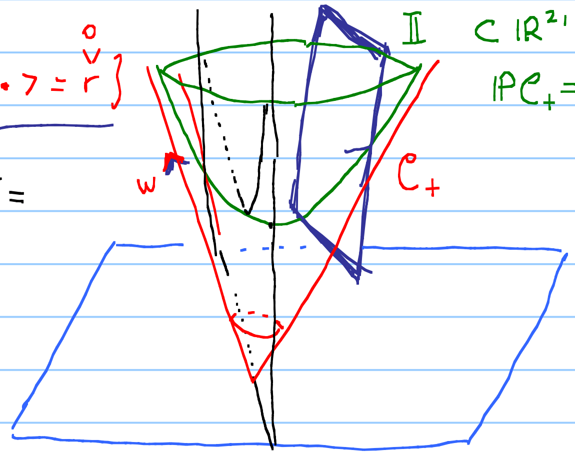
Hyperbolic Manifolds - Lecture 6

Note Title 18/11/2020

Summary: • The hyperbolic plane $\mathbb{H}^2$

• Isometries $\simeq PSL_2\mathbb{R}$, triangles, area and curvature

$\mathbb{H}^2 = \underbrace{\{\langle w,\cdot\rangle = \overset{0<}{r}\}}$

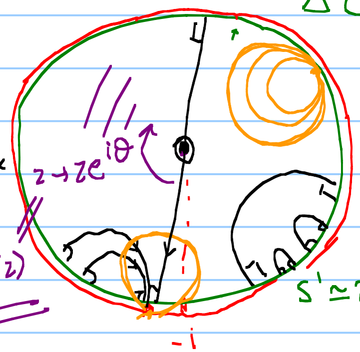


$\simeq$ stereographic projection

ϕ

$\phi(\mathcal{H})$

$\{z\in\mathbb{C} \mid |z|<1\} = \Delta \subset \mathbb{C}$

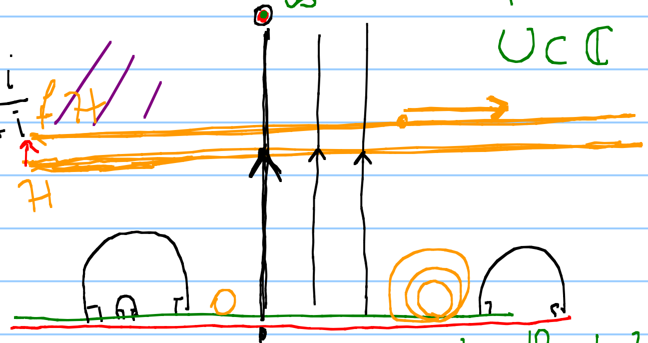


$I(z) = \dfrac{\bar z + i}{i\bar z + i}$

$\simeq$ inversion

$I(-i)$

$\{z\in\mathbb{C} \mid \mathrm{Im}(z)>0\} = U \subset \mathbb{C}$



$g_{\mathbb{I}}(x) = \langle\cdot,\cdot\rangle_{(2,1)}\big|_{x^\perp}$

$g_\Delta(z) = \dfrac{4}{(1-|z|^2)^2}\, g_{Eucl}(z)$

$g_U(x) = \dfrac{1}{\mathrm{Im}(z)^2}\, g_{Eucl}(z)$

Isometries and $PSL_2\mathbb{R}$

Normal forms of isometries in $U = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}$

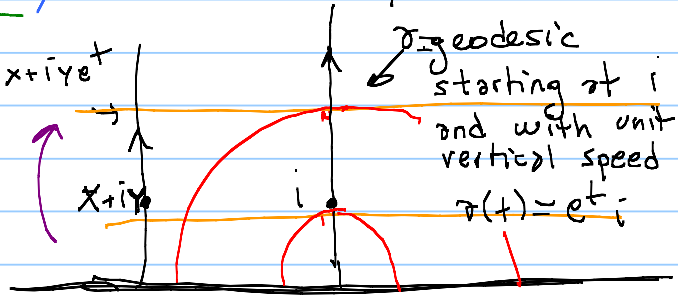
$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$

- parabolic isometries that fix ∞ : $\tau_b := (z \longrightarrow z+b)$ with $b \in \mathbb{R} \setminus \{0\}$
- loxodromic isometries that fix $\{\infty, 0\}$: $h_\lambda := (z \longrightarrow \lambda^2 z)$ with $\lambda \in (0,\infty)$ $\lambda \neq 1$
- elliptic isometries: $I R_\theta I^{-1}$ where

