Lecture 6 – Addendum

Note Title 19/11/2020

A

$\boxed{\text{Lemma: } \quad \mathrm{Isom}^+(U) = PSL_2\mathbb{R}}$

In the class we have seen that the transformations

$$\begin{aligned} \tau_b(z) &= z+b \qquad b\in\mathbb{R} \\ h_\lambda(z) &= \lambda^2 z \\ r(z) &= -1/z \end{aligned}$$

, which are orientation preserving isometries of U, generate $PSL_2\mathbb{R}$.

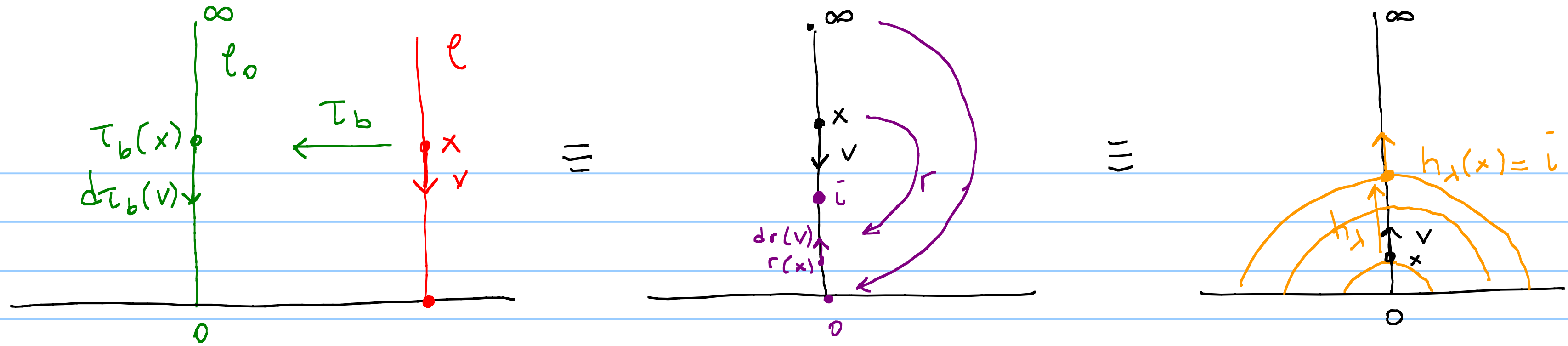
We see $PSL_2\mathbb{R}$ as a group of fractional linear transformations of $\mathbb{CP}^1 = \mathbb{C}\cup\{\infty\}$: the matrix $\begin{bmatrix} a & b \\ c & d\end{bmatrix}$ acts as $[x_0 : x_1] \in \mathbb{CP}^1 \longrightarrow [ax_0+bx_1 : cx_0+dx_1] \in \mathbb{CP}^1$ and as $z \longrightarrow \dfrac{az+b}{cz+d}$ in the affine chart $\mathbb{C} \cong \{[x_0:x_1] \mid x_1 \neq 0\} \subset \mathbb{CP}^1$ with affine coordinate $z = x_0/x_1$.

In order to conclude that $\mathrm{Isom}^+(U) = PSL_2\mathbb{R}$ we only need to show that the transformations $\{\tau_b\}_{b\in\mathbb{R}}$ $\{h_\lambda\}_{\lambda>0}$ and r generate $\mathrm{Isom}^+(U)$

