HYPERBOLIC HEEGAARD SPLITTINGS AND DEHN TWISTS

PETER FELLER, ALESSANDRO SISTO, AND GABRIELE VIAGGI

Abstract. We consider the family of Heegaard splittings of genus g at least three which are defined via a glueing map that is the n-th power of the Dehn twist along a curve that satisfies a natural topological assumption, namely pared acylindricity. We show that if n is at least 14, then the Heegaard splitting has a hyperbolic metric for which the simple closed curve defining the Dehn twist is a closed geodesic of length at least $0.7/(n^2g^2)$ and at most $34.3/n^2$.

1. Introduction

A Heegaard splitting is a closed orientable 3-manifold

$$M_f = H \cup_{f:\partial H\to\partial H} H$$

obtained by glueing two copies of a handlebody H along an orientation preserving diffeomorphism of the boundary $f : \partial H \to \partial H$. A classical result due to Heegaard states that every closed connected orientable 3-manifold is diffeomorphic to some M_f. A general, challenging problem is to convert the topological and dynamical information about the glueing map f into topological and geometric properties of M_f (most relevantly for this paper, see e.g. [Hem01, Yos14] for the topological side, and [Nam05, NS09, FSVb, HV19, HJ] for the geometric side).

In this article, we consider glueing maps f that are some power of a Dehn twist along a simple closed curve $\gamma \subset \partial H$, and specifically we prove the following.

Theorem. *Let H be a handlebody of genus $g \geq 3$. Let $\gamma \subset \partial H$ be an essential simple closed curve such that (H, γ) is* pared acylindrical*. Let τ_γ be a (left or right) Dehn twist along γ. If $n \geq 14$, then $M_{\tau_\gamma^n}$ admits a hyperbolic metric such that the curve $\gamma \subset M_{\gamma,n}$ is a geodesic and its length $\ell_{M_{\tau_\gamma^n}}(\gamma)$ satisfies*

$$\frac{0.7}{g^2n^2} < \ell_{M_{\tau_\gamma^n}}(\gamma) < \frac{34.3}{n^2}.$$

Pared acylindricity is a topological condition introduced by Thurston in the study of deformations spaces of hyperbolic metrics. We provide the exact definition below, and at this point only note that it is satisfied when, in the curve graph of the boundary surface of the handlebody, the curve γ is at distance at least 3 from the disk set of the handlebody; see [Hem01] for definitions of these terms. In fact, in the distance at least 3 case, the manifold $M_{\tau_\gamma^n}$ is hyperbolic for $n \geq 3$, by [Yos14, Lemma 5.3] and in view of Thurston's hyperbolization and [Hem01], but no bounds were previously known on the length of γ.

The theorem is curated to please the eye. For a more precise, albeit more technical version which also covers the case $g = 2$, see Theorem 2.1 below.

In future work [FSVa], we establish hyperbolicity and length bounds for a much larger class of Heegaard splittings using subsurface projections. The reason why we deal with the case of powers of Dehn twists in this separate paper rather than as part of [FSVa] are the more elementary nature of the arguments for the powers

2020 *Mathematics Subject Classification.* 57K32.

of Dehn twist case, which allows a short exposition of the key idea, and, more importantly, the fact that the constants in the present case are explicit. In the more general setup, the strategy cannot yield explicit constants; see also the last bullet point of the next paragraph.

We list some features of Theorem 2.1.

- The lower bound on the power on the Dehn twist and the upper bound on the length of the geodesic are independent of the genus of the handlebody.
- There are lots of simple closed curves $\gamma \subset \partial H$ that satisfy the assumption of Theorem 2.1. For example, there is an open dense subset $\mathcal{D} \subset \mathcal{PML}$ of full Lebesgue measure (the so-called Masur domain, see Lecuire [Lec06]) of the space of projective measured laminations on ∂H, which is a certain completion of the space of simple closed curves on ∂H homeomorphic to a sphere of dimension $6g-7$ where g is the genus of H, such that every curve $\gamma \in \mathcal{D}$ is pared acylindrical.
- In a separate paper [FSVa], we combine the ideas of this article and of [FSVb] with tools introduced by Masur and Minsky [MM99], [MM00], Minsky [Min10], and Brock, Canary, and Minsky [BCM12] around the solution of the Ending Lamination Conjecture to establish hyperbolicity and length bounds (as given by the Length Bound Theorem in [BCM12]) for a vast class of Heegaard splittings with large subsurface projections. The ideas presented in the present paper correspond to the most basic case of large annular projections.

