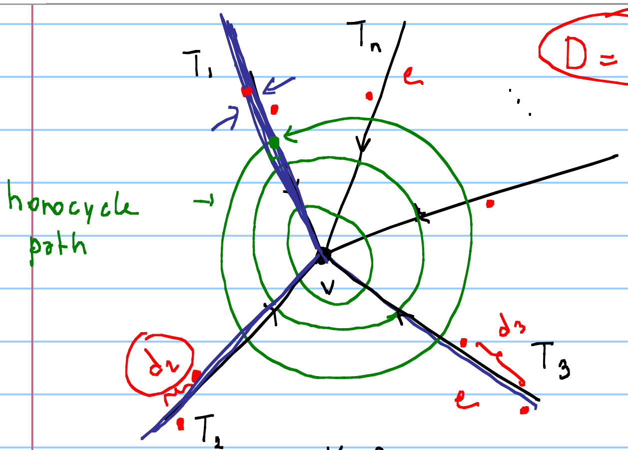
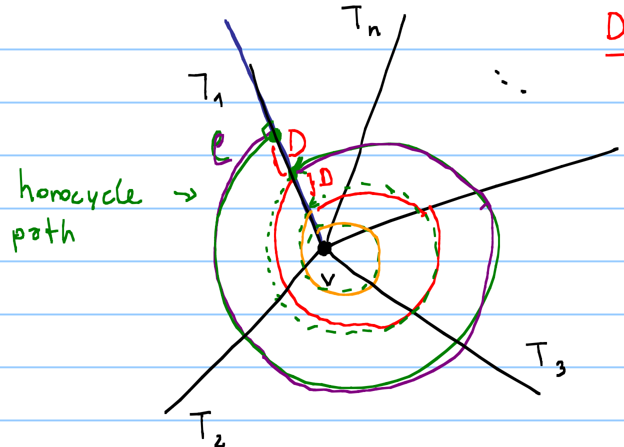
Hyperbolic Manifolds - Lecture 9

