VOLUMES OF RANDOM 3-MANIFOLDS

GABRIELE VIAGGI

Abstract. We prove a law of large numbers for the volumes of families of random hyperbolic mapping tori and Heegaard splittings providing a sharp answer to a conjecture of Dunfield and Thurston.

1. Introduction

Every orientation preserving diffeomorphism $f \in \mathrm{Diff}^+(\Sigma)$ of a closed orientable surface $\Sigma = \Sigma_g$ of genus $g \geq 2$ can be used to define 3-manifolds in two natural ways: We can construct the *mapping torus*

$$T_f := \Sigma \times [0,1] \,/\, (x,0) \sim (f(x),1)\,,$$

and we can form the *Heegaard splitting*

$$M_f := H_g \cup_{f:\partial H_g \to \partial H_g} H_g.$$

The latter is obtained by gluing together two copies of the handlebody H_g of genus g along the boundary $\partial H_g = \Sigma$. In both cases the diffeomorphism type of the 3-manifold only depends on the isotopy class of f, which means that it is well-defined for the *mapping class* $[f] \in \mathrm{Mod}\,(\Sigma) := \mathrm{Diff}^+(\Sigma)/\mathrm{Diff}^+_0(\Sigma)$ in the *mapping class group*. We use X_f to denote either T_f or M_f.

Invariants of the 3-manifold X_f give rise to well-defined invariants of the mapping class $[f]$. For example, if X_f supports a *hyperbolic metric*, then we can use the geometry to define invariants of $[f]$: By Mostow rigidity, if such hyperbolic metric exists, then it is unique up to isometry.

After Perelman's solution of Thurston's geometrization conjecture, the only obstruction to the existence of a hyperbolic metric on X_f can be phrased in topological terms: A closed orientable 3-manifold is hyperbolic if and only if it has no essential spheres, tori and Klein bottles and it is not Seifert fibered. Mapping classes that are sufficiently complicated in an appropriate sense (see Thurston [45] and Hempel [21]) give rise to manifolds that satisfy these properties.

For a closed hyperbolic 3-manifold X_f, a good measure of its complexity is provided by the *volume* $\mathrm{vol}(X_f)$. According to a celebrated theorem by Gromov and Thurston, it equals a universal multiple of the *simplicial volume* of X_f, a topologically defined invariant (see for example Chapter C of [2]). As X_f is not always hyperbolic, in general we define $\mathrm{vol}(X_f)$ to be this

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universal multiple of the simplicial volume, a quantity that always makes sense.

The purpose of this article is to study the *growth* of the volume for families of *random 3-manifolds* or, equivalently, *random mapping classes.*

A random mapping class is the result of a *random walk* generated by a probability measure on the mapping class group, and a random 3-manifold is one of the form X_f where f is a random mapping class. Such notion of random 3-manifolds has been introduced in the foundational work by Dunfield and Thurston [16]. They conjectured that a random 3-manifold is hyperbolic and that its volume grows linearly with the step length of the random walk (Conjecture 2.11 of [16]).

The existence of a hyperbolic metric has been settled by Maher for both mapping tori [27] and Heegaard splittings [28].

Here we prove Dunfield and Thurston volume conjecture interpreting it in a strict way (see also Conjecture 9.2 in Rivin [43]). Our main result is the following *law of large numbers*: Let μ be a probability measure on $\mathrm{Mod}(\Sigma)$ whose support is a finite symmetric generating set. Let $\omega = (\omega_n)_{n\in\mathbb{N}}$ be the associated random walk

Theorem 1. *There exists $v = v(\mu) > 0$ such that for almost every $\omega = (\omega_n)_{n\in\mathbb{N}}$ the following holds*

$$\lim_{n\to\infty} \frac{\mathrm{vol}(X_{\omega_n})}{n} = v.$$

Here $(X_{\omega_n})_{n\in\mathbb{N}}$ is either the family of mapping tori or Heegaard splittings.

We observe that the asymptotic is the same for both mapping tori and Heegaard splittings. We also remark that the important part is the existence of an *exact asymptotic* for the volume as the *coarsely linear* behaviour follows from previous work. In the case of mapping tori, it is a consequence of work of Brock [8] and the theory of random walks on weakly hyperbolic groups: The former shows that there exists a constant $c(g) > 0$, only depending on the genus g of Σ, such that for every pseudo-Anosov f

$$\frac{1}{c(g)} d_{\mathrm{WP}}(f) \le \mathrm{vol}(T_f) \le c(g) d_{\mathrm{WP}}(f)$$

where $d_{\mathrm{WP}}(f)$ is the Weil-Petersson translation length of f. The latter provides a linear asymptotic for $d_{\mathrm{WP}}(f)$ (see for example Maher-Tiozzo [30]).

The coarsely linear behaviour for the volume of a random Heegaard splitting follows from results by Maher [28] combined with an unpublished work of Brock and Souto. We refer to the introduction of [28] for more details.

Theorem 1 will be derived from the more technical Theorem 2 concerning *quasi-fuchsian* manifolds. We recall that a quasi-fuchsian manifold is a hyperbolic 3-manifold Q homeomorphic to $\Sigma \times \mathbb{R}$ that has a *compact* subset, the *convex core* $\mathcal{CC}(Q) \subset Q$, that contains all geodesics of Q joining two of its points. The asymptotic geometry of Q is captured by two conformal classes on Σ, i.e. two points in the Teichmüller space $\mathcal{T} = \mathcal{T}(\Sigma)$. Bers [3] showed that for every ordered pair $X, Y \in \mathcal{T}$ there exists a unique quasi-fuchsian manifold, which we denote by $Q(X, Y)$, realizing those asymptotic data.

Theorem 2. *There exists $v = v(\mu) > 0$ such that for every $o \in \mathcal{T}$ and for almost every $\omega = (\omega_n)_{n\in\mathbb{N}}$ the following limit exists:*

$$\lim_{n\to\infty} \frac{\mathrm{vol}\left(\mathcal{CC}(Q(o, \omega_n o))\right)}{n} = v.$$

We remark that $v(\mu)$ is the same as in Theorem 1. Once again, the coarsely linear behaviour of the quantity in Theorem 2 was known before: The technology developed around the solution of the ending lamination conjecture by Minsky [38] and Brock-Canary-Minsky [12], with fundamental contributions by Masur-Minsky [34], [35], gives a combinatorial description of the internal geometry of the convex core of a quasi-fuchsian manifold. This combinatorial picture is a key ingredient in Brock's proof [7] of the following coarse estimate: There exists a constant $k(g) > 0$, only depending on the genus g of Σ, such that

$$\frac{1}{k(g)} d_{\mathrm{WP}}(X, Y) - k(g) \leq \mathrm{vol}\left(\mathcal{CC}(Q(X, Y))\right) \leq k(g) d_{\mathrm{WP}}(X, Y) + k(g).$$

This link between volumes of hyperbolic 3-manifolds and Weil-Petersson geometry of Teichmüller space, as in the case of random mapping tori, leads to the coarsely linear behaviour for the volume of the convex cores of $Q(o, \omega_n o)$, but does not give, by itself, a law of large numbers. The main novelty in this paper is that we work directly with the geometry of the quasi-fuchsian manifolds rather than their combinatorial counterparts. This allows us to get exact asymptotics rather than coarse ones.

The relation between Theorem 1 and Theorem 2 is provided by a *model manifold* construction similar to Namazi [39], Namazi-Souto [40], Brock-Minsky-Namazi-Souto [13]. In the case of random 3-manifolds the heuristic picture is the following: The geometry of X_{ω_n} largely resembles the geometry of the convex core of $Q(o, \omega_n o)$. More precisely, as far as the volume is concerned, we have

$$\left|\mathrm{vol}\left(X_{\omega_n}\right) - \mathrm{vol}\left(\mathcal{CC}(Q(o, \omega_n o))\right)\right| = o(n).$$

We now describe the basic ideas behind Theorem 2: Suppose that the support of μ equals a finite symmetric generating set S and consider $f = s_1 \ldots s_n$, a long random word in the generators $s_i \in S$. Then f corresponds to a quasi-fuchsian manifold $Q(o, fo)$. Fix N large, and assume $n = Nm$ for simplicity. We can split f into smaller blocks of size N

$$f = (s_1 \ldots s_N) \cdots (s_{N(m-1)+1} \ldots s_{Nm})$$

which we also denote by $f_j := s_{jN+1} \cdots s_{(j+1)N}$. Each block corresponds to a quasi-fuchsian manifold $Q(o, f_j o)$ as well. The main idea is that the geometry of the convex core $\mathcal{CC}(Q(o, fo))$ can be roughly described by juxtaposing, one after the other, the convex cores of the single blocks $\mathcal{CC}(Q(o, f_j o))$. In particular, the volume $\mathrm{vol}(\mathcal{CC}(Q(o, fo)))$ can be well approximated by the *ergodic sum*

$$\sum_{1 \leq j \leq m} \mathrm{vol}\left(\mathcal{CC}(Q(o, f_j o))\right)$$

which, when divided by $n = mN$, converges by the Birkhoff ergodic theorem.

In the paper, we will make this heuristic picture more accurate. Our three main ingredients are the model manifold, bridging between the geometry of the Teichmüller space $\mathcal{T}$ and the internal geometry of quasi-fuchsian manifolds [38],[12], a recurrence property for random walks [1] and the method of natural maps from Besson-Courtois-Gallot [4]. They correspond respectively to Proposition 3.10, Theorem 4.3 and Theorem 3.11. Proposition 3.10 and Theorem 4.3 are used to construct a geometric object, i.e. a negatively curved model for T_f, associated to the ergodic sum written above. Theorem 3.11 let us compare this model to the underlying hyperbolic structure.

As an application of the same techniques, along the way, we give another proof of the following well-known result [24], [10] relating iterations of pseudo-Anosovs, volumes of quasi-fuchsian manifolds and mapping tori

Proposition 3. *Let ϕ be a pseudo-Anosov mapping class. For every $o \in \mathcal{T}$ the following holds:*

$$\lim_{n \to \infty} \frac{\mathrm{vol}\left(\mathcal{CC}(Q(o, \phi^n o))\right)}{n} = \mathrm{vol}\left(T_\phi\right).$$

Outline. The paper is organized as follows.

In Section 2 we introduce quasi-fuchsian manifolds. They are the building blocks for the cut-and-glue construction of Section 3. We prove that, under suitable assumptions, we can glue together a family of quasi-fuchsian manifolds in a geometrically controlled way. The geometric control on the glued manifold is good enough for the application of volume comparison results.

As an application of the cut-and-glue construction we show that the volume of a random gluing is essentially the volume of a quasi-fuchsian manifold (Proposition 3 follows from the same principle). As a consequence, in Section 5, we deduce Theorem 1 from Theorem 2 whose proof is carried out shortly after.

In Section 4 we discuss random walks on the mapping class group and on Teichmüller space. The goal is to describe the picture of a random Teichmüller ray and state the main recurrence property.

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2. Quasi-Fuchsan manifolds

We start by introducing *quasi-fuchsian* manifolds and their geometry.

2.1. **Marked hyperbolic 3-manifolds.** Let M be a compact, connected, oriented 3-manifold. A marked hyperbolic structure on M is a complete Riemannian metric on $\operatorname{int}(M)$ of constant sectional curvature $\sec \equiv -1$. We regard two Riemannian metrics as equivalent if they are isometric via a diffeomorphism homotopic to the identity.

Every marked hyperbolic structure corresponds to a quotient $\mathbb{H}^3/\Gamma$ of the hyperbolic 3-space $\mathbb{H}^3$ by a discrete and torsion free group of isometries $\Gamma < \operatorname{Isom}^+\left(\mathbb{H}^3\right) = \operatorname{PSL}_2(\mathbb{C})$ together with an identification of $\pi_1(M)$ with Γ, called the *holonomy representation* $\rho : \pi_1(M) \longrightarrow \operatorname{PSL}_2(\mathbb{C})$.

We are mostly interested in the cases where $M = \Sigma \times [-1, 1]$ is a trivial I-bundle over a surface and when M is closed. By Mostow Rigidity, if M is closed and admits a hyperbolic metric, then the metric is unique up to isometries. In this case we denote by $\operatorname{vol}(M)$ the volume of such a metric.

2.2. **Quasi-fuchsian manifolds.** A particularly flexible class of structures is provided by the so-called *quasi-fuchsian* manifolds

Definition (Quasi-Fuchsian). A marked hyperbolic structure Q on $\Sigma \times [-1, 1]$ is *quasi-fuchsian* if $\mathbb{H}^3/\rho(\pi_1(\Sigma))$ if it admits a non-empty *compact* subset which is *convex*, that is, a compact subset that contains every geodesic joining two of its points. The smallest such convex subset is called the *convex core* and is denoted by $\mathcal{CC}(Q)$.

The convex core $\mathcal{CC}(Q)$ is always a topological submanifold. If it has codimension 1 then it is a totally geodesic surface and we are in the *fuchsian* case, the group $\Gamma < \operatorname{Isom}^+\left(\mathbb{H}^3\right)$ stabilizes a totally geodesic $\mathbb{H}^2 \subset \mathbb{H}^3$. In the generic case it has codimension 0 and is homeomorphic to $\Sigma \times [-1, 1]$. The inclusion $\mathcal{CC}(Q) \subset Q$ is always a homotopy equivalence.

We denote by

$$\operatorname{vol}(Q) := \operatorname{vol}(\mathcal{CC}(Q)) \in [0, \infty)$$

the volume of the convex core of the quasi-fuchsian manifold Q.

2.3. **Deformation space.** We denote by $\mathcal{T}$ the Teichmüller space of Σ, that is, the space of marked hyperbolic structures on Σ up to isometries homotopic to the identity. We equip $\mathcal{T}$ with the Teichmüller metric $d_{\mathcal{T}}$.

To every quasi-fuchsian manifold Q one can associate the *conformal boundary* $\partial_c Q$ in the following way: The surface group $\pi_1(\Sigma)$ acts on $\mathbb{H}^3$ by isometries and on $\mathbb{CP}^1 = \partial\mathbb{H}^3$ by Möbius transformations. It also preserves a convex set, the lift of $\mathcal{CC}(Q)$ to the universal cover, on which it acts cocompactly. By Milnor-Švarc, for any fixed basepoint $o \in \mathbb{H}^3$, the orbit map $\gamma \in \pi_1(\Sigma) \to \gamma o \in \mathbb{H}^3$ is a quasi-isometric embedding and extends to a topological embedding on the boundary $\partial\pi_1(\Sigma) \hookrightarrow \mathbb{CP}^1$. The image is a topological circle Λ, called the *limit set*, that divides the Riemann sphere $\mathbb{CP}^1$ into a union of two topological disks $\Omega = \mathbb{CP}^1 \backslash \Lambda$. The action $\pi_1(\Sigma) \curvearrowright \Omega$ preserves the connected components, and is free, properly discontinuous and conformal. The quotient $\partial_c Q = \Omega/\pi_1(\Sigma) = X \sqcup Y$ is a disjoint union of two marked oriented Riemann surfaces, homeomorphic to Σ, and it is called the *conformal boundary* of Q. We endow X and Y with the unique hyperbolic metrics in their conformal classes. The quotient $\bar{Q} := (\mathbb{H}^3 \cup \Omega)/\Gamma$ compactifies Q.

Theorem 2.1 (Double Uniformization, Bers [3]). *For every ordered pair of marked hyperbolic surfaces $(X,Y) \in \mathcal{T} \times \mathcal{T}$ there exists a unique equivalence class of quasi-fuchsian manifolds, denoted by $Q(X,Y)$, realizing the conformal boundary $\partial_c Q(X,Y) = X \sqcup Y$.*

The mapping class group $\mathrm{Mod}\,(\Sigma)$ acts on quasi-fuchsian manifolds by precomposition with the marking. In Bers coordinates it plainly translates into $\phi Q(X,Y) = Q\,(\phi X, \phi Y)$.

2.4. **Teichmüller geometry and volumes.** Later, it will be very important for us to quantify the price we have to pay in terms of volume if we want to replace a quasi-fuchsian manifold Q with another one Q'. We would like to express $|\mathrm{vol}\,(Q) - \mathrm{vol}\,(Q')|$ in terms of the geometry of the conformal boundary.

Despite the fact that the Weil-Petersson geometry is more natural when considering questions about volumes, we will mainly use the Teichmüller metric $d_{\mathcal{T}}$. The reason is that we are mostly concerned with *upper bounds* for the volumes of the convex cores. It is a classical result of Linch [25] that the Teichmüller distance is bigger than the Weil-Petersson distance $d_{\mathrm{WP}} \leq \sqrt{2\pi|\chi(\Sigma)|}\,d_{\mathcal{T}}$. The following is our main tool:

Proposition 2.2 (Proposition 2.7 in Kojima-McShane [24], see also Schlenker [44]). *There exists $\kappa = \kappa(\Sigma) > 0$ such that*

$$\left|\mathrm{vol}\,(Q(X,Y)) - \mathrm{vol}\left(Q(X',Y')\right)\right| \leq \kappa\left(d_{\mathcal{T}}(X,X') + d_{\mathcal{T}}(Y,Y')\right) + \kappa.$$

This formulation is not literally Proposition 2.7 of [24] so we spend a couple of words to explain the two differences. Firstly, the estimate in Proposition 2.7 of [24] concerns the *renormalized volume* and not volume of the convex core. However, the two quantities only differ by a uniform additive constant (see Theorem 1.1 in [44]). Secondly, their statement is limited to the case where $X = X' = Y'$, but their proof extends word by word to the more general setting: It suffices to apply their argument to the one parameter families $Q(X,Y_t)$ and $Q(X_t,Y')$, where X_t and Y_t are the Teichmüller geodesics joining X to X' and Y to Y'.