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ $\gamma'(t) = x + iye^t$

$$I(z) = \frac{z+i}{iz+i}$$

$$R_\theta(z) = e^{i\theta} z$$



for example : $\theta = \pi$: $r_\pi := \left(z \longrightarrow -\frac{1}{z}\right)$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

τ_b, h_λ, r_π are isometries of U

Observe that each τ_b, h_λ, r_π is the restriction of a globally defined (projective / fractional linear) transformation of $\mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$ that preserves the real projective line $\mathbb{R} \cup \{\infty\} = \mathbb{RP}^1 \subset \mathbb{CP}^1$

In particular the group generated by $\boxed{\{\tau_b, h_\lambda, r_\pi\} = \mathrm{Isom}^+(U)}$ is also a subgroup of the group of projective transformations of $\mathbb{CP}^1$ that preserve the real projective line $\mathbb{RP}^1 \subset \mathbb{CP}^1$ $= PSL_2\mathbb{R}$

$$\mathrm{Isom}^+(U) < PSL_2\mathbb{R} < \underset{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}{\underset{\Psi}{PSL_2\mathbb{C}}}$$

$$\underset{[x_0 : x_1]}{\underset{\Psi}{\mathbb{CP}^1}} = \mathbb{P}(\mathbb{C}^2_{x_0,x_1}) = \underset{\{[x_0,x_1] : x_1 \neq 0\}}{\underset{\|}{\mathbb{C} \cup \{\infty\}}}$$

$$z = \frac{x_0}{x_1}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} a x_0 + b x_1 \\ c x_0 + d x_1 \end{bmatrix}$$

$$z \longrightarrow \frac{az+b}{cz+d}$$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL_2\mathbb{R} \curvearrowright U$ as $z \in U \longrightarrow \dfrac{az+b}{cz+d} \in U$ $\|$

$Isom^+(U) < PSL_2\mathbb{R}$

Lemma: $Isom^+(U) = PSL_2\mathbb{R}$

$\tau_{a/c \in \mathbb{R}} \circ \cdots$

$\|$

because $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$bc - ad = -1$

$\uparrow$

$SL_2\mathbb{R}$

Pf. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL_2\mathbb{R}$ $\quad z \longrightarrow \dfrac{az+b}{cz+d} = \dfrac{a}{c} + \dfrac{b - \frac{ad}{c}}{cz+d}$

$$= \frac{a}{c} - \frac{1}{c^2 z + dc} = \tau_{a/c} \circ r_\pi \circ \cdots$$

$$= \tau_{a/c} \circ r_\pi \circ \tau_{cd} \circ h_{c^2}(z)$$

$\square$

A matrix criterion

Lemma: $A \in PSL_2\mathbb{R} \setminus \{Id\}$, then A is

" r_θ $\simeq$ • Elliptic $\iff |tr(A)| < 2$

" T_b $\simeq$ • parabolic $\iff |tr(A)| = 2$

" h_λ $\simeq$ • loxodromic $\iff |tr(A)| > 2$

Pf. The characteristic polynomial of $A \in SL_2\mathbb{R}$ is $p_A(T) = T^2 - tr(A)T + \underset{\overset{\|}{1}}{det(A)}$

$$= T^2 - tr(A)T + 1$$

the discriminant is $D = tr(A)^2 - 4$, so

$|tr(A)| > 2 \iff$ If $D > 0$ then A has real eigenvalues λ, λ^{-1} and is conjugate to the matrix $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \Rightarrow$ it is conjugate to h_λ

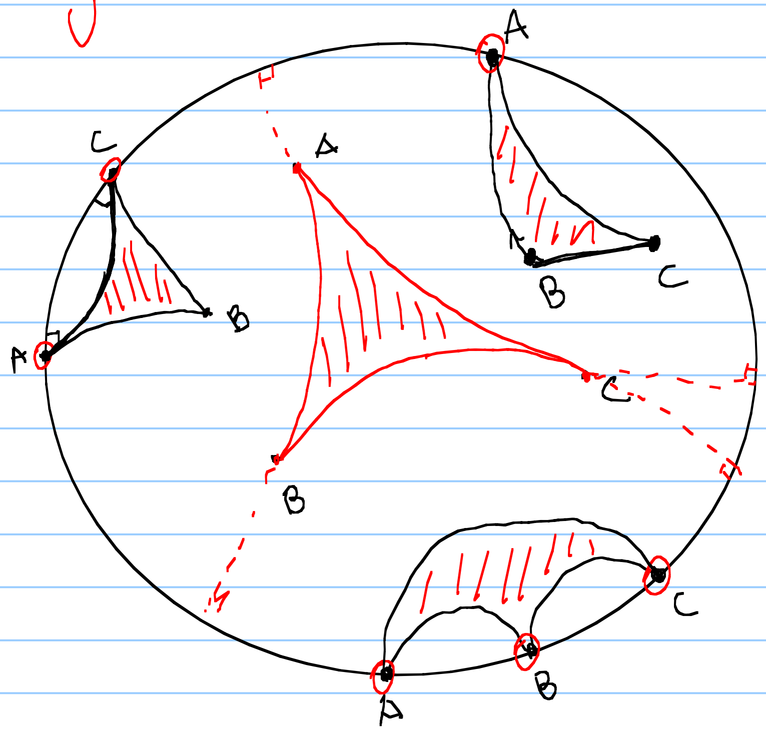
$|tr(A)|<2$ $\Rightarrow$ If $D<0$ $\Rightarrow$ A has eigenvalues $e^{i\theta}, e^{-i\theta}$ so it is conj to $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\simeq r_\theta$