Note Title 01/12/2020



$D = d_1 + \dots + d_n = 0$

complete

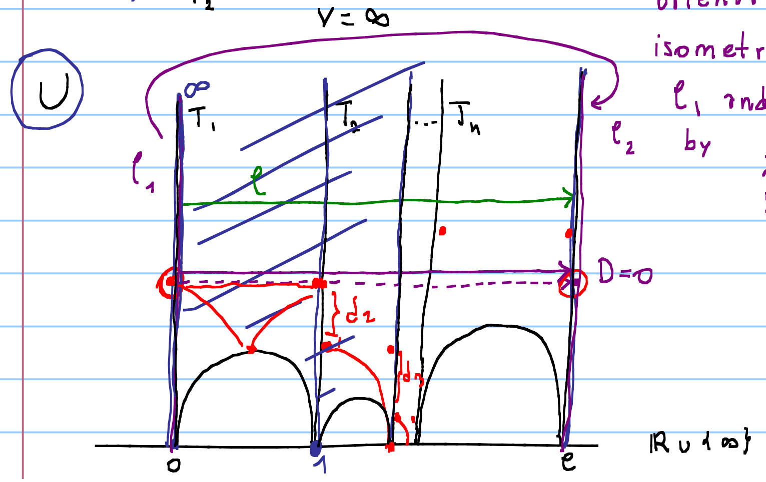


$D = d_1 + \dots + d_n > 0$

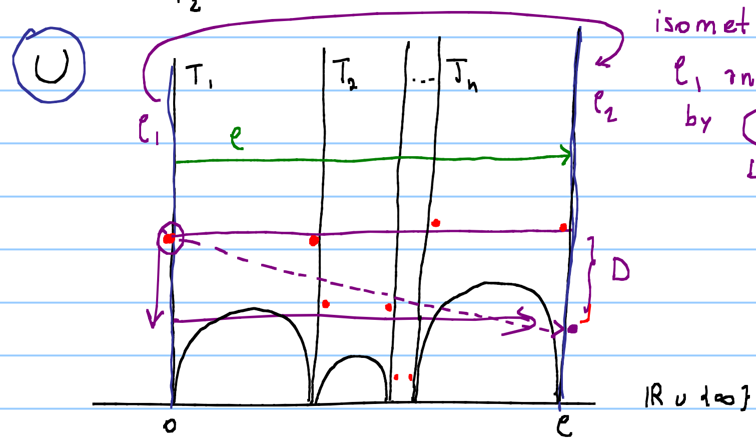
incomplete

total length of the horocycle path

$$L = \sum_{n \in \mathbb{N}} e^{-nD} \ell < \infty$$



orientation preserving isometry between ℓ_1 and ℓ_2 given by $z \longrightarrow z + \ell$

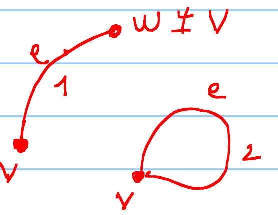


orientation preserving isometry between ℓ_1 and ℓ_2 given by $z \longrightarrow e^D z + \ell$

number of edges of $\Sigma = 3T/2$

$$\text{Complete structures} = \left\{ \{d(e)\}_{e \,\in\, \text{edge pairs}} \in \mathbb{R}^{\mathcal{E}} \;\middle|\; \begin{array}{l} \text{For every vertex } v \\ \displaystyle\sum_{e \text{ incident to } v} d(e) = 0 \end{array} \right\}$$

counted with multiplicity



Proposition: The vertex equations are linearly independent.

Corollary: The dimension of the space of complete solutions is $\dim = 3T/2 - V$.

number of triangles

number of vertices

In particular, if Σ, the resulting oriented surface with punctures, is connected, then $\dim = 6g(\Sigma) - 6 + 2p(\Sigma)$.

genus

number of punctures

Pf. $\sum_{e \text{ incident to } v} d(e) = 0$

For every vertex $v \Rightarrow x_v = ($ j-th entry $=$ number of endpoints of e_j that represent v $)$ (e_j: j-th edge)

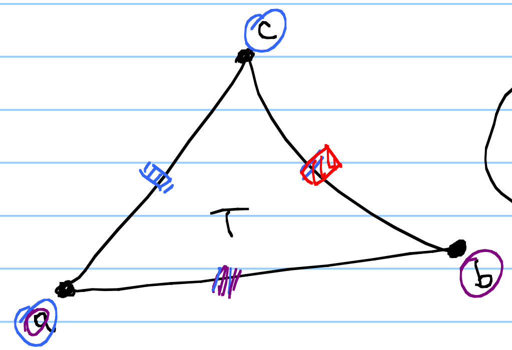
$\underline{d} \in \mathbb{R}^{\mathcal{E}}$ $\qquad \sum d(e) = x_v \cdot \underline{d}$

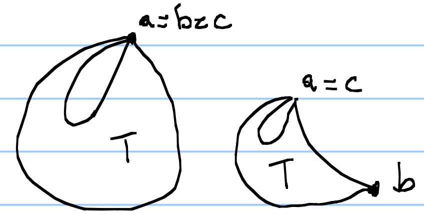
$\Rightarrow$ we need to check that the family $\{x_v\}_{v \in \mathcal{V}}$ is linearly indep.

$$\sum n_v x_v = 0$$

$$\Rightarrow \begin{cases} n_a + n_b = 0 \\ n_b + n_c = 0 \\ n_c + n_a = 0 \end{cases} \Rightarrow n_a = n_b = n_c = 0$$