2.5. **Geometry of the convex core.** We associate to the quasi-fuchsian manifold $Q = Q(X, Y)$ the Teichmüller geodesic $l : [0, d] \to \mathcal{T}$ joining X to Y where $d = d_{\mathcal{T}}(X, Y)$. Work of Minsky [38] and Brock-Canary-Minsky [12] relates the geometry of the Teichmüller geodesic l to the internal geometry of $\mathcal{CC}(Q)$. In the next section we will use this information to glue together convex cores of quasi-fuchsian manifolds in a controlled way.

As a preparation, we start with a description of the boundary $\partial\mathcal{CC}(Q)$ and introduce some useful notation. We recall that, topologically, $\mathcal{CC}(Q) \simeq \Sigma \times [-1, 1]$. The convex core separates $\bar{Q} = Q \cup \partial_c Q$ into two connected components, containing, respectively, X and Y. We denote by $\partial_X \mathcal{CC}(Q)$ and $\partial_Y \mathcal{CC}(Q)$ the components of $\partial\mathcal{CC}(Q)$ that are contained, respectively, in the same component of $\bar{Q} - \mathrm{int}(\mathcal{CC}(Q))$ as X and Y. As observed by Thurston, the surfaces $\partial_X \mathcal{CC}(Q)$ and $\partial_Y \mathcal{CC}(Q)$, equipped with the induced path metric, are hyperbolic. By a result of Sullivan, they are also uniformly bilipschitz equivalent to X and Y (see Chapter II.2 of [14]).

3. Gluing and Volume

This section describes a gluing construction (Proposition 3.10) which is a major technical tool in the article. It allows us to cut and glue together quasi-fuchsian manifolds in a sufficiently controlled way. The control on the *models* obtained with this procedure is then exploited to get volume comparisons via the method of natural maps (Theorem 3.11 as in [4]) which is the second major tool of the section.

Along the way we recover a well-known result (Proposition 3) relating iterations of pseudo-Anosov maps and volumes of quasi-fuchsian manifolds.

3.1. **Product regions and Cut and Glue construction.** The cut and glue construction we are going to describe is a standard way to glue Riemannian 3-manifolds. Here we import the discussion and some of the observations of Section 5 of [20] and adapt them to our special setting. We start with a few definitions.

Definition (Product Region). Let Q be a quasi-fuchsian manifold with convex core $\mathcal{CC}(Q)$. A *product region* $U \subset \mathcal{CC}(Q)$ is a codimension 0 submanifold of $\mathcal{CC}(Q)$ homeomorphic to $\Sigma \times [0, 1]$ whose inclusion in Q is a homotopy equivalence.

Notice that, by a standard fact in 3-dimensional topology (see Proposition 3.1 in [48]), every π_1-injective closed surface in $Q \simeq \Sigma \times \mathbb{R}$ is isotopic to $\Sigma \times \{0\}$. Hence, every product region $U \subset Q$ is isotopic to $\Sigma \times [0, 1]$.

Using the product structure of $Q \simeq \Sigma \times \mathbb{R}$ we can define a *top boundary* $\partial_+ U$ and a *bottom boundary* $\partial_- U$. Since they are both isotopic to $\Sigma \times \{0\}$, the boundaries $\partial_\pm U$ separate Q. In particular, it makes sense to say that a point $x \in Q$ is *below* or *above* $\partial_\pm U$.

A product region comes together with a *marking*, an identification $j_U : \pi_1(\Sigma) \xrightarrow{\sim} \pi_1(U)$, defined as follows: The data of a marked hyperbolic structure Q gives us an identification $\pi_1(\Sigma) \simeq \pi_1(Q)$ and the inclusion $U \subset Q$, being a homotopy equivalence, gives $\pi_1(Q) \simeq \pi_1(U)$. The marking allows us to talk about the homotopy class of a map between product regions.

Any orientation preserving diffeomorphism $k : U \to V$ induces a well-defined mapping class $[k] \in \mathrm{Mod}\,(\Sigma) \simeq \mathrm{Out}^+(\pi_1(\Sigma))$ (Dehn-Nielsen-Baer, Theorem 8.1 in [17]), namely, the one corresponding to the outer automorphism

$$\pi_1(\Sigma) \overset{j_U}{\simeq} \pi_1(U) \overset{k}{\simeq} \pi_1(V) \overset{j_V}{\simeq} \pi_1(\Sigma).$$

We also want to quantify the geometric quality of a map between product regions. Since we want compare the curvature tensors of ρ_U and $k^*\rho_V$, a good measurement for us is provided by the $\mathcal{C}^2$-norm on tensors.

Definition ($\mathcal{C}^2$-norm). Let $U \subset Q$ be a product region. Denote by ∇ the Levi-Civita connection on (Q, ρ_Q). Denote by $|\bullet|_x$ the norm induced by the inner product $\rho_Q(x)$ of T_xU on tensors on T_xU. Let τ be a tensor on U. The $\mathcal{C}^2$-norm of τ at x is the quantity

$$||\tau||_{\mathcal{C}^2,x} := |\tau(x)|_x + |\nabla\tau(x)|_x + \left|\nabla^2\tau(x)\right|_x.$$

Similarly, the $\mathcal{C}^2$-norm of τ on U is given by

$$||\tau||_{\mathcal{C}^2(U)} := \sup_{x\in U} |\tau(x)|_x + \sup_{x\in U} |\nabla\tau(x)|_x + \sup_{x\in U} \left|\nabla^2\tau(x)\right|_x.$$

Definition (Almost-Isometric). Let $k : U \to V$ be a diffeomorphism between product regions $U \subset Q$ and $V \subset Q'$. We compare the metric ρ_Q with the pull-back metric $k^*\rho_{Q'}$ by measuring their distance in the $\mathcal{C}^2$-norm on tensors on U, that is $||\rho_U - k^*\rho_V||_{\mathcal{C}^2(U)}$.

For $\xi > 0$ we say that k is ξ-*almost isometric* if $||\rho_U - k^*\rho_V||_{\mathcal{C}^2(U)} < \xi$.

The elementary blocks that we are going to glue together along almost isometric diffeomorphisms of product regions are pieces of quasi-fuchsian manifolds bounded by product regions.

Definition (Gluing Block). Let $U^-, U^+ \subset Q$ be disjoint product regions in a quasi-fuchsian manifold Q such that U^- lies below ∂_+U^+. The associated *gluing block* is the submanifold $Q_0 \subset Q$ bounded by ∂_+U^+ and ∂_-U^-. We call U^+ the *top collar* and U^- the *bottom collar* of the gluing block Q_0.

Topologically, a gluing block is again a product region.

Lemma 3.1. *Let $U^+, U^- \subset Q$ be disjoint product regions. The gluing block Q_0 bounded by ∂_+U^+ and ∂_-U^- is diffeomorphic to $\Sigma \times [-1, 1]$.*

Proof. Identify $Q \simeq \Sigma \times \mathbb{R}$ with the interior of the manifold $\Sigma \times [-1, 1]$. By Proposition 3.1 and Corollary 3.2 in [48], the surfaces ∂_+U^+ and $\partial_-U^- \subset Q$ are parallel to $\Sigma \times \{1\}$ and $\Sigma \times \{-1\}$, thus the region they cobound is diffeomorphic again to $\Sigma \times [-1, 1]$. $\square$

The main reason why we want to make a distinction between product regions and gluing blocks is of geometric nature: Later on, we will allow a gluing block to have arbitrarily complicated geometry while we will always keep the geometry of a product region uniformly controlled.

The following lemma is what we refer to as the cut-and-glue construction.

LEMMA 3.2. *Let Q, Q' be quasi-fuchsian manifolds. Denote by $\rho_Q, \rho_{Q'}$ their Riemannian metrics. Consider a pair of gluing blocks $Q_0 \subset Q$ and $Q_0' \subset Q'$ where the top collar of Q_0 is the product region $U \subset Q_0$ and the bottom collar of Q_0' is the product region $U' \subset Q_0'$. Suppose we have an orientation preserving diffeomorphism $k : U \to U'$ between them. Suppose also that $\theta : U \to [0,1]$ is a smooth function with $\theta \equiv 0$ on a collar of $\partial_- U$ and $\theta \equiv 1$ on a collar of $\partial_+ U$. Then we can form the 3-manifold*

$$Q'' = Q_0 \cup_{k:U\to U'} Q_0'$$

and endow it with the Riemannian metric

$$\rho := \begin{cases} \rho_Q & \text{on } Q_0 \setminus U \\ (1-\theta)\rho_Q + \theta k^* \rho_{Q'} & \text{on } U \\ \rho_{Q'} & \text{on } Q_0' \setminus U'. \end{cases}$$

If k is ξ-almost isometric, then the identity map $(U, \rho_Q) \to (U, \rho)$ is $(1+\xi)^{1/2}$-Lipschitz. In particular, we have the following volume bound

$$\mathrm{vol}_\rho(U) \leq (1+\xi)^{3/2} \mathrm{vol}_{\rho_Q}(U).$$

Thus, if $\mathrm{vol}_{\rho_Q}(U)$ is uniformly bounded, the same is true for $\mathrm{vol}_\rho(U)$. If moreover $||\theta||_{\mathcal{C}^2(U)} \cdot \left|\left|\rho_Q - k^*\rho_{Q'}\right|\right|_{\mathcal{C}^2(U)}$ is sufficiently small, then, on $U \subset Q''$, we have the following sectional curvature bound

$$\left|1 + \sec_{Q''}\right| \leq c_3 \, ||\theta||_{\mathcal{C}^2(U)} \cdot \left|\left|\rho_Q - k^*\rho_{Q'}\right|\right|_{\mathcal{C}^2(U)}$$

for some universal constant $c_3 > 0$.

Proof. For simplicity, let us denote by $\tau := \rho_Q - k^*\rho_{Q'}$ the difference between the two metrics. Write $\rho_Q - \rho = \theta(\rho_Q - k^*\rho_{Q'}) = \theta\tau$.

Let $u, v \in T_x U$ be a pair of unit vectors for ρ_Q. We have

$$|\rho(u,v) - \rho_Q(u,v)| = \theta(x)|\tau(u,v)| \leq \theta(x)|\tau(x)|_x < \xi.$$

In particular, we have $1 - \xi \leq \rho(u,u) \leq 1 + \xi$, so that the identity between $(U, \rho_Q) \to (U, \rho)$ is $(1+\xi)^{1/2}$-Lipschitz which implies the volume bound.

As for the curvature bounds, we proceed as follows: Denote by Riem_ρ and Riem_Q the Riemann curvature tensors of ρ and ρ_Q respectively. We show that their difference can be bounded pointwise by $||\theta\tau||_{\mathcal{C}^2(U)}$. Notice that the first and second covariant derivatives of $\theta\tau$ are $\nabla\theta\tau = \nabla\theta \otimes \tau + \theta\nabla\tau$ and $\nabla^2\theta\tau = (\nabla^2\theta) \otimes \tau + 2(\nabla\theta) \otimes (\nabla\tau) + \theta\nabla^2\tau$, so that we get

$$||\theta\tau||_{\mathcal{C}^2(U)} \leq 2||\theta||_{\mathcal{C}^2(U)} \cdot ||\tau||_{\mathcal{C}^2(U)}.$$

Let $x \in U$ be any point. In order to get a bound on $|\mathrm{Riem}_\rho(x) - \mathrm{Riem}_Q(x)|$, we work in local normal coordinates for ρ_Q: We first identify a small metric ball around x with a small ball around the origin in T_xU via the exponential map $\exp_x^Q : T_xU \to U$ given by ρ_Q and then we isometrically identify $(T_xU, \rho_Q(x))$ with the standard Euclidean 3-space $\mathbb{R}^3$. Call (x^1, x^2, x^3) the resulting local coordinates.

Notice that, since ρ_Q is a hyperbolic metric, the expression of $(\exp_x^Q)^*\rho_Q$ in such coordinates is universal. In particular, an explicit computation shows that there exists a universal constant $a_3 > 0$ such that the following holds: For every $(2,0)$-tensor $\tau = \tau_{ij}dx^i dx^j$ and every $1 \le i, j, \alpha, \beta \le 3$ we have

$$|\tau_{ij}(0)|, |\partial_\alpha \tau_{ij}(0)|, |\partial_\alpha \partial_\beta \tau_{ij}(0)| \le a_3 ||\tau||_{\mathcal{C}^2,x}.$$

In local coordinates the Riemann curvature tensors $\mathrm{Riem}_\rho(x)$, $\mathrm{Riem}_Q(x)$ are both determined by the same algebraic rational expression $R(\bullet)$ only involving the coeffiecients of the metric $\rho_{ij}(0)$ and their first and second derivatives $\partial_\alpha \rho_{ij}(0)$ and $\partial_\alpha \partial_\beta \rho_{ij}(0)$. For the metric ρ_Q, the coefficients are $(\rho_Q)_{ij}(0) = \delta_{ij}$ and their derivatives are $\partial_\alpha(\rho_Q)_{ij}(0) = 0$ and $\partial_\alpha\partial_\beta(\rho_Q)_{ij}(0) = \delta_{i\beta}\delta_{j\alpha} - \delta_{ij}\delta_{\alpha\beta}$. In particular, there exists a universal constant $b_3 > 0$, only depending on the local behaviour of the rational function $R(\bullet)$ in a neighbourhod of the point determined by $(\rho_Q)_{ij}(0), \partial_\alpha(\rho_Q)_{ij}(0), \partial_\alpha\partial_\beta(\rho_Q)_{ij}(0)$, such that, if $||\tau||_{\mathcal{C}^2,x}$ is sufficiently small, then

$$|\mathrm{Riem}_\rho(x) - \mathrm{Riem}_Q(x)| \le b_3 \sup_{1 \le i,j,\alpha,\beta \le 3} \{|\tau_{ij}(0)|, |\partial_\alpha \tau_{ij}(0)|, |\partial_\alpha\partial_\beta \tau_{ij}(0)|\}.$$

Putting together the previous inequalities we get the desired curvature bound by setting $c_3 := 2a_3b_3$. □

In all our applications of Lemma 3.2 we will always be in the situation where the bump function θ on U has uniformly bounded $\mathcal{C}^2$-norm $||\theta||_{\mathcal{C}^2(U)} \le K$. In particular, in order to fulfill the condition on $||\theta||_{\mathcal{C}^2(U)} \cdot ||\rho_Q - k^*\rho_{Q'}||_{\mathcal{C}^2(U)}$, it will be enough to ask that $k : U \to U'$ is a ξ-almost isometric diffeomorphism with ξ small enough compared to K.

We now produce uniform bump functions on certain product regions: To each product region we associate two parameters, *diameter* and *width*

$$\mathrm{diam}(U) := \sup\{d_U(x,y)\,|x, y \in U\},$$
$$\mathrm{width}(U) := \inf\{d_Q(x,y)\,|x \in \partial_+U, y \in \partial_-U\}.$$

If a product region has width at least D and diameter at most $2D$ we say that it has *size* D. The Margulis Lemma implies that the injectivity radius of a product region of size D, defined as

$$\mathrm{inj}(U) := \inf_{x\in U}\{\mathrm{inj}_x(Q)\},$$

is bounded from below in terms of D:

Lemma 3.3. *For every $D > 0$ there exists $\epsilon_0(D) > 0$ such that a product region U of size D has* $\mathrm{inj}(U) > \epsilon_0$.

Proof. The inclusion of U in Q is π_1-surjective. Having diameter bounded by $2D$, the region U cannot intersect too deeply any very thin Margulis tube $\mathbb{T}_\gamma$ otherwise $\pi_1(U) \to \pi_1(Q)$ would factor through $\pi_1(U) \to \pi_1(\mathbb{T}_\gamma)$. □

In particular, a compactness argument with the geometric topology on pointed hyperbolic manifolds gives us the following property: Once we fix the size of a product region we can produce a uniform bump function on it.

LEMMA 3.4. *For every $D > 0$ there exists $K > 0$ such that the following holds: Let $U \simeq \Sigma \times [0,1]$ be a product region of size D. Then there exists a smooth function $\theta : U \to [0,1]$ with the following properties:*

- *Near the boundaries it is constant: $\theta|_{\partial_- U} \equiv 0$ and $\theta|_{\partial_+ U} \equiv 1$.*
- *θ has uniformly bounded $\mathcal{C}^2$-norm $||\theta||_{\mathcal{C}^2} \leq K$.*

For a proof see for example [20].

3.2. **Almost-isometric embeddings of product regions.**

For a fixed $\eta > 0$ we denote by $\mathcal{T}_\eta$ the *η-thick part* of Teichmüller space consisting of those hyperbolic structures with no closed geodesic shorter than η.

The following is a consequence of the *model manifold* technology developed by Minsky [38] around the solution of the Ending Lamination Conjecture (completed then in Brock-Canary-Minsky [12]).

PROPOSITION 3.5. *For every $\eta, \xi, \delta, D > 0$ there exist $\epsilon_1 = \epsilon_1(\eta, g), D_0(\eta, g)$ and $h = h(\eta, \xi, \delta, D) > 0$ such that the following holds: Let Q_1, Q_2 be quasi-fuchsian manifolds with associated Teichmüller geodesics $l_i : I_i \subseteq \mathbb{R} \to \mathcal{T}$ with $i = 1, 2$. Suppose that l_1, l_2 δ-fellow travel on a subsegment J of length at least h and entirely contained in $\mathcal{T}_\eta$. Then there exist product regions $U_i \subset \mathcal{CC}(Q_i)$ of size D and injectivity radius $\mathrm{inj}(U_j) \geq \epsilon_1$ and a ξ-almost isometric orientation preserving diffeomorphism $k : U_1 \to U_2$ in the homotopy class of the identity. Moreover, if D is much larger than D_0, we can assume that U_i contains a closed geodesic α_i^* with the following properties: It lies far away from the boundary, that is $d_{Q_i}(\alpha_i^*, \partial U_i) \geq D_0$, and represents a simple closed curve $\alpha \subset \Sigma$ whose length in $T \in J$, the midpoint of the segment J, is bounded by $L_T(\alpha) \leq D_0$.*

In the statement and in the next section we use the following notation:

> **Notation**. If $\alpha : S^1 \to Q$ is a closed loop in a hyperbolic 3-manifold, we denote by $l(\alpha)$ its length and by $l_Q(\alpha)$ the length of the unique geodesic representative in the homotopy class. If the target instead is a hyperbolic surface $\alpha : S^1 \to X$, we use the notations $L(\alpha)$ and $L_X(\alpha)$.

