!_{1-q}(\rho_L)$ with Fourier coefficients $c(m,\gamma)$ satisfying $c(m,\gamma) \in \mathbb{Z}$ for $m < 0$ and $c(0,0) \in 2\mathbb{Z}$, there is a meromorphic function $\Xi_f : \mathscr{H}_{\mathrm{U}} \to \bar{\mathbb{C}}$ with the following properties:

i) *Ξ_f is an automorphic form of weight $c(0,0)/2$ for Γ_L^{U}, with a character χ of finite order.*

ii) *The zeros and poles of Ξ_f lie on Heegner divisors. The divisor of Ξ_f on $\mathscr{H}_{\mathrm{U}}$ is given by*

$$\operatorname{div}(\Xi_f) = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(m,\beta)\, \mathbf{H}(m,\beta).$$

The multiplicities of the $\mathbf{H}(m,\beta)$ are 2, if $2\beta = 0$ in L'/L, and 1 otherwise. Note that $c(m,\beta) = c(m,-\beta)$ and $\mathbf{H}(m,\beta) = \mathbf{H}(m,-\beta)$.

iii) *For the cusp corresponding to ℓ and for each Weyl chamber W, $\Xi_f(z)$ has an infinite product expansion of the form*

$$\Xi_f(z) = C e\left(\frac{\langle z, \rho_f(W)\rangle}{\langle \ell', \ell \rangle}\right) \cdot \prod_{\substack{\lambda \in K' \\ (\lambda,W)>0}} \prod_{\substack{\beta \in L_0'/L \\ p(\beta)=\lambda+K}} \left[1 - e\left(\frac{\langle z, \lambda\rangle}{\langle \ell', \ell\rangle} + 2\Re\left[\xi \frac{\langle \beta, \ell'\rangle}{\langle \ell', \ell\rangle}\right]\right)\right]^{c(q(\lambda),\beta)},$$

$$\text{where} \quad \xi = \frac{\zeta}{\delta} = \begin{cases} \frac{1}{2} & \text{if } D_{\mathbb{F}} \equiv 0 \mod 2, \\ \frac{1}{2}\left(1+\delta^{-1}\right) & \text{if } D_{\mathbb{F}} \equiv 1 \mod 2, \end{cases}$$

with a constant C of absolute value 1 and a factor depending on the Weyl vector $\rho_f(W) \in K \otimes_{\mathbb{Z}} \mathbb{R}$. The product converges normally for any z lying in the complement of the set of poles of Ξ_f, and satisfying $\langle z, z \rangle > 4|\langle \ell', \ell \rangle|^2 |m_0|$, where $m_0 = min\{n \in \mathbb{Z};\, c(n,\gamma) \neq 0\}$.

iv) *The lift is multiplicative: $\Xi(z; f+g) = \Xi(z;f) \cdot \Xi(z;g)$.*

Remark 4.2.2. *The Borcherds lift is equivariant under the automorphism group $\mathrm{O}_f^+(L)$ attached to f, recall definition 2.1.9. Thus, Ξ_f is actually automorphic under $\mathrm{O}_f^+(L) \cap \mathrm{SU}(L)$. This group contains Γ_L^{U} as a subgroup, since $\Gamma_L^{\mathrm{U}} \subset \Gamma_L^{\mathrm{O}}$ and $\Gamma_L^{\mathrm{O}} \subseteq \mathrm{O}_f^+(L)$.*

Remark 4.2.3. *The theorem can be phrased without the assumption $c(0,0) \in 2\mathbb{Z}$. However, in this case, the lift gives rise to automorphic forms of non integral weight, and the character χ is replaced by a multiplier system of finite order, compare remark 2.2.5.*

Further, the assumption that ℓ' be isotropic can also be relaxed, if the number field $\mathbb{F}$ has even discriminant, see remark 3.2.10.

Corollary 4.2.4. *We use the notation of the theorem. Suppose L is the direct sum of a hyperbolic plane $H \simeq \mathscr{O}_{\mathbb{F}} \oplus \mathscr{D}_{\mathbb{F}}^{-1}$ and a definite part D, with $\langle D, H \rangle = 0$. Then, for a cusp corresponding to $\ell \in H$ and every Weyl chamber W, the lift $\Xi_f(z)$ has a Borcherds product expansion of the form*

$$\Xi_f(z) = C\, e\left(\frac{\langle z, \rho_f(W)\rangle}{\langle \ell', \ell\rangle}\right) \prod_{\substack{\lambda \in K' \\ (\lambda,W)>0}} \left[1 - e\left(\frac{\langle z, \lambda \rangle}{\langle \ell', \ell \rangle}\right)\right]^{c(q(\lambda),\lambda)}. \tag{4.2.1}$$

In particular, if $\langle \ell, \ell' \rangle = -\delta^{-1}$, *the Borcherds product of* $\Xi_f(z)$ *can be written as*

$$\Xi_f(z) = C\, e\left(\delta \left\langle z, \rho_f(W) \right\rangle\right) \prod_{\substack{\lambda \in \mathbb{Z}\zeta\ell \oplus \mathbb{Z}\ell' \oplus D' \\ (\lambda, W) > 0}} \Big(1 - e\big(\delta \langle z, \lambda \rangle\big)\Big)^{c(q(\lambda),\lambda)}, \tag{4.2.2}$$

where, as usual, $\zeta = \frac{1}{2}\delta$ *if* $D_{\mathbb{F}}$ *is even and* $\zeta = \frac{1}{2}(1+\delta)$, *if* $D_{\mathbb{F}}$ *is odd.*

Remark 4.2.5. *If the lattice* L *is unimodular, it can be written in the form required by the corollary,* $L = H \oplus D$, *because of lemma 1.1.15. Also, for a lattice of this type, it always possible to choose* $\ell' \in H$ *with* $\langle \ell, \ell' \rangle = -\delta^{-1}$, *recall lemma 1.1.16.*

Proof. By assumption, the hyperbolic apart of L is unimodular, so $\ell' \in L$. Further, by proposition 3.1.9, $e_2 = \frac{\zeta}{\delta\langle \ell', \ell \rangle}\ell'$ is contained in L' and since $e_2 \in H \otimes_{\mathbb{Z}} \mathbb{F}$, clearly e_2 is contained in H, and in particular $e_2 \in L$.

Now, $(v, \ell) = 0$ for any $v \in L \otimes_{\mathbb{Z}} \mathbb{Q}$ with $v \notin \mathbb{Q}e_2$, thus, with $e_2 \in L$, $\big(e_2, \ell\big) = 1$ the evenness of L implies $(L, \ell) = \mathbb{Z}$. So, the level of the cusp, N_ℓ is equal to one. Then, the lattice L_0', can then be identified with L' and thus $L_0'/L = L'/L$. Moreover, $N_\ell = 1$ also implies that the discriminant groups K'/K and L'/L are isomorphic. In fact, we can consider $\lambda \in K'/K$ as an element of L'/L. This follows from the description of the set of representatives $\beta \in L'/L$ with $p(\beta) \in \lambda + K$ given on p. 90 [9], p. 45, namely $\beta = \lambda - \big(\lambda, e_2\big)\,\ell/N_\ell + b\ell/N_\ell$, where b runs modulo N_ℓ, compare also [9], p. 45 (Note however the difference in notation, as in [9] $\lambda \in L_0'/L$, taking the role of β here.). Since $N_\ell = 1$ and $e_2 \perp K$, we can set $\beta = \lambda$ (i.e. $\beta = \lambda + 0\ell$, with $b = 0$).

If $\langle \ell, \ell' \rangle = -\delta^{-1}$, the hyperbolic plane H can be written in the form $\mathcal{O}_{\mathbb{F}}\ell \oplus \mathcal{O}_{\mathbb{F}}\ell'$. By (3.1.1), we have $e_2 = -\ell'$, $e_3 = -\zeta\ell$ and $e_4 = -\ell'$. Hence, $H \cap \ell^{\perp} \cap e_2^{\perp} = \mathbb{Z}\zeta\ell \oplus \mathbb{Z}\ell'$. □

The proof of theorem 4.2.1 is based on theorem 2.2.3 and its proof given in [5]. We will use the pull back under the embedding α constructed in chapter 3 to rewrite the unfolded regularized integral giving the singular theta lift $\Phi_L(Z, f)$ as an expression in τ and σ (or in the attached representative $z \in \widetilde{\mathcal{K}_{\mathrm{U}}^1}$). Recall briefly, how the embedding α induces embeddings between the different models for the symmetric domains of $\mathrm{SU}(1,q)$ and $\mathrm{SO}(1,q)$, and can in turn be described by any of these, as illustrated by the following diagram:

$$\begin{array}{ccccc}
 & \mathrm{Gr}_{\mathrm{U}} & \hookrightarrow & \mathrm{Gr}_{\mathrm{O}} \ni \mathbb{R}\mu \oplus w & \\
 & \updownarrow & & \updownarrow & \\
z \in \widetilde{\mathcal{K}_{\mathrm{U}}^1} \xrightarrow{\pi_{\mathrm{U}}} & \mathcal{K}_{\mathrm{U}} & \overset{\alpha}{\hookrightarrow} & \mathcal{K}_{\mathrm{O}} \xleftarrow{\pi_{\mathrm{O}}} \widetilde{\mathcal{K}_{\mathrm{O},1}^{+}} \ni Z_L = X_L + iY_L & \\
 & \updownarrow & & \updownarrow & \\
(\tau, \sigma) \in & \mathcal{H}_{\mathrm{U}} & \hookrightarrow & \mathcal{H}_{\mathrm{O}} \ni Z = X + iY &
\end{array}$$

Here the canonical projections are denoted π_{U} for $V_{\mathbb{R}} \to \mathbb{P}(V_{\mathbb{R}})$ and π_{O} for $V_{\mathbb{C}} \to \mathbb{P}(V_{\mathbb{C}})$. Using the notation developed in chapter 3, the horizontal map marked with α in the diagram can be written in the form

$$z \longmapsto Z_L = \widehat{\left(\frac{1}{2\langle \ell', \ell \rangle}\right)} z + i\,\widehat{\left(\frac{-\iota}{2\langle \ell', \ell \rangle}\right)} z,$$

as in (3.2.8). A description in terms of an explicit basis can be read off from (3.2.12). For the map between $\mathscr{H}_{\mathrm{U}}$ to $\mathscr{H}_{\mathrm{O}}$ see (3.2.13). The map from Gr_{U} to Gr_{O} can be described explicitly using remark 3.2.9.

For later reference, we give the pullback for some expressions occurring in [5], which will be needed in the course of the proof. To facilitate notation, in what follows, denote $e_1 = e$.

$$e_v = \frac{(e, X_L)}{X_L^2} X_L + \frac{(e, Y_L)}{Y_L^2} Y_L = \frac{X_L}{Y^2} = \frac{\langle \ell, \ell' \rangle^\frown z}{\langle z, z\rangle}, \quad e_v^2 = \frac{1}{Y^2} = \frac{2|\langle \ell, \ell' \rangle|^2}{\langle z, z\rangle} \tag{4.2.3a}$$

$$\lambda_w = \frac{(Y,\lambda)}{Y^2} Y = \Im \frac{\langle z, \lambda\rangle}{\langle \ell', \ell\rangle} \cdot \frac{-\hat{\imath}\, \langle \ell', \ell \rangle^\frown z}{\langle z, z\rangle}, \quad |\lambda_w| = \left| \Im \frac{\langle z,\lambda\rangle}{\langle \ell', \ell\rangle}\right| \cdot \frac{\sqrt{2} |\langle \ell', \ell\rangle|}{|\langle z, z\rangle|^{1/2}} \tag{4.2.3b}$$

$$\frac{|\lambda_w|}{|e_v|} = |(Y,\lambda)| = \left| \Im \frac{\langle z,\lambda\rangle}{\langle \ell', \ell\rangle}\right|. \tag{4.2.3c}$$

Proof of theorem 4.2.1. The proof is mostly based on the proof of Borcherds' theorem 13.3 from [5]. We reproduce the main steps of the proof in our setting. Pulling back Borcherds' Ψ_L to our Ξ is carried out by rewriting all expressions involving μ, w and the tube domain coordinate Z in terms of τ and σ or of the attached vector $z \in \widetilde{\mathscr{K}_{\mathrm{U}}^1}$, given by $z = \ell' - \langle \ell', \ell \rangle\, \delta\tau\ell + \sigma$. We will do this gradually, to make the structure of the proof more transparent.

iii) The largest part of the proof goes into showing that an infinite product expansion of the claimed form can be obtained from the Fourier expansion of the theta lift.

By theorem 7.1 in [5], the lift $\Phi_L(v, f)$ is given by the constant term at $s = 0$ of (the analytic continuation of) the following expression:

$$\frac{1}{\sqrt{2}|e_v|}\Phi_K(w, f_K) + \frac{\sqrt{2}}{|e_v|} \sum_{\lambda\in K'} \sum_{n>0} e\left((n\lambda, \mu)\right) \sum_{\substack{\beta\in L_0'/L\\ p(\beta)=\lambda}} e\left(n\left(\beta, e_2\right)\right)$$
$$\times \int_{y>0} c(q(\lambda), \beta) \exp\left(-\frac{\pi n^2}{2 y e_v^2} - 2\pi y \lambda_w^2 \right) y^{1-s-\frac{5}{2}} dy.$$

Now, for $\lambda \in K'$, we have $(\lambda, \mu) = (X, \lambda)$ which is equal to $\Re \frac{\langle z,\lambda\rangle}{\langle \ell', \ell\rangle}$. Further, by (4.2.3a), $|e_v|^2 = 2\frac{|\langle \ell', \ell\rangle|}{\langle z, z\rangle}$. Also, recall that $\langle z, z\rangle > 0$. Thus,

$$\frac{\sqrt{\langle z,z\rangle}}{|\langle \ell', \ell\rangle|} \Phi_K(Y_L, f_K) + \frac{\sqrt{\langle z,z\rangle}}{|\langle \ell', \ell\rangle|} \sum_{\lambda\in K'} \sum_{n>0} e\left(n\,\Re \frac{\langle z,\lambda\rangle}{\langle \ell', \ell\rangle}\right) \sum_{\substack{\beta\in L_0'/L\\ p(\beta)=\lambda}} e\left(n\left(\beta, e_2\right)\right)$$
$$\times \int_{y>0} c\left(q(\lambda), \beta\right) \exp\left(-\frac{\pi n^2}{2y} \frac{\langle z,z\rangle}{2|\langle \ell', \ell\rangle|} - 2\pi y \lambda_w^2 \right) y^{1-s-\frac{5}{2}} dy.$$

We fix the Weyl chamber W under consideration. We assume $Y \in W$, or equivalently, $z \in W$. The theta lift for the smaller lattice Φ_K can be expressed in terms of the Weyl vector $\rho_f(W)$ associated to W

$$\frac{\sqrt{\langle z,z\rangle}}{|\langle \ell', \ell\rangle|} \Phi_K(Y, f_K) = 8\pi \left(\rho_f(W), \frac{Y}{|Y|}\right) = 8\pi \left(\rho_f(W), p(Y_L)\right),$$

by the definition of the Weyl vector in section 10 of [5].

Now we consider the sums involving the integral. The condition $Y \in W$ assures that (λ, v_1) takes the same sign as (λ, Y) for all $v_1 \in W$. Since $w_k = \mathbb{R}Y$, it follows that $\lambda_w = 0$ exactly if $\lambda = 0$. So we rewrite the sum over λ as a term involving $\lambda = 0$ and a sum over $\lambda \neq 0$ for which $\lambda_w \neq 0$, which can further be split up into sums involving all λ with $(\lambda, W) > 0$ and $(\lambda, W) < 0$.

We will treat the term with $\lambda = 0$ first. This is given by

$$\frac{\sqrt{\langle z,z\rangle}}{|\langle \ell',\ell\rangle|}\cdot\sum_{n>0}\sum_{\beta\in\mathbb{Z}/N\mathbb{Z}} e\left(n\left(\beta,e_2\right)\right)\int_{y>0} c(0,\beta\ell/N)\exp\left(-\frac{\pi n^2}{y}\frac{\langle z,z\rangle}{4|\langle \ell',\ell\rangle|^2}\right)y^{1-s-\frac{5}{2}}dy.$$

After applying Borcherds' lemma 7.3, which evaluates the integral, and cancelling factors, the result is

$$2\sum_{\beta\in\mathbb{Z}/N\mathbb{Z}} e\left(n\left(\beta,e_2\right)\right)c(0,\beta\ell/N)\Gamma\left(s+\tfrac{1}{2}\right)\left[\frac{4|\left\langle \ell',\ell\right\rangle|^2}{\langle z,z\rangle}\right]^s\sum_{n>0}\frac{e\left(\beta/N\right)}{n^{2s+1}}.$$

The constant term at $s = 0$ of this expression can now be determined by lemma 13.1 of [5] which gives

$$\begin{aligned}2\,c(0,0)\cdot\left(-\log\frac{2|\langle \ell',\ell\rangle|^2}{|\langle z,z\rangle|}+\tfrac{1}{2}\Gamma'(1)-\log\sqrt{2\pi}\right)\\ +\,2\sum_{\substack{\beta\in\mathbb{Z}/N\mathbb{Z}\\ \beta\neq 0}} c(0,\beta\ell/N)\Big[-\log\left(1-e\left(\tfrac{\beta}{N}\right)\right)\Big].\end{aligned}$$

The remaining sum over $\lambda \neq 0$ can be evaluated using lemma 7.2 of [5], which directly gives the expression for the limit at $s = 0$. We get

$$\begin{aligned}&\frac{\sqrt{\langle z,z\rangle}}{|\langle \ell',\ell\rangle|}\sum_{\substack{\lambda\in K'\\ \lambda\neq 0}}\sum_{n>0} e\left(n\,\Re\frac{\langle z,\lambda\rangle}{\langle \ell',\ell\rangle}\right)\sum_{\substack{\beta\in L_0'/L\\ p(\beta)=\lambda}} e\left(n\left(\beta,e_2\right)\right)\\ &\qquad\qquad\times\, c(q(\lambda),\beta)\left(n\frac{\sqrt{\langle z,z\rangle}}{2|\langle \ell',\ell\rangle|}\right)^{-1}\exp\left(-2\pi n\frac{|\lambda_w|}{|\ell_v|}\right)\\ &=2\sum_{\substack{\lambda\in K'\\ \lambda\neq 0}}\sum_{\substack{\beta\in L_0'/L\\ p(\beta)=\lambda}}\sum_{n>0}\frac{c(q(\lambda),\beta)}{n}\cdot e\left(n\left(\Re\frac{\langle z,\lambda\rangle}{\langle \ell',\ell\rangle}+i\left|\Im\frac{\langle z,\lambda\rangle}{\langle \ell',\ell\rangle}\right|+(\beta,e_2)\right)\right),\end{aligned}$$

where for the last equality we have made use of $\frac{|\lambda_w|}{|\ell_v|} = |\Im\frac{\langle z,\lambda\rangle}{\langle \ell',\ell\rangle}|$ from (4.2.3c).