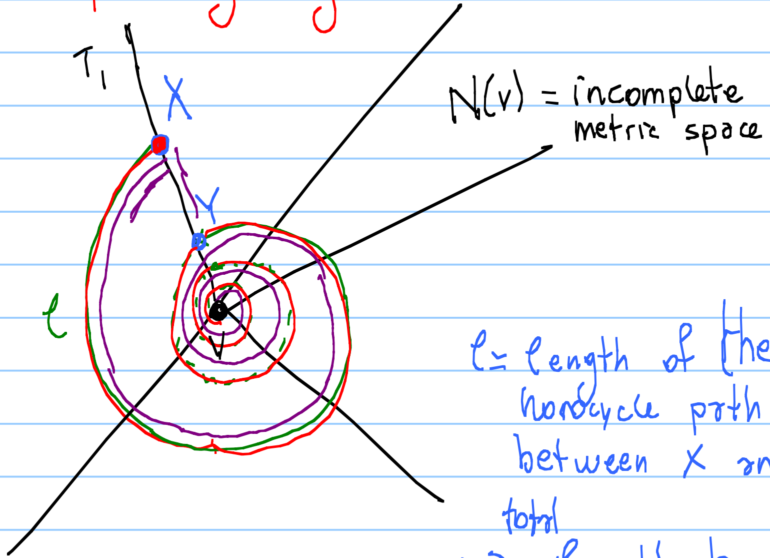
the weigth given by $\sum n_v x_v$ to the edges ab, bc, ca