Proof. We argue by contradiction: Fix $\eta, \xi, \delta > 0$ and D sufficiently large. Consider, a sequence of Teichmüller geodesics l_1^n, l_2^n that are defined on intervals $I_1^n, I_2^n \subset \mathbb{R}$ containing a common subsegment $J_n = [-n, n] \subset I_1^n \cap I_2^n$ along which they δ-fellow travel in the η-thick part of Teichmüller space.

Assume by contradiction that there is no ξ-almost isometric embedding of a product region of size D of Q_1^n into Q_n^2 as in the statement of the proposition.

Since the mapping class group acts cocompactly on $\mathcal{T}_\eta$ (see Theorem 12.6 of [17]), we can assume that, up to remarking, $l_1^n(0)$ and $l_2^n(0)$ lie in a fixed compact set $\mathcal{T}_\eta$. Thus, up to subsequences, the geodesics l_j^n converge uniformly on compact subsets to bi-infinite geodesics $l_j^\infty : \mathbb{R} \to \mathcal{T}$ entirely contained in $\mathcal{T}_\eta$ for $j = 1, 2$. Moreover, l_1^∞ and l_2^∞ δ-fellow travel along the whole line $\mathbb{R}$.

By work of Masur [33], a Teichmüller geodesic $l : \mathbb{R} \to \mathcal{T}$ that stays in $\mathcal{T}_\eta$ is defined by a quadratic differential with horizontal and vertical foliations λ^- and λ^+ that have the following three properties: They are *transverse*, which means that $i(\lambda^+, \lambda^-) > 0$ and $i(\lambda^+, \nu) + i(\lambda^-, \nu) > 0$ for every measured lamination $\nu \in \mathcal{ML}$, *minimal*, that is $i(\lambda^+, \alpha), i(\lambda^-, \alpha) > 0$ for every simple closed curve $\alpha \subset \Sigma$, and *uniquely ergodic*. Furthermore, by another result of Masur [31], such Teichmüller geodesic converges in the forward and backward directions to the projective classes determined by λ^+ and λ^- in the Thurston compactification of Teichmüller space $\mathcal{T} \cup \mathcal{PML}$. This means that l has well defined endpoints at infinity on $\mathcal{PML}$.

In particular, the previous discussion applies to both l_1^∞ and l_2^∞. Since these geodesics converge in the forward and backward directions to minimal uniquely ergodic laminations and they δ-fellow travel all the time, it follows that they have the same endpoints at infinity (see for example Lemma 1.4.1 of [22]). Since the endpoints at infinity are the horizontal and vertical foliations of the quadratic differentials defining the geodesics, we conclude, by work of Gardiner-Masur [18], that l_1^∞ and l_2^∞ coincide.

We denote by $l := l_1^\infty = l_2^\infty$ the limit geodesic and by $\lambda^+, \lambda^- \in \mathcal{ML}$ some representatives of its endpoints.

We now return to the finite Teichmüller segments $l_j^n(I_j^n)$ for $j = 1, 2$. Notice that the endpoints of such segments can be very far away from the subsegments $l_j^n(J_n)$ on which we have uniform convergence to l_j^∞. However, it is still true that the forward and backward endpoints of $l_j^n(I_j^n)$ converge to λ^+ and λ^- respectively (see for example [32]).

As the endpoints of $l_j^n(I_j^n)$ onverge to λ^+ and λ^- and $i(\lambda^+, \nu) + i(\lambda^-, \nu) > 0$ for every measured lamination $\nu \in \mathcal{ML}$, we can apply Thurston's Double Limit Theorem [45] to the sequence of quasi-fuchsian manifolds Q_j^n. Thus, up to subsequences, we can assume that Q_1^n and Q_2^n converge in the algebraic topology (see Chapter 9 of [46]). The limits are hyperbolic structures Q_1^∞ and Q_2^∞ on $\Sigma \times \mathbb{R}$ (see Chapter 9 of [46] or Bonahon [5]). We now argue that Q_1^∞ and Q_2^∞ are isometric to each other.

By the solution of the Ending Lamination Conjecture by Minsky [38] and Brock-Canary-Minsky [12], it is enough to show that the end invariants of Q_1^∞ and Q_2^∞ are equal. In this case, the computation of the end invariants of Q_j^∞ could be carried out using just the technology developed by Thurston in

Chapter 9 of [46] and continuity properties of length functions, as discussed by Brock [6]. However, in order to shorten the argument, we will use work of Brock-Bromberg-Canary-Minsky [11] that covers a much more general setup and gives us a simple criterion to check.

Let us consider Q_1^∞, the arguments for Q_2^∞ are completely analogous. Let α_n be a shortest geodesic on X_1^n, the forward endpoint of $l_1^n(I_n^1)$. Since $X_1^n \to [\lambda^+]$ and λ^+ is minimal, by a result of Klarreich (see Theorem 1.2 of [23]) we have that λ^+ defines a point on the boundary at infinity $\partial_\infty \mathcal{C}$ of the *curve graph* $\mathcal{C}$ and $\alpha_n \to \lambda^+$ in $\mathcal{C} \cup \partial_\infty \mathcal{C}$. By Theorem 1.1 of [11], it follows that λ^+ is contained in the collection of end invariants of the positive end of Q_1^∞. Since λ^+ is minimal, the only possibility is that it coincides with the end invariant of the positive end. Similarly, λ^- is the end invariant of the negative end. Thus Q_1^∞ is a doubly degenerate structure on $\Sigma \times \mathbb{R}$ with ending laminations λ^+ and λ^-.

By the solution of the Ending Lamination Conjecture [38], [12], the two manifolds Q_1^∞ and Q_2^∞ are isometric via an orientation preserving isometry in the homotopy class of the identity. We identify them and denote both structures by $Q := Q_1^\infty = Q_2^\infty$.

Since the Teichmüller geodesic l is contained in $\mathcal{T}_\eta$, by work of Rafi [41] the manifold Q has a positive lower bound on the injectivity radius $\mathrm{inj}(Q) \geq \epsilon_1$, where $\epsilon_1 = \epsilon_1(\eta, g) > 0$ is a uniform constant.

Consider a curve α on $X = l(0)$ with length smaller than a Bers constant $B = B(g) > 0$. Notice that, since l_1^n and l_2^n converge to l, we have that α, for n large enough, has length bounded by $2B$ on both $l_1^n(0)$ and $l_2^n(0)$ which are the midpoints of the fellow traveling segments J_n. Denote by α^* the geodesic representative of α in Q.

It is a standard fact that α^* lies on the image of a 1-Lipschitz map $f : Y \to Q$ from a hyperbolic surface X to Q (see for example [14]). Since f is 1-Lipschitz and $\mathrm{inj}(Q) \geq \epsilon_1$, we must have $Y \in \mathcal{T}_{2\epsilon_1}$. Thus, there is a uniform upper bound on the diameter of X. As a consequence, since f is 1-Lipschitz, α^* lies in a subset $f(Y) \subset Q$ of uniformly bounded diameter.

If D is much larger than D_0 and D_0 is large enough, then α^*, being conatined in a subset of uniformly bounded diameter, lies in the middle of a product region U of size D and satisfies $d_Q(\alpha^*, \partial U) \geq D_0$. This follows from standard compactness arguments. It can also be deduced from the structure of the Model Manifold described in [38] and [12] in the case where there is a uniform lower bound on the injectivity radius.

We now show how to embed almost isometrically such a product region in both Q_j^n for every n large enough.

Since the algebraic limit Q is a doubly degenerate structure on $\Sigma \times \mathbb{R}$, it follows (see Theorem 9.2 in [46]) that the convergence $Q_j^n \to Q$ is *strong*, meaning that the manifolds converge both algebraically and *geometrically* to the same limit. For an extensive treatment of algebraic, geometric and strong convergence we refer to Chapter 9 of Thurston [46].

For hyperbolic manifolds, geometric convergence coincides with Cheeger-Gromov convergence (see the definition of the *geometric topology* and Theorem E.1.13 in Chapter E of [2]), thus, for every large n, we have smooth embeddings $f_j^n : U \subset Q \to Q_j^n$ that converge to isometries (in the sense described in Chapter E of [2]) as n goes to ∞.

As the inclusion $U \subset Q$ is a homotopy equivalence and the convergence is strong, the submanifolds $f_j^n(U) \subset Q_j^n$ are again product regions and the embeddings f_j^n are in the homotopy class of the identity. In particular, the curves $f_j^n(\alpha^*)$ represent α in Q_j^n.

Since f_j^n is arbitrarily close to an isometry, the size of $f_j^n(U)$ is comparable to the size of U and the curves $f_j^n(\alpha^*)$ have very small geodesic curvature. This implies that such curves are very close to their geodesic representatives. Therefore, we can also assume that the geodesic representative of α in Q_j^n lies in the product region $f_j^n(U)$ and is far enough from its boundary.

We can now conclude: Consider the composition $k_n := (f_1^n)^{-1} f_2^n$. Since both f_1^n and $(f_2^n)^{-1}$ are converging to isometries (see Lemma E.1.11 of [2]) we have that, for n large enough, k_n is a ξ-almost isometric diffeomorphism between the product regions $f_1^n(U) \subset Q_1^n$ and $f_2^n(U) \subset Q_2^n$. Both product regions have size D and contain the geodesic representative of α in the middle. This provides the desired contradiction and finishes the proof. □

We remark that, even if we will not need this additional property, the closed geodesics α_i^* can also be assumed to have uniformly bounded length: Notations as in the proof. By Theorem A of Minsky [36], the surface $X := l(0)$ as well as each surface $l(t)$ admits a map $f : X \to Q$ with *energy* $E(f) := \int_X ||\mathrm{d}f||^2 \mathrm{dvol}_X$ uniformly bounded by $A = A(\eta, g) > 0$. From here, standard estimates give us $l_Q(\alpha)^2 \leq A \sinh(L_X(\alpha)/2)/2 \leq A \sinh(B/2)/2$.

The geodesic representatives of α are used to locate the product regions inside the convex cores. We explain that in the following section. For now we observe the following immediate consequence of Proposition 3.5:

Definition (η-Height). Let $l : I \to \mathcal{T}$ be a Teichmüller geodesic. The *η-height* of l is the length of the maximal connected subsegment of I whose image is entirely contained in $\mathcal{T}_\eta$.

Corollary 3.6. *Fix $\eta > 0$. There exists a function $\rho : (0, \infty) \to (0, \infty)$ with $\rho(h) \uparrow \infty$ as $h \uparrow \infty$ and the following property: Let $Q = Q(X, Y)$ be a quasi-fuchsian manifold with associated geodesic $l : I \to \mathcal{T}$. If the η-height is at least h, then $d_Q(\partial_X \mathcal{CC}(Q), \partial_Y \mathcal{CC}(Q)) \geq \rho(h)$.*

We remark that, using Theorem 7.16 of Brock-Bromberg [9], it is possible to replace $\rho(\bullet)$ with the effective estimate $d_Q(\partial_X \mathcal{CC}(Q), \partial_Y \mathcal{CC}(Q)) \geq \frac{1}{A} d_{\mathcal{C}}(\alpha_X, \alpha_Y) - A$ where $A > 0$ is a uniform constant, $d_{\mathcal{C}}$ is the distance in the *curve graph* of Σ, and α_X, α_Y are shortest closed geodesics on X, Y. By work of Hamenstädt [19], the distance $d_{\mathcal{C}}(\alpha_X, \alpha_Y)$ up to multiplicative and additive constants is bounded from below by the η-height of $[X, Y]$.

3.3. **Position of the product region.** We make the following choice:

> **Standing assumption**. From now on we fix once and for all a sufficiently large size $D_1 \geq D_0$ and an injectivity radius lower bound $\epsilon_1 > 0$ for the product regions we consider. Here $D_0(\eta, g) > 0$ and $\epsilon_1 = \epsilon_1(\eta, g)$ are the thresholds provided by Proposition 3.5.

Let $\alpha : S^1 \to Q$ be a non-trivial closed curve in a hyperbolic 3-manifold Q that has a geodesic representative $\alpha^* \subset Q$. By basic hyperbolic geometry

$$\cosh\left(d_Q(\alpha, \alpha^*)\right) l_Q(\alpha) \leq l(\alpha).$$

Suppose that $Q = Q(X, Y)$ is a quasi-fuchsian manifold. Let $U \subset Q$ be a product region of size D_1 and injectivity radius $\mathrm{inj}(U) \geq \epsilon_1$ containing the geodesic representative α^* of $\alpha \subset \Sigma$. Recall that $\partial_X \mathcal{CC}(Q)$ denotes the boundary of the convex core that faces the conformal boundary X. By a Theorem due to Sullivan (see Chapter II.2 and in particular Theorem II.2.3.1 in [14]), there exists a universal constant K such that $\partial_X \mathcal{CC}(Q)$ and X are K-bilipschitz equivalent via a homeomorphism in the homotopy class of the identity. We have

$$d_Q(\partial_X \mathcal{CC}(Q), \alpha^*) \leq \operatorname{arccosh}\left(\frac{L_{\partial_X \mathcal{CC}(Q)}(\alpha)}{l_Q(\alpha)}\right) \leq \operatorname{arccosh}\left(\frac{K L_X(\alpha)}{2\epsilon_1}\right).$$

Let $T \in \mathcal{T}$ be a hyperbolic structure for which $L_T(\alpha) \leq D_0 = D_0(\eta, g)$. Wolpert's inequality $L_X(\alpha) \leq L_T(\alpha) e^{2d_{\mathcal{T}}(X,T)}$ (see Lemma 12.5 in [17]) allows us to continue the chain of inequalities to the following:

$$d_Q(\partial_X \mathcal{CC}(Q), \alpha^*) \leq \operatorname{arccosh}\left(\frac{K D_0}{2\epsilon_1} e^{2d_{\mathcal{T}}(X,T)}\right).$$

Let us introduce the auxiliary function F defined by

$$F(t) = \operatorname{arccosh}\left(\frac{K D_0}{2\epsilon_1} e^{2t}\right).$$

Notice that $F(t) \sim 2t$ for t large. With this notation we have

LEMMA 3.7. *Let $U \subset Q(X, Y)$ be a product region of size D_1 that contains the geodesic representative of the curve $\alpha \subset \Sigma$. Let $T \in \mathcal{T}$ be a surface such that $L_T(\alpha) \leq D_0$. Then*

$$d_Q(\partial_X \mathcal{CC}(Q), U) \leq F(d_{\mathcal{T}}(X, T)).$$

Combining Corollary 3.6 and Lemma 3.7 we can ensure that a pair of product regions is well separated. To this purpose we introduce another auxiliary function G defined by

$$G(t) = \inf_{r>0}\left\{\text{for every } s > r \text{ we have } \rho(s) > 2F(t) + 4D_1\right\}.$$

Notice that both F and G are increasing functions.

LEMMA 3.8. *Let U^-, U^+ be product regions of size D_1 and injectivity radius at least ϵ_1 in $Q = Q(X^-, X^+)$. Suppose they contain, respectively, the geodesic representatives of $\alpha^-, \alpha^+ \subset \Sigma$. Let $T^-, T^+ \in \mathcal{T}$ be surfaces such that $L_{T^-}(\alpha^-), L_{T^+}(\alpha^+) \leq D_0$. Consider*

$$d := \max\{d_{\mathcal{T}}(X^-, T^-), d_{\mathcal{T}}(X^+, T^+)\}.$$

If the η-height h of Q is at least $h \geq G(d)$ then the product regions are disjoint and cobound gluing block $Q^0 \subset Q$ for which U^-, U^+ are, respectively the bottom and top collars.

Proof. By Corollary 3.6 we have $d_Q(\partial_{X^-}\mathcal{CC}(Q), \partial_{X^+}\mathcal{CC}(Q)) \geq \rho(h)$ and, by Lemma 3.7, $d_Q(\partial_{X^\pm}\mathcal{CC}(Q), U^\pm) \leq F(d)$. If $\rho(h) - F(d) - 2D_1 \geq F(d) + 2D_1$, then the product regions U^-, U^+ are disjoint and, therefore, cobound a gluing block Q_0. By definition of G, if $h > G(d)$, the previous inequality holds. Furthermore, since U^+ is closer to $\partial_{X^+}\mathcal{CC}(Q)$ than U^-, we have that U^- lies below U^+. Therefore U^+ and U^- are, respectively, the top and bottom collars of Q_0. □

Finally, we quantify how much volume we lose if we replace the convex core of the quasi-fuchsian manifold Q with the gluing block $Q_0 \subset \mathcal{CC}(Q)$.

LEMMA 3.9. *Assumptions and notations as in Lemma 3.8. Suppose that the geodesic representatives of α^+ and α^- lie at distance at least D_0 from ∂U^+ and ∂U^-. There exists $V_1(D_1, \eta, d)$ such that*

$$\left|\mathrm{vol}\,(Q) - \mathrm{vol}\left(Q^0\right)\right| \leq V_0.$$

Proof. Recall that by Proposition 3.1 and Corollary 3.2 of [48] $\partial_+ U^+$ is parallel to $\partial_{X^+}\mathcal{CC}(Q)$. Denote by $R^+ \simeq \Sigma \times [0,1]$ the region they cobound. Using a construction due to Brock [7], we now find a singular chain that covers R^+ and has volume uniformly bounded by $d_{\mathcal{T}}(X^+, T^+) \leq d$. We briefly sketch the argument.