If $D=0$ $\Rightarrow$ A has eigenvalue 1 with mult 2 and it is conj to $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \tau_1$

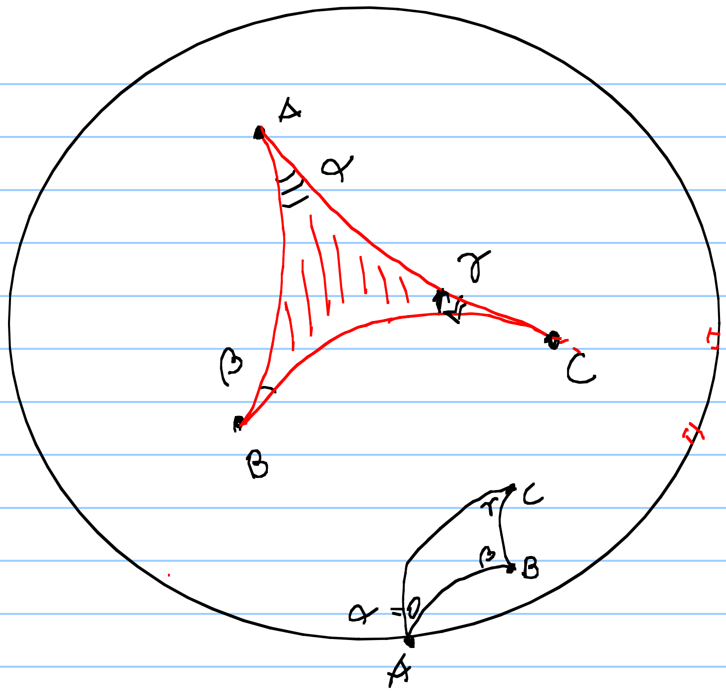
$\varnothing$

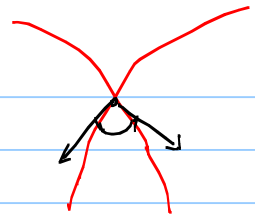
Triangles in $\mathbb{H}^2$



Every triple $A, B, C \in \mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$ of pairwise distinct pts determines a unique triangle Δ with vertices A, B, C in $\mathbb{H}^2$ bounded by the geodesic AB, BC, CA

We call those vertices that lie on $\partial_\infty \mathbb{H}^2$ <u>ideal vertices</u>



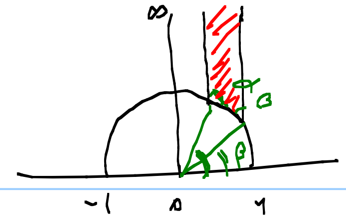


Call α, β, γ the angles at A, B, C

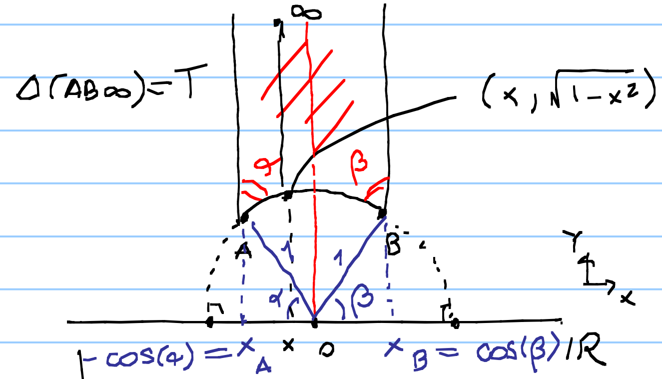
(angle formed by the lines AB, AC) ...