Since for $\lambda \neq 0$, (λ, W) is either > 0 or < 0, we may as well take the sum over vectors with fixed sign and their negatives:

$$\begin{aligned}2\sum_{\substack{\lambda\in K'\\ (\lambda,W)>0}}\sum_{\substack{\beta\in L_0'/L\\ p(\beta)=\lambda}}\sum_{n>0}\frac{c(q(\lambda),\beta)}{n}\Bigg[\, e\left(n\left(\Re\frac{\langle z,\lambda\rangle}{\langle \ell',\ell\rangle}+i\Im\frac{\langle z,\lambda\rangle}{\langle \ell',\ell\rangle}+(\beta,e_2)\right)\right)\\ +\, e\left(n\left(-\Re\frac{\langle z,\lambda\rangle}{\langle \ell',\ell\rangle}+-i\Im\frac{\langle z,\lambda\rangle}{\langle \ell',\ell\rangle}-(\beta,e_2)\right)\right)\Bigg].\end{aligned}$$

Recall that $c(m,\beta) = c(m,-\beta)$, since the Borcherds input has weight $1 - b/2 = 1 - q$, see remark 2.1.8. The sum over n takes the form

$$\sum_{n>0} \frac{1}{n} \left[e\left(n \cdot \left(\Re \frac{\langle z,\lambda\rangle}{\langle \ell',\ell\rangle} + i\Im \frac{\langle z,\lambda\rangle}{\langle \ell',\ell\rangle} + (\beta, e_2) \right)\right) \right.$$
$$\left. + \; e\left(n \cdot \overline{\left(\Re \frac{\langle z,\lambda\rangle}{\langle \ell',\ell\rangle} + i\Im \frac{\langle z,\lambda\rangle}{\langle \ell',\ell\rangle} + (\beta, e_2) \right)} \right) \right],$$

which is just a sum over two logarithmic series. Summing these gives

$$2 \sum_{\substack{\lambda \in K' \\ (\lambda, W)>0}} \sum_{\substack{\beta \in L_0'/L \\ p(\beta)=\lambda}} -c\big(q(\lambda),\beta\big) \, \log\left[\left(1 - e\left(\frac{\langle z,\lambda\rangle}{\langle \ell',\ell\rangle} + (\beta, f) \right)\right) \right.$$
$$\left. \times \left(1 - e\overline{\left(\frac{\langle z,\lambda\rangle}{\langle \ell',\ell\rangle} + (\beta, e_2) \right)}\right) \right]$$
$$= -4 \sum_{\substack{\lambda \in K' \\ (\lambda, W)>0}} \sum_{\substack{\beta \in L_0'/L \\ p(\beta)=\lambda}} c(q(\lambda),\beta) \log \left| 1 - e\left(\frac{\langle z,\lambda\rangle}{\langle \ell',\ell\rangle} + (\beta, e_2) \right) \right|.$$

Finally, e_2 is given by $\frac{1}{2}\left\langle \ell',\ell \right\rangle^{-1} \ell'$ if the discriminant $D_{\mathbb{F}}$ of the number field $\mathbb{F}$ is even and by $\frac{1}{2}\frac{1+\delta}{\delta}\left\langle \ell',\ell \right\rangle^{-1} \ell'$, if the number field has odd discriminant. Thus, we can rewrite $\big(\beta, e_2\big)$ as $2\Re \bar{\xi} \frac{\langle \delta, \ell' \rangle}{\langle \ell, \ell' \rangle}$, with ξ as defined above. After gathering all contributions the regularized theta lift is given by

$$\begin{aligned}(\alpha^* \Phi(f))(z) = {} & 8\pi \left(\rho_f(W), p(Y(z)) \right) + c(0,0) \cdot \left(\log \frac{|\langle z,z\rangle|}{2|\langle \ell',\ell\rangle|^2} - \Gamma'(1) - \log(2\pi) \right) \\ & - 2 \sum_{\substack{\beta \in \mathbb{Z}/N\mathbb{Z} \\ \beta \neq 0}} c(0, \beta\ell/N) \cdot \log\left(1 - e\left(\frac{\beta}{N}\right)\right) \\ & - 4 \sum_{\substack{\lambda \in K' \\ (\lambda, W)>0}} \sum_{\substack{\beta \in L_0'/L \\ p(\beta)=\lambda}} c(\langle \lambda,\lambda\rangle, \beta) \cdot \log \left| 1 - e\left(\frac{\langle z,\lambda\rangle}{\langle \ell',\ell\rangle} + 2\Re\left[\bar{\xi}\frac{\langle \beta, \ell'\rangle}{\langle \ell, \ell'\rangle}\right]\right)\right|.\end{aligned}$$

Since $\Xi = \alpha^* \Psi_L$, by the definition of Ψ_L in part iv) of Borcherds' theorem, see (2.2.4) in theorem 2.2.3, it is equal to

$$-4\log\left|\,\Xi(z;f)\,\right| - 4\frac{c_0(0)}{2}\left(\log|p(Y_L)| + \frac{1}{2}\Gamma'(1) + \log\sqrt{2\pi}\right).$$

From this, we get exactly the claimed form of the product expansion for each Weyl chamber W, recalling that

$$|Y| = |p(Y_L)| = \left| \frac{\langle z,z\rangle}{2|\langle \ell',\ell\rangle|^2} \right|.$$

Thus, $\Xi(z)$ is given by the exponential of (one quarter of) the Weyl chamber term times that of the sum over λ. A constant factor is further contributed by the exponential of the finite sum over $\beta \in \mathbb{Z}/N\mathbb{Z}$ (or more precisely, its square root, due to the factor of 4 in the definition). For z in the Weyl chamber attached to $\rho_f(W)$, $(\lambda, W) > 0$ is equivalent to $\Im \langle \ell', \ell \rangle^{-1} \langle z, \lambda \rangle > 0$.

A precise criterion for the normal convergence of the Borcherds product is given [9], see theorem 3.22 on p. 88f: The product converges normally on the complement of the set of poles if $q(Y) > |m|$, with $m = \min\{n \in \mathbb{Z};\, c(n,\gamma) \neq 0\}$. On $\mathscr{H}_\mathrm{U}$, we thus have normal convergence if $\langle -iz, -iz \rangle = \langle z, z \rangle$ is greater than $4|m||\langle \ell', \ell \rangle|^2$. This is the case if the point $(\tau, \sigma) \in \mathscr{H}_\mathrm{U}$ lies in a neighborhood of infinity $\mathscr{H}_\mathrm{U}^C$ as introduced in definition 1.1.23, with $C = 4|m||\langle \ell', \ell \rangle|^2$.

i) Finally, lemma 13.1 of [5] assures that Ψ_L is in fact automorphic of weight $c(0,0)/2$ and holomorphic on $\mathscr{H}_\mathrm{O}$, from which automorphy and holomorphicity of Ξ on $\mathscr{H}_\mathrm{U}$ follow through the properties of the pull-back.

That the multiplier system χ is of finite order, and, in particular, that under the assumption $c(0,0) \in 2\mathbb{Z}$, χ is a character of finite order, also follows from theorem 2.2.3 by pull-back.

ii) Our treatment of Heegner divisors follows more closely Bruinier in [9] than Borcherds in [5]. In theorem 3.22 of [9], the divisor of Ψ_L is already given in the form reproduced in part ii) of our statement of theorem 2.2.3,

$$\mathrm{div}(\Psi_L) = \frac{1}{2} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(m,\beta) H(m,\beta),$$

with the $H(m,\beta)$ Heegner divisors of discriminant (m,β) on $\mathscr{H}_\mathrm{O}$, as introduced in definition 2.1.13. As formulated in corollary 4.1.3, we can pull this back to a Heegner divisor on $\mathscr{H}_\mathrm{U}$, as for each discriminant (m,β), the divisor $H(m,\beta)$ on $\mathscr{H}_\mathrm{O}$ induces the Heegner divisor $\mathbf{H}(m,\beta)$ as defined in 4.1.1 on $\mathscr{H}_\mathrm{U}$. By (4.1.4) each point of $\mathbf{H}(m,\beta)$ is contained in the intersection of $H(m,\beta)$ with the image $\alpha(\mathscr{H}_\mathrm{U})$ in $\mathscr{H}_\mathrm{O}$. The restriction to $\alpha(\mathscr{H}_\mathrm{U})$ is needed only, because the definitions of Heegner divisors on $\mathscr{H}_\mathrm{U}$ and on $\mathscr{H}_\mathrm{O}$ are derived from the sub-Grassmannians $\lambda^\perp$ of Gr_O and $\mathbf{H}_\lambda$ of Gr_U; the image $\alpha(\mathbf{H}_\lambda)$ as such is, of course, only a subset of $\lambda^\perp$, and not a full sub-Grassmannian of codimension one.

iv) This follows directly from the multiplicativity of Borcherds' lift $\Psi_L(Z,f)$. □

4.3 Values of Borcherds products at the cusps

In the present section, we study the behavior of Borcherds products as in theorem 4.2.1 on the boundary of $\mathscr{H}_\mathrm{U}$. Rather than working with the infinite product expansion of $\Xi_f(z)$ directly, we first consider the behavior of the orthogonal automorphic Borcherds lift $\Psi_L(Z,f)$ on the one-dimensional boundary component of $\mathscr{H}_\mathrm{O}$ associated with the two-dimensional totally isotropic space over $\mathbb{Q}$ given by the $\mathbb{F}$-span of ℓ. We then pull this back under the embedding α constructed in chapter 3 in the usual manner.

We will assume regularity on the boundary. Under the assumptions of the Koecher principle, this means no restriction, of course. However, the Koecher principle does not apply to the case where $V_\mathbb{R}$, $\langle \cdot, \cdot \rangle$ has signature $(1,1)$.

4.3.1 The Borcherds lift $\Psi_L(Z,f)$ on a one-dimensional boundary component

We want to evaluate $\Psi_L(Z,f)$ on a fixed one-dimensional boundary component. Let W be a fixed Weyl chamber containing a neighborhood of the boundary component under consideration.

Off the poles, $\Psi_L(Z,f)$ is a holomorphic automorphic form and can be expanded as a Fourier series, summed over lattice vectors fulfilling the following condition of semi-positivity:

$$\mu \in K' \quad \text{is semi-positive iff} \quad \mu_1 \geq 0,\, \mu_2 \geq 0 \text{ and } q(\mu) \geq 0. \tag{4.3.1}$$

Equivalently, if μ is semi-positive we can also say that μ lies in the closure $\overline{\mathscr{C}_+}$ of the positive cone $\mathscr{C}_+$, as in (1.2.20).

Thus, at least for Y sufficiently large, we have

$$\Psi_L(Z,f) = e\left(\left(\rho_f(W), Z\right)\right) \prod_{\substack{\lambda\in K' \\ (\lambda,W)>0}} \left(1 - e\left((\lambda, Z)\right)\right)^{c(q(\lambda),\lambda)} \overset{!}{=} \sum_{\substack{\mu\in K' \\ \mu \text{ s.p.}}} a(\mu) e\left((\mu, Z)\right). \tag{4.3.2}$$

Denote by Φ the Siegel operator introduced in section 1.2.6. Assuming the regularity of $\Psi(Z,f)$ on the boundary component under consideration, we will show the following lemma.

Lemma 4.3.1. *Consider a one-dimensional boundary component of $\mathscr{H}_O$. Let W be a Weyl chamber which 'runs into' this boundary component. Assume regularity of $\Psi(Z,f)$ on the boundary. Then, $\left(\Phi \mid \Psi_L(Z,f)\right)(Z)$ either vanishes identically or has an infinite product expansion of the form*

$$\left(\Phi \mid \Psi_L(Z,f)\right)(Z) = e\left(\rho_2 z_1\right) \prod_{\substack{\lambda=\lambda_2 e_3\in K' \\ \lambda_2>0}} \left(1 - e\left(\lambda_2 z_1\right)\right)^{c(0,\lambda)}. \tag{4.3.3}$$

Note that $\Im z_1 > 0$ since Z is contained in the positive cone of $\mathscr{H}_O$.

To prepare the proof of this lemma, we first consider the properties of the Weyl vector $\rho_f(W)$.

Lemma 4.3.2. *Under the assumptions of lemma 4.3.1, assume further that $\Phi \mid \Psi_L(Z,f)$ is not identically* 0*. Then, the Weyl vector $\rho_f(W)$ is semi-positive and consists only of the e_4-component, thus $\rho_f(W) = \left(0, \rho_2, 0\right)$, with $\rho_2 \geq 0$.*

Proof. By the definition of the Siegel operator Φ and under the assumptions made, for the Fourier expansion of Ψ_L around the cusp corresponding to e_1, we have

$$\begin{aligned}\left(\Phi \mid \Psi_L(Z,f)\right)(Z) &= \lim_{z_2\to i\infty} \sum_{\substack{\mu\in K' \\ \mu \text{ s.p.}}} a(\mu)\, e\left((\mu, Z)\right) \\ &= \lim_{z_2\to i\infty} \sum_{\substack{\mu=(\mu_1,\mu_2,\mu_D)\in K' \\ \mu \text{ s.p.}}} a(\mu)\, e\left(z_1\mu_2 + z_2\mu_1 + \left(\mu_D, \mathfrak{z}\right)\right).\end{aligned}$$

Only terms with $\mu_1 = 0$ contribute to the limit, as in view of (4.3.1), we have $\mu_1 \geq 0$, but $\lim_{z_2 \to i\infty} e\big((\mu, Z)\big) = 0$ if $\mu_1 > 0$, so only terms with $\mu_1 = 0$ remain.

Now, one way to expand the Borcherds product as a sum is by multiplying all factors in the infinite product part and expanding – each factor may first be expressed as a sum using binomial series. We carry this out more precisely in the proof of lemma 4.3.1 below. For now, we are only interested in the leading term:

$$e\big((\rho, Z)\big) \prod_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} \Big(1 - e\big((\lambda, Z)\big)\Big)^{c\left(q(\lambda), \lambda\right)} = e\big((\rho, Z)\big)\Big[1 + \dots\Big].$$

By comparing with the Fourier expansion of $\Psi_L(Z, f)$, it follows that $e\left((\rho, Z)\right)$ occurs as a term in the Fourier expansion, with coefficient $a(\rho) = 1$. Thus, necessarily by the assumption of regularity, it follows that ρ is semi-positive, too.

If $\rho_1 \neq 0$. the limit of $e\left(z_1 \rho_2 + z_2 \rho_1 + \left(\rho_D, \mathfrak{z}\right)\right)$ vanishes. Then however, $\Phi \mid \Psi_L(Z, f) \equiv 0$.

If on the other hand, $\rho_1 = 0$, the semi-positivity condition takes the form $q\left(\rho\right) = q\left(\rho_D\right) > 0$, forcing $\rho_D = 0$. □

Now for the proof of lemma 4.3.1.

Proof of lemma 4.3.1. Assume $\Phi \mid \Psi_L(Z, f)$ does not vanish identically. We write the Borcherds product as an infinite series, using the binomial series (or the binomial theorem if $c(q(\lambda), \lambda)$ is an integer):

$$\begin{aligned}\big(\Phi \mid \Psi_L(Z, f)\big)(z_1, z_2, \mathfrak{z}) &= e\big(\rho_2 z_1\big) \prod_{\substack{\lambda \in K' \\ (\lambda, W) > 0 \\ \lambda_1 = 0}} \Big(1 - e\left(z_1 \lambda_2 + \left(\lambda_D, \mathfrak{z}\right)\right)\Big)^{c\left(q(\lambda_D), \lambda\right)} \\ &= e\big(\rho_2 z_1\big) \prod_{\substack{\lambda \in K' \\ (\lambda, W) > 0 \\ \lambda_1 = 0}} \sum_{n \geq 0} (-1)^n \binom{c\left(q\left(\lambda_D\right), \lambda\right)}{n} e\big(z_1 \lambda_2 + \left(\lambda_D, \mathfrak{z}\right)\big).\end{aligned}$$

After multiplying all factors in the remaining product we get

$$e\big(\rho_2 z_1\big) + e\big(\rho_2 z_1\big) \sum_{k > 0} \sum_{\substack{\lambda_1, \dots, \lambda_k \in K' \\ (\lambda_i, W) > 0,\, \lambda_{i,1} = 0 \\ n_1, \dots, n_k \in \mathbb{Z},\, n_i > 0}} b\Big(\sum_{i=1}^k n_i \lambda_i\Big) e\left(\Big(\sum_{i=1}^k n_i \lambda_i\,,\, Z\Big)\right),$$

$$\text{where we have set} \qquad b\Big(\sum_{i=1}^k n_i \lambda^i\Big) := \prod_{i=1}^k (-1)^{n_i} \binom{c\left(q(\lambda_{i,D}), \lambda_i\right)}{n_i}.$$

Now, on the one hand, since $k = 1$ and $n_1 = 1$ are permissible, every $\lambda \in K'$ with $(\lambda, W) > 0$ occurs in the sum. On the other hand, set $\tilde{\lambda} = \sum_{i=1}^k n_i \lambda_i$. Clearly, $\tilde{\lambda} \in K'$, $\tilde{\lambda}_1 = 0$ and

$(\tilde{\lambda}, W) > 0$ by linearity, since the n_i are positive. Also, if $\tilde{\lambda}$ occurs several times in the sum, say m times, we can replace $\tilde{\lambda}$ by $m\tilde{\lambda}$. We thus have

$$\begin{aligned}\left(\Phi \mid \Psi_L(Z,f)\right)(z_1,z_2,\mathfrak{z}) = e\left(\rho_2 z_1\right) + \sum_{\substack{\tilde{\lambda}\in K' \\ (\tilde{\lambda},W)>0}} b(\tilde{\lambda}) e\left(\left(\tilde{\lambda}+\rho_2 e_3\,,\, Z\right)\right) \\ \overset{!}{=} \sum_{\mu \text{ s.p.}} a(\mu) e\left((\mu, Z)\right).\end{aligned}$$

This implies that $\tilde{\lambda}+\rho_2 e_3$ is semi-positive, as well, which is the case exactly if $\tilde{\lambda}$ is semi-positive, since $\rho_2 \geq 0$ and $\tilde{\lambda}_1 = 0$. Then, $q(\tilde{\lambda}_D) \geq 0$ implies $\lambda_D = 0$.

Finally, as we assumed $\Phi \mid \Psi_L(Z,f) \not\equiv 0$, we must have $\tilde{\lambda}_2 > 0$ for every $\tilde{\lambda}$.

Since this is valid for all $\tilde{\lambda}$ in the sum, by the definition of $\tilde{\lambda}$ above, it follows that $\lambda_D = 0$ and (since the n_i are non-negative) $\lambda_2 > 0$ for all λ occurring in the product. □

4.3.2 The behavior of $\Xi_f(z)$ on the boundary of $\mathscr{H}_{\mathrm{U}}$

The way we choose to calculate $\lim_{\tau\to i\infty} \Xi(\tau,\sigma;f)$, is to pull back (4.3.3) as given in lemma 4.3.1. With the embedding α constructed in chapter 3, with the basis vector e_3 from (3.1.1) and with the image $Z(\tau,\sigma)$ given by (3.2.13), we immediately get the following proposition.

Theorem 4.3.3. *Assume that* $\lim_{\tau\to i\infty} \Xi_f(\tau,\sigma)$ *exists, i.e.* $\Xi_f(\tau,\sigma)$ *is regular at the cusp of* $\mathscr{H}_{\mathrm{U}}$*. Then, either* $\lim_{\tau\to i\infty} \Xi_f(\tau,\sigma)$ *vanishes at the cusp or*

$$\lim_{\tau\to i\infty} \Xi_f(\tau,\sigma) = e\left(-\bar{\zeta}\rho^W_{f,1}\right) \prod_{\substack{\lambda=\lambda_1\zeta\ell\in K' \\ \lambda_1\in\mathbb{Q}_{>0}}} \left(1 - e\left(-\lambda_1\bar{\zeta}\right)\right)^{c(0,\lambda)}.$$

Where, as usual, ζ *denotes a generator of* $\mathcal{O}_{\mathbb{F}}$*, while* $\rho^W_{f,1}$ *denotes the* e_3*-component of the Weyl vector* $\rho_f(W)$ *(the only component by lemma 4.3.2).*

Remark. *Of course, a proof of this proposition could also be obtained by working directly with the product expansions of* Ξ_f*, proceeding as in the proof of lemma 4.3.1.*

Remark 4.3.4. *The expression in* (4.3.3) *can be interpreted as a CM-value of a generalized eta-quotient. The infinite product in* (4.3.3) *can be identified with an (in general meromorphic) elliptic modular form on the half-plane* $\mathbb{H}$ *contained in the boundary of* $\mathscr{H}_{\mathrm{O}}$*, which generalizes classical eta-quotients, see [59] for an introduction to these. The evaluation at the cusp* ∞ *of* $\mathscr{H}_{\mathrm{U}}$ *comes down to evaluating this product at an* $\mathbb{F}$*-rational point* $-\bar{\zeta} \in \mathcal{O}_{\mathbb{F}}$*, fixed by the construction of the embedding.*

5 Lifting forms from $\mathscr{M}_0^!$ – Borcherds products for $\mathrm{SU}(1,1)$

In this chapter, as an application of the main theorem 4.2.1, we carry out the lift in the special case where $V_{\mathbb{R}}$, $\langle\cdot,\cdot\rangle$ has signature $(1,1)$. In this case, the input space of the multiplicative lift consists of weakly holomorphic forms with weight 0, while the target functions are automorphic products on an arithmetic subgroup in $\mathrm{SU}(1,1)$.

