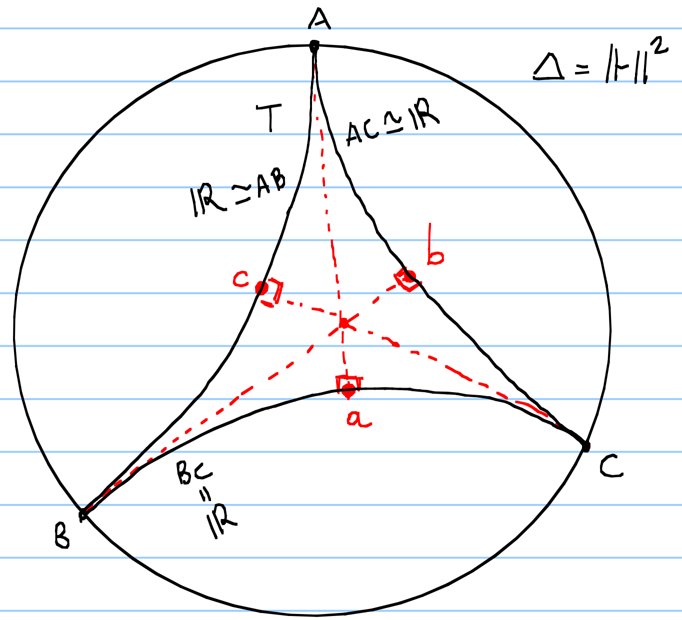
Hyperbolic Manifolds – Lecture 8

Note Title 25/11/2020

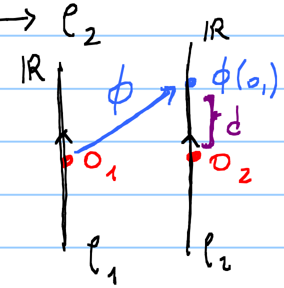


Geometric ideal triangulations:

① Take $2n$ ideal triangles $T_1, \dots, T_{2n} \subset \mathbb{H}^2$
(recall: Up to isom. $\exists!$ ideal triangle in $\mathbb{H}^2$)
with $T_i = \Delta(A_i, B_i, C_i)$.

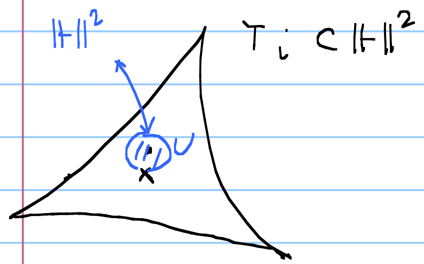
② Mark each side with a basepoint
$a_i \in \overrightarrow{B_iC_i}$, $b_i \in \overrightarrow{A_iC_i}$, $c_i \in \overrightarrow{A_iB_i}$ as in the picture

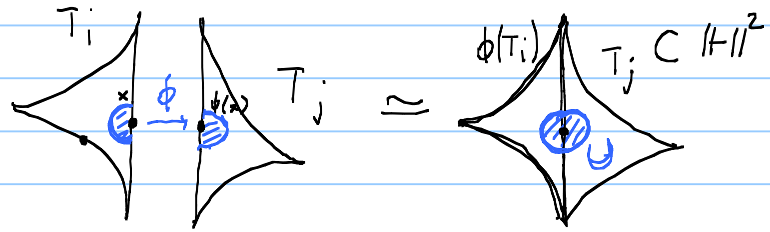
③ choose a pairing of the sides and, for each pair choose an isometry $\phi : \ell_1 \longrightarrow \ell_2$ between the corresponding geodesics.



(Recall: this can be encoded in a single real parameter d that measure the signed dist. between o_2 and $\phi(o_1)$

Note : Given $o_1 \in \vec{l}_1 \subset \mathbb{H}^2$, $o_2 \in \vec{l}_2 \subset \mathbb{H}^2$ and $d \in \mathbb{R}$ $\exists!\, \phi \in \mathrm{Isom}^+(\mathbb{H}^2)$ s.t. $\phi(\vec{l}_1) = \vec{l}_2$ (as oriented lines) and the signed distance between $\phi(o_1)$ and o_2 is $\underline{d}$)





