UNIFORM MODELS AND SHORT CURVES FOR RANDOM 3-MANIFOLDS

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Abstract. We give two constructions of hyperbolic metrics on Heegaard splittings satisfying certain conditions that only use tools from the deformation theory of Kleinian groups. In particular, we do not rely on the solution of the Geometrization Conjecture by Perelman. Both constructions apply to random Heegaard splitting with asymptotic probability 1. The first construction provides explicit uniform bilipschitz models for the hyperbolic metric. The second one gives a general criterion for a curve on a Heegaard surface to be a short geodesic for the hyperbolic structure, such curves are abundant in a random setting. As an application of the model metrics, we discuss the coarse growth rate of geometric invariants, such as diameter and injectivity radius, and questions about arithmeticity and commensurability in families of random 3-manifolds.

1. Introduction

Every closed orientable 3-manifold M can be presented as a *Heegaard splitting*. This means that M is diffeomorphic to a 3-manifold M_f obtained by gluing together two handlebodies (taking the second one with opposite orientation) of the same genus H_g along an orientation preserving diffeomorphism $f \in \mathrm{Diff}^+(\Sigma)$ of their boundaries $\Sigma := \partial H_g$

$$M_f = H_g \cup_{f:\partial H_g \to \partial H_g} H_g.$$

If we fix the genus g, however, not all 3-manifolds can arise. In this article, we restrict our attention to the family of those 3-manifolds that can be described as Heegaard splittings of a *fixed genus* $g \geq 2$.

The problem of finding hyperbolic structures on *most* 3-manifolds with a splitting of a fixed genus $g \geq 2$ was originally raised by Thurston (as Problem 24 in [Thu82]) and made more precise by Dunfield and Thurston (see Conjecture 2.11 of [DT06]) via the introduction of the notion of *random Heegaard splittings*.

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Such notion is based on the observation that the diffeomorphism type of M_f only depends on the isotopy class of the gluing map f, so it is well-defined for elements in the *mapping class group*

$$[f] \in \mathrm{Mod}(\Sigma) := \mathrm{Diff}^+(\Sigma)/\mathrm{Diff}_0(\Sigma).$$

Therefore, Heegaard splittings of genus $g \geq 2$ are naturally parameterized by mapping classes $[f] \in \mathrm{Mod}(\Sigma)$.

A family $(M_n)_{n\in\mathbb{N}}$ of random Heegaard splittings of genus $g \geq 2$, or *random 3-manifolds*, is one of the form $M_n = M_{f_n}$ where $(f_n)_{n\in\mathbb{N}}$ is a random walk on the mapping class group $\mathrm{Mod}(\Sigma)$ driven by some initial probability measure μ whose finite support generates $\mathrm{Mod}(\Sigma)$. If $(f_n)_{n\in\mathbb{N}}$ is such a random walk, we will denote by $\mathbb{P}_n$ the distribution of the n-th step f_n and by $\mathbb{P}$ the distribution of the path $(f_n)_{n\in\mathbb{N}}$.

Exploiting work of Hempel [Hem01] and the solution of the geometrization conjecture by Perelman, Maher showed in [Mah10b] that a random Heegaard splitting M_f of genus $g \geq 2$ admits a hyperbolic metric, thus answering Dunfield and Thurston's conjecture.

The main goal of this article is to provide a *constructive* and *effective* approach to the hyperbolization of random 3-manifolds. Before describing our main contributions (Theorems 2 and 3), we note that this approach yields a proof of Maher's result that does not rely on Perelman's work.

Theorem 1. *There is a Ricci flow free hyperbolization for random 3-manifolds.*

By Ricci flow free hyperbolization we mean that we construct the hyperbolic metric only using tools from the deformation theory of Kleinian groups. Specifically, we use the model manifold technology by Minsky [Min10] and Brock, Canary and Minsky [BCM12], as well as the effective version of Thurston's Hyperbolic Dehn Surgery by Hodgson and Kerckhoff [HK05] and Brock and Bromberg's Drilling Theorem [BB04].

We remark that, even though we do not rely on Perelman's solution of the geometrization conjecture, we do use the main result from Maher [Mah10b], namely, the fact that the *Hempel distance* of the Heegaard splittings (see Hempel [Hem01]) grows coarsely linearly along the random walk.

We develop two different approaches to Theorem 1, both bringing new and more refined information than the mere existence of a hyperbolic metric.

In fact, with the first construction we provide a *model manifold* that captures, up to *uniform bilipschitz distortion*, the geometry of the random 3-manifold and allows the computation of its geometric invariants. The hyperbolic metric is constructed explicitly gluing elementary building blocks.

With the second construction, instead, we give a purely topological criterion for the Heegaard splitting to admit a hyperbolic metric for which a simple closed curve on the Heegaard surface is a *short closed geodesic*. This is a general criterion of hyperbolicity and applies to a large class of Heegaard splittings. The existence of the hyperbolic metric will be guaranteed by appealing to Thurston's Hyperbolization Theorem.

Uniform bilipschitz models for random 3-manifolds. The notion of model manifold that we use is similar to the ones considered by Brock, Minsky, Namazi and Souto in [Nam05], [NS09], [BMNS16] and is depicted in the following definition of *ε-model metric*: A Riemannian metric (M_f, ρ) is a ε-model metric for $\varepsilon < 1/2$ if there is a decomposition into five pieces $M_f = H_1 \cup \Omega_1 \cup Q \cup \Omega_2 \cup H_2$ satisfying the three requirements

(1) Topologically, H_1 and H_2 are homeomorphic to genus g handlebodies while Q, Ω_1 and Ω_2 are homeomorphic to $\Sigma \times [0, 1]$.
(2) Geometrically, ρ has negative curvature $\sec \in (-1-\varepsilon, -1+\varepsilon)$, but outside the region $\Omega = \Omega_1 \cup \Omega_2$ the metric is purely hyperbolic, i.e. $\sec = -1$.
(3) The piece Q is almost isometrically embeddable in a complete hyperbolic 3-manifold diffeomorphic to $\Sigma \times \mathbb{R}$.

The importance of the last requirement resides in the fact that we understand explicitly hyperbolic 3-manifolds diffeomorphic to $\Sigma \times \mathbb{R}$ thanks to the work of Minsky [Min10] and Brock, Canary and Minsky [BCM12] which provides a detailed combinatorial description of their internal geometry.

The following is our more precise version of Theorem 1.

Theorem 2. *For every $\varepsilon > 0$ and $K > 1$ we have*

$$\mathbb{P}_n[M_f \text{ has a hyperbolic metric } K\text{-bilipschitz to a } \varepsilon\text{-model metric}] \overset{n\to\infty}{\longrightarrow} 1.$$

We remark that ε-model metrics on random Heegaard splittings, similar to the ones that we build here, are constructed in [HV19]. There, the existence of a underlying hyperbolic metric is guaranteed by Maher's result and it is unclear whether the ε-model metrics are uniformly bilipschitz to it.

However, we also mention that, using an unpublished result by Tian [Tia], the mere fact that a metric ρ is a ε-model metric and that the regions Ω_1, Ω_2 where it is not hyperbolic have uniformly bounded diameter (as follows from [HV19]), implies, if $\varepsilon > 0$ is sufficiently small, that ρ is uniformly close up to third derivatives to a hyperbolic metric. Here, we do not rely on Tian's result. Instead, in order to provide a uniform bilipschitz control, we exploit ergodic properties of the random walk and drilling and filling theorems by Hodgson and Kerckhoff [HK05] and Brock and Bromberg [BB04].

Our methods follow closely [BMNS16] and [BD15] where uniform ε-model metrics are constructed for special classes of 3-manifolds.

The idea is the following: We can obtain a hyperbolic metric on M_f by a hyperbolic cone manifold deformation from a finite volume metric on a *drilled* manifold $\mathbb{M}$ which has the following form: Let $\Sigma \times [1, 4]$ be a tubular neighborhood of $\Sigma \subset M_f$. We consider 3-manifolds diffeomorphic to

$$\mathbb{M} = M_f - (P_1 \times \{1\} \cup P_2 \times \{2\} \cup P_3 \times \{3\} \cup P_4 \times \{4\})$$

where P_j is a pants decomposition of the surface $\Sigma \times \{j\}$. A finite volume hyperbolic metric on such a manifold can be constructed explicitly by gluing together the convex cores of two maximally cusped handlebodies H_1, H_2 and three maximally cusped I-bundles Ω_1, Q, Ω_2.

$$\mathbb{M} = H_1 \cup \Omega_1 \cup Q \cup \Omega_2 \cup H_2.$$

Most of our work consists of finding suitable pants decompositions for which the Dehn surgery slopes needed to pass from $\mathbb{M}$ to M_f satisfy the assumptions of the effective Hyperbolic Dehn Surgery Theorem [HK05]. In order to find them we crucially need two major tools: The work of [HV19] on the geometry of hyperbolic handlebodies and ergodic properties of the random walks proved by Baik, Gekhtman and Hamenstädt [BGH20].

We stress the fact that for both Theorem 1 and Theorem 2 we assume that the support of μ is finite and generates the *entire* mapping class group.

Short curves via knots on Heegaard surfaces. We now discuss *short closed geodesics* in hyperbolic Heegaard splittings and random 3-manifolds: We identify purely topological conditions on a simple closed curve on the Heegaard surface $\gamma \subset \Sigma$ that ensure that M_f has a hyperbolic structure and γ is a very short geodesic in it.

The criterion that we find builds upon the groundbreaking work of Minsky [Min10] on hyperbolic structures on $\Sigma \times \mathbb{R}$. In that setting, the collection of simple closed curves on Σ that are isotopic to very short closed geodesics on a hyperbolic structure on $\Sigma \times \mathbb{R}$ can be read off the list of *subsurface coefficients* associated to the *end invariants* of such a structure.

To some extent, in a complicated Heegaard splitting M_f, the role of the end invariants can be, in a first approximation, replaced by the *disk sets* $\mathcal{D}$ and $f\mathcal{D}$ of the splitting. Those $\mathcal{D}$ and $f\mathcal{D}$ are the subsets of the *curve graph* $\mathcal{C}$ of the Heegaard surface Σ given by the essential simple closed curves $\delta \subset \Sigma$ that *compress* to in the first and second handlebody of the Heegaard surface, respectively.

For complicated hyperbolic Heegaard splittings we have the following conjectural picture: A curve $\gamma \subset \Sigma$ is isotopic to a short geodesic if and only if it lies on the boundary $\gamma \subset \partial W$ of a proper essential subsurface $W \subset \Sigma$ where the disk sets $\mathcal{D}$ and $f\mathcal{D}$ have large *subsurface projection* $d_W(\mathcal{D}, f\mathcal{D})$.

In this direction we prove the following result.

Theorem 3. *Let $\Sigma := \partial H_g$, $g \geq 2$, be fixed. There exists a constant $C_\Sigma > 0$ such that the following holds. Let $\gamma \subset \Sigma$ be a non-separating simple closed curve with complement $W := \Sigma - \gamma$. Let $f \in \mathrm{Mod}(\Sigma)$ be a mapping class. Suppose that*

(a) Both (H_g, γ) and $(H_g, f(\gamma))$ are pared acylindrical handlebodies*.*

(b) We have a large subsurface projection $d_W(\mathcal{D}, f\mathcal{D}) \geq C_\Sigma$.

Then M_f has a hyperbolic metric. Moreover, the length of γ in M_f is bounded by

$$\ell_{M_f}(\gamma) \leq C_\Sigma / d_W(\mathcal{D}, f\mathcal{D}).$$

We recall that, informally speaking, the pair (H_g, γ) is called a *pared acylindrical handlebody* if $\Sigma - \gamma$ is incompressible and there are no non-trivial essential cylinders in (H_g, γ) and $(H_g, \Sigma - \gamma)$. These objects arise naturally in the study of cusped hyperbolic structures on H_g (see Thurston [Thu86a]). Many pairs (H_g, γ) have this property. For example, if $\gamma \subset \Sigma$ satisfies $d_{\mathcal{C}}(\gamma, \mathcal{D}) \geq 3$, then (H_g, γ) is pared acylindrical. As the disk set $\mathcal{D}$ is a small quasi-convex subset of $\mathcal{C}$ by Masur and Minsky [MM04], non-separating curves $\gamma \in \mathcal{C}$ that are far from $\mathcal{D}$ are abundant.

The idea for the proof of Theorem 3 is the following: We associate to $\gamma \subset \Sigma$ the 3-manifold $M_f - \gamma$. Notice that it decomposes as

$$M_f - \gamma = (H_g - \gamma) \cup (H_g - f(\gamma)).$$

If both (H_g, γ) and $(H_g, f(\gamma))$ are *pared acylindrical handlebodies*, then the JSJ theory tells us that the complement $M_f - \gamma$ is irreducible and atoroidal. Moreover it is also *Haken* since we can choose $\Sigma - \gamma \subset M_f - \gamma$ as a Haken surface. Hence, by Thurston's Hyperbolization Theorem, the 3-manifold $M_f - \gamma$ admits a complete finite volume hyperbolic structure.

As in the proof of Theorem 2, we can deform, via hyperbolic cone manifold structures, the metric on $M_f - \gamma$ to a hyperbolic metric on M_f for which γ is a short curve provided that the filling slope satisfies the assumptions of the effective Hyperbolic Dehn Surgery Theorem [HK05]. To this extent we argue that the size of a standard torus horosection of the cusp of $M_f - \gamma$ is comparable with the subsurface projection $d_W(\mathcal{D}, f\mathcal{D})$. In order to check this we use some tools from the model manifold technology by Minsky [Min10] and Brock, Canary and Minsky [BCM12]. The upper bound on the length immediately follows from Hodgson and Kerckhoff's deformation theory of hyperbolic cone manifolds.

Ergodic properties of random walks on the mapping class group imply that the condition of large subsurface projection of $\mathcal{D}$ and $f_n\mathcal{D}$ on the complement W_n of some non-separating curves $\gamma_n \subset \Sigma$ holds with asymptotic probability one, that is, with $\mathbb{P}_n \to 1$. Thus, Theorem 3 applies to random 3-manifolds and gives a proof of Theorem 1 that does not rely on Perelman's work.

Notice that the curve γ will be short in M_f, namely, we have a uniform upper bound on the length $\ell_{M_f}(\gamma) < \varepsilon_\Sigma$. In a direction opposite to Theorem 3, one can ask whether every very short curve on a complicated hyperbolic Heegaard splitting arises from a cone manifold deformation of a manifold of the form $M_f - \gamma$ with $\gamma \subset \Sigma$ a simple closed curve. This is the case for *strongly irreducible* hyperbolic Heegaard splittings M_f as proved by Souto [Sou08] and Breslin [Bre11]: There is a constant $\varepsilon_\Sigma > 0$ such that, every closed geodesic of length at most ε_Σ in M_f is isotopic to a simple closed curve γ on the Heegaard surface $\Sigma \subset M_f$.

We briefly comment on the assumptions of Theorem 3.

As for condition (a), its use is twofold: It implies that the complement $M_f - \gamma$ is hyperbolizable and it also implies that the inclusions $\Sigma - \gamma, \Sigma - f(\gamma) \subset H_g$ are *doubly incompressible* (as defined by Thurston [Thu86a]). This fact plays a central role in the proof via Thurston's Uniform Injectivity [Thu86a]. As mentioned above, since $\mathcal{D}$ and $f\mathcal{D}$ are a small quasi-convex subsets of the curve graph, there are plenty of non-separating curves that satisfy $d_{\mathcal{C}}(\gamma, \mathcal{D} \cup f\mathcal{D}) \geq 3$ and, hence, condition (a). They all yield cusped hyperbolic manifolds $M_f - \gamma$.

Concerning, instead, the restriction on W and condition (b) we remark that the choice of W as the complement of a non-separating simple closed curve $\gamma \subset \Sigma$ is certainly artificial, but allows some simplifications and is enough for our application to random 3-manifolds. More important is the fact that W is a *non-annular* subsurface because, in this case, a large subsurface projection is reflected in the size of a standard torus horosection of the cusp of $M_f - \gamma$.

Applications. We describe some consequences of Theorem 2.

We start with a geometric application: We exploit the geometric control given by the ε-model metric to compute the coarse growth or decay rate of the geometric invariants along the family $(M_{f_n})_{n\in\mathbb{N}}$.

The general strategy is very simple: We use the model manifold technology [Min10], [BCM12] and compute the geometric invariants for the middle piece Q of the ε-model metric. Then, we argue that the invariants of Q are uniformly comparable with those of M_f.

For example, combined with a result of Brock [Bro03], Theorem 2 allows the computation of the coarse growth rate of the volume, which is well-known to be linear as explained in [Mah10b] (see also [HV19]). Combined with results of Baik, Gekhtman and Hamenstädt [BGH20] it shows that the smallest positive eigenvalue of the Laplacian behaves like $1/n^2$ as computed in [HV19]. We notice that Theorem 2 allows a uniform approach to those result.

Here we do not carry out those computations because they are already well established. Instead, we use the model metric to control the diameter growth rate and systole decay.

Theorem 4. *There exists $c > 0$ such that*

$$\mathbb{P}_n[\mathrm{diam}(M_f) \in [n/c, cn]] \overset{n\to\infty}{\longrightarrow} 1.$$

The ingredients of the proof are Theorem 2 and a result by White [Whi01].

Using work of the second author and Taylor [ST19], instead, we get

Theorem 5. *There exists $c > 0$ such that*

$$\mathbb{P}_n\left[\mathrm{inj}(M_f) \leq c/\log(n)^2\right] \overset{n\to\infty}{\longrightarrow} 1.$$

As described in [ST19], $1/\log(n)^2$ is *exactly* the coarse decay rate for the length of the shortest curve in random mapping tori. Our methods, however, only give an upper bound in the case of random Heegaard splittings.

In a completely different direction we use Theorem 2 to prove the following

Theorem 6. *With asymptotic probability 1 the following holds*

(1) M_f is not arithmetic.
(2) M_f is not in a fixed commensurability class $\mathcal{R}$.

The proof combines a study of geometric limits of random 3-manifolds, Proposition 7.3, with arguments from Biringer and Souto [BS11].

Overview. This article is divided into four parts that correspond to: The construction of the model metric (Sections 2 and 3). Short curves on Heegaard splittings (Sections 4 and 5). The application to random 3-manifolds (Section 6). The computation of the coarse growth rate of geometric invariants (Section 7). Next, we briefly describe the content of each section.

In Section 2 we outline the construction of the ε-model metric. In Section 3 we develop the two main technical tools that we need and use them to build many examples to which the model metric construction applies.

In Section 4 we describe the topological part of the proof of Theorem 3, in particular, we check that $M_f - \gamma$ is hyperbolizable. The geometric part is developed, instead, in Section 5 where we use the model manifold technology to check that the cusp of $M_f - \gamma$ has a large horosection.

In Section 6 we prove Theorems 1 and 2 by showing that the examples of Section 3 and the ones described by Theorem 3 are generic from the point of view of a random walk.

Lastly, in Section 7 we prove Theorems 4, 5 and 6.

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PF and AS want to follow up on GV's acknowledgement above. A previous arxiv version [Via19] of this article contained GV's path to Theorem 1, a result PF and AS had independently announced in [FMTS18]. PF and AS want to acknowledge that the GV had independently pursued and worked out a complete proof of Theorem 1 via Theorem 2 as part of his impressive PhD-work. They are most thankful for GV's insight and flexibility in merging the two works into a coherent work, which exceeds the sum of its parts and which, in particular, includes further applications we would not have found separately.

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2. A gluing scheme

Here we outline a construction for the ε-model metric which follows closely ideas of Brock and Dunfield [BD15] and Brock, Minsky, Namazi and Souto [BMNS16]. At the end of the discussion we formulate a criterion of applicability.

2.1. **Assembling simple pieces.** The construction is somehow implicit in the description of an ε-model metric. It has two steps. We start with five building blocks H_1, H_2 and Q, Ω_1, Ω_2 which are the *convex cores* of *geometrically finite maximally cusped* complete hyperbolic structures on H_g and $\Sigma \times [1,2]$ respectively. The pieces Ω_1 and Ω_2 will play the role of the collars of the other structures as we are going to explain later on.

For convenience of the reader, we briefly describe the geometry of H_1, H_2 and Q, Ω_1, Ω_2. The convex core Q of a geometrically finite maximally cusped structure on $\Sigma \times [1,2]$ is diffeomorphic to the drilled product

$$Q \cong \Sigma \times [1,2] - (P_1 \times \{1\} \cup P_2 \times \{2\})$$

where P_1, P_2 are pants decompositions of Σ such that no curve in P_1 is isotopic to a curve in P_2. The drilled product is endowed with a complete finite volume hyperbolic metric with totally geodesic boundary

$$\partial Q = \partial_1 Q \sqcup \partial_2 Q = (\Sigma \times \{1\} - P_1 \times \{1\}) \sqcup (\Sigma \times \{2\} - P_2 \times \{2\}).$$

and rank one cusps at $P_1 \cup P_2$. If we fix the isotopy class of the identification of Q with the drilled product, there exists a unique maximally cusped structure with cusp data $P_1 \cup P_2$. We denote it by $Q(P_1, P_2)$.

Analogously, the convex core H of a geometrically finite maximally cusped structure on H_g is diffeomorphic to the drilled handlebody

$$H \cong H_g - P$$

where P is a pants decomposition of $\partial H_g = \Sigma$ (throughout this article we keep this identification fixed) with the property that every curve in P is not compressible and no two curves in P are isotopic within H_g. Again, H is endowed with a complete finite volume hyperbolic metric with totally geodesic boundary

$$\partial H \cong \partial H_g - P$$

and rank one cusps at P. If we keep track of the isotopy class of the identification between H and the drilled handlebody, there exists a unique maximally cusped structure with cusp data P. We denote it by $H(P)$.

Each component of the boundaries $\partial Q, \partial H$ is a three punctured sphere. It inherits a complete finite area hyperbolic metric. Such a structure is unique up to isometries isotopic to the identity. Hence, once we decided a pairing of the components of ∂H_1, ∂H_2 with $\partial_1 \Omega_1$, $\partial_2 \Omega_2$ and of $\partial_2 \Omega_1$, $\partial_1 \Omega_2$

with $\partial_1 Q$, $\partial_2 Q$, there is no ambiguity in implementing it to an isometric diffeomorphism. Gluing the pieces together along such a diffeomorphism we get a 3-manifold

$$\mathbb{M} := H_1 \cup_{\partial H_1 \cong \partial_1 \Omega_1} \Omega_1 \cup_{\partial_2 \Omega_1 \cong \partial_1 Q} Q \cup_{\partial_2 Q \cong \partial_1 \Omega_2} \Omega_2 \cup_{\partial_2 \Omega_2 \cong \partial H_2} H_2$$

which is non-compact and has a naturally defined complete finite volume hyperbolic structure.

In our case the pairing is natural as our structures are of the form

$$\begin{aligned} H_1 &= H(P_1), \\ \Omega_1 &= Q(P_1, P_2), \\ Q &= Q(P_2, P_3), \\ \Omega_2 &= Q(P_3, P_4), \\ H_2 &= H(f^{-1} P_4). \end{aligned}$$

We think of Ω_1 and Ω_2 as the collar structures of the boundaries of the three larger pieces $\mathbb{N}_1 = H_1 \cup \Omega_1$, $\mathbb{Q} = \Omega_1 \cup Q \cup \Omega_2$ and $\mathbb{N}_2 = \Omega_2 \cup H_2$.