Acknowledgements. PF acknowledges financial support of the by the SNSF Grant 181199. GV acknowledges financial support of the DFG 427903332 and of the DFG 390900948.

2. A precise version of the theorem

We recall the definition of pared acylindricity and state a refined, albeit more technical version of the theorem from the introduction.

Definition (Pared Acylindrical [Thu86a, Thu86b]). Let H be a handlebody of genus $g \geq 2$. Let $\gamma \subset \partial H$ be an essential simple closed curve. Let A be a tubular neighborhood of γ in ∂H. We say that (H, γ) is a *pared acylindrical* if

- the inclusion $\partial H - \gamma \subset H$ is π_1-injective,
- every essential map $(S^1 \times [0,1], S^1 \times \{0,1\}) \to (H, A)$ is homotopic as a map of pairs into A, and
- every essential map $(S^1 \times [0,1], S^1 \times \{0,1\}) \to (H, \partial H - A)$ is homotopic as a map of pairs into ∂H.

Theorem 2.1. *Let H be a handlebody of genus $g \geq 2$. Let $\gamma \subset \partial H$ be an essential simple closed curve such that (H, γ) is* pared acylindrical. *Let τ_γ be a (left or right) Dehn twist along γ.*

If $g \geq 3$ and $n \geq 14$, then $M_{\tau_\gamma^n}$ admits a hyperbolic metric for which the curve $\gamma \subset M_{\tau_\gamma^n}$ is a geodesic of length

(1)
$$\frac{2\pi}{\frac{(2\pi n(g-1))+5)^2}{2\sqrt{3}\left(2\sinh\left(\frac{\ln(3)}{2}\right)\right)^2}+16.17} < \ell_{M_{\tau_\gamma^n}}(\gamma) < \frac{2\pi}{n^2\frac{2\sinh\left(\frac{\ln(3)}{2}\right)}{2.5}-2n+\frac{2.5}{2\sinh\left(\frac{\ln(3)}{2}\right)}-28.78}.$$

If $g = 2$ and $n \geq 21$, then $M_{\tau_\gamma^n}$ admits a hyperbolic metric for which $\gamma \subset M_{\tau_\gamma^n}$ is a geodesic and (1) *holds with all occurrences of* $2\sinh\left(\frac{\ln(3)}{2}\right)$ *replaced by* $\frac{\sqrt[4]{2}}{2}$.

Theorem 2.1 recovers the theorem from the introduction.

Proof of the theorem from the introduction. For the lower bound, we calculate

$$\begin{aligned}\frac{0.7}{n^2g^2} &< \frac{2\pi}{\frac{(2\pi ng)^2}{2\sqrt{3}(2\sinh(\ln(3)/2))^2}} \\ &< \frac{2\pi}{\frac{(2\pi ng)^2-(2\pi n-5)^2}{2\sqrt{3}(2\sinh(\ln(3)/2))^2}+16.17} \\ &< \frac{2\pi}{\frac{(2\pi ng-(2\pi n-5))^2}{2\sqrt{3}(2\sinh(\ln(3)/2))^2}+16.17} \\ &= \frac{2\pi}{\frac{(2\pi n(g-1))+5)^2}{2\sqrt{3}(2\sinh(\ln(3)/2))^2}+16.17},\end{aligned}$$

where in the third inequality we use $\frac{(2\pi n-5)^2}{2\sqrt{3}(2\sinh(\ln(3)/2))^2} > 16.17$ for $n \geq 14$.

For the upper bound, for $n \geq 14$, we have

$$\frac{2\pi}{n^2\frac{2\sinh\left(\frac{\ln(3)}{2}\right)}{2.5} - 2n + \frac{2.5}{2\sinh\left(\frac{\ln(3)}{2}\right)} - 28.78} < \frac{34.3}{n^2}.$$

Thus, the theorem from the introduction is implied by Theorem 2.1. □

In the rest of this section, we prove Theorem 2.1.

2.1. **Hyperbolization of the drilled double.** Observe that $M_{\tau_\gamma^n} - \gamma$ can be understood as the double of $H - \gamma$. By Thurston's Hyperbolization for Haken manifolds ([Kap09, Theorem 1.42]), as (H, γ) is pared acylindrical, the manifold $M_{\tau_\gamma^n} - \gamma$ admits a complete finite volume hyperbolic metric which is itself the double of a complete hyperbolic metric with finite volume and totally geodesic boundary on $H - \gamma$ (see also [BO04, Theorem 2]). We fix such a hyperbolic structure on $M_{\tau_\gamma^n} - \gamma$ and denote it by $\mathbb{M}_\gamma$.

