Hyperbolic Manifolds – Lecture 4

Note Title 11/11/2020

Summary:
- loxodromic and parabolic isometries ||
- conformal models of $\mathbb{H}^n$: Poincaré disk Δ_n and upper half space U_n ||

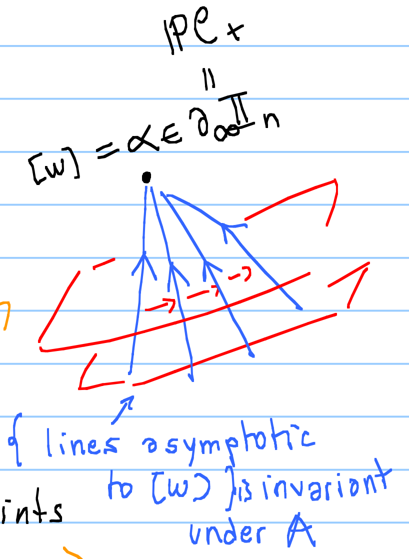
($\Rightarrow$ $\boxed{\text{dictionary between } \mathbb{I}_n \text{ and } \Delta_n}$)

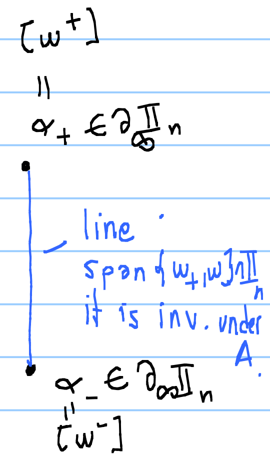
$Aw = \lambda w$

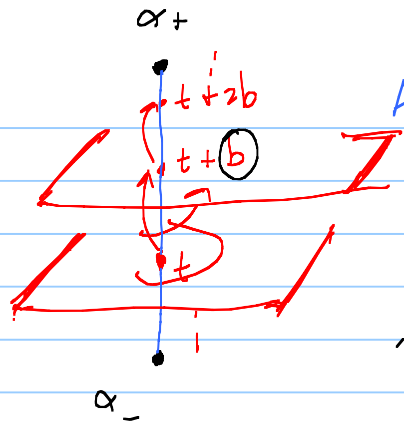
Geometric description of loxodromic and parabolic isometries

Recall

Def: A non-elliptic isometry $A \in \mathrm{Isom}(\mathbb{I}_n)$ is
- parabolic if it fixes exactly one boundary point
- loxodromic if it fixes exactly two distinct boundary points







the geodesic
A leaves invariant $[\alpha_+, \alpha_-] = \mathrm{Span}\{\omega_+, \omega_-\} \cap \mathbb{I}_n$ and acts on it as a translation $t \to t+b$

called the axis of A

The number b is the translation length of A

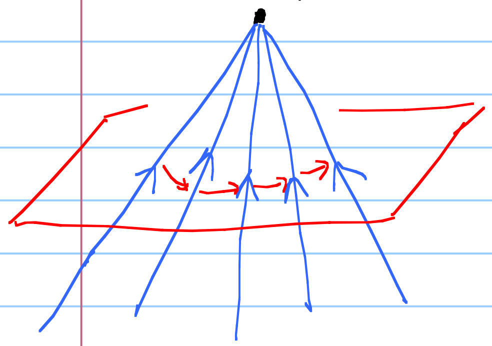
The axis of A is the unique A-invariant line in $\mathbb{I}_n$

Pf. If A leaves invariant l then it acts on it a $t \longrightarrow \pm t + b$ (just like an isometry of $\mathbb{R}$) since A is not elliptic and each $t \longrightarrow -t+b$ has a fixed point, the only possibility is that A acts by translations $t \to t+b$

$\Longrightarrow$ A fixes $l(\infty)$ and $l(-\infty)$

$\Longrightarrow \{l(\infty), l(-\infty)\} = \{\alpha_+, \alpha_-\} \Longrightarrow l = [\alpha_+, \alpha_-]$. $\square$

$q = [w] \in \partial_\infty \mathbb{I}_n = \mathbb{P}\mathcal{C}_+$



A parabolic motion leaves invariant the family of lines asymptotic to q and also the family of affine subspaces orthogonal to w

Def (Horosphere): A horosphere $\mathcal{H} \subset \mathbb{I}_n$ centered at $[w] \in \partial_\infty \mathbb{I}_n$ is a $\neq \emptyset$ subspace of $\mathbb{I}_n$ of the form

$$\mathcal{H} = \{ x \in \mathbb{I}_n \mid \langle w, x \rangle_{(n,1)} = r \}.$$