In what follows we import some terminology from [7].

Let P_X and P_T be pants decompositions on Σ whose components have length bounded by D_0 on X^+ and T^+ respectively and such that $\alpha^+ \subset P_T$.

The construction by Brock, see Theorem 5.7 of [7], provides us a manifold $N = \mathrm{cap}_{X^+} \cup N_\Delta$ homeomorphic to $\Sigma \times [0,2]$, where N_Δ identifies with $\Sigma \times [0,1]$ and cap_{X^+} with $\Sigma \times [1,2]$, and a homotopy equivalence $f : N \to Q$ with the following properties:

(1) N_Δ is the *triangulated part* of N. It is obtained by gluing *triangulated blocks* associated to a geodesic sequence $P_1 \to \cdots \to P_n$ of elementary moves joining $P_1 = P_T$ to $P_n = P_X$ in the *pant graph*.
(2) The map f is $\mathcal{C}^1$ and simplicial on N_Δ, i.e. it maps each simplex in N_Δ to a *straight simplex* in Q.
(3) The triangulatation of the top and bottom boundary surfaces of N_Δ are *suited* to P_X and P_T and the restrictions of f to those boudary

componensts are *simplicial hyperbolic surfaces* that map each curve in P_X and P_T to its geodesic representative in Q.

(4) f restricted to cap_{X^+} is a homotopy from the boundary of a small collar neighbourhood of $\partial_{X^+}\mathcal{CC}(Q)$ in $Q - \mathrm{int}(\mathcal{CC}(Q))$ to the restriction of f to the top boundary of N_Δ.

Proposition 5.8 and Lemma 5.11 of [7] then show that f can be taken so that the volume of $f(N)$ is bounded by the distance in the pant graph between P_X, P_T. Notice that, by Theorem 1.1 of [7], the distance in the pant graph can be coarsely replaced by the Weil-Petersson distance $d_{\mathrm{WP}}(X^+, T^+)$ and, in turn, we always have $d_{\mathrm{WP}} \le \sqrt{2\pi|\chi(\Sigma)|}d_{\mathcal{T}}$ by [25].

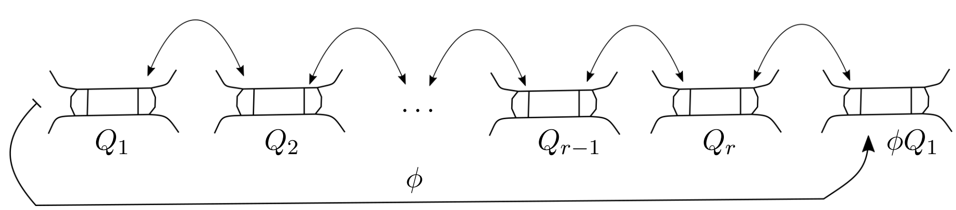
In order to conclude, we only need to show that f covers R^+. As $f(\Sigma \times \{2\})$ lies above R^+, it is enough to prove that $f(\Sigma \times \{0\})$ lies below R^+.

Claim: We have $f(\Sigma \times \{0\}) \subset U^+$.

For simplicity we introduce the notation $\Sigma_0 := \Sigma \times \{0\}$. The simplicial map f induces on Σ_0 a singular hyperbolic metric with cone points at the vertices of the triangulation and cone angles at least 2π. For every $\epsilon > 0$ the *ϵ-thin part* of Σ_0 is the set of points on the singular surface around which there is a homotopically non-trivial loop of length at most 2ϵ. It is a standard fact (see Lemma 1.10 in [5]) that there exists $L = L(g) > 0$ such that for every $x, y \in \Sigma_0$ there is a path γ joining them such that the total length of the subsegments of γ outside the ϵ-thin part is bounded by L.

Suppose now that $f(\Sigma_0)$ is not contained in U^+. Recall that f maps $\alpha^+ \subset \Sigma_0$ isometrically to its geodesic representative in Q which lies in the middle of U^+. Consider an arbitrary path $\gamma \subset \Sigma_0$ joining $x \in \alpha^+$ to a point y such that $f(y) \notin U^+$. Since $f(\gamma)$ connects the interior to the exterior of U^+, there exists an initial segment γ_0 such that $f(\gamma_0) \subset U^+$ joins $f(x)$ to a point on ∂U^+. Notice that f, being 1-Lipschitz, sends the ϵ-thin part to the the region of Q where the injectivity radius is at most ϵ. In particular, as $\mathrm{inj}(U^+) \ge \epsilon_1$, we have that γ_0 does not intersect the ϵ_1-thin part. Moreover, the length of γ_0 is at least $d_Q(f(x), \partial U) \ge D_0$. Since this holds for every path γ between x and y, if $D_0 \ge 2L$ (which we can assume), this would violate the above L-bounded diameter property of Σ_0. $\square$

3.4. **A gluing theorem.** Recall that we fixed $D_1 > 0$ sufficiently large once and for all. The following is our first crucial technical tool.



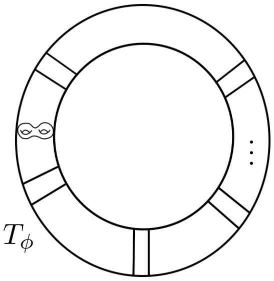


FIGURE 1. Gluing.

PROPOSITION 3.10. *Fix $\eta, \delta > 0$ and $\xi \in (0,1)$ small enough. There exists $h_0(\eta, \xi, \delta) > 0$ such that the following holds: Let ϕ be a mapping class. Let $\{Q_i = Q(X_i^-, X_i^+)\}_{i=1}^r$ be a family of quasi-fuchsian manifolds with corresponding Teichmüller geodesics $\{l_i : I_i \to \mathcal{T}\}_{i=1}^r$. Suppose that the following holds:*

- *For every $i < r$, the geodesics l_i, l_{i+1} δ-fellow travel when restricted to $J_i^+ \subset I_i$ and $J_{i+1}^- \subset I_{i+1}$. For $i = r$, the geodesics l_r and ϕl_1 δ-fellow travel along $J_r^+ \subset I_r$ and $J_1^- \subset I_1$.*
- *For every $i \leq r$, the segments J_i^- and J_i^+ are respectively terminal and initial for I_i, have length $\left|J_i^-\right|$, $\left|J_i^+\right| \in [h_0, 2h_0]$ and are entirely contained in $\mathcal{T}_\eta$.*
- *The η-height of l_i is at least $G(2h_0)$ for all $i \leq r$.*

Then, for every $i \leq r$ there are product regions $U_i^-, U_i^+ \subset \mathcal{CC}(Q_i)$ and ξ-almost isometric orientation preserving diffeomorphisms $k_i : U_i^+ \to U_{i+1}^-$ for $i < r$ and $k_r : U_r^+ \to U_1^-$ such that:

- *For every $i \leq r$, the product regions U_i^- and U_i^+ have size D_1 and are disjoint with U_i^- lying below U_i^+.*
- *For $i < r$ the diffeomorphism k_i is in the homotopy class of the identity, while k_r is in the homotopy class of ϕ.*

Let $Q_i^0 \subset \mathcal{CC}(Q_i)$ be the gluing block bounded by $\partial_- U_i^-$ and $\partial_+ U_i^+$ for which U_i^-, U_i^+ are the bottom and top collars as in Lemma 3.8. Then we can form the Riemannian manifold

$$X_\phi := Q_1^0 \cup_{k_1 : U_1^+ \to U_2^-} Q_2^0 \cup \cdots \cup Q_{r-1}^0 \cup_{k_{r-1} : U_{r-1}^+ \to U_r^-} Q_r^0 / k_r : U_r^+ \to U_1^-$$

using the cut and glue construction Lemma 3.2. The metric is given by

$$\rho := \begin{cases} \rho_{Q_i} & \text{on } Q_i^0 \setminus (U_i^- \cup U_i^+) \\ (1-\theta_i)\rho_{Q_i} + \theta_i k_i^* \rho_{Q_{i+1}} & \text{on } U_i^+ = k_i(U_{i+1}^-) \text{ with } i < r \\ (1-\theta_i)\rho_{Q_r} + \theta_r k_r^* \rho_{Q_1} & \text{on } U_r^+ = k_r(U_1^-), \end{cases}$$

where $\theta_i : U_i^+ \to [0,1]$ are uniform bump functions with $||\theta_i||_{\mathcal{C}^2(U_i^+)} \leq K$ for some uniform $K = K(D_1) > 0$ as given by Lemma 3.4. The 3-manifold X_ϕ has the following properties:

(i) *Topology: X_ϕ is diffeomorphic to T_ϕ.*
(ii) *Curvature: $|1 + \sec_X| \leq c_3 K \xi$ where $c_3 > 0$ is as in Lemma 3.2.*
(iii) *The inclusions $Q_i^0 \setminus (U_i^- \cup U_i^+) \subset X$ are isometric.*
(iv) *Volume: There exists $V_1 = V_1(\eta, D_1, h_0)$ such that*

$$\left| \mathrm{vol}\,(X_\phi) - \sum_{i \leq r} \mathrm{vol}\,(Q_i) \right| \leq r V_1.$$

Proof. Proposition 3.10 follows directly from several applications of Proposition 3.5 and the cut and glue construction Lemma 3.2 combined with the control provided by Lemma 3.8 and Lemma 3.9.

As a first step, we produce product regions and diffeomorphisms. For every $i < r$, since l_i and l_{i+1} δ-fellow travel in the η-thick part $\mathcal{T}_\eta$ along segments of length at least $h_0 \geq h(\eta, \xi, \delta, D_1)$, Proposition 3.5 produces product regions $U_i^+ \subset \mathcal{CC}(Q_i)$ and $U_{i+1}^- \subset \mathcal{CC}(Q_{i+1})$ of size D_1 and a ξ-almost isometric diffeomorphism $k_i : U_i^+ \to U_{i+1}^-$ in the homotopy class of the identity between them. Analogouly, the ξ-almost isometric embedding k_r is obtained as the composition of the one provided by Proposition 3.5 for the fellow traveling of l_r and ϕl_1 and the isometric remarking $\phi Q_1 \to Q_1$ in the isotopy class of ϕ (see Figure 1).

As a second step, we control the relative position of the product regions U_i^- and U_i^+ inside $\mathcal{CC}(Q_i)$. Notice that, by Proposition 3.5, we can also assume that each product region $U_i^\pm \subset \mathcal{CC}(Q_i)$ contains, at a distance of at least D_0 from the boundary $\partial U_i^\pm$, the geodesic representative of $\alpha_i^\pm$, a curve that has moderate length for $T_i^\pm$, the midpoint of the segment $J_i^\pm$. We use this curves to check that U_i^+ and U_i^- are disjoint so that they cobound a gluing block $Q_i^0 \subset \mathcal{CC}(Q_i)$ of which they are, respectively, the top and bottom collars.

In order to check that, by Lemma 3.8, it is enough to check that the η-height of l_i is at least $G(d_i)$ where $d_i = \max\{d_{\mathcal{T}}(T_i^+, X_i^+), d_{\mathcal{T}}(T_i^-, X_i^-)\}$. Notice that X_i^- and X_i^+ are, respectively, endpoints of J_i^- and J_i^+ because these segments are, respectively, initial and terminal for l_i. Hence, $d_i \leq \max\{|J_i^-|, |J_i^+|\} \leq 2h_0$. Since G is increasing, if the η-height of l_i is at least $G(2h_0)$, then the condition of Lemma 3.8 is satisfied.

At this point, we are ready to apply the cut and glue construction simultaneously to the family of gluing blocks Q_i^0 with gluing maps k_i and form the Riemannian manifold (X_ϕ, ρ) where ρ is obtained by interpolating the metrics on the various pieces. By Lemma 3.2, the sectional curvatures of X_ϕ are controlled by $|1 + \sec_{X_\phi}| \leq c_3 K \xi$.

Next, we take care of the volume of X_ϕ. It is given by

$$\mathrm{vol}(X_\phi) = \sum_{j\le r}\Big(\mathrm{vol}(Q_i^0) - \mathrm{vol}_{\rho_{Q_i}}(U_i^-) - \mathrm{vol}_{\rho_{Q_i}}(U_i^+) + \mathrm{vol}_\rho(U_i^+)\Big).$$

Therefore

$$\left|\mathrm{vol}(X_\phi) - \sum_{j\le r}\mathrm{vol}(Q_i)\right| \le \sum_{j\le r}\left|\mathrm{vol}(Q_i) - \mathrm{vol}(Q_i^0)\right| + \sum_{j\le r}\left|\mathrm{vol}_{\rho_{Q_i}}(U_i^-) - \mathrm{vol}_{\rho_{Q_i}}(U_i^+) + \mathrm{vol}_\rho(U_i^+)\right|.$$

By Lemma 3.9, we have $|\mathrm{vol}(Q_i) - \mathrm{vol}(Q_i^0)| \le V_0$ where $V_0 = V_0(D_1, \eta, 2h_0)$. The other terms can be estimated as follows: Since the product regions $U_i^\pm$ have size D_1, their volume is bounded by $\mathrm{vol}_{\rho_{Q_i}}(U_i^\pm) \le V_0'$ where $V_0' = V_0'(2D_1)$ is the volume of a ball of radius $2D_1$ in $\mathbb{H}^3$. By Lemma 3.2, we have $\mathrm{vol}_\rho(U_i^+) \le (1+\xi)^{3/2}\mathrm{vol}_{\rho_{Q_i}}(U_i^+) \le 2V_0'$. Thus $|\mathrm{vol}_{\rho_{Q_i}}(U_i^-) - \mathrm{vol}_{\rho_{Q_i}}(U_i^+) + \mathrm{vol}_\rho(U_i^+)| \le 3V_0'$. The volume bound follows by setting $V_1(D_1, \eta, h_0) := \max\{V_0, 3V_0'\}/2$.

Lastly, we check that X_ϕ is diffeomorphic to the mapping torus T_ϕ. This follows from the fact that the gluing blocks Q_i^0 are diffeomorphic to $\Sigma\times[0,1]$ (by Lemma 3.1) and that the gluing maps k_i are in the homotopy class of the identity for $i < r$ while k_r is in the homotopy class of ϕ. $\square$

We remark that, by a celebrated theorem of Thurston [45], if ϕ is a *pseudo-Anosov* mapping class, then the mapping torus T_ϕ admits a hyperbolic metric. A pseudo-Anosov element ϕ is one that acts as a hyperbolic isometry of Teichmüller space: It preserves a unique Teichmüller geodesic $l : \mathbb{R} \to \mathcal{T}$ on which it acts by translations $\phi l(t) = l(t + L(\phi))$. The quantity $L(\phi) > 0$ is called the *translation length* of ϕ (see Chapter 13 of [17]).

3.5. **Comparing the volume.** The second fundamental ingredient is a volume comparison result. If we have two Riemannian metrics ρ_0 and ρ on the same 3-manifold M we can compare their volumes using the method of *natural maps* introduced by Besson, Courtois and Gallot. We mainly refer to their work [4] as we use some consequences of it. Given a map $f : N \to M$ between Riemannian manifolds satisfying certain curvature conditions, the method produces families of natural maps $F : N \to M$ homotopic to f and with Jacobian bounded in terms of the *volume entropies* of the manifolds. We need the following result:

Theorem 3.11 (Besson-Courtois-Gallot [4]). *Let (M, ρ) and (M_0, ρ_0) be closed orientable Riemannian 3-manifolds such that there exists:*

- *A lower bound for the Ricci curvature of the source* $\mathrm{Ric}_\rho \ge -2\rho$.
- *A uniform bound for the sectional curvatures of the target $-k \le \sec_{\rho_0} \le -1$ for some $k \ge 1$.*

Then for every continuous map $f : M \longrightarrow M_0$ we have

$$\mathrm{vol}\,(M) \geq |\deg(f)|\,\mathrm{vol}\,(M_0)\,.$$

We now describe some applications.

The first one is to the models constructed in Proposition 3.10:

Corollary 3.12. *Let $\xi \in (0,1)$ be small enough. If ϕ is a pseudo-Anosov mapping class and X_ϕ is as in Proposition 3.10, then*

$$|\mathrm{vol}\,(T_\phi) - \mathrm{vol}\,(X_\phi)| \leq 3c_3K\xi\mathrm{vol}\,(X_\phi)\,.$$

Proof. The mapping torus $T_\phi = X_\phi$ of ϕ admits a purely hyperbolic Riemannian metric (T_ϕ, ρ_0) and the metric (X_ϕ, ρ) with sectional curvature $\sec_\rho \in (-1 - c_3K\xi, -1 + c_3K\xi)$. We apply Theorem 3.11 to the identity map in both directions after suitably rescaling the metric on X_ϕ so that it fulfills the Ricci and sectional curvature bounds. This gives

$$(1 - c_3K\xi)^{3/2}\mathrm{vol}\,(X_\phi) \leq \mathrm{vol}\,(T_\phi) \leq (1 + c_3K\xi)^{3/2}\mathrm{vol}\,(X_\phi)\,.$$

Since $(1 \pm x)^{3/2} = 1 \pm \frac{3}{2}x + o(x)$ for x small, the conclusion follows. □

The second application is a construction of a very peculiar model of a mapping torus T_ϕ of a pseudo-Anosov diffeomorphism ϕ. Recall that ϕ acts on its axis by translating points by $L(\phi)$.