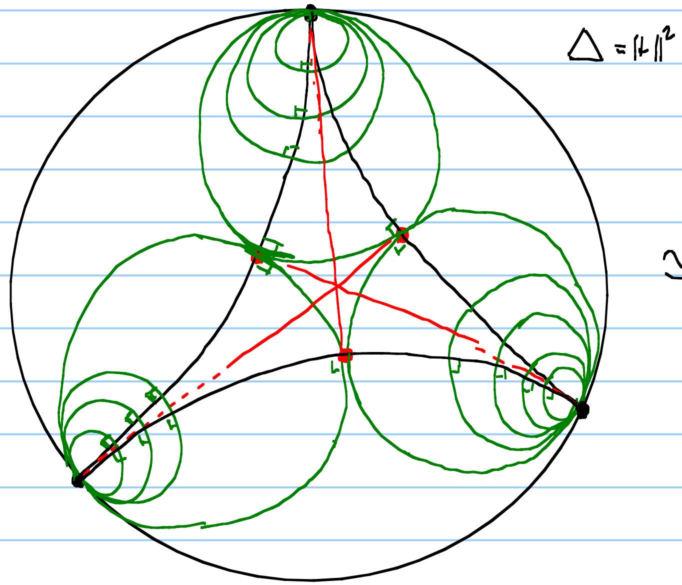
④ Form the surface with punctures

$$\Sigma = \bigsqcup_{i=1}^{2n} T_i \Big/ \bigsqcup_{e \in \text{pair of sides}} \phi_e$$

It carries a natural atlas to $\mathbb{H}^2$ (as in the picture) which gives Σ a hyperbolic mfd structure (possibly incomplete)

Today: • completeness equations
• space of parameters

Completeness



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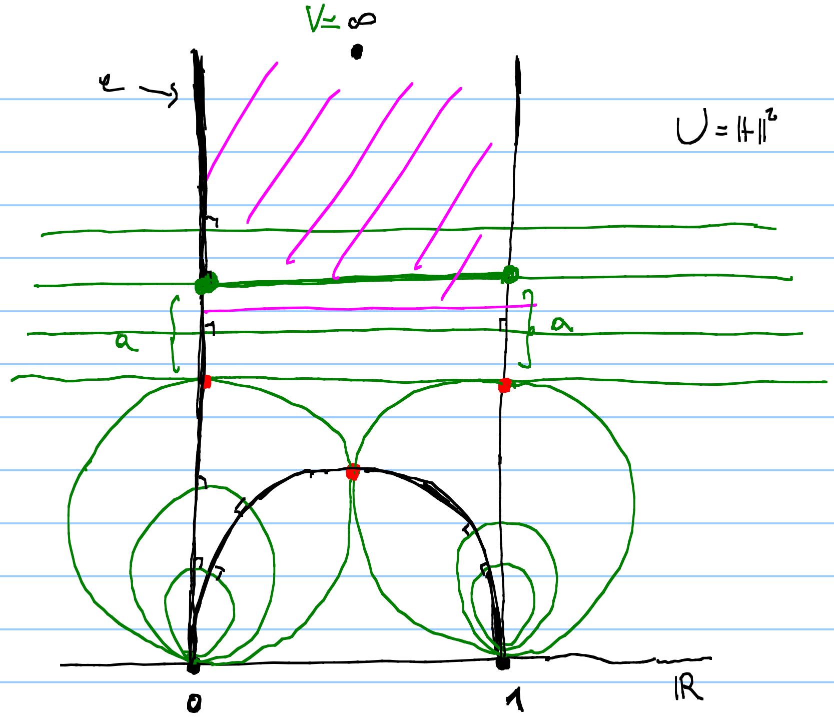


Proposition: Consider an ideal vertex v of $\Sigma = \bigsqcup T_i \,/\, \bigsqcup_{e \in \text{edge pairs}} \phi_e$.

Let $N(v)$ be a small closed neighbourhood of it in Σ

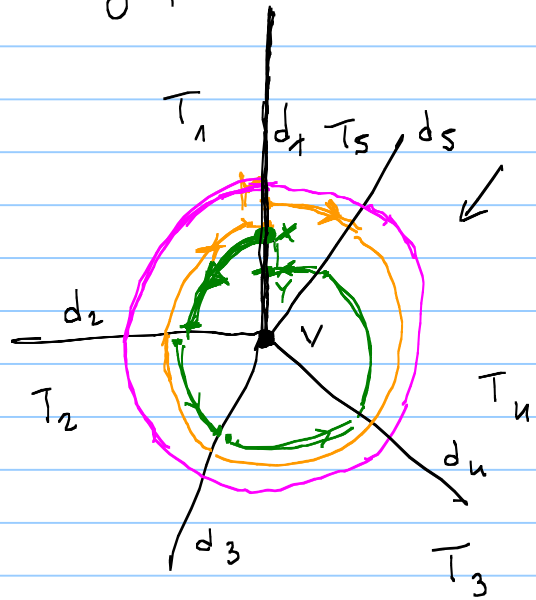
TFAE:

① $N(v)$ is complete (as a metric space)

② Every horocycle path that starts suff. close to v (and goes in the positive direction) is closed

③ There exists a closed horocycle path that starts close to v (and goes in the positive direction).