The family of horospheres centered at $[w] = \mathrm{Fix}_{\mathbb{I}_n \cup \partial_\infty \mathbb{I}_n}(A)$ is invariant under A $(Aw = \lambda w)$

$$\langle w, x \rangle = r \Rightarrow \langle w, Ax \rangle = \langle A^{-1} w, x \rangle = \frac{1}{\lambda} \langle w, x \rangle = \frac{1}{\lambda} r \Rightarrow A \{ \langle w, \cdot \rangle = r \} = \{ \langle w, \cdot \rangle = \tfrac{r}{\lambda} \}$$

Since A is not elliptic and not loxodromic, then each horosphere centered at $[w]$ is invariant

Pf. $Aw = \lambda w$ $\quad \lambda$ eigenvalue of $A \Longrightarrow$ also λ^{-1} is an eigenvalue for A

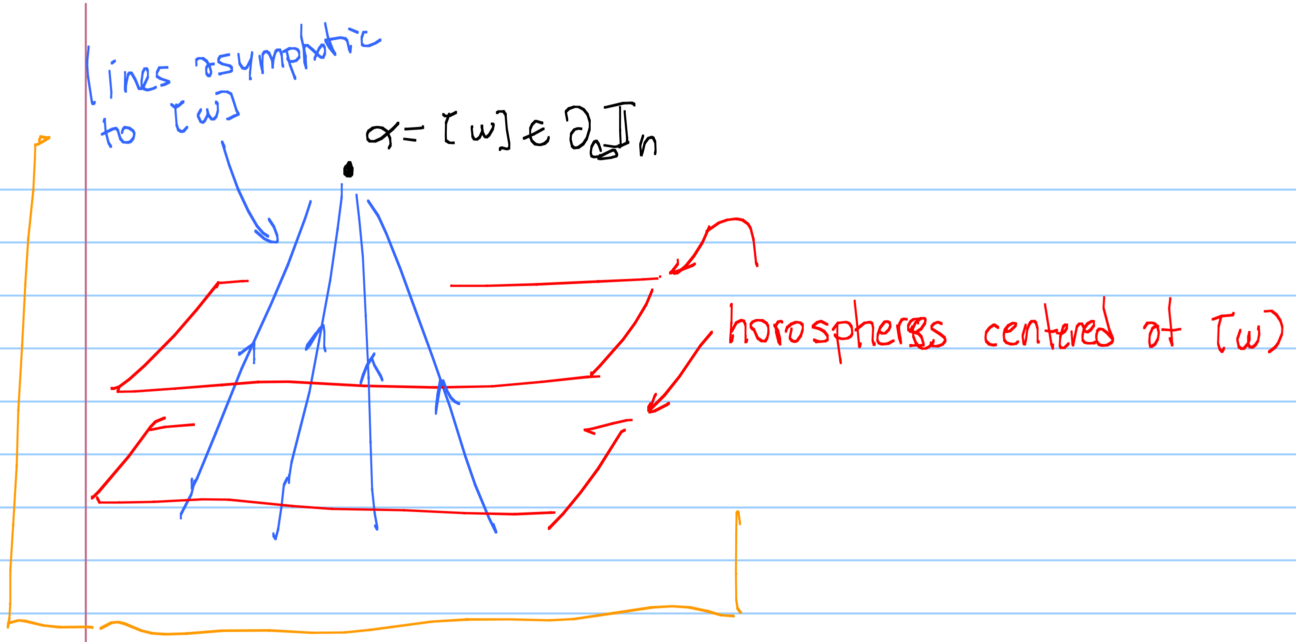
Claim: $\lambda = 1$

Suppose that this is not the case, then $\lambda \neq \lambda^{-1} \Rightarrow \lambda^{-1}$ corresponds to another eigenvector w' $\quad w \neq w'$. We also have

$$\underline{\lambda^{-2}\langle w', w'\rangle} = \langle Aw', Aw'\rangle = \underline{\langle w', w'\rangle} \Longrightarrow \langle w', w'\rangle = 0$$

$\Rightarrow$ $w' \in C_+$ and its class $[w']$ is fixed by A contradicting the fact that $Fix(A) = \{[w]\}$
$I_n \cup \partial_\infty I_n$

$\emptyset$



A few words on horospheres: Geometrically we have the following characterization

Lemma: A connected hypersurface $\mathcal{H}' \subset \mathbb{I}_n$ is orthogonal to all geodesics asymptotic to $[w] \in \partial_\infty \mathbb{I}_n \iff \mathcal{H}'$ is (contained in) a horosphere