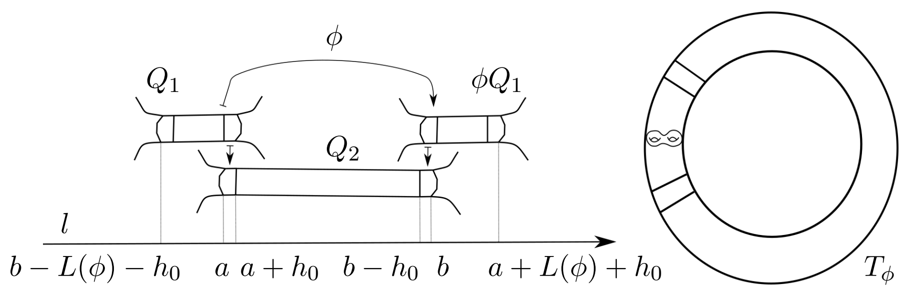


Figure 2. Model for a mapping torus.

Corollary 3.13. *Fix $\eta > 0$ and $\xi \in (0,1)$. There exists $h = h(\xi, \eta) > 0$ such that the following holds: Let ϕ be a pseudo-Anosov with axis $l : \mathbb{R} \to \mathcal{T}$. Suppose that there are $a, b \in \mathbb{R}$ such that $a < a + h < b - h < b < a + L(\phi)$ and $l([a, a+h]), l([b-h, b]) \subset \mathcal{T}_\eta$. Then*

$$|\mathrm{vol}\,(T_\phi) - \mathrm{vol}\,(Q(l(a), l(b))| \leq (1 + \xi)\kappa L(\phi) - \kappa(b - a) + \mathrm{const}$$

where $\kappa > 0$ is as in Proposition 2.2 and const *only depends on η, h, D_1.*

Proof. Let $h_0(\eta, \xi/3c_3K, 0)$ be as in Proposition 3.10. If $h \geq \max\{h_0, G(2h_0)\}$, then the quasi-fuchsian manifolds (see Figure 2)

$$\{Q_1 = Q(l(b - L(\phi) - h_0), l(a + h_0)), Q_2 = Q(l(a), l(b))\}$$

satisfy the assumption of Proposition 3.10, with respect to the mapping class ϕ, gluing parameters η, $\xi/3c_3K$ and $\delta = 0$ and where the intervals are $J_1^- = [b - L(\phi) - h_0, b - L(\phi)]$, $J_1^+ = J_2^- = [a, a + h_0]$ and $J_2^+ = [b - h_0, b]$. In particular, we can glue Q_1 and Q_2 to form the Riemannian manifold X_ϕ.

We use the model X_ϕ to compare the volumes of T_ϕ and Q_2: We have

$$|\mathrm{vol}\,(T_\phi) - \mathrm{vol}\,(Q_2)| \leq |\mathrm{vol}\,(T_\phi) - \mathrm{vol}\,(X_\phi)| + |\mathrm{vol}\,(X_\phi) - \mathrm{vol}\,(Q_2)|\,.$$

By Corollary 3.12, the first term is bounded by

$$|\mathrm{vol}\,(T_\phi) - \mathrm{vol}\,(X_\phi)| \leq 3c_3K\frac{\xi}{3c_3K}\mathrm{vol}\,(X_\phi) = \xi\mathrm{vol}\,(X_\phi)$$

and, by Proposition 3.10, we can bound $\mathrm{vol}\,(X_\phi)$ by

$$\mathrm{vol}\,(X_\phi) \leq \mathrm{vol}\,(Q_1) + \mathrm{vol}\,(Q_2) + 2V_1.$$

Proposition 3.10 also implies that the second term is bounded by

$$|\mathrm{vol}\,(X_\phi) - \mathrm{vol}\,(Q_2)| \leq \mathrm{vol}\,(Q_1) + 2V_1.$$

Putting together the previous inequalities we get

$$|\mathrm{vol}\,(T_\phi) - \mathrm{vol}\,(Q_2)| \leq \xi\mathrm{vol}\,(Q_2) + (1+\xi)\mathrm{vol}\,(Q_1) + (1+\xi)2V_1.$$

The conclusion now follows from Proposition 2.2 which gives

$$\begin{aligned}\mathrm{vol}\,(Q_1) &\leq \kappa(L(\phi) - b + a + 2h_0) + \kappa,\\ \mathrm{vol}\,(Q_2) &\leq \kappa(b-a) + \kappa.\end{aligned}$$

□

Using this estimates we recover the following well-known result (see for example [10], [24]):

Proposition 3. *Let ϕ be a pseudo-Anosov mapping class. Then for every $o \in \mathcal{T}$ we have*

$$\lim \frac{\mathrm{vol}\,(Q(o, \phi^n o))}{n} = \mathrm{vol}\,(T_\phi)\,.$$

Proof. There exists $\eta_\phi > 0$ such that $l_\phi : \mathbb{R} \to \mathcal{T}$, the Teichmüller axis of ϕ, lies in $\mathcal{T}_{\eta_\phi}$. Fix $\xi > 0$ small enough and consider $h = h(\eta_\phi, \xi)$ as given by Corollary 3.13. For n large enough $a_n = 0$ and $b_n = nL(\phi)$ fulfill the assumption of Corollary 3.13 with respect to ϕ^n which has $L(\phi^n) = nL(\phi)$. Hence, for all large n,

$$\begin{aligned}|\mathrm{vol}\,(Q(l_\phi(a_n), l_\phi(b_n))) - \mathrm{vol}\,(T_{\phi^n})| &\leq (1+\xi)\kappa L(\phi^n) - \kappa(b_n - a_n) + \mathrm{const}\\ &= \xi\kappa n L(\phi) + \mathrm{const}.\end{aligned}$$

Observe that, as T_{ϕ^n} is a degree n covering of T_ϕ, we have $\mathrm{vol}\,(T_{\phi^n}) = n\mathrm{vol}\,(T_\phi)$. Also, observe that $l_\phi(b_n) = l_\phi(nL(\phi)) = \phi^n l_\phi(0)$. Denote $l_\phi(0)$ by o_1. Dividing by $n\mathrm{vol}\,(T_\phi)$ and passing to the limit we get

$$1 - \xi\kappa L(\phi) \leq \liminf \frac{\mathrm{vol}\,(Q(o_1, \phi^n o_1))}{n\mathrm{vol}\,(T_\phi)} \leq \limsup \frac{\mathrm{vol}\,(Q(o_1, \phi^n o_1))}{n\mathrm{vol}\,(T_\phi)} \leq 1 + \xi\kappa L(\phi).$$

As ξ is arbitrary, the claim for o_1 follows. For a general o, it suffices to notice that, by Proposition 2.2, the difference $|\mathrm{vol}\,(Q(o,\phi^n o)) - \mathrm{vol}\,(Q(o_1,\phi^n o_1))|$ is uniformly bounded by $\kappa(d_{\mathcal{T}}(o,o_1)+d_{\mathcal{T}}(\phi^n o,\phi^n o_1))+\kappa = 2\kappa d_{\mathcal{T}}(o,o_1)+\kappa$. $\square$

We remark that the results mentioned above [10], [24] prove something stronger, that is $|2n\mathrm{vol}\,(T_\phi) - \mathrm{vol}\,(Q(\phi^{-n}o,\phi^n o))| = O(1)$.

4. Random Walks

We start talking about random walks on the mapping class group. We set up terminology, notations and first observations. The goal of the section is to introduce the third and last major technical tool of the paper which is a recurrence property (Theorem 4.3).

4.1. **Random walks on the mapping class group.**

We will work in the following generalities:

> **Standing assumption**. Let $S \subset \mathrm{Mod}\,(\Sigma)$ be a finite symmetric set S generating the group $G = \langle S\rangle$. Let μ be a probability measure which is *symmetric*, that is $\mu(s) = \mu(s^{-1})$, and whose support equals S. We only consider random walks driven by probability measures arising this way with $G = \mathrm{Mod}(\Sigma)$.

Let us start with the most basic definition:

Definition (Random Walk). A *random walk on G driven by μ* is given by the following data: Let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence of random variables with values in S which are independent and have the same distribution μ. The *n-th step of the random walk* is the random variable $\omega_n := s_1 \dots s_n$. The random walk is the process $\omega := (\omega_n)_{n\in\mathbb{N}}$.

> **Notation**. We will always denote by $s = (s_n)_{n\in\mathbb{N}}$ the sequence of labels and by $\omega = (\omega_n := s_1 \dots s_n)_{n\in\mathbb{N}}$ the path traced by the sequence of labels.

We denote by $\mathbb{P}_n$ the distribution of the n-th step of the random walk coincides with the n-th fold convolution of μ with itself.

We denote by $\mathbb{P}$ the distribution of $(\omega_n)_{n\in\mathbb{N}}$. This can be described as the measure on $\Omega^+ := G^{\mathbb{N}}$ (endowed with the σ-algebra $\mathcal{E}^+$ of cylinder sets) given by the push-forward $\mathbb{P} := T^+_*\mu^{\mathbb{N}}$ of the product measure $\mu^{\mathbb{N}}$ under the following measurable transformation:

$$T^+ : \Omega^+ \to \Omega^+ \quad \text{defined by} \quad T^+(s) = \omega.$$

Notice that $\mathbb{P}_n = (\pi_n)_*\mathbb{P}$, where $\pi_n : \Omega^+ \to G$ is the projection to the n-th factor. We call $(\Omega^+, \mathcal{E}^+, \mathbb{P})$ the *space of unilateral sample paths*.

Let $\mathcal{P}$ be a property of mapping classes $f \in \mathrm{Mod}\,(\Sigma)$. We call it *typical* if it is very likely that a random mapping class has it, that is

$$\mathbb{P}_n\left[f \in \mathrm{Mod}\,(\Sigma) \mid f \text{ has } \mathcal{P}\right] \overset{n\to\infty}{\longrightarrow} 1.$$

The starting point of our discussion are two results by Maher [27], [28] that ensure that the property "X_f is hyperbolic" is typical and hence it makes sense to consider the hyperbolic volume of X_f.

Moreover, as the convergence $\mathbb{P}_n\left[f \mid X_f \text{ is hyperbolic}\right] \to 1$ happens exponentially fast (see for example Maher-Tiozzo [30], Lubotzky-Maher-Wu [26], Maher-Schleimer [29]), we also get that for $\mathbb{P}$-almost every $\omega = (\omega_n)_{n\in\mathbb{N}}$ there exists n_ω such that X_{ω_n} is hyperbolic for every $n \geq n_\omega$.

Later it will be convenient for us to work with *bilateral sample paths* instead of unilateral ones. We recall the relevant definitions and properties:

Definition (Bilateral Paths). The space of bilateral sample paths is

$$\Omega := \{(s_j)_{j\in\mathbb{Z}} \in G^{\mathbb{Z}} \mid s_0 = 1\}$$

endowed with the σ-algebra $\mathcal{E}$ generated by cylinder sets and the probability $P := T_*\mu^{\mathbb{Z}}$ where $T : G^{\mathbb{Z}} \to \Omega$ is the invertible measurable transformation defined by extending T^+ as follows

$$T(s)_j = \omega_j = \begin{cases} s_1 \dots s_j & \text{if } j > 0, \\ 1 & \text{if } j = 0, \\ s_0^{-1} \dots s_{j+1}^{-1} & \text{if } j < 0. \end{cases}$$

For symmetric probability measures, $(\Omega, \mathcal{E}, P)$ is canonically identified as a measure space with $(\Omega^+, \mathcal{E}^+, \mathbb{P}) \otimes (\Omega^+, \mathcal{E}^+, \mathbb{P})$, via the map

$$(s_j)_{j\in\mathbb{Z}} \in \Omega \to ((s_{-j})_{j\in\mathbb{N}}, (s_j)_{j\in\mathbb{N}}) \in G^{\mathbb{N}} \times G^{\mathbb{N}}.$$

Notice that, if $\mathcal{P}$ is a property only depending on the forward unilateral paths $(\omega_n)_{n\in\mathbb{N}}$, then $\mathcal{P}$ holds for $\mathbb{P}$-almost every $(\omega_n)_{n\in\mathbb{N}} \in \Omega^+$ if and only if it holds for P-almost every $(\omega_n)_{n\in\mathbb{Z}} \in \Omega$.

Definition (Shift Operator). On the space $G^{\mathbb{Z}}$ there is a natural *shift operator* $U : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ defined by

$$\left(U\left(s_i\right)_{i\in\mathbb{Z}}\right)_j = s_{j+1}.$$

It is a standard computation to check that U preserves $\mu^{\mathbb{Z}}$ and that $(G^{\mathbb{Z}}, \mu^{\mathbb{Z}}, U)$ is mixing and hence *ergodic*. The same properties hold for (Ω, P, σ) where $\sigma := TUT^{-1}$ as the two systems are conjugate via T.

If $\omega = T(s) \in \Omega$ is the bilateral path traced by a random walk, then we can write the shifted forward unilateral path as $(\sigma^i\omega)_j = \omega_i^{-1}\omega_{i+j}$.

4.2. **Linear drift and sublinear tracking.** Consider the action on Teichmüller space $G \curvearrowright \mathcal{T}$ and fix a basepoint $o \in \mathcal{T}$. Every random walk $\omega = (\omega_n)_{n\in\mathbb{N}} \in \Omega^+$ traces an orbit $\{\omega_n o\}_{n\in\mathbb{N}} \subset \mathcal{T}$.

It follows from the triangle inequality that the random variables $d_{\mathcal{T}}(o, \omega_n o)$ are subadditive with respect to the shift map σ. By Kingman's subadditive ergodic theorem and ergodicity of (Ω, P, σ), there exists a constant $L_{\mathcal{T}} \geq 0$, called the *drift* of the random walk on Teichmüller space, such that for $\mathbb{P}$-almost every sample path $\omega \in \Omega^+$ we have

$$\frac{d_{\mathcal{T}}(o, \omega_n o)}{n} \xrightarrow{n\to\infty} L_{\mathcal{T}}.$$

It is natural to ask whether the orbit $\{\omega_n o\}_{n\in\mathbb{N}}$ converges to some point on the Thurston compactification of Teichmüller space $\mathcal{PML}$. This property was first established by Kaimanovich-Masur [22].

THEOREM 4.1 (Kaimanovich-Masur [22]). *We have $L_{\mathcal{T}} > 0$. For $\mathbb{P}$-almost every sample path $\omega = (\omega_n)_{n\in\mathbb{N}} \in \Omega^+$ and for every basepoint $o \in \mathcal{T}$, the sequence $\{\omega_n o\}_{n\in\mathbb{N}}$ converges to a point $\mathrm{bnd}(\omega) \in \mathcal{PML}$ which is independent of $o \in \mathcal{T}$. The map $\mathrm{bnd} : \Omega^+ \to \mathcal{PML}$ is Borel measurable. Moreover, $\mathbb{P}$-almost surely, the point $\mathrm{bnd}(\omega)$ is uniquely ergodic, minimal and filling.*

Moreover, Tiozzo [47] showed that the orbit $\{\omega_n o\}_{n\in\mathbb{N}}$ can also be *tracked* by a Teichmüller ray in the following sense:

THEOREM 4.2 (Tiozzo [47]). *For $\mathbb{P}$-almost every sample path $\omega = (\omega_n)_{n\in\mathbb{N}} \in \Omega^+$ and for every basepoint $o \in \mathcal{T}$, there exists a unit speed Teichmüller ray $\tau_\omega : [0, +\infty)$ starting at $\tau_\omega(0) = o$ and ending at $\tau_\omega(\infty) = \mathrm{bnd}(\omega)$ such that*

$$\lim_{n\to\infty} \frac{d_{\mathcal{T}}(\omega_n o, \tau(L_{\mathcal{T}} n))}{n} = 0.$$

The ray τ_ω is called the *tracking ray* of ω.

4.3. **Recurrence.** Now we can present our last fundamental ingredient which is the following recurrence property:

THEOREM 4.3 (Baik-Gekhtman-Hamenstädt, Propositions 6.9 and 6.11 of [1]). *Let $o \in \mathcal{T}$ be a basepoint. We have:*

- Recurrence*: For every $\eta > 0$ sufficiently small, for every $0 < a < b$ and $h > 0$, for $\mathbb{P}$-almost every ω with tracking ray τ_ω there exists $N = N(\omega) > 0$ such that for every $n \geq N$ the segment $\tau_\omega[an, bn]$ has a connected subsegment of length h entirely contained in $\mathcal{T}_\eta$.*
- Fellow-Traveling*: There exists $\delta > 0$ such that for every $\epsilon > 0$ and for $\mathbb{P}$-almost every sample path ω there exists $N = N(\omega) > 0$ such that for every $n \geq N$, the element ω_n is pseudo-Anosov with translation length $L(\omega_n) \in [(1-\epsilon)L_{\mathcal{T}} n, (1+\epsilon)L_{\mathcal{T}} n]$. Its axis l_n δ-fellow-travels the tracking ray τ_ω on $[\epsilon L_{\mathcal{T}} n, (1-\epsilon)L_{\mathcal{T}} n]$, i.e. for every $t \in [\epsilon L_{\mathcal{T}} n, (1-\epsilon)L_{\mathcal{T}} n]$ we have $d_{\mathcal{T}}(\tau_\omega(t), l_n) < \delta$.*

For the convergence $L(\omega_n)/n \to L_{\mathcal{T}}$ see also Dahmani-Horbez [15].

4.4. **A larger class of random walks.** As stated at the beginning of the section, in this paper we only work with a symmetric probability measures μ with finite support S that generates the full mapping class group $G = \mathrm{Mod}(\Sigma)$. This allows us to keep the statements uniform and to avoid distinguishing between different families of random 3-manifolds.

However, at the price of making a distinction between mapping tori, quasi-fuchsian manifolds and Heegaard splittings, the assumptions on μ can be considerably relaxed and still obtain the convergence results in Theorems 1 and 2. We briefly describe, without details, two larger classes of random walks to which our results can be extended.