We take the lattice L to be the hyperbolic plane $\mathscr{O}_{\mathbb{F}} \oplus \mathscr{D}_{\mathbb{F}}^{-1}$ over the imaginary quadratic number field $\mathbb{F}$. Since L is unimodular, the discriminant group L'/L is trivial and the Weil representation ρ_L restricts to the usual multiplier system of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}$. Thus, $\mathscr{M}_0^!(\rho_L)$ is the space of scalar valued weight 0 weakly holomorphic modular forms, $\mathscr{M}_0^! = \mathscr{M}_0^!(\Gamma(1))$.

In the following section 5.1, we first consider the Borcherds lift for the orthogonal group, using an extended version of Borcherds theory due to Bruinier, see [9]. This permits us to recover the Weyl vector $\rho_f(W)$ from theorem 4.2.1 in explicit form. For this, it suffices to calculate the lift Φ_K on the signature $(1,1)$ subspace $K \otimes_{\mathbb{Z}} \mathbb{R}$ of the quadratic $V'_{\mathbb{R}}$, $(\cdot,\cdot)$, where K is the Lorentzian lattice $L \cap e^{\perp} \cap e'^{\perp}$. While our primary aim is thus served, we also give the Fourier expansion of the singular theta lift for $\mathrm{SO}(2,2)$ and determine some examples of the resulting Borcherds products.

Subsequently, in section 5.2, we calculate Borcherds products on the unitary group $\mathrm{SU}(L)$, for every number field $\mathbb{F}$, using corollary 4.2.4 of the main theorem of chapter 4 and inserting the Weyl vectors from section 5.1. We also give the pull-back to $\mathscr{H}_{\mathrm{U}}$ of the Fourier expansion of the singular theta lift under the embedding from chapter 3.

In the present case of signature $(1,1)$ the symmetric domain $\mathscr{H}_{\mathrm{U}}$ can be identified with the classical upper half-plane $\mathbb{H}$. Peculiar to this situation, the shape of the Weyl chambers can be easily described: Determined by inequalities only in the imaginary part of the complex variable τ, they are 'stacked', with the 'topmost' Weyl chamber a neighborhood of the cusp $i\infty$ and the 'lower' Weyl chambers as stripe-shaped regions in $\mathbb{H}$.

The Heegner-divisors, given by points in $\mathbb{H}$, are a further object of our consideration. These are CM-points and their CM-orders are contained in $\mathscr{O}_{\mathbb{F}}$. As it turns out, see p. 116, the conductor of such a CM-order depends on the index of the Heegner divisor. Thus, in a sense, the CM-orders can be prescribed, through the choice of $\mathbb{F}$, on the one hand, and the input function, on the other.

5.1 The Borcherds lift for $\mathrm{SO}(1,1)$ and $\mathrm{SO}(2,2)$

We consider a lattice L given by the orthogonal sum of two hyperbolic planes over $\mathbb{Z}$, thus $L \simeq \mathbb{Z}^4$ with the quadratic form $q((k,l,m,n)) = kl + mn$. Then, the Lorentzian lattice K is simply a hyperbolic plane, $K \simeq \mathbb{Z}^2$ with the quadratic form q restricted to K, $q((m,n)) = mn$. The modular group Γ_L^{O} is an arithmetic subgroup of $\mathrm{SO}(2,2)$.

The inputs for the Borcherds lift are weakly holomorphic modular forms of weight 0 for the elliptic modular group $\mathrm{SL}_2(\mathbb{Z})$. The space $\mathscr{M}_0^!$ of such forms is isomorphic to $\mathbb{C}[j]$, where j is the usual modular invariant, and is spanned by the functions $J_b(\tau) = q^{-b} + O(q)$, for $b = 1, 2, \dots$, together with the constant function.

We use the theorem 2.2.3 of Borcherds, from [5], to calculate the main part of the product expansion. However, for the evaluation of the factor associated to the Weyl vector, we use results of Bruinier, from [9]. This gives us more precise information concerning the Weyl chambers and Heegner divisors. As an additional benefit, we can also give the Fourier expansion of the lifted functions in more explicit terms than usual.

However, the method of [9] is based on non-holomorphic Poincaré series $F_m(\tau,s)$. The special values $F_m(\tau,1)$ differ from by $J_b(\tau)$ by the constant term $b_b(0,1)$, see [16]. For this reason, the product expansions obtained this way differ from the Borcherds lift by an eta-factor, the lift of the constant term. In our setting, it is the contribution of this to the Weyl vector term that we must divide out to recover the actual multiplicative Borcherds lift of $J_b(\tau)$.

5.1.1 Prelude: Lifting constants

In view of the considerations just made, we start by considering the Borcherds lift of (integer) constants. Since the lift is multiplicative, it suffices to determine the lift of the constant function $1 = q^0 \in \mathbb{C}[j]$.

There is no principal part to this function and the only Weyl chamber is the positive cone. By Borcherds' theorem the lift is a modular form of weight $1/2$, possibly with multiplier system for a modular group in $\mathrm{O}^+(2,2)$.

The only non-zero Fourier coefficient, 1, occurs for $q(\lambda)=0$, so the Borcherds product takes the form

$$\Psi_L(Z,1) = e\left(\left(\rho_f, Z\right)\right) \prod_{\substack{m\in\mathbb{Z}\\ m>0}} \left(1-e(mz_2)\right) \prod_{\substack{n\in\mathbb{Z}\\ n>0}} \left(1-e(nz_1)\right), \tag{5.1.1}$$

since for $\lambda=(m,n)\in\mathbb{Z}^2$, the equation $q(\lambda)=mn=0$ implies that either m or n is zero, and further $(m,n)\neq(0,0)$, since $(\lambda,Y)>0$ holds for every Y in the positive cone. (Assuming the positive cone has been chosen so that y_1, $y_2>0$.)

Under the action of $\mathrm{SL}_2(\mathbb{Z})\times\mathrm{SL}_2(\mathbb{Z})$ induced by the standard identification $\mathbb{H}\times\mathbb{H}\to\mathscr{H}_{\mathrm{O}}$,

$$(\tau_1,\tau_2) \longmapsto \begin{pmatrix} \tau_2\tau_1 & \tau_1 \\ \tau_2 & 1 \end{pmatrix},$$

$\Psi_L(Z,1)$ becomes a form of parallel weight $(\frac{1}{2},\frac{1}{2})$ in the variables $\tau_1=z_1$ and $\tau_2=z_2$.

We now proceed to show the following:

Lemma 5.1.1. *As a function on $\mathbb{H}\times\mathbb{H}$, the multiplicative Borcherds lift of the constant 1 is equal to the product of two eta functions:*

$$\Psi_L\left((\tau_1,\tau_2),1\right) = \eta(\tau_1)\cdot\eta(\tau_2). \tag{5.1.2}$$

Proof. Clearly, by (5.1.1), the infinite product expansion of $\Psi_L(Z,1)$ matches that of the eta product, except possibly for the factor involving ρ_f. Thus, we have

$$e\left(\frac{1}{24}\tau_1\right)\cdot e\left(\frac{1}{24}\tau_2\right)\cdot\frac{\Psi_L\left((\tau_1,\tau_2),1\right)}{e\left(\tau_1\rho_2+\tau_2\rho_1\right)} = \eta(\tau_1)\cdot\eta(\tau_2).$$

As the right hand side transforms with weight $1/2$ and the usual eta multiplier system in both variables, while $\Psi_L(Z,1)$ also transforms with parallel weight $1/2$ it follows that $\rho_1=\rho_2=1/24$. □

By comparing (5.1.2) with (5.1.1) we see that the Weyl vector ρ_f for the constant function $f = 1$ is given by

$$\rho_f = \frac{1}{24}(1,1).$$

We now define the non-holomorphic Poincaré series $F_m(\tau,s)$ of weight 0 and index m already mentioned above. For the following see [16] and [9] for further details.

Definition 5.1.2. *Let m be a negative integer and denote by $I_v(z)$ be the usual modified Bessel function as in [66], §10. For $s \in \mathbb{C}$ and $y \in \mathbb{R} \setminus \{0\}$ write*

$$\mathscr{I}_s(y) = \sqrt{\frac{\pi|y|}{2}} I_{s-1/2}(|y|).$$

Then, the non-holomorphic Poincaré series $F_m(\tau,s)$ of index m is defined as

$$F_m(\tau,s) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \mathscr{I}_s(2\pi|m|\Im(\gamma\tau)) e\left(-|m|\Re(\gamma\tau)\right),$$

where we have written Γ for $\mathrm{SL}_2(\mathbb{Z})$ and Γ_∞ for the subset $\{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)\,;\, n \in \mathbb{Z}\}$.

It is well known that $F_m(\tau,s)$ is holomorphic at $s = 1$ and that

$$F_m(\tau,1) = J_{|m|}(\tau) + b_m(0,1), \quad \text{with the constant term} \quad b_m(0,1) = 24\,\sigma_{|m|}(1),$$

where the divisor sum $\sigma_b(s)$ is defined more generally as $\sigma_n(s) = n^{(1-s)/2} \sum_{d|n} d^s$.

The contribution of the constant term $b_m(0,1)$ to the Weyl vector ρ_{F_m} for the Borcherds lift of $F_m(0,1)$ is given by

$$\rho_{F_m} = b_m(0,1) \cdot \rho_1 = 24\,\sigma_{|m|} \cdot \frac{1}{24}(1,1) = \left(\sigma_{|m|}, \sigma_{|m|}\right), \qquad \text{with} \quad \sigma_{|m|} = \sum_{d||m|} d. \qquad (5.1.3)$$

Thus, upon determining the lifting of a weight zero modular form expressed as a linear combination of $J_b(\tau)$ for different bs, we must, for the additive lift, subtract a correction term consisting of (suitable multiples of) (5.1.3), with $m = -b$, for each b.

5.1.2 The Weyl vector term for $J_b(\tau)$

We can now direct our attention to the normalized weight 0 modular forms

$$J_b(\tau) = q^{-b} + O(q).$$

First, we need to calculate the Weyl vector terms for these forms. Since we are working on the smaller lattice $K \simeq \mathbb{Z}^2$, we are in the setting of signature $(1,1)$ which is not completely covered by the approach in [9], so our starting point is the integral expression for the regularized lift.

The integral

Denote by Φ_m^K, with $m \leq -1$ the regularized theta lift of the non-holomorphic Poincaré series F_m. We start with (2.31) on p. 56 of [9], wherein $\mathrm{M}_{\nu,\mu}(y)$ denotes the M-Whittaker function as in [66], p. 190:

$$\Phi_m^K(v,s) = \frac{2(4\pi|m|)^{-k/2}}{\Gamma(2s)} \times \sum_{\substack{\lambda\in\beta+L\\ q(\lambda)=m}} \int_0^\infty \mathbf{M}_{-k/2,s-1/2}(4\pi|m|y)\, y^{b^-/4+b^+/4-2} \exp(-4\pi y q(\lambda_v) + 2\pi m)\, dy.$$

The negative integer m is the index of a non-zero Fourier coefficient in the principal part of the function to lift. For the $J_b(\tau)$, m equals $-b$.

The numbers l, k and s_0 used in [9] depend on the signature of K. In the present case of signature $(1,1)$, they are given by.

$$l = 2,\ k = 1 - l/2 = 0,\ s_0 = 1 - k/2 = 1.$$

Now, the integral simplifies considerably:

$$\begin{aligned}\int_0^\infty & \mathbf{M}_{0,1/2}(4\pi|m|y) y^{-3/2} e^{-4\pi y q(\lambda_v) - 2\pi y|m|} dy \\ &= \int_0^\infty \left(1 - e^{-4\pi|m|y}\right) e^{-4\pi y q(\lambda_v)} y^{-3/2} dy \\ &= 4\pi\left(\sqrt{q(\lambda_v)+|m|} - \sqrt{q(\lambda_v)}\right).\end{aligned} \tag{5.1.4}$$

Here, from the second to the third line, we have made use of the identity

$$\mathbf{M}_{0,1/2}(z) = \left(1 - e^{-z}\right) e^{\frac{1}{2}z},$$

which follows from a relation between Kummer's function $M(a,b,z)$, as in [66], p. 189, and the M-Whittaker function,

$$\mathbf{M}_{\kappa,\mu}(z) = e^{-\frac{1}{2}z} z^{\frac{1}{2}+\mu-\kappa} M(1/2+\mu-\kappa, 1+2\mu, z), \quad \text{see [66], p. 190,}$$
$$\text{and the special case} \quad M(1,2,z) = \frac{1}{z}\left(e^z - 1\right), \quad \text{see [66], p. 194.}$$

while the last line of 5.1.4 is obtained by applying the following lemma.

Lemma 5.1.3. *For a, b positive real numbers the following identity holds*

$$\int_0^\infty (1 - e^{-ay}) e^{-by} y^{-3/2} dy = 2\sqrt{\pi}\left(\sqrt{a+b} - \sqrt{b}\right).$$

Proof. Expand the integrand as follows:

$$\int_0^\infty \left(1-e^{-ay}\right) e^{-by} y^{-3/2} dy = -\int_0^\infty \sum_{k=1}^\infty \frac{(-a)^k}{k!} y^{(k-1/2)-1} e^{-by} dy.$$

The sum converges absolutely, so we can switch the sum with the integral and get

$$-\sum_{k=1}^\infty \frac{(-a)^k}{k!} \int_0^\infty y^{(k-1/2)-1} e^{-by} dy = -\sum_{k=1}^\infty \frac{(-1)^k a^k}{k!} \frac{\Gamma(k-1/2)}{b^{k-1/2}},$$

with Euler's integral. The values of the Gamma-function at the half-integral places are given by

$$\Gamma(k-1/2) = \frac{1\cdot 3\cdot 5\cdots(2k-3)}{2^{k-1}}\Gamma(1/2), \quad \text{with} \quad \Gamma(1/2) = \sqrt{\pi},$$

see [66], p. 76. Now, comparing this series with the power series expansion around the origin of the function $\sqrt{1+x}$,

$$\sqrt{1+x} = 1 + \frac{1}{2}\sum_{k=1}^\infty \frac{(-1)^{k-1} 1\cdot 3\cdot 5\cdots(2k-3)}{2^{k-1}} \frac{x^k}{k!},$$

for $x = a/b$, we recover the claimed identity:

$$\int_0^\infty \left(1-e^{-ay}\right) e^{-by} y^{-3/2} dy = 2\cdot\Gamma(1/2)\sqrt{b}\left(\sqrt{1+a/b}-1\right).$$

Note that the above series can also be interpreted as a special elementary case of a hypergeometric function of the type $F(\alpha,\beta;\beta;z) = (1-z)^{-\alpha}$, cf. [66] p. 213, with $\alpha = -1/2$, $z = -a/b$ and suitable β. □

Remark 5.1.4. *In [9] the integral expression for $\Phi_m^K(v,s)$ is evaluated using a formula for the Laplace transform of the Whittaker function from [25], p. 215, according to which*

$$f(t) = t^{\nu-1} M_{\kappa,\mu}(at), \qquad \textit{with} \quad \Re(\mu+\nu) > -\frac{1}{2},$$

has the following Laplace transform $\int_0^\infty f(t) e^{-pt}\, dt$:

$$a^{\mu+1/2}\Gamma(\mu+\nu+1/2)\left(p+\frac{a}{2}\right)^{-\mu-\nu-1/2} \cdot F\left(\mu+\nu+\frac{1}{2}, \mu-\kappa+\frac{1}{2}; 2\mu+1; \frac{a}{(p+1/2a)}\right).$$

Unfortunately, this formula does not give the correct result in the present case.

Indeed, inserting $\kappa = -k/2$, $\mu = s-1/2$, $a = 4\pi|m|$ and $\nu = b^+/4 + b^-/4 - 1 = 0$, where in general (b^+, b^-) is the signature of V', and setting $p = 4\pi q(\lambda_v) + 2\pi|m|$ leads to the special case $F(1/2,1;2;z)$ of the hypergeometric function. However, we have (e.g. by equation 15.1.13 of [66])

$$F(1/2,1;2;z) = \frac{2}{1+\sqrt{1-z}},$$

which is not correct. (Compare this to (5.1.4) *for $z = m/q(\lambda_v)$.)*

Negative norm vectors and calculation of $\Phi_m^K(v)$

Next, let us rewrite the previous result (5.1.4) somewhat, expressing v through Y and writing everything in hyperbolic coordinates. We have

$$\lambda_v = \frac{(\lambda, Y)Y}{Y^2}, \quad \sqrt{q(\lambda_v)} = \frac{1}{\sqrt{2}|Y|}|(\lambda, Y)|.$$

We need to consider vectors λ with $q(\lambda) = m$, with m a negative integer (equal to $-b$ for the lift of $J_b(\tau)$). For these,

$$q(\lambda_v) + q(\lambda_{v^\perp}) = m = q(\lambda).$$

Further, since $K \otimes_{\mathbb{Z}} \mathbb{R}$ is 2-dimensional, there is only one direction perpendicular to Y, so we may choose a vector Y' with $|Y'| = |Y|$, $(Y, Y') = 0$ and calculate $\lambda_{v^\perp}$ by projecting onto this vector.

We set for $Y' = (y_1, -y_2)$, where Y is given by (y_1, y_2), in hyperbolic coordinates. With this convention,

$$\Phi_{-1}^K(Y) = \frac{4\sqrt{2}\pi}{|Y|} \sum_{\substack{\lambda \in \beta + L \\ q(\lambda) = m}} \left(|-\lambda_1 y_2 + \lambda_2 y_1| - |\lambda_1 y_2 + \lambda_2 y_1|\right).$$

Since $K \simeq \mathbb{Z}^2$ is simply a hyperbolic plane over the integers, we have $q(\lambda) = \lambda_1 \lambda_2$, so

$$\left\{\lambda \in \mathbb{Z}^2 \,;\, q(\lambda) = m\right\} = \left\{\pm(k, -l), \quad k, l \in \mathbb{Z}_{>0} \,;\, k \cdot l = -m\right\}.$$

Introducing a factor of 2 to take into consideration the two signs for each pair k, l, we get

$$\begin{aligned} \Phi_m^{\mathbb{Z}^2}(Y) &= \frac{4\sqrt{2}\pi}{|Y|} \sum_{\substack{k,l \in \mathbb{Z}_{>0} \\ kl = -m}} \big(|-ky_2 - ly_1| - |ky_2 - ly_1| \\ &\qquad\qquad + |ky_2 + ly_1| - |-ky_2 + ly_1|\big) \\ &= \frac{8\sqrt{2}\pi}{|Y|} \sum_{\substack{k,l \in \mathbb{Z}_{>0} \\ kl = -m}} \left(|-ky_2 - ly_1| - |ky_2 - ly_1|\right). \end{aligned}$$

Note that we can also write this in the form

$$\Phi_m^{\mathbb{Z}^2}(Y) = \frac{8\sqrt{2}\pi}{|Y|} \sum_{\substack{k \in \mathbb{Z}_{>0} \\ k|m}} \left(|ky_2 - \frac{m}{k} y_1| - |ky_2 + \frac{m}{k} y_1|\right). \tag{5.1.5}$$

Example 5.1.5 ($m = -1$)**.** *If $m = -1$, the only pair of positive integers with product 1 is $k = l = 1$. So there is only one pair of vectors, $\lambda = (1, -1)$ and $-\lambda = (-1, 1)$. Thus,*

$$\Phi_{-1}^{\mathbb{Z}^2}(Y) = \frac{8\sqrt{2}\pi}{|Y|}\left(|y_1 + y_2| - |y_1 - y_2|\right).$$

Example 5.1.6 ($m = -p$)**.** *Let m be the negative of a prime number p. There are only two positive integers with product p, which leads to two pairs of vectors:*

$$\pm\lambda_1 = \pm(1,-p) \qquad \text{and} \quad \pm\lambda_2 = \pm(p,-1).$$

Now, $\Phi_{-p}^{\mathbb{Z}^2}(Y)$ is easy to calculate, too:

$$\Phi_{-p}^{\mathbb{Z}^2}(Y) = \frac{8\sqrt{2}\pi}{|Y|}\big(|py_2+y_1| - |py_2-y_1| + |y_2+py_1| - |y_2-py_1|\big).$$

Heegner divisors and Weyl chambers

The orthogonal complements of λ and $-\lambda$ are the same, so each of the pairs k, l considered above defines exactly one direction (we have only two dimensions!) perpendicular to both negative norm vectors associated with k, l. Thus, we get one prime divisor, by considering the sub-Grassmannian of $\mathscr{G}(K)$ given by

$$\lambda^\perp = (-\lambda)^\perp = (k,-l)^\perp = \mathbb{R}(k,l),$$

Note that the pairs $\pm(k,-l)$ and the pairs $\pm(l,-k)$ define two divisors which, viewed as lines in $K \otimes \mathbb{R}$, are symmetric to each other under reflection through the diagonal. Of course, if $k = l$, in other words if $-m$ is a square or $m = -1$, the diagonal also corresponds to a divisor of this form.