Topologically, $\mathbb{M}$ is diffeomorphic to a drilled M_f, namely, let $\Sigma \times [1,4]$ denote a tubular neighborhood of the Heegaard surface $\Sigma \subset M_f$, then

$$\mathbb{M} \cong M_f - (P_1 \times \{1\} \cup P_2 \times \{2\} \cup P_3 \times \{3\} \cup P_4 \times \{4\}).$$

The pieces Ω_1, Q, Ω_2 are identified with

$$\begin{aligned} \Omega_1 &= \Sigma \times [1,2] - (P_1 \times \{1\} \sqcup P_2 \times \{2\}), \\ Q &= \Sigma \times [2,3] - (P_2 \times \{2\} \sqcup P_3 \times \{3\}), \\ \Omega_2 &= \Sigma \times [3,4] - (P_3 \times \{3\} \sqcup P_4 \times \{4\}). \end{aligned}$$

The curves in $P_1 \cup P_2 \cup P_3 \cup P_4$ represent the rank two cusps of $\mathbb{M}$.

In order to pass from $\mathbb{M}$ to the closed 3-manifold M_f we have to perform Dehn fillings on each cusp. This is the second step of the construction. The filling slopes are completely determined by the identification of $\mathbb{M}$ with the drilled M_f: They are the meridians γ of small tubular neighborhoods of the curves in $\alpha \times \{j\} \subset P_j \times \{j\}$ inside $\Sigma \times [1,4]$.

Under such circumstances, the Hodgson and Kerckhoff effective version [HK05] of Thurston's Hyperbolic Dehn Surgery Theorem gives us sufficient conditions to guarantee that M_f has a hyperbolic metric obtained via a hyperbolic *cone manifold deformation* of the metric $\mathbb{M}$.

The condition is as follows: For every cusp of $\mathbb{M}$ we fix a torus horosection $T \subset \mathbb{M}$ on the boundary of the η_M-thin part where $\eta_M > 0$ is some fixed Margulis constant.

On each such horosection we have the slope $\gamma \subset T$, determined by the gluing. We represent it as a simple closed geodesic for the intrinsic flat metric of T. Hodgson and Kerckhoff deformation theory requires that the flat geodesic γ has sufficiently large *normalized length*, a quantity defined by

$$\mathrm{nl}(\gamma) := l(\gamma)/\sqrt{\mathrm{Area}(T)}.$$

We have:

THEOREM 2.1 (Hodgson-Kerckhoff [HK05]). *Let $\mathbb{M}$ be a complete finite volume hyperbolic 3-manifold with n cusps. Let γ_j be flat geodesic slopes on torus horosections of the cusps. Suppose that the normalized length of each γ_j is at least $\mathrm{nl}_{HK} = 10.6273$. Then, there is a family $(M_t)_{t\in[0,2\pi]}$ of hyperbolic cone manifold structures on the Dehn filled manifold M whose singular loci are the core curves of the added tori and such that the cone angles of M_t equal t. The final hyperbolic cone manifold $M_{2\pi}$ is non singular. Moreover, the length of the core geodesic α_j is controlled by $l_{M_{2\pi}}(\alpha_j) \leq a/\mathrm{nl}(\gamma_j)^2$ for some universal constant $a > 0$.*

The proof we provide below allows to make the constant a explicit. However, we only use Theorem 2.1 in combination with statements where no explicit constants are available (see the argument below and proof of Theorem 3).

Proof. The "moreover" part is not explicitly stated in [HK05], so we now explain how it follows from the proofs. All references are from [HK05].

We will explain the case of one cusp, the necessary modifications for more cusps are explained above Theorem 5.11 of [HK05].

Throughout the deformation of the cone manifold structures, certain quantities need to be controlled. For consistency with [HK05], we denote the cone angle by α. The authors define:

- The length of the singular geodesic, ℓ, which for $\alpha = 2\pi$ is the length of the core geodesic that we want to control (see Remark 5.8 in [HK05]).
- A function $h(r) = 3.3957 \tanh(r)/\cosh(2r)$, for $r \geq 0.531$, which is decreasing (Lemma 5.2 of [HK05]).
- The parameter ρ so that $h(\rho) = \alpha\ell$.
- The function $F(w) = -\frac{(1+4z+6z^2+z^4)}{(z+1)(1+z^2)^2}$.

Just as right below equation (47) of [HK05], for any $z \geq 0.4862$, the cone-angle can be increased all the way to 2π maintaining $\tanh(\rho) \geq z$ provided

$$\hat{L}^2 \geq \frac{(2\pi)^2}{3.3957(1-z)} \exp\left(\int_z^1 F(w)dw\right),$$

where $\hat{L}$ is the normalized length of our filling slope.

Setting $c := \frac{(2\pi)^2}{3.3957} \exp\left(\int_{0.4862}^1 F(w)dw\right)$, the inequality is satisfied for

$$z = 1 - \frac{c}{\hat{L}^2}.$$

Hence, we will maintain $\tanh(\rho) \geq 1 - \frac{c}{\hat{L}^2}$, which can rewritten as

$$2\frac{e^{-\rho}}{e^{\rho}+e^{-\rho}} \leq \frac{c}{\hat{L}^2}.$$

In particular, $e^{-2\rho} \leq \frac{c}{\hat{L}^2}$. Finally, at $\alpha = 2\pi$, from $h(\rho) = 2\pi\ell$ we get

$$\ell \leq \frac{3.3957}{2\pi}\frac{\tanh(\rho)}{\cosh(\rho)} \leq \frac{3.3957}{2\pi}\frac{2}{e^{2\rho}} \leq \frac{3.3957c}{\pi}\frac{1}{\hat{L}^2}. \qquad \square$$

We want to guarantee that these conditions are fulfilled. This is where most of our work lies.

Once we know that $\mathbb{M}$ and M_f are connected by a family of hyperbolic cone manifolds, an application of Brock and Bromberg's Drilling Theorem [BB04] ensures that $\mathbb{M}$ is K-bilipschitz to M_f away from its cusps. The constant K only depends on the length $l_{M_f}(\alpha_j)$ which, by Theorem 2.1, is again controlled by the normalized length $\mathrm{nl}(\gamma_j)^2$.

Theorem 2.2 (Brock-Bromberg [BB04]). *Let $\eta_M > 0$ be a Margulis constant. For every $n > 0$ and $\xi > 0$ there exists $0 < \eta_B(\xi) < \eta_M$ such that the following holds: Let M be a geometrically finite hyperbolic 3-manifold. Let $\Gamma = \alpha_1 \sqcup \cdots \sqcup \alpha_n \subset M$ be a collection of simple closed geodesics of length $l_M(\alpha_j) < \eta_B(\xi)$ for all $j \leq n$. Let N be the unique geometrically finite hyperbolic structure on $M - \Gamma$ with the same conformal boundary as M. Then, there exists a $(1+\xi)$-bilipschitz diffeomorphism*

$$\left(N - \bigsqcup_{j\leq n}\mathbb{T}_{\eta_M}(\alpha_j), \bigsqcup_{j\leq n}\partial\mathbb{T}_{\eta_M}(\alpha_j)\right) \longrightarrow \left(M - \bigsqcup_{j\leq n}\mathbb{T}_{\eta_M}(\alpha_j), \bigsqcup_{j\leq n}\partial\mathbb{T}_{\eta_M}(\alpha_j)\right)$$

where $\mathbb{T}_{\eta_M}(\alpha)$ denotes a standard η_M-Margulis neighborhood for α.

2.2. **Two criteria for Dehn filling with long slopes.** Certifying that the filling slopes have large normalized length is the main point that we have to address. We now discuss two criteria to check this condition.

The argument branches in two cases: We consider separately the filling slopes in $\mathbb{Q} = \Omega_1 \cup Q \cup \Omega_2$ and the ones in $\mathbb{N}_j = H_j \cup \Omega_j$. The two cases are similar in spirit, but the second one is technically more involved than the first one. However, the ideas are the same, so we will explain them with more details in the easier setting.

The I-bundle case. Consider first the hyperbolic manifold

$$\begin{aligned}\mathbb{Q} &= \Omega_1 \cup Q \cup \Omega_2\\ &= Q(P_1, P_2) \cup Q(P_2, P_3) \cup Q(P_3, P_4).\end{aligned}$$

Topologically it is diffeomorphic to

$$\Sigma \times [1,4] - (P_1 \times \{1\} \sqcup P_2 \times \{2\} \sqcup P_3 \times \{3\} \sqcup P_4 \times \{4\})$$

The curves in P_1 and P_4 represent rank one cusps on $\partial\mathbb{Q}$ while the curves in P_2 and P_3 represent rank two cusps. We now try to understand what happens when we Dehn fill only the *rank two* cusps.

The filling slopes are chosen so that after the Dehn surgery, the natural inclusions $\Sigma \hookrightarrow \Omega_1, Q, \Omega_2$ become isotopic in the filled manifold so that it is naturally identified with

$$\mathbb{Q}^{\text{fill}} \cong \Sigma \times [1,4] - (P_1 \times \{1\} \sqcup P_4 \times \{4\}).$$

We observe that there exists a unique marked maximally cusped structure on $\mathbb{Q}^{\text{fill}}$ where the rank one cusps are precisely given by $P_1 \times \{1\} \sqcup P_4 \times \{4\}$ (we assume that no curve in P_i is isotopic to a curve in P_j if $i \neq j$). We denote such a structure by $Q(P_1, P_4)$.

We are now ready to explain the main idea. Recall that our goal is to show that the filling slopes we singled out on the rank two cusps of $\mathbb{Q}$ have very large normalized length. This can be checked also in $\mathbb{Q}^{\text{fill}}$ once we know that $\mathbb{Q}$ uniformly bilipschitz embeds in $\mathbb{Q}^{\text{fill}}$ away from standard cusp neighborhoods.

The strategy is as follows: Consider the maximally cusped structure $Q(P_1, P_4)$ and denote by Γ the collection of geodesic representatives of P_2 and P_3. Suppose that the collection Γ consists of extremely short simple closed geodesics, say of length at most $\eta < \eta_B(1/2)$, and that it is isotopic to $P_2 \times \{2\} \cup P_3 \times \{3\}$ under the identification with the drilled product.

Under such assumptions, we have the following.

Topologically, since the diffeomorphism type of $Q(P_1, P_4) - \Gamma$ only depends on the isotopy class of Γ, the manifold $Q(P_1, P_4) - \Gamma$ is diffeomorphic to $\mathbb{Q}$.

Geometrically, by Theorem 2.2, we can replace, up to 3/2-bilipschitz distortion away from standard Margulis neighborhoods of Γ, the hyperbolic metric on $Q(P_1, P_4) - \Gamma$ with the unique geometrically finite structure with the same conformal boundary and rank two cusps instead of Γ. By uniqueness, such a geometrically finite structure structure on $Q(P_1, P_4) - \Gamma$ is precisely our initial manifold $\mathbb{Q} = \Omega_1 \cup Q \cup \Omega_2$.

In conclusion, $\mathbb{Q} = \Omega_1 \cup Q \cup \Omega_2$ uniformly bilipschitz embeds, away from the cusps, in $Q(P_1, P_4) - \Gamma$ and the filling slopes are mapped to meridians of large Margulis tubes. Comparing the normalized length of a filling slope γ in the two metrics we deduce that it must be very large because in $Q(P_1, P_4)$ the curve γ is the meridian on the boundary of a very large Margulis tube. In fact

LEMMA 2.3. *Let $\mathbb{T}_{\eta_M}(\alpha)$ be a Margulis tube of radius R around a simple closed geodesic α of length $l(\alpha) < \eta_M$. Let γ be the flat geodesic representing the meridian on $\partial\mathbb{T}_{\eta_M}(\alpha)$. Then the normalized length is*

$$\text{nl}(\gamma) = \sqrt{\frac{2\pi \tanh(R)}{l(\alpha)}}.$$

In particular $\mathrm{nl}(\gamma) \to \infty$ *as* $l(\alpha) \to 0$ *independently of the radius* R.

For example, there exists $\eta > 0$ such that if $l(\alpha) < \eta$ then $\mathrm{nl}(\gamma)$ is much bigger than nl_{HK}, the Hodgson-Kerckhoff constant.

Proof. The metric on $\mathbb{T}_{\eta_M}(\alpha)$ can be written in Fermi coordinates as

$$ds^2 = dr^2 + \cosh(r)^2 dl^2 + \sinh(r)^2 d\theta^2$$

where $(r, l, \theta) \in [0, R] \times [0, l(\gamma)] \times [0, 2\pi]$ are, respectively, the distance from α, the length along α and the angle around α parameters. The flat torus on the boundary has area $\mathrm{Area}(\partial\mathbb{T}_{\eta_M}(\alpha)) = 2\pi l(\alpha)\cosh(R)\sinh(R)$. The flat meridian $\gamma \subset \partial\mathbb{T}_{\eta_M}(\alpha)$ is represented by the curve $\theta \to (R, 0, \theta)$ of length $l(\gamma) = 2\pi\sinh(R)$. Hence the formula for the normalized length.

Notice that $\tanh(R)$ is roughly 1 when R is very large so that, in this case, the normalized length is approximately $\mathrm{nl}(\gamma) \approx l(\alpha)^{-1/2}$. It follows from work of Brooks and Matelski [BM82] that the radius of the Margulis tube $\mathbb{T}_{\eta_M}(\alpha)$ is at least $R \geq \frac{1}{2}\log(\eta_M/l(\alpha)) - R_0$ where $R_0 > 0$ is some universal constant. Hence the second claim of the lemma when $l(\alpha)$ is very small. □

Applying Lemma 2.3 to the previous situation, we can conclude the following criterion

> **Criterion for I-bundles**: Fix $\mathrm{nl}_0 > \mathrm{nl}_{HK}$. The normalized length of the filling slopes corresponding to P_2 and P_3 is at least nl_0 provided that the collection of geodesic representatives in $Q(P_1, P_4)$ of the curves in $P_2 \cup P_3$ consists of simple geodesics of length at most η, where η only depends on nl_0, and is isotopic to $P_2 \times \{2\} \cup P_3 \times \{3\}$.

This concludes the I-bundle case.

The handlebody case. The second part consists of the same analysis for $\mathbb{N}_j = H_j \cup \Omega_j$ and $j = 1, 2$. The strategy is exactly the same. We only consider $\mathbb{N}_1 = H_1 \cup \Omega_1$ as the case of $\mathbb{N}_2 = \Omega_2 \cup H_2$ is completely analogous.

Parameterize a collar neighborhood of $\Sigma = \partial H_g$ in H_g as $\Sigma \times [1, 2]$ with $\partial H_g = \Sigma \times \{2\}$. Topologically we have

$$\mathbb{N}_1 = H_g - (P_1 \times \{1\} \sqcup P_2 \times \{2\}).$$

Geometrically, the curves in P_2 correspond to rank one cusps while the one in P_1 correspond to rank two cusps. We are interested in filling in the rank two cusps. As before, the filling slopes are determined by the gluing.

After filling we have

$$\mathbb{N}_1^{\mathrm{fill}} = H_g - P_2.$$

Again, there is a unique maximally cusped structure on $\mathbb{N}_1^{\mathrm{fill}}$ whose cusps are given by P_2. We denote it by $H(P_2)$. We argue as before and assume that the collection Γ of geodesic representatives of P_1 consists of very short curves and is isotopic to $P_1 \times \{1\}$. Using the Drilling Theorem we compare the normalized length in $\mathbb{N}_1$ and $H(P_2)$.

Again, relying on Lemma 2.3, we will use the following criterion.

> **Criterion for handlebodies**: Fix $\mathrm{nl}_0 > \mathrm{nl}_{HK}$. The normalized length of the filling slopes corresponding to P_1 is at least nl_0 provided that the collection of the geodesic representatives in $H(P_2)$ of the curves in P_1 consists of simple closed geodesic of length at most η, where η only depends on nl_0, and is isotopic to $P_1 \times \{1\}$.

When considering $\mathbb{N}_2 = \Omega_2 \cup H_2 = Q(P_3, P_4) \cup H(f^{-1}P_4)$, we ask the same requirements replacing P_1 with $f^{-1}P_4$ and P_2 with $f^{-1}P_3$.

This concludes the handlebody case.

Thus, in the previous discussion we established the following

Proposition 2.4. *Fix $K \in (1, 2)$. Suppose that there are four pants decompositions P_1, P_2, P_3, P_4 such that the* I-bundle *and the* handlebody criteria *are satisfied with parameter η sufficiently small only depending on K. Then, M_f admits a hyperbolic metric and a model metric $\mathbb{M}$. Furthermore, $\mathbb{M}$ and M_f can be connected by a family of hyperbolic cone manifolds and we have a K-bilipschitz diffeomorphism*

$$\left(\mathbb{M} - \bigsqcup_{\alpha \in P_1 \cup P_2 \cup P_3 \cup P_4} \mathbb{T}_{\eta_M}(\alpha)\right) \cong \left(M_f - \bigsqcup_{\alpha \in P_1 \cup P_2 \cup P_3 \cup P_4} \mathbb{T}_{\eta_M}(\alpha)\right).$$

We conclude with a small remark. The model manifold technology of Minsky [Min10] and Brock, Canary and Minsky [BCM12], provides several tools to locate and measure the length of the geodesic representatives of P_2 and P_3 in $Q(P_1, P_4)$. However, the same technology is not available for handlebodies. This is the place where the difficulties arise.

3. A family of examples

In this section we construct many examples satisfying the I-bundle and handlebody criteria. Later, in Section 6, we will show that this family is *generic* from the point of view of random walks.

We need two ingredients: The first one is a model for a collar of the boundary of a maximally cusped handlebody H or I-bundle Q. Following [HV19], we have that, in certain cases, it is possible to force a H and Q to look exactly like a maximally cusped I-bundle Ω near the boundary ∂H and $\partial_1 Q$ or $\partial_2 Q$. This is roughly the content of Propositions 3.1 and 3.2.

The second ingredient is a family of hyperbolic mapping tori T_ψ on which we want to model the collars Ω. These mapping tori have a distinguished fiber $\Sigma \subset T_\psi$ with a pants decomposition P consisting of extremely short geodesics. The collars Ω will look like a large portion of the infinite cyclic covering of T_ψ. See Theorem 3.3 and its corollaries, in particular Corollary 3.7.

In the end we will be able to detect whether M_f can be described as one of the examples we constructed simply by staring at the geometry of the Teichmüller segment $[o, fo]$ where $o \in \mathcal{T}$ is some base point that we will carefully fix once and for all. This is the content of Proposition 3.9.

3.1. **The geometry of the collars.** We discuss now the first main tool, that is, Propositions 3.1 and 3.2. For the statements we need to introduce some terminology and facts from the deformation theory of geometrically finite structures on handlebodies and I-bundles (for the general deformation theory, we refer to Chapter 7 of [CM04]). We also need a suitable definition of collars for the boundary of such structures which is not just purely topological, but also geometrically significant.

We start by describing the deformation spaces of geometrically finite metrics. Even if we are mainly interested in maximal cusps, we begin with the more flexible class of *convex cocompact* structures.

A convex cocompact hyperbolic metric on a handlebodies H_g or an I-bundle $\Sigma \times [1, 2]$ is a complete hyperbolic metric on the interior, $\text{int}(H_g)$ or $\Sigma \times (1, 2)$, that has a compact subset which is convex in a strong sense. This means that it contains all the geodesics joining two of its points. The minimal such subset is called the *convex core*. It is always a topological submanifold homeomorphic to the ambient manifold (except in the *fuchsian* case which we ignore). Its boundary is parallel to the boundary of the ambient manifold.

The Ahlfors-Bers theory associates to each convex cocompact metric a conformal structure on each boundary component. The deformation spaces of such metrics are parameterized by those conformal structures. Hence, they are identified with the Teichmüller space of the boundary. For each $Y \in \mathcal{T}(\partial H_g)$ and $(X, Y) \in \mathcal{T}(\Sigma \times \{1\}) \times \mathcal{T}(\Sigma \times \{2\})$ there are convex cocompact structures $H(Y)$ on H_g and $Q(X, Y)$ on $\Sigma \times [1, 2]$, unique up to isometries isotopic to the identity, realizing those boundary data. For a comprehensive account see Chapter 7 of [CM04].

Geometrically finite *maximally cusped* hyperbolic structures on H_g or $\Sigma \times [1, 2]$ can be thought as lying on the boundary of the deformation spaces. For every pair of pants P on ∂H_g such that no curve in P is compressible and no two curves in P are isotopic in H_g there exists a unique maximally cusped handlebody $H(P)$ with rank one cusps at P. Similarly, for every pants decomposition $P_1 \cup P_2$ of $\Sigma \times \{1\} \cup \Sigma \times \{2\}$ such that no curve in P_1 is isotopic to a curve in P_2, there exists a unique maximally cusped structure $Q(P_1, P_2)$ on $\Sigma \times [1, 2]$ realizing those cusp data. We refer to Maskit [Mas83] for the existence of such structures and to Keen, Maskit and Series [KMS93] for the uniqueness part and the description of the geometry of their convex cores (as given in Section 2).

By slight abuse of notations, sometimes we will denote both the complete convex cocompact or maximally cusped structure and the corresponding convex core in the same way. However, it will be clear from the context which one we are using.

The internal geometry of the convex cores of geometrically finite I-bundles has a rich structure. It is captured by the combinatorics and geometry of the *curve graph* $\mathcal{C} = \mathcal{C}(\Sigma)$ by the groundbreaking work of Minsky [Min10] and Brock, Canary and Minsky [BCM12] with fundamental contributions by Masur and Minsky [MM99], [MM00].

This is the second piece of deformation theory that we need, it goes under the name of *model manifold technology.* Our use of this technology will not be heavy as we only need a few concepts and consequences, but we mostly hide the relation between the two. We briefly explain what we need.

The starting point is the following: To every convex cocompact structure Q on $\Sigma \times [1, 2]$ we have an associated pair of curve graph invariants P_1 and P_2. They are pants decompositions on $\Sigma \times \{1\}$ and $\Sigma \times \{2\}$ that are the shortest for the conformal structure on the boundary. They might not be uniquely defined, in such case we just pick two. Similarly, for a maximally cusped structure Q we associate to it the cusp data P_1 and P_2. We think of these pants decompositions as subsets of the curve graph $\mathcal{C}$.

Recall now that for every proper essential subsurface $W \subsetneq \Sigma$ which is not a three punctured sphere there is a *subsurface projection*, as defined by Masur and Minsky in [MM00]. It associates to each curve $\alpha \in \mathcal{C}$ the subset $\pi_W(\alpha)$ (possibly empty) of the curve graph $\mathcal{C}(W)$ of all possible essential surgeries of $\alpha \cap W$. The definition is slightly different for annuli. We associate to the curve graph invariants P_1 and P_2 the collection of coefficients

$$\{d_W(P_1, P_2) = \mathrm{diam}_{\mathcal{C}(W)}(\pi_W(P_1) \cup \pi_W(P_2))\}_{W \subsetneq \Sigma}.$$

As established by Minsky [Min10], the pants decompositions P_1 and P_2 together with the list $\{d_W(P_1, P_2)\}_{W \subsetneq \Sigma}$ allow to determine and locate the collection of *short curves* in Q. A special case, which is important for Propositions 3.1 and 3.2, is when the subsurface coefficients are all uniformly bounded. It corresponds to the situation where the only possible very short curves are the geodesic representatives of P_1 and P_2. For each other closed geodesic there is a positive uniform lower bound for the length.

The following notion was introduced by Minsky in [Min01] (see also [BMNS16]).