For mapping tori and quasi-fuchsian manifolds it is enough that μ is symmetric with a finite support S that generates a subgroup G containing two pseudo-Anosov elements that act as independent loxodromics on the *curve graph* (see [30] for the definitions). All the theorems in this section hold in these generalities.

For Heegaard splittings, we further require that the two pseudo-Anosov elements also act as independent loxodromics on the *handlebody graph* (see [29] for a definition). Crucially, the condition implies, by work of Maher-Schleimer [29] and Maher-Tiozzo [30], that random walks on G have a *positive drift* on the handlebody graph. This ensures that a random Heegaard splitting is hyperbolic and plays a role also in the construction of the model metric from [20] used in the next section.

With these caveats, the proofs can be extended by following word-by-word the same lines, no change is needed.

5. A Law of Large Numbers for the Volume

We are ready to prove the law of large numbers for the volumes of random 3-manifolds.

Theorem 1. *$\mathbb{P}$-almost surely the following limit exists*

$$\lim_{n\to\infty}\frac{\mathrm{vol}\,(X_{\omega_n})}{n}=v.$$

The family of 3-manifold $\{X_{\omega_n}\}_{n\in\mathbb{N}}$ can denote either the mapping tori or the Heegaard splittings defined by ω_n.

We will deduce it from the following analogue concerning quasi-fuchsian manifolds. The idea is that, according to the geometric models, the volume of a random 3-manifold is always captured by a quasi-fucshian manifold.

Theorem 2. *For every $o\in\mathcal{T}$ and for $\mathbb{P}$-almost every $\omega\in\Omega^+$ the following limit exists:*

$$\lim_{n\to\infty}\frac{\mathrm{vol}\,(Q(o,\omega_n o))}{n}=v.$$

Let us remark again that $v=v(\mu)>0$ is the same as in Theorem 1.

5.1. Mapping tori and Heegaard splittings.

Let us assume Theorem 2 and prove the result for random 3-manifolds:

Proof of Theorem 1. Fix $\epsilon>0$. Let $\tau_\omega:[0,\infty)\to\mathcal{T}$ be the ray tracking ω.

Mapping tori. We use a model for T_{ω_n} coming from Corollary 3.13 (see also Figure 2): By Theorem 4.3, if n is large enough, we can find on τ_ω

points x_n, w_n with $x_n < x_n + h < w_n - h < w_n < x_n + L(\omega_n)$ such that the intervals $[x_n, x_n + h]$ and $[w_n - h, w_n]$ satisfy the following:

- They are contained in $[\epsilon L_{\mathcal{T}} n, 2\epsilon L_{\mathcal{T}} n]$ and $[(1-2\epsilon)L_{\mathcal{T}} n, (1-\epsilon)L_{\mathcal{T}} n]$ respectively and their images are η-thick.
- The mapping class ω_n is pseudo-Anosov and its translation length satisfies $(1-\epsilon)L_{\mathcal{T}} n \le L(\omega_n) \le (1+\epsilon)L_{\mathcal{T}} n$.
- The restriction of τ_ω to $[x_n, w_n]$ δ-fellow travels the Teichmüller axis $l_n : \mathbb{R} \to \mathcal{T}$ of ω_n along the subsegment $l_n[a_n, b_n]$.

In particular, on l_n there are segments of the form $[a_n, a_n+h']$ and $[b_n-h'', b_n]$ with $h', h'' \ge h - 2\delta$ that δ-fellow travel $[x_n, x_n + h]$ and $[w_n - h, w_n]$. Such segments are contained in a uniformly thick part of Teichmüller space only depending on η and δ. Thus, if h is sufficiently large, they satisfy the assumptions of Corollary 3.13 with parameter $\xi \in (0, \epsilon)$. Therefore, we get

Lemma 5.1. *For $\mathbb{P}$-almost every ω and every large enough $n \ge n_\omega$ we have*

$$|\mathrm{vol}\,(Q(\tau_\omega(x_n), \tau_\omega(w_n))) - \mathrm{vol}\,(T_{\omega_n})| \le \epsilon n.$$

and

$$|\mathrm{vol}\,(Q_{\omega_n}) - \mathrm{vol}\,(Q(\tau_\omega(x_n), \tau_\omega(w_n)))| \le \epsilon n.$$

Proof of Lemma 5.1. By Proposition 2.2, we have

$$|\mathrm{vol}(Q(l_n(a_n), l_n(b_n))) - \mathrm{vol}(Q(\tau_\omega(x_n), \tau_\omega(w_n)))| \le \mathrm{const}.$$

By Corollary 3.13 we have

$$\begin{aligned}
&|\mathrm{vol}\,(T_{\omega_n}) - \mathrm{vol}\,(Q(l_n(a_n), l_n(b_n)))| \\
&\le (1+\xi)\kappa L(\omega_n) - \kappa(b_n - a_n) + \mathrm{const} \\
&\le (1+\xi)\kappa(1+\epsilon)L_{\mathcal{T}} n - \kappa(1-4\epsilon)L_{\mathcal{T}} n + \mathrm{const} \\
&= \kappa(\xi + 4\epsilon)L_{\mathcal{T}} n + \mathrm{const}.
\end{aligned}$$

This proves the first part of the statement up to adjusting ϵ and ξ and taking n large enough. As for the second part, we have: By Proposition 2.2

$$\begin{aligned}
&|\mathrm{vol}\,(Q(o, \omega_n o)) - \mathrm{vol}\,(Q(\tau_\omega(x_n), \tau_\omega(w_n)))| \\
&\le \kappa(d_{\mathcal{T}}(o, \tau_\omega(x_n)) + d_{\mathcal{T}}(\tau_\omega(w_n), \omega_n o)) + \kappa.
\end{aligned}$$

By our choice of x_n we have that $d_{\mathcal{T}}(o, \tau_\omega(x_n)) \le 2\epsilon L_{\mathcal{T}} n$. Similarly, by our choice of w_n and Tiozzo's sublinear tracking (Theorem 4.2), if n is large enough we also have

$$d_{\mathcal{T}}(\omega_n o, l_n(w_n)) \le d_{\mathcal{T}}(\omega_n o, \tau_\omega(L_{\mathcal{T}} n)) + d_{\mathcal{T}}(\tau_\omega(w_n), \tau_\omega(L_{\mathcal{T}} n)) \le \epsilon n + 2\epsilon L_{\mathcal{T}} n.$$

Putting together the previous inequalities and adjusting ϵ and n concludes the proof. $\square$

Lemma 5.1 and Theorem 2 imply that $|\mathrm{vol}\,(T_{\omega_n}) - nv| = o(n)$ which concludes the proof for mapping tori.

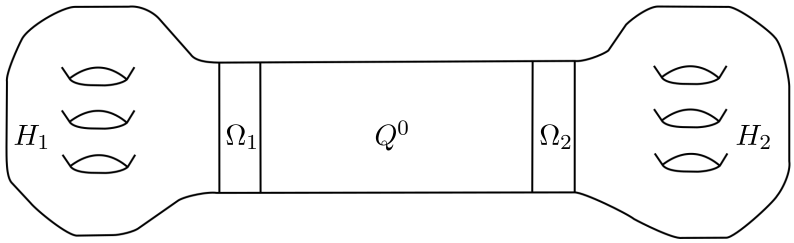


FIGURE 3. Model for a random Heegaard splitting.

Heegaard splittings. The argument is completely analogous to the previous one, but the model is different. We use the one constructed in [20], in particular Proposition 7.1. For convenience of the reader we give a brief description of it: Recall that $\epsilon > 0$ is fixed. A random Heegaard splitting M_{ω_n} admits a negatively curved Riemannian metric ρ with the following properties (see Figure 3): It is purely hyperbolic outside two regions $\Omega := \Omega_1 \sqcup \Omega_2$ which have uniformly bounded diameter and where the sectional curvatures lie in the interval $(-1-\epsilon, -1+\epsilon)$. The complement $M_{\omega_n} - \Omega$ decomposes into three connected pieces $H_1 \sqcup Q^0 \sqcup H_2$. The pieces H_1, H_2 are homeomorphic to handlebodies and have small volume $\mathrm{vol}(H_1 \sqcup H_2 \sqcup \Omega) \leq \epsilon n$. The middle piece Q^0 embeds isometrically in the convex core of $Q(o, \omega_n o)$, moreover $\mathrm{vol}(Q(o,\omega_n o)) - \mathrm{vol}(Q^0) \leq \epsilon n$. Hence we can apply again Theorem 3.11 and Theorem 2. □

We now proceed with the proof of Theorem 2.

5.2. **Strategy overview.** Denote by Q_ϕ the manifold $Q(o, \phi o)$.

We want to show that for $\mathbb{P}$-almost every $\omega \in \Omega^+$ the sequence $\mathrm{vol}(Q_{\omega_n})/n$ converges. Recall that this is equivalent to prove that the same happens for P-almost every $\omega \in \Omega$. Suppose this is not the case. Then there exists a set Ω_{bad} with positive measure $P[\Omega_{\mathrm{bad}}] > 0$ on which

$$\limsup_{n\to\infty} \frac{\mathrm{vol}(Q_{\omega_n})}{n} - \liminf_{n\to\infty} \frac{\mathrm{vol}(Q_{\omega_n})}{n} > 0.$$

We can as well assume that there is a small $\epsilon_0 > 0$ and a set $\Omega_{\mathrm{bad}}^{\epsilon_0}$ with positive measure $\zeta_0 := P\left[\Omega_{\mathrm{bad}}^{\epsilon_0}\right] > 0$ on which the difference is at least $\epsilon_0 > 0$. Hence, in order to get a contradiction, it is enough to prove that for every $\epsilon, \zeta > 0$ there exists a set $\Omega_{\epsilon,\zeta}$ with measure $P[\Omega_{\epsilon,\zeta}] \geq 1 - \zeta$ on which the difference between limsup and liminf is smaller than ϵ.

First we observe that we can exploit a *neighbour approximation property* of the volumes (Lemma 5.3). This allows a convenient technical reduction: We can make the random walk *faster* and still keep under control the asymptotic behaviour (Lemma 5.4). The faster we make the random walk the more properties we can prescribe, a feature that will be important in Proposition 5.5. The central step of the proof consists in finding a set on which the variables $\mathrm{vol}(Q_{\omega_{nN}})$ and the *ergodic sum* $\sum_{j<n} \mathrm{vol}(Q_{\sigma^{jN}(\omega)_N})$ are comparable (Proposition 5.5). Finally, we use the *ergodic theorem* to conclude the proof.

5.3. **A faster random walk.** For every $N \in \mathbb{N}$ we can replace the random walk ω with $(\omega_{jN})_{j\in\mathbb{N}}$ and the shift map σ with σ^N. The dynamical system (Ω, P, σ^N) is still ergodic. As we wish to apply the ergodic theorem, we discuss the integrability condition of the volume function and the relations between the asymptotics of the faster random walk and the original one. Recall that S, the finite symmetric support of μ, generates $G = \mathrm{Mod}(\Sigma)$.

LEMMA 5.2. *There exists $C > 0$ such that for every $\phi \in G$ we have* $\mathrm{vol}(Q_\phi) \leq C|\phi|_S + C$ *where $|\phi|_S$ is the word length in the generating set S.*

Proof. Let $\phi = s_1 \dots s_n$ with $s_i \in S$. By Proposition 2.2 we have $\mathrm{vol}(Q_\phi) \leq \kappa d_{\mathcal{T}}(o, \phi o) + \kappa$. By the triangle inequality $d_{\mathcal{T}}(o, s_1 \dots s_n o) \leq \sum_{j<n} d_{\mathcal{T}}(o, s_j o) \leq \max_{s\in S} \{d_{\mathcal{T}}(o, so)\} n$. □

In particular, for any fixed $n \in \mathbb{N}$, the function $\mathrm{vol}(Q_{\omega_n})$ is integrable on $(\Omega, \mathcal{E}, P)$ and we can apply the Birkhoff ergodic theorem. Moreover, we have the following neighbour approximation property.

LEMMA 5.3. *For P-almost every sample path $\omega \in \Omega$, for every n, m we have*

$$\left|\mathrm{vol}\left(Q_{\omega_{n+m}}\right) - \mathrm{vol}\left(Q_{\omega_n}\right)\right| \leq Cm + C$$

where $C > 0$ is the same as in Lemma 5.2.

Proof. By Proposition 2.2 $\left|\mathrm{vol}\left(Q_{\omega_{n+m}}\right) - \mathrm{vol}\left(Q_{\omega_n}\right)\right| \leq \kappa d_{\mathcal{T}}(\omega_n o, \omega_{n+m} o) + \kappa$. From the triangle inequality $d_{\mathcal{T}}(\omega_n o, \omega_{n+m} o) \leq \max_{s\in S} \{d_{\mathcal{T}}(o, so)\} m$. □

The next completely elementary lemma illustrates why the neighbour approximation property allows to speed up the random walk without loosing control on the asymptotic behaviour.

LEMMA 5.4. *Consider a sequence $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and an integer $N \in \mathbb{N}$. Suppose that, for some $C > 0$, the sequence satisfies $|a_{n+m} - a_n| \leq Cm + C$ for every n, m. Assume that $A := \limsup_{j\to\infty} \frac{a_{jN}}{jN}$ and $a := \liminf_{j\to\infty} \frac{a_{jN}}{jN}$ are finite. Then*

$$a \leq \liminf_{n\to\infty} \frac{a_n}{n} \leq \limsup_{n\to\infty} \frac{a_n}{n} \leq A.$$

5.4. **Comparison with ergodic sums.** The following is our main estimate

PROPOSITION 5.5. *Fix $\epsilon, \zeta > 0$. There exist $N = N(\epsilon, \zeta) > 0$ and a set $\Omega_{\epsilon,\zeta,N}$ with $P[\Omega_{\epsilon,\zeta,N}] \geq 1 - \zeta$ such that for every $\omega \in \Omega_{\epsilon,\zeta,N}$ and $n \in \mathbb{N}$ large* *enough we have*

$$\left|\mathrm{vol}\left(Q_{\omega_{nN}}\right)-\sum_{0\le j<n}\mathrm{vol}\left(Q_{(\sigma^{jN}\omega)_N}\right)\right|\le \mathrm{const}\cdot\epsilon nN$$

for some uniform const >0 *only depending on* Σ *and* μ*.*

We will show that, for a suitably chosen N, both families $\{Q_{\omega_{nN}}\}$ and $\{Q_{(\sigma^{jN}\omega)_N}\}_{j<n}$ can be *refined* to construct *models*, via Proposition 3.10, for the hyperbolic mapping torus $T_{\omega_{nN}}$. The central property of the models is that they *nearly compute the volume* $\mathrm{vol}(T_{\omega_{nN}})$. This suffices to conclude.

Dependence of the constants. For convenience of the reader, before describing the arguments, we summarize the dependence of the various constants that will appear in the proof. Here is a schematic description.

A priori choices. Recall that we fixed once and for all the thickness threshold η for Teichmüller space and a fellow traveling parameter δ as they appear in Theorem 4.3. They determine a uniform size D_1 and a uniform $\mathcal{C}^2$-bound K (as in Proposition 3.10). They will also give, in Lemma 5.6, another fellow traveling parameter $\delta'=\delta'(\delta,\eta)$.

Choices that depend on ϵ. We proceed backwards: In order to get a bound of the order $O(\epsilon)$ we will need: A small error on $|1+\sec|$ in the gluing construction (as in Proposition 3.10 and Corollary 3.12). This requires a small almost isometric parameter ξ (as in Proposition 3.10). In turn, a small ξ necessitates a minimal amount of fellow traveling $h_0=h_0(\xi,\eta,\delta+\delta')$ and η-height $G(2h_0)$ (as in Proposition 3.10). Such quantities correspond to an error in the volume bound proportional to $V_1(\eta,D_1,h_0)$ (as in Proposition 3.10). This gives a restriction on the smallest $N\in\mathbb{N}$ for which $V_1/N<\epsilon$.

Some simplifications. Since the value of $L_{\mathcal{T}}>0$ is irrelevant and only complicates some formulas below by affecting the value of some constants, we are going to assume $L_{\mathcal{T}}=1$. In the course of the proof, specifically in the inequalities (1)-(13), we will get several uniform constants which depend on previous steps and whose explicit expressions are irrelevant for the argument. In order to simplify the exposition we will always denote these different constants by const >0.

Proof. Let $h>0$ be a very large height. For every N denote by $\Omega_{\epsilon,N}$ the set of paths satisfying the following properties

(1) ω_n is pseudo-Anosov and $L(\omega_n)/n\in(1-\epsilon,1+\epsilon)$ for every $n\ge N$.
(2) l_n, the axis of ω_n, δ-fellow travels $\tau_\omega[\epsilon n,(1-\epsilon)n]$ for every $n\ge N$.
(3) $\omega_n\tau_{\sigma^n(\omega)}[\epsilon n,\infty]$ δ-fellow travels $\tau_\omega[(1+\epsilon)n,\infty]$ for every $n\ge N$.
(4) $\tau_\omega[\epsilon n,2\epsilon n]$ and $\tau_\omega[(1\pm\epsilon)n,(1\pm 2\epsilon)n]$ contain η-thick subsegments of length at least h for every $n\ge N$.
(5) The conclusions of Lemma 5.1 hold for every $n\ge N$.
(6) $d_{\mathcal{T}}(o,\omega_n o)/n\in(1-\epsilon,1+\epsilon)$ for all $n\ge N$.