Denote by M the number of distinct pairs $(k,-l)$. It is clear that M is finite. For example, if $-m$ is square-free, M is just the number of divisors of $-m$, see 5.2.4 below. Then, the Weyl chambers are defined by M conditions of the form either

$$((k,-l),W) > 0 \quad \text{or} \quad ((k,-l),W) < 0.$$

Equivalently, either $ky_2 > ly_1$ or $ky_2 < ly_1$ for every $Y = (y_1, y_2)$ in W. As W is a subset of the positive cone, we have $y_1y_2 > 0$ and may thus assume y_1, $y_2 > 0$. Since M is finite, we see that there are only finitely many (at most $2^M + 1$) Weyl chambers.

On each Weyl chamber, $\Phi_m^{\mathbb{Z}^2}(Y;W)$ is given by

$$\begin{aligned}\Phi_m^{\mathbb{Z}^2}(Y;W) &= \frac{8\sqrt{2}\pi}{|Y|}\Bigg(\sum_{\substack{k,l\in\mathbb{Z}_{>0}\\ kl=-m\\ ((k,-l),W)>0}} (ky_2+ly_1)-(ky_2-ly_1)\\ &\qquad + \sum_{\substack{k,l\in\mathbb{Z}_{>0}\\ kl=-m\\ ((k,-l),W)<0}} (ky_2+ly_1)+(ky_2-ly_1)\Bigg)\\ &= \frac{8\sqrt{2}\pi}{|Y|}\Bigg(\sum_{\substack{k,l\in\mathbb{Z}_{>0}\\ kl=-m\\ ((k,-l),W)>0}} 2ly_1 + \sum_{\substack{k,l\in\mathbb{Z}_{>0}\\ kl=-m\\ ((k,-l),W)<0}} 2ky_2\Bigg).\end{aligned} \tag{5.1.6}$$

The Weyl vector of index m, is defined through

$$\Phi_m^K(Y;W) = 8\sqrt{2}\pi\ \big(Y, \rho_m(W)\big)\,.$$

It can be read off from (5.1.6); we get

$$\rho_m(W) = 2\Big(\sum_{\substack{k,l\in\mathbb{Z}_{>0}\\ kl=-m\\ ky_2>ly_1\\ \forall(y_1,y_2)\in W}} l v_2 \quad + \sum_{\substack{k,l\in\mathbb{Z}_{>0}\\ kl=-m\\ ky_2<ly_1\\ \forall(y_1,y_2)\in W}} k v_1\Big), \quad \text{where} \quad v_1=(1,0),\ v_2=(0,1), \quad (5.1.7)$$

are the standard hyperbolic basis vectors, and where the conditions $ky_2 > ly_1$ and $ky_2 < ly_1$ must be satisfied for every $Y = (y_1, y_2) \in W$. (It suffices, however, to check them for one arbitrary $Y_0 \in W$, recall lemma 2.1.16.)

Example 5.1.7 $(m = -1)$. *In the simplest case, $m = -1$, the diagonal, spanned by $(1,1) = (1,-1)^\perp$, is the only Heegner divisor. It dissects the positive cone into two Weyl chambers of equal shape, defined by the conditions $y_2 \gtrless y_1$. For these chambers, denoted $W_>$ and $W_<$, we have*

$$\Phi_{-1}^{\mathbb{Z}^2}(Y;W_>) = \frac{8\sqrt{2}\pi}{|Y|}2y_1, \quad \textit{and} \quad \rho_{-1}(W_<) = 2v_2,$$
$$\Phi_{-1}^{\mathbb{Z}^2}(Y;W_<) = \frac{8\sqrt{2}\pi}{|Y|}2y_2, \quad \textit{and} \quad \rho_{-1}(W_>) = 2v_1.$$

Example 5.1.8 $(m = -p)$. *Let m be the negative of a prime. In this case the two pairs of lattice vectors $\pm\lambda_{1/2}$ bring forth two Heegner divisors, the corresponding lines are spanned by $(1,p)$ and $(p,1)$.*

The positive cone is split into three Weyl chambers W_{++},W_{-+} and W_{--}, determined by whether $y_2 > py_1$, $py_1 > y_2 > \frac{1}{p}y_1$ or $y_2 < \frac{1}{p}y_1$.

$$\Phi_{-p}^{\mathbb{Z}^2}(Y;W) = \frac{8\sqrt{2}\pi}{|Y|} \times \begin{cases} (2p+2)y_1 & \textit{on } W_{++}, \\ 2y_1+2y_2 & \textit{on } W_{-+}, \\ (2p+2)y_2 & \textit{on } W_{--}. \end{cases}$$

For the Weyl vectors, we have

$$\rho_{-p}(W_{++}) = 2(p+1)v_2, \quad \rho_{-p}(W_{-+}) = 2v_1 + 2v_2,$$
$$\rho_{-p}(W_{--}) = 2(p+1)v_1.$$

5.1.3 Intermezzo: The Borcherds lift for SO(2, 2)

Our primary aim, determining the Weyl vector contribution for each Weyl chamber of the Grassmannian $\mathscr{G}(K)$, has been fulfilled by (5.1.7). As a supplement to this, on the following pages, we give the Fourier expansion of the singular theta lift Φ_L, as a special case of the additive Borcherds lift for SO(2, 2). Further, we determine some Borcherds products on the orthogonal group.

Fourier expansions of Borcherds lift

Given a function $f \in \mathscr{M}_0^!$, the Fourier expansion of its additive Borcherds lift with respect to a given Weyl chamber W, denoted $\Phi_L^W(Z,f)$, consists of two main contributions – one, $\Phi_K^W(Z,f)$, coming from the principal part of f, involving negative norm vectors in the smaller lattice $K \simeq \mathbb{Z}^2$, and consisting of logarithms of Euler factors for positive terms in the Fourier expansion of the input function.

Up to now, we have calculated the Fourier expansion of Φ_m^K which the contribution corresponding to the principal part in the lift of the non-holomorphic Poincaré series $F_m(\tau,1)$.

Assume that f has a Fourier expansion of the form

$$f = \sum_{m \gg -\infty} c(m) q^m.$$

The principal part of f can be written as a linear combination of the principal parts of the $J_d(\tau)$, so the Fourier expansion of $\Phi_K^W(Z,f)$ is given by

$$\Phi_K^W(Z,f) = \sum_{\substack{m<0\\ c(m)\neq 0}} c(m)\cdot\Big(\frac{1}{2}\Phi_m^K(Z;W) - \sigma_m\cdot\big((1,1),Y\big)\Big),$$

where we have taken into account the correction due to the constant Fourier coefficient $b_m(0,1)$ of $F_m(\tau,1)$.

Note that the factor $1/2$ is needed when passing from the Weyl vectors of the chambers of a given index containing the smaller chamber W to the Weyl vector of W. This factor may not be applied to the correction term.

In the end, we have

$$\Phi_K^W(Z,f) = \frac{8\pi\sqrt{2}}{|Y|}\Big[\sum_{\substack{m<0\\ c(m)\neq 0}} c(m)\Big(\sum_{\substack{k,l\in\mathbb{Z}_{>0}\\ kl=-m\\ ((k,-l),W)>0}} l y_1 + \sum_{\substack{k,l\in\mathbb{Z}_{>0}\\ kl=-m\\ ((k,-l),W)<0}} k y_2 - \sigma_{-m}(y_1+y_2)\Big) + \frac{c(0)}{24}(y_1+y_2)\Big].$$

The Fourier expansion of the Borcherds lift $\Phi_L^W(Z,f)$ is now given by

$$\begin{aligned} &8\pi\Big(\sum_{\substack{m<0\\ c(m)\neq 0}} c(m)\Big(\sum_{\substack{k,l\in\mathbb{Z}_{>0}\\ kl=-m\\ ((k,-l),W)>0}} l y_1 + \sum_{\substack{k,l\in\mathbb{Z}_{>0}\\ kl=-m\\ ((k,-l),W)<0}} k y_2 - \sigma_{-m}(y_1+y_2)\Big) + \frac{c(0)}{24}(y_1+y_2)\Big)\\ &\quad + c(0)\big(-\log(Y^2) - \Gamma'(1) - \log(2\pi)\big)\\ &\quad + 4\sum_{\substack{\lambda=(m,n)\in\mathbb{Z}^2\\ (\lambda,W)>0}} c(mn)\log\big|1-e(mz_2+nz_1)\big|. \end{aligned} \tag{5.1.8}$$

The second line does not contribute to the Borcherds product, whereas the remaining lines give the logarithm of (the absolute value of) the multiplicative lift.

Remark 5.1.9. *The convergence of* (5.1.8) *can be assured for* $Y^2 > 2|m_0|$*, where* m_0 *is the lowest exponent in the principal part,* $m_0 = \min\{m \in \mathbb{Z}\,;\ c(m) \neq 0\}$*. This follows from some estimates on the growth of Fourier coefficients, see [9], p. 84.*

For $J_b(\tau)$ as input function

We write out the Fourier expansion for the lift of $J_b(\tau)$. Since here the principal part is just q^{-b} and there is no constant term, the above expression simplifies somewhat.

$$\begin{aligned}\Phi_L^W(Z,J_b) = 8\pi\Big(\sum_{\substack{k,l\in\mathbb{Z}_{>0}\\ kl=b\\ ((k,-l),W)>0}} l y_1 + \sum_{\substack{k,l\in\mathbb{Z}_{>0}\\ kl=b\\ ((k,-l),W)<0}} k y_2 - \sigma_b(y_1+y_2)\Big)\\ + 4\sum_{\substack{\lambda=(m,n)\in\mathbb{Z}^2\\ (\lambda,W)>0}} c(mn)\log\left|1-e(mz_2+nz_1)\right|.\end{aligned} \tag{5.1.9}$$

We specialize this to two examples, $J_1(\tau)$ and $J_p(\tau)$, with p a prime number.

Example 5.1.10. *We begin with the additive Borcherds lift of*

$$J_1(\tau) = j(\tau) - 744 = q^{-1} + 196884q + \dots.$$

Here, $m = -1$*, so this takes up the examples 5.1.5 and 5.1.7. Since* $\sigma_1 = 1$*, we must subtract the correction term* $-8\pi(y_1+y_2)$ *from* $\Phi_{-1}^{\mathbb{Z}^2}(Y;W)$ *to get* $\Phi_{J_1}^K(W)$*.*

For the Weyl chamber $W_>$*, the Fourier expansion is given by*

$$\Phi_L^{W_>}(Z,J_1) = -8\pi y_2 + 4\sum_{\substack{m,n\in\mathbb{Z}\\ m>0,n\geq -1}} c(mn)\log\left|1-e(mz_2+nz_1)\right|,$$

while for $W_<$*, we get*

$$\Phi_L^{W_<}(Z,J_1) = -8\pi y_1 + 4\sum_{\substack{m,n\in\mathbb{Z}\\ n>0,m\geq -1}} c(mn)\log\left|1-e(mz_2+nz_1)\right|.$$

Example 5.1.11. *Next, we consider the lift of* $J_p(\tau) = q^{-p} + O(q)$*, with a prime* p*. This takes up examples 5.1.6 and 5.1.8. The divisor sum* σ_p *is* $1+p$*, so we subtract* $-8\pi(1+p)(y_1+y_2)$ *from the result in 5.1.8 to get the contribution of* $\Phi_K^W(Z,J_p)$*.*

All together, we get

$$\begin{aligned}\Phi_L^W(Z,J_p) = 8\pi\begin{cases}-(p+1)y_2 & \text{on } W_{++}\\ -py_2 - py_1 & \text{on } W_{+-}\\ -(p+1)y_1 & \text{on } W_{--}\end{cases}\\ + 4\sum_{\substack{m,n\in\mathbb{Z}_{>0}\cup\{-p\}\\ (W,(m,n))>0}} c(mn)\log\left|1-e(mz_2+nz_1)\right|.\end{aligned}$$

Depending on whether $W = W_{++}$*,* $W = W_{+-}$ *or* $W = W_{--}$*.*

Borcherds products for $J_b(\tau)$

The Borcherds product consists of an infinite product part, and a factor depending on the Weyl vector, which is determined by the principal part of the input function. Write the Fourier expansion of $J_b(\tau)$ in the form $q^{-b} + \sum_{m>0} c(m)q^m$.

We first put together all contributions to the Weyl vector. Denote the Weyl vector calculated using [9], p. 88, as $\rho(J_b;W)'$. Since the principal part of $J_b(\tau)$ is given by q^{-b} we need only to consider the Weyl chambers of index $m=-b$. Then,

$$\rho(J_b;W)' = \frac{1}{2}\rho_{-b}(W),$$

where $\rho_{-b}(W)$ is the Weyl vector from (5.1.7).

To recover the Weyl vector for the actual Borcherds lift of $J_b(\tau)$, we need to correct for the constant term $b_m(0,1)$ in the Fourier expansion of the Poincaré series $F_m(\tau,1)$. This is done by subtracting the term given in (5.1.3). So, we get

$$\rho_{J_b}(W) = \rho(J_b;W)' - \sigma_b \cdot (1,1). \tag{5.1.10}$$

For the multiplicative lift, by theorem 2.2.3, Borcherds' theorem 13.3 in [5], we have

$$\begin{aligned}\Psi_L^W(Z,J_b) &= e\left(\left(\rho_{J_b}(W),Z\right)\right) \prod_{\substack{\lambda\in K\\(\lambda,W)>0}} \left(1-e((\lambda,Z))\right)^{c(mn)}\\ &= e\left(\rho_1 z_2 + \rho_2 z_1\right) \prod_{\substack{m,n\in\mathbb{Z}\\((m,n),W)>0}} \left(1-e\left(mz_2+nz_1\right)\right)^{c(mn)}.\end{aligned} \tag{5.1.11}$$

Here, ρ_i and z_i, for $i=1,2$ denote the components of $\rho_{J_b}(W)$ and Z. The product converges for $q(Y)=\Im(z_1)\cdot\Im(z_2)>b$.

We take up the examples considered above.

Example 5.1.12. *We begin with the Borcherds lift of*

$$J_1(\tau) = j(\tau) - 744 = q^{-1} + 196884q + \dots .$$

Since $d=1$, as in example 5.1.10, we make use of the results from examples 5.1.5 and 5.1.7. Since $\sigma_1 = 1$, the Weyl vector is given by

$$\rho(J_1;W) = \frac{1}{2}\rho_1(W) - (1,1).$$

For the Weyl chamber $W_>$, defined by $y_2>y_1$, the Weyl vector equals $(-1,0)$. Thus, the Borcherds product takes the form

$$\Psi_L^{W_>}((z_1,z_2),J_1) = e(-z_2) \prod_{\substack{m,n\in\mathbb{Z}\\m>0}} \left(1-e(mz_2+nz_1)\right)^{c(mn)}.$$

The condition $m>0$ is obtained thus: Since $c(mn)\neq 0$ only for $mn>0$ or $mn=-1$, we must consider whether $(1,-1)$ or $(-1,1)$ has positive inner product with any $Y\in W_>$.

Similarly, for $W_<$, which is defined by $y_1>y_2$, the product is given by

$$\Psi_L^{W_<}((z_1,z_2),J_1) = e(-z_1) \prod_{\substack{m,n\in\mathbb{Z}\\n>0}} \left(1-e(mz_2+nz_1)\right)^{c(mn)}.$$

Example 5.1.13. *Next, we lift $J_p(\tau) = q^{-p} + O(q)$, with p a rational prime. This takes up the examples 5.1.6 and 5.1.8, while the Fourier expansion has already been considered in example 5.1.11.*

The divisor sum σ_p is $1 + p$. So the Weyl vectors are given by

$$\rho(J_p; W) = \frac{1}{2}\rho_{-p}(W) - (1+p)(1,1),$$

where $\rho_{-p}(W)$ can be read off from example 5.1.8.

The Borcherds product in this case is given by

$$\Psi_L^W\big((z_1,z_2),J_p\big) = \begin{Bmatrix} e(-(p+1)z_2) \\ e(-pz_1)e(-pz_2) \\ e(-(p+1)z_1) \end{Bmatrix} \cdot \prod_{\substack{m,n\in\mathbb{Z} \\ ((m,n),W)>0}} \big(1 - e(mz_2 + nz_1)\big)^{c(mn)},$$

depending on whether $W = W_{++}$, $W = W_{-+}$ or $W = W_{--}$.

5.2 The lift to $\mathrm{SU}(1,1)$

In this section we will calculate the lift of the weight zero modular forms J_b to Borcherds products Ξ_{J_b} on $\mathrm{SU}(1,1)$. We use our main theorem 4.2.1 from chapter 4. The results on the Borcherds lift to $\mathrm{SO}(1,1)$ and $\mathrm{SO}(2,2)$ from the previous section will help us to recover the Weyl vector term and to give a geometric description of the Weyl chambers. For this, we use the embedding α constructed in chapter 3.

In the present setting, the Siegel domain $\mathscr{H}_{\mathrm{U}}$ is isomorphic to the classical complex upper half plane $\mathbb{H} = \{z \in \mathbb{C};\ \Im z > 0\}$. The automorphic products we obtain can be identified with meromorphic functions on $\mathbb{H}$. Since the functions J_b have no constant term in their Fourier expansion, the automorphic products Ξ_{J_b} have weight 0, so they are classical modular functions. An explicit description of the embedding α will also help us to write our results in a form making the identification between $\mathscr{H}_{\mathrm{U}}$ and $\mathbb{H}$ apparent.

5.2.1 The lattice and the upper half-planes

For an imaginary quadratic number field $\mathbb{F} = \mathbb{Q}(\sqrt{d})$, denote by L the hermitian hyperbolic plane over $\mathscr{O}_{\mathbb{F}}$,

$$L = \mathscr{O}_{\mathbb{F}} \oplus \mathscr{D}_{\mathbb{F}}^{-1},$$

equipped with the usual hermitian form

$$\langle \lambda, \lambda' \rangle = \lambda_1 \bar{\lambda}'_2 + \lambda_2 \bar{\lambda}'_1, \qquad \text{where} \quad \lambda_1, \lambda'_1 \in \mathscr{O}_{\mathbb{F}},\ \lambda_2, \lambda'_2 \in \mathscr{D}_{\mathbb{F}}^{-1}.$$

Then, L is an even unimodular $\mathscr{O}_{\mathbb{F}}$-lattice and $L \otimes_{\mathscr{O}_{\mathbb{F}}} \mathbb{F} = V$ is a two-dimensional vector space over $\mathbb{F}$. After $\mathbb{F}$-sesquilinear extension of the hermitian form, V carries the structure of a hermitian space $V, \langle\cdot,\cdot\rangle$. After further extending scalars, $V_{\mathbb{R}} = V \otimes_{\mathbb{F}} \mathbb{R}$ is a complex hermitian space of signature $(1,1)$.

Write $\mathcal{O}_{\mathbb{F}}$ in the form $\mathbb{Z}+\mathbb{Z}\zeta$ and denote by δ the square root, with the principal branch of the complex square root, of the discriminant $D_{\mathbb{F}}$ of $\mathbb{F}$. Then, as a $\mathbb{Z}$-module, L can be written in the form

$$L = (\mathbb{Z}+\mathbb{Z}\zeta) \oplus \left(\tfrac{1}{\delta}\mathbb{Z} + \tfrac{\zeta}{\delta}\mathbb{Z}\right).$$

With the bilinear form $(\cdot,\cdot) = \mathrm{Tr}_{\mathbb{F}|\mathbb{Q}} \langle\cdot,\cdot\rangle$, L is a quadratic $\mathbb{Z}$-module of signature $(2,2)$, which is isomorphic to the orthogonal sum of two hyperbolic planes over $\mathbb{Z}$.

As an $\mathbb{F}$-basis of V, we choose two lattice vectors, $\ell \in \mathcal{O}_{\mathbb{F}}$, and $\ell' \in \mathcal{D}_{\mathbb{F}}^{-1}$ both isotropic with $\langle \ell, \ell' \rangle \neq 0$. Further, without loss of generality, we require ℓ and ℓ' to be generators of L as an $\mathcal{O}_{\mathbb{F}}$-module, so L can be written as $L = \mathcal{O}_{\mathbb{F}}\ell \oplus \mathcal{O}_{\mathbb{F}}\ell'$.