Pf. $\mathcal{H}$ horosphere centered at $[w]$, $x \in \mathcal{H}$ $\Rightarrow$ $T_x\mathcal{H} = \underset{\overset{\parallel}{T_x \mathbb{I}_n}}{x^\perp} \cap w^\perp = \operatorname{span}\{w, x\}^\perp$

($\Leftarrow$) γ geodesic asymptotic to w $\Rightarrow$ can always write it as a combination of the starting point and w

$\Rightarrow$ $\gamma'(t) \in \operatorname{Span}\{\gamma(t), w\}$

$\Rightarrow$ γ is orthogonal to all horospheres centered at $[w]$ that it intersects.

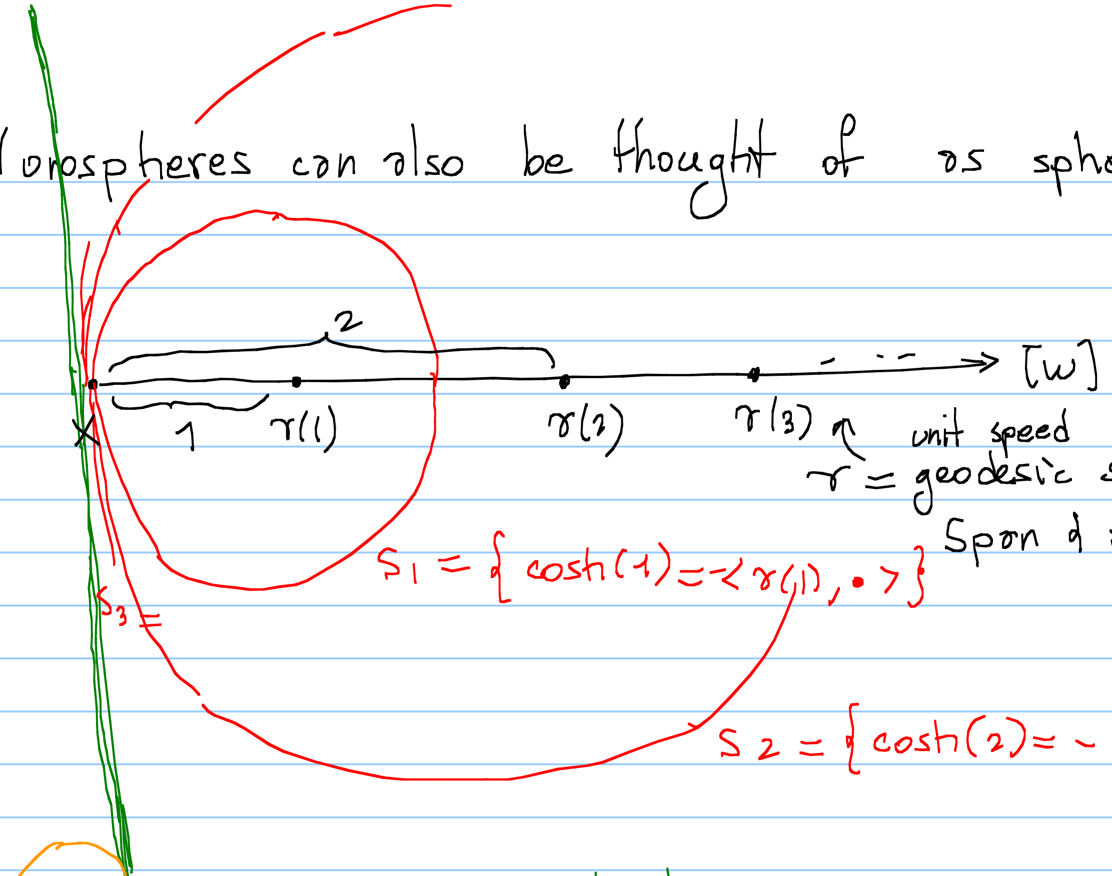
($\Rightarrow$) Let $\mathcal{H}'$ be a connected hypersurface orthogonal to all geodesics asymptotic to w

$\alpha : I \to \mathcal{H}'$ smooth path on $\mathcal{H}'$ $\quad \frac{d}{dt} \langle \alpha(t), w \rangle = \langle \alpha'(t), w \rangle = 0$

$\Rightarrow$ $\langle \alpha(t), w \rangle \equiv \text{const}$ $\Rightarrow$ since $\mathcal{H}'$ is connected we conclude that $\langle w, \bullet \rangle \equiv \underset{\neq 0}{\text{const}}$ on the whole $\mathcal{H}'$!