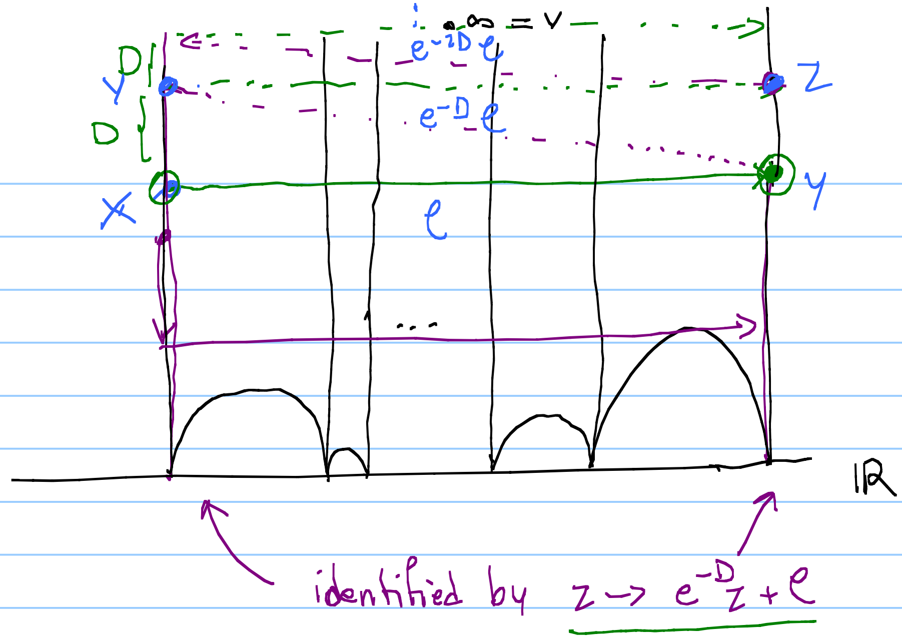




$\emptyset$

Non-complete gluings

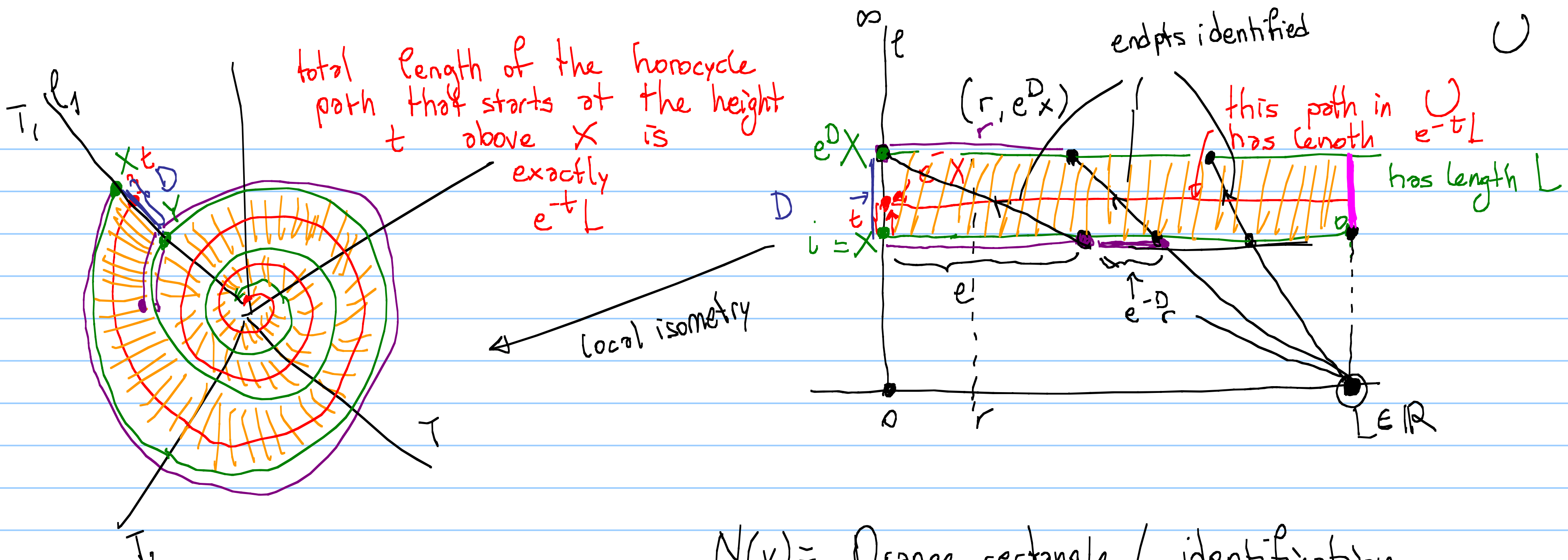




ℓ = length of the horocycle path between X and Y

$\Rightarrow$ total length horocycle path $= \sum_{n \in \mathbb{N}} e^{-nD}\ell = L < \infty$

$\Rightarrow$ When we take the completion of $N(v)$ (denoted by $\overline{N(v)}$) every horocycle path converges somewhere.



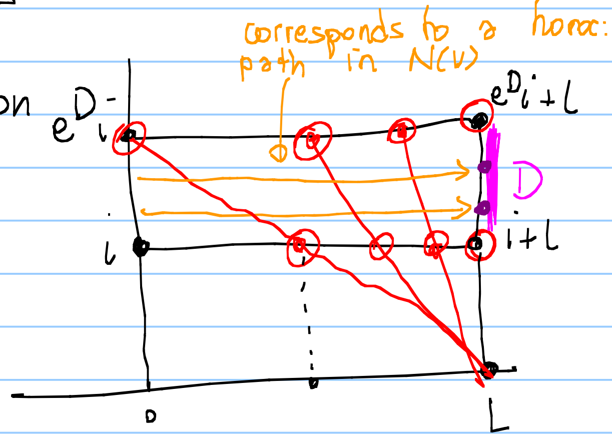
$N(v) =$ Orange rectangle / identification of the two horizontal sides

$$(r, e^D x) \simeq (e^{-D} r + \ell, x)$$

Proposition: The completion of $N(v)$ embeds isometrically in the space obtained from the closed rectangle with vertices

$R\left(i, e^D i, \; i+L, \; e^D i + L \right)$ under the identification

$$i + L = \sum_{n\in\mathbb{N}} e^{-Dn} \ell$$



on the horizontal sides given by

$$r + e^D i \sim e^{-D} r + \ell + i$$

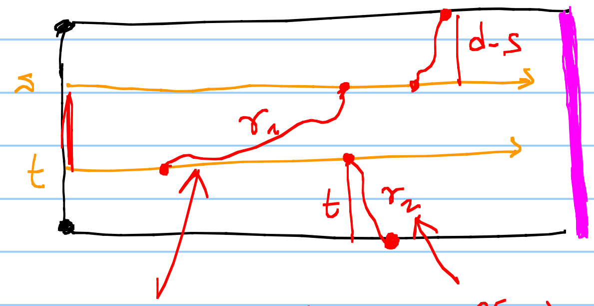
as the subset with horizontal coord $\neq L$

The metric on the quotient is the natural one induced by g_0

In particular $\overline{N(v)} - N(v) =$ closed geodesic of length $D = |d_1 + \dots + d_n|$.