For example, we can put $\ell = 1$ and $\ell' = -\delta^{-1} \in \mathcal{D}_{\mathbb{F}}^{-1}$, as in example 1.1.14.

The positive projective cone $\mathcal{K}_{\mathrm{U}} \subset \mathbb{P}(V_{\mathbb{R}})$ is a model for the symmetric domain of $\mathrm{SU}(1,1)$. Denote by $\widetilde{\mathcal{K}_{\mathrm{U}}}$ the preimage under the canonical projection π. There is a unique set of representatives $\widetilde{\mathcal{K}_{\mathrm{U}}^1} \subset \widetilde{\mathcal{K}_{\mathrm{U}}}$ of the form

$$\widetilde{\mathcal{K}_{\mathrm{U}}^1} = \{ z = \ell' - \tau\delta \langle \ell', \ell \rangle \ell \}.$$

The Siegel domain $\mathcal{H}_{\mathrm{U}}$ is then given by the upper half-plane

$$\mathcal{H}_{\mathrm{U}} = \{\tau \in \mathbb{C};\ \Im\tau > 0\}.$$

The identification $\mathcal{H}_{\mathrm{U}} \simeq \mathbb{H}$ and the group action.

Since the Siegel domain $\mathcal{H}_{\mathrm{U}}$ is defined as the set of $\tau \in \mathbb{C}$ with $\Im\tau > 0$, clearly as sets, $\mathcal{H}_{\mathrm{U}}$ and the classical upper half plane $\mathbb{H}$ are the same. Since $\mathcal{H}_{\mathrm{U}}$ and $\mathbb{H}$ can also defined as symmetric domains, for the operation of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SU}(1,1)(\mathbb{R})$, respectively, we will briefly describe the isomorphism between $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SU}(1,1)$.

The special unitary group $\mathrm{SU}(1,1)$ operates on $\mathcal{H}_{\mathrm{U}}$ by fractional linear transformations as follows. Let γ in $\mathrm{SU}(1,1)$, and $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be the matrix representing γ as an element of $\mathrm{SU}(V)(\mathbb{R})$ in the basis ℓ and ℓ'. Abbreviate $\epsilon = -\delta \langle \ell', \ell \rangle$. Then,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \tau \longmapsto \frac{A\tau - B\epsilon^{-1}}{C\epsilon\tau + D}.$$

This can be verified easily by considering the operation of γ on the projective line $[z] = [(\epsilon\tau, 1)^t] \in \mathbb{P}(V_{\mathbb{R}})$.

Proposition 5.2.1. *An isomorphism between* $\mathrm{SL}_2(\mathbb{R})$ *and* $\mathrm{SU}(1,1)$ *is given by*

$$\mathrm{SL}_2(\mathbb{R}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & b\epsilon \\ c\epsilon^{-1} & d \end{pmatrix} \in \mathrm{SU}(V)(\mathbb{R}),$$

where $\epsilon = -\delta \langle \ell', \ell \rangle$*. In particular, for* $\langle \ell, \ell' \rangle = \delta^{-1}$*, the mapping between* $\mathrm{SL}_2(\mathbb{R})$ *and* $\mathrm{SU}(V)(\mathbb{R})$ *is the identity. The parabolic subgroup* Γ_∞ *of* $\mathrm{SL}_2(\mathbb{Z})$*, consisting of matrices* $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$*, with* $n \in \mathbb{Z}$*, is sent to the set of translations* $[n] = \left(\begin{smallmatrix} 1 & \epsilon n \\ 0 & 1 \end{smallmatrix}\right)$*, with* $n \in \mathbb{Z}$*, see* (1.1.5) *(these are all the integral points in the parabolic subgroup* $P(\ell)$ *as there are no Eichler elements).*

Heegner divisors and Weyl chambers

Heegner divisors

For the following recall section 4.1.1. Let λ be a lattice vector of negative norm, $\lambda \in L$ with $q(\lambda) = m < 0$, $m \in \mathbb{Z}$.

Through definition in (4.1.2), we have associated to λ the subset $\mathbf{H}_\lambda$ of $\mathscr{K}_U \simeq \mathrm{Gr}_U$,

$$\mathbf{H}_\lambda = \left\{ [z] \in \mathscr{K}_U;\ \langle z, \lambda \rangle = 0 \right\}, \tag{5.2.1}$$

a codimension one sub-Grassmannian of Gr_U, which can be identified with a subset of the Siegel domain $\mathscr{H}_U$, described by (4.1.3).

When $\mathscr{K}_U$ is embedded into $\mathscr{K}_O$, on the image, $\mathbf{H}_\lambda$ agrees with the intersection of $\alpha(\mathscr{H}_U)$ with the analogously defined subset $\lambda^\perp$ of $\mathscr{K}_O$, attached to λ, recall lemma 4.1.2,

$$\alpha(\mathbf{H}_\lambda) = \lambda^\perp \cap \alpha(\mathscr{H}_U), \quad \text{with} \quad \lambda^\perp = \left\{ [Z_L] \in \mathscr{K}_O;\ \left(Z_L, \lambda\right) = 0 \right\}.$$

Finally, for a negative integer m the Heegner divisor of discriminant m is defined as the sum

$$\mathbf{H}(m) = \sum_{\substack{\lambda \in L \\ q(\lambda)=m}} \mathbf{H}_\lambda, \qquad \text{supported on the set} \quad \bigcup_{\substack{\lambda \in L \\ q(\lambda)=m}} \mathbf{H}_\lambda \subset \mathscr{H}_U.$$

We say that a point τ in $\mathbb{H}$ lies on the Heegner divisor $\mathbf{H}(m)$ if, assuming the identification of $\mathscr{H}_U$ with $\mathbb{H}$, the line $[z(\tau)] \in \mathscr{K}_U$ is contained in $\mathbf{H}(m)$.

Weyl chambers

Weyl chambers in $\mathscr{H}_O$ are subsets defined by inequalities involving only the imaginary part of elements $Z \in \mathscr{H}_O$, recall defintion 2.1.15. Given a Weyl chamber W in $\mathscr{H}_O$, by definition 4.1.4 the corresponding Weyl chamber $\mathscr{H}_U$ is the set of τ for which the image in $\mathscr{H}_O$ lies in W.

In the present setting, a Weyl chamber of $\mathscr{H}_U$ can easily be described through inequalities satisfied by the imaginary part of τ. We recover these from the inequalities defining W in $\mathscr{H}_O$, depending on the coordinates used. For this reason, we use the same notation for a Weyl chamber in $\mathscr{H}_O$ and the corresponding Weyl chamber in $\mathscr{H}_U$, at least when the context is clear.

CM-orders of Heegner divisors

Next, we want to describe explicitly the points τ in the upper half plane model $\mathscr{H}_U \simeq \mathbb{H}$ which constitute Heegner divisors and then determine their CM-orders.

The description depends on the choice of the basis vectors ℓ and ℓ' and the realization of the half-plane model $\mathscr{H}_U$ as a subset of the cone $\mathscr{K}_U$. We put $\ell = 1 \in \mathscr{O}_\mathbb{F}$ and $\ell' = -\delta^{-1} \in \mathscr{D}_\mathbb{F}^{-1}$. Then, the lattice L is given by $\mathscr{O}_\mathbb{F}\ell \oplus \mathscr{O}_\mathbb{F}\ell'$ and the representative $z(\tau) \in \widetilde{\mathscr{K}_U^1}$ associated to $\tau \in \mathscr{H}_U$ takes the form $z = \ell' + \tau\ell$.

Consider a lattice vector of negative norm, that is $\lambda \in L$, with $q(\lambda) = \langle \lambda, \lambda \rangle = m$, with m a negative integer. We write λ as

$$\lambda = a\ell' + b\ell, \quad \text{with} \quad a, b \in \mathscr{O}_\mathbb{F}.$$

Then, $m = \langle \lambda, \lambda \rangle = -|\delta|^{-1} 2\Im(a\bar{b})$.

The condition for the point $\tau \in \mathscr{H}_{\mathrm{U}}$ to be an element of the primitive divisor $\mathbf{H}_\lambda$ associated to λ can be phrased as

$$0 = \langle z(\tau), \lambda \rangle = \left\langle \ell' + \tau \ell, a\ell' + b\ell \right\rangle,$$

whence, since $\delta^{-1} = \left\langle \ell, \ell' \right\rangle = -\left\langle \ell', \ell \right\rangle$, we get

$$\tau = \frac{\bar{b}}{\bar{a}}. \tag{5.2.2}$$

The Heegner divisor of index m on $\mathscr{H}_{\mathrm{O}}$ is given by a formal linear combination of points (with multiplicities)

$$\mathbf{H}(m) = \sum_{\substack{\lambda \in L \\ \langle \lambda, \lambda \rangle = m}} \mathbf{H}_\lambda = \sum_{\substack{\lambda = a\ell' + b\ell \\ q(\lambda) = m}} (\tau_\lambda = \bar{b}/\bar{a}).$$

CM-orders

Given a Heegner point τ in the upper half-plane $\mathbb{H} \simeq \mathscr{H}_{\mathrm{U}}$, we consider the lattice $\mathbb{Z} + \tau\mathbb{Z} = \Lambda_\tau$. Since $\tau \in \mathbb{F}$, the elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$ has complex multiplication by an order $\mathscr{O}_\tau$ in $\mathscr{O}_{\mathbb{F}}$. Equivalently, $\mathscr{O}_\tau$ is the multiplier system of the hermitian lattice Λ_τ.

The minimal quadratic equation defining τ as an element of $\mathbb{F}$ is given by

$$\begin{aligned} 0 &= \tau^2 - \mathrm{Tr}(\tau)\tau + N(\tau) \\ &= \tau^2 - \left(\frac{b}{a} + \frac{\bar{b}}{\bar{a}} \right) \tau + N\left(\frac{b}{a} \right) \\ &= \tau^2 - \frac{\mathrm{Tr}\big(a\bar{b}\big)}{N(a)} \tau + \frac{N(b)}{N(a)}. \end{aligned}$$

Now, once we multiply with $N(a)$, we get an equation with integral coefficients:

$$0 = N(a)\tau^2 - \mathrm{Tr}\big(a\bar{b}\big)\tau + N(\bar{b}). \tag{5.2.3}$$

If the equation is reduced, its discriminant is the discriminant of the CM-point τ, see for example [45]. In general, however, we can not assume the coefficients to be coprime – even if λ is primitive. Thus, let $q = \gcd(N(a), N(b), \mathrm{Tr}(a\bar{b}))$.

We rewrite $N(\bar{b})$ as $N(a\bar{b}) \cdot N(a)^{-1}$. Since we have

$$\Im(a\bar{b}) = -\frac{|\delta|}{2} m,$$

this results in the following equation, which has integral coefficients (and is reduced exactly if $q = 1$)

$$0 = N(\alpha)\tau^2 - 2\Re\big(a\bar{b}\big)\tau + N(a)^{-1} \left(\Re\big(a\bar{b}\big)^2 + \tfrac{1}{4}|\delta|^2 m^2 \right). \tag{5.2.4}$$

The discriminant of this equations calculates to

$$D = 4\Re\big(a\bar{b}\big)^2 - 4N(a)N(a)^{-1} \left(\Re\big(a\bar{b}\big)^2 + \tfrac{1}{4}|\delta|^2 m^2 \right) = -|\delta|^2 m^2 = m^2 \delta^2 = m^2 D_{\mathbb{F}}.$$

Now, if $q \neq 1$, we have $q \mid N(a)$ and $q \mid N(b)$, thus, $q^2 \mid N(a\bar{b})$ and since $q \mid \mathrm{Tr}(a\bar{b})$, clearly $q^2 \mid \delta^2 m^2$. Thus, the reduced equation

$$0 = \frac{1}{q}\left(N(a)\tau^2 - \mathrm{Tr}\left(a\bar{b}\right)\tau + N(\bar{b})\right) \tag{5.2.5}$$

has discriminant $D' = D/q^2 = \delta^2(m/q)^2 = D_{\mathbb{F}}(m/q)^2$.

Since $f^2 - f \equiv 0 \mod 2$ for $f \in \mathbb{Z}$, the $\mathbb{Z}$-module $\mathscr{O}_\tau$ can be written in the form

$$\mathscr{O}_\tau = \mathbb{Z} + \frac{D' + \sqrt{D'}}{2}\mathbb{Z} = \mathbb{Z} + \frac{|m|}{q}\frac{D_{\mathbb{F}} + \sqrt{D_{\mathbb{F}}}}{2}\mathbb{Z} = \mathbb{Z} + \frac{|m|}{q}\mathscr{O}_{\mathbb{F}}.$$

Thus the conductor of the CM-order $\mathscr{O}_\tau$, cf. [45] chapter 8 §1 theorems 1 and 3, is given by $|m|/q$, with $q = \gcd(N(a), N(b), \mathrm{Tr}\,(a\bar{b}))$, as above.

Remark 5.2.2. *Note that the discriminant of a CM-points remain invariant under the operation of* $\mathrm{SU}(L)$. *Indeed with the isomorphism between* $\mathrm{SL}_2(\mathbb{Z})$ *and* $\mathrm{SU}(L)$ *given in proposition 5.2.1 it is easily verified that for* τ *as given by* (5.2.2) *we have*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\tau = \frac{\bar{b}'}{\bar{a}'}, \qquad \text{with} \quad \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

$$\text{Also,} \quad q' = \gcd\left(N(a'), N(b'), \mathrm{Tr}(a'\bar{b}')\right) = \gcd\left(N(a), N(b), \mathrm{Tr}(a\bar{b})\right) = q.$$

Embedding and choice of basis

We now recall the embedding from chapter 3 as a preparation for the calculations in the rest of this chapter. Generally, the embedding of symmetric domains takes the form,

$$z \longmapsto \frac{z}{2\,\widehat{\langle \ell', \ell\rangle}} + i\left(\frac{-\hat{\imath}z}{\widehat{\langle \ell', \ell\rangle}}\right) = Z_L,$$

with $z = \ell' - \delta\,\langle \ell', \ell\rangle\,\tau\ell \in \widetilde{\mathscr{K}_{\mathrm{U}}^{1}}$ and where $Z_L \in \widetilde{\mathscr{K}_{\mathrm{O},1}^{+}}$ is the standard representative in $V_{\mathbb{C}}$ associated to the tube domain coordinate Z. In terms of τ and Z, the embedding $\mathscr{H}_{\mathrm{U}} \hookrightarrow \mathscr{H}_{\mathrm{O}}$ is given by

$$\tau \longmapsto Z = \begin{cases} \tau e_3 + \frac{\delta}{2}e_4 & \text{if } \mathscr{D}_{\mathbb{F}} \text{ is even,} \\ \tau e_3 - \bar{\omega}e_4 & \text{if } \mathscr{D}_{\mathbb{F}} \text{ is odd,} \end{cases}$$

with $\omega = \frac{1}{2}(1+\delta)$ and e_3, e_4 basis vectors as determined in (3.1.1) for the signature $(1,1)$ quadratic subspace $K \otimes_{\mathbb{Z}} \mathbb{C} \subset V_{\mathbb{C}}$, $(\cdot,\cdot) = \mathrm{Tr}_{\mathbb{F}/\mathbb{Q}}\langle \cdot,\cdot\rangle$.

We briefly recall some of the construction in the present setting: The $\mathbb{F}$-vector space $V = L \otimes_{\mathscr{O}_{\mathbb{F}}} \mathbb{F}$, when considered as a rational quadratic space over $\mathbb{Q}$, can be decomposed into two hyperbolic planes over $\mathbb{Q}$. In order to get a decomposition of L into two perpendicular hyperbolic planes over $\mathbb{Z}$, we decompose the quadratic space V', $(\cdot,\cdot)$ into (complementary) maximal isotropic subspaces.

As a basis for the first of these, put $e_1 = \ell$ and $e_3 = -\hat{\zeta}\ell$. For the second maximal isotropic subspace, basis elements e_2 and e_4, with $e_2, e_4 \in L$ are determined as in (3.1.1). Then, since $e_4 \perp e_1, e_2$ and $e_2 \perp e_3, e_4$ the lattice is the direct sum of two perpendicular hyperbolic planes spanned by e_1, e_2 and e_3, e_4, respectively.

We denote the latter hyperbolic plane by K, as usual, since $K = L \cap e_1^\perp \cap e_2^\perp$. Further, the basis is normalized to ensure that $(e_1, e_2) = (e_3, e_4) = 1$, so e_1 and e_2 take the role of Borcherds' z and z', while $K \simeq \mathbb{Z}^2$.

While it is possible to treat number fields of both odd and even discriminant consistently, as in chapter 3, we will treat them separately:

- In section 5.2.2, the number field $\mathbb{F}$ has even discriminant $D_\mathbb{F}$ equal to $4d$, with $d \equiv 2, 3 \bmod 4$. Then, $\delta = 2\sqrt{d}$, the ring of integers $\mathcal{O}_\mathbb{F}$ is given by $\mathbb{Z} + \delta/2\mathbb{Z}$.
- On the other hand, in section 5.2.3, $\mathbb{F}$ has odd discriminant. In this case, $D_\mathbb{F}$ is equal to d, with $d \equiv 1 \bmod 4$, δ is given by $\sqrt{d}$, and $\mathcal{O}_\mathbb{F} = \mathbb{Z} + \omega\mathbb{Z}$, where ω denotes $\frac{1}{2}(1+\delta)$.

5.2.2 Number fields with even discriminant

Let d be a negative squarefree integer, $d \equiv 2, 3 \mod 4$. In this case,

$$L = \mathcal{O}_\mathbb{F} \oplus \delta^{-1}\mathcal{O}_\mathbb{F} = \left(\mathbb{Z} + \tfrac{\delta}{2}\mathbb{Z}\right) \oplus \left(\delta^{-1}\mathbb{Z} + \tfrac{1}{2}\mathbb{Z}\right), \quad \text{with} \quad \delta = 2\sqrt{d}.$$

As usual, we choose elements ℓ and ℓ' from L with ℓ primitive isotropic and $\langle \ell, \ell' \rangle \neq 0$ and take these as basis vectors for $V = L \otimes_{\mathcal{O}_\mathbb{F}} \mathbb{F}$. Without loss of generality, we assume $L = \mathcal{O}_\mathbb{F}\ell \oplus \mathcal{O}_\mathbb{F}\ell'$.

Now, a maximal isotropic subspace of $V, (\cdot,\cdot)$ is spanned by $e_1 = \ell$ and $e_3 = -\frac{1}{2}\hat{\delta}\ell = -\hat{\zeta}\ell$. A basis e_2, e_4 for the complementary maximal isotropic subspace as considered in 3.1.2 is given by

$$e_2 = \widehat{\left(\frac{1}{2\langle \ell', \ell\rangle}\right)}\ell', \qquad e_4 = \widehat{\left(\frac{1}{\delta\langle \ell', \ell\rangle}\right)}\ell'. \tag{5.2.6}$$

For the embedding from chapter 3, we have the following description. The image in Gr_O of $[z] \in \mathcal{H}_\mathrm{U}$, represented by $z = \ell' - \langle \ell', \ell\rangle \tau\delta\ell \in \widetilde{\mathcal{H}_\mathrm{U}^1}$, is spanned by vectors $X_L(z)$ and $Y_L(z)$ given by

$$X_L(z) = \widehat{\left(\frac{1}{\langle \ell', \ell\rangle}\right)} z, \qquad Y_L(z) = \widehat{\left(\frac{-i}{\langle \ell', \ell\rangle}\right)} z.$$

Independent of the value of $\langle \ell, \ell'\rangle$, the image $Z_L(z) = X_L + iY_L \in V_\mathbb{C}$ is given by

$$Z_L(z) = e_2 + \frac{\delta}{2}e_4 - \frac{\delta\tau}{2}e_1 + \tau e_3, \qquad \text{while} \quad Z(\tau) = \tau e_3 + \frac{\delta}{2}e_4 \tag{5.2.7}$$

is the corresponding point in $\mathcal{H}_\mathrm{O}$. Its imaginary part Y is given by $\frac{1}{2}|\delta|e_4 + \Im\tau e_3$ and its real part X by $\Re\tau e_3$.

In order to use the results from 5.1 previously calculated, we must choose an ordering of the basis for $K \otimes_\mathbb{Z} \mathbb{R}$. *Thus, put $v_1 = e_4$ and $v_2 = e_3$.*

Now, the stage is set to actually lift modular forms from $\mathcal{M}_0^!$ to automorphic forms for a subgroup of $\mathrm{SU}(L)$.