If a vertex is an ideal vertex then we define its angle to be <u>0</u>.

Proposition: $\text{Area}(\Delta(A,B,C)) = \pi - (\alpha+\beta+\gamma)$



Pf. We work with $U = \mathbb{H}^2$



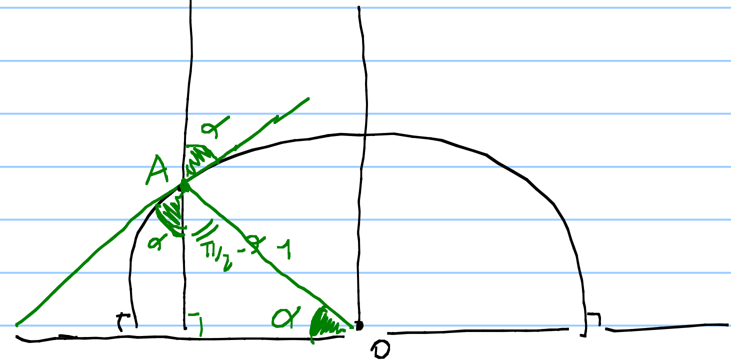
Case $C \in \partial_\infty U$: Up to isometries we can assume that $C = \infty$ and that $\overset{\frown}{AB}$ is an arc of the unit circle centered at O

$\gamma = 0$

$$\sqrt{\det(g)}\,dx\,dy = \frac{1}{y^2}\,dx\,dy$$

$$\text{Area}(\Delta(AB\infty)) = \int_T \underset{\parallel}{d\text{vol}_{g_U}} = \int_T \frac{1}{y^2}\,dx\,dy$$

$$= \int_{-\cos\alpha}^{\cos\beta} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}\,dy\,dx$$

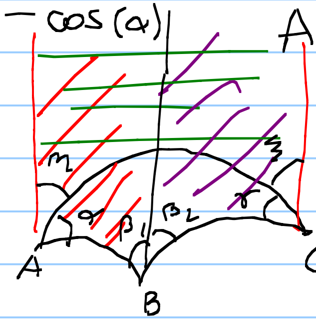


$$= \int_{-\cos(\alpha)}^{\cos(\beta)} \left[-\frac{1}{Y} \right]_{\sqrt{1-x^2}}^{\infty} dx$$

$$= \int_{-\cos(\alpha)}^{\cos(\beta)} \frac{1}{\sqrt{1-x^2}}\, dx = \Big[-\arccos(x) \Big]_{\cos(\pi-\alpha)}^{\cos(\beta)} = \pi - (\alpha+\beta)$$

$$\cos(\pi-\alpha) = -\cos(\alpha)$$

$$D = \infty \qquad \pi - (\alpha+\beta) = \text{Area}$$



General case :

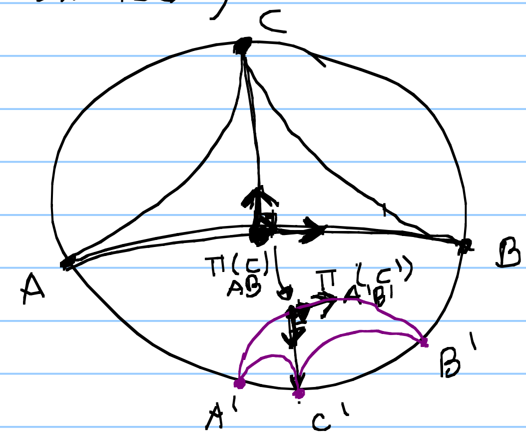
$$\text{Area}(\Delta(ABC))$$
$$=$$
$$\text{Area}(ABD) + \text{Area}(BCD) - \text{Area}(ACD)$$

$$\text{Area}(ABD) = \pi - (\beta_1 + \alpha + \eta), \quad \text{Area}(BCD) = \pi - (\beta_2 + \gamma + \xi), \quad \text{Area}(ACD) = \pi - (\eta + \xi)$$

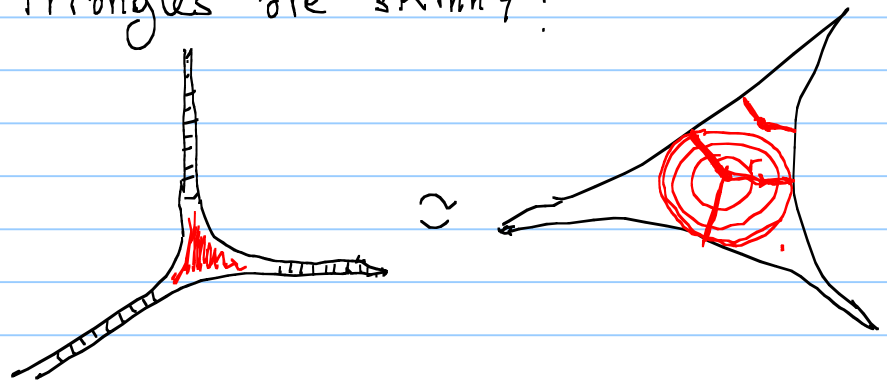
$$= \pi - (\alpha + \beta + \gamma) \qquad \square$$