$\square$

Horospheres can also be thought of as spheres centered at a point at infinity



unit speed

γ = geodesic starting at x and asymptotic to ω

$\mathrm{Span}\{x, \omega\} \cap \mathbb{I}_n$

$S_1 = \{\cosh(1) = -\langle \gamma(1), \bullet \rangle\}$

$S_3 =$

$x, y \in \mathbb{I}_n$

$S_2 = \{\cosh(2) = -\langle \gamma(2), \bullet \rangle\}$ $\quad \cosh(d(x,y)) = -\langle x, y \rangle_{(n,1)}$

$\mathcal{H} =$ horosphere centered at ω and passing through x $= \{ y \in \mathbb{I}_n \mid \langle \omega, y \rangle = \langle \omega, x \rangle \}$

"$\uparrow$"
S_n

$S_t = \{ \cosh(t) = -\langle \gamma(t), \cdot \rangle \}$

$\gamma(t) = \cosh(t) x + \sinh(t) y$ $\qquad$ $[w] \ni x + y$

$w = x + v$

$= \{ 1 = -\langle x + \tanh(t) y, \cdot \rangle \}$

$\downarrow t \to \infty$

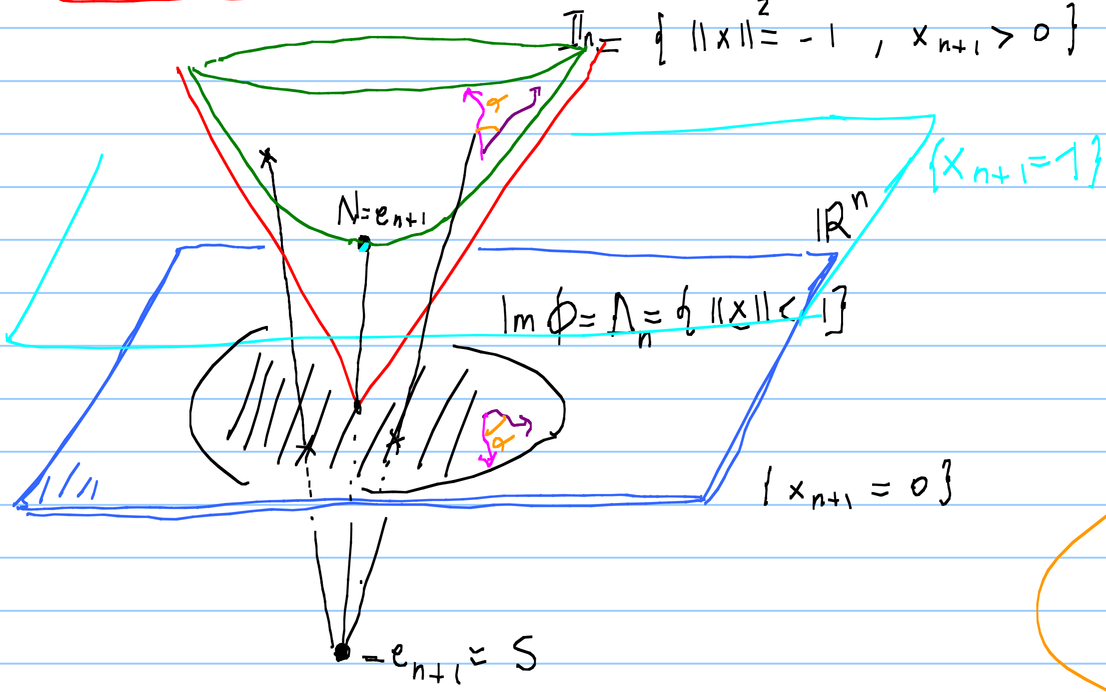
1

$\downarrow$

$\{ 1 = -\langle \underset{w}{\underset{\parallel}{x + v}}, \cdot \rangle \}$ ← this is the horosphere centered at $[w]$ and passing through x.

later we will see that horospheres are all isometric to some $(\mathbb{R}^{n-1}, \lambda g_{Eucl})$.