Definition (Bounded Combinatorics and Height). We say that two pants decompositions P_1, P_2 of Σ have *R-bounded combinatorics* if for every proper subsurface $W \subsetneq \Sigma$ we have $d_W(P_1, P_2) \leq R$. We say that they have *height* at least h if we have $d_{\mathcal{C}}(P_1, P_2) \geq h$.

As for the internal geometry of a geometrically finite handlebody the situation is more complicated as the compressibility of the boundary brings in several issues. We will restrict our attention to the geometry of some *collars* of the the boundary of the convex core.

We still choose for every convex cocompact structure on H_g a curve graph invariant, namely, a pants decomposition P on $\Sigma = \partial H_g$ which is the shortest when measured with the conformal boundary. In a similar way we associate to every maximally cusped structure the cusp data P.

Definition (Disk Set). The *disk set* $\mathcal{D}$ associated to the handlebody H_g is the subset of the curve graph $\mathcal{C}$ of the boundary $\Sigma = \partial H_g$ defined by

$$\mathcal{D} = \{\delta \in \mathcal{C} \mid \delta \text{ compressible in } H_g\}.$$

In order to construct a model for the collar of a geometrically finite handlebody we will have to keep track of how the curve graph invariant P of the geometrically finite structure interacts with the disk set $\mathcal{D}$.

The idea is the following: If P is far away from $\mathcal{D}$ then a large collar of the boundary of the convex core looks like a geometrically finite I-bundle.

We are almost ready for the statements of Propositions 3.1 and 3.2, we only need one last definition, the one of a geometrically controlled collar of the boundary of a geometrically finite structure. For convenient technical simplifications, it will be better for us to work with *quasi collars* (see below for the definition) instead of using directly collars. The reason is that we might wish to allow ourselves to throw away a uniform initial piece from a collar and still call the result a collar.

Let $M = H$ or Q denote the convex core of either a convex cocompact or a maximally cusped structure on either H_g or $\Sigma \times [1, 2]$. Consider

$$M^{\text{nc}} = M - \bigcup_{\alpha \in \text{cusp}(M)} \mathbb{T}_{\eta_M}(\alpha).$$

the *non cuspidal part* of M. As before, $\mathbb{T}_{\eta_M}(\alpha)$ denotes a standard η_M-Margulis neighborhood of the cusp α. We have that M^{nc} is homeomorphic to M. Its boundary ∂M^{nc} is parallel to the boundary of the ambient manifold, that is ∂H_g or $\Sigma \times \{1\} \cup \Sigma \times \{2\}$. Hence, it is naturally identified with it, up to isotopy. In particular, each component of ∂M^{nc} is always naturally identified with Σ.

The definition of quasi collar is analogous to the one of *product region* given in [HV19]: Consider a component Σ_0 of ∂M^{nc} and identify it with $\Sigma_0 \cong \Sigma$ as above.

Definition (Quasi Collar). A *quasi collar* of *size* (D, W, K) of the component $\Sigma_0 \subset \partial M^{\text{nc}}$, denoted by

$$\text{collar}_{D,W,K}(\Sigma_0),$$

is a subset of a topological collar of Σ_0 in M^{nc}, denoted by $\text{collar}(\Sigma_0)$. We require the following additional geometric properties: There exists a parametrization $\text{collar}(\Sigma_0) = \Sigma \times [0, 3]$ such that Σ_0 is identified with $\Sigma \times \{0\}$ and $\text{collar}_{D,W,K}(\Sigma_0)$ corresponds to $\Sigma \times [1, 2]$. Furthermore we have

- The diameter of $\Sigma \times \{1\}$ and $\Sigma \times \{2\}$, measured with the intrinsic metric, is at most D.
- The *width* of $\text{collar}_{D,W,K}(\Sigma_0)$, that is the distance between $\Sigma \times \{1\}$ and $\Sigma \times \{2\}$, is at least W and at most $2W + 2D$.
- The distance of $\Sigma \times \{1\}$ from the distinguished boundary $\Sigma \times \{0\} = \Sigma_0$ is at least K and at most $2K + 2D$.

Notice that each quasi collar $\text{collar}_{D,W,K}(\Sigma_0)$ is *marked* with the isotopy class of an inclusion of Σ. Using this marking we can associate to every homotopy equivalence f between quasi collars a homotopy class $[f] \in \text{Mod}(\Sigma)$.

We are ready to state Propositions 3.1 and 3.2.

Proposition 3.1 (Propositions 4.1 and 6.1 of [HV19]). *For every $R, \varepsilon, \xi > 0$ there exist $D_0 = D_0(R, \varepsilon) > 0$ and $K_0, W_0 > 0$ such that for every $W \geq W_0$ there exists $h = h(\varepsilon, R, \xi, W) > 0$ such that the following holds: Consider $(Y, Z) \in \mathcal{T} \times \mathcal{T}$ and $X \in \mathcal{T}$. Suppose that $X, Y \in \mathcal{T}_\varepsilon$. Let P_X, P_Y and P_Z be short pants decompositions for X, Y and Z respectively. Consider the convex cores of the convex cocompact structures $Q(X, Y)$ and $H(Y), Q(Z, Y)$. Suppose that*

- *P_X, P_Y have R-*bounded combinatorics *and* height *at least h.*
- *In the handlebody case H_g:*

$$d_{\mathcal{C}}(P_Y, \mathcal{D}) \geq d_{\mathcal{C}}(P_Y, P_X) + d_{\mathcal{C}}(P_X, \mathcal{D}) - R.$$

- *In the I-bundle case $\Sigma \times [1, 2]$:*

$$d_{\mathcal{C}}(P_Y, P_Z) \geq d_{\mathcal{C}}(P_Y, P_X) + d_{\mathcal{C}}(P_X, P_Z) - R.$$

Then, there exist $(1 + \xi)$-bilipschitz diffeomorphisms of quasi collars

$$f : \text{collar}_{D_0,W,K_0}(\partial_2 Q(X, Y)) \to \text{collar}_{D_1,W_1,K_1}(\partial H(Y))$$

and

$$f : \text{collar}_{D_0,W,K_0}(\partial_2 Q(X, Y)) \to \text{collar}_{D_1,W_1,K_1}(\partial_2 Q(Z, Y))$$

for some slightly perturbed parameters D_1, W_1, K_1. The diffeomorphisms are in the homotopy class of the identity with respect to the natural markings.

Proposition 3.1 follows from [HV19].

We will use the following mild variation for maximally cusped structures.

Proposition 3.2. *For every R, ξ there exist $D_0 = D_0(R) > 0$ and $W_0, K_0 > 0$ such that for every $W \geq W_0$ there exists $h = h(\xi, R, W) > 0$ such that the following holds: Consider pants decompositions P_Y, P_X and P_Z of Σ. Consider the convex cores of the maximally cusped structures $Q(P_X, P_Y)$ and $H(P_Y), Q(P_Z, P_Y)$. Suppose that*

- *P_X, P_Y have R-*bounded combinatorics *and* height *at least h.*
- *In the handlebody case H_g:*

$$d_{\mathcal{C}}(P_Y, \mathcal{D}) \geq d_{\mathcal{C}}(P_X, P_Y) + d_{\mathcal{C}}(P_X, \mathcal{D}) - R.$$

- *In the I-bundle case* $\Sigma \times [1,2]$*:*

$$d_{\mathcal{C}}(P_Y, P_Z) \geq d_{\mathcal{C}}(P_Y, P_X) + d_{\mathcal{C}}(P_X, P_Z) - R.$$

Then, there exist $(1+\xi)$*-bilipschitz diffeomorphism between quasi collars*

$$f : \mathrm{collar}_{D_0,W,K_0}(\partial_2 Q(P_X, P_Y)) \to \mathrm{collar}_{D_1,W_1,K_1}(\partial H(P_Y))$$

and

$$f : \mathrm{collar}_{D_0,W,K_0}(\partial_2 Q(P_X, P_Y)) \to \mathrm{collar}_{D_1,W_1,K_1}(\partial_2 Q(P_Z, P_Y))$$

for some slightly perturbed parameters D_1, W_1, K_1*. The diffeomorphisms are in the homotopy class of the identity with respect to the natural markings.*

Sketch of proof. Using Theorem 2.2 it is possible to quickly reduce Proposition 3.2 to the previous one. We only sketch the proof. We only treat the handlebody case as the I-bundle case is completely analogous.

First, we approximate P_X, P_Y with hyperbolic surfaces X, Y on which the pair of pants decompositions consist of very short geodesics, say of length contained in the interval $[\varepsilon, 2\varepsilon]$ with ε much smaller than a Margulis constant. Such surfaces are contained in $\mathcal{T}_\varepsilon$.

By results of Canary [Can01] and Otal [Ota03], the collections of geodesic representatives $\Gamma_X \cup \Gamma_Y$ and Γ_Y of the curves $P_X \cup P_Y$ and P_Y in $Q(X,Y)$ and $H(Y)$ have length $O(\varepsilon)$ and are isotopic to $P_X \cup P_Y \subset \partial_1 Q(X,Y) \cup \partial_2 Q(X,Y)$ and $P_Y \subset \partial H(Y)$. Hence, by Theorem 2.2, if ε is small enough, we have $(1+\xi)$-bilipschitz embeddings of the non cuspidal part of the convex core of the maximally cusped structures in the corresponding complete convex cocompact hyperbolic 3-manifolds

$$\phi_Q : \left(Q(P_X, P_Y) - \bigcup_{\alpha \in P_X \cup P_Y} \mathbb{T}_{\eta_M}(\alpha) \right) \to \left(Q(X,Y) - \bigcup_{\alpha \in P_X \cup P_Y} \mathbb{T}_{\eta_M}(\alpha) \right)$$

and

$$\phi_H : \left(H(P_Y) - \bigcup_{\alpha \in P_Y} \mathbb{T}_{\eta_M}(\alpha) \right) \to \left(H(Y) - \bigcup_{\alpha \in P_Y} \mathbb{T}_{\eta_M}(\alpha) \right).$$

Now, if h is large enough only depending on ε, ξ, R, and W, we can apply Proposition 3.1 and find a $(1+\xi)$-bilipschitz diffeomorphism

$$g : \mathrm{collar}_{D_0,W,K}(\partial_2 Q(X,Y)) \to \mathrm{collar}_{D_1,W_1,K_1}(\partial H(Y)).$$

If K, W are sufficiently large, then both quasi collars will be contained in the images of ϕ_Q and ϕ_H. We just compose those with g, that is $f := \phi_H g \phi_Q^{-1}$. □

3.2. **Models for the collars.** As anticipated, we use Proposition 3.2 to construct a very particular class of maximally cusped handlebodies with a simple collar structure. This is our second main ingredient.

Recall that our goal is to construct examples that satisfy the criteria for handlebodies and for I-bundles. Also, recall that these criteria require to control the length and the isotopy class of the collection of geodesic representatives of a pants decomposition of the boundary. The examples we are going to describe are exactly tailored for that goal.

The idea is as follows: We first construct maximally cusped I-bundles Ω for which the length and isotopy class conditions are satisfied almost by definition for many pants decompositions. If a maximally cusped structure on H_g or $\Sigma \times [1,2]$ has a collar that is geometrically very close to Ω, then it also satisfies the criteria.

We now develop the strategy in more details.

The structure of the collar Ω will be modeled on the geometry of a *hyperbolic mapping torus*, or pseudo-Anosov mapping class, *with a short pants decomposition*. These are, respectively, T_ψ and ψ as described below.

Definition (Pseudo-Anosov with a Short Pants Decomposition). A pseudo-Anosov mapping class ψ or a hyperbolic mapping torus T_ψ with a short pants decomposition are the objects obtained from the following procedure.

Let P be a pants decomposition of Σ. Let $\phi \in \mathrm{Mod}(\Sigma)$ be a mapping class such that no curve in P is isotopic to a curve in ϕP. For example, a large power of any pseudo-Anosov suffices. Consider the convex core Q of the maximally cusped structure $Q(P, \phi P)$. The boundary ∂Q consists of totally geodesic hyperbolic three punctured spheres that are paired according to ϕ. We glue them together isometrically as prescribed by the pairing. The glued manifold is a finite volume hyperbolic 3-manifold diffeomorphic to

$$T_\phi - P \times \{0\} = (\Sigma \times [0,1]/(x,0) \sim (\phi x, 1)) - P \times \{0\}.$$

The curves in $P \times \{0\}$ represent rank two cusps. By Thurston's Hyperbolic Dehn Surgery (see Chapter E.6 of [BP92]) we can do Dehn surgery on the cusps such that the resulting manifold still carries a hyperbolic metric for which the core curves of the added solid tori are very short geodesics.

Furthermore, we can restrict ourselves to Dehn fillings for which the filled manifold still fibers over S^1 in a way compatible with the restriction of the fibering of T_ϕ to $T_\phi - P \times \{0\}$. In fact, observe that for each $\alpha \in P$ corresponding to a boundary torus $\mathbb{T}_\alpha$ we have a preferred meridian m_α and longitude l_α coming from the fibering of T_ϕ. If we perform Dehn surgeries with slopes $m_\alpha + k l_\alpha$, the filled manifold will be diffeomorphic to the mapping torus T_ψ where $\psi = \phi \delta_P^k$ and $\delta_P \in \mathrm{Mod}(\Sigma)$ is a Dehn twist about the pants decomposition P.

We call ψ and T_ψ a pseudo-Anosov mapping class and a hyperbolic mapping torus with a short pants decomposition P respectively.

Consider the infinite cyclic covering $\hat{T}_\psi$ of T_ψ. Topologically, we can identify it with $\Sigma \times \mathbb{R}$ where the level sets $\Sigma_n := \Sigma \times \{n\}$ correspond to all the lifts of the fiber $\Sigma \times \{0\} \subset T_\psi$ and in such a way that the curves in

$$\bigcup_{n\in\mathbb{Z}} \psi^n P \times \{n\} \subset \bigcup_{n\in\mathbb{Z}} \Sigma \times \{n\}$$

are very short geodesics. A fundamental domain for the deck group action on $\hat{T}_\psi$ is given by the submanifold $[\Sigma_0, \Sigma_1]$ bounded by Σ_0 and Σ_1. The region $[\Sigma_n, \Sigma_m]$ bounded by Σ_n and Σ_m with $n < m$ is a stack of $m - n$ isometric copies of $[\Sigma_0, \Sigma_1]$.

We now approximate $\hat{T}_\psi$ with a maximally cusped I-bundle $Q(P, \psi^n P)$. Our collars will be of the form $\Omega = Q(P, \psi^m P)$ for some suitably chosen m.

We will use the following from [BD15], see also Figure 3.7 of the same article.

Theorem 3.3 (Theorem 3.5 of [BD15]). *Let ψ be a mapping class with a short pants decomposition P. For every $\xi > 0$ there exist $k > 0$ and $d > 0$ such that for every $n > 0$ sufficiently large the non-cuspidal part of $Q_n = Q(P, \psi^n P)$ admits a decomposition*

$$Q_n^{\text{nc}} = A_n \cup B_n \cup C_n$$

where A_n and C_n have diameter bounded by d while B_n is the image of a $(1 + \xi)$-bilipschitz embedding with a quasi collar image

$$f : [\Sigma_k, \Sigma_{n-k}] \subset \hat{T}_\psi \to Q_n^{\text{nc}}.$$

The embedding f is in the homotopy class of the identity with respect to the natural markings. Moreover, we can parameterize Q_n^{nc} as $\Sigma \times [0, 3]$ in such a way that A_n, B_n and C_n correspond respectively to $\Sigma \times [0, 1], \Sigma \times [1, 2]$ and $\Sigma \times [2, 3]$.

Observe that in the maximally cusped I-bundles $Q(P, \psi^n P)$ we have, by default, many pants decompositions whose length and isotopy class are well controlled.

In order to be able to exploit such control to check I-bundles and handlebody criteria we will need three consequences of Theorem 3.3. For them, we use the following fact proved in the Appendix B.

Lemma 3.4. *For every $\eta < \eta_M/2$ there exists $\xi > 0$ such that the following holds: Let $\mathbb{T}_{\eta_M}(\alpha)$ be a Margulis tube with core geodesic α of length $l(\alpha) \in [\eta, \eta_M/2]$. Suppose that there exists a $(1+\xi)$-bilipschitz embedding of the tube in a hyperbolic 3-manifold $f : \mathbb{T}_{\eta_M}(\alpha) \to M$. Then $f(\alpha)$ is homotopically non-trivial and it is isotopic to its geodesic representative within $f(\mathbb{T}_{\eta_M}(\alpha))$.*

Consider the $(1 + \xi)$-bilipschitz embedding given by Theorem 3.3

$$f : [\Sigma_k, \Sigma_{n-k}] \subset \hat{T}_\psi \to Q_n^{\text{nc}}.$$

Recall that $[\Sigma_k, \Sigma_{n-k}] = \Sigma \times [k, n-k]$ and that the curves $\psi^j P \times \{j\} \subset \Sigma_j = \Sigma \times \{j\}$ are short geodesics in the infinite cyclic covering and have length in the interval $[\eta, \eta_M]$. Denote by Γ_j the collection of geodesic representatives of $f(\psi^j P \times \{j\})$ in Q_n. By Lemma 3.4, if ξ is small compared to η, we get

Corollary 3.5. *The collection* $\Gamma = \Gamma_k \cup \cdots \cup \Gamma_{n-k}$ *is isotopic to*

$$\bigcup_{k<j<n-k} f(\psi^j P \times \{j\}) \subset \bigcup_{k<j<n-k} f(\Sigma_j).$$

via an isotopy supported on $\bigsqcup_{\alpha \in \psi^{k+1}P \cup \cdots \cup \psi^{n-k-1}P} f(\mathbb{T}_{\eta_M}(\alpha))$.

We now locate suitable quasi collars inside $Q(P, \psi^n P)$. First, notice that

$$[\Sigma_k, \Sigma_{n-k}] = \bigcup_{k<j\le n-k} [\Sigma_{j-1}, \Sigma_j]$$

and each $[\Sigma_{j-1}, \Sigma_j]$ is an isometric copy of the fundamental domain $[\Sigma_0, \Sigma_1]$. Each $f[\Sigma_{j-1}, \Sigma_{j+1}] \subset Q$ is a quasi collar for $\partial_1 Q^{\mathrm{nc}}$ for every $k < j < n-k$. We now estimate the quasi collar size (D, W, K).

By Theorem 3.3, we also have that each component of

$$Q^{\mathrm{nc}} - f[\Sigma_k, \Sigma_{n-k}]$$

has diameter bounded by $d = d(\psi, \xi)$. Denote by $w = w(\psi) > 0$ the width of the fundamental domain $[\Sigma_0, \Sigma_1]$. Denote, instead, by $a = a(\psi) > 0$ the intrinsic diameter of the isometric surfaces Σ_j. Notice that, up to replacing ψ with a power (a change that does not seriously affect any of the arguments), we can as well assume that $2a$ is much smaller than w. Since f is $(1+\xi)$-bilipschitz, up to increasing a little and uniformly a and w, those are also the diameter and width parameters for each $f[\Sigma_{j-1}, \Sigma_j]$. We have for $j \ge k$

$$w(j-k) \le d_Q(f(\Sigma_j), \partial_1 Q^{\mathrm{nc}}) \le (w+a)(j-k) + d.$$

Therefore the size of the quasi collar $f[\Sigma_{j-1}, \Sigma_{j+1}]$ can be chosen to be

$$\begin{aligned} D &= a, \\ W &= 2w, \\ K_j &= (w+a)(j-k) + d + 2w. \end{aligned}$$

Analogous estimates hold for $\partial_2 Q^{\mathrm{nc}}$. Hence

Corollary 3.6. *There exists* $w, a > 0$ *and only depending on* ψ *such that for every* $k < j < n-k$ *the surface* $f(\Sigma_j)$ *is contained in*

$$\mathrm{collar}_{a,2w,K_j}(\partial_1 Q)$$

and, similarly, the surface $f(\Sigma_{n-j})$ *is contained in*

$$\mathrm{collar}_{a,2w,K_j}(\partial_2 Q).$$

The Corollaries 3.5 and 3.6 combined with Proposition 3.2 help us in checking that the handlebody and I-bundle criteria are satisfied. In fact, we have the following: With the same notation as before, consider again the $(1+\xi)$-bilipschitz embedding as a quasi collar

$$f : [\Sigma_k, \Sigma_{n-k}] \subset \hat{T}_\psi \to Q_n = Q(P, \psi^n P).$$

The bilipschitz parameter ξ can be chosen to be arbitrarily small provided that n is sufficiently large.

COROLLARY 3.7. *Let ψ be a mapping class with a short pants decomposition P of length η. Consider $Q_n = Q(P, \psi^n P)$. Let Σ_0 be a component of ∂M^{nc} where M is a maximally cusped handlebody or I-bundle. Suppose that we have a $(1+\xi)$-bilipschitz diffeomorphism*

$$g : \mathrm{collar}_{a,2w,K_j}(\partial_1 Q_n) \to \mathrm{collar}_{D,W,K}(\Sigma_0)$$

or

$$g : \mathrm{collar}_{a,2w,K_j}(\partial_2 Q_n) \to \mathrm{collar}_{D,W,K}(\Sigma_0)$$

for some D, W, K. If ξ is small enough (only depending on ψ), n is large enough (only depending on ψ and ξ) and $k < j < n - k$, then the collection of geodesic representatives of

$$gf(\psi^j P \times \{j\}) \subset gf(\Sigma_j)$$

or

$$gf(\psi^{n-j} P \times \{n-j\}) \subset gf(\Sigma_{n-j})$$

has length $O(\eta)$, is contained in the image of g, and is isotopic within it to $gf(\psi^j P \times \{j\})$ or $gf(\psi^{n-j} P \times \{n-j\})$.

Proof. We only treat the first case, the other one is analogous. By Corollary 3.5 and Corollary 3.6, we can assume that the geodesic representatives Γ_j of $\psi^j P$ in Q_n are contained in $\mathrm{collar}_{a,2w,K_j}(\partial_1 Q_n)$ and isotopic within it to $f(\psi^j P \times \{j\}) \subset f(\Sigma_j)$. Their length is $O(\eta)$. Since g is a $(1+\xi)$-bilipschitz diffeomorphism, if ξ is small enough compared to η, by Lemma 3.4, we can assume that the geodesic representatives of $g(\Gamma_j)$ in M are contained in $\mathrm{collar}_{D,W,K}(\partial H)$ and isotopic within it to $g(\Gamma_j)$ which, in turn, is isotopic to $gf(\psi^j P \times \{j\}) \subset gf(\Sigma_j)$. □

3.3. **Criteria for I-bundles and handlebodies revised.** We are now ready to give a more manageable version of the criteria for I-bundles and handlebodies and construct many example that satisfy those conditions. This is the goal of Proposition 3.8.

PROPOSITION 3.8. *Let ψ, ϕ be mapping classes with short pants decompositions P, P' of length in $[\eta, \eta_M/2]$. There exists $j = j(\psi, \phi)$ such that the following holds: Consider*

$$(P_1, P_2) = (\psi^{-j} P, P) \text{ and } (P_3, P_4) = (P', \phi^j P').$$

Suppose that for some n very large we have respectively

(1) $d_{\mathcal{C}}(P, \mathcal{D}) \geq d_{\mathcal{C}}(P, \psi^{-n} P) + d_{\mathcal{C}}(\psi^{-n} P, \mathcal{D}) - R$,
(2) $d_{\mathcal{C}}(\psi^{-j} P, \phi^j P') \geq d_{\mathcal{C}}(\psi^{-j} P, \psi^n P) + d_{\mathcal{C}}(\psi^n P, \phi^j P') - R$,

and

(3) $d_{\mathcal{C}}(P', f\mathcal{D}) \geq d_{\mathcal{C}}(P', \phi^n P') + d_{\mathcal{C}}(\phi^n P', f\mathcal{D}) - R$,
(4) $d_{\mathcal{C}}(\psi^{-j} P, \phi^j P') \geq d_{\mathcal{C}}(\phi^j P', \phi^{-n} P') + d_{\mathcal{C}}(\phi^{-n} P', \psi^{-j} P) - R$.