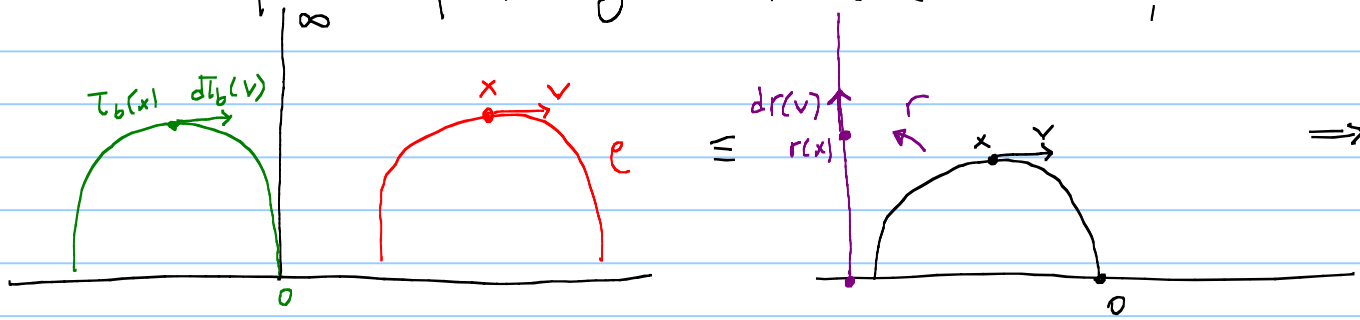
Denote by $G < \mathrm{Isom}^+(U)$ the subgroup these transformations generate.

Claim: G acts transitively on unit tangent vectors of U

Let $x \in U$ be a point and $v \in T_x U$ a unit tangent vector, denote by ℓ the geodesic starting at x with velocity v. First we consider the case where ℓ is vertical. In this case we can move the support of ℓ to the vertical line ℓ_0 joining 0 to ∞ via a translation τ_b. If after this transformation the vector v points toward 0, we use the π-rotation r centered at i to reverse the direction so that it points toward ∞. Finally, using the homothety h_λ, which is a translation along ℓ_0 we move v to the tangent vector of ℓ_0 at i pointing upwords. (See picture below)



Suppose now that ℓ is not vertical. Then, using a translation τ_b we can assume that the forward endpoint of ℓ is 0. Using the "rotation" r centered at i sending $0 \to \infty$ we can transform ℓ to a vertical line on which v is an upward pointing vector. We conclude, as in the previous case



$\Longrightarrow$ use τ_b and h_λ to bring (x, v) to $(i$, unit upward pointing vector$)$.

$\square$

We now conclude the proof of $G = \mathrm{Isom}^+(U)$ by noticing that

Claim $\Rightarrow$ G acts transitively on positively oriented base frames of U

Thus: Fix a base frame $x \in U$, (v_1, v_2) positive o.n. basis of $T_x U$.
Take $f \in \mathrm{Isom}^+(U)$. By the claim there is $g \in G$ with

$$g(x) = f(x), \quad dg_x(v_1) = df_x(v_1), \quad dg_x(v_2) = df_x(v_2)$$

but isometries that coincide at the first order coincide everywhere, so $f = g$.
$\Rightarrow$ $\mathrm{Isom}^+(U) = G$. $\square$

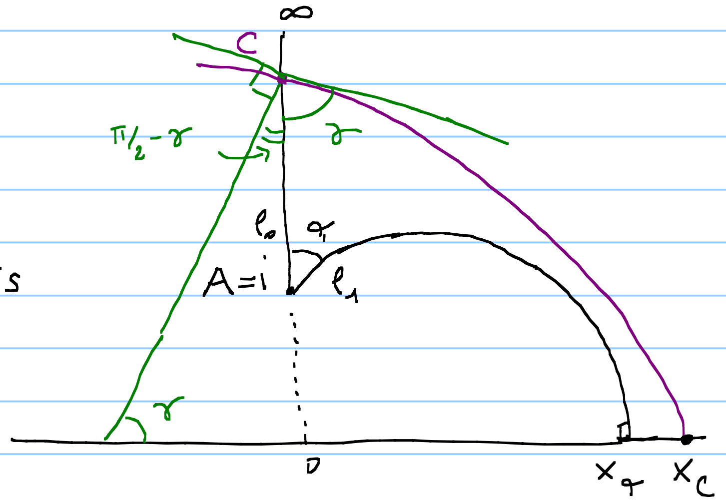
B

Proposition: $\forall\ 0 \le \alpha, \beta, \gamma \le \pi$ with $\alpha+\beta+\gamma \le \pi$ there exists a unique triangle $T \subset \mathbb{H}^2$ (up to isometries) with angles α, β, γ.