To begin with, we pull back the Fourier expansions of the additive lifts $\Phi_L(Z, f)$ determined in previous section from $\mathcal{H}_\mathrm{O}$ to $\mathcal{H}_\mathrm{U}$, the Borcherds products $\Xi_{J_b}(\tau)$ will be determined on p. 121ff.

The pullback of the Fourier expansion (5.1.8) of $\Phi_L(f)$ for $f \in \mathscr{M}_0^!$ is given by

$$\begin{aligned}\big(\alpha^*\Phi_L(f)\big)(\tau) = 8\pi \sum_{\substack{m<0\\ c(m)\neq 0}} c(m)\Big(&\sum_{\substack{k,l\in\mathbb{Z}_{>0}\\ kl=-m\\ k\Im\tau_0 - l|\delta|/2>0}} l\frac{|\delta|}{2} + \sum_{\substack{k,l\in\mathbb{Z}_{>0}\\ kl=-m\\ k\Im\tau_0 - l|\delta|/2>0}} k\Im\tau - (\frac{1}{2}|\delta| + \Im\tau)\sigma_{-m}\Big)\\ &+ c(0)\big(-\log(|\delta|\Im\tau) - \Gamma'(1) - \log(2\pi)\big)\\ &+ 4\sum_{\substack{\lambda=(m,n)\in\mathbb{Z}^2\\ (\lambda,W)>0}} c(mn)\log\Big|1 - e\big(m\tau + \pi\tfrac{\delta}{2}n\big)\Big|.\end{aligned}$$

The condition $(\lambda, W) > 0$ for $\lambda = me_4 + ne_3 \in K$ is equivalent to $(Y, \lambda) > 0$ for every $Y = \Im Z$ in the Weyl chamber, which in turn is satisfied if $\big(Y_0, \lambda\big) > 0$ for one Y_0 in the Weyl chamber. Since $Y = \Im\tau e_3 + \frac{1}{2}|\delta|e_4$, we can replace $((m,n), W) > 0$ with the condition

$$\Im\tau m + \frac{1}{2}|\delta|n > 0, \quad \text{for all } \tau \in W, \text{ and further with} \quad \Im\tau_0 m + \frac{1}{2}|\delta|n > 0, \quad \text{for } \tau_0 \in W,$$

since it suffices to verify the Weyl chamber condition for one arbitrary $\tau_0 \in W$. As usual, we consider $\tau \in \mathscr{H}_{\mathrm{U}}$ as contained in W, if the image Z of τ in the tube domain has imaginary part Y with $Y \in W$. If W is defined by a set of conditions of the form $ay_2 + by_1 > 0$, then such are satisfied by τ with $2a\Im\tau + b|\delta| > 0$.

Remark. *We can also write the Weyl chamber condition using $z \in \widetilde{\mathscr{K}_{\mathrm{U}}^1}$, as in chapter 4. The condition $(\lambda, W) > 0$ can be reformulated*

$$0 < \Im\tau_0 m + \frac{|\delta|}{2}n = \big(Y_0, \lambda\big) = \big(Y_{0,L}, \lambda\big) = \Im\frac{\langle z_0, \lambda\rangle}{\langle \ell', \ell\rangle},$$

where the first and third equalities hold by the construction of the embedding, see (3.2.11) *and* (3.2.7)*, while the second holds since $\lambda \in K \otimes_{\mathbb{Z}} \mathbb{R}$ is perpendicular to e_1 and e_2.*

Similarly, the inequalities defining a Weyl chamber can be reformulated in terms of z, see remark 4.1.5 on p. 93.

Now, for $f = J_b$, there is no constant term $c(0)$ and the principal part consists only of q^{-b}. For the pullback $\alpha^*\Phi_L(J_b)$, we get the following

$$\begin{aligned}(\alpha^*\Phi_L(J_b))(\tau) = 8\pi\Big(&\sum_{\substack{k,l\in\mathbb{Z}_{>0}\\ kl=b\\ 2k\Im\tau_0 - l|\delta|>0}} \frac{l|\delta|}{2} + \sum_{\substack{k,l\in\mathbb{Z}_{>0}\\ kl=b\\ 2k\Im\tau_0 - l|\delta|<0}} k\Im\tau - \big(\tfrac{1}{2}|\delta| - \Im\tau\big)\,\sigma_b\Big)\\ &+ 4\sum_{\substack{k,l\in\mathbb{Z}_{>0}\cup\{-b\}\\ \lambda=ke_4+le_3\\ \Im\left(\langle\ell',\ell\rangle^{-1}\langle z(\tau_0),\lambda\rangle\right)>0}} c(kl)\log\Big|1 - e\big(k\tau + \pi\tfrac{\delta}{2}l\big)\Big|.\end{aligned}$$

Remark 5.2.3. *By remark 5.1.9, the Fourier expansion of $\Phi_L(f)$ converges for $q(Y) > 2|m_0|$. Thus, for $\alpha^*\Phi_L(f)$ convergence is assured when $\frac{1}{2}\Im\tau\delta > |m_0|$. In particular, when $\Im\tau\delta > 2b$ for $f = J_b$. As this condition depends on δ, for a fixed input, we get different domains of convergence, as we vary the number field $\mathbb{F}$.*

Example 5.2.4 ($J_1(\tau)$)**.** *Consider the Fourier expansion of $J_1(\tau)$, given in example 5.1.10. The Fourier expansion of the pullback $\alpha^*\Phi_L(J_1)$ takes the following form.*

$$\big(\alpha^*\Phi_L(J_1)\big)(\tau;W) = -8\pi\begin{cases}\Im\tau & \text{for } W_>\\ \frac{1}{2}|\delta| & \text{for } W_<\end{cases} + 4\sum_{\substack{m,n\in\mathbb{Z}_{>0}\cup\{-1\}\\ m>0 \text{ on } W_>\\ n>0 \text{ on } W_<}} c(mn)\log\Big|1-e\big(m\tau+\tfrac{1}{2}n\delta\big)\Big|.$$

The Weyl chambers $W_>$ and $W_<$ are obtained by pulling back the ones introduced in example 5.1.7. They are given by the conditions $2\Im\tau > |\delta|$ and $2\Im\tau < |\delta|$, respectively.

Example 5.2.5 ($J_p(\tau)$)**.** *With the Fourier expansion of $\Phi^L_{J_p}$ from example 5.1.11, the Fourier expansion of the pullback can be given immediately:*

$$\big(\alpha^*\Phi_L(J_p)\big)(\tau;W) = 8\pi\begin{cases}-(p+1)\Im\tau & \text{on } W_{++}\\ -p\Im\tau - p|\delta|/2 & \text{on } W_{+-}\\ -(p+1)|\delta|/2 & \text{on } W_{--}\end{cases} + 4\sum_{\substack{m,n\in\mathbb{Z}_{>0}\cup\{-p\}\\ \lambda=me_4+ne_3\\ \Im\left(\langle\ell',\ell\rangle^{-1}\langle z_0,\lambda\rangle\right)>0}} c(mn)\log\Big|1-e\left(m\tau+n\tfrac{1}{2}\delta\right)\Big|.$$

The Weyl chambers are defined as in example 5.1.8, with $y_2 = \Im\tau$ and $y_1 = \frac{1}{2}|\delta|$ inserted into to defining conditions.

Borcherds products

For the multiplicative lift $\Xi_f(\tau)$ of a weight 0 weakly holomorphic modular form $f \in \mathscr{M}^!_0$, the Borcherds product expansion on $\mathscr{H}_U$ is given by the general formula in our main theorem 4.2.1 of chapter 4 specialized to the case of a unimodular lattices of signature $(1,1)$, in which corollary 4.2.4 simplifies further:

$$\Xi_f(\tau) = e\left(\frac{\langle z,\rho_f(W)\rangle}{\langle\ell',\ell\rangle}\right)\prod_{\substack{\lambda\in K\\ (W,\lambda)>0}}\left(1-e\left(\frac{\langle z,\lambda\rangle}{\langle\ell',\ell\rangle}\right)\right)^{c(q(\lambda))}, \tag{5.2.8}$$

$$\text{where}\quad (W,\lambda)>0 \quad\text{means}\quad \Im\frac{\langle z,\lambda\rangle}{\langle\ell',\ell\rangle} > 0,$$

with $z = z(\tau)$, for every $\tau \in W$ – the notation for the Weyl chamber condition introduced in definition 4.1.6. Recall that it suffices to check this condition for one arbitrary $\tau_0 \in W$, see

We then determine the Weyl vector using the results previously calculated in section 5.1. If W is a Weyl chamber with respect to f, the Weyl vector $\rho_f(W)$ can be obtained from Weyl vectors of the form (5.1.7) as follows, cf. [9], p. 88. If for $m<0$ the coefficient $c(m)$ in the principal part of the Fourier expansion of f is non-zero, there is a Weyl chamber of index m containing W, say W_m. The Weyl vector of this chamber $\rho_m(W_m)$ is multiplied with a factor of $1/2$, when passing to the smaller chamber W, further, we must add the correction term $\sigma_{|m|}\cdot(1,1)$ to take into account for the constant term of the Poincaré series $F_b(\tau,1)$, recall p. 103. Contributions for all $m<0$ with $c(m)\neq 0$ are summed, weighted by the Fourier coefficients $c(m)$, thus,

$$\rho_f(W)=\sum_{\substack{m<0\\ c(m)\neq 0}}\frac{c(m)}{2}\big(\rho_m(W_m)-\sigma_{|m|}\cdot(1,1)\big)+\frac{c(0)}{24}. \tag{5.2.9}$$

Remark. *Since we have already calculated some of the relevant Borcherds products in the previous section 5.1, rather then using theorem 4.2.1, we could alternatively determine Borcherds products on* SU(1,1) *through the pull back under the embedding α from* (5.2.7). *It should be clear that the pullback $\big(\alpha^*\Psi_L\big)(\tau)$ would give the same expression for $\Xi_f(\tau)$ as* (5.2.8).

This can also be verified through a calculation, by showing that $\big(Z_L,v\big)\left\langle \ell,\ell'\right\rangle=\left\langle z,v\right\rangle$ for every $v\in V_{\mathbb{R}}$ and $\left\langle z,\kappa\right\rangle=\left\langle \ell,\ell'\right\rangle\big(Z,\kappa\big)$ for all $\kappa\in K\otimes_{\mathbb{Z}}\mathbb{R}$.

Example 5.2.6 (Constant functions)**.** *To determine the lift $\Xi_1(\tau)$ of the constant function* 1*, we can mimic the considerations used in 5.1.1 to determine $\Psi_L(Z,1)$ and find that Ξ_1 is given by an eta function times some constant. However, in this particular case, it is more straightforward to determine $\Xi_1(\tau)$ from the earlier result for $\Psi_L(Z,1)$ by pulling back under the embedding* (5.2.7)*. This gives*

$$\begin{aligned}\Xi_1(\tau)&=e\left(\frac{1}{24}(\tau+\tfrac{1}{2}\delta)\right)\prod_{n>0}\Big(1-e\big(\tfrac{1}{2}\delta n\big)\Big)\prod_{m>0}\big(1-e(m\tau)\big)\\&=\eta\big(\tfrac{\delta}{2}\big)\,\eta(\tau).\end{aligned}$$

The expression for Borcherds products in (5.2.8) can be brought into a simpler form, showing no dependence on how $\mathscr{H}_{\mathrm{U}}$ is constructed as a subset of $\widetilde{\mathscr{K}_{\mathrm{U}}}$. This way, the interpretation of $\Xi_f(\tau)$ as a function on the upper half plane $\mathbb{H}\simeq\mathscr{H}_{\mathrm{U}}$ is more apparent:

$$\Xi_f(\tau)=e\left(\rho_2\tau+\rho_1\frac{\delta}{2}\right)\prod_{\substack{m,n\in\mathbb{Z}\\ 2n\Im\tau+|\delta|m>0\\ \forall\tau\in W}}\left(1-e\left(\tau n+\frac{\delta}{2}m\right)\right)^{c(mn)}, \tag{5.2.10}$$

with the Weyl vector $(\rho_1,\rho_2)=\rho_f(W)$ given by (5.2.9).

It suffices to verify

$$\begin{aligned}\left\langle z,\lambda\right\rangle&=\left\langle \ell',\ell\right\rangle\left(\tau\lambda_2+\tfrac{1}{2}\delta\lambda_1\right),\quad\text{for}\quad\lambda=(\lambda_1,\lambda_2)\in K\simeq\mathbb{Z}^2\quad\text{and}\\ \left\langle z,\rho_f\right\rangle&=\left\langle \ell',\ell\right\rangle\left(\rho_2\tau+\rho_1\tfrac{1}{2}\delta\right).\end{aligned}$$

Replacing τ by τ_0 and taking the imaginary part, it also follows that the Weyl chamber condition $(W,\lambda)>0$ can be written in the form $2\lambda_2\Im\tau+|\delta|\lambda_1>0$.

In fact, since both λ and ρ_f are contained in the real subspace $K \otimes_{\mathbb{Z}} \mathbb{R}$ of the real space $V'_{\mathbb{R}}$ underlying $V_{\mathbb{R}}$, it suffices to consider an arbitrary vector $\kappa \in K \otimes_{\mathbb{Z}} \mathbb{R}$. Write κ as $\kappa = (\kappa_1, \kappa_2) = \kappa_1 e_3 + \kappa_2 e_4$, with $e_3 = -\frac{1}{2}\delta\ell$ and e_4 given by (5.2.6). Then,

$$\begin{aligned}\langle z, \kappa\rangle &= \frac{1}{2}\delta\kappa_1 \langle \ell', \ell\rangle - \tau\delta \langle \ell', \ell\rangle \kappa_2 \langle \ell, e_4\rangle \\ &= \langle \ell', \ell\rangle \left(\tfrac{1}{2}\delta\kappa_1 - \tau\delta\kappa_2 \big(-\delta\overline{\langle \ell', \ell\rangle}\big)^{-1} \langle \ell, \ell'\rangle\right) \\ &= \langle \ell', \ell\rangle \left(\tfrac{1}{2}\delta\kappa_1 + \tau\kappa_2\right),\end{aligned}$$

as claimed.

Borcherds products with inputs $J_b(\tau)$

For the Borcherds products of Ξ_{J_b}, $b = 1, 2, \ldots$, since $J_b = q^{-b} + O(q)$, we need only consider Weyl chambers of index $m = -b$. Further, the coefficient $c(-b) = 1$ and there is no constant term. Thus, (5.2.9) simplifies and the Weyl vector $\rho_{J_b}(W)$ is given by

$$\rho_{J_b}(W) = \frac{1}{2}\rho_m(W) - \sigma_{|m|}(1,1). \tag{5.2.11}$$

We consider two examples for $b = 1$ and for $b = p$ with p a prime number. Further somewhat more involved examples are given in section 5.2.4.

Example 5.2.7. *We begin by considering the Borcherds lift of*

$$J_1(\tau) = j(\tau) - 744 = q^{-1} + 196884\,q + 21493760\,q^2 + \cdots$$

Here, we are in the case where $m = -1$. The Weyl chambers and corresponding Weyl vectors have been determined in example 5.1.7. There are two Weyl chambers in $\mathscr{G}(K)$, $W_>$ and $W_<$ defined through $y_2 > y_1$ and $y_2 < y_1$. As Weyl chambers in $\mathscr{H}_{\mathrm{U}}$, they are defined by $2\Im\tau > |\delta|$ and $2\Im\tau < |\delta|$. (Keep in mind that we have fixed the isomorphism $\mathbb{Z}^2 \simeq K$ by putting $v_1 = e_4$ and $v_2 = e_3$.)

For the Weyl chamber $W_>$, the Weyl vector is given by

$$\rho_{J_1}(W_>) = \frac{1}{2}\rho_{-1}(W_>) - (1,1) = e_3 - e_3 - e_4 = (0,-1).$$

Thus, the Borcherds product for Ξ_{J_1} takes the form

$$\Xi_{J_1}(\tau)(W_>) = e(-\tau) \prod_{\substack{m,n\in\mathbb{Z}\\ n>0}} \left(1 - e\big(n\tau + \tfrac{1}{2}\delta m\big)\right)^{c(mn)}.$$

The condition $n > 0$ is obtained as follows; We have $c(mn) = 0$ for $mn \notin \mathbb{Z}_{>0} \cup \{-1\}$. Of the two vectors $\lambda = (m,n)$ with $q(\lambda) = -1$, only $(-1,1)$ satisfies the Weyl chamber condition $\Im\tau n + m|\delta|/2 > 0$ for $\tau \in W_>$.

For the Weyl chamber $W_<$, we have $\rho_{J_1}(W_<) = \frac{1}{2} \cdot 2e_4 - (1,1) = (-1,0)$. The attached Borcherds product is given by

$$\Xi_{J_1}(\tau)(W_<) = e\big(-\tfrac{1}{2}\delta\big) \prod_{\substack{m,n\in\mathbb{Z}\\ n>0}} \left(1 - e\big(m\tau + n\tfrac{1}{2}\delta\big)\right)^{c(mn)}.$$

The condition for absolute convergence of either product is $\frac{1}{2}\Im\tau|\delta| > 1$. So, for example $W_>$ is contained in the domain of convergence, which, if $d \neq -1$, also includes part of $W_<$.

Example 5.2.8. *We now consider the case where m is the negative of a prime integer, that is the lift of $J_p(\tau) = q^{-p} + O(p)$. Recall from example 5.1.8 that there are three Weyl chambers of $\mathcal{G}(K)$, W_{++}, W_{-+} and W_{--}, which are defined by the inequalities $y_2 > py_1$, $py_1 > y_2 > \frac{1}{p}y_1$ and $y_2 < \frac{1}{p}y_1$, respectively.*

The three resulting Weyl chambers of $\mathcal{H}_U$ are then described as follows. Every τ contained in W_{++} satisfies the condition $2\Im\tau > |\delta|p$, while $\tau \in W_{-+}$ satisfies $|\delta|p > 2\Im\tau > |\delta|/p$ and, finally, $\tau \in W_{--}$ if $|\delta|/p > 2\Im\tau$.

The divisor sum σ_p is equal to $p+1$, thus, from (5.2.11), the Weyl vectors for $W = W_{++}, W_{-+}, W_{--}$ are given by

$$\rho_{J_p}(W) = \frac{1}{2}\rho_{-p}(W) - (p+1)\cdot(1,1) = \begin{cases} -(p+1)e_4 & \text{if} \quad W = W_{++}, \\ -pe_3 - pe_4 & \text{if} \quad W = W_{-+}, \\ -(p+1)e_3 & \text{if} \quad W = W_{--}, \end{cases}$$

with the Weyl vectors $\rho_{-p}(W)$ from example 5.1.8 with v_1 replaced by e_4 and v_2 by e_3.

The Borcherds product expansion of Ξ_{J_p} is given by

$$\Xi_{J_p}(\tau)(W) = \prod_{\substack{m,n\in\mathbb{Z} \\ 2m\Im\tau+|\delta|n>0 \\ \forall\tau\in W}} \left(1 - e\left(m\tau + \tfrac{1}{2}n\delta\tau\right)\right)^{c(mn)}$$
$$\times \begin{cases} e\left(-(p+1)\tau\right) \\ e\left(-\frac{1}{2}p\delta - p\tau\right) \\ e\left(-\frac{1}{2}(p+1)\delta\right), \end{cases}$$

depending on whether $W = W_{++}$, $W = W_{-+}$ or $W = W_{--}$.

The condition for absolute convergence is given by $\Im\tau > \frac{2p}{|\delta|}$. As we vary the input J_p, we see that this condition is, for example, always satisfied on W_{++}, and that if $|d| > p^2$, the domain of convergence encompasses all of W_{-+} and part of W_{--}. Thus, the Borcherds product attached to W_{--} converges on W_{++} but may fail to converge on W_{--} itself, namely if $|d| < p^2$.

