Hyperbolic Manifolds – Lecture 7

Note Title 24/11/2020

Hyperbolic surfaces

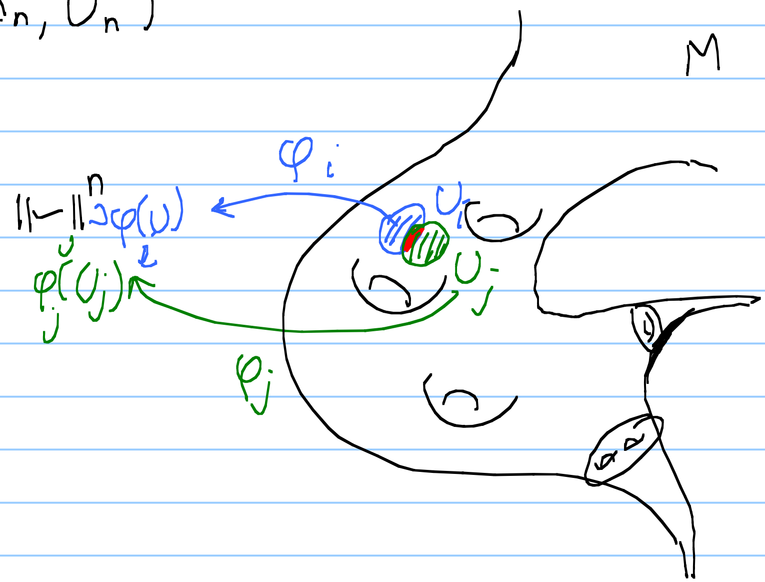
Def 1: A (complete) hyperbolic n-mfd is a (complete) Riemannian n-mfd (M,g) locally isometric to $\mathbb{H}^n$ $(= \Pi_n, \Delta_n, U_n)$

i.e. there exists an atlas

$$\mathcal{A} = \{ \varphi_j : U_j \longrightarrow \mathbb{H}^n \}_{j \in J}$$

where each φ_j is an isometry between

$$(U_j, g) \longrightarrow (\varphi(U_j), g_{\mathbb{H}^n})$$



Consider the change of charts in $\mathcal{A}$ given by

$$\begin{tikzcd} & \varphi_i(V) \subset \mathbb{H}^n \\ V = U_i \cap U_j \arrow[ur, "\varphi_i"] \arrow[dr, "\varphi_j"'] & \\ & \varphi_j(V) \subset \mathbb{H}^n \arrow[uu] \end{tikzcd}$$

$\varphi_i \varphi_j^{-1}$ is an isometry between open sets in $\mathbb{H}^n$

Obs: Any isometry $f: V \to V'$ between open sets in $\mathbb{H}^n$ is the restriction of a global isometry of $\mathbb{H}^n$

$x \in V$
$(v_1, \dots, v_n)$ o.n. basis of $T_x \mathbb{H}^n$
$\downarrow$
$f(x)$
$df_x(v_1, \dots, v_n)$

this assigment be realized also by $F \in \mathrm{Isom}(\mathbb{H}^n)$

but, since isom. are determined by the first order, we have $f = F|_V$ ← connected

Thus Def 1 is equiv to

Def 2 A (complete) hyperbolic n-mfd is a n-mfd M together with an atlas

$$\mathcal{A} = \{ \varphi_j : U_j \longrightarrow \varphi_j(U_j) \subset \mathbb{H}^n \}$$

s.t. the change of charts $\varphi_i \varphi_j^{-1}$ are restrictions of global isometries of $\mathbb{H}^n$. Denote by g the Riemannian metric on M defined locally by $\varphi_j^* g_{\mathbb{H}^n}$.

ex. check that this does not depend on the choice of local charts

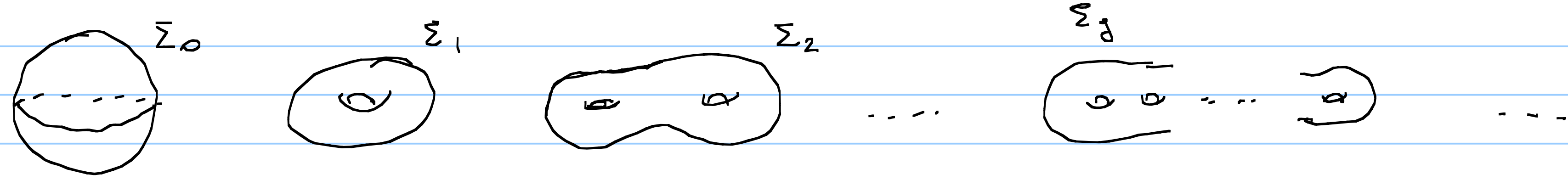
The hyp. mfd M is complete if (M, g) is complete.