Then P_1, P_2, P_3, P_4 *satisfies the* I-bundle *and* handlebody criteria *with parameter* $O(\eta)$.

Proof. We have to check two handlebody and one I-bundle criteria. The arguments for the three different cases follow the same lines. In order to avoid repetitions, we only prove in details that there exists $j = j(\phi, \psi)$ such that the pair $(\psi^{-j}P, P)$ satisfies the handlebody criterion if n is large enough. The other cases are completely analogous and require no new ideas. In the end of the proof, we briefly discuss the adjustments needed for the I-bundle criterion.

The handlebody criterion. In order to check the handlebody criterion for $(\psi^{-j}P, P)$, by Corollary 3.7, we just need to get a $(1+\xi)$-bilipschitz diffeomorphism

$$g : \text{collar}_{a,2w,K_j}(\partial_2 Q(\psi^{-n}P, P)) \to \text{collar}_{D,W,K}(\partial H(P))$$

in the homotopy class of the identity. Such a diffeomorphism will be provided by Proposition 3.2. Notice at this point that $Q(\psi^{-n}P, P)$ and $Q(P, \psi^n P)$ are isometric as they only differ by the marking.

In order to apply Proposition 3.2, observe that, by work of Minsky [Min01], the pairs $(\psi^n P, \psi^m P)$ and $(\phi^m P', \phi^n P')$ satisfy for all $n,\ m \in \mathbb{Z}$ the R-bounded combinatorics condition for some $R = R(\psi, \phi) > 0$. Furthermore, for any fixed h, if $|n - m|$ is large, again, depending only on ψ, ϕ and h, they also satisfy the large height assumption as pseudo-Anosov elements act as loxodromic motions on the curve graph by Masur and Minsky [MM99]. Property (1) from our assumptions is exactly the last one needed to guarantee that Proposition 3.2 can be applied.

Before applying Proposition 3.2 we have to be a bit careful with the various constants and their dependence. We pause for a moment and discuss this delicate point. The mapping classes ψ and ϕ determine η and R and also the parameters a and w of Corollary 3.7 and D_0 of Proposition 3.2. Furthermore, the mapping classes together with the choice of ξ determine k and d in Proposition 3.3. In turn, k determines the allowable range $k < j < n - k$.

So, we want to choose ξ much smaller than the one, only depending on the mapping classes, required by Corollary 3.7 to hold. Once this is fixed we have a collection of potential candidates for the quasi collars

$$\text{collar}_{a,2w,K_j}(\partial_2 Q(\psi^{-n}P, P))$$

with $k < j < n - k$ for any n very large.

Once we fixed ξ, we have also fixed $K_0, W_0 > 0$ of Proposition 3.2. So, for every $W \geq W_0$ we have a $(1+\xi)$-bilipschitz diffeomorphism of quasi collars

$$f : \text{collar}_{D_0,W,K_0}(\partial_2 Q(\psi^{-n}P, P)) \to \text{collar}_{D_1,W_1,K_1}(\partial H(P)).$$

In order to get the desired embeddings, we just have to choose $k < j < n-k$ and W such that one of our candidate quasi-collars is contained in the domain of definition of f

$$\text{collar}_{a,2w,K_j}(\partial_2 Q(\psi^{-n}P,P)) \subset \text{collar}_{D_0,W,K_0}(\partial_2 Q(\psi^{-n}P,P)).$$

It suffices to do the following: We first choose j such that $K_j - a > K_0 + D_0$. This determines a minimal $j = j(\psi,\phi)$ as required by the statement of Proposition 3.8. Then, we choose W such that $K_0 + W - D_0 > 2K_j + 4a + 2w$. This finally determines a final threshold for h and n.

The I-bundle criterion. The proof is word by word the same as in the handlebody case, one only has to replace the collar of $\partial H(P)$ with $\partial_1 Q(\psi^{-j}P, \phi^j P')$ and $\partial_2 Q(\psi^{-j}P, \phi^j P')$. Again, if n is very large, Proposition 3.2 furnishes $(1+\xi)$-bilipschitz diffeomorphisms

$$\text{collar}_{a,2w,K_j}(\partial_1 Q(\psi^{-j}P, \psi^{n-j}P)) \to \text{collar}_{D_1,W_1,K_1}(\partial_1 Q(\psi^{-j}P, \phi^j P'))$$

and

$$\text{collar}_{a,2w,K_j}(\partial_2 Q(\phi^{j-n}P, \phi^j P)) \to \text{collar}_{D_1,W_1,K_1}(\partial_2 Q(\psi^{-j}P, \phi^j P')).$$

Notice that the constant $j = j(\phi,\psi)$ is the same as before. Corollary 3.7 together with a careful bookkeeping of the markings concludes the proof. □

3.4. **From the curve graph to Teichmüller space.** We now translate the curve graph conditions (1) - (4) in terms of Teichmüller geometry in such a way that it will not be hard to check them for a random segment $[o, fo]$.

It is convenient to recall now a few facts due to Masur and Minsky [MM99], [MM00] about the relation between Teichmüller space $\mathcal{T}$ endowed with the Teichmüller metric $d_{\mathcal{T}}$ and the curve graph $\mathcal{C}$.

The connection is established via the *shortest curves projection* $\Upsilon : \mathcal{T} \to \mathcal{C}$, a coarsely defined map that associates to every marked hyperbolic surface $X \in \mathcal{T}$ a shortest geodesic pants decomposition on it $\Upsilon(X)$. By classical work of Bers (see e.g. [Par14] and references therein) there is a uniform upper bound, only depending on the topology of Σ, on the length of such pants decomposition. We choose $o \in \mathcal{T}$ with the following property

> **Standing assumption**: The base point $o \in \mathcal{T}$ is chosen such that its projection to the curve graph lies on the disk set of the handlebody $\Upsilon(o) \in \mathcal{D}$.

From a geometric point of view, Masur and Minsky proved that the curve graph $\mathcal{C}$ is a Gromov hyperbolic space (see [MM99]) and the disk set $\mathcal{D} \subset \mathcal{C}$ is a uniformly quasi-convex subspace (see [MM04]).

It also follows from [MM99] that Υ is Lipschitz and sends Teichmüller geodesics to uniform *unparameterized* quasi-geodesics. The latter means that there is a constant B only depending on Σ such that $d_{\mathcal{C}}(\Upsilon(Y),\Upsilon(Z)) \geq d_{\mathcal{C}}(\Upsilon(Y),\Upsilon(X)) + d_{\mathcal{C}}(\Upsilon(X),\Upsilon(Z)) - B$ for every $Z < X < Y$ aligned on a Teichmüller geodesic. In particular, by hyperbolicity of $\mathcal{C}$, the image $\Upsilon[Z,Y] \subset \mathcal{C}$ is a uniformly quasi-convex subset.

We have the following:

PROPOSITION 3.9. *Let ψ, ϕ be pseudo-Anosov mapping classes with short pants decompositions P, P'. Let $l_\psi, l_\phi : \mathbb{R} \to \mathcal{T}$ be their Teichmüller geodesics. For every $\delta > 0$ there exists $h > 0$ such that the following holds: Suppose that on the segment $[o, fo]$ there are four points $o < S_1 < S_2 < S_3 < S_4 < fo$ with the following properties*

(i) *$[S_1, S_2]$ and $[S_3, S_4]$ have length at least h and δ-fellow travel l_ψ and l_ϕ respectively.*
(ii) *We have $d_{\mathcal{C}}(\Upsilon[S_1, S_4], \mathcal{D}) \geq h$ and $d_{\mathcal{C}}(\Upsilon[S_1, S_4], f\mathcal{D}) \geq h$.*

Then, up to perhaps replacing P, P' with $\psi^r P, \phi^{r'} P'$, (1) - (4) hold.

The proof uses the arguments from Proposition 7.1 of [HV19].

Proof. We have to show that (i) and (ii) imply (1) - (4).

We start with Properties (1) and (3). Let us only consider (1) as (3) is completely analogous. Recall that we fixed o such that $\Upsilon(o) \in \mathcal{D}$ and hence $\Upsilon(fo) = f\Upsilon(o) \in f\mathcal{D}$.

As $\Upsilon[o, fo]$ is a uniformly quasi-convex subset of the Gromov hyperbolic space $\mathcal{C}$, there is a coarsely well defined nearest point projection $\pi : \mathcal{C} \to \Upsilon[o, fo]$. Since $\mathcal{D}, f\mathcal{D}$ are also uniformly quasi-convex and $\Upsilon[S_1, S_4]$ is very far from both while the endpoints satisfy $\Upsilon(o) \in \mathcal{D}$ and $f\Upsilon(o) \in f\mathcal{D}$, we conclude that $\pi(\mathcal{D})$ and $\pi(f\mathcal{D})$ lie on opposite sides of $\Upsilon[S_1, S_4]$ and are far from it.

Consider $S_1 \leq a < b \leq S_4$ and the projections $\alpha := \Upsilon(a)$ and $\beta := \Upsilon(b)$. By hyperbolicity of $\mathcal{C}$ and uniform quasi-convexity of $\Upsilon[o, fo]$, any geodesic joining $\delta_0 \in \mathcal{D}$ to β can be broken into two subsegments $[\delta_0, \beta_0] \cup [\beta_0, \beta]$ where β_0 is uniformly close to $\pi(\delta_0)$, the nearest point projection of δ_0, which has the form $\Upsilon(t)$ for some $o < t < S_1$. By the uniform unparameterized quasi-geodesic image property of Υ, the segment $[\beta_0, \beta]$ passes uniformly close to α. Therefore, we have

$$d_{\mathcal{C}}(\beta, \delta_0) \geq d_{\mathcal{C}}(\beta, \alpha) + d_{\mathcal{C}}(\alpha, \delta_0) - R_0$$

for some uniform R_0. Taking the minimum over all $\delta_0 \in \mathcal{D}$ we get

$$d_{\mathcal{C}}(\Upsilon(b), \mathcal{D}) \geq d_{\mathcal{C}}(\Upsilon(a), \Upsilon(b)) + d_{\mathcal{C}}(\Upsilon(a), \mathcal{D}) - R_0.$$

The last ingredient that we need is the fact that the sequence of curves $\{\psi^n P\}_{n\in\mathbb{Z}}$ lie uniformly close, only depending on ψ, to the uniform quasi-axis of ψ given by the composition Υl_ψ. This follows from work of Minsky [Min01]. Notice that Υl_ψ lies uniformly close to $\Upsilon[S_1, S_2]$ by fellow traveling assumption. In particular, there are $\psi^{r+n}P$, $\psi^r P$ and $\psi^{r-n}P$ that lie uniformly close, only depending on ψ, to $\Upsilon(S_1)$, $\Upsilon(S_\psi)$ and $\Upsilon(S_2)$ where $S_1 < S_\psi < S_2$. The difference $(r+n) - (r-n) = 2n$ is bounded from below by some linear function of h of the form $\kappa h - \kappa$ with $\kappa > 0$ only depending on ψ.

Therefore, since $S_1 < S_\psi$, we have

$$d_{\mathcal{C}}(\psi^r P, \mathcal{D}) \geq d_{\mathcal{C}}(\psi^r P, \psi^{r-n} P) + d_{\mathcal{C}}(\psi^{r-n} P, \mathcal{D}) - R$$

for some uniform R, only depending on ψ.

For simplicity, we replace P with $\psi^r P$ and still denote it by P.

We now move on to Properties (2) and (4). Again, we consider only (2) as (4) uses the same arguments. Property (2) just follows from the uniform unparameterized quasi-geodesic property of $\Upsilon[S_1, S_4]$.

In more details we proceed as follows: As before, up to replacing P and P' with $\psi^r P$ and $\phi^{r'} P'$, we can assume that $\psi^{-n} P, \psi^n P$ (resp. $\phi^{-n'} P', \phi^{n'} P'$) are uniformly close, only depending on ψ (resp. ϕ), to $\Upsilon(S_1), \Upsilon(S_2)$ (resp. $\Upsilon(S_3), \Upsilon(S_4)$).

Recall that P_1 and P_4 are of the form $P_1 = \psi^{-j} P$ and $P_4 = \phi^j P'$ for some uniform $j = j(\phi, \psi)$. So, up to a uniform error, we can replace them with P and P'. For them we have

$$d_{\mathcal{C}}(\Upsilon(S_\psi), \Upsilon(S_\phi)) \geq d_{\mathcal{C}}(\Upsilon(S_\psi), \Upsilon(S_2)) + d_{\mathcal{C}}(\Upsilon(S_2), \Upsilon(S_\phi)) - R.$$

□

4. Short curves on Heegaard splittings

In this and the next section we move to the second construction and discuss short curves on Heegaard splittings. The goal is to prove Theorem 3. The family of examples that arise from Theorem 3 is shown to be generic from the point of view of the random walk in Section 6.

4.1. **Outline.**

We now outline the strategy to produce a hyperbolic metric on M_f for which $\gamma \subset \Sigma$ is a short geodesic.

We start by associating to γ the 3-manifold

$$M_f - \gamma = (H_g - \gamma) \cup_f (H_g - f(\gamma)).$$

The Heegaard splitting M_f is obtained from $M_f - \gamma$ by Dehn filling along a filling slope which is completely determined by the topology.

According to Thurston's Hyperbolization Theorem, the manifold $M_f - \gamma$ admits a complete hyperbolic metric provided that it is *irreducible*, *atoroidal* and *Haken*; see, for example, Theorem 1.42 in [Kap09]. We take [Kap09] as our general reference for basic 3-manifold topology. Notice, however, that, since it is the interior of a compact manifold with non-empty boundary, by basic 3-manifold topology, if $M_f - \gamma$ is irreducible, then it will also be automatically Haken (see Corollary 1.24 in [Kap09]).

Both irreducibility and atoroidality will follow from our assumption that both (H_g, γ) and $(H_g, f(\gamma))$ are *pared acylindrical handlebodies*. This is condition (a) of Theorem 3 and is borrowed from the theory of Jaco, Shalen and Johannson (see Chapter 5 of [CM04]).

Once we have a hyperbolic structure on $M_f - \gamma$ we deform it to a hyperbolic metric on M_f via a cone manifold deformation. Again, as in Section 2, the tool for such an operation is Hodgson and Kerckhoff effective version of the Hyperbolic Dehn Surgery Theorem (see Theorem 2.1). We need to certify that the canonical filling slope has large normalized length on a standard torus horosection $T := \partial\mathbb{T}_{\eta_M}(\gamma)$ of the cusp of $M_f - \gamma$.

Actually, we will check something stronger, namely that, if condition (b) of Theorem 3 holds, then the flat torus T itself will be long and skinny so that *every* filling slope different from the one corresponding to $(\Sigma - \gamma) \cap T$ will have large normalized length.

The central point of the proof is to show that T is long and skinny provided that the subsurface projection $d_{\Sigma-\gamma}(\mathcal{D}, f\mathcal{D})$ is large. The main ingredient for the argument is the model manifold technology by Minsky [Min10] and Brock, Canary and Minsky [BCM12].

At this point, before going on, we need a further consequence of the condition (a): If (H_g, γ) is pared acylindrical, then the inclusion $\Sigma - \gamma \subset H_g$ is *doubly incompressible*. Double incompressibility allows to use Thurston's Uniform Injectivity for Pleated Surfaces [Thu86a] and its consequences. In particular, we show that there are two simple closed curves $\alpha \subset \Sigma - \gamma$ and $\beta \subset \Sigma - f(\gamma)$ which are represented by geodesics in $M_f - \gamma$ with moderate length and which are combinatorially close to the disk set projections as follows: $d_{\Sigma-\gamma}(\mathcal{D}, \alpha) \leq 2$ and $d_{\Sigma-\gamma}(f\mathcal{D}, \beta) \leq 2$.

Notice that, since $d_{\Sigma-\gamma}(\mathcal{D}, f\mathcal{D})$ is large, also $d_{\Sigma-\gamma}(\alpha, \beta)$ will be large.

In order to produce the moderate length curves α and β it is convenient to work with the geometrically finite coverings of $M_f - \gamma$ corresponding to $\pi_1(H_g - \gamma)$ and $\pi_1(H_g - f(\gamma))$. Such coverings are hyperbolic structures on $H_g - \gamma$ and $H_g - f(\gamma)$ with rank one cusps at γ and $f(\gamma)$ respectively.

As a last step, incompressibility of $\Sigma - \gamma \subset M_f - \gamma$ and the fact that the curves $\alpha, \beta \subset \Sigma - \gamma$ are represented by moderate length curves in $M_f - \gamma$ and are combinatorially far apart in the curve graph $\mathcal{C}(\Sigma - \gamma)$ implies, via the model manifold technology, that $T = \partial\mathbb{T}_{\eta_M}(\gamma)$ is long and skinny.

This concludes the outline of the strategy.

The remainder of this section is dedicated to the topological part of the proof, the geometric part of the argument, instead, will be discussed in the next section.

4.2. **Pared acylindrical handlebodies.** We start by introducing some basic topological objects, namely a simple collection of *pared acylindrical handlebodies*. Such objects occur naturally in the study of cusped hyperbolic structures on H_g (see Thurston [Thu86a], [Thu86c])

Definition (Pared Acylindrical). Let $\gamma \subset \Sigma$ be a simple closed curve with a regular neighborhood $N(\gamma) \subset \Sigma$. We say that (H_g, γ) is a *pared handlebody* if the following conditions hold

(1) The inclusion $\Sigma - \gamma \subset H_g$ is π_1-injective.
(2) Every essential map $(A, \partial A) \to (H_g, N(\gamma))$ is homotopic as a map of pairs into $N(\gamma)$.

We say that (H_g, γ) is also *acylindrical* if furthermore

(3) Every essential map $(A, \partial A) \to (H_g, \Sigma - N(\gamma))$ is homotopic as a map of pairs into Σ.

From a curve graph point of view, we have the following useful criterion

LEMMA 4.1. *Let $\gamma \subset \Sigma$ be an essential simple closed curve. We have*

(i) *If $d_{\mathcal{C}}(\gamma, \mathcal{D}) \geq 2$ then (H_g, γ) is pared.*
(ii) *If $d_{\mathcal{C}}(\gamma, \mathcal{D}) \geq 3$ then (H_g, γ) is pared acylindrical.*

Proof. Let us prove (i). By Dehn's Lemma, if $\Sigma - \gamma$ is not π_1-injective, then $\Sigma - \gamma$ admits a compression which implies $d_{\mathcal{C}}(\gamma, \mathcal{D}) \leq 1$. Therefore, if $d_{\mathcal{C}}(\gamma, \mathcal{D}) \geq 2$, the inclusion $\Sigma - \gamma \subset H_g$ satisfies property (1).

As for property (2), we proceed as follows: Consider an essential annulus $f : (A, \partial A) \to (H_g, N(\gamma))$ which cannot be properly homotoped into $N(\gamma)$. The boundary curves $\alpha_j = f(\partial_j A)$, for $j = 1, 2$, are homotopic to a power of the simple core curve $\gamma \subset N(\gamma)$, say $\alpha_j \simeq \gamma^{n_j}$ with $n_j \neq 0$ because f is essential. We freely homotope f such that $\alpha_j = \gamma^{n_j}$ and a simple arc $\eta \subset A$ joining the two boundary components of ∂A maps to a loop $\beta = f(\eta)$ in H_g.

By assumption α_1 and α_2 are homotopic in H_g. At the level of fundamental groups, we have $\gamma^{n_1} = \beta\gamma^{n_2}\beta^{-1}$ in $\pi_1(H_g)$. Since $\pi_1(H_g) = \mathbb{F}_g$ is a free group, necessarily $\beta = \gamma^k$ for some $k \in \mathbb{Z}$. Hence we can homotope f such that it maps a regular neighborhood U of $\eta \cup \partial A$ to $N(\gamma)$. The complement of U in A is a disk with boundary δ mapped to $N(\gamma)$. Since $N(\gamma)$ maps π_1-injectively to H_g, the loop $f(\delta)$ is also homotopically trivial in $N(\gamma)$. Since H_g is aspherical we can homotope f restricted to the complement of U to a nullhomotopy of $f(\delta)$ in $N(\gamma)$. Hence $\Sigma - \gamma \subset H_g$ satisfies (2).

We now prove (ii). We need to check property (3). Consider an essential annulus $f : (A, \partial A) \to (H_g, \Sigma - \gamma)$ which cannot be properly homotoped into $\Sigma - \gamma$. By the Annulus Theorem, we can assume that f is an embedding. We conclude using the following.

Claim. $H_g - \gamma$ does not contain any properly embedded essential annulus $(A, \partial A) \subset (H_g, \Sigma - \gamma)$.

Proof of the claim. Since handlebodies do not contain incompressible and ∂-incompressible surfaces, the annulus A admits a boundary compression. This means that we find an embedded disk $D^2 \subset H_g$ whose boundary is divided into two segments $\partial D^2 = \alpha \cup \beta$ with $\alpha \subset A$ and $\beta \subset \Sigma$, both joining the two components of the boundary ∂A. The boundary δ of a tubular neighborhood of $\partial A \cup \beta$ is a disk bounding curve. By construction it has distance at most 2 from γ. This concludes the proof. □

4.3. **Hyperbolizable drilled Heegaard splittings.** Gluing together two pared acylindrical handlebodies produces an irreducible and atoroidal 3-manifold. We give a proof of this fact

Proposition 4.2. *If (H_g, γ) and $(H_g, f(\gamma))$ are both pared acylindrical then $M_f - \gamma$ is irreducible, atoroidal and Haken.*

In particular, by Thurston's Hyperbolization Theorem for Haken manifolds, the drilled splitting $M_f - \gamma$ admits a hyperbolic structure, which is also unique by Mostow-Prasaad rigidity.

We split the proof of Proposition 4.2 into small steps, Lemmas 4.3, 4.4 and 4.5. First we observe that $\Sigma - \gamma \subset M_f - \gamma$ is incompressible.

Lemma 4.3. *The surface $\Sigma - \gamma \subset M_f - \gamma$ is π_1-injective. The fundamental group $\pi_1(M_f - \gamma)$ decomposes as $\pi_1(H_g - \gamma) *_{\pi_1(\Sigma-\gamma)} \pi_1(H_g - f(\gamma))$.*

Proof. This is a consequence of Dehn's Lemma and Seifert–van Kampen's Theorem. □

We now use transversality arguments together with acylindricity of the handlebodies to establish irreducibility and atoroidality of $M_f - \gamma$.