2.2. **The cusp.** Let us briefly recall the thick-thin decomposition of a complete finite volume hyperbolic 3-manifold (see e.g. [BP92, Chapter D]). By standard consequences of the Margulis Lemma, there exists a universal constant $\epsilon > 0$, called the Margulis constant, as follows. For every finite volume hyperbolic 3-manifold $\mathbb{M}$, every unbounded connected component of the ϵ-*thin part* of $\mathbb{M}$, defined by,

$$\mathbb{M}(\epsilon) := \{x \in \mathbb{M} \mid \mathrm{inj}_x(\mathbb{M}) < \epsilon\}$$

is isometric to a quotient of a horoball $\mathcal{O} \subset \mathbb{H}^3$ by a discrete torsion free group of isometries that stabilize $\mathcal{O}$. Such a connected component is called a *cusp*.

In our case, the thin part of $\mathbb{M}_\gamma$ has a single unbounded component, which we denote by $\mathbb{T}(\gamma)$. We note that the first homology (with rational coefficients) of $\mathbb{M}_\gamma$ has rank either $g+1$ or g. Indeed, since $\mathbb{M}_\gamma$ is obtained from a connect sum of g copies of $S^1 \times S^2$ by removing a curve γ, a Mayer-Vietoris argument applied to writing the connected sum as a union of $\mathbb{M}_\gamma$ and a neighborhood of γ reveals that the rank is as claimed. In particular, if $g \geq 3$, we can and do choose $\epsilon = \ln(3)$ by [FPS22, Theorem 1.5 (3)] (which has [CS92, Theorem 9.1] as its key input). If instead $g = 2$, the rank of the first homology of $\mathbb{M}_\gamma$ is either 3 or 2. Due to the latter possibility, we pick $\epsilon = 2\sinh^{-1}\left(\frac{\sqrt[4]{2}/2}{2}\right)$ in case $g = 2$, which we may do by [Ada19].

As $\mathbb{M}_\gamma$ is the double of $H-\gamma$, the cusp $\mathbb{T}(\gamma)$ is itself the double of $U = \mathbb{T}(\gamma)\cap(H-\gamma)$. The boundary of U decomposes as a Heegaard surface part $A = U \cap (\partial H - \gamma)$, which is the union $A = A_1 \cup A_2$ of two cuspidal neighborhoods of the ends of the finite area hyperbolic surface $\partial H - \gamma$, and a handlebody part $\partial U - A$ which is a properly embedded annulus in $H - \gamma$.

The boundary of $\mathbb{T}(\gamma)$ is the double of the annulus $\partial U - A$. We choose and fix for the rest of the paper two simple closed curves in $\partial\mathbb{T}(\gamma)$ whose homology classes form a basis for the first integer homology of $\partial\mathbb{T}(\gamma)$:

- The curve α is a component of ∂A.
- The curve β is the double of an arc κ joining the two boundary components of the annulus $\partial U - A$. Note that the homology class of β does not depend on the choice of the arc.

2.3. **Dehn Filling.** $\mathbb{T}(\gamma) \cup \gamma$ is an open solid torus and a tubular neighborhood of γ in $M_{\tau_\gamma^n}$ and, hence, $M_{\tau_\gamma^n}$ is obtained from $\mathbb{M}_\gamma - \mathbb{T}(\gamma)$ by Dehn filling along a suitable slope. This means that we recover $M_{\tau_\gamma^n}$, up to diffeomorphism, as the result of glueing together $\mathbb{M}_\gamma - \mathbb{T}(\gamma)$ with a standard solid torus $D^2 \times S^1$ along a diffeomorphism $\phi_n : \partial(D^2 \times S^1) \to \partial\mathbb{T}(\gamma)$. The diffeomorphism type of the resulting manifold is completely determined by the homotopy class of the curve $\mu_n = \phi_n(\partial D^2 \times \{\star\})$ on the torus $\partial\mathbb{T}(\gamma)$. This homotopy class is called the *slope* of the Dehn filling. In our case, the slope is the (unique up to orientation reversal) non-trivial homotopy class $\mu_n \subset \partial\mathbb{T}(\gamma)$ given by curves that are null-homotopic in the solid torus $\mathbb{T}(\gamma) \cup \gamma$.