5.2.3 Number fields with odd discriminant

Now let d be a negative squarefree integer, with $d \equiv 1 \mod 4$. Then, $\mathbb{F} = \mathbb{Q}(\sqrt{d})$ has discriminant $D_{\mathbb{F}} = d$ and $\delta = \sqrt{d}$. In this case, the lattice L is given by

$$L = \mathcal{O}_{\mathbb{F}} \oplus \delta^{-1}\mathcal{O}_{\mathbb{F}} = (\mathbb{Z} + \omega\mathbb{Z}) \oplus \left(\frac{1}{\delta}\mathbb{Z} + \frac{\omega}{\delta}\mathbb{Z}\right), \quad \text{where} \quad \omega = \frac{1}{2}(1+\delta).$$

As before, we select elements $\ell \in \mathcal{O}_{\mathbb{F}}$ and $\ell' \in \mathcal{D}_{\mathbb{F}}^{-1}$, which we take as a basis of $L \otimes_{\mathcal{O}_{\mathbb{F}}} \mathbb{F}$. We require ℓ and ℓ' to be generators of L as an $\mathcal{O}_{\mathbb{F}}$-module.

In the rational quadratic space $V, (\cdot,\cdot)$, a maximal isotropic subspace is spanned by $e_1 = \ell$ and $e_3 = -\hat{\ell}$. As determined in section 3.1.2, a basis e_2, e_4 for a second maximal isotropic subspace, where e_1, e_2 and e_3, e_4 span two orthogonal hyperbolic planes, consists of

$$e_2 = \frac{1+\delta^{-1}}{2\langle\ell',\ell\rangle}\ell' = \frac{1}{\langle\ell',\ell\rangle}\frac{\omega}{\delta}\ell', \qquad e_4 = \frac{1}{\delta\langle\ell',\ell\rangle}\ell'. \tag{5.2.12}$$

Under the embedding $\alpha : \mathscr{H}_\mathrm{U} \to \mathscr{H}_\mathrm{O}$, as examined in chapter 3, the image $Z(\tau)$ in $\mathscr{H}_\mathrm{O}$ of $\tau \in \mathscr{H}_\mathrm{U}$ and the corresponding representative in $\widetilde{\mathscr{K}_{\mathrm{O},1}^{+}}$ are given by

$$\begin{aligned} Z(\tau) &= (-\bar{\omega})e_4 + \tau e_3, \\ Z_L(\tau) &= e_2 + (-\bar{\omega})e_4 + \tau e_3 + \bar{\omega}\tau e_1, \end{aligned}$$

as in (3.2.12) and (3.2.13), with σ omitted in the current setting, of course.

Weyl chambers

In the present case, conditions of the form $ay_2 + by_1 > 0$ defining a Weyl chamber W can be reformulated as $2a\Im\tau + b|\delta| > 0$, since $Y = \Im\tau e_3 + \frac{1}{2}|\delta|e_4$. (Recall that we identify e_3 with v_2 and e_4 with v_1.) For example, the Weyl chamber $W_>$ from example 5.1.7 defined through $y_2 > y_1$ corresponds to a Weyl chamber $2\Im\tau > |\delta|$.

A condition of the form $(Y, \lambda) > 0$, for all Y in W, in turn is satisfied by $\lambda = me_4 + ne_3 \in K$ with

$$2m\Im\tau + n|\delta| > 0,$$

for all $\tau \in W$, implied by $2m\Im\tau_0 + n|\delta| > 0$ for one arbitrary, fixed $\tau_0 \in W$.

Finally, the domain of convergence for the lift is described by an inequality of the form

$$q(Y) = \Im\tau \cdot \Im(-\bar{\omega}) = \frac{1}{2}|\delta|\Im\tau > |m_0|.$$

The Borcherds lift

Pulled back Fourier expansion

As in the case of number fields with even discriminant, we apply the pull-back under α to the Fourier expansion of the Borcherds lift. We limit this to two examples.

The Fourier expansion of $\alpha^*\Phi_L(J_b)$ for a general J_b, with $b = 1, 2, 3, \dots$ is given by

$$\begin{aligned} \big(\alpha^*\Phi_L(J_b)\big)(\tau) = 8\pi\Big(&\sum_{\substack{k,l \in \mathbb{Z}_{>0} \\ kl = -b \\ 2k\Im\tau_0 - l|\delta| > 0}} \frac{l|\delta|}{2} + \sum_{\substack{k,l \in \mathbb{Z}_{>0} \\ kl = -b \\ 2k\Im\tau_0 - l|\delta| > 0}} k\Im\tau - \big(\tfrac{1}{2}|\delta| - \Im\tau\big)\sigma_d \Big) \\ &+ 4 \sum_{\substack{k,l \in \mathbb{Z}_{>0} \cup \{b\} \\ 2k\Im\tau_0 + l|\delta| > 0}} c(kl) \log\big|1 - e\big(k\tau - \bar{\omega}l\big)\big|. \end{aligned}$$

For the simplest example, the lift of J_1, we have

$$\begin{aligned} \big(\alpha^*\Phi_L(J_1)\big)(\tau) = &\begin{cases} -8\pi\Im\tau & \text{on } W_> \\ -2\pi|\delta| & \text{on } W_< \end{cases} \\ &+ 4 \sum_{\substack{m,n \in \mathbb{Z} \\ m > 0 \text{ for } W_> \\ n > 0 \text{ for } W_<}} c(mn) \log\big|1 - e\big(-m\bar{\omega} + n\tau\big)\big|. \end{aligned}$$

Borcherds products

For the constant function 1, the Borcherds product takes the form of an eta-product. It is most easily calculated by pulling back $\Psi_L(Z,1)$, as calculated in section 5.1.1. We get, quite simply

$$\Xi_1(\tau) = e\left(\frac{\tau-\bar{\omega}}{24}\right)\prod_{n>0}\big(1-e(-n\bar{\omega})\big)\prod_{m>0}\big(1-e(m\tau)\big) = \eta(-\bar{\omega})\,\eta(\tau).$$

As in (5.2.10), for a Weyl chamber W, we can write the Borcherds product expansion of Ξ_{J_b} in the form

$$\Xi_{J_b}(\tau) = e\big(\rho_2\tau-\rho_1\bar{\omega}\big)\prod_{\substack{k,l\in\mathbb{Z}\\ 2k\Im\tau+l|\delta|>0\\ \forall\tau\in W}}\big(1-e\big(\tau k-l\bar{\omega}\big)\big)^{c(kl)}, \tag{5.2.13}$$

with $\rho_{J_b}(W)=\rho_1 e_3+\rho_2 e_4$ given by

$$\rho_{J_b}(W) = \frac{1}{2}\rho_{-b}(W)-\sigma_b(1,1),$$

as in (5.2.11). The product expansion (5.2.13) follows from theorem 4.2.1 and its simplified version (5.2.8) through a similar calculation as for (5.2.10): For $\kappa = e_3\kappa_1+e_4\kappa_2 \in K\otimes_{\mathbb{Z}}\mathbb{R}$, we have

$$\begin{aligned}\langle z,\kappa\rangle &= -\bar{\omega}\kappa_1\,\langle\ell',\ell\rangle - \tau\delta\,\langle\ell',\ell\rangle\,\kappa_2\,\langle\ell,e_4\rangle\\ &= \langle\ell',\ell\rangle\Big(-\bar{\omega}\kappa_1-\tau\delta\kappa_2\big(-\delta\langle\ell,\ell'\rangle\big)^{-1}\langle\ell,\ell'\rangle\Big)\\ &= \langle\ell',\ell\rangle\left(-\bar{\omega}\kappa_1+\tau\kappa_2\right).\end{aligned}$$

It follows that

$$\frac{\langle z,\rho_f(w)\rangle}{\langle\ell,\ell'\rangle} = \tau\rho_2-\bar{\omega}\rho_1, \quad \frac{\langle z,\lambda\rangle}{\langle\ell,\ell'\rangle} = \tau\lambda_2-\bar{\omega}\lambda_1.$$

The Weyl chamber condition $\Im\big(\langle z_0,\lambda\rangle\,\langle\ell',\ell\rangle^{-1}\big)>0$ for $\lambda=\lambda_1 e_3+\lambda_2 e_4$, with $\lambda_i\in\mathbb{Z}$, $i=1,2$, can be rephrased as $\Im\tau_0\lambda_2+\frac{1}{2}|\delta|\lambda_1>0$, as the imaginary part of $-\bar{\omega}$ is given by $\frac{1}{2}|\delta|$.

The condition for absolute convergence of the product in (5.2.13) can be formulated as $\frac{1}{2}\Im\tau|\delta|>b$.

Finally, we consider the standard examples $b=1$ and $b=p$, with p a prime, as before.

Example 5.2.9. *We consider the lift of $J_1(\tau)$, as above in 5.2.7, but now for number fields with odd discriminant. The Weyl chambers $W_>$ and $W_<$ are given by $2\Im\tau>|\delta|$ and $2\Im\tau<|\delta|$. The Borcherds product for $W_<$ is given by*

$$\Xi_{J_1}(\tau)(W_<) = e(+\bar{\omega})\prod_{\substack{m,n\in\mathbb{Z}\\ n>0}}\big(1-e\big(m\tau-n\bar{\omega}\big)\big)^{c(mn)}.$$

whereas for $W_>$ we get

$$\Xi_{J_1}(\tau)(W_>) = e(-\tau)\prod_{\substack{m,n\in\mathbb{Z}\\ m>0}}\big(1-e\big(m\tau-n\bar{\omega}\big)\big)^{c(mn)}.$$

Example 5.2.10. *As for number fields of even discriminant, see example 5.2.8, we consider the lift of $J_p(\tau)$, for a prime p. There are three Weyl chambers W_{++}, W_{-+} and W_{--}. Of these, W_{++} is defined by the condition $2\Im\tau > |\delta|p$, while elements of W_{-+} must satisfy $|\delta|p > 2\Im\tau > |\delta|/p$. For W_{--} the requirement is $|\delta|/p > 2\Im\tau$.*

The product expansion of Ξ_{J_p} is then given by

$$\Xi_{J_p}(\tau)(W) = \prod_{\substack{m,n\in\mathbb{Z} \\ 2m\Im\tau+n|\delta|>0 \\ \forall\tau\in W}} \left(1 - e\left(m\tau - n\bar{\omega}\right)\right)^{c(mn)}$$

$$\times \begin{cases} e\left(-(p+1)\tau\right) & \text{on } W_{++} \\ e\left(p\bar{\omega} - p\tau\right) & \text{on } W_{-+} \\ e\left(+(p+1)\bar{\omega}\right) & \text{on } W_{--} \end{cases}.$$

The product expansion converges for τ with $\Im\tau > 2p|\sqrt{d}|^{-1}$. The domain of convergence varies in extent depending both on J_p and on $\mathbb{F}$. For example, if $|d| \geq 4$ it encompasses all of W_{++} and also part of W_{-+}. (As $d \equiv 1 \mod 4$ this implies $|d| \geq 7$.) Convergence on (part of) W_{--} requires $|d| > 4p^2$.

5.2.4 Further examples: Lift of $J_b(\tau)$ with squarefree b

We work out one further, slightly more involved example case for the lifting, which encompasses our previous standard example of J_p, with p a prime.

In the following, let m be a squarefree negative integer, $m \neq -1$. Denote by P the set of divisors of $|m| = -m$, including 1 and $|m|$,

$$P = \left\{d \in \mathbb{Z}_{>0}\,;\, d \mid -m\right\}.$$

Heegner divisors and Weyl chambers

For any $t \in P$, we have lattice vectors $\pm\lambda_\pm(t)$ in $\mathbb{Z}^2$ with norm m,

$$\lambda_+(t) = \left(\tfrac{|m|}{t}, -t\right), \quad \text{and} \quad \lambda_-(t) = \left(t, -\tfrac{|m|}{t}\right).$$

Note that $\lambda_\pm(t) = \lambda_{mp}(|m|/t)$. The two lattice vectors attached to t define Heegner 'divisors', the lines

$$\lambda_+(t)^\perp = \mathbb{R}\left(\tfrac{|m|}{t}, t\right), \text{ and } \quad \lambda_-(t)^\perp = \mathbb{R}\left(t, \tfrac{|m|}{t}\right),$$

which are reflections of each other along the diagonal. Such lines, for $t \in P$, mark the boundary of the Weyl chambers of index m. For each t, to $\lambda_+(t)$ we attach the condition $\left(\lambda_+(t), Y\right) > 0$. This can be written as an inequality of the form

$$y_2 > \left(\frac{|m|}{t}\right)^{-1} t y_1 = \frac{t^2}{|m|} y_1.$$

Now, let t, t' be elements of P. Clearly, $t > t'$ implies $t^2/|m| > t'^2/|m|$. Thus, if Y satisfies the condition attached to t, it satisfies the condition attached to $\lambda_+(t')$ for any t' less than t.

Similarly, for $\lambda_-(t)$ the condition $\big(\lambda_-(t), Y\big) < 0$, defines the inequality

$$y_2 < \frac{|m|}{t^2} y_1.$$

The set of $Y \in \mathbb{R}^2$ satisfying this condition is the reflection through the diagonal of the set of Y satisfying $\big(\lambda_+(t), Y\big) > 0$. If Y satisfies the condition attached to $\lambda_-(t)$ it will also satisfy the corresponding conditions for all t' *smaller* than t.

We can parametrize the set lines $\lambda_+(t)^\perp$ and $\lambda_-(t)^\perp$ by a subset Q of P defined as

$$Q = \left\{ d \in P\,;\ d > \frac{|m|}{d} \right\}.$$

That the association of t to the pair $\lambda_+(t)$, $\lambda_-(t)$ is unique for $t \in Q$.

To each $t \in Q$, we associate two mirror-image subsets of the Y-plane, denoted W_t^+ and W_t^-,

$$W_t^+ = \left\{ Y = (y_1, y_2) \in \mathbb{R}^2\,;\ y_2 > \tfrac{t^2}{|m|} y_1 \quad \text{and} \quad \tfrac{t'^2}{|m|} y_1 > y_2, \forall t' \in Q, t' > t \right\},$$
$$W_t^- = \left\{ Y = (y_1, y_2) \in \mathbb{R}^2\,;\ y_2 < \tfrac{|m|}{t^2} y_1 \quad \text{and} \quad y_2 > \tfrac{|m|}{t'^2} y_1, \forall t' \in Q, t' > t \right\}.$$

It follows from the above considerations that W_t^+ and W_t^- are Weyl chambers. Besides the $W_t^\pm$ for $t \in Q$, there is one further Weyl chamber, which contains the diagonal. We denote this by W_1, it can be described as the set

$$W_1 = \left\{ Y = (y_1, y_2) \in \mathbb{R}^2\,;\ \frac{q^2}{|m|} y_1 > y_2 > \frac{|m|}{q^2} y_1 \right\},$$

with q the smallest element of Q. While this is the 'middle' Weyl chamber, the two 'outermost' Weyl chambers are $W_{|m|}^+$ and $W_{|m|}^-$. For $t \in \mathbb{Q}$ with $t \neq |m|$ the Weyl chamber W_t^+ is bounded by the lines

$$\lambda_{+,t}^\perp = \mathbb{R}\left(t, \tfrac{|m|}{t}\right) = \lambda_{-,|m|/t}^\perp$$
$$\lambda_{+,u}^\perp = \mathbb{R}\left(r, \tfrac{|m|}{u}\right) = \lambda_{-,|m|/u}^\perp,$$

where u is the smallest element of Q larger than t, thus $u = \min\{d \in Q; d > t\}$.

Remark 5.2.11. *If we define W_t^+ and W_t^- for all $t \in P \setminus Q$, as well, then each Weyl chamber $W \neq W_{|m|}^\pm$ has two descriptions: Either as W_t^+ for some t or as W_r^- for $r = \frac{|m|}{u}$ where u is next-largest to t in P. In particular, $W_1 = W_q^- = W_{|m|/q}^+$. The two Weyl chambers W_t^+ and W_s^- with $s = \frac{|m|}{t}$, share a boundary.*

Determining Φ_m^K.

Next, we calculate $\Phi_m^{\mathbb{Z}^2}(Y;W)$ for each Weyl chamber. Since the vectors in $\mathbb{Z}^2$ with norm m are given by the set $\{\lambda_\pm(t)\,;\, t\in Q\}$, the expression for Φ_m^K from (5.1.6) takes the form

$$\Phi_m^{\mathbb{Z}^2}(Y;W) = \frac{8\sqrt{2}\pi}{|Y|}\cdot 2\Bigg(\sum_{\substack{d\in Q\\ (\lambda_+(d),Y_0)>0}} d y_1 + \sum_{\substack{d\in Q\\ (\lambda_-(d),Y_0)>0}} \frac{|m|}{d} y_1 + \sum_{\substack{d\in Q\\ (\lambda_+(d),Y_0)<0}} \frac{|m|}{d} y_2 + \sum_{\substack{d\in Q\\ (\lambda_-(d),Y_0)<0}} d y_2\Bigg),$$

with Y_0 an arbitrary element in W. We now work out what this is for one of the Weyl chambers $W_t^\pm$, so let t be an element of Q. For $Y\in W_t^+$ and $d\in Q$, the condition $\big(\lambda_+(d),Y\big)>0$ holds iff $d\le t$, while $\big(\lambda_-(d),Y\big)>0$ for all d. So,

$$\begin{aligned}\Phi_m^{\mathbb{Z}^2}(Y;W_t^+) &= \frac{8\sqrt{2}\pi}{|Y|}\cdot 2\Bigg(\sum_{\substack{d\in Q\\ d\le t}} d y_1 + \sum_{d\in \mathbb{Q}} \frac{|m|}{d} y_1 + \sum_{\substack{d\in Q\\ d>t}} \frac{|m|}{d} y_2\Bigg)\\ &= \frac{8\sqrt{2}\pi}{|Y|}\cdot 2\Bigg(\sum_{\substack{d\in P\\ d\le t}} d y_1 + \sum_{\substack{d\in P\\ d>t}} \frac{|m|}{d} y_2\Bigg).\end{aligned}$$

In contrast, for $Y\in W_t^-$, we have $\big(\lambda_-(d),Y\big)<0$ precisely if $d\le t$, while $\big(\lambda_+(d),Y\big)<0$ for all $d\in Q$. Thus

$$\begin{aligned}\Phi_m^{\mathbb{Z}^2}(Y;W_t^-) &= \frac{8\sqrt{2}\pi}{|Y|}\cdot 2\Bigg(\sum_{\substack{d\in Q\\ d>t}} \frac{|m|}{d} y_1 + \sum_{d\in Q} \frac{|m|}{d} y_2 + \sum_{\substack{d\in Q\\ d\le t}} d y_2\Bigg)\\ &= \frac{8\sqrt{2}\pi}{|Y|}\cdot 2\Bigg(\sum_{\substack{d\in P\\ d>t}} \frac{|m|}{d} y_1 + \sum_{\substack{d\in P\\ d\le t}} d y_2\Bigg).\end{aligned}$$

The Weyl vectors attached to W_t^+ and W_t^- are given by

$$\rho_m(W_t^+) = 2\sum_{\substack{d\in P\\ d\le t}} \frac{|m|}{d} v_1 + 2\sum_{\substack{d\in P\\ d\le t}} d v_2, \quad\text{and}\quad \rho_m(W_t^-) = 2\sum_{\substack{d\in P\\ d\le t}} d v_1 + 2\sum_{\substack{d\in P\\ d\le t}} \frac{|m|}{d} v_2. \tag{5.2.14}$$

In particular, for $\Phi_m^{\mathbb{Z}^2}$ on the two 'outer-most' Weyl chambers $W_{|m|}^\pm$, we have

$$\Phi_m^{\mathbb{Z}^2}(Y;W_{|m|}^+) = \frac{8\sqrt{2}\pi}{|Y|}\cdot 2\big(\sigma_{|m|} y_1\big) \quad\text{and}\quad \Phi_m^{\mathbb{Z}^2}(Y;W_{|m|}^-) = \frac{8\sqrt{2}\pi}{|Y|}\cdot 2\big(\sigma_{|m|} y_2\big).$$

Finally, for the 'middle' Weyl chamber W_1, for all d, $\big(\lambda_+(d),Y\big)<0$ and $\big(\lambda_-(d),Y\big)>0$, thus

$$\Phi_m^{\mathbb{Z}^2}(Y;W_1) = \frac{8\sqrt{2}\pi}{|Y|}\cdot 2\sum_{d\in Q}\frac{|m|}{d}(y_1+y_2)$$

The attached Weyl vector $\rho_m(W_1)$ takes the form

$$\rho_m(W_1) = \Big(2\sum_{d\in P\setminus Q} d\Big)\cdot\big(v_1+v_2\big). \tag{5.2.15}$$

Borcherds products $\Xi_{J_{|m|}}$

We are now ready to deal with the lift to SU(1,1). In contrast to the previous sections of this chapter, we will treat both number fields of even and of odd discriminant at same time, as in chapter 3.