Lemma 4.4. *$M_f - \gamma$ is irreducible.*

Proof. Assume towards a contradiction that $M_f - \gamma$ is reducible, and let S be an embedded 2-sphere that does not bound a ball. We may and do assume that S intersects $\Sigma - \gamma$ transversally and such that any component of $(\Sigma - \gamma) \cap S$ that is innermost in S is not homotopically trivial in $\Sigma - \gamma$. Indeed, otherwise this innermost component bounds a disc in $\Sigma - \gamma$ and a disk in S such that their union is contained in one of the two handlebodies. However, this sphere bounds a ball B in that handlebody by irreducibility of handlebodies, and thus B can be used to guide an isotopy of S that removes that component of $(\Sigma - \gamma) \cap S$. Note that $\Sigma \cap S$ is nonempty since otherwise S would be contained in one handlebody and bound a ball, again by irreducibility of handlebodies. Hence, any component α of $\Sigma \cap S$ that is innermost in S compresses in one of the two handlebodies via the disc D in S with $D \cap \Sigma = \alpha$. This contradicts π_1-injectivity of $\Sigma - \gamma$ into both handlebodies. □

Lemma 4.5. *$M_f - \gamma$ is atoroidal.*

Proof. We show that $M_f - \gamma$ is topologically atoroidal (i.e. every incompressible torus is boundary parallel), which implies atoroidal for Haken manifolds (which $M_f - \gamma$ is); see Section 1.2 in [Kap09]. Let T be an incompressible torus in $M_f - \gamma$. We show that T is boundary parallel. Arrange that T is in general position with respect to $\Sigma - \gamma$, and take $\{\alpha_i\}_{i=1,\dots,n}$ to be the simple closed curves that are the components of the intersection $(\Sigma - \gamma) \cap T$. By an innermost argument, if some α_i was null-homotopic in T, then some α_j would bound a disk in one of the handlebodies H bounded by Σ. However, this implies that α_j is trivial in $\Sigma - \gamma$ since $\Sigma - \gamma$ maps π_1-injectively to both handlebodies. Then, the union of the disks in S_K and T with boundary α_j bounds a ball in H by irreducibility of H, and we can use this ball to reduce the number of components of $(\Sigma - \gamma) \cap T$. In view of this argument, we can assume that each α_i is an essential curve on T. Moreover, notice that each α_i is then automatically an essential curve on $\Sigma - \gamma$, whence on Σ, too, for otherwise some α_j would bound a compressing disk. We reindex the α_i to make sure that consecutive ones (modulo n) bound an annulus in T.

Consider now some α_i, and let H be the handlebody containing the annulus $A \subseteq T$ bounded by α_i and α_{i+1} (notice that if there is one α_i, then in fact there are at least two because $\Sigma - \gamma$ separates $M_f - \gamma$, so that $(\Sigma - \gamma) \cap T$ needs to have at least two connected components).

Claim: A is boundary parallel.

Proof of the Claim. Using that (H, γ) is pared acylindrical, we note that α_i and α_{i+1} are two boundaries of an embedded annulus A' in Σ that is homotopic rel boundary to A in H. Since $\partial H \setminus (\partial A)$ is disconnected, also $H \setminus A$ is disconnected. Let N be the 3-manifold given as the closure of the component with boundary $A \cup A'$. To establish that A is boundary parallel, we show that N is a solid torus.

Let $I \subset A$ be a properly embedded interval connecting the two boundary components of A. And let $I' \subset A'$ be an arc that is homotopic rel boundary to I in H. An innermost circle argument gives that $I' \cup I$ is also null-homotopic in N, hence by Dehn's Lemma there exists a disc $D \subset N$ with boundary $I' \cup I$. Let S be the sphere obtained as the union of the disc given by boundary compression of A along D and the disc in Σ with the same boundary. This sphere S bounds a ball in H, and thus N, by irreducibility of handlebodies. Thus N compresses to a ball; hence, N is a solid torus as desired. □

Let A' be the annulus in Σ that union A forms a torus that bounds a solid torus in H. If A' does not contain γ, then we can reduce the number of components of $(\Sigma - \gamma) \cap T$ by isotoping A in $M_f - \gamma$ to the other handlebody. Otherwise, A' is (up to isotopy in Σ) $N(\gamma)$, and we can apply an isotopy in M_f to move A inside a regular neighborhood N of γ in M_f. Notice that there is at least one γ_i for otherwise T would be contained in one of the handlebodies; however, handlebodies do not contain incompressible tori since incompressible tori are π_1-injective and the fundamental groups of handlebodies do not contain $\mathbb{Z}^2$ subgroups. In particular, T is a union of annuli as above, and hence, after applying finitely many isotopies, we reduce to the case that every annulus as T is entirely contained in N (we can assume that the isotopies we found above move N inside itself). Hence, T can be thought of as an incompressible surface in $N - \gamma$. There is a classification of incompressible surfaces in S^1–bundles; see [Wal67, Satz 2.8], which in our case implies that T is boundary parallel, as required. □

Hence $M_f - \gamma$ is irreducible and atoroidal. As mentioned before, irreducibility is already enough to ensure that it is also Haken. However, we observe that also $\Sigma - \gamma \subset M_f - \gamma$ would work as Haken surface.

Combining with Lemma 4.1, we get

COROLLARY 4.6. *Let $\gamma \subset \Sigma$ be a non-separating simple closed curve. If $d_{\mathcal{C}}(\gamma, \mathcal{D} \cup f\mathcal{D}) \geq 3$, then $M_f - \gamma$ is hyperbolizable.*

4.4. **Double incompressibility.** The second crucial topological property of a pared acylindrical handlebody (H_g, γ) that we need is the fact that the inclusion of the boundary $\Sigma - \gamma \subset H_g$ is *doubly incompressible.* The following definition is due to Thurston [Thu86a].

DEFINITION (Doubly Incompressible). Let $\gamma \subset \Sigma$ be an essential simple closed curve with a tubular neighborhood $N(\gamma)$ in Σ and $U(\gamma)$ in H_g. The inclusion $\Sigma - \gamma \subset H_g$ is *doubly incompressible* if it satisfies

(a) The inclusion $\Sigma - \gamma \subset H_g$ is π_1-injective.
(b) Essential relative homotopy classes of maps $(I, \partial I) \to (\Sigma - N(\gamma), \partial N(\gamma))$ are mapped injectively to relative homotopy classes of maps $(I, \partial I) \to (H_g - U(\gamma), \partial U(\gamma))$.
(c) There are no essential cylinders in $\Sigma - N(\gamma)$: This means that every essential map $f : (A, \partial A) \to (H_g, \Sigma - N(\gamma))$ is either homotopic into $U(\gamma)$ or the restriction of f to ∂A extends to a map into $\Sigma - N(\gamma)$.
(d) Each maximal abelian subgroup of $\pi_1(\Sigma - \gamma)$ is mapped to a maximal abelian subgroup of $\pi_1(H_g)$.

Since maximal abelian subgroups of a free group are infinite cyclic group, the last condition is equivalent to

(e) each maximal cyclic subgroup of $\pi_1(\Sigma - \gamma)$ is mapped to a maximal cyclic subgroup of $\pi_1(H_g)$.

As Thurston observes (see Section 7 of [Thu86a]) we have the following

PROPOSITION 4.7. *If (H_g, γ) is pared acylindrical, then the inclusion $\Sigma - \gamma \subset H_g$ is doubly incompressible.*

The proposition is probably well-known to experts and follows from JSJ theory. However, it might not be easy to extract from the literature. For this reason, and for the sake of being more self-contained, we include a proof in Appendix A.

5. LONG SKINNY CUSP HOROSECTION

In this section we show that a standard torus horosection T of the cusp of $M_f - \gamma$ is *long* and *skinny*, so that *every* filling slope different from the one coming from $(\Sigma - \gamma) \cap T$ will have large normalized length, provided that $d_W(\mathcal{D}, f\mathcal{D})$ is sufficiently large. This is the geometric part of the proof of Theorem 3 and rests on the model manifold technology of Minsky [Min10] and Brock, Canary and Minsky [BCM12].

Here, the standard torus horosection T is $\partial\mathbb{T}_{\eta_M}(\gamma)$, where $\eta_M > 0$ is a fixed Margulis constant and $\mathbb{T}_{\eta_M} \subset M_f - \gamma$ is the cusp of $M_f - \gamma$ that forms a connected component of the η_M-thin part of $M_f - \gamma$.

The proof is divided into two steps. The first one consists of finding simple closed curves $\alpha, \beta \subset W := \Sigma - \gamma$ that are represented by closed geodesics in $M_f - \gamma$ with moderate length and such that $d_W(\alpha, \mathcal{D})$, $d_W(\beta, f\mathcal{D}) \leq 2$. This is the content of Proposition 5.2 and Corollary 5.3. As a second step, once we have such curves α and β, we argue that $d_W(\alpha, \beta)$ gives a coarse lower bound for the length of any slope on T that does not come from the Heegaard surface. We prove this in Proposition 5.7.

5.1. **Handlebody covering.** In order to find the curves α and β, we work with *handlebody coverings* as we now describe.

Consider first the pared handlebody (H_g, γ). The fundamental group of $H_g - \gamma \subset M_f - \gamma$ injects into $\pi_1(M_f - \gamma)$ (see Lemma 4.3) and determines a covering N of $M_f - \gamma$ to which $H_g - \gamma$ lifts homeomorphically. By slight abuse of notation we will not distinguish between $H_g - \gamma$ and its homeomorphic lift to N. Recall from the outline, that we want to find a simple closed curve $\alpha \subset \Sigma - \gamma$ that has moderate length representative in N and satisfies $d_W(\alpha, \mathcal{D}) \leq 2$.

As both $H_g - \gamma$ and N are aspherical, the inclusion $H_g - \gamma \subset N$ is a homotopy equivalence. Since the pair $(N, H_g - \gamma)$ has also the homotopy extension property (e.g. it can be give the structure of a CW-pair), it follows that the manifold N deformation retracts to $H_g - \gamma$ (this is a general fact; see, for example, [Hat02, Corollary 0.20]). Therefore, according to Proposition 4.7, the inclusion $W = \Sigma - \gamma \subset N$ is doubly incompressible.

Incidentally, even though we will not need it, for the sake of clarity, we describe a more complete picture of the covering. The manifold N is a geometrically finite structure on H_g with a rank one cusp at γ and the submanifold $H_g - \gamma$ is a *relative Scott core* for N [Sco73, McC86, KS89]. The fact that N is homeomorphic to the interior of H_g follows from Bonahon's Tameness Theorem [Bon86]. Geometric finiteness is, instead, a consequence of Canary and Thurston's Covering Theorem [Can96].

We now come back to double incompressibility. Crucially, it allows to use Thurston's Uniform Injectivity for Pleated Surfaces [Thu86a].

Theorem 5.1 (Uniform Injectivity, [Thu86a, Theorem 5.7]). *Fix $\eta > 0$, a Margulis constant. For every $\varepsilon > 0$ there exists $\delta > 0$ such that for any type preserving doubly incompressible pleated surface $g : W \to N$ with pleating locus λ and induced metric σ, if $x, y \in \lambda$ lie in the η-thick part of (W, σ), then*

$$d_{\mathbf{P}(N)}(\mathbf{p}_g(x), \mathbf{p}_g(y)) \leq \delta \Longrightarrow d_\sigma(x, y) \leq \varepsilon.$$

Here $\mathbf{p}_g : \lambda \to \mathbf{P}(N)$ denotes the map induced by g from the lamination λ to the projective unit tangent bundle of N.

We use Theorem 5.1 to prove the following

PROPOSITION 5.2. *There exists $L > 0$ such that the following holds: For every $\delta \in \mathcal{D}$ there exists a pleated surface $g : (W, \sigma) \to N$ in the proper homotopy class of the inclusion $W = \Sigma - \gamma \subset N$ that realizes $\lambda := \pi_W(\delta)$ as a sublamination of its pleating locus and such that one of the arcs δ_0 of $\lambda \cap W_0$ satisfies $\ell_\sigma(\delta_0) \le L$. Here W_0 is the η_0-non cuspidal part of (W, σ) where $\eta_0 > 0$ is a universal constant.*

Finally, the moderate length segments provided by Proposition 5.2 can be promoted to moderate length curves α and β in a simple way

COROLLARY 5.3. *There exists $L > 0$, only depending on Σ such that there are simple closed curves $\alpha, \beta \subset W$ with $d_W(\alpha, \pi_W(\mathcal{D})), d_W(\beta, \pi_W(f\mathcal{D})) \le 2$ and satisfying $\ell_{M_f - \gamma}(\alpha^*), \ell_{M_f-\gamma}(\beta^*) \le L$, where α^*, β^* are the geodesic representatives of α, β in $M_f - \gamma$.*

Proof. We only provide the argument for α since the one for β is completely analogous. Choose $\delta \in \mathcal{D}$ arbitrarily. Let $g : (W, \sigma) \to N$ and δ_0 be the pleated surfaces and the moderate length arc in $\lambda \cap W_0$ provided by Proposition 5.2. The length of ∂W_0 is bounded by $2\eta_0$. One of the boundary components of a regular neighborhood of $\delta_0 \cup \partial W_0$ is an essential simple closed curve α of length at most $2L + 2\eta_0$ with the property $d_W(\alpha, \pi_W(\delta)) \le 2$. □

The proof of Proposition 5.2 is where we fully exploit the assumption that W is the complement of a *non-separating* simple closed curve to make the arguments elementary. The main ingredients of the proof are Thurston's Uniform Injectivity and a technical lemma about quasi geodesic concatenations in $\mathbb{H}^3$. Of some use in the proof will be also the following general property of pleated surfaces observed by Thurston in [Thu86a]:

LEMMA 5.4 (Lemma 3.1 of [Min00]). *For every Margulis constant $\eta_0 > 0$ there exists η_M, only depending on η_0 and the topological type of W, such that, if $g : (W, \sigma) \to N$ is a π_1-injective pleated surface, then only the η_0-thin part of (W, σ) can enter the η_M-thin part of N.*

5.2. **Quasi geodesic concatenations.**

Beside the use of Uniform Injectivity, our proof of Proposition 5.2 is elementary and rests on the following fact about piecewise broken geodesics in $\mathbb{H}^3$, which we state without proof.

LEMMA 5.5. *There exists $L > 0$ and $A = A(L) > 0$ such that the following holds: Let $\gamma : I \to \mathbb{H}^3$ be a broken piecewise geodesic $\gamma = \gamma_1 * \cdots * \gamma_m$ with breaking angles $\angle\gamma_{i-1}\gamma_i$ contained in $(0, \pi/2)$ and geodesic segments of length at least L. Then γ is a A-quasi geodesic.*

In particular, if the length of the geodesic segments is large enough, only depending on A, then γ cannot be a loop. Lemma 5.5 follows from the fact that, in a hyperbolic space, local quasi geodesics, such as γ (as is not difficult to check), are global quasi geodesics.

Using Lemma 5.5, we see that

LEMMA 5.6. *For each $\varepsilon > 0$ there exists $L_0 > 0$ such that the following holds: Let $X = \mathbb{H} - \sqcup_{1\leq j\leq n}\mathcal{O}_j$ be the complement in $\mathbb{H}^3$ of a family of pairwise disjoint open horoballs. Let $\gamma = \gamma_1 * \cdots * \gamma_{2n}$ be a concatenation of paths in X such that*

- *γ_{2j+1} is a geodesic of length at least ε on the horosphere $\mathcal{H}_j = \partial\mathcal{O}_j$.*
- *γ_{2j} is the orthogonal segment connecting $\mathcal{H}_{j-1}$ and $\mathcal{H}_j$. We require that γ_{2j} has length at least L_0.*

Then γ is not a loop.

Proof. We use the following fact that can be easily checked in the upper half space model of $\mathbb{H}^3$. If x, y lie on the same horosphere $\mathcal{H}$, then we have

$$\sinh(d_{\mathbb{H}^3}(x,y)) = d_{\mathcal{H}}(x,y).$$

Denote by $\mathcal{H}_j = \partial\mathcal{O}_j$ the boundary horosphere of the horoball $\mathcal{O}_j$. Observe that, by assumption, the flat geodesic $\gamma_{2j} \subset \mathcal{H}_j$ has length $\ell(\gamma_{2j}) \geq \varepsilon$.

If we expand $\mathcal{H}_j$ radially from its center at infinity, the intrinsic geometry of the horosphere expands exponentially with exponent equal to the increase in the radius. Hence, if we inflate all the horoballs $\mathcal{O}_j$ by r, then each γ_{2j+1} is shortened to an arc γ^r_{2j+1} of length

$$\ell(\gamma^r_{2j+1}) = \ell(\gamma_{2j+1}) - 2r \geq L - 2r,$$

while the length of the inflated γ_{2j}, denoted by γ^r_{2j}, becomes

$$\ell(\gamma^r_{2j}) = e^r\ell(\gamma_{2j}).$$

We now straighten all the γ^r_{2j} relative to the endpoints and obtain geodesic arcs α^r_{2j} of length $\ell(\alpha^r_{2j}) = \sinh^{-1}(e^r\ell(\gamma_{2j})) \geq \sinh^{-1}(e^r\varepsilon)$.

Let us now consider the angles between the segments γ_{2j-1} and α_{2j}. Observe that, by assumption, γ^r_{2j-1} is orthogonal to $\mathcal{H}^r_j$, the inflated horosphere. Therefore, as, by convexity of horoballs, α_{2j} is contained in $\mathcal{O}^r_j$, the angle $\angle\gamma^r_{2j-1}\alpha^r_{2j}$ between the two geodesics γ_{2j-1} and α_{2j} is in $(0, \pi/2)$.

In conclusion, if the length of each γ_{2j+1} is much larger than $L + 2r$ and $\sinh^{-1}(e^r\varepsilon) \geq L$, where L is the constant of Lemma 5.5, then the broken piecewise geodesic $\gamma = \gamma^r_1 * \alpha^r_2 * \cdots * \gamma^r_{2n-1} * \alpha^r_{2n}$ is a A-quasi geodesic. If it is also sufficiently long compared to A, which can be again achieved by assuming that each γ_{2j+1} is long enough, it cannot be a loop. □

5.3. **Moderate length surgeries.** We now prove Proposition 5.2.

Proof of Proposition 5.2. Pick $\delta \in \mathcal{D}$ arbitrarily. Since (H_g, γ) is pared acylindrical, δ intersects γ essentially. Consider $\lambda = \delta \cap W$, it is a multi-arc in $W = \Sigma - \gamma$.

Now, after collapsing parallel components to a single one, λ can be realized as a sub-lamination of the pleating locus of some pleated surface $g : (\Sigma -$

$\gamma, \sigma) \to N$ in the proper (relative to cusps) homotopy class of the inclusion $\Sigma - \gamma \subset N$ (see for example Theorems I.5.3.6 and I.5.3.9 in [CEG06]).

When regarding δ as a simple closed curve on Σ, we can think of it as a concatenation $\delta = \beta_1 * \cdots * \beta_{2n}$ of arcs β_{2j} in W and arcs β_{2j+1} crossing a regular annular neighborhood of γ. Here we are using the fact that W is the complement of a non-separating simple closed curve.

Using the proper homotopy between the inclusion $W = \Sigma - \gamma \subset N$ and g, we simultaneously straighten all the β_{2j}'s to subarcs α_{2j} of the geodesic leaves of $g(\lambda)$ that start and end in the standard horosection $T = \partial \mathbb{T}_{\eta_M}(\gamma)$. Then, again, using the proper homotopy between $\Sigma - \gamma \subset N$ and the deformation retraction of N to $N - \mathbb{T}_{\eta_M}(\gamma)$, we replace the arcs β_{2j+1} with arcs α_{2j+1} on the horosection T that are geodesic with respect to the intrinsic flat metric and join the endpoints of the previously obtained α_{2j} and $\alpha_{2(j+1)}$.

We have that $\delta \subset \Sigma$ is homotopic in N to a concatenation $\alpha_1 * \cdots * \alpha_{2n}$ of closed arcs α_j, where each α_{2j+1} is a geodesic in the boundary of the standard horosection T, while each α_{2j} is a proper geodesic arc in $N - \mathbb{T}_{\eta_M}(\gamma)$ contained in the pleating locus $g(\lambda)$.

Notice that, by Lemma 5.4, if η_M is sufficiently small, then, when we look at α_{2j} in the intrinsic metric of the pleated surface, it will join two points of the η_0-cuspidal part for some universal $\eta_0 > 0$.

Now, by Theorem 5.1, which applies to g because it is doubly incompressible, there is a uniform $\varepsilon > 0$ such that the length of each α_{2j+1}, is at least ε. Let L_0 be as in Lemma 5.6, for the given ε. Notice that this constant only depends on Σ. Being homotopically trivial in N, the loop $\alpha_1 * \cdots * \alpha_{2n}$ lifts to a closed loop in $\mathbb{H}^3$. Hence, by Lemma 5.6, some α_{2i} must have length less than L_0. □

5.4. **Size of the standard horosection.** Consider $T = \partial \mathbb{T}_{\eta_M}$, the boundary of the standard horosection. We show that T is long and skinny, meaning that the length of any slope $\mu \subset T$ different from the one coming from $(\Sigma - \gamma) \cap T$ is very long, provided that $d_W(\alpha, \beta)$ is sufficiently large.

PROPOSITION 5.7. *There exists $c = c(\Sigma) > 0$ such that*

$$d_W(\alpha, \beta) \leq c\ell_T(\mu) + c$$

for every slope μ in T that is not homotopic to a component of $(\Sigma - \gamma) \cap T$.

Proof. Notice that there is a bound D', depending only on Σ, on the length of a component of $(\Sigma - \gamma) \cap T$. In fact, the boundary of the cusp of a complete hyperbolic surface of finite area can be bounded in terms of the topological type of the surface only, as the area of the cusp is an increasing function of the length of its boundary, while the total area of the surface only depends on the topological type (e.g. by the Gauss-Bonnet theorem for non-compact surfaces).

Observe now that the intrinsic diameter of T satisfies

$$\mathrm{diam}(T) \leq D := (\ell_T(\mu) + D')/2;$$

hence, any two points on T can be joined by a flat geodesic of length at most D.

Consider the covering $p : Q \to M_f - \gamma$ corresponding to $\pi_1(W = \Sigma - \gamma) < \pi_1(M_f - \gamma)$. By Bonahon's Tameness [Bon86] Q is geometrically and topologically tame. Moreover, since $\Sigma - \gamma$ is not a virtual fiber (for example, because it separates $M_f - \gamma$), the covering Q is a geometrically finite hyperbolic structure on $W \times \mathbb{R}$ (without accidental parabolics) by Thurston-Canary Covering Theorem [Can96]. Denote by $X \sqcup Y = \partial\mathcal{CC}(Q)$ the boundary of the convex core.

By work of Minsky [Min10] (see also Theorem 2.1.3 of Bowditch [Bow11]), there exists a uniform quasi-geodesic $l := \alpha_0, \alpha_1, \dots, \alpha_n$ in $\mathcal{C}(W)$, that is

$$\frac{1}{C}|i-j| - C \leq d_W(\alpha_i, \alpha_j) \leq C|i-j| + C$$

for some uniform constant $C > 0$, with the following properties

(i) Every α_j has a geodesic representative α_j^* in Q of moderate length, that is $\ell(\alpha_j^*) \leq L$ for some uniform $L > 0$.
(ii) The initial and terminal curves have moderate length on the X and Y boundary components, that is $\ell_X(\alpha_0), \ell_Y(\alpha_n) \leq L$, with L as before.
(iii) Every curve $\beta \in \mathcal{C}(W)$ such that $\ell_Q(\beta) \leq L$ lies uniformly close to the quasi geodesic l, that is $d_W(\beta, l) \leq R$ for some uniform $R > 0$.

We notice that

$$n \geq c_0 d_W(\alpha, \beta) - c_0$$

for some uniform $c_0 > 0$. In fact, on one hand, by property (iii), we have $d_W(\alpha, l), d_W(\beta, l) \leq R$. On the other hand, the uniform quasi geodesic property of $l := \alpha_0, \dots, \alpha_n$ and hyperbolicity of the curve graph $\mathcal{C}(W)$ gives $d_W(\alpha_0, \alpha_n) \geq d_W(\alpha, \beta) - R_1$ for some uniform $R_1 > 0$. Combined together the two properties give the estimate above.

