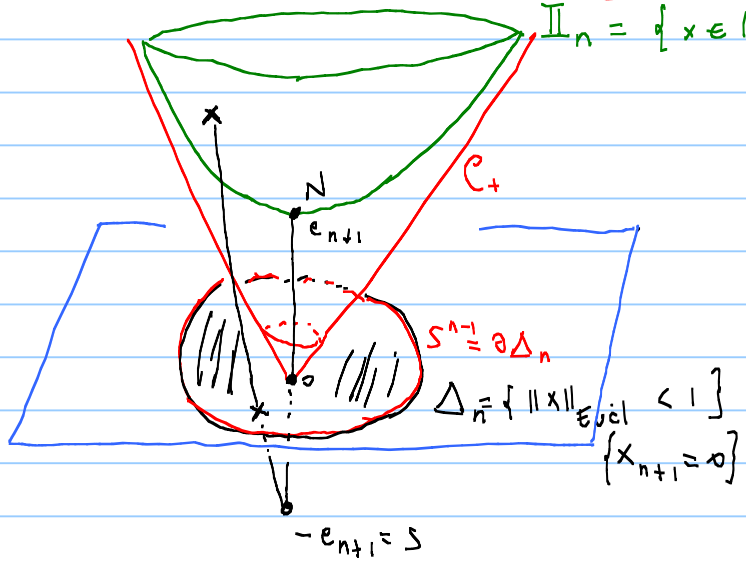
Hyperbolic Manifolds - Lecture 5

Note Title 17/11/2020

Summary: Poincaré disk and Upper half space models

$\mathbb{I}_n = \{ x \in \mathbb{R}^{n,1} \mid \|x\|^2_{(n,1)} = -1, \ x_{n+1} > 0 \}$



- $\phi : \mathbb{I}_n \longrightarrow \Delta_n$

$$\phi(\underline{x}, x_{n+1}) = \frac{1}{1 + x_{n+1}} \underline{x}$$

$\underline{x} = (x_1, \dots, x_n)$

- $\phi^{-1} : \Delta_n \longrightarrow \mathbb{I}_n$

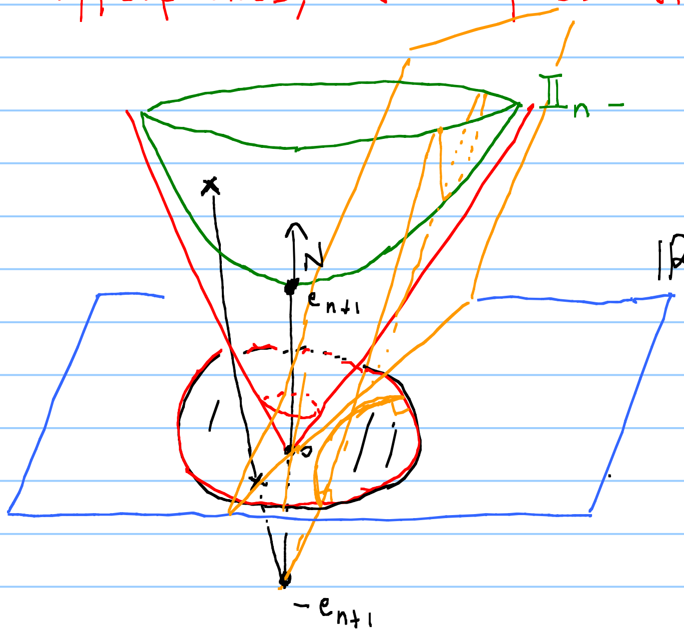
extends to a homeomorphism $\Delta_n \cup S^{n-1} \longrightarrow \mathbb{I}_n \cup \partial_\infty \mathbb{I}_n$

$$\phi^{-1}(\underline{x}) = \frac{1}{1 - \|\underline{x}\|^2} \left(2\underline{x}, \ 1 + \|\underline{x}\|^2 \right)$$

$$\left[(\phi^{-1})^* g_{\mathbb{I}_n} = \frac{4}{(1 - \|\underline{x}\|^2)^2} \, g_{Eucl} =: g_{\Delta_n} \right]$$

conformal: they measure the same angles

Hyperplanes, K-subspaces and geodesics in Δ_n



Lemma: Hyperplanes in Δ_n are intersections of hyperplanes or $(n-1)$-spheres of $\mathbb{R}^n = \{x_{n+1}=0\}$ orthogonal to S^{n-1} with Δ_n