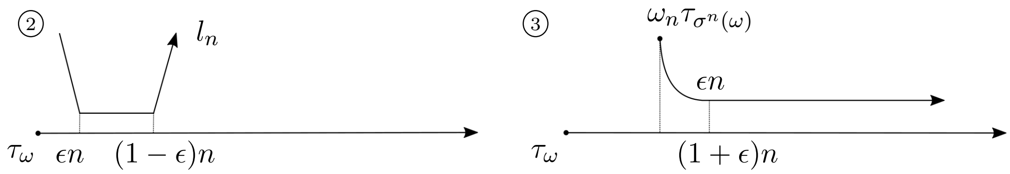


Figure 4. Properties 2 and 3.

Observe that if $N_1 \geq N_2$ then $\Omega_{\epsilon,N_2} \subseteq \Omega_{\epsilon,N_1}$, if we enlarge N the set can only get bigger. We reserve ourselves the right to determine later a suitable N. Since all the properties are satisfied asymptotically with probability one, for fixed $\epsilon, \zeta > 0$ there exists some $N(\epsilon, \zeta, h)$ such that $\Omega_{\epsilon,N}$ has measure at least $1-\zeta$. Fix N larger than this threshold and speed up the random walk, that is replace ω with $(\omega_{jN})_{j\in\mathbb{N}}$ and σ with σ^N.

By ergodicity of (Ω, P, σ^N), the orbits $\{\sigma^{jN}\omega\}_{j\in\mathbb{N}}$ will visit the set $\Omega_{\epsilon,N}$ very often, the number of hitting times being proportional to the measure of the set $\geq 1-\zeta$. We record the hitting times by subdividing the interval $[n] = \{0, \dots, n\}$ into a disjoint union of consecutive intervals $[n] = I_1 \sqcup J_1 \sqcup \cdots \sqcup I_k \sqcup J_k$ where the I_i's contain the indices j for which $\sigma^{jN}\omega \in \Omega_{\epsilon,N}$, whereas the J_i's are the bad indices (J_k might be empty). By the ergodic theorem the total length of the bad intervals is controlled by

$$\frac{1}{n}\sum_{j<n} \mathbb{1}_{\Omega\setminus\Omega_{\epsilon,N}}(\sigma^{jN}\omega) = \frac{1}{n}\sum_{i\leq k}|J_i| \overset{n\to\infty}{\longrightarrow} P\left[\Omega\setminus\Omega_{\epsilon,N}\right] \leq \zeta.$$

Basic case. We start by proving the proposition assuming that all indices are good. We are going to define two families of quasi-fuchsian manifolds that satisfy the hypotheses of Proposition 3.10 and can be glued to form a model for $T_{\omega_{nN}}$ that nearly computes its volume.

The two families consist of:

I Quasi-fuchsian manifolds related to $Q_{\sigma^{jN}(\omega)_N}$ for every $j \in [n]$.

II A pair of quasi-fuchsian manifolds related to $Q_{\omega_{nN}}$ as in Lemma 5.1.

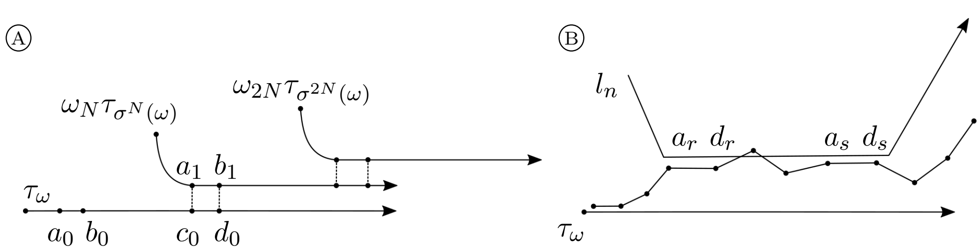


Figure 5. Basic case.

Family I. Proceed inductively. Begin with $i = 0$ and the two Teichmüller rays τ_ω and $\omega_N\tau_{\sigma^N(\omega)}$ (see Figure 5 A). By property (3), the restrictions $\omega_N\tau_{\sigma^N(\omega)}|_{[\epsilon N,\infty)}$ and $\tau_\omega|_{[(1+\epsilon)N,\infty)}$ are δ-fellow travelers. By property (4), the ray τ_ω contains four points $a_0 < b_0 < c_0 < d_0$ such that $[a_0, b_0] \subset [\epsilon N, 2\epsilon N]$ and $[c_0, d_0] \subset [(1+\epsilon)N, (1+2\epsilon)N]$, their image is η-thick and their length is at least h. The segment $[c_0, d_0]$ determines $[a_1, b_1]$ by the condition that $\omega_N \tau_{\sigma^N(\omega)}[a_1, b_1]$ δ-fellow travels $\tau_\omega[c_0, d_0]$ and $[a_1, b_1] \subset [\epsilon N, 2\epsilon N]$. As $1 \in [n]$ is good, we can go on and find $[c_1, d_1] \subset [(1+\epsilon)N, (1+2\epsilon)N]$ of length at least h and with $\tau_{\sigma^N(\omega)}$-image in $\mathcal{T}_\eta$. Inductively we determine $a_i < b_i < c_i < d_i$ for every $i \leq n$. Before going on, let us simplify a little the notation by introducing

$$A_i = \omega_{iN} \tau_{\sigma^{iN}(\omega)}(a_i), \qquad B_i = \omega_{iN} \tau_{\sigma^{iN}(\omega)}(b_i),$$
$$C_i = \omega_{iN} \tau_{\sigma^{iN}(\omega)}(c_i), \qquad D_i = \omega_{iN} \tau_{\sigma^{iN}(\omega)}(d_i).$$

We also denote the iN-th point in the orbit of o by $O_i = \omega_{iN} o$.

We associate to the index $i \leq n$ the quasi-fuchsian manifold $Q(A_i, D_i)$.

Observe that, by Proposition 2.2, the volume of $Q_{\sigma^{iN}(\omega)_N}$ is uniformly comparable with the one of $Q(A_i, D_i)$. In fact, we have

$$\begin{aligned} &\left| \mathrm{vol}\,(Q(A_i, D_i)) - \mathrm{vol}\left(Q_{\sigma^{iN}(\omega)_N}\right)\right| \\ &\leq \kappa(d_{\mathcal{T}}(O_i, A_i) + d_{\mathcal{T}}(D_i, O_{i+1})) + \kappa \leq \kappa 4\epsilon N + \mathrm{const.} \end{aligned} \tag{1}$$

The family of quasi-fuchsian manifolds $Q(A_i, D_i)$ and their Teichmüller geodesics $[A_i, D_i]$ satisfy the long fellow traveling along initial and terminal segments and the large height conditions of Proposition 3.10 with respect to the mapping class ω_{nN} except perhaps for the requirement on $\omega_{nN}[A_0, D_0]$ and $[A_{n-1}, D_{n-1}]$.

In order to make sure that this last fellow-traveling requirement is fulfilled we proceed as follows: First, we refine the collection of manifolds $\{Q(A_i, D_i)\}_{i \leq n}$ to a smaller on $\{Q(A_i, D_i)\}_{r \leq i \leq s}$ so that each $[A_i, D_i]$ in the family uniformly fellow travels l_n, the axis of ω_{nN}. Then we add a suitable initial additional quasi-fuchsian manifold with the right fellow traveling properties.

The first step of the process can be carried out using property (2) that says that $[O, O_n]$ δ-fellow travels l_n along a long central subsegment and the fact that sequences of consecutive good segments satisfy a stronger fellow traveling property, namely:

Lemma 5.6. *The segment $[A_i, D_i]$ δ'-fellow travels $[O, O_n]$ for some uniform $\delta' = \delta'(\delta, \eta) > 0$.*

Proof. Let $\mathcal{C}$ be the curve graph of Σ. Consider the coarsely well-defined shortest curve projection $\Upsilon : \mathcal{T} \to \mathcal{C}$ that associates to a marked hyperbolic surface X a shortest geodesic $\Upsilon(X)$ on it. By Masur-Minsky [34] we have the following: The curve graph $\mathcal{C}$ is hyperbolic and the projection is coarsely Lipschitz and sends Teichmüller geodesics to *unparametrized* uniform quasi geodesics. In particular, by stability of quasi geodesics, $\Upsilon[A_i, D_i]$ is uniformly Hausdorff close to the geodesic segment $[\Upsilon(A_i), \Upsilon(C_i)]$. The same holds true for $\Upsilon[O, O_n]$ and $[\Upsilon(O), \Upsilon(O_n)]$.

Since the composition of Υ with a parametrized, η-thick and sufficiently long Teichmüller geodesic is a uniform *parametrized* quasi geodesic (see [19]), we also have the following: If the δ-fellow traveling h between $[C_{i-1}, D_{i-1}]$ and $[A_i, B_i]$ is sufficiently long, then the geodesics $[\Upsilon(A_{i-1}), \Upsilon(D_{i-1})]$ and $[\Upsilon(A_i), \Upsilon(D_i)]$ uniformly fellow travel along a segment, terminal for the first and initial for the second, which is as long as we wish.

In particular this implies that, if h is large enough, then the concatenation of the geodesic segments

$$[\Upsilon(O), \Upsilon(C_0)] \cup [\Upsilon(A_1), \Upsilon(C_1)] \cup \cdots \cup [\Upsilon(A_{n-1}), \Upsilon(C_{n-1})] \cup [\Upsilon(A_n), \Upsilon(O_n)]$$

is a uniform $(1, K)$ local quasi geodesic. By the stability of uniform local quasi geodesics in hyperbolic spaces, we conclude that every segment $[\Upsilon(A_i), \Upsilon(D_i)]$ lies uniformly Hausdorff close to $[\Upsilon(O), \Upsilon(O_n)]$.

Therefore, there are intervals $[U_i, U_i'], [V_i, V_i'] \subset [O, O_n]$ for which the Υ-projections are uniformly close to the projections of $[A_i, B_i]$ and $[C_i, D_i]$. Notice that, since $[A_i, B_i]$ and $[C_i, D_i]$ are long segments and Υ is Lipschitz, we must also have that the segments $[U_i, U_i'], [V_i, V_i']$ are long.

We now show that there are points $P_i \in [U_i, U_i']$ and $Q_i \in [V_i, V_i']$ that are uniformly close to $[A_i, D_i]$ and $[C_i, D_i]$ in Teichmüller space. In order to prove this, we need the following:

Claim: For every $\eta > 0$ and $\nu > 0$ there exists $T = T(\eta, \nu) > 0$ and $\delta_1 = \delta_1(\eta, \nu) > 0$ such that if γ, γ' are two Teichmüller geodesics defined on an interval $J = [0, T]$ with $\gamma(J) \subset \mathcal{T}_\eta$ and the Haudorff distance between the images $\Upsilon\gamma$ and $\Upsilon\gamma'$ is at most ν, then $d_{\mathcal{T}}(\gamma(t), \gamma'(t)) < \delta_1$ for some $t \in J$.

The claim is consequence of the contracting properties of Teichmüller geodesics. For the sake of completeness, we now explain how to deduce it from the work of Minsky [37] and Hamenstädt [19].

Proof of the Claim. Let $\pi_{\mathcal{T}} : \mathcal{T} \to \gamma(J)$ denote the nearest point projection to the η-thick Teichmüller segment $\gamma(J)$. If $P := \pi_{\mathcal{T}}\gamma'(0)$ and $Q := \pi_T\gamma'(T)$ are sufficiently far apart, then, by [37], γ' uniformly fellow travels γ along the subsegment $[P, Q]$. We now show that this is the case with our assumptions. More precisely, we show that P and Q are uniformly close to $\gamma(0)$ and $\gamma(T)$ respectively, so that their distance is roughly T.

Let us denote the shortest curve projections of the endpoints of γ and γ' by $\alpha := \Upsilon\gamma(0), \beta := \Upsilon\gamma(T)$ and $\alpha' := \Upsilon\gamma'(0), \beta' := \Upsilon\gamma'(T)$.

By stability of quasi-geodesics, $\Upsilon\gamma$ and $\Upsilon\gamma'$ are uniformly Hausdorff close to the geodesics $[\alpha, \beta]$ and $[\alpha', \beta']$. Furthermore, by assumption, $\Upsilon\gamma'$ is uniformly Hausdorff close to $\Upsilon\gamma$. Thus $[\alpha, \beta]$ and $[\alpha', \beta']$ are uniformly Hausdorff close.

Connect $\gamma'(0)$ to $\gamma(0)$ with a Teichmüller geodesic θ and consider the projection $\Upsilon\theta$. By [37], the geodesic θ passes uniformly close to the nearest point projection P. Since Υ is coarsely Lipschitz, $\Upsilon\theta$ passes unifromly close to $\Upsilon(P)$. In particular, up to a uniform additive error, the diameter of $\Upsilon\theta$ is bounded from below by $d_{\mathcal{C}}(\Upsilon(P), \Upsilon\gamma(0))$.

By [19], the restriction of Υ to a η-thick Teichmüller geodesic, such as $[\gamma(0), P]$, is a parametrized uniform quasi-geodesic. Thus, $d_{\mathcal{C}}(\Upsilon(P), \Upsilon\gamma(0))$ is coarsely bounded from below by $d_{\mathcal{T}}(P, \gamma(0))$.

Since $\Upsilon\theta$ is a uniform unparametrized quasi-geodesic it lies in a uniform neighbourhood of the the geodesic $[\alpha, \alpha']$ connecting its endpoints. Hence, as $[\alpha, \alpha']$ has uniformly bounded length, the diamenter of $\Upsilon\theta$ is uniformly bounded. As a consequence, also $d_{\mathcal{T}}(P, \gamma(0))$ must be uniformly bounded.

Similarly, also $d_{\mathcal{T}}(Q, \gamma(T))$ is uniformly bounded. This finishes the proof of the claim. □

The claim, applied to $\gamma = [A_i, B_i], [C_i, D_i]$ and $\gamma' = [U_i, U_i'], [V_i, V_i']$ with ν determined by the fellow traveling constant δ and the Lipschitz constant of Υ, provides us points $P_i, Q_i \in [O, O_n]$ that are uniformly close to the thick subsegments $[A_i, B_i], [C_i, D_i]$. By Theorem 7.1 of [42] the segment $[P_i, Q_i]$ uniformly fellow travels $[A_i, D_i]$. □

We now use Lemma 5.6 and property (2) to refine the collection of quasi-fuchsian manifolds $\{Q(A_i, D_i)\}_{i \leq n}$ to one that can be used in Proposition 3.10 with respect to the mapping class ω_{nN} with a sufficiently small almost isometric constant ξ: By Lemma 5.6 and property (2), there is a subsegment $[r, s] \subset [n]$ of size $s - r \geq (1-\epsilon)n$, obtained by discarding an initial and a terminal subsegment of length proportional to ϵn, such that for all $r \leq i \leq s$ $[A_i, D_i]$ uniformly fellow travels l_n (see Figure 5 B).

By Lemma 5.2 and the fact that $|[n] \setminus [r, s]| \leq \epsilon n$, we have

$$\sum_{j \notin [r,s]} \mathrm{vol}\left(Q_{(\sigma^{jN}\omega)_N}\right) \leq \sum_{j \notin [r,s]} CN + C \leq \mathrm{const} \cdot \epsilon n N. \tag{2}$$

We add to the collection $\{Q(A_i, D_i)\}_{i \in [r,s]}$ an initial quasi-fuchsian manifold $Q(\omega_{nN}^{-1} C_s, B_r)$. Using Proposition 2.2 we see that

$$\mathrm{vol}\left(Q(C_s, \omega_{nN} B_r)\right) \leq \kappa d_{\mathcal{T}}(C_s, \omega_{nN} B_r) + \kappa \leq \mathrm{const} \cdot \epsilon n N. \tag{3}$$

In fact: By our choice of the interval $[r, s]$, the points B_r, C_s are uniformly close to points $l_n(t_r), l_n(t_s)$. In particular, the point $\omega_{nN} B_r$ is uniformly close to $l_n(t_r + L(\omega_{nN}))$. Thus

$$d_{\mathcal{T}}(C_s, \omega_{nN} B_r) \simeq d_{\mathcal{T}}(l_n(t_s), \omega_{nN} l_n(t_r)) = L(\omega_{nN}) - (t_s - t_r).$$

We now estimate $L(\omega_{nN}) - (t_s - t_r)$: We have $t_s - t_r \simeq d_{\mathcal{T}}(B_r, C_s)$. By the definition of B_r and C_s $d_{\mathcal{T}}(B_r, C_s) = d_{\mathcal{T}}(O_r, O_s) + O(\epsilon N)$. By property (6), $d_{\mathcal{T}}(O_r, O_s) \geq (1-\epsilon)(s-r)N$. Therefore, as $s - r \geq (1-\epsilon)n$, we get

$t_s - t_r \geq (1-\epsilon)^2 nN + O(\epsilon N)$. By property (1) we have $L(\omega_{nN}) \leq (1+\epsilon)nN$ so $L(\omega_{nN}) - (t_s - t_r) \simeq (1+\epsilon)nN - (1-\epsilon)^2 nN$ whence inequality (2).

By construction, the family $\{Q(\omega_{nN}^{-1} C_s, B_r)\} \sqcup \{Q(A_i, D_i)\}_{r \leq i \leq s}$ satisfies the gluing conditions of Proposition 3.10 provided that h is very large. As a result it can be glued to form a model manifold $X_{\omega_{nN}}$ which, by Proposition 3.10 and Corollary 3.12, can be used to obtain the following volume estimate

$$\begin{aligned} &\left|\mathrm{vol}\left(T_{\omega_{nN}}\right) - \sum_{i \in [r,s]} \mathrm{vol}\left(Q(A_i, D_i)\right) - \mathrm{vol}\left(Q(\omega_{nN}^{-1} C_s, B_r)\right)\right| \\ &\qquad \leq \left|\mathrm{vol}\left(T_{\omega_{nN}}\right) - \mathrm{vol}\left(X_{\omega_{nN}}\right)\right| + \\ &+ \left|\mathrm{vol}\left(X_{\omega_{nN}}\right) - \sum_{i \in [r,s]} \mathrm{vol}\left(Q(A_i, D_i)\right) - \mathrm{vol}\left(Q(\omega_{nN}^{-1} C_s, B_r)\right)\right| \\ &\qquad \leq nV_1 + \mathrm{const} \cdot \epsilon nN \end{aligned} \tag{4}$$

where $V_1 = V_1(\eta, h, D_1)$ is as in Proposition 3.10.