Now we want to construct hyp 2-mfds, that is hyp. surfaces

Recall

Thm (Classification of surface): Any closed (cpt. + no boundary) connected orientable 2-mfd Σ is diffeomorfic to one of the following standard ones (called Σ_g)

Σ_g



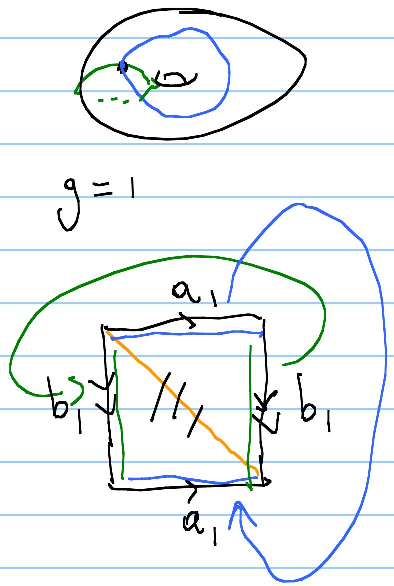
$g=0$ $\chi=2$ $\quad$ $g=1$ $\chi=0$ $\quad$ $g=2$ $\chi=-2$ $\quad$ g $\chi=2-2g$ ← genus of the surface (complete topological invariant)

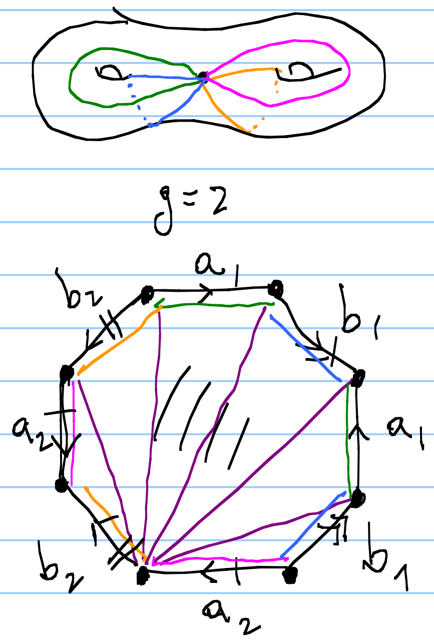
$\chi = 2-2g$

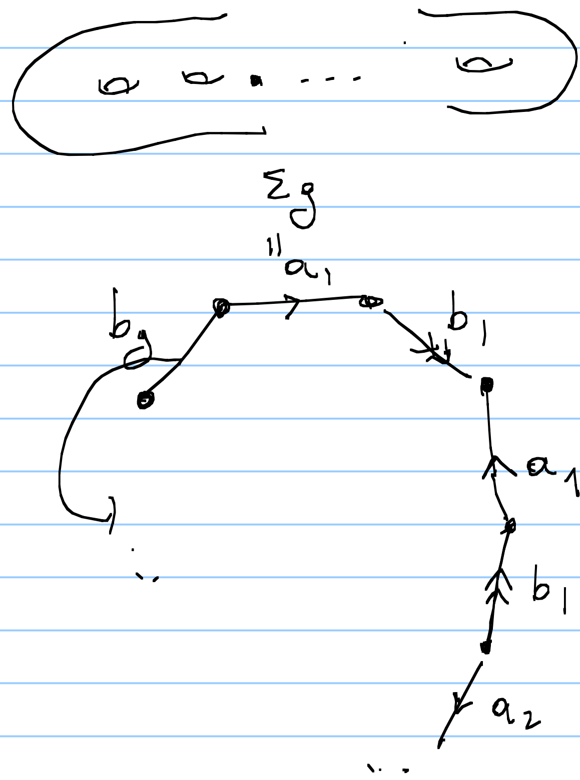
‖

Euler characteristic ← can be easily computed from a triangulation $\mathcal{T}$

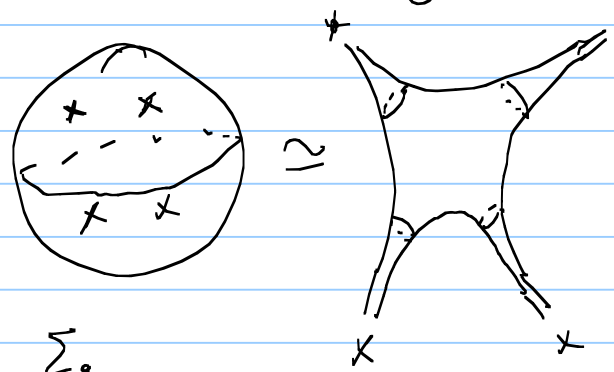
Equivalently we can present $\Sigma_{g\geq 1}$ as a quotient of a $4g$-gon with identifications of the sides