Pf. Let us prove existence first

Existence: Case $0 < \alpha, \beta, \gamma < \pi$

Consider the configuration in the picture of a vertical ray ℓ_0 starting at $A = i$ and another ray ℓ_1 on its right forming an angle of α with it. This second ray intersects the real line at $x_\alpha \in \mathbb{R}$.



For every C on the vertical ray consider the geodesic ℓ_C starting at C and forming with ℓ_0 an angle of γ. It can be constructed as follows: Take the Euclidean line passing through C and forming an angle of $\pi/2 - \gamma$ with the vertical line. It intersects the horizontal line at a point O_c with angle γ (see picture). Take ℓ_c the circle with center O_c and radius $R = \overline{O_c C}$. The geodesic ℓ_c form an angle of γ with the vertical line (because the upper half plane model is conformal).

The intersection of ℓ_c with the positive real line $(0, \infty)$ is given by

$$x_c = R - |O_c O| = |O_c C| - |O_c O| = \frac{|C|}{\sin(\gamma)} - \frac{|C|}{\tan(\gamma)}$$

There is an interval of values for $|C|$ for which ℓ_c intersects ℓ_1 and so determines a triangle T_c bounded by ℓ_0, ℓ_1 and ℓ_c. Notice that for $|C| = 1$, T_c degenerates to the point $A = i$ and hence $\text{Area}(T_{c=A}) = 0$. For $x_c = x_\alpha$ instead, T_c has an ideal vertex at x_c, so $\text{Area}(T_{x_c = x_\alpha}) = \pi - (\alpha + \gamma)$

Denote by $\beta(c)$ the third angle of the triangle T_c.

$$\boxed{\text{Claim: If } c > c' \text{, then } T_{c'} \subsetneq T_c \text{ so that } \operatorname{Area}(T_c) > \operatorname{Area}(T_{c'})}$$

Proof of claim: It is enough to check that

$$\overset{\text{distance centers}}{\overset{\shortparallel}{|O_c O_{c'}|}} + \overset{\text{radius}(c')}{\overset{\shortparallel}{|O_{c'} C'|}} < \overset{\text{radius}(C)}{\overset{\shortparallel}{|O_c C|}}$$

$$\underset{\frac{|c|}{\tan(\gamma)} - \frac{|c'|}{\tan(\gamma)}}{\underset{/\!/}{}} \qquad \underset{\frac{|c'|}{\sin(\gamma)}}{\underset{\shortparallel}{}} \qquad \underset{\frac{|c|}{\sin(\gamma)}}{\underset{\shortparallel}{}}$$

$$\iff \quad |c| - |c'| < \frac{1}{\cos(\gamma)}\left(|c| - |c'|\right) \qquad \emptyset$$

From the area computation we know that $\operatorname{Area}(T_c) = \pi - (\alpha + \gamma + \overset{\in [0,\, \pi - (\alpha+\gamma)]}{\beta(c)})$, so, by the monotonicity of the claim we obtain that, for every $0 < \beta \leq \pi - (\alpha + \gamma)$ there exists a unique C above A s.t. $\beta(C) = \beta$.

$\boxed{\text{Uniqueness:}}$ Let T be a triangle with angles $0 < \alpha, \beta, \gamma < \pi$ at A, B, C respectively.

Up to isometries we can assume that $A = i$, AC is vertical with C above A and AB on the right of AC. But then ABC is one of the triangles T_C that we constructed before. We conclude by the uniqueness part of the Existence construction ($\exists !\ C$ s.t. $\beta(C) = \beta$).

$\square$