Homotopy classes of simple closed curves on a torus $\partial\mathbb{T}(\gamma)$ are parametrized by the primitive homology classes in $H_1(\partial\mathbb{T}(\gamma), \mathbb{Z})$. From here on, for curves on a torus, we do not distinguish between them, their homotopy classes and their homology classes in our notation. The slope defined by μ_n along which Dehn filling of $\mathbb{M}_\gamma$ yields $M_{\tau_\gamma^n}$ has, up to correct choice of orientations which we suppress, a simple expression in terms of the homology basis α, β, namely $\mu_n = n\alpha + \beta$. To see this, note that if $n = 0$, indeed $\mu_0 = \beta$ since β bounds a disc in the double of H. For general n, μ_n is obtained from μ_0 by applying n Dehn twists along α as e.g. argued in [Lic62, Proof of Theorem 2]; hence, $\mu_n = n\alpha + \mu_0 = n\alpha + \beta$ as desired.

In order to prove our theorem, we apply the universal Dehn filling theorem of Hodgson and Kerckhoff [HK05] to deform the hyperbolic structure of $\mathbb{M}_\gamma$ to a hyperbolic structure on $M_{\tau_\gamma^n}$. We use the following statement of the filling theorem obtained by combining the work of Hodgson and Kerckhoff and results of Futer, Purcell, and Schleimer. Recall that the *normalized length* of a curve $\mu \subset \partial\mathbb{T}(\gamma)$, where $\mathbb{T}(\gamma)$ is a cusp of some finite volume hyperbolic 3-manifold, is defined by

$$L(\mu) := \frac{\mathrm{Length}(\bar{\mu})}{\sqrt{\mathrm{Area}(\partial\mathbb{T}(\gamma))}}$$

where $\bar{\mu}$ is a (flat) geodesic representative of μ on $\partial\mathbb{T}(\gamma)$ (the intrinsic metric of $\partial\mathbb{T}(\gamma)$ is locally isometric to a horosphere $\mathcal{H}$ in $\mathbb{H}^3$ which is isometric to $\mathbb{R}^2$).

Theorem 2.2 ([HK05, Theorem 1.1] and [FPS22, Corollary 6.13]). *Let $\Gamma \subset M$ be a knot in a 3-manifold M. Suppose that $M - \Gamma$ has a complete finite volume hyperbolic metric for which the* normalized length *L of the meridian of Γ is at least $L \geq 7.823$. Then M has a complete hyperbolic metric for which Γ is a geodesic of length*

$$\frac{2\pi}{L^2 + 16.17} < \ell_M(\gamma) < \frac{2\pi}{L^2 - 28.78}.$$

To establish Theorem 2.1, we provide bounds on the normalized length of μ_n.

2.4. **Normalized length computation.** Let us work in the upper half space model $\mathbb{H}^3 = \mathbb{C} \times (0, \infty)$. Denote by $\rho : \pi_1(\partial\mathbb{T}(\gamma)) \to \mathrm{Isom}^+(\mathbb{H}^3)$ the holonomy homomorphism. We normalize the configuration so that the image fixes ∞. Under this assumption, each $\rho(\eta)$ is a parabolic transformation of the form $(z, t) \to (z + \tau(\eta), t)$, where $\tau : \pi_1(\partial\mathbb{T}(\gamma)) \to \mathbb{C}$ is a group homomorphism, and we can identify $\mathbb{T}(\gamma)$ with the quotient by $\rho(\pi_1(\mathbb{T}(\gamma)))$ of a horoball

$$\mathcal{O} = \{(z, t) \in \mathbb{H}^3 \mid t \geq T\}$$

whose boundary is the horosphere $\mathcal{H} = \mathbb{C} \times \{T\}$. Note that the metric on $\mathcal{H}$ coincides with the standard flat metric of $\mathbb{C}$ rescaled by $1/T$. Thus, the length of the flat geodesic representative of any $\eta \in \pi_1(\partial\mathbb{T}(\gamma))$ coincides with $|\tau(\eta)|/T$.