Pf. We have a natural map from $\mathcal{R}/_\sim \xrightarrow{\Phi} \overline{N(v)}$. $\subset \mathcal{R} \xrightarrow{\Phi} N(v)$

We have to check that Φ is injective:

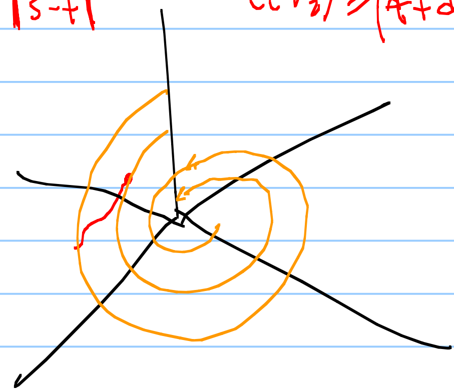


$\ell(\gamma_1) \geq |s-t|$ $\ell(\gamma_2) \geq |t+d-s|$

In both cases we have a definite distance between the two horocycles in $N(v)$

$\Downarrow$

The two horocycles cannot converge to the same point in the completion.

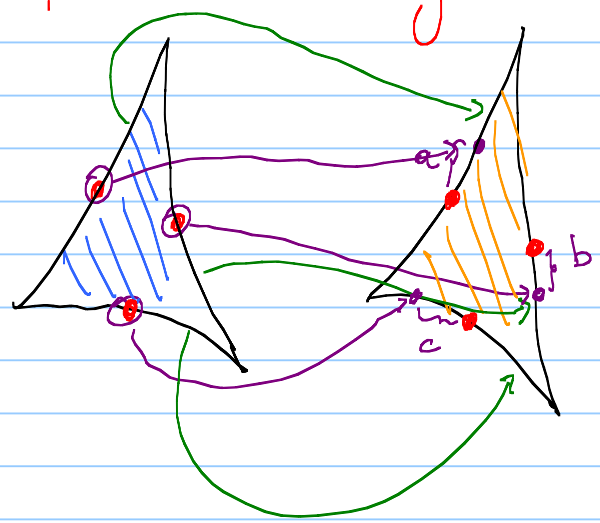


$\Rightarrow \quad R/\sim \xrightarrow{\phi} \overline{N(v)}$ is a homeomorphism

The metric on $\overline{N(v)}$ matches the natural metric on $R/\sim$ induced by g_0. $\emptyset$

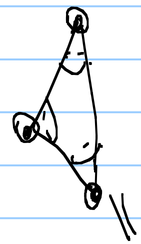
As a consequence, each point $\{d(e)\}_{e \in \text{edge pairs}} \in \mathbb{R}^{\mathcal{E}}$ corresponds to a complete hyperbolic surface with (possibly empty) totally geodesic boundary (and the boundary length at a vertex v is given by $\left|\sum_{e \text{ incident to } v} d(e)\right|$).