Write $\mathscr{O}_{\mathbb{F}}$ in the form $\mathbb{Z}+\zeta\mathbb{Z}$ with $\zeta=\frac{1}{2}\delta$ if $D_{\mathbb{F}}$ is even and $\zeta=\omega=\frac{1}{2}(1+\delta)$ if $D_{\mathbb{F}}$ is odd – keeping in mind that $\delta=2\sqrt{d}$ in the former and $\delta=\sqrt{d}$ in the latter case. The embedding $\alpha:\mathscr{H}_{\mathrm{U}}\to\mathscr{H}_{\mathrm{O}}$ is then given by $\tau\longmapsto Z_L=-\bar{\zeta}e_4+\tau e_3$, with the basis vectors $e_1,\dots,e_4$ from (3.1.1).

The conditions defining the Weyl chambers now take the form of inequalities on $\Im\tau$, namely

$$\Im\tau>\frac{t^2|\delta|}{2|m|}\qquad\text{for }\lambda_+(t)\qquad\text{and}\quad\Im\tau<\frac{|m|\,|\delta|}{2t^2}\qquad\text{for }\lambda_-(t).$$

So, for example $W^+_{|m|}$ is the half-plane defined by $2\Im\tau>|m||\delta|$, W_1 is the strip $\frac{q^2}{|m|}|\delta|>2\Im\tau>\frac{|m|}{q^2}|\delta|$ and $W^-_{|m|}$ is given by $\frac{|\delta|}{|m|}>2\Im\tau>0$.

The corrected Weyl vectors $\rho_{J_{|m|}}(W)$ can be recovered from (5.2.14) by way of (5.2.11) with the identification $v_1=e_4$, $v_2=e_3$. Thus,

$$\begin{aligned}\rho_{J_{|m|}}(W_t^+)&=\sum_{\substack{s\in P\\ s\le t}}\big(s,\tfrac{|m|}{s}\big)-\sigma_{|m|}(1,1)&&=\sum_{\substack{s\in P\\ s>t}}(-s)\cdot e_3+\sum_{\substack{s\in P\\ s>|m|/t}}(-s)\cdot e_4,\\ \rho_{J_{|m|}}(W_t^-)&=\sum_{\substack{s\in P\\ s\le t}}\big(\tfrac{|m|}{s},s\big)-\sigma_{|m|}(1,1)&&=\sum_{\substack{s\in P\\ s>|m|/t}}(-s)\cdot e_3+\sum_{\substack{s\in P\\ s>t}}(-s)\cdot e_4.\end{aligned}$$

For the Weyl chamber W_1, with (5.2.15) we get

$$\rho_{J_{|m|}}(W_1)=\sum_{s\in P\setminus Q}(s,s)-\sigma_{|m|}\cdot(1,1)=\sum_{s\in Q}(-s,-s).$$

The Borcherds product expansion of $\Xi_{J_{|m|}}$ now takes the fom

$$\begin{aligned}\Xi_{J_{|m|}}(\tau)=&\prod_{\substack{k,l\in\mathbb{Z}\\ ((l,k),W)>0}}\Big(1-e\big(k\tau-l\bar{\zeta}\big)\Big)^{c(mn)}\\ &\times\begin{cases}e\Big(+\sum_{\substack{s\in P\\ s>t}}s\bar{\zeta}-\sum_{\substack{s\in P\\ s>|m|/t}}s\tau\Big)&\text{if }W=W_t^+\text{ for }t\in Q.\\ e\Big(-\sum_{s\in Q}s(-\bar{\zeta}+\tau)\Big)&\text{if }W=W_1,\\ e\Big(+\sum_{\substack{s\in P\\ s>|m|/t}}s\bar{\zeta}-\sum_{\substack{s\in P\\ s>t}}s\tau\Big)&\text{if }W=W_t^-\text{ for }t\in Q.\end{cases}\end{aligned}\tag{5.2.16}$$

For the special cases $W=W^{\pm}_{|m|}$ of the topmost and bottom-most Weyl chambers, we have the following Borcherds products:

$$\Xi_{J_{|m|}}(\tau)=\begin{Bmatrix}e\big(-\sigma_{|m|}\tau\big)\\ e\big(+\sigma_{|m|}\bar{\zeta}\big)\end{Bmatrix}\cdot\prod_{\substack{k,l\in\mathbb{Z}\\ ((l,k),W)>0}}\Big(1-e\big(k\tau-l\bar{\zeta}\big)\Big)^{c(kl)}.$$

The condition for absolute convergence of the product expansions is $\Im\tau > 2|\delta|^{-1}|m|$. Certainly, this is fulfilled on a part of the topmost Weyl chamber $W^+_{|m|}$, so at least there, the Borcherds product converges.

Generally, whether the product expansion converges on a given Weyl chamber W depends on δ, in other words, on the number field $\mathbb{F}$. To find out if W is contained in the domain of convergence, we can examine the convergence condition case by case, for $W = W^+_t$, $W = W^-_t$, with $t \in Q$, and for $W = W_1$. In the first case, for example, assume $W = W^+_t$. Then, $2|m|\Im\tau > t^2|\delta|$. For $t \neq |m|$ denote by u the next largest element in Q. Then, we also have $2|m|\Im\tau < u^2|\delta|$. Thus, we have convergence on all of W if $t^2|\delta|^2 > 4m^2$ and at least on part of W if either $W = W^+_{|m|}$ or otherwise, for $t \neq |m|$, if $u^2|\delta|^2 = u^2|D_{\mathbb{F}}| > 4m^2$.

By varying $\mathbb{F}$, we find that the domain of absolute convergence encompasses only part of the Weyl chamber $W^+_{|m|}$ for $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$, but all of $W^+_{|m|}$ for $\mathbb{Q}(\sqrt{-2})$ and all number fields $\mathbb{Q}(\sqrt{d})$ with $|d| > 3$. In contrast, convergence on even part of $W^-_{|m|}$ requires $|D_{\mathbb{F}}| > 4m^2$.

Taking $m = -2$, for example, the Borcherds product $\Xi_{J_2}(\tau; W^-_2)$ for J_2 converges on part of 'its' Weyl chamber W^-_2 for $\mathbb{Q}(\sqrt{-6})$, and also on W_1 and on W^+_t for every $t \in Q$.

Remark. *If as in remark 5.2.11 we define W^+_t for all $t \in P$, we can describe all Weyl chambers as W^+_t for some t, except for $W^-_{|m|}$. The defining inequalities can then all be written in the same form, except in the case of $W^-_{|m|}$, which is given by $0 < 2\Im\tau < |\delta||m|^{-1}$. A similar remark applies if all Weyl chambers, except for $W^+_{|m|}$, are described as W^-_t for some $t \in P$.*

List of Notation

$\mathbb{F}$	An imaginary quadratic number field, page 19
$\mathscr{D}_{\mathbb{F}}^{-1}$	Inverse different ideal, page 20
$\mathscr{O}_{\mathbb{F}}$	The ring of integers in $\mathbb{F}$, page 19
ζ	A generator of $\mathscr{O}_{\mathbb{F}} = \mathbb{Z} + \zeta\mathbb{Z}$, page 20
$\langle \cdot, \cdot \rangle$	A non-degenerate, indefinite hermitian form, page 20
V, $V_{\mathbb{R}}$	In section 1.1 and from chapter 3 on: hermitian spaces over $\mathbb{F}$ and over $\mathbb{C}$, respectively, page 20
$\mathscr{O}_L$	The ring of multipliers of a lattice L, page 22
$\mathrm{U}(V)$	The unitary group of V, page 23
$\mathrm{SU}(V)$	The special unitary group of V, page 23
Γ_L^{U}	The discriminant kernel in $\mathrm{SU}(L)$ of the lattice L, page 23
L'	The dual of a lattice L, page 22
Gr_{U}	The Grassmannian model for the symmetric domain of $\mathrm{SU}(V)(\mathbb{R})$, page 23
$\mathscr{K}_{\mathrm{U}}$	A projective cone with $\mathscr{K}_{\mathrm{U}} \simeq \mathrm{Gr}_{\mathrm{U}}$ if V has signature $(1, q)$, page 24
ℓ, ℓ'	Lattice vectors with $\ell \in L$ primitive isotropic and ℓ' with $\langle \ell, \ell' \rangle \neq 0$, page 24
D	A definite lattice $\subset L$ with $\langle \ell, D \rangle = \langle \ell', D \rangle = 0$, page 24
$\widetilde{\mathscr{K}_{\mathrm{U}}^1}$	A set of normalized representatives z for $\mathscr{K}_{\mathrm{U}}$, page 25
z	An element of $\widetilde{\mathscr{K}_{\mathrm{U}}^1}$, $z = \ell' - \tau\delta \langle \ell', \ell \rangle \ell$, page 25
$\mathscr{H}_{\mathrm{U}}$	The Siegel domain model for the symmetric domain of $\mathrm{SU}(V)$, page 26
τ, σ	Coordinates on $\mathscr{H}_{\mathrm{U}}$, page 26
X_Γ	A unitary modular variety, page 26
$P(\ell)$	the stabilizer of the cusp in $\mathrm{SU}(V)$, page 27
$[h, t]$	An element of $H(\ell)$.
$H(\ell)$	The unitary Heisenberg group in $P(\ell)$, page 28
$\Gamma(\ell)$	the stabilizer of the cusp in an arithmetic group $\Gamma \subset \mathrm{SU}(L)$, page 28
$\mathscr{H}_{\mathrm{U}}^C$	A system of neighborhoods of ∞ in $\mathscr{H}_{\mathrm{U}}$, page 29
$j(\gamma, z)$	An automorphy factor for the action of $\mathrm{SU}(V)$ on $\mathscr{H}_{\mathrm{U}}$, page 31

$a_n(\sigma)$	Fourier-Jacobi coefficients, page 33
f_ℓ	For $f : \widetilde{\mathcal{K}_\mathrm{U}} \to \bar{\mathbb{C}}$, the corresponding function on $\mathcal{H}_\mathrm{U}$, page 32
V, $V_\mathbb{R}$	In chapter 1.2 and chapter 2: Quadratic spaces over $\mathbb{Q}$ and $\mathbb{R}$, respectively, page 34
$(\cdot,\cdot)$	A non-degenerate bilinear form of signature $(2,b)$, page 35
$q(\cdot)$	The quadratic form attached to $(\cdot,\cdot)$, page 35
$\mathrm{O}(V)$	The orthogonal group of V, page 35
$\mathrm{SO}(V)$	The special orthogonal group of V, page 35
$\mathrm{O}^+(V)$	The spinor kernel in $\mathrm{O}(V)$, page 35
L, L'	An even lattice in V and its dual, page 35
Γ_L^O	The discriminant kernel in $\mathrm{SO}(L)$ of the lattice L, page 35
Gr_O	A Grassmannian model for the symmetric domain of $\mathrm{SO}(V)(\mathbb{R})$, page 35
e, e'	Lattice vectors with $e \in L$ primitive isotopic and $e' \in L'$ with $\left(e, e'\right) = 1$, page 37
K	A Lorentzian lattice, $K = L \cap e^\perp \cap e'^\perp$, page 37
p_K	A projection from $V_\mathbb{R} = L \otimes_\mathbb{Z} \mathbb{R}$ to $W_\mathbb{R} = K \otimes_\mathbb{Z} \mathbb{R}$, page 37
$e_1, \ldots, e_4$	A set of vectors spanning two hyperbolic planes, with $e_1 = e$ and $e_2 = e'$, page 37
μ, w	Coordinates on the Grassmannian Gr_O, page 38
$V_\mathbb{C}$	The quadratic space $V_\mathbb{R} \otimes \mathbb{C}$, page 38
$\mathcal{N}$, $\mathcal{K}_\mathrm{O}$	Subsets of the projective space $\mathbb{P}(V_\mathbb{C})$, with $\mathcal{N} \supset \mathcal{K}_\mathrm{O}$, page 39
$\mathcal{K}_\mathrm{O}^+$	One of the two connected components of $\mathcal{K}_\mathrm{O}$, with $\mathrm{Gr}_\mathrm{O} \simeq \mathcal{K}_\mathrm{O}$, page 39
Z_L	A normalized representative in $\widetilde{\mathcal{K}_\mathrm{O}}$, page 39
$\mathcal{H}_\mathrm{O}^\pm$	The set of $Z \in K \otimes_\mathbb{Z} \mathbb{C}$, which bijects to $\widetilde{\mathcal{K}_\mathrm{O}}$, , page 40
$\mathcal{H}_\mathrm{O}$	The tube domain model, a connected component of $\mathcal{H}_\mathrm{O}^\pm$ mapped to $\widetilde{\mathcal{K}_\mathrm{O}^+}$, page 40
$\widetilde{\mathcal{K}_{\mathrm{O},1}^+}$	A normalized set of representatives Z_L for $\mathcal{K}_\mathrm{O}^+$, page 40
X_L, Y_L	The real and imaginary parts of $Z_L = X_L + iY_L \in \widetilde{\mathcal{K}_{\mathrm{O},1}^+}$, page 40
$Z = (z_1, z_2, \mathfrak{z})$	Refined tube domain coordinates defined if V contains 2 isotropic vectors, page 41
$\mathfrak{X}_\Gamma$	An orthogonal modular variety, page 41
$E(u, v)$	An Eichler element in $\mathrm{O}^+(V)(\mathbb{R})$, page 44
$\mathrm{H}(V_0)$	The Heisenberg group of the definite subspace $V_0 \subset V$, page 46

$[\lambda, \mu, \tau]$	An element of $\mathrm{H}(V_0)$, page 46
$\mathrm{J}(V_0)$	The Jacobi group of V_0, $\mathrm{J}(V_0) = \mathrm{SL}_2(\mathbb{Q}) \ltimes \mathrm{H}(V_0)$, page 47
f_e	For $f : \widetilde{\mathscr{K}_{\mathrm{O}}} \to \bar{\mathbb{C}}$, the corresponding function on $\mathscr{H}_{\mathrm{O}}$, page 50
$J(\gamma, Z_L)$	Automorphy factor for the action of $\mathrm{SO}(V)(\mathbb{R})$ on $\widetilde{\mathscr{K}}^{+}_{\mathrm{O},1}$, page 48
$\mathscr{C}_+$; $\overline{\mathscr{C}}_+$	The (open) positive cone in $K \otimes_{\mathbb{Z}} \mathbb{R}$; its closure, page 52
$\phi(\tau, \mathfrak{z})$	A Jacobi-form for an arithmetic subgroup of $\mathrm{J}(D)$, page 54
$\phi_m(z_2, \mathfrak{z})$	A Jacobi-form induced from an orthogonal modular form , page 54
Φ	The Siegel operator, page 55
$\Gamma(N)$	A principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, page 58
$\vert_k$	The weight k Petersson slash operator, page 58
$\mathscr{M}^!_k(\Gamma, \chi)$	Weakly holomorphic scalar valued elliptic modular forms , page 59
$(M, \phi(\tau))$	An element of $\mathrm{Mp}_2(\mathbb{R})$, page 60
$\mathscr{M}_k(\Gamma, \chi)$	Holomorphic (scalar valued) elliptic modular forms , page 59
$\mathrm{Mp}_2(\mathbb{R})$	A metaplectic group, page 60
$(\mathfrak{e}_\gamma)_\gamma$	Standard basis elements of $\mathbb{C}[L'/L]$, page 60
ρ_L	The Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$ attached to L'/L, page 60
ρ^*_L	The dual representation, page 61
$\vert_\kappa$, $\vert^*_\kappa$	The weight κ slash operators for ρ_L, ρ^*_L , page 61
$\mathscr{M}^!_\kappa(\rho_L)$	Weakly holomorphic modular forms of weight κ with respect to ρ_L, page 61
$\mathrm{O}^+_f(L)$	The automorphism group in $\mathrm{O}^+(L)$ of $f \in \mathscr{M}^!_{1-b/2}(\rho_L)$, page 62
L_0	$L_0 = \{\lambda \in L' ; (\lambda, e) \equiv 0 \mod N\}$, page 63
N	The level of the cusp e, page 63
p	A projection $p : L'_0 \to K'$ with $p(L) = K$, page 63
$\lambda^\perp$	A sub-Grassmannian, the corresponding prime divisors on $\mathscr{H}_{\mathrm{O}}$, page 64
$S(m, \beta, U)$	See page 65
$H(m, \beta)$	The Heegner divisor of discriminant (m, β), page 65
$\mathscr{G}(K)$	The Grassmannian attached to K, page 65
$\mathscr{H}$, $\mathscr{C}_K$	Realizations of $\mathscr{G}(K)$, page 65
W	A Weyl chamber of $\mathscr{G}(K)$; also of $\mathscr{H}_{\mathrm{O}}$, page 66

$\Phi_L(Z,f)$	The regularized theta lift, page 67
$\Theta_L(\tau,Z)$	The Siegel theta function, page 68
$\theta_\gamma(\tau,Z)$	One of the component functions of $\Theta_L(\tau,Z)$, page 68
$\mathscr{C}_{s=0}[f(s)]$	The constant term of Laurant expansion of $f(s)$ at $s=0$, page 68
$\Psi_L(Z,f)$	The multiplicative Borcherds lift, page 68
$\rho_f(W)$	The Weyl vector attached to $\Psi_L(Z,f)$ and W , page 69
V'	The rational quadratic space underlying the hermitian space V, page 71
$V'_\mathbb{R}$	The real quadratic space underlying the complex hermitian space $V_\mathbb{R}$, page 71
$\hat{w}$	A scalar $w\in\mathbb{C}$ of $V_\mathbb{R}$ acting on $V'_\mathbb{R}$, page 74
$\mathscr{E}(e,t)$	See page 74
$\mathscr{T}(e,h)$	See page 74
α	The embedding constructed in chapter 3, page 79
$Z_L(z)$	The image $Z_L=\alpha(z)\in\widetilde{\mathscr{H}_\mathrm{O}}$, page 85
Γ'	$=\Gamma\cap\mathrm{SU}(L)$, where $\Gamma\subset\mathrm{O}^+(L)$ with $\mathrm{SU}(L)\hookrightarrow\mathrm{O}^+(L)$, page 86
$V_\epsilon(x)$, $U_\epsilon(x)$	Neighborhoods of a point $x\in\partial\mathscr{H}_\mathrm{O}$.
N_ℓ	The level of the cusp ℓ, page 89, see also page 63
L_0	in chapter 4: a hermitian lattice attached to N_ℓ and L', page 89
$\mathbf{H}(m,\beta)$	The Heegner divisors of discriminant (m,β) on $\mathscr{H}_\mathrm{U}$, page 91
$\mathbf{H}_\lambda$	A sub-Grassmannian of Gr_U attached to λ, page 90
m_0	$=\min\{n\in\mathbb{Z};\ c(n,\gamma)\neq 0\}$, page 94
W	In chapter 3: A Weyl chamber of $\mathscr{H}_\mathrm{U}$, page 92
$(\lambda,W)>0$	A positivity condition with respect to a Weyl chamber , page 93
Ξ_f	The unitary Borcherds lifting, page 94
$\mathscr{M}_0^!$	$=\mathscr{M}_0^!(\Gamma(1))$, page 103
$J_b(\tau)$	A modular form with $J_b(\tau)=q^{-b}+O(q)\in\mathscr{M}_0^!$, page 104
$\eta(\tau)$	The usual eta-function on $\mathbb{H}$.
$\sigma_n(s)$	A generalized divisor sum, $\sigma_n(1)=\sum_{d\mid n}d$, page 105
$F_m(\tau,s)$	Non-holomorphic Poincaré series, page 105
$\Phi_m^K(v,s)$	The regularized theta lift to $\mathrm{SO}(1,1)$ of $F_m(\tau,s)$, page 106

$\Phi_m^{\mathbb{Z}^2}(Y)$	Φ_m^K for $K \simeq \mathbb{Z}^2$, page 108
$\rho_m(W)$	Weyl vector for a Weyl chamber of $K \simeq \mathbb{Z}^2$, page 110
Φ_L^W	The regularized Borcherds lift for SO(2, 2), page 111
$\mathbf{H}(m)$	Heegner divisor of index m on $\mathscr{H}_{\mathrm{U}} \simeq \mathbb{H}$, page 116
$\mathscr{O}_\tau$	The CM-oder of a Heegner point τ, page 117

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