The moderate length geodesics α_j^* are well spaced in Q. This is a consequence of the model manifold technology, which we use in the form of the following result of Bowditch [Bow11] and Brock and Bromberg [BB11].

Theorem 5.8 (see Theorem 2.1.4 of [Bow11] and Theorem 7.16 of [BB11]). *For every $L > 0$ there exists $A > 1$ such that the following holds. Let Q a hyperbolic structure on $W \times \mathbb{R}$ for which the boundary ∂W is parabolic. Let $Q_0 = Q - \mathbb{T}_{\eta_M}(\partial W)$ be the complement of the standard cusp neighborhoods. Suppose that $\alpha, \beta \in \mathcal{C}(W)$ are simple closed curves represented in Q by closed geodesics of length at most L. Then*

$$\frac{1}{A} d_{\mathcal{C}(W)}(\alpha, \beta) - A \leq \rho_{Q_0}\left(\mathbb{T}_{\eta_M}(\alpha), \mathbb{T}_{\eta_M}(\beta)\right) \leq A d_{\mathcal{C}(W)}(\alpha, \beta) + A$$

*Where ρ_{Q_0} denotes the η_M-*electric distance *in Q_0.*

The η_M*-electric distance* $\rho_{Q_0}(x, y)$ between two points $x, y \in Q_0$ is defined to be the infimum of the *electric lengths* of all paths joining them in Q_0. The electric length of a path δ in Q_0 is the length of the portion of δ that lies in the η_M-thick part of Q_0. Observe that, by definition, the electric distance satisfies $\rho_{Q_0} \leq d_{Q_0}$.

The electric distance is also defined on hyperbolic surfaces (W, σ), and it is a fact (bounded diameter lemma; see Lemma 1.10 of [Bon86]) that for any fixed Margulis constant $\eta > 0$, if (W, σ) has finite area, then the η-electric diameter of (W, σ) is uniformly bounded only in terms of η and $\chi(W)$. This fact applies in our setting: The η_M-electric diameter of any pleated surface is uniformly bounded.

It follows from Theorem 5.8 and the fact that the sequence $\alpha_0, \cdots, \alpha_n$ is a uniform quasi-geodesic that

$$\frac{1}{B}|i-j| - B \leq \rho_{Q_0}(\alpha_i^*, \alpha_j^*) \leq B|i-j| + B.$$

for some uniform $B > 0$. Also notice that X and Y are uniformly close to α_0^* and α_n^* respectively, meaning that

$$\rho_{Q_0}(\alpha_0^*, X), \rho_{Q_0}(\alpha_n^*, Y) \leq \log(2L/\eta_M).$$

This follows from the following standard fact, which we state without proof.

LEMMA 5.9. *Let $\alpha \subset Q$ be a closed curve homotopic to a geodesic α^*. Then*

$$d_Q(\alpha, \mathbb{T}_{\eta_M}(\alpha^*)) \leq \log(2\ell(\alpha)/\eta_M).$$

For simplicity, from now on, we assume $n = 2m$ since it does not affect the argument and simplifies the notation. Consider a pleated surface $G : (W, \sigma) \to Q$ realizing the middle curve $\alpha_{m=n/2}$ in Q. Under the covering projection G descends to the pleated surface $g := pG : (W, \sigma) \to M_f - \gamma$ realizing α_m in $M_f - \gamma$.

The manifold Q has two cusps that cover $\mathbb{T}_{\eta_M}(\gamma)$; we fix one of those and denote it by $\mathbb{T}_Q^1$, with boundary $T_Q^1 = \partial \mathbb{T}_Q^1$. Then, we choose points $x \in T_Q^1 \cap X$ and $w \in W$ such that $G(w) \in T_Q^1$. Observe that $\mathrm{inj}_x(X), \mathrm{inj}_w(W, \sigma) \geq \eta_M$. As a consequence, since the electric diameter of pleated surfaces is uniformly bounded, we have

$$\rho_{Q_0}(G(w), \alpha_m^*), \rho_{Q_0}(x, \alpha_0^*) \leq K$$

for some uniform $K > 0$. Now connect $p(x)$ to $g(w)$ via a shortest flat geodesic ξ between them on T. Denote by δ the lift of ξ to Q with basepoint $G(w)$. We have $\ell(\delta) \leq D = (\ell_T(\mu) + D')/2$.

In order to conclude the proof of Proposition 5.7 it remains to show that

Claim: We have $\ell(\delta) \geq cn - c$ for some uniform constant $c > 0$.

We divide the proof of this claim into two cases.

Case I. The endpoint z of δ different from $G(w)$ coincides with x.

In this case, we have

$$\ell(\delta) \geq \rho_{Q_0}(x, G(w)) \geq \rho_{Q_0}(\alpha_0^*, \alpha_m^*) - 2K \geq \frac{m}{B} - B - 2K,$$

as desired.

Case II. The endpoint z of δ different from $G(w)$ differs from x.

In this case, let τ be a non peripheral loop on X based at x that has moderate length, say $\ell_X(\tau) \leq L_1$ for some uniform $L_1 > 0$ only depending on W.

We now observe that $p(\tau)$ does not lift to a loop based at z in Q. In fact, we claim that $p(\tau)$ admits a unique lift which is a loop based at a point on the chosen cusp of Q and such a lift is τ, which is based at $x \neq z$. In general, lifts of $p(\tau)$ based at a point on $p^{-1}(p(x)) \cap T_Q^1$ correspond to elements $\kappa \in \pi_1(T, p(x))$ such that $\kappa p(\tau)\kappa^{-1} \in p_*\pi_1(X, x)$.

LEMMA 5.10. *If $c \in \pi_1(\Sigma - \gamma)$ is not peripheral and $\kappa \in \pi_1(M_f - \gamma) - \pi_1(\Sigma - \gamma)$, then*

$$\kappa c \kappa^{-1} \notin \pi_1(\Sigma - \gamma).$$

Proof. Recall that $\pi_1(M_f - \gamma)$ is a free product with amalgamation

$$\pi_1(M_f - \gamma) = \pi_1(H_g - \gamma) *_{\pi_1(\Sigma - \gamma)} \pi_1(H_g - f(\gamma)).$$

Write κ as a reduced word $\kappa = a_1 b_1 \dots a_n b_n$ with $a_j \in \pi_1(H_g - \gamma) - \pi_1(\Sigma - f(\gamma))$ for $j > 1$ and $b_j \in \pi_1(H_g - \gamma) - \pi_1(\Sigma - \gamma)$ for $j < n$. Since $\kappa \notin \pi_1(\Sigma - \gamma)$, either $b_n \notin \pi_1(\Sigma - \gamma)$, or we can take b_n to be the identity and $a_n \notin \pi_1(\Sigma - \gamma)$.

The two cases can be dealt with in the same way, so we only consider the first case, that is, we assume that $b_n \notin \pi_1(\Sigma - \gamma)$. We have

$$\kappa c \kappa^{-1} = a_1 b_1 \dots a_n b_n c b_n^{-1} a_n^{-1} \dots b_1^{-1} a_1^{-1}.$$

We claim that $b_n c b_n^{-1} \in \pi_1(H_g - \gamma)$ is not in $\pi_1(\Sigma - \gamma)$ provided that c is not a peripheral element: This follows from the fact that (H_g, γ) is pared acylindrical. In fact, suppose that $b_n c b_n^{-1} \in \pi_1(\Sigma - \gamma)$ and consider the homotopy between c and $b_n c b_n^{-1}$ which takes place in $H_g - \gamma$. We have the following possibilities: If the homotopy is deformable into the cusp, then c would be peripheral, which is ruled out by our initial assumption. As (H_g, γ) is pared acylindrical, if the homotopy is not deformable to the cusp, then it is deformable to the boundary $\Sigma - \gamma$. In this case b_n would be contained in $\pi_1(\Sigma - \gamma)$, which is again a contradiction. Therefore, if c is not peripheral, the word $\kappa c \kappa^{-1} = a_1 b_1 \dots a_n (b_n c b_n^{-1}) a_n^{-1} \dots b_1^{-1} a_1^{-1}$ is still reduced, and it contains a term not in $\pi_1(\Sigma - \gamma)$. Hence it cannot represent an element in $\pi_1(\Sigma - \gamma)$. □

We now return to the main argument for Proposition 5.7. By Lemma 5.10, the loop $g_n(\tau)$ lifts to an arc η with basepoint z on the preferred cusp and another endpoint u on a different component of $p^{-1}(\mathbb{T}_{\eta_M}(\gamma))$. We now observe that, if η_M has been chosen sufficiently short in the beginning, no component of $p^{-1}(\mathbb{T}_{\eta_M}(\gamma))$ different from the cusps of Q intersects the convex core $\mathcal{CC}(Q)$.

LEMMA 5.11. *If η_M is sufficiently small, only depending on the topological type of W, then*

$$p^{-1}\mathbb{T}_{\eta_M}(\gamma) \cap \mathcal{CC}(Q) = \mathrm{cusp}(Q).$$

Proof. Let $p^{-1}\mathbb{T}_{\eta_M}(\gamma) = \bigsqcup_{j\in I} \mathcal{O}_j$ be the full preimage of the cusp under the covering projection $p : Q \to M_f - \gamma$. Suppose that a component $X \subset \partial\mathcal{CC}(Q)$ of the boundary of the convex core intersects one of the components $\mathcal{O}_j$ of the lift of the Margulis tube. Note that $p : X \to M_f - \gamma$ is a type preserving pleated surface in the homotopy class of the inclusion $\Sigma - \gamma \subset M_f - \gamma$ and that $p(\mathcal{O}_j) = \mathbb{T}_{\eta_M}(\gamma)$. By Lemma 5.4, the pleated surface $p(X)$ can only intersect $\mathbb{T}_{\eta_M}(\gamma)$ in its η_0-cuspidal part, for some uniform η_0, if η_M has been chosen sufficiently small in the beginning. This means that $\mathcal{O}_j$ intersects the cuspidal part on X and, hence, $\mathcal{O}_j$ is one of the cusps of Q. □

We are now able to conclude: By Lemmas 5.10 and 5.11, the arc $\delta * \eta$ has an endpoint $G(w) \in T^1_Q \cap \mathcal{CC}(Q)$ and another one $u \in p^{-1}(\mathbb{T}_{\eta_M}(\gamma)) - \mathcal{CC}(Q)$ outside the convex core. Therefore, it must intersect $\partial\mathcal{CC}(Q) = X \sqcup Y$. Say it intersects X. In particular $\rho_{Q_0}(G(w), X) \leq d_{Q_0}(G(w), X) \leq \ell(\delta * \eta)$, which, combined with the previously established inequalities gives us

$$\begin{aligned}
\ell(\delta) + L_1 &\geq \ell(\delta * \eta) \\
&\geq \rho_{Q_0}(G(w), X) \\
&\geq \rho_{Q_0}(\alpha^*_m, \alpha^*_0) - \rho_{Q_0}(X, \alpha^*_0) - 2K \\
&\geq m/B - B - 2K - \log(2L/\eta_M).
\end{aligned}$$

This concludes the proof of the claim. □

5.5. **The proof of Theorem 3.** Combining Proposition 5.7 and Theorem 2.1, we complete the proof of Theorem 3 as follows.

Proof of Theorem 3. We endow $M_f - \gamma$ with a complete finite volume hyperbolic structure, which exists by the assumption of (H_g, γ) and $(H_g, f(\gamma))$ being pared acylindrical.

Let μ be the flat geodesic on $T = \partial\mathbb{T}_{\eta_M}(\gamma)$ that represents the filling slope needed to pass from $M_f - \gamma$ to M_f (also known as the meridian of γ). By Theorem 2.1, if $\mathrm{nl}(\mu) \geq \mathrm{nl}_{HK}$, then there is a cone manifold deformation that brings the hyperbolic structure on $M_f - \gamma$ to a hyperbolic metric on M_f for which γ is a geodesic of length $\ell_{M_f}(\gamma) \leq a/\mathrm{nl}(\mu)^2$ for some universal constant $a > 0$.

By Proposition 5.7, we have

$$\ell(\mu) \geq c d_W(\alpha, \beta) - c,$$

which is larger than $\frac{1}{2}cd_W(\mathcal{D}, f\mathcal{D})$ provided that $d_W(\mathcal{D}, f\mathcal{D})$ is sufficiently large. Notice that $\mathrm{Area}(T) \le \eta_M \ell(\mu)$ whenever $\eta_M < \ell(\mu)$ (which we have given that $d_W(\mathcal{D}, f\mathcal{D})$) is sufficiently large); hence,

$$\mathrm{nl}(\mu) = \ell(\mu)/\sqrt{\mathrm{Area}(T)} \ge \sqrt{\ell(\mu)/\eta_M}.$$

Thus $\mathrm{nl}(\mu) \ge \mathrm{nl}_{HK}$ if the subsurface projection of the disk sets to W is sufficiently large. This shows that M_f is hyperbolic.

In order to conclude, it remains to bound the length of γ in M_f. This follows again from Theorem 2.1:

$$\ell(\gamma) \le \frac{a}{\mathrm{nl}(\mu)^2} \le \frac{a}{\ell(\mu)/\eta_M} \le \frac{2a\eta_M/c}{d_W(\mathcal{D}, f\mathcal{D})}. \qquad \square$$

6. The proof of Theorems 1 and 2

In this section we prove a precise form of Theorem 2 about the structure of random 3-manifolds. Before we state the theorem, we recall some background and set some notation regarding random walks.

6.1. **Random walks.**

We start by recalling some background material on random walks on the mapping class group. We crucially consider only random walks driven by probability measures μ whose support S is a finite symmetric generating set for the *entire* mapping class group.

Definition (Random Walk). Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables with values in S and distribution μ. The *n-th step of the random walk* is the random variable $f_n := s_1 \dots s_n$. We denote by $\mathbb{P}_n$ its distribution. The *random walk driven by* μ is the process $(f_n = s_1 \dots s_n)_{n\in\mathbb{N}} \in \mathrm{Mod}(\Sigma)^{\mathbb{N}}$. It has a distribution which we denote by $\mathbb{P}$.

The mapping class group acts on Teichmüller space $\mathrm{Mod}(\Sigma) \curvearrowright \mathcal{T}$. If we fix a base point $o \in \mathcal{T}$ we can associate to every random walk $(f_n)_{n\in\mathbb{N}}$ an orbit $\{f_n o\}_{n\in\mathbb{N}} \subset \mathcal{T}$.

It is a standard consequence of the subadditive ergodic theorem that there exists a constant $L \ge 0$, called the *drift* of the random walk on Teichmüller space, such that for $\mathbb{P}$-almost every sample path $(f_n)_{n\in\mathbb{N}}$ we have

$$\frac{d_{\mathcal{T}}(o, f_n o)}{n} \overset{n\to\infty}{\longrightarrow} L.$$

In general, the drift can be 0. However, it has been established by Kaimanovich and Masur [KM96] that, in our case, $L > 0$.

6.2. **Statement and discussion.**

We are now ready to state the precise version of Theorem 2, using the above setup. We say that a sequence of events has asymptotic probability 1 if the probability of the events goes to 1 as n tends to infinity.

Theorem 2. *Fix $K > 1$ and $\varepsilon > 0$. Let $o \in \mathcal{T}$ be a fixed basepoint. Denote by $L > 0$ the drift of the random walk and by τ_n the parameterization of the geodesic segment $[o, f_n o]$ by arc length. With asymptotic probability* 1, *we have the following: There exist pants decompositions P_2^n and P_3^n of Σ that are the shortest pants decompositions of some $S_2^n \in \tau_n[\varepsilon Ln, 2\varepsilon Ln]$ and $S_3^n \in \tau_n[(1-2\varepsilon)Ln, (1-\varepsilon)Ln]$ respectively, that is $P_j^n = \Upsilon(S_j^n)$, such that*

(a) The convex core of the maximally cusped I-bundle $Q(P_2^n, P_3^n)$ K-bilipschitz embeds in M_{f_n} away from the cusps.
(b) The maximal cusp $Q(P_2^n, P_3^n)$ is also K-bilipschitz to $Q(S_2^n, S_3^n)$ away from its cusps (as in Proposition 2.4).

In Subsection 6.5 at the end of this section we also describe how to derive Theorem 1 from Theorem 3 and properties of the random walk.

The proof of Theorem 2 does not use 3-dimensional hyperbolic geometry anymore. Rather, via Proposition 3.9, we will only have to work with the dynamics of a random walk on Teichmüller space and the curve graph.

In more detail: Thanks to the work done in the previous sections, namely Proposition 2.4, Proposition 3.8, and Proposition 3.9, we only need to check that the Teichmüller segment $\tau_n = [o, f_n o]$ contains four points $o < S_1 < S_2 < S_3 < S_4 < f_n o$ satisfying the conditions (i) and (ii) of Proposition 3.9.

The heuristic picture is the following. Consider the curve graph projection $\Upsilon[o, f_n o]$ of the segment $[o, f_n o]$. The endpoints $\delta = \Upsilon(o)$ and $f_n\delta = \Upsilon(f_n o)$ lie on the disk sets $\mathcal{D}$ and $f\mathcal{D}$. Hyperbolicity of the curve graph, quasi-convexity of the disk sets and the fact that Υ in monotone along geodesics together imply that, if $\mathcal{D}$ and $f\mathcal{D}$ are sufficiently far away, then the path $\Upsilon[o, f_n o]$ roughly decomposes into three parts: Initially, it fellow travels $\mathcal{D}$. Then, it follows a shortest geodesic between $\mathcal{D}$ and $f_n\mathcal{D}$. Lastly, it fellow travels $f_n\mathcal{D}$.

Any subsegment of the middle piece automatically satisfies property (ii).

Property (i) follows, instead, from ergodic properties of the random walk, see below for discussion and references. In particular, we will use that for any pseudo-Anosov ϕ, the segment $[o, f_n o]$ often fellow travels a translate of the axis l_ϕ of the pseudo-Anosov. Therefore, we just have to make sure that the two needed long fellow travelings happen on the subsegment that projects to the middle piece of $\Upsilon[o, f_n o]$.

We will deduce this combining the aforementioned ergodic properties of random walks with work of Maher [Mah10b] who proved that, with asymptotic probability 1, the distance between $\mathcal{D}$ and $f_n\mathcal{D}$ increases linearly and up to a sublinear error is the distance between the endpoints δ and $f_n\delta$. Hence, the middle piece in the above description, takes up almost all of $\Upsilon[o, f_n o]$.

6.3. **Ergodic properties of random walks.** We can now state the ergodic property of random walks that we need. It is inspired by [BGH20, Proposition 6.9]. In fact, we believe that the following statement can be extracted from its proof, with the exception, perhaps, of the logarithmic size of the fellow traveling. We include a complete proof of the precise form that we need.

THEOREM 6.1. *Let $\phi \in \mathrm{Mod}(\Sigma)$ be a pseudo-Anosov with axis l_ϕ in Teichmüller space, and let $0 < a < b < 1$. Denote by $L > 0$ the Teichmüller drift of the random walk. There exists $\varepsilon_0 > 0$ such that with asymptotic probability 1 the following holds: Denote by τ_n the segment $[o, f_n o]$. Then l_ϕ has a subsegment of length $\varepsilon_0 \log(n)$ one of whose translates uniformly fellow-travels a subsegment of $\tau_n[aLn, bLn]$.*

Proof. For g in $\mathrm{Mod}(\Sigma)$, we denote by π^g the closest-point projection to gl_ϕ.

Such projections have strong contraction properties described in the Contraction Theorem of [Min96]. In particular, it is well-known that they imply the following: There exists a constant $D > 0$, depending on ϕ, such that if $d_{\mathcal{T}}(\pi^g(x), \pi^g(y)) \geq D$ then the geodesic $[x, y]$ has a subsegment $[x_1, y_1]$ with $d_{\mathcal{T}}(x_1, \pi^g(x))$, $d_{\mathcal{T}}(y_1, \pi^g(y)) \leq D$. By Theorem 4.2 of [Min96], the segment $[x_1, y_1]$ δ-fellow travels l_ϕ for some δ only depending on l_ϕ.

Therefore, in order to conclude, it suffices to prove the following claim (the theorem follows up to moving a, b an arbitrarily small amount and modifying ε).

Claim: Given a, b, ϕ as in the statement, there exists $\varepsilon > 0$ such that with asymptotic probability 1, there exists g such that $d_{\mathcal{T}}(\pi^g(o), \pi^g(f_n o)) \geq \varepsilon \log(n)$ and $d_{\mathcal{T}}(o, \pi^g(o)), d_{\mathcal{T}}(o, \pi^g(f_n o)) \in [aLn, bLn]$.

The claim is a consequence of the following properties, which can be found in the existing literature as we explain below: There exist $a' < b'$, $\varepsilon \in (0, (b-a)/10)$ and $C > 0$ such that the following hold with asymptotic probability 1

(i) $d_{\mathcal{T}}(f_j o, \tau_n) \leq C \log(n)$ for every $j \leq n$.
(ii) $d_{\mathcal{T}}(f_{\lfloor a'n \rfloor} o, o) \in [(a+\varepsilon)Ln, (a+2\varepsilon)Ln)]$ and $d_{\mathcal{T}}(f_{\lfloor b'n \rfloor} o, o) \in [(b-2\varepsilon)Ln, (b-\varepsilon)Ln)]$.
(iii) There are g and $\varepsilon > 0$ such that $d_{\mathcal{T}}(\pi^g(f_{\lfloor a'n \rfloor} o), \pi^g(f_{\lfloor b'n \rfloor} o)) \geq \varepsilon \log(n)$.
(iv) For the same g of (iii), we have $d_{\mathcal{T}}(\pi^g(f_{\lfloor a'n \rfloor} o), \pi^g(o)) \leq \varepsilon \log(n)/3$ and $d_{\mathcal{T}}(\pi^g(f_{\lfloor b'n \rfloor} o), \pi^g(f_n o)) \leq \varepsilon \log(n)/3$.

Assuming (i)-(iv) we prove the claim. Afterwards, we give the references to the literature.

Proof of the claim. In the whole proof, all statements and inequalities are meant to hold with asymptotic probability 1. By properties (iii) and (iv), it follows that $d(\pi^g(o), \pi^g(f_n o)) \geq \varepsilon \log(n)/3$, whence the first part of the claim. We now argue that $\pi^g(o)$ and $\pi^g(f_n o)$ have distance within the desired interval from o.

Observe that the geodesic joining $f_{\lfloor a'n\rfloor}o$ to $f_{\lfloor b'n\rfloor}o$ fellow travels gl_ϕ along the subsegment connecting $x_n := \pi^g(f_{\lfloor a'n\rfloor}o)$ to $y_n := \pi^g(f_{\lfloor a'n\rfloor}o)$, because the projections are very far apart by property (iii). In particular, x_n and y_n are uniformly close to points p_n and q_n on $[f_{\lfloor a'n\rfloor}o, f_{\lfloor b'n\rfloor}o]$ respectively. By property (iv), the projections x_n and y_n are also logarithmically close to $\pi^g(o)$ and $\pi^g(f_n o)$. Therefore, we have

$$d_{\mathcal{T}}(o, \pi^g(o)) = d_{\mathcal{T}}(o, p_n) + O(\log(n)),$$
$$d_{\mathcal{T}}(o, \pi^g(f_n o)) = d_{\mathcal{T}}(o, q_n) + O(\log(n)).$$

Hence, we can focus on estimating $d_{\mathcal{T}}(o, p_n)$ and $d_{\mathcal{T}}(o, q_n)$. In fact, we will provide an estimate on $d_{\mathcal{T}}(o, p)$ for any point $p \in [f_{\lfloor a'n\rfloor}o, f_{\lfloor b'n\rfloor}o]$.