④ $d_1 + d_2 + \dots + d_n = 0$.



RmK: Σ is complete $\Longleftarrow$ $(\Longleftrightarrow)$ It is complete around each vertex v

$\Longleftrightarrow \sum_{d_i \text{ incident to } v} d_i = 0 \quad \forall\, v \in \text{Vertices}$

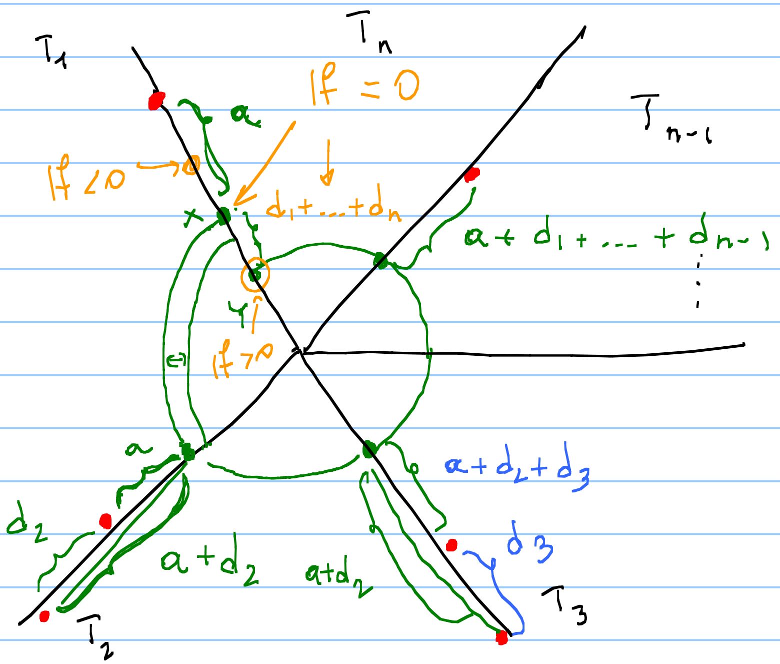
(The point is : $\Sigma - \bigsqcup_{v \in \text{Vertices}} N(v)$ is cpt)

Proof.

④ $\Longleftrightarrow$ ③ See

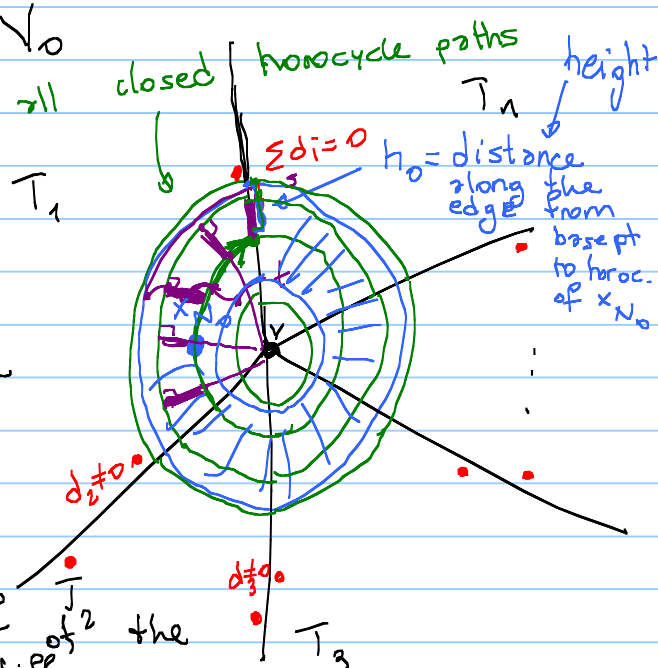
$\Updownarrow$

①



② ⇒ ① Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $N(v)$

suppose that $d(x_n, x_m) \leq \varepsilon \quad \forall \quad n,m \geq N_0$



Notice that horocycle paths separate the neigh $N(v)$ and are nested according to their height

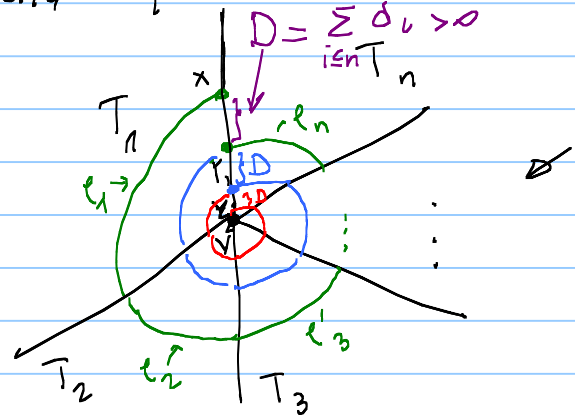
Therefore if we want to go from the horocycle of height t to the horocycle of heigth s then we must travel at least for a length of $|s-t|$