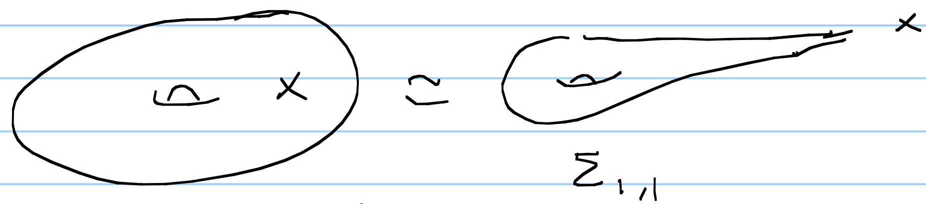


The surfaces we are going to work with have also <u>punctures</u>, that is they are obtained from a closed surface by removing a finite collection of points. The pair (genus, number of punctures) = (g,p) is a complete topological invariant of a surface with punctures.



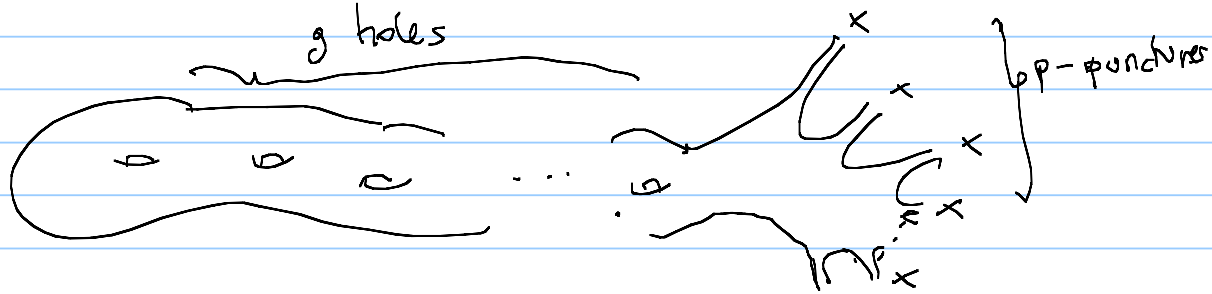
$\Sigma_{0,4}$

$\Sigma_{g,p}$



$\Sigma_{1,1}$

In general $\Sigma_{g,p}$ =

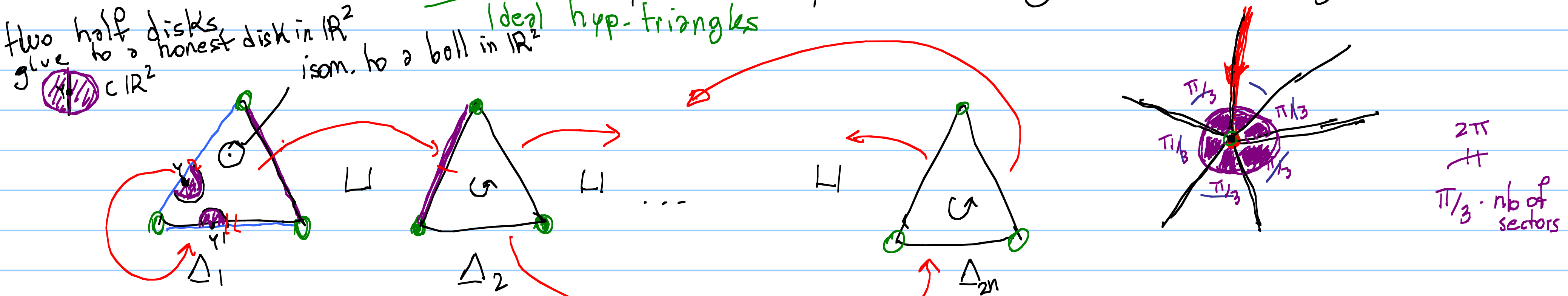


Ideal triangulations

A way to construct surfaces with punctures is the following:

① Start with $2n$ ~~Euclidean simplices~~ Ideal hyp. triangles (equilateral triangles with side length $=1$)

two half disks glue to a honest disk in $\mathbb{R}^2$ $\subset \mathbb{R}^2$ isom. to a ball in $\mathbb{R}^2$



② glue their sides in pairs with isom. (orientation reversing) — ex. check that this is a 2-mfd around every pt.
(After step 2 we get a closed orientable surface Σ, not necessarily connected)

~~③ remove the vertices from Σ = quotient~~

The resulting object is a surface with punctures

If we assume that Σ = quotient is connected, then $\Sigma \simeq \Sigma_g$
and we can compute the genus by

$$
\begin{aligned}
2-2g = \chi = & \ \# \text{ nb of triangles} \quad T = 2n \\
& - \# \text{ nb of edges} \quad E = 3T/2 = 3n \\
& + \# \text{ nb of vertices} \quad V = p
\end{aligned}
$$

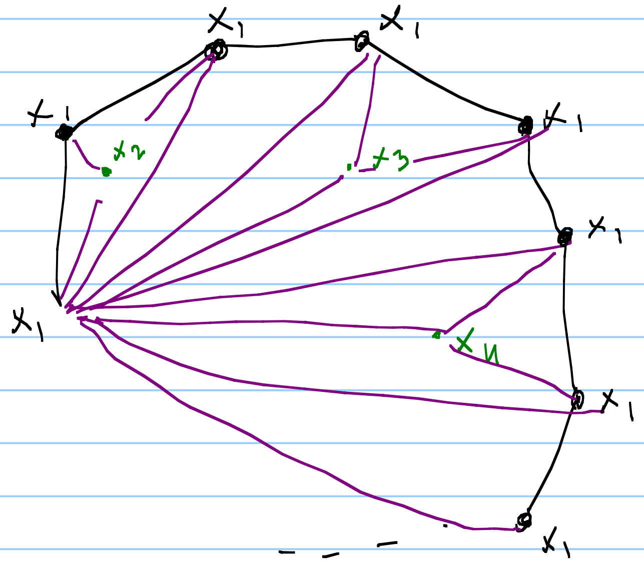
$$= V - n \quad \Rightarrow \quad g = \frac{n+2-V}{2} \qquad p = V$$

$$\Sigma - \text{vertices} = \Sigma_{g,p} = \Sigma_{\frac{n+2-V}{2}, V}$$

Proposition: Every surface with punctures has an ideal triangulation

Pf. $\Sigma_{g,p} = \Sigma_g - \{x_1, \dots, x_p\}$

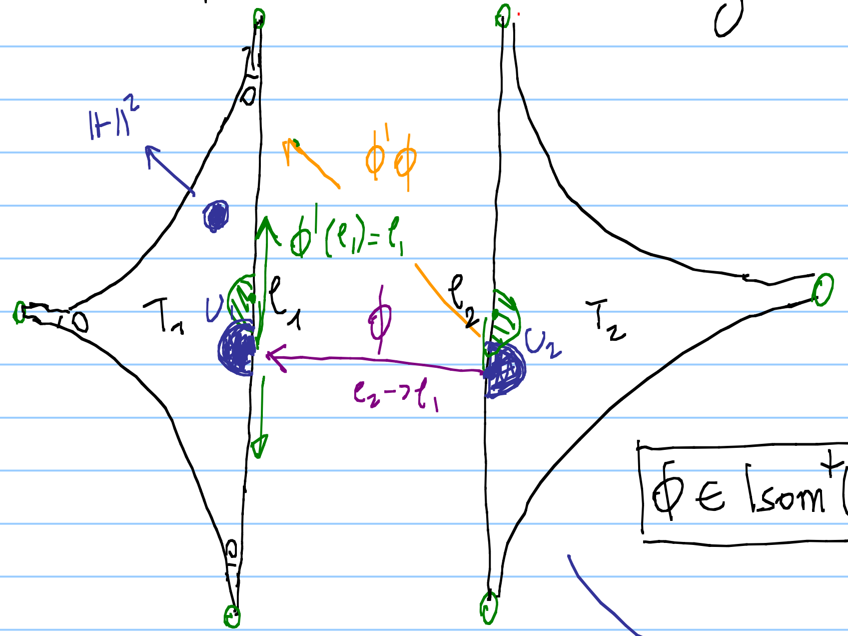
$=$

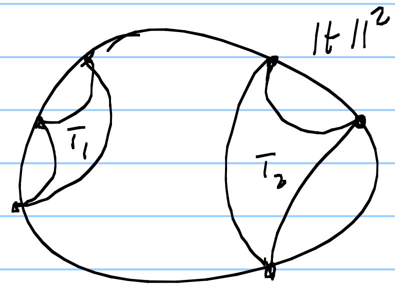


$\square$

Geometric ideal triangulations

We can replace Euclidean triangles Δ_i with <u>ideal</u> hyperbolic triangles T_i