Pf. $H = V \cap \mathbb{I}_n$ hyperplane

If $\underline{e_{n+1} \in V} \Rightarrow \phi(H) = V \cap \mathbb{R}^n \perp S^{n-1}$

Notice that ϕ commutes with $O(n)$ $\left(\text{seen as } \begin{pmatrix} O(n) & 0 \\ 0 & 1 \end{pmatrix} \right)$

Also notice that $O(n)$ sends $(n-1)$-spheres in $\mathbb{R}^n \perp S^{n-1}$ to $(n-1)$-spheres $\perp S^{n-1}$.

$\cap$

$\underline{\mathrm{Isom}(\mathbb{I}_n)}$

So it suffices to show the claim for a particular $V = w^\perp$

$\uparrow$ space like vector $w = V^\perp$

Up to $O(n)$ we can assume that $w = (\alpha, 0, \dots, 0, 1)$.

$$V = \{ \alpha x_1 - x_{n+1} = 0 \} = w^\perp$$

$$\underset{\substack{\| \\ V \cap \mathbb{I}_n}}{} \quad \phi^{-1}(\underline{x}) \in H \iff \frac{1+\|\underline{x}\|^2}{\cancel{1-\|\underline{x}\|^2}} = 2\alpha \frac{x_1}{\cancel{1-\|\underline{x}\|^2}} \iff (x_1-\alpha)^2 + x_2^2 + \dots + x_n^2 = \alpha^2 - 1 > 0$$

$$\phi^{-1}(\underline{x}) = \frac{1}{1-\|\underline{x}\|^2}\left(2\underline{x}, 1+\|\underline{x}\|^2\right)$$

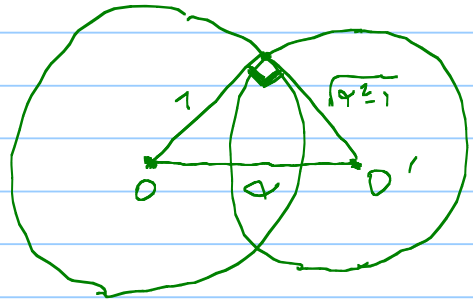
$\|\underline{x}\| < 1$

w is space-like

$0 < \|w\|^2 = \alpha^2 - 1$

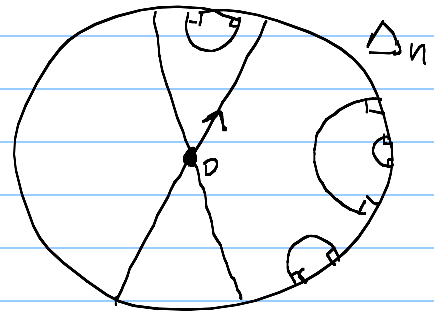
This is the equation of a $(n-1)$-sphere centered at $(\alpha, 0, \dots, 0) = O'$ with radius $R^2 = \alpha^2 - 1$ which is $\perp S^{n-1}$

(because $R^2 + 1 = (\text{distance between centers})^2$ $\boxtimes$



Corollary: k-subspaces in Δ_n are intersections of k-planes and k-spheres of $\mathbb{R}^n$ orthogonal to S^{n-1} with Δ_n.