The length of the flat representative of μ_n on the torus $\partial\mathbb{T}(\gamma)$ is given by $\mathrm{Length}(\bar{\mu}_n) = |\tau(\mu_n)|/T = (|\tau(n\alpha + \beta)|)/T$. By the triangle inequality,

$$\frac{n|\tau(\alpha)| - |\tau(\beta)|}{T} \leq \mathrm{Length}(\bar{\mu}_n) \leq \frac{n|\tau(\alpha)| + |\tau(\beta)|}{T}.$$

The area of the torus $\partial\mathbb{T}(\gamma)$ is given by $\mathrm{Area}(\partial\mathbb{T}(\gamma)) = \frac{|\tau(\alpha)||\tau(\beta)|}{T^2}\sin(\theta)$ where θ is the angle between $\tau(\alpha), \tau(\beta) \in \mathbb{C}$. Notice that the fact that the injectivity radius of $\partial\mathbb{T}(\gamma)$ is bounded from below by $r_\epsilon \coloneqq 2\sinh\left(\frac{\epsilon}{2}\right)$ implies that the area of $\partial\mathbb{T}(\gamma)$ is bounded from below by $2\sqrt{3}r_\epsilon^2$ since the minimal area among flat tori of injectivity radius r is $2\sqrt{3}r^2$. To see the latter one may argue as follows. Consider a flat torus as the result of glueing up a parallelogram given as the union of two similar acute triangles. Taking $\pi/3 \leq \varphi \leq \pi/2$ to be the largest angle of the acute triangle and s_1 and s_2 to be the length of the two sides that form φ, we have that the area equals $\sin(\varphi)s_1 s_2 \geq \frac{\sqrt{3}}{2}(2r)^2$ as desired.

By the last paragraph, we have $2\sqrt{3}r_\epsilon^2 \leq \mathrm{Area}(\partial\mathbb{T}(\gamma)) \leq \frac{|\tau(\alpha)||\tau(\beta)|}{T^2}$. Therefore, the normalized length of μ_n is bounded by

$$\frac{(n|\tau(\alpha)| - |\tau(\beta)|)^2}{|\tau(\alpha)||\tau(\beta)|} \leq L(\mu_n)^2 \leq \frac{(n|\tau(\alpha)| + |\tau(\beta)|)^2}{2\sqrt{3}r_\epsilon^2 T^2}. \tag{2}$$

As a consequence, we want to bound the lengths of geodesic representatives of α and β. Of course $2r_\epsilon$ is a lower bound. We will use the following upper bounds.

Proposition 2.3. *Let H be a handlebody of genus at least 2. For all essential simple closed curve $\gamma \subset \partial H$ such that (H, γ) is pared acylindrical, we have*

$$\mathrm{Length}(\bar{\alpha}) < 2\pi(g-1) \text{ and } \mathrm{Length}(\bar{\beta}) \leq 5,$$

where $\bar{\alpha}$ and $\bar{\beta}$ are geodesic representatives of the curves $\alpha, \beta \subset \partial\mathbb{T}(\gamma)$ as defined in Subsection 2.2.

Using Proposition 2.3, which we prove in the next section, we prove Theorem 2.1.

Proof of Theorem 2.1. We set $a \coloneqq \mathrm{Length}(\bar{\alpha}) = \frac{|\tau(\alpha)|}{T}$ and $b \coloneqq \mathrm{Length}(\bar{\beta}) = \frac{|\tau(\beta)|}{T}$. We want to apply Theorem 2.2, hence we check that $L \coloneqq L(\mu_n) \geq 7.823$. Since that the lower bound of (2) reads $n^2\frac{a}{b} - 2n + \frac{b}{a} \leq L^2$, it suffices to have $n^2\frac{a}{b} - 2n + \frac{b}{a} \geq 7.823^2$. By Proposition 2.3 and $2r_\epsilon \leq a$, we have $\frac{a}{b} \geq \frac{2r_\epsilon}{5} = \frac{r_\epsilon}{2.5}$. Hence, since $n^2t + t^{-1}$ increases after $t = \frac{1}{n}$, we find $n^2\frac{r_\epsilon}{2.5} - 2n + \frac{2.5}{r_\epsilon} \leq n^2\frac{a}{b} - 2n + \frac{b}{a} \leq L^2$ whenever n is at least 5. Therefore, we know that $L > 7.823$ if we choose $n \geq 5$ such that $n^2\frac{\epsilon}{2.5} - 2n + \frac{2.5}{\epsilon} > 7.823^2$. If $g \geq 3$, in which case $r_\epsilon = 2\sinh\left(\frac{\ln(3)}{2}\right)$, the latter is true for all $n \geq 14$. In case $g = 2$, using $r_\epsilon = \sqrt[4]{2}/2$ in place of $2\sinh\left(\frac{\ln(3)}{2}\right)$, one finds that the inequality holds whenever $n \geq 21$. From now on we assume that either $g = 2$ and $n \geq 21$ or $g \geq 3$ and $n \geq 14$.