An example: Two triangles



green lines = the pairing

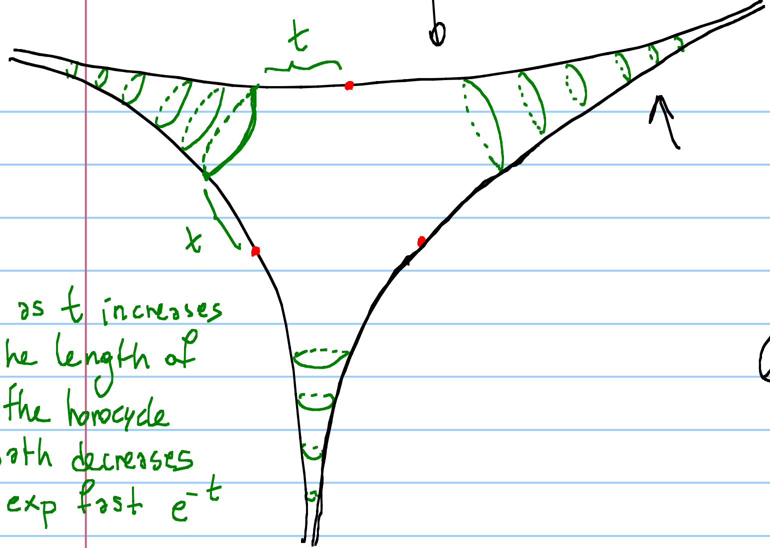
$\forall (a,b,c) \in \mathbb{R}^3$ we get a hyperbolic surface by gluing the two triangles with those parameters



quotient $\simeq \Sigma_{0,3}$

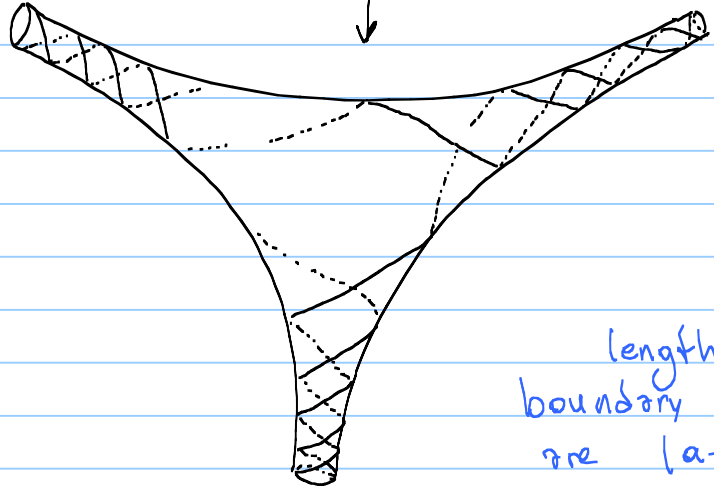
Which gluings are complete? By dim. count there is only one $a=b=c=0$.

this is the complete structure $(a,b,c)=\underline{0}\in\mathbb{R}^3$

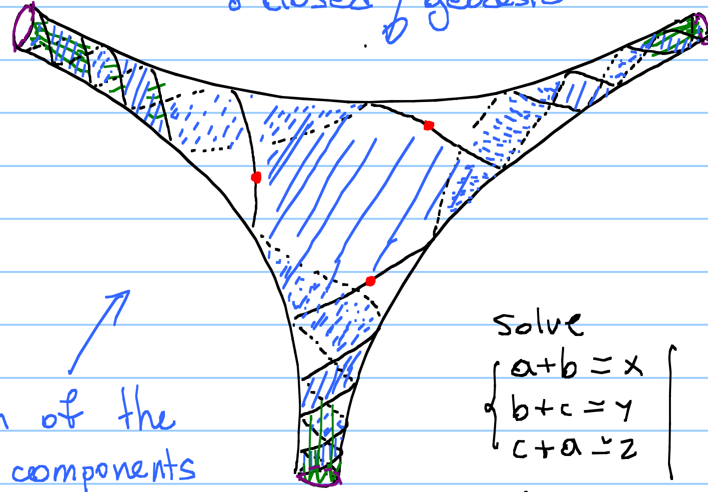


as t increases the length of the horocycle path decreases exp fast e^{-t}

generically every vertex will be incomplete



every edge of the triangle spirals around the boundary which is a closed geodesic



length of the boundary components are $|a+b|$, $|b+c|$, $|c+a|$

solve
$$\begin{cases} a+b=x \\ b+c=y \\ c+a=z \end{cases}$$

Lemma: $\forall\, x,y,z\in(0,\infty)$ $\exists$ a hyperbolic pair of pants with totally geodesic boundary of length x,y,z.