Family II. By property (5) and Lemma 5.1, we can find on τ_ω a pair of points $x_n \in [\epsilon nN, 2\epsilon nN]$ and $w_n \in [(1-2\epsilon)nN, (1-\epsilon)nN]$ which define a quasi-fuchsian manifold whose volume approximates simultaneously the volume of the mapping torus $T_{\omega_{nN}}$ and the volume of the quasi-fuchsian manifold $Q_{\omega_{nN}}$

$$\left|\mathrm{vol}\left(T_{\omega_{nN}}\right) - \mathrm{vol}\left(Q(\tau_\omega(x_n), \tau_\omega(w_n))\right)\right| \leq \mathrm{const} \cdot \epsilon nN \tag{5}$$

and

$$\left|\mathrm{vol}\left(Q_{\omega_{nN}}\right) - \mathrm{vol}\left(Q(\tau_\omega(x_n), \tau_\omega(w_n))\right)\right| \leq \mathrm{const} \cdot \epsilon nN. \tag{6}$$

Notice that inequalities (5) and (6) hold also in the presence of bad intervals as we only used property (5). We will use them in the general case as well.

Putting together the previous estimates (1)-(5) we get

$$\left|\mathrm{vol}\left(Q(\tau_\omega(x_n), \tau_\omega(w_n))\right) - \sum_{j \in [n]} \mathrm{vol}\left(Q_{(\sigma^{jN}\omega)_N}\right)\right| \leq \mathrm{const} \cdot \epsilon nN$$

Together with (6) this settles the basic case.

General case. We now allow the presence of bad intervals. First, let us observe that the basic case immediately implies that if $I = [i,t] \subset [n]$ is an interval consisting entirely of good indices then we can represent the part of the ergodic sum coming from I as the volume of a quasi-fuchsian manifold whose geodesic is a subsegment of $\tau_{\sigma^{iN}(\omega)}$. This means that there is a pair of points $\epsilon|I|N < x < 2\epsilon|I|N$ and $(1-2\epsilon)|I|N < w < (1-\epsilon)|I|N$ such that

$$\left|\mathrm{vol}\left(Q(\tau_{\sigma^{iN}(\omega)}(x), \tau_{\sigma^{iN}(\omega)}(w))\right) - \sum_{j \in I} \mathrm{vol}\left(Q_{(\sigma^{jN}\omega)_N}\right)\right| \leq \mathrm{const} \cdot \epsilon|I|N. \tag{7}$$

The idea of the general case is to proceed as in the basic case but with different building blocks and use the ergodic theorem to keep under control the the contribution of the bad intervals.

The presence of bad intervals brings in some issues, whose nature is related to the way the random walk deviates from the tracking ray, that we have to address. However, no new ingredients are needed, only a more careful choice of the interval subdivision.

The problem can be summarized as follows: Consider a good interval I_j and the adjacent bad interval J_j. Look at the deviation from the tracking ray of I_j introduced by J_j. It might happen that the quasi-fuchsian manifold associated to the good interval I_{j+1} is too small compared to the deviation and we are uncertain whether or not to include it in the gluing family. In order to get around the issue, we wait until the first time when the fellow traveling between the tracking rays of I_j and I_{j+1} is restored, discard all the good small intervals in between and replace the quasi-fuchsian manifold associated to I_j. So we start by refining the interval subdivision.

Refinement of the interval subdivision. Denote by $i_j < t_j$ the initial and the terminal indices in the j-th good interval $I_j = [i_j, t_j]$. We proceed inductively. Start with $I_1 = [i_1 = 0, t_1]$ and $J_1 = [t_1 + 1, i_2 - 1]$. Consider $I_2 = [i_2, t_2]$. We determine a new i_3^{new} by the following condition

$$i_3^{\text{new}} := \min \{ i > t_2 + \epsilon(|I_1| + |J_1|) \text{ and } i \text{ is good} \} .$$

This requirement restores, by property (3), the fellow traveling between $\omega_{i_1 N} \tau_{\sigma^{i_1 N}(\omega)}$ and $\omega_{i_2 N} \tau_{\sigma^{i_2 N}(\omega)}$. That is $\omega_{i_1 N} \tau_{\sigma^{i_1 N}(\omega)}[(1+\epsilon)(|I_1|+|J_1|)N, \infty)$ and $\omega_{i_2 N} \tau_{\sigma^{i_2 N}(\omega)}[\epsilon(|I_1|+|J_1|)N, \infty)$ are δ-fellow travelers (property (3)). The index i_3^{new} lies in some good interval I_{j_3}. We make the following replacement

$$\begin{aligned} I_3 \longrightarrow & I_3^{\text{new}} := [i_3^{\text{new}}, t_{j_3}] \\ J_2 \longrightarrow & J_2^{\text{new}} := [t_2 + 1, i_3^{\text{new}} - 1] \\ & \quad = J_2^{\text{old}} \sqcup I_3 \sqcup \cdots \sqcup J_{j_3 - 1} \sqcup [i_{j_3}, i_3^{\text{new}} - 1]. \end{aligned}$$

By our choice, if $j_3 > 3$, then the sum of the lengths $|J_2^{\text{old}}| + |I_3| + \cdots + |I_{j_3-1}|$ and $i_3^{\text{new}} - i_{j_3}$ are controlled by $\epsilon(|I_1| + |J_1|)$. The length of $|J_{j_3-1}|$ can be, instead, arbitrarily long. Furthermore $|I_3^{\text{new}}| \le |I_{j_3}|$. Observe that, for the new J_2 we have $|J_2^{\text{new}}| = i_3^{\text{new}} - t_2 \le \epsilon(|I_1| + |J_1|) + |J_{j_3-1}|$. We leave untouched all the intervals after I_{j_3}, but we shift back the remaining indices $j \to 3 + j - j_3$ for all $j > j_3$. We repeat the process and get inductively the new set of indices

$$i_r^{\text{new}} := \min \left\{ i > t_{r-1}^{\text{new}} + \epsilon(|I_{r-2}^{\text{new}}| + |J_{r-2}^{\text{new}}|) \text{ and } i \text{ is good} \right\}$$

and intervals

$$\begin{aligned} I_r \longrightarrow & I_r^{\text{new}} := [i_r^{\text{new}}, t_{j_r}] \\ J_{r-1} \longrightarrow & J_{r-1}^{\text{new}} := [t_{r-1} + 1, i_r^{\text{new}} - 1] \end{aligned}$$

that satisfy $|J_r^{\text{new}}| \le \epsilon(|I_{r-2}^{\text{new}}| + |J_{r-2}^{\text{new}}|) + |J_{j_{r+1}-1}|$. We end up with a new subdivision $[n] = I_1^{\text{new}} \sqcup J_1^{\text{new}} \sqcup \cdots \sqcup I_{k'}^{\text{new}} \sqcup J_{k'}^{\text{new}}$ that still has the property

$$\sum_{t\le k'} |J_t^{\text{new}}| \le \sum_{t\le k'} \epsilon(|I_{t-2}^{\text{new}}| + |J_{t-2}^{\text{new}}|) + |J_{j_{t+1}-1}^{\text{old}}| \le \epsilon \sum_{t\le k'} |J_{t-2}^{\text{new}}| + \epsilon n + \zeta n.$$

Hence $\sum_{t\le k'} |J_t^{\text{new}}| \le (\epsilon n + \zeta n)/(1-\epsilon) \le 4\epsilon n$ if $\zeta < \epsilon < 1/2$. In particular the volumes corresponding to the new bad indices still add up to a small amount. In fact, by Lemma 5.2, we have

$$\sum_{i\in\bigsqcup J_j^{\text{new}}} \text{vol}\left(Q_{\sigma^{iN}(\omega)_N}\right) \le (CN + C)\sum_{i<k'} |J_i^{\text{new}}| < \text{const}\cdot \epsilon n N. \tag{8}$$

For the sake of simplicity, after the refinement, we return to the previous notation $i_j := i_j^{\text{new}}$, $t_j := t_j^{\text{new}}$ and $I_j := I_j^{\text{new}}$, $J_j := J_j^{\text{new}}$, but assume the new properties.

Family III. The proof can now proceed parallel to the basic case, so we only sketch the arguments. We define a family of quasi-fuchsian manifolds, one for every pair of adjacent intervals $I_j \sqcup J_j$, that can be glued to form a model for $T_{\omega_{nN}}$ that nearly computes its volume.

Proceed inductively. Start with $I_1 \sqcup J_1 = [0, t_1 = |I_1| - 1] \sqcup [t_1 + 1, i_2 - 1 = |I_1| + |J_1|]$. Since τ_ω is a good ray, we can find segments $[a_1, b_1] \subset [\epsilon|I_1|N, 2\epsilon|I_1|N]$ and $[c_1, d_1] \subset [(1+\epsilon)(|I_1| + |J_1|)N, (1+2\epsilon)(|I_1| + |J_1|)N]$ which are η-thick and have length at least h. Now consider $I_j \sqcup J_j$ for $j > 1$. As in the basic case, we single out a pair of segments $[a_j, b_j]$, $[c_j, d_j]$ on the tracking ray of $\sigma^{i_j N}(\omega)$ normalized so that it starts at O_{i_j}. The first one, $[a_j, b_j]$, is determined by the condition that it is a δ-fellow traveler of $[c_{j-1}, d_{j-1}]$ contained in $[\epsilon(|I_j| + |J_j|)N, 2\epsilon(|I_j| + |J_j|)N]$ (see Figure 5 A). Here we are using in an essential way the properties of the refined interval and property (3) of good rays. The second one, $[c_j, d_j]$, is a η-thick h-long subsegment of $[(1+\epsilon)(|I_j| + |J_j|)N, (1+2\epsilon)(|I_j| + |J_j|)N]$. We simplify the notation by introducing

$$\begin{aligned} A_j &= \omega_{i_j N}\tau_{\sigma^{i_j N}(\omega)}(a_j), & B_j &= \omega_{i_j N}\tau_{\sigma^{i_j N}(\omega)}(b_j),\\ C_j &= \omega_{i_j N}\tau_{\sigma^{i_j N}(\omega)}(c_j), & D_i &= \omega_{i_j N}\tau_{\sigma^{i_j N}(\omega)}(d_j). \end{aligned}$$

We associate to $I_j \sqcup J_j$ the manifold $Q(A_j, D_j)$.

The analogue of Lemma 5.6 holds word by word if we replace the old segments with the new ones, that is $[A_i, D_i]$ uniformly fellow travels $[O, O_n]$.

By property (2), the latter uniformly fellow travels l_n, the axis of ω_{nN}, along $\tau_\omega[\epsilon nN, (1-\epsilon)nN]$. In particular we can find $0 < r < s < n$ such that $[A_r, D_r]$ and $[A_s, D_s]$ are, respectively, the first and the last segments that fellow travel $\tau_\omega[\epsilon nN, (1-\epsilon)nN]$ along some subsegments, which is terminal for the first and initial for the second.

Up to discarding an initial (resp. terminal) segment of $[A_r, D_r]$ (resp. $[A_s, D_s]$) of length smaller than $\epsilon|A_r D_r|$ (resp. $\epsilon|A_s D_s|$) we can assume that

$[A_r, D_r]$ (resp. $[A_s, D_s]$) uniformly fellow travels subsegments of $\tau_\omega[\epsilon nN, (1-\epsilon)nN]$ and l_n (as in Figure 5 B). The volumes of the associated quasi-fuchsian manifolds change at most by $\mathrm{const} \cdot \epsilon nN$ according to Proposition 2.2.

We can also assume, by recurrence, that $[A_r, D_r]$ (resp. $[A_s, D_s]$) contains an initial (resp. terminal) η-thick subsegment $[A_r, B_r]$ (resp. $[C_s, D_s]$) of length at least h. We add the quasi-fuchsian manifold $Q(\omega_{nN}^{-1} C_s, B_r)$ to the family. As in the basic case we have

$$\mathrm{vol}\left(Q(\omega_{nN}^{-1} C_s, B_r)\right) \leq \mathrm{const} \cdot \epsilon nN. \tag{9}$$

Applying Proposition 3.10 to the family $\{Q(\omega_{nN}^{-1} C_s, B_r)\} \sqcup \{Q(A_j, D_j)\}_{j \in [r,s]}$ we can perform the cut and glue construction and get a manifold diffeomorphic to $T_{\omega_{nN}}$ with volume

$$\begin{aligned} \left| \mathrm{vol}\left(T_{\omega_{nN}}\right) - \sum_{i \in [r,s]} \mathrm{vol}\left(Q(A_i, D_i)\right) - \mathrm{vol}\left(Q(\omega_{nN}^{-1} C_s, B_r)\right) \right| & \\ \leq \left|\mathrm{vol}\left(T_{\omega_{nN}}\right) - \mathrm{vol}\left(X_{\omega_{nN}}\right)\right| + & \\ + \left| \mathrm{vol}\left(X_{\omega_{nN}}\right) - \sum_{i \in [r,s]} \mathrm{vol}\left(Q(A_i, D_i)\right) - \mathrm{vol}\left(Q(\omega_{nN}^{-1} C_s, B_r)\right) \right| & \\ \leq nV_1 + \mathrm{const} \cdot \epsilon nN. & \end{aligned} \tag{10}$$

The fellow traveling property of $\bigsqcup_{i<r}[A_i, D_i]$ (resp. $\bigsqcup_{i>s}[A_i, D_i]$) with $\tau_\omega[0, 2\epsilon nN]$ (resp. $[\tau_\omega((1-\epsilon)nN), O_n]$) implies that $\sum_{i \notin [r,s]} d_{\mathcal{T}}(A_i, D_i) \leq 2\epsilon nN$ and, by Lemma 2.2,

$$\sum_{i \notin [r,s]} \mathrm{vol}\left(Q(A_i, D_i)\right) \leq \mathrm{const} \cdot \epsilon nN. \tag{11}$$

We compare now the volume of $Q(A_i, D_i)$ with the ergodic sum over the good interval I_i. Since the interval I_j is good, we find on $\tau_{\sigma^{i_j N}(\omega)}$ two points $\epsilon|I_j|N < x_j < 2\epsilon|I_j|N$ and $(1-2\epsilon)|I_j|N < w_j < (1-\epsilon)|I_j|N$ such that inequality (7) holds for $I = I_j$. Before going on, let us simplify the notation, by introducing $X_j = \omega_{i_j N} \tau_{\sigma^{i_j N}(\omega)}(x_j)$ and $W_j = \omega_{i_j N} \tau_{\sigma^{i_j N}(\omega)}(w_j)$. We have

$$\left| \mathrm{vol}\left(Q(X_j, W_j)\right) - \sum_{i \in I_j} \mathrm{vol}\left(Q_{\sigma^{iN}(\omega)_N}\right) \right| \leq \mathrm{const} \cdot \epsilon |I_j|. \tag{12}$$

By Proposition 2.2, we have

$$|\mathrm{vol}\left(Q(A_j, D_j)\right) - \mathrm{vol}\left(Q(X_j, W_j)\right)| \leq \kappa(d_{\mathcal{T}}(A_j, X_j) + d_{\mathcal{T}}(D_j, W_j)) + \kappa.$$

As $a_j, x_j \in [0, \epsilon(|I_j| + |J_j|)N]$ and $d_j, w_j \in [(1-\epsilon)|I_j|N, (1+2\epsilon)(|I_j| + |J_j|)N]$ we can continue the chain of inequalities with

$$\leq \mathrm{const} \cdot \epsilon |I_j| N + \mathrm{const} \cdot |J_j| N.$$

Adding all the contributions we get

$$\left|\sum_{j\leq k} \mathrm{vol}\left(Q(A_j, D_j)\right) - \sum_{j\leq k} \mathrm{vol}\left(Q(X_j, W_j)\right)\right| \tag{13}$$
$$\leq N \sum_{j\leq k} \mathrm{const}\cdot \epsilon |I_j| + \mathrm{const}\cdot |J_j| \leq \mathrm{const}\cdot \epsilon n N + \mathrm{const}\cdot \zeta n N.$$

Putting together inequalities (10)-(13) and (5), (6) concludes the proof. □

Theorem 2 is now reduced to an application of the ergodic theorem which says that for P-almost every ω the following limit exists and is finite

$$\lim_{n\to\infty} \frac{1}{nN} \sum_{j<n} \mathrm{vol}\left(Q_{(\sigma^{jN}\omega)_N}\right) = v_N.$$

If N and $\Omega_{\epsilon,\zeta,N}$ are as in Proposition 5.5 then

$$\limsup_{j\to\infty} \frac{\mathrm{vol}\left(Q_{\omega_{jN}}\right)}{jN} - \liminf_{j\to\infty} \frac{\mathrm{vol}\left(Q_{\omega_{jN}}\right)}{jN} \leq \epsilon$$

on $\Omega_{\epsilon,\zeta,N}$ which has measure at least $1-\zeta$. Applying Lemma 5.4 we get

$$\limsup_{n\to\infty} \frac{\mathrm{vol}\left(Q_{\omega_n}\right)}{n} - \liminf_{n\to\infty} \frac{\mathrm{vol}\left(Q_{\omega_n}\right)}{n} \leq \epsilon.$$

This concludes the proof of Theorem 2.

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Mathematical Institute, Heidelberg University
Im Neuenheimer feld 205, 69120 Heidelberg, Germany

e-mail: gviaggi@mathi.uni-heidelberg.de