Corollary: Geodesics are either diameters of Δ_n or arcs of circles orthogonal to S^{n-1}



The parametrization of those passing through the origin is

$$\exp_0(tv) = \phi\left(\langle e_{n+1}, v \rangle \cap \mathbb{I}_n \right)$$

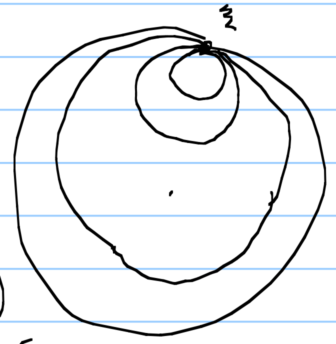
$\|v\|=1$

$$= \phi\left(\cosh(t) e_{n+1} + \sinh(t) \underset{\mathbb{R}^n}{\overset{}{\underset{\Psi}{v}}} \right)$$

$$= \frac{\sinh(t)}{1+\cosh(t)} v = \tanh(t/2)\, v.$$

Spheres and horospheres

Lemma: Horospheres in Δ_n are spheres tangent to S^{n-1} (and the tangent point is their center at infinity $S^{n-1} = \partial_\infty \Delta_n$)



Pf. $\mathcal{H} = \{ \langle w, \cdot \rangle_{(n,1)} = r \}$ $\quad r<0$

$w \in C_+ =$ isotropic cone

$[w] \in \mathbb{P} C_+ = \partial_\infty \mathbb{I}_n$

By $O(n)$-equiv. of ϕ (notice that $O(n)$ sends spheres tangent to S^{n-1} to spheres tangent to S^{n-1})

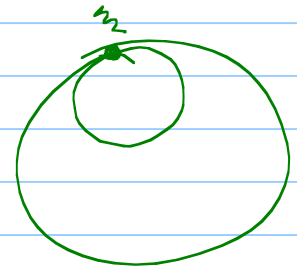
we can assume $w = (1, 0, \dots, 0, 1) \Rightarrow \mathcal{H} = \{ x_1 - x_{n+1} = \underset{\wedge_0}{r} \}$ $\quad (\underline{x}, x_{n+1}) \in \mathbb{I}_n$

$$\phi^{-1}(\underline{x}) \in \mathcal{H} \iff \frac{1 + \|\underline{x}\|^2}{1 - \|\underline{x}\|^2} - \frac{2x_1}{1 - \|\underline{x}\|^2} = r$$

$$\iff \quad 1+\|\underline{x}\|^2 - 2x_1 = r\left(1-\|\underline{x}\|^2\right)$$

$$\iff \quad \|\underline{x}\|^2 + \frac{2}{r-1}\,x_1 = \frac{r+1}{r-1}$$

$$\iff \quad \left(x_1+\frac{1}{r-1}\right)^2 + x_2^2 + \dots + x_n^2 = \frac{r^2}{(r-1)^2}$$



This is a sphere S centered at $O' = \left(-\frac{1}{r-1}, 0, \dots, 0\right)$ of radius $\frac{|r|}{|r-1|} = R$

notice that $d_{Eucl}(O', O) + R = 1$ so S is tangent at S^{n-1} at $\xi = (-1, 0, \dots, 0)$

$$\Updownarrow$$

$$\underbrace{[(\xi,1)]}_{=[w]} \in \underbrace{\partial_\infty \Pi_n}_{=S^{n-1}}$$

$\square$

Lemma: Metric balls in Δ_n are Euclidean n-disks
(Warning: the center is not the metric center)

$$\cosh(d(x,y)) = -\langle x,y\rangle_{(n,1)}$$

Pf. $B(x \in \mathbb{I}_n, r) = \{ y \in \mathbb{I}_n \mid -\langle x,y\rangle \leq \cosh(r) \}$

$$\partial B = \{ y \in \mathbb{I}_n \mid \langle x,y\rangle = -\cosh(r) \}$$

Up to $O(n)$ we can assume $x = (\alpha, 0, \dots, 0, \beta)$ with $\alpha^2 - \beta^2 = -1$