Remark: • Area $\leq \pi$ always! (And it is equal to π exactly when A, B, C are all ideal)

All triangles with all ideal vertices are isometric to each other!



• Triangles are skinny!



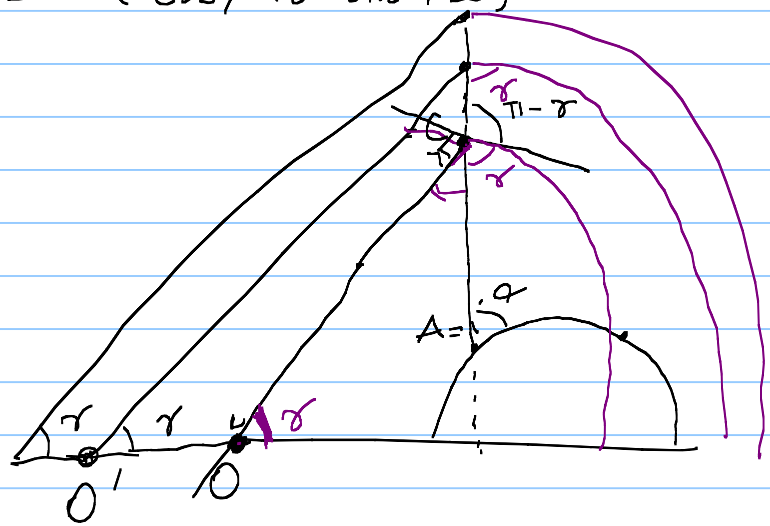
Since $\text{area}(B(r)) \uparrow \infty$ as $r \uparrow \infty$ we have a uniform upper bound on the radius of a ball that can be put inside a triangle!

Proposition: For every $0 \le \alpha, \beta, \gamma \le \pi$ with $\alpha+\beta+\gamma < \pi$
There exists a unique triangle $\Delta(ABC) \subset \mathbb{H}^2$ (up to isometries)
with angles α, β, γ at A, B, C.

Pf. Let us not worry about degenerate cases (easy to analyze)

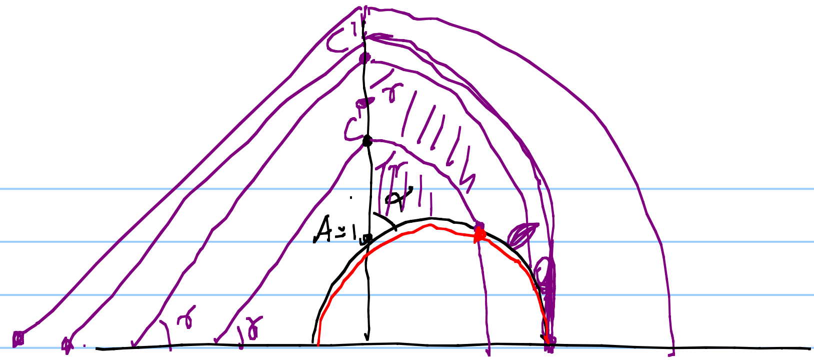
Assume: $0 < \alpha, \beta, \gamma < \pi$ (all vertices in $\mathbb{H}^2$)

We work in $U = \mathbb{H}^2$. Up to isometries we can assume $A = i$ and C lies above A on a vertical line



$\Rightarrow$ B will lie on a circle that form an angle of α with the vertical line

$\Rightarrow$ get a family of triangles indexed by points above A on the vertical line with $\angle A = \alpha$ $\angle C = \gamma$



$$\text{Area}(T_c) = \pi - (\alpha + \gamma + \beta(c))$$

Since the triangles are monotone $T_c \subset T_{c'}$ if $c' > c$

$\Rightarrow$ $\text{Area}(T_c) \uparrow$ as $c \uparrow$

$\Rightarrow$ $\beta(c) \downarrow$ and the extremal values that it reaches are 0 and $\pi - (\alpha + \gamma)$

$\Rightarrow$ $\exists!\, c$ s.t $\beta(c) = \beta$. $\emptyset$