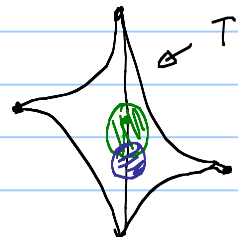


There are many isometries that carry ℓ_2 to ℓ_1

$$\boxed{\phi \in \mathrm{Isom}^+(\mathbb{H}^2)}$$

$\phi(\ell_2) = \ell_2$

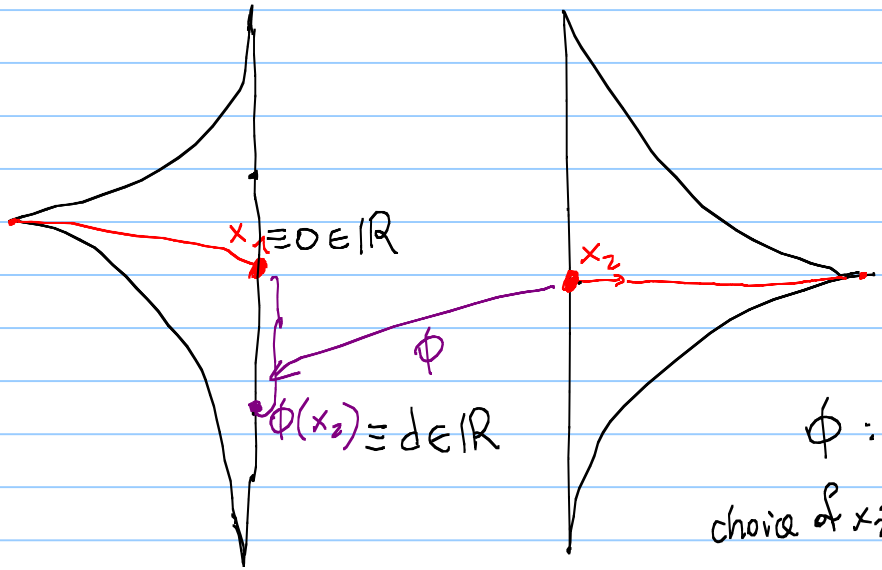
$T_1 \cup \phi(T_2) \subset \mathbb{H}^2$ a quadrilateral

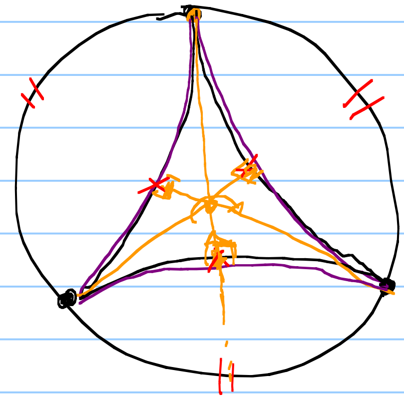


$U_1 \cup U_2 \to \mathbb{H}^2$

$\mathbb{R} \longrightarrow \mathbb{R}$

Need: For every pair of edges that must be identified we need an isometry that identifies them





$$\begin{array}{ccc} \phi : \ell_2 & \to & \ell_1 \\ \parallel & & \parallel \\ \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x + d \end{array}$$

choice of x_2 — choice of x_1

d uniquely determines ϕ!

$\Rightarrow$ For every edge pair we need a real number d that says where the basepoint of one side is mapped (by the isometry) on the other side

In conclusion we have $\underbrace{\mathbb{C}}_{\in} \subseteq \mathbb{R}^{E = \#\text{ edge pairs}}$

Proposition: For every choice of parameters $\{d(e)\}_{e \in \text{edge pairs}}$ the metrics on the triangles T_i define a Riem. metric on $\bigsqcup T_i \big/ \text{isometric pairings determined by } \{d(e)\}_{e \in \text{edge pairs}}$ which is locally isometric to $\mathbb{H}^2$.