We argued above that $n^2\frac{r_\epsilon}{2.5}-2n+\frac{2.5}{r_\epsilon}$ provides a lower bound for L^2. Combined with $\ell_{M_{\tau_\gamma^n}}(\gamma)<\frac{2\pi}{L^2-28.78}$ from Theorem 2.2, we have that

$$\ell_{M_{\tau_\gamma^n}}(\gamma)<\frac{2\pi}{n^2\frac{r_\epsilon}{2.5}-2n+\frac{2.5}{r_\epsilon}-28.78}.$$

Finally, we discuss the lower bound for $\ell_{M_{\tau_\gamma^n}}(\gamma)$. We use $\frac{2\pi}{L^2+16.17}<\ell_{M_{\tau_\gamma^n}}(\gamma)$ from Theorem 2.2 in combination with

$$L^2\overset{(2)}{\leq}\frac{(na+b)^2}{2\sqrt{3}r_\epsilon^2}\overset{\text{Proposition 2.3}}{<}\frac{\left(2\pi n(g-1))+5\right)^2}{2\sqrt{3}r_\epsilon^2}.$$

Therefore, we have $\ell_{M_{\tau_\gamma^n}}(\gamma)>\frac{2\pi}{\frac{(2\pi n(g-1))+5)^2}{2\sqrt{3}r_\epsilon^2}+16.17}$, as desired. □

3. The proof of Proposition 2.3

We use the following elementary fact about 3-dimensional hyperbolic space.

Lemma 3.1. *Consider a path γ in $\mathbb{H}^3$ consisting of a concatenation $\alpha_0\star\beta_0...\alpha_n\star\beta_n$ and horospheres $H_0,\ldots,H_{n+1}$ where*

- *α_i is a geodesic on the horosphere H_i of length >2.5, where H_i and H_{i+1} are disjoint, and*
- *β_i is a geodesic that meets H_i and H_{i+1} orthogonally.*

Then γ is not a loop.

Proof. Denote by π_i the map assigning, to any $x\in\mathbb{H}^3\cup\partial\mathbb{H}^3$ not contained in the open horoball whose boundary is H_i, the entrance point in H_i of a geodesic from x to the point p_i at infinity of H_i. We will use the following two facts.

- If H is a horosphere disjoint from the horosphere H_i with point at infinity p, then $\pi_i(H)$ is a ball around $\pi_i(p)$ of radius at most $1/2$ in the intrinsic metric of H_i.
- if $d_{H_i}(\pi_i(x),\pi_i(y))\geq 2$, then any geodesic from x to y intersects H_i.

To see these two facts, we work in the Poincaré half-space model and w.l.o.g. set $H_i=\{(z,t)\in\mathbb{H}^3\mid t=1\}$. To see the first fact, note that any H is an Euclidean sphere (minus its ideal point $p=(z_0,0)\in\partial\mathbb{H}^3$) contained in $\{(z,t)\in\mathbb{H}^3\mid 0<t\leq 1\}$. Denote by r the Euclidean sphere's radius, which is at most $\frac{1}{2}$, and note that $\pi_i(H)=\pi_i\left(\{(z,t)\in\mathbb{H}^3\mid |z-z_0|\leq r\text{ and }t\leq 1\}\right)$, which is the ball around $\pi_i(p)$ of radius $r\leq 1/2$ as claimed. To see the second fact, write $x=(z_x,t_x)$ and $x=(z_y,t_y)$ and note that $|z_y-z_x|=d_{H_i}(\pi_i(x),\pi_i(y))\geq 2$ by assumption. The geodesic from x to y is a subarc of a unique infinite geodesic with ideal end points at infinity, which is an Euclidean half-circle meeting $\partial\mathbb{H}^3$ orthogonally in two points of Euclidean distance $d\geq|z_y-z_x|\geq 2$. Hence the half-circle's highest point has Euclidean height $d/2\geq 1$ and must be contained on the arc between any two points $x=(z_x,t_x)$ and $x=(z_y,t_y)$ on the half-circle with $t_x,t_y\leq 1$ and $|z_y-z_x|\geq 2$. This shows that the arc must meet H_i as desired.

Let x_i,y_i be the starting and final point of α_i. We will show that $\pi_0(p_i)$ is contained in the ball of radius $1/2$ around y_0 for all $i>0$ by induction on the number of pieces of a concatenation as in the statement. For $i=1$, we have $\pi_0(p_1)=y_0$ since y_0 lies on the geodesic from p_1 to p_0 because β_i meets H_0 and H_1 orthogonally.