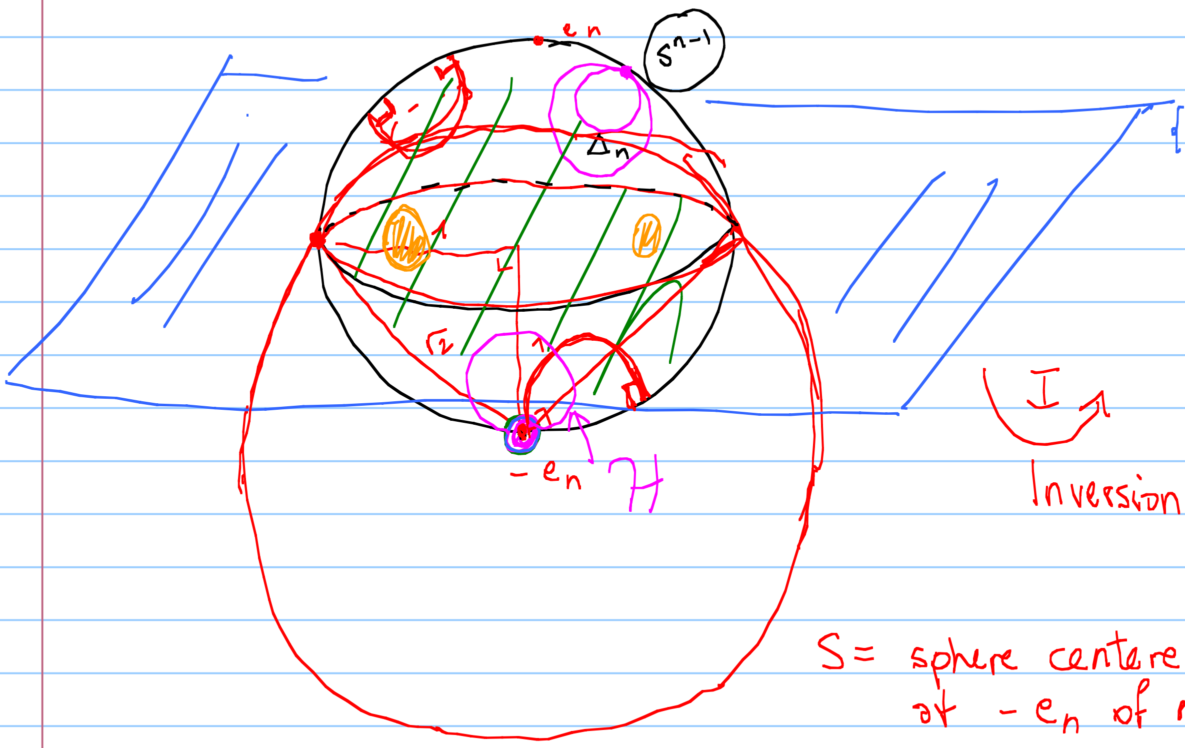
$$\phi^{-1}(\underline{x}) \in \partial B \iff \frac{2\alpha x_1}{1 - \|\underline{x}\|^2} - \beta \frac{1 + \|\underline{x}\|^2}{1 - \|\underline{x}\|^2} = -\cosh(r) = -R$$