In particular if $d(x_n, x_{N_0}) < \varepsilon$ (recall the def of the path metric) the height of the horocycle path through x_n differs from the one of x_{N_0} by at most ε

$\Rightarrow$ Thus $\{x_n\}_{n\geq N_0}$ is trapped in the cpt set bounded by the horocycles of height $h_0-\varepsilon,\ h_0+\varepsilon$ $\Rightarrow$ x_n converges up to subsequences $\Rightarrow$ $N(v)$ is complete. $\varnothing$

① $\Rightarrow$ ④ Suppose that $d_1+\dots+d_n\neq 0$. Wlog $d_1+\dots+d_n>0$.

We show that we can construct a Cauchy sequence that does not converge.



$L_0 := \sum_{j\leq n}\ell_j$ = length of the horocycle path

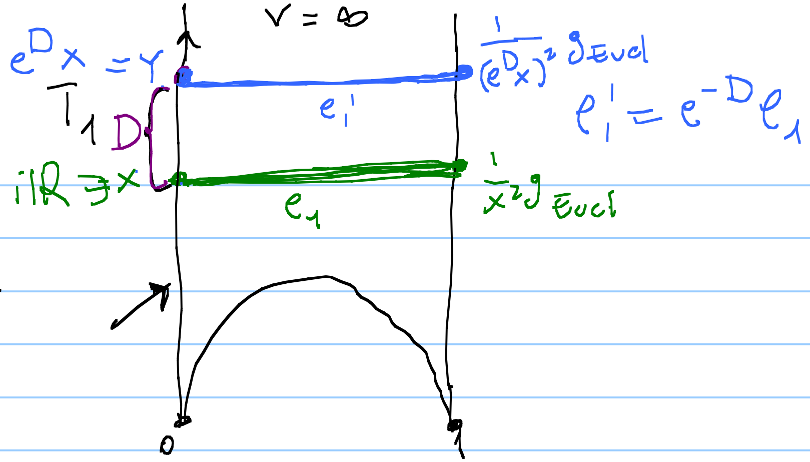
$L_1 = \sum_{j\leq n}\ell_j' = e^{-D}L_0$

$L_2 = e^{-D}L_1 = e^{-2D}L_0$

$\vdots$

$$L_n = e^{-nD} L_0$$

$$g_{|t|^2=0} = \frac{1}{\mathrm{Im}(z)^2}\, g_{\mathrm{Eucl.}}$$



$\gamma(t) = x e^t$

$D = \log \frac{y}{x}$

$d(x,y) \le$ length fo path joining them (for example the horocycle path)

$\le L_0$

$d(y_1, y_2) \le e^{-D} L_0$

$d(y_2, y_3) \le e^{-2D} L_0$

$\vdots$

(just notice that $\sum e^{-nD} L_0 < \infty$)

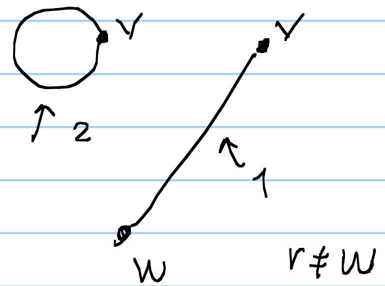
$\Rightarrow$ the sequence $\{y_n\}_{n\in\mathbb{N}}$ is Cauchy but it does not converge in $N(v)$ (as it should conv. to $v \notin N(v)$)

Space of parameters

$\mathcal{E}$ = nb of edge pairs (= number of edges in Σ $= 3(2n)/2 = 3n$)

The gluing parameters live in $\mathbb{R}^{\mathcal{E}}$

The ones that give rise to complete hyperbolic structures are singled out by the vertex equations $\{d_1 + \dots + d_n = 0\}_{v \in \text{vertices}}$ where $\{d_i\}$ are the edges incident to the vertex v with multiplicity.



Proposition: The vertex conditions are linearly independent.

Corollary: The space of complete solutions has dimension

assuming that Σ is connected

$$\dim = 3n - \underset{\text{nb of vertices}}{V} = 6g - 6 + 2p$$

g = genus of Σ $= \dfrac{n+2-V}{2}$ $\quad p = V =$ punctures