Suppose that the claim holds for a given i. We have $\pi_1(p_0)=x_1$, by the base case of the induction, and $d_{H_1}(\pi_1(p_{i+1}),y_1)\leq 1/2$, by the induction hypothesis applied to the concatenation $\alpha_1\star\beta_1\ldots\beta_{i+1}$. Since $d_{H_1}(x_1,y_1)>2.5$, we have

$d_{H_1}(\pi_1(p_0), \pi_1(p_{i+1})) \geq 2$, so that the geodesic from p_{i+1} to p_0 intersects H_1. Therefore $\pi_0(p_{i+1})$ is contained in $\pi_0(H_1)$, which is contained in the ball of radius $1/2$ around y_0, as required. □

Proof of Proposition 2.3. We first prove the upper bound for $a := \mathrm{Length}(\bar{\alpha})$ using that the hyperbolic surface $\partial H - \gamma$ has area $-2\pi\chi(\partial H - \gamma) = 4\pi(g-1)$. We have that $A \subset \partial H - \gamma$ is the union of two cusps with boundary of length a. Since the area of a cusp of a hyperbolic surfaces is equal to the length of its boundary (as can e.g. be checked by an explicit calculation in the upper-half plane model), the area of $\partial H - A$, which is positive, is $4\pi g - 4\pi - 2a$. Therefore, we have $a < 2\pi(g-2)$.

Next we apply Lemma 3.1 to get an upper bound for $\mathrm{Length}(\bar{\beta})$. Consider $\delta \subset \partial H$ an (arbitrary) essential simple closed curve bounding a properly embedded disk $\Delta \subset H$. By applying an isotopy if necessary, we can and do assume that δ is in minimal position with respect to γ (note that δ must intersect γ because of the pared acylindrical assumption). We write δ as a concatenation

$$\delta = \delta_1 * \cdots * \delta_v,$$

where each δ_j is the closure of a component of $\delta - \gamma$. We label the intersections $\delta \cap \gamma = p_1, \cdots, p_v$ such that δ_j is the arc joining p_j to p_{j+1} (indices modulo v).

Each arc δ_j is properly homotopic within $\partial H - \gamma$ to a bi-infinite complete geodesic $\bar{\delta}_j$. By the homotopy extension property, we can extend the homotopy on $\partial\Delta - \{p_1, \cdots, p_v\}$ to a proper homotopy of the whole (punctured) disk $\Delta - \{p_1, \cdots, p_v\}$. Properness of the map $\Delta \to \mathbb{M}_\gamma$ implies that every vertex p_j has a small neighborhood W_j that maps to $\mathbb{T}(\gamma)$.

We lift the resulting ideal disk with totally geodesic boundary to $\mathbb{H}^3$ and, by a slight abuse of notation, we will keep denoting its side by $\bar{\delta}_j$. Note that each $\bar{\delta}_j$ joins the centers ξ_j, ξ_j' of different horoball components $\mathcal{O}_j, \mathcal{O}_j'$ of the pre-image of the cusp $\mathbb{T}(\gamma)$ to the universal cover $\mathbb{H}^3$. Note also that we must have $\xi_j' = \xi_{j+1}$ (indices modulo v) since every small neighborhood W_j is mapped to the same horoball.

For each side $\bar{\delta}_j$ consider the horospheres $\mathcal{H}_j = \partial\mathcal{O}_j$ and $\mathcal{H}_j' = \partial\mathcal{O}_j'$ centered at ξ_j and ξ_j'. Let x_j, x_j' be the intersections of $\mathcal{H}_j, \mathcal{H}_j'$ with $\bar{\delta}_j$.

As x_j, x_{j+1}' lie on the same horosphere $\mathcal{H}_j = \mathcal{H}_{j+1}'$ they are connected by a horospherical geodesic h_j. Observe that the projection of h_j to $\mathbb{M}_\gamma$ lies on $\partial U - A$ so, the double of the arc represents the homology class of β.

In order to conclude, it is enough to bound the length of at least one of the h_j. Indeed, since β is isotopic to the double of any arc h_j, we have that $\mathrm{Length}(\bar{\beta})$ is less than or equal to twice the length of h_j for all $1 \leq j \leq v$.

Since, by Lemma 3.1, the path $[x_1', x_1] * h_i * \cdots * [x_v', x_v] * h_v$ is closed, there is a j such that the flat arc h_j has length at most 2.5. Thus, $\mathrm{Length}(\bar{\beta})$ is at most 5. □

References

[Ada19] C. Adams. Waist size for cusps in hyperbolic 3-manifolds II. *Geom. Dedicata*, 203:53–66, 2019.