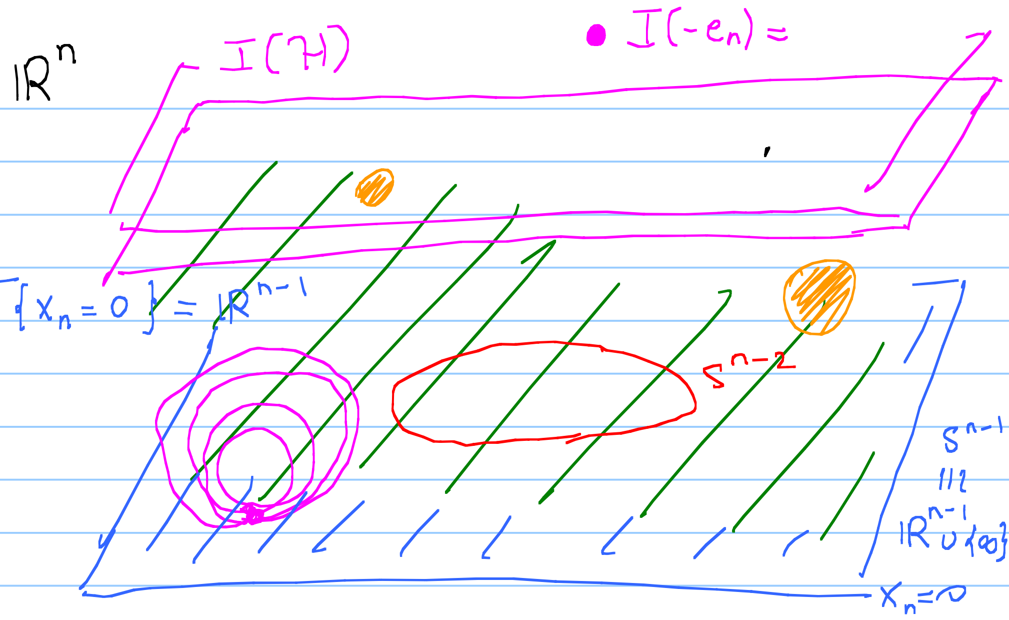
$$\vdots$$

This is a sphere with center $\left(\frac{\alpha}{R+\beta}, 0, \dots, 0 \right)$ radius $\frac{\sqrt{R^2-1}}{R+\beta}$. $\varnothing$

Upper half space model

$\mathbb{R}^n$



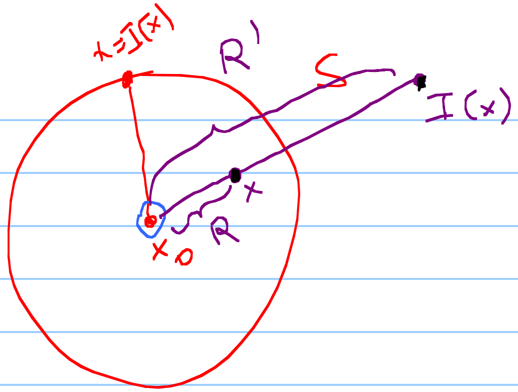


$\overset{I}{\curvearrowright}$

Inversion around S

$S =$ sphere centered at $-e_n$ of radius $R = \sqrt{2}$

- $I(\Delta_n) = \{x_n > 0\}$
- $I(S^{n-1}) = \{x_n = 0\} = \mathbb{R}^{n-1}$
- $I(-e_n) = \infty$



$$\boxed{RR' = r(S)^2}$$

$x_0 = -e_n \qquad r(S) = \sqrt{2}$

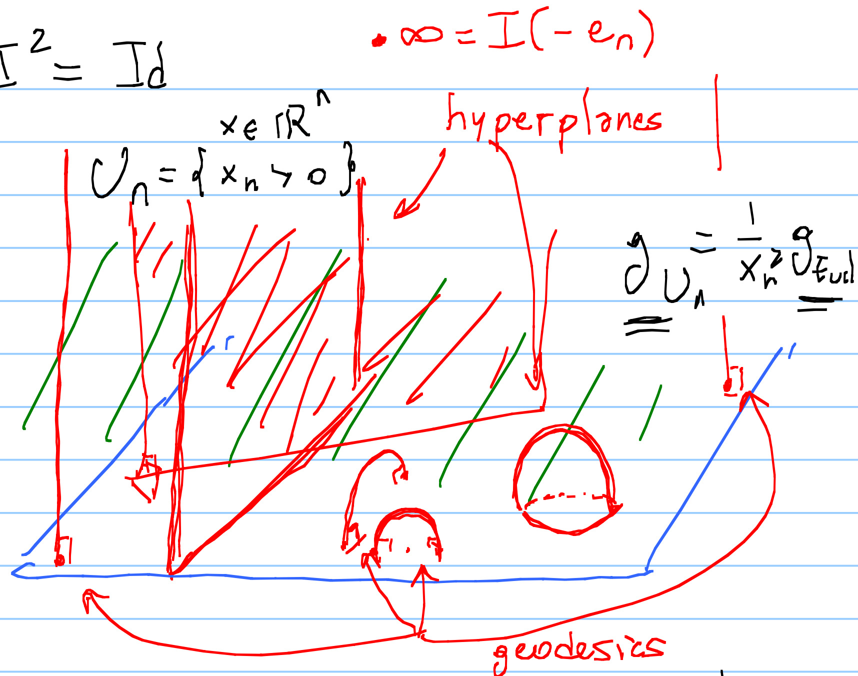
$$I(x) = -e_n + \frac{2}{|x+e_n|^2}(x+e_n)$$