By the triangle inequality, for any point $p \in [f_{\lfloor a'n\rfloor}o, f_{\lfloor b'n\rfloor}o]$, we have

$$d_{\mathcal{T}}(o, f_{\lfloor b'n\rfloor}o) - d_{\mathcal{T}}(f_{\lfloor a'n\rfloor}o, f_{\lfloor b'n\rfloor}o) \le d_{\mathcal{T}}(o, p),$$
$$d_{\mathcal{T}}(o, p) \le d_{\mathcal{T}}(o, f_{\lfloor a'n\rfloor}o) + d_{\mathcal{T}}(f_{\lfloor a'n\rfloor}o, f_{\lfloor b'n\rfloor}o).$$

We now estimate distances using properties (i) and (ii).

In view of (ii) and the inequalities above, for our purposes it suffices to show that

$$d_{\mathcal{T}}(f_{\lfloor a'n\rfloor}o, f_{\lfloor b'n\rfloor}o) \le d_{\mathcal{T}}(o, f_{\lfloor b'n\rfloor}o) - d_{\mathcal{T}}(o, f_{\lfloor a'n\rfloor}o) + O(\log(n)).$$

We obtain this inequality as follows: Let $r_n, s_n \in \tau_n$ be provided by property (i), so that $d_{\mathcal{T}}(r_n, f_{\lfloor a'n\rfloor}o), d_{\mathcal{T}}(s_n, f_{\lfloor b'n\rfloor}o) = O(\log(n))$. Using that r_n and s_n lie on a geodesic originating at o, we have

$$\begin{aligned} d_{\mathcal{T}}(f_{\lfloor a'n\rfloor}o, f_{\lfloor b'n\rfloor}o) &\le d_{\mathcal{T}}(r_n, s_n) + O(\log(n)) \\ &= |d_{\mathcal{T}}(o, s_n) - d_{\mathcal{T}}(o, r_n)| + O(\log(n)) \\ &\le |d_{\mathcal{T}}(o, f_{\lfloor b'n\rfloor}o) - d_{\mathcal{T}}(o, f_{\lfloor a'n\rfloor}o)| + O(\log(n)). \end{aligned}$$

Since $\varepsilon < (b-a)/10$, in view of (ii) we can remove the absolute value, and obtain the required estimate. □

Now we provide references for the properties (i)-(iv).

Property (i) is a corollary of Theorem 10.7 [MS20] obtained summing the probabilities that each step of the walk is logarithmically far from τ_n.

Property (ii) follows from positivity of the Teichmüller drift, which implies that for any $\varepsilon > 0$, with probability going to 1 as k tends to infinity we have $d_{\mathcal{T}}(o, f_k o) \in [(L-\varepsilon)k, (L+\varepsilon)k]$ (see the argument for [KM96, Theorem 4.3(i)]).

This easily allows us to choose appropriate a', b'.

For later purposes, we also note that (i) and the aforementioned property imply the following proposition, which is a version of a theorem of Tiozzo [Tio15] and could also be deduced from said theorem.

THEOREM 6.2. *In the setting of the theorem, for any $\varepsilon > 0$ with asymptotic probability 1 we have $d_{\mathcal{T}}(f_m o, \tau_n(Lm)) \le \varepsilon m$ for all $\varepsilon n \le m \le (1-\varepsilon)n$.*

Properties (iii) and (iv) follow from Theorem 2.3 and Proposition 3.2 of [ST19], where a general framework is provided to show that random walk create logarithmically large projections. We explain how: In the terminology of [ST19] we want to show that

$$\left(\mathcal{S} := \{gl_\phi\}_{g\in\mathrm{Mod}(\Sigma)}, Y_0 := l_\phi, \{\pi^g\}_{g\in\mathrm{Mod}(\Sigma)}, \pitchfork\right),$$

where we define the projections on the group $\pi^g : \mathrm{Mod}(\Sigma) \to gl_\phi$ to be

$$\pi^g(h) := \pi^g(ho),$$

forms a *projection system* (as in [ST19, Definition 2.1]), where $\pitchfork$ is the relation on the translates of having bounded projection to each other, and that the probability measure μ is *admissible* (as in [ST19, Definition 2.2]).

The fact that the 4-tuple is a projection system follows from the contraction property of the projections π^g and well-known arguments. More specifically, referring to the requirements (1)-(5) of [ST19, Definition 2.1], we have that: Properties (1)-(3) are straightforward. Property (4) follows from the contraction property and, e.g., [Sis17, Lemma 2.5]. Property (5) follows instead from the fact that there are finitely many cosets of $\langle g\rangle$ such that if the projection of hl_ϕ on l_ϕ is unbounded, then h belongs to one of these cosets. This follows from, e.g., [Sis17, Corollary 4.4].

The fact that μ is admissible is also not difficult to be checked: Among the requirements perhaps only property (4) is not immediate. This property says, in our context, that the probability that the random walk ends up in one of the cosets of $\langle g\rangle$ for which the projection of hl_ϕ on l_ϕ is unbounded is exponentially small in the length of the walk. This holds because after n steps the random walk can only possibly visit linearly many of the elements of those cosets (they are undistorted), while the probability of ending up at any one of them is exponentially small, just because $\mathrm{Mod}(\Sigma)$ is non-amenable. Now that we explained why [ST19] applies, the third item follows from the [ST19, Theorem 2.3], while the fourth one follows from [ST19, Proposition 3.2], with $R = 0$. □

As a different application of the same projection systems framework, we have the following statement whose proof is rather similar to the previous one. It will be used in the proof of Theorem 1 via Theorem 3 and in the application to the decay rate of the shortest geodesics for random 3-manifolds. Recall that the projection $\Upsilon : \mathcal{T} \to \mathcal{C}$ sends $f_j o$ to $f_j\delta$ with $\delta \in \mathcal{D}$.

Denote by $L_{\mathcal{C}} = \lim d_{\mathcal{C}}(\delta, f_n\delta)/n > 0$ the drift of the random walk on the curve graph, which is positive by a result of Maher [Mah10a].

Theorem 6.3. *Denote by $L_{\mathcal{C}}$ the curve graph drift of the random walk, and let $0 < a < b < 1$. Then, there exist $\varepsilon_0 > 0$ and $C > 0$ such that with asymptotic probability 1 the following holds: There exists a non-separating simple closed curve $\gamma_n \subset \Sigma$ such that*

- $d_{\gamma_n}(\delta, f_n\delta) \geq \varepsilon_0 \log(n)$.

- $d_Y(\delta, f_n\delta) \leq C$ *for every proper subsurface* $Y \subset \Sigma - \gamma_n$.
- $d_{\mathcal{C}}(\delta, \gamma_n) \in [aL_{\mathcal{C}}n, bL_{\mathcal{C}}n]$.

Proof. If we exclude the location of the curve γ_n with respect to $[\delta, f_n\delta]$, that is, the third requirement in the list, then the statement of the theorem is exactly the content of Proposition 7.1 of [ST19]. Here we want to control simultaneously the presence of a curve γ_n with large annular projection and bounded projections to the subsurfaces disjoint from it together with the position of γ_n on the segment $[\delta, f_n\delta]$ in order to make sure that it lies far away from the disk sets $\mathcal{D}$ and $f_n\mathcal{D}$.

As for the case of Teichmüller space, we have that the following properties hold with asymptotic probability 1:

(i)' $d_{\mathcal{C}}(f_j\delta, [\delta, f_n\delta]) \leq C\log(n)$.
(ii)' $d_{\mathcal{C}}(f_{\lfloor a'n\rfloor}\delta, \delta) \in [(a+\varepsilon)L_{\mathcal{C}}n, (a+2\varepsilon)L_{\mathcal{C}}n)]$ and $d_{\mathcal{C}}(f_{\lfloor b'n\rfloor}\delta, \delta) \in [(b-2\varepsilon)L_{\mathcal{C}}n, (b-\varepsilon)L_{\mathcal{C}}n)]$.

Observe that, if there is a large annular subsurface projection between $f_{\lfloor a'n\rfloor}\delta$ and $f_{\lfloor b'n\rfloor}\delta$ on some γ_n, then the curve γ_n lies on the 1-neighborhood of $[f_{\lfloor a'n\rfloor}\delta, f_{\lfloor b'n\rfloor}\delta]$ by the Bounded Geodesic Image Theorem [MM99]. Just like in the proof of the claim in Theorem 6.1, properties (i)' and (ii)' ensure then that γ_n is at the appropriate distance from δ. That is, it satisfies the last requirement of Theorem 6.3.

Regarding the size of the subsurfaces projections we proceed as follows. We need three "buffer projections", as in Proposition 7.1 [ST19], whose proof yields the following: There are $\varepsilon_1 > 0$ and $C_1 > 0$ such that the following holds with asymptotic probability 1. There exist non-separating curves $\gamma_1^n, \gamma_2^n, \gamma_3^n$ such that

(iii)' $d_{\gamma_j^n}(f_{\lfloor a'n\rfloor}\delta, f_{\lfloor b'n\rfloor}\delta) \geq \varepsilon_1\log(n)$.
(iv)' $d_Y(f_{\lfloor a'n\rfloor}\delta, f_{\lfloor b'n\rfloor}\delta) \leq C_1$ for all subsurfaces $Y \subset \Sigma - \gamma_2^n$.
(v)' $d_{\gamma_1^n}(f_{\lfloor b'n\rfloor}\delta, \gamma_2^n), d_{\gamma_3^n}(f_{\lfloor a'n\rfloor}\delta, \gamma_2^n), d_{\gamma_2^n}(f_{\lfloor a'n\rfloor}\delta, \gamma_1^n), d_{\gamma_2^n}(f_{\lfloor b'n\rfloor}\delta, \gamma_3^n) \leq C_1$.

Similar to property (iv) used in the proof of Theorem 6.1, we have the following replacement which holds with asymptotic probability 1 and follows again from Proposition 3.2 of [ST19]

(vi)' $d_{\gamma_1^n}(f_{\lfloor a'n\rfloor}\delta, \delta) \leq \varepsilon_1\log(n)/3$ and $d_{\gamma_3^n}(f_{\lfloor b'n\rfloor}\delta, \delta) \leq \varepsilon\log(n)/3$.

A consequence of these properties is that the annular projections $\pi_{\gamma_2^n}(\delta)$ and $\pi_{\gamma_2^n}(f_{\lfloor a'n\rfloor}\delta)$ coarsely coincide. This is a routine application of the Behrstock Inequality [Beh06], which states that there is a constant B such that for all curves α, β, γ we have $\min\{d_\gamma(\alpha, \beta), d_\alpha(\gamma, \beta)\} \leq B$ (provided that the quantities are well-defined). Here is the argument: By properties (iii)' and (v)', $d_{\gamma_1^n}(f_{\lfloor a'n\rfloor}\delta, \gamma_2^n)$ is large. Hence, by property (vi)', $d_{\gamma_1^n}(\delta, \gamma_2^n)$ is also large. Therefore, by the Behrstock Inequality, $d_{\gamma_2^n}(\delta, \gamma_1^n)$ is bounded and, by property (v)', the same holds for $d_{\gamma_2^n}(\delta, f_{\lfloor a'n\rfloor}\delta)$ as required.

The same argument also applies to projections $\pi_Y(\delta)$ and $\pi_Y(f_{\lfloor a'n \rfloor}\delta)$ for all subsurfaces Y in the complement of γ_2^n. In fact, we have the following: Since $d_{\gamma_1^n}(\delta, \gamma_2^n)$ and $d_{\gamma_1^n}(f_{\lfloor a'n \rfloor}\delta, \gamma_2^n)$ are both large and Y is a subsurface of the complement of γ_2^n, also $d_{\gamma_1^n}(\delta, \partial Y)$ and $d_{\gamma_1^n}(f_{\lfloor a'n \rfloor}\delta, \partial Y)$ are large. Hence, by the Behrstock Inequality, $d_Y(\delta, \gamma_n^1)$ and $d_Y(f_{\lfloor a'n \rfloor}\delta, \gamma_n^1)$ are both uniformly bounded.

Changing the roles of δ and $f_{\lfloor a'n \rfloor}\delta$ with $f_n\delta$ and $f_{\lfloor b'n \rfloor}\delta$ concludes the proof. □

6.4. **The proof of Theorem 2.** Consider the Teichmüller segment $[o, f_n o]$. Fix $\delta > 0$ large enough. We need to find two pseudo Anosov mapping classes ψ and ψ' with short pants decompositions and four surfaces $o < S_1 < S_2 < S_3 < S_4 < f_n o$ such that

(i) $[S_1, S_2]$, $[S_3, S_4]$ have length at least h, depending only on δ, ψ, ψ', and δ-fellow travel $l_\psi, l_{\psi'}$.
(ii) $d_{\mathcal{C}}(\Upsilon[S_1, S_4], \mathcal{D}) \geq h$ and $d_{\mathcal{C}}(\Upsilon[S_1, S_4], f_n\mathcal{D}) \geq h$

We first prove the second property: Recall that we chose $o \in \mathcal{T}$ such that $\delta := \Upsilon(o) \in \mathcal{D}$ (and hence $f_n\delta = \Upsilon(f_n o) \in f_n\mathcal{D}$).

We show for every fixed $\rho > 0$, the probability that these properties hold is at least $1 - \rho$ for every n sufficiently large.

Claim: For every $h > 0$ and $\varepsilon > 0$, with asymptotic probability 1 we have that both $d_{\mathcal{C}}(\Upsilon\tau_n[\varepsilon Ln, (1-\varepsilon)Ln], \mathcal{D})$ and $d_{\mathcal{C}}(\Upsilon\tau_n[\varepsilon Ln, (1-\varepsilon)Ln], f_n\mathcal{D})$ are greater than h.

Proof of the claim. The claim is a consequence of

Theorem 6.4 (Maher [Mah10b]). *For every $\varepsilon > 0$ we have*

$$\mathbb{P}_n \left[d_{\mathcal{C}}(\mathcal{D}, f\mathcal{D}) \in [(L_{\mathcal{C}} - \varepsilon)n, (L_{\mathcal{C}} + \varepsilon)n] \right] \overset{n\to\infty}{\longrightarrow} 1.$$

Choose $\varepsilon_2 > \varepsilon_1 > 0$ much smaller than ε. By Theorem 6.4 we have $d_{\mathcal{C}}(\mathcal{D}, f_n\delta) \geq (L_{\mathcal{C}} - \varepsilon_1)n$ with probability at least $1 - \rho$ for n large.

By Theorem 6.2 we can assume $d_{\mathcal{T}}(f_m o, \tau_n(Lm)) \leq \varepsilon_1 m$ for every $\varepsilon_1 n < m < (1-\varepsilon_1)n$ with probability $\geq 1-\rho$ for n large. We also assume $L_{\mathcal{C}} - \varepsilon_1 < d_{\mathcal{C}}(\delta, f_n\delta)/n < L_{\mathcal{C}} + \varepsilon_1$ with probability $\geq 1 - \rho$ for every n large.

Consider $m \in [\varepsilon_1 n, (1 - \varepsilon_2)n]$.

We have the following estimate on the distance from $\mathcal{D}$: Let $B > 0$ be the Lipschitz constant of $\Upsilon : \mathcal{T} \to \mathcal{C}$. Recall that $f_m\delta = \Upsilon(f_m o)$

$$\begin{aligned} d_{\mathcal{C}}(\Upsilon\tau_n(Lm), \mathcal{D}) &\geq d_{\mathcal{C}}(f_m\delta, \mathcal{D}) - d_{\mathcal{C}}(f_m\delta, \Upsilon\tau_n(Lm)) \\ &\geq (L_{\mathcal{C}} - \varepsilon_1)m - B\varepsilon_1 m \\ &\geq (L_{\mathcal{C}} - \varepsilon_1 - B\varepsilon_1)\varepsilon_1 n. \end{aligned}$$

Notice that if ε_1 is small enough, the right hand side increases linearly in n with uniform constants.

As for the other disk set $f_n\mathcal{D}$, we also get

$$\begin{aligned} d_{\mathcal{C}}(\Upsilon\tau_n(Lm), f_n\mathcal{D}) &\geq d_{\mathcal{C}}(\delta, f_n\mathcal{D}) - d_{\mathcal{C}}(\delta, f_m\delta) - d_{\mathcal{C}}(f_m\delta, \Upsilon\tau_n(Lm)) \\ &\geq (L_{\mathcal{C}} - \varepsilon_1)n - (L_{\mathcal{C}} + \varepsilon_1)m - B\varepsilon_1 m \\ &\geq [(L_{\mathcal{C}} - \varepsilon_1) - (L_{\mathcal{C}} + \varepsilon_1)(1-\varepsilon_2) - B\varepsilon_1(1-\varepsilon_2)]n. \end{aligned}$$

As before, if ε_1 is very small compared to ε_2, the right hand side increases linearly in n with uniform constants. In conclusion, if ε_1 is small enough and n is large enough, the claim holds as $[\varepsilon Ln, (1-\varepsilon)Ln] \subset [\varepsilon_1 n, (1-\varepsilon_2)n]$.

This settles the proof of property (ii) for the segment $\tau_n[\varepsilon Ln, (1-\varepsilon)Ln]$. Observe that any subsegment $[S_1^n, S_4^n]$ will enjoy the same property.

We now take care of (i).

Claim: Let ϕ be a pseudo-Anosov element with a short pants decomposition. Let $l_\phi : \mathbb{R} \to \mathcal{T}$ be its axis. For every $\varepsilon > 0$, for every $h > 0$, with asymptotic probability 1, the Teichmüller segments $\tau_n[\varepsilon Ln, 2\varepsilon Ln]$ and $\tau_n[(1-2\varepsilon)Ln, (1-\varepsilon)Ln]$ ξ-fellow travel (with ξ only depending on ϕ), along subsegments $\tau_n[t_1^n, t_2^n]$ and $\tau_n[t_3^n, t_4^n]$ of length at least h, some translates $\psi_n = g_n l_\phi$ and $\psi_n' = g_n' l_\phi$ of the axis l_ϕ.

Proof of the claim. We just need to apply Theorem 6.1 with parameters $0 < a < b$ given by $0 < L\varepsilon < 2\varepsilon L$ and $0 < (1-2\varepsilon)L < (1-\varepsilon)L$ respectively.

Conclusion of the proof: For a fixed $\varepsilon > 0$ we define $o < S_1^n < S_2^n < S_3^n < S_4^n < f_n o$ to be the four surfaces $\tau_n(t_1^n) < \tau_n(t_2^n) < \tau_n(t_3^n) < \tau_n(t_4^n)$ as given by the second claim. By construction they satisfy the properties (i) and (ii) of Proposition 3.9 and, hence, can be used in the model metric construction of Proposition 2.4. This concludes the proof of Theorem 2.

6.5. **The proof of Theorem 1 via short curves.** We sketch now a proof of Theorem 1 that uses the construction of Theorem 3.

The argument also gives that M_{f_n} contains a curve of length $\leq 1/\log(n)$ (later on, we improve this estimate using model metric, see Theorem 5).

There is yet another version of Theorems 6.1 and 6.3 which says that, with high probability, there is a curve γ_n such that $d_{\Sigma-\gamma_n}(\Upsilon(o), \Upsilon(f_n o))$ has size at least $\log(n)$, and γ_n lies close to the middle of a geodesic in $\mathcal{C}$ from $\Upsilon(o)$ to $\Upsilon(f_n o)$. Similarly to the first Claim in the proof of Theorem 2, we have that γ_n also lies on a shortest geodesic connecting $\mathcal{D}$ and $f_n\mathcal{D}$, and far from the endpoints of said geodesic. Using that $\mathcal{D}$ is quasiconvex and the Bounded Geodesic Image Theorem [MM99], we have that the subsurface projection to $\Sigma - \gamma_n$ of $\mathcal{D}$ is bounded and coarsely coincides with that of $\Upsilon(o)$. A similar statement holds for $f_n\mathcal{D}$. Hence, we get that $d_{\Sigma-\gamma_n}(\mathcal{D}, f_n\mathcal{D})$ is logarithmically large, and also that (H_g, γ_n) and $(H_g, f_n(\gamma_n))$ are both pared acylindrical. We can therefore use Theorem 3.

7. Four applications

We describe four applications of Theorem 2.

We recall that the model metric decomposition consists of five pieces

$$\mathbb{M}_n = H_1^n \cup \Omega_1^n \cup Q_n \cup \Omega_2^n \cup H_2^n,$$

but, for our applications, we will mainly focus on the maximally cusped structure $Q_n = Q(P_2^n, P_3^n)$, as given by Theorem 2. We recall that it bilipschitz embeds, away from its cusps, into M_{f_n} with bilipschitz constant arbitrarily close to 1 as n goes to ∞.

7.1. **Diameter growth.** As a first geometric application, we compute the coarse growth rate for the diameter of random 3-manifolds.

Theorem 4. *There exists $c > 0$ such that*

$$\mathbb{P}_n[\mathrm{diam}(M_f) \in [n/c, cn]] \overset{n\to\infty}{\longrightarrow} 1.$$

The proof of Theorem 4 has two different arguments, one for the coarse upper bound and one for the coarse lower bound. The upper bound comes from a result by White [Whi01] that relates the diameter to the *presentation length* of the fundamental group, a topological and algebraic invariant. Of a different nature is the coarse lower bound where we heavily use the ε-model metric structure of Theorem 2 and the relation with the model manifold.

We start with the upper bound. We need the following definition

Definition (Presentation Length). Let G be a finitely presented group. The length of a finite presentation $G = \langle F \,|\, R\rangle$ is given by

$$l(F, R) = \sum_{r\in R} |r|_F - 2$$

where $|r|_F$ denotes the word length of the relator $r \in R$ with respect to the generating set F. The presentation length of G is defined to be

$$l(G) := \min \{l(F, R) \mid G = \langle F \,|\, R\rangle \text{ finite presentation}\}.$$

We also recall that a relator $r \in R$ is triangular if $|r|_F \le 3$.

Theorem 7.1 (White [Whi01]). *There exists $c > 0$ such that for every closed hyperbolic 3-manifold M we have*

$$\mathrm{diam}(M) \le c \cdot l(\pi_1 M).$$

Let $S \subset \mathrm{Mod}(\Sigma)$ be the finite support of the probability measure μ.

Lemma 7.2. *There exists $C(S) > 0$ such that for every $f \in \mathrm{Mod}(\Sigma)$ we have*

$$l(\pi_1(M_f)) \le C|f|_S.$$

In particular $\mathrm{diam}(M_f) \le K|f|_S$ *where* $K = c \cdot C$.

Proof. The 3-manifold M_f admits a triangulation T with a number of simplices uniformly proportional, depending on S, to the word length $|f|_S$. We have $\pi_1(M_f) = \pi_1(T_2)$ where T_2 denotes the 2-skeleton of T. By van Kampen, the fundamental group of a 2-dimensional connected simplicial complex X admits a presentation $\pi_1(X) = \langle F \,|\, R\rangle$ where every relation is triangular and the number of relations $|R|$ is roughly the number of 2-simplices. □

As a corollary, we get

$$\operatorname{diam}(M_{f_n}) \leq K|f_n = s_1 \ldots s_n|_S \leq Kn$$

thus proving the upper bound in Theorem 4.

The coarse lower bound follows from the structure of the model metric and the estimate of Theorem 5.8 that comes from the model manifold technology of Minsky [Min10].