[BCM12] J. Brock, R. Canary, and Y. Minsky. The classification of Kleinian surface groups, II: The ending lamination conjecture. *Ann. of Math. (2)*, **176**(1):1–149, 2012.

[BO04] F. Bonahon and J.-P. Otal. Laminations measurées de plissage des variétés hyperboliques de dimension 3. *Ann. of Math. (2)*, **160**(3):1013–1055, 2004.

[BP92] R. Benedetti and C. Petronio. *Lectures on hyperbolic geometry*. Universitext. Springer-Verlag, Berlin, 1992.

[CS92] M. Culler and P. Shalen. Paradoxical decompositions, 2-generator Kleinian groups, and volumes of hyperbolic 3-manifolds. *J. Amer. Math. Soc.*, **5**(2):231–288, 1992.

[FPS22] D. Futer, J. Purcell, and S. Schleimer. Effective bilipschitz bounds on drilling and filling. *Geom. Topol.*, **26**(3):1077–1188, 2022.

[FSVa] P. Feller, A. Sisto, and G. Viaggi. Length bounds for Heegaard splittings. In preparation.

[FSVb] P. Feller, A. Sisto, and G. Viaggi. Uniform models and short curves for random 3-manifolds. *ArXiv e-print.* ArXiv:1910.09486v3 [math.GT].

[Hem01] J. Hempel. 3-manifolds as viewed from the curve complex. *Topology*, **40**(3):631–657, 2001.

[HJ] U. Hamenstädt and F. Jäckel. Stability of einstein metrics and effective hyperbolization in large hempel distance. *ArXiv e-print.* ArXiv:2206.10438 [math.GT].

[HK05] C. Hodgson and S. Kerckhoff. Universal bounds for hyperbolic Dehn surgery. *Ann. of Math.*, **162**(1):367–421, 2005.

[HV19] U. Hamenstädt and G. Viaggi. Small eigenvalues of random 3-manifolds. *ArXiv e-print*, 2019. ArXiv:1903.08031 [math.GT].

[Kap09] M. Kapovich. *Hyperbolic manifolds and discrete groups*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2009. Reprint of the 2001 edition.

[Lec06] C. Lecuire. An extension of the Masur domain. In *Spaces of Kleinian groups*, volume 329 of *London Math. Soc. Lecture Note Ser.*, pages 49–73. Cambridge Univ. Press, Cambridge, 2006.

[Lic62] W. B. R. Lickorish. A representation of orientable combinatorial 3-manifolds. *Ann. of Math. (2)*, 76:531–540, 1962.

[Min10] Y. Minsky. The classification of Kleinian surface groups I: Models and bounds. *Ann. of Math.*, **171**(1):1–107, 2010.

[MM99] H. Masur and Y. Minsky. Geometry of the complex of curves I: Hyperbolicity. *Invent. Math.*, **138**(1):103–149, 1999.

[MM00] H. Masur and Y. Minsky. Geometry of the complex of curves II: Hierarchical structure. *Geom. Funct. Anal.*, **10**(4):902–974, 2000.

[Nam05] H. Namazi. *Heegaard splittings and hyperbolic geometry*. ProQuest LLC, Ann Arbor, MI, 2005. Thesis (Ph.D.)–State University of New York at Stony Brook.

[NS09] H. Namazi and J. Souto. Heegaard splittings and pseudo-Anosov maps. *Geom. Funct. Anal.*, **19**(4):1195–1228, 2009.

[Thu86a] W. Thurston. Hyperbolic structures on 3-manifolds, I: Deformation of acylindrical manifolds. *Ann. of Math.*, **124**(2):203–246, 1986.

[Thu86b] W. Thurston. Hyperbolic structures on 3-manifolds, III: Deformations of 3-manifolds with incompressible boundary. *ArXiv e-print*, 1986. ArXiv:9801058 [math.GT].

[Yos14] M. Yoshizawa. High distance Heegaard splittings via Dehn twists. *Algebr. Geom. Topol.*, 14(2):979–1004, 2014.

Department of Mathematics, ETH Zürich, Switzerland

Email address: `peter.feller@math.ch`

Department of Mathematics, Heriot-Watt University, Edinburgh, UK

Email address: `a.sisto@hw.ac.uk`

Department of Mathematics, Sapienza University of Rome, Italy

Email address: `gabriele.viaggi@uniroma1.it`