Conformal models of $\mathbb{H}^n$



$(x_1, \dots, x_n, x_{n+1}) = (\underline{x}, x_{n+1})$

Stereographic projection

$\phi : \mathbb{I}_n \longrightarrow \Delta_n$

$$\phi(\underline{x}, x_{n+1}) = \frac{1}{1 + x_{n+1}} \underline{x}$$

$\phi^{-1} : \Delta_n \longrightarrow \mathbb{I}_n$

$$\phi^{-1}(\underline{x}) = \frac{1}{1 - \|\underline{x}\|^2} \left(2\underline{x} , 1 + \|\underline{x}\|^2 \right)$$

Def (Conformal) $f : (M, g) \longrightarrow (M', g')$ smooth diffeomorphism is conformal if it does not distort angles, that is, there exists $\varphi : M \longrightarrow (0, \infty)$ smooth s.t. $\forall x \in M$, $\forall u, v \in T_x M$ we have $g'(df(u), df(v)) = \varphi(x)\, g(u, v)$.

The stereographic projection is conformal between $(\mathbb{I}_n, g_{\mathbb{I}_n})$ and (Δ_n, g_{Eucl})

Lemma: $(\phi^{-1})^* g_{\mathbb{I}_n} = \dfrac{4}{(1 - \|\underline{x}\|^2)^2} g_{Eucl} =: g_{\Delta_n}$

Poincaré Disk model of $\mathbb{H}^n$ $:= (\Delta_n, g_{\Delta_n})$

Lemma: $O(n) < \mathrm{Isom}(\Delta_n, g_{\Delta_n})$

Lemma: $\phi^{\pm 1}$ extends continuously to a diffeomorphism between

$$\mathbb{I}_n \cup \partial_\infty \mathbb{I}_n \longrightarrow \Delta_n \cup S^{n-1} \quad (\text{and identifies } \partial_\infty \mathbb{I}_n \text{ with } S^{n-1})$$

Pf.

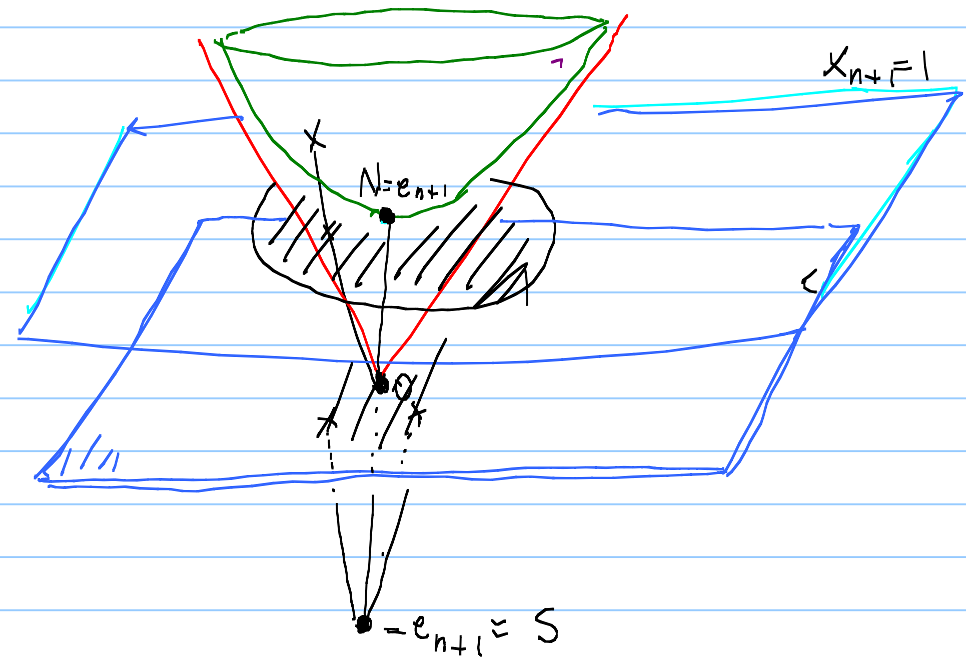
$$\underline{x} \in \Delta_n \xrightarrow{\phi^{-1}} \frac{1}{1-\|\underline{x}\|^2}\left(2\underline{x},\ 1+\|\underline{x}\|^2\right) \simeq \left(\frac{2}{1-\|\underline{x}\|^2}\underline{x},\ \frac{1+\|\underline{x}\|^2}{1-\|\underline{x}\|^2}\right)$$

$$\downarrow$$

$$\frac{2}{1+\|\underline{x}\|^2}\underline{x}$$

If $x \longrightarrow \xi \in S^{n-1}$, then $\frac{2}{1+\|\underline{x}\|^2}\underline{x} \longrightarrow \xi \Longleftrightarrow [(\xi, 1)] \in \partial_\infty \mathbb{I}_n$

$\square$



$x_{n+1}=1$
$N=e_{n+1}$
0
x
$-e_{n+1}=S$