In particular, by Theorem 5.8, if $Q_n = Q(P_2^n, P_3^n)$ is a maximal cusp then the distance between the boundary components of its non-cuspidal part Q^{nc} is at least $Ad_{\mathcal{C}}(P_2^n, P_3^n) - A$. In the case of random 3-manifolds we have

$$\begin{aligned} d_{\mathcal{C}}(P_2^n, P_3^n) &= d_{\mathcal{C}}(\Upsilon(S_2^n), \Upsilon(S_3^n)) \\ &\geq d_{\mathcal{C}}(\Upsilon(o), \Upsilon(f_n o)) - d_{\mathcal{C}}(\Upsilon(o), \Upsilon(S_2^n)) - d_{\mathcal{C}}(\Upsilon(S_3^n), f_n\Upsilon(o)) \\ &\asymp L_{\mathcal{C}} n - o(n). \end{aligned}$$

7.2. **Injectivity radius decay.** As a second geometric application, we give a coarse upper bound to the decay rate of the length of the shortest geodesic of random 3-manifolds.

Theorem 5. *There exists $c > 0$ such that*

$$\mathbb{P}_n \left[\operatorname{inj}(M_f) \leq c/\log(n)^2\right] \overset{n\to\infty}{\longrightarrow} 1.$$

Proof. By Theorem 2, it is enough to show that $Q(P_2^n, P_3^n)$ satisfies

$$\operatorname{systole}(Q(P_2^n, P_3^n)) \leq c/\log(n)^2.$$

We recall that P_2^n and P_3^n are short pants decompositions on surfaces $S_2^n \in \tau_n[\varepsilon Ln, 2\varepsilon Ln]$ and $S_3^n \in \tau_n[(1-2\varepsilon)Ln, (1-\varepsilon)Ln]$ where τ_n is the parameterized Teichmüller segment $[o, f_n o]$, for a fixed basepoint $o \in \mathcal{T}$.

By Minsky [Min00], a curve $\gamma \in \mathcal{C}$ not contained in P_2^n or P_3^n is short in $Q(P_2^n, P_3^n)$ if and only if there is a large subsurface projection $d_Y(P_2^n, P_3^n)$ on a proper subsurface $Y \subset \Sigma$ with $\gamma \subset \partial Y$. Furthermore, by the Length Bound Theorem [BCM12], its length will be bounded by

$$\ell_{Q(P_2^n, P_3^n)}(\gamma) \leq D \frac{S_\gamma(P_2^n, P_3^n)}{d_\gamma(P_2^n, P_3^n)^2 + S_\gamma(P_2^n, P_3^n)^2}$$

for some uniform constant $D > 0$ and where $d_\gamma(P_2^n, P_3^n)$ is the annular projection corresponding to γ and

$$S_\gamma(P_2^n, P_3^n) = 1 + \sum_{Y \in \mathcal{Y}_\gamma} \{\{d_Y(P_2^n, P_3^n)\}\}_K.$$

Here $\mathcal{Y}_\gamma$ denotes the collection of essential subsurfaces of $\Sigma - \gamma$, $K > 0$ is, again, some uniform constant, and $\{\{\bullet\}\}_K$ is the function defined by $\{\{x\}\}_K = x$ if $x > K$ and 0 otherwise.

Putting things together, it is enough to show that there exists a curve $\gamma \subset \Sigma$ for which $d_\gamma(P_2^n, P_3^n) \geq \varepsilon_0 \log(n)$ and $S_\gamma(P_2^n, P_3^n) \leq C$ for some uniform $\varepsilon_0 > 0$ and C. For the purposes of the argument below, we note that the latter condition is equivalent to having uniformly bounded projections on all Y contained in $\Sigma - \gamma$, in view of the distance formula of [MM00].

Replacing P_2^n with $\delta = \Upsilon(o)$ and P_3^n with $f_n\delta = \Upsilon(f_n o)$, the aforementioned property is contained the statement of Theorem 6.3. To conclude, we only have to argue that δ and P_2^n have coarsely the same subsurface projections to the annulus corresponding to γ and to all subsurfaces contained in $\Sigma - \gamma$ (and similarly for $f_n\delta$ and P_n^3). But this holds provided that we choose ε small enough in Theorem 2, and a, b sufficiently close to $1/2$ in Theorem 6.3. In fact, in this case we have that geodesics in $\mathcal{C}$ from δ to P_2^n cannot pass 2-close to γ_n, just because they are much shorter than the distance from δ to γ_n. We can therefore apply the Bounded Geodesic Image Theorem [MM99] and conclude (since a similar argument also applies to $f_n\delta$ and P_3^n). □

We conclude the discussion with a couple of remarks on the lower bound for the injectivity radius. If we consider only the three middle pieces $\Omega_n^1 \cup Q_n \cup \Omega_n^2$ of the ε-model metric, the rate of $1/\log(n)^2$ is exactly the coarse decay rate of the systole. This is again an adaptation of the arguments of [ST19]. Hence, in order to get a precise lower bound, we have to understand the systole of the handlebody pieces H_n^1 and H_2^n. Such computation would be possible, for example, in the presence of a model manifold technology for handlebodies analogue to the one of Minsky [Min10] and Brock, Canary and Minsky [BCM12] for hyperbolic manifolds diffeomorphic to $\Sigma \times \mathbb{R}$.

7.3. **Geometric limits of random 3-manifolds.** We now exploit the model metric structure to establish the existence of certain geometric limits (see Chapter E.1 of [BP92] for the definition of the pointed geometric topology) for families of random 3-manifolds.

These limits will be used in our last application concerning the arithmeticity and the commensurability class of random 3-manifolds.

Proposition 7.3. *Let $\phi \in \mathrm{Mod}(\Sigma)$ be a pseudo-Anosov mapping class. Consider a sequence $A_n \subset \mathrm{Mod}(\Sigma)$ such that $\limsup \mathbb{P}_n[A_n] > 0$. Then, we can find a sequence $n_j \uparrow \infty$ and elements $f_{n_j} \in A_{n_j}$ such that $M_{f_{n_j}}$ are hyperbolic 3-manifolds and the sequence $M_{f_{n_j}}$ converges to the infinite cyclic covering of hyperbolic mapping torus T_ϕ in the pointed geometric topology for a suitable choice of base points $x_{n_j} \in M_{f_{n_j}}$.*

Proof. By assumption, there exist $\delta > 0$ and a sequence $m_j \uparrow \infty$ such that $\mathbb{P}_{m_j}[A_{m_j}] \geq \delta$. We choose $(n_j)_{j\in\mathbb{N}}$ by inductively refining $(m_j)_{j\in\mathbb{N}}$.

By Theorem 6.1 and Theorem 2, for every $k \in \mathbb{N}$ we have that the event

$$G_{n,k} := \left\{ \begin{array}{c} M_{f_n} \text{ satisfies Theorem 2 with parameters } K := 1 + 1/k \text{ and } \varepsilon \\ \tau_n = [o, f_n o] \text{ satisfies Theorem 6.1 with parameters } \varepsilon \text{ and } \phi \end{array} \right\}$$

has probability at least $1 - \delta/10$ for every sufficiently large n, say for $n \geq R_k$. In particular, if $m_i \geq R_k$, we have $A_{m_i} \cap G_{m_i,k} \neq \emptyset$.

We define now inductively the sequence $(n_j)_{j\in\mathbb{N}}$. Suppose that we have already chosen $n_1, \cdots, n_{j-1}$. The next element will be

$$n_j := \min\{m_i \,|m_i > \max\{n_{j-1}, R_j\}\}.$$

As $n_j > R_j$, we have $A_{n_j} \cap G_{n_j,j} \neq \emptyset$, so we can choose $f_{n_j} \in A_{n_j} \cap G_{n_j,j}$.

We recall that $\tau_n = [o, f_n o]$ satisfies Theorem 6.1 with parameters ϕ and ε, so it has a subsegment $\tau_n[3\varepsilon Ln, (1-3\varepsilon)Ln]$ that uniformly travels a translate $g_n l_\phi$ along a subsegment of length $\varepsilon_0 \log(n)$.

As $\tau_{n_j}[3\varepsilon L n_j, (1-3\varepsilon)L n_j] \subset [S_2^{n_j}, S_3^{n_j}]$, up to remarking $[S_2^{n_j}, S_3^{n_j}]$, an operation that does not change the isometry type of $Q(S_2^{n_j}, S_3^{n_j})$, we can assume that $[S_2^{n_j}, S_3^{n_j}]$ uniformly fellow travels l_ϕ along the subsegment $l_\phi[-a_{n_j}, a_{n_j}]$ with $a_{n_j} = \varepsilon_0 \log(n_j) \uparrow \infty$. Hence, the sequence of Teichmüller segments $[S_2^{n_j}, S_3^{n_j}]$ is converging uniformly on compact subsets to a geodesic l that uniformly fellow travels the axis l_ϕ.

Notice that l_ϕ converges in the forward and backward directions to the projective classes of the invariant laminations λ^+ and λ^- of the pseudo-Anosov mapping class ϕ. Since l fellow travels l_ϕ and λ^+ and λ^- are are minimal, filling and uniquely ergodic (see Exposé 9 and Exposé 12 of [FLP12]), we conclude that also l converges in the forward and backward direction to the same laminations and so do the sequences of endpoints $(S_3^{n_j})_{j\in\mathbb{N}}$ and $(S_2^{n_j})_{j\in\mathbb{N}}$ respectively (see, for example, Lemma 1.4.2 of [KM96]).

By Thurston's Double Limit Theorem [Thu86b] and the solution of the Ending Lamination Conjecture by Minsky [Min10] and Brock, Canary and Minsky [BCM12], this implies that, if we take suitable base points, the sequence of convex cocompact manifolds $Q(S_2^{n_j}, S_3^{n_j})$ and the sequence of maximally cusped manifolds $Q(P_2^{n_j}, P_3^{n_j})$ both converge in the geometric topology to the infinite cyclic covering of T_ϕ. As $Q(P_2^{n_j}, P_3^{n_j})$ becomes geometrically arbitrarily close to $M_{f_{n_j}}$, the claim follows. □

7.4. **Commensurability and arithmeticity.** Dunfield and Thurston, using a simple homology computation, have shown in [DT06] that their notion of random 3-manifold is not biased towards a certain fixed set of 3-manifolds. This means that for every fixed 3-manifold M, with asymptotic probability 1, M_f is not diffeomorphic to M.

Using geometric tools it is possible to strengthen this conclusions and show that Dunfield and Thurston's notion of random 3-manifolds is also transverse, in a sense made precise in the theorem below, to the class of arithmetic hyperbolic 3-manifolds and to the class of 3-manifolds which are commensurable to a fixed 3-manifold M.

Theorem 6. *With asymptotic probability* 1 *the following holds*

(1) M_f is not arithmetic.
(2) M_f is not in a fixed commensurability class $\mathcal{R}$.

Proof. The argument is mostly borrowed from Biringer-Souto [BS11].

The proof of both points starts from the following observation: Each M_{f_n} finitely covers a maximal orbifold $M_{f_n} \to \mathcal{O}_n$.

We first prove the non-arithmeticity. We argue by contradiction: Suppose that $\mathbb{P}_n[M_f$ is arithmetic$]$ does not go to 0. Combining with Theorem 5, we also have

$$\limsup \mathbb{P}_n[M_f \text{ is arithmetic and } \mathrm{inj}(M_f) \le c/\log(n)^2] > 0.$$

By Proposition 7.3, up to passing to a subsequence, say the whole sequence for simplicity, we can pick M_{f_n} such that M_{f_n} is arithmetic, has injectivity radius $\mathrm{inj}(M_{f_n}) \le c/\log(n)^2$ and there are base points $x_n \in M_{f_n}$ such that the sequence (M_{f_n}, x_n) converges geometrically to (Q_∞, x_∞) where Q_∞ is a doubly degenerate structure on $\Sigma \times \mathbb{R}$ with $\mathrm{inj}(Q_\infty) > 0$.

Since M_{f_n} are arithmetic, the orbifolds $\mathcal{O}_n$ are congruence and have $\lambda_1(\mathcal{O}_n) \ge 3/4$ (see [BS91] or Theorem 7.1 in [BS11]). By Proposition 4.3 of [BS11], the orbifolds $\mathcal{O}_n$ cannot be all different, hence we can assume that they are fixed all the time $\mathcal{O}_n = \mathcal{O}$. We get a contradiction by observing that $\mathcal{O}$ is covered by closed 3-manifolds M_{f_n} with arbitrarily small injectivity radius.

We now discuss commensurability. Proceed again by contradiction and assume that $\mathbb{P}_n[M_f$ is in the commensurability class $\mathcal{R}]$ does not go to 0. By the arithmetic part we know that we also have

$$\limsup \mathbb{P}_n[M_f \in \mathcal{R},\ M_f \text{ not arithmetic, and } \mathrm{inj}(M_f) \le c/\log(n)^2] > 0.$$

As before, using Proposition 7.3, choose a geometrically convergent sequence $(M_{f_n}, x_n) \to (Q_\infty, x_\infty)$ of non-arithmetic, commensurable hyperbolic 3-manifolds with $\mathrm{inj}(M_{f_n}) \downarrow 0$. Commensurability and non-arithmeticity imply together that $\mathcal{O}_n = \mathcal{O}$ is fixed all the time: It is the orbifold corresponding to the commensurator $\mathrm{Comm}(\pi_1(M_{f_n}))$, which is a discrete subgroup of $\mathrm{PSL}_2\mathbb{C}$ by Margulis (see Theorem 10.3.5 in [MR03]) and is an invariant of the commensurability class. We conclude with the same argument as before. □

Appendix A. Double incompressibility for pared handlebodies

We give a proof of Proposition 4.7 whose statement we recall

Proposition 4.7. *If (H_g, γ) is pared acylindrical, then the inclusion $\Sigma - \gamma \subset H_g$ is doubly incompressible.*

We have to prove that the conditions (a)-(e) of the definition of double incompressibility hold. We proceed step by step by checking one condition at a time. Note that conditions (a) and (c) both follow immediately from the defining properties of pared acylindrical handlebodies. Hence, we only focus on (b) and (e).

A.1. **Homotopy classes of arcs.** We check condition (b).

LEMMA. *Essential relative homotopy classes of arcs $(I, \partial I) \to (\Sigma - \gamma, N(\gamma))$ map injectively into relative homotopy classes of arcs $(I, \partial I) \to (H_g, U(\gamma))$.*

Proof. For simplicity denote $A := N(\gamma)$ and $U := U(\gamma)$. Consider two arcs α, β with endpoints in $\mathrm{int}(A)$, each intersecting ∂A transversely in exactly two points.

Suppose that they are homotopic as maps into (H_g, A). Then, we can find arcs ξ, δ in $\mathrm{int}(A)$, each joining an endpoint of α and an endpoint of β, such that the concatenation $\kappa = \xi * \alpha * \delta^{-1} * \beta^{-1} \subset \Sigma$ is nullhomotopic in H_g.

Either κ is nullhomotopic in Σ, in which case α and β represent the same homotopy class $(I, \partial I) \to (\Sigma, A)$, or κ is essential in Σ.

Suppose we are in the second case. Up to a little perturbation we can assume that κ has only transverse self intersections and intersects ∂A exactly in $(\alpha \cap \partial A) \cup (\beta \cap \partial A)$. By the Loop Theorem there is a diskbounding curve η in $\kappa \cup U$ where U is a tiny neighborhood of the singular set of transverse self intersections of κ. Such a curve η has geometric intersection at most 2 with ∂A and hence with γ.

Claim. If $i(\eta, \gamma) \leq 2$, then $\Sigma - \gamma$ has either an essential disk or an essential annulus.

In particular, the existence of η contradicts the assumption on (H_g, γ) being pared acylindrical.

Proof of the claim. The curve η bounds an essential disk $\eta = \partial D^2$ in H_g.

If $i(\eta, \gamma) = 0$, then D^2 is an essential disk disjoint from γ.

If $i(\eta, \gamma) = 1$, then the boundary of a regular neighborhood of $D^2 \cup \gamma$ in H_g is an essential disk disjoint from γ.

If $i(\eta, \gamma) = 2$, then the boundary of a regular neighborhood of $\gamma \cup D^2$ in H_g contains an essential annulus disjoint from γ. □

A.2. **Maximal abelian subgroups.** We check condition (e).

LEMMA. *Maximal cyclic subgroups of $\pi_1(\Sigma - \gamma)$ are mapped to maximal cyclic subgroups of $\pi_1(H_g)$.*

Proof. We need to check that every primitive element of $\pi_1(\Sigma - \gamma)$ is also primitive in $\pi_1(H_g)$. We proceed as in Canary-McCullogh (see Lemma 5.1.1 in [CM04]). Suppose this is not the case, then there exists an essential map $f : A = S^1 \times I \to H_g$ such that $f(\partial_1 A) = \alpha$, a loop representing a primitive element in $\pi_1(\Sigma - \gamma)$, and $f(\partial_2 A) = \beta^k$ for some $k \geq 2$ and $\beta \notin \pi_1(\Sigma - \gamma)$.

The map $f : A \to H_g$ factors through $f_0 : A_0 \to H_g$ where A_0 is the quotient space obtained by identifying points on $\partial_2 A$ that differ by a $2\pi/k$-rotation. We have $f_0(\partial_1 A_0) = \alpha$ and $f_0(\partial_2 A_0) = \beta$. Notice that A_0 embeds in a solid torus $\mathbb{T} = D^2 \times S^1$ in such a way that $\partial_1 A_0$ is a simple closed curve on $T := \partial\mathbb{T}$ and $\partial_2 A_0$ is the core curve $0 \times S^1$ and moreover $\mathbb{T}$ deformation retracts to A_0. By the last property we can extend f_0 to a map $F_0 : \mathbb{T} \to H_g$.

We show that F_0 can be homotoped relative to $\partial_1 A_0$ such that $F_0(\mathbb{T}) \subset \Sigma - N(\gamma)$. This implies that $\alpha = F_0(\partial_1 A_0)$ is homotopic in $\Sigma - \gamma$ to $F_0(\partial_2 A_0)^k$ and, hence, it could not have been primitive.

The boundary $T = \partial\mathbb{T}$ is divided into two annuli $T = U \cup V$: A tubular neighborhood U of $\partial_1 A_0$ and the complement V. Up to a small homotopy we can assume $F_0(U) \subset \Sigma - N(\gamma)$. Consider the restriction of F_0 to the annulus V. We claim that we can homotope it into $\Sigma - N(\gamma)$. In fact, if this were not the case, then by the Annulus Theorem we would find an essential embedded annulus $(A, \partial A) \subset (H_g, \Sigma - N(\gamma))$ contradicting the fact that (H_g, γ) is pared acylindrical. Therefore we can homotope F_0 relative to U in such a way that $F_0(T) \subset \Sigma - N(\gamma)$.

We finally show that we can homotope F_0 such that $F_0(\mathbb{T}) \subset \Sigma - \gamma$. The meridian $\mu = \partial D^2 \times \{\star\}$ of the solid torus $\mathbb{T}$ is now mapped to a loop in $\Sigma - N(\gamma)$ which is nullhomotopic in H_g. Since $\Sigma - \gamma$ is π_1-injective, the loop $F_0(\mu)$ is also trivial in $\Sigma - \gamma$. As H_g is aspherical, we can homotope the restriction of F_0 to $D^2 \times \{\star\}$ to a nullhomotopy that takes place in $\Sigma - N(\gamma)$. Finally, as the the complement of $T \cup D^2 \times \{\star\}$ is a 3-ball B, using again the fact that H_g is aspherical we can homotope F_0 restricted to B such that the image of the entire solid torus $\mathbb{T}$ lies in $\Sigma - N(\gamma)$. ☐

Appendix B. Isotopies of Margulis tubes

We prove the following

Lemma 3.4. *For every $\eta < \eta_M/2$ there exists $\xi > 0$ such that the following holds: Let $\mathbb{T}_{\eta_M}(\alpha)$ be a Margulis tube with core geodesic α of length $l(\alpha) \in [\eta, \eta_M/2]$. Suppose that there exists a $(1+\xi)$-bilipschitz embedding of the tube in a hyperbolic 3-manifold $f : \mathbb{T}_{\eta_M}(\alpha) \to M$. Then $f(\alpha)$ is homotopically non-trivial and it is isotopic to its geodesic representative within $f(\mathbb{T}_{\eta_M}(\alpha))$.*

Proof. The universal cover of $\mathbb{T}_{\eta_M}(\alpha)$ is a a-neighborhood $N_a(l)$ of a geodesic $l \subset \mathbb{H}^3$. Denote by $F : N_a(l) \to \mathbb{H}^3$ the lift of f to the universal coverings.

By basic hyperbolic geometry, we have that for every subsegment $[p, q] \subset l$ of length $l([p, q]) \leq \eta$, the image $F[p, q]$ is contained in the ε-neighborhood of the geodesic $[F(p), F(q)]$ with $\varepsilon = O(\xi)$. This implies, if ξ is sufficiently small, that F restricted to l is a uniform quasi-geodesic. As a consequence f is π_1-injective and $f(\alpha)$ is homotopic its geodesic representative β within $N_\varepsilon(\beta)$ with $\varepsilon = O(\xi)$. We want to show that $f(\alpha)$ is actually isotopic to β.

The proof can now be concluded using topological tools.

Up to a very small isotopy we can assume that $f(\alpha)$ is disjoint from β and still contained in $N_\varepsilon(\beta)$. For safety, we assume that an entire metric tubular neighborhood of $f(\alpha)$ of the form $f(N_\delta(\alpha))$ for some tiny δ is disjoint from β and contained in $N_\varepsilon(\beta)$.

Since the radius of the tube $f(\mathbb{T}_{\eta_M}(\alpha))$ is large, we can assume that a metric tubular neighborhood of β of the form $N_r(\beta)$ with $r > \varepsilon$ is contained in $f(\mathbb{T}_{\eta_M}(\alpha))$. Denote by $T_\beta = \partial N_r(\beta)$ its boundary and observe that $T_\beta \subset f(\mathbb{T}_{\eta_M}(\alpha)) - f(N_\delta(\alpha))$. The complementary region $f(\mathbb{T}_{\eta_M}(\alpha)) - f(N_\delta(\alpha))$ is diffeomorphic to $T_\alpha \times [0,1]$ where T_α is a 2-dimensional torus.

Notice that T_β is incompressible in $T_\alpha \times [0,1]$. In fact, the only possible compressible curve on T_β is the boundary ∂D_β of the compressing disk D_β of the tubular neighborhood of $N_r(\beta)$. Every other simple closed curve is homotopic in $f(\mathbb{T}_{\eta_M}(\alpha))$ to a multiple of $\beta \simeq f(\alpha)$ and hence it is not trivial. However, the curve ∂D_β cannot be compressible in $T_\alpha \times [0,1]$ otherwise it would bound a disk D'_β with interior disjoint from D_β and together they would give a 2-sphere $S^2 \cong D_\beta \cup D'_\beta$ intersecting once β. Such a sphere is homologically non trivial in $f(\mathbb{T}_{\eta_M}(\alpha))$, but a solid torus does not contain such an object.

By standard 3-dimensional topology, incompressibility implies that T_β is parallel to $T_\alpha \times \{1\} = f(\mathbb{T}_{\eta_M}(\alpha))$. Therefore, β is the core curve $\beta \cong 0 \times S^1$ for another product structure $f(\mathbb{T}_{\eta_M}(\alpha)) \cong D^2 \times S^1$ or, in other words, there exists an orientation preserving self diffeomorphism of $f(\mathbb{T}_{\eta_M}(\alpha))$ that sends $f(\alpha)$ to β. Such a diffeomorphism is isotopic to a power of the Dehn twist along the meridian disk of the solid torus, hence it does not change the isotopy class of the core curve.

This concludes the proof. □

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