Fact: **Inversions** send:

1. • spheres not passing through the center x_0 to spheres not passing through the center
2. • spheres passing through the center x_0 to hyperplanes not passing through x_0 and viceversa
3. • leave invariant all hyperplanes passing through x_0

Inversions are (anti)conformal and involutive $I^2 = Id$

Lemma: $(I^{-1})g_{\Delta_n} = \frac{1}{x_n^2} g_{Eucl} = g_{U_n}$



By properties ①, ② and ③ of inversions, we have

Lemma: k-subspaces of U_n are upper half k-spheres with center on $\mathbb{R}^{n-1}$ ($\Leftrightarrow \perp \mathbb{R}^{n-1}$) and upper half vertical k-planes (orthogonal to $\mathbb{R}^{n-1}$)

Lemma: Horospheres not centered at ∞ are spheres tangent to $\mathbb{R}^{n-1}$ at their center. Horospheres centered at ∞ are horizontal Euclidean affine hyperplanes.

Corollary: Horospheres, with their intrinsic metric are flat

Pf. $\mathcal{H}$ centered at ∞, $\mathcal{H} = \{x_n = t\}$ $\quad g_{U_n}\big|_{\mathcal{H}} = \frac{1}{t^2}\, g_{Eucl}$

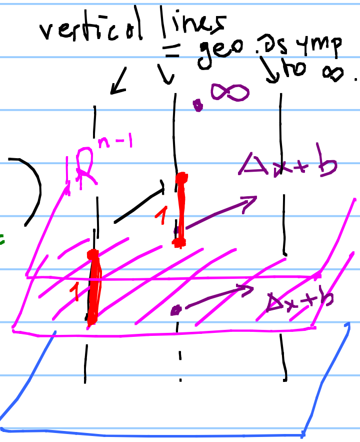
$$\left(\mathcal{H}, g_{U_n}|_{\mathcal{H}}\right) \underset{\text{isometric}}{\cong} \left(\mathbb{R}^{n-1}, \frac{1}{t^2} g_{Eucl}\right). \quad \varnothing$$

Lemma: Metric balls in U_n are Euclidean disks.

Normal forms for parabolic and loxodromic isometries in $U_n = \{(x,t) \mid {x \in \mathbb{R}^{n-1} \atop t>0}\}$

Proposition: $f \in \mathrm{Isom}(U_n, g_{U_n})$ with $\infty \in \mathrm{fix}(f)$. We have

- (i) If f is parabolic then $f(x,t) = (Ax+b, t)$ for some $A \in O(n-1)$, $b \in \mathbb{R}^{n-1} \setminus \{0\}$,



- (ii) If f is loxodromic with $\mathrm{fix}(f) = \{0, \underset{I(-e_n)}{\infty}\}$, then $f(x,t) = (\lambda A x, \lambda t)$ with $A \in O(n-1)$, $\lambda > 0$, $\lambda \neq 1$.