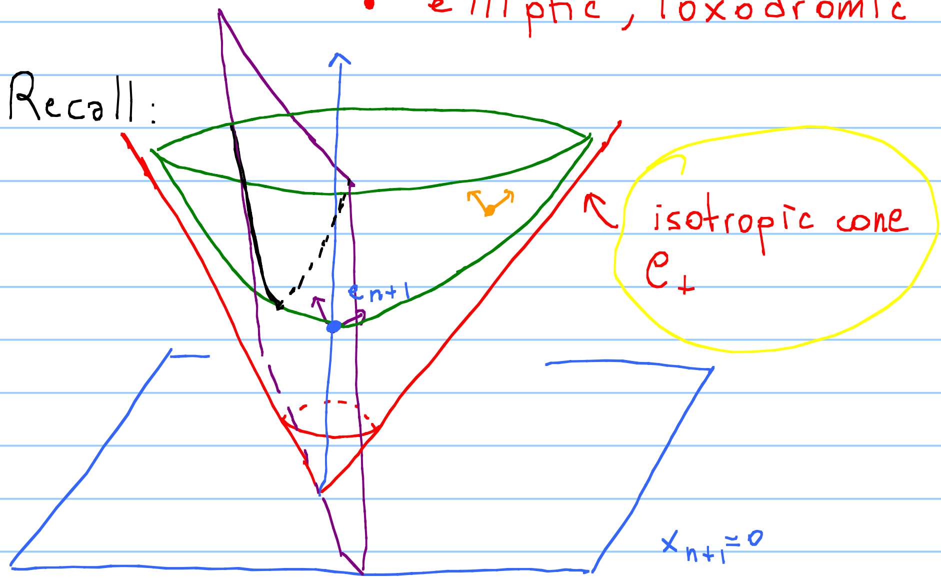
Hyperbolic Manifolds - Lecture 3

Summary:
- discussion on geodesics and metric structure.
- the boundary of $\mathbb{I}_n$,
- elliptic, loxodromic and parabolic isometries.

Recall:

$n=2$



① $\mathbb{I}_n = \{ x \in \mathbb{R}^{n,1} \mid \|x\|^2_{(n,1)} = x_1^2 + \dots + x_n^2 - x_{n+1}^2 = -1 ,\ x_{n+1} > 0 \}$

② $\mathrm{Isom}(\mathbb{I}_n) \curvearrowright \mathbb{I}_n$ as transitively as isometries are allowed to (prescribed first order)

③ many totally geodesic k-subspaces $V \cap \mathbb{I}_n$ isometric to $\mathbb{I}_k$ (1-subspaces = support of geodesics)

Geodesics: $V \subset \mathbb{R}^{n,1}$ 2-dim subspace with sign $(1,1) \Rightarrow V \cap \mathbb{I}_n$ is totally geodesic

geodesic starting at $x \in \mathbb{I}_n$ with velocity $v \in T_x\mathbb{I}_n = x^\perp$ $= \gamma_v$

$V = \operatorname{span}\{x, v\}$ sign $(1,1)$ $\quad V \cap \mathbb{I}_n =$ support of γ_v

Up to $\operatorname{Isom}(\mathbb{I}_n)$ we can reduce to: geodesics starting at $\boxed{\begin{matrix} x \\ \shortparallel \\ e_{n+1}\end{matrix}}$ with velocity $\boxed{v = e_1}$

$\gamma : \mathbb{R} \longrightarrow \mathbb{I}_n$ parametrization of $\operatorname{Span}\{e_{n+1}, e_1\} \cap \mathbb{I}_n$

$\gamma(t) = \gamma_1(t) e_1 + \gamma_{n+1}(t) e_{n+1}$

$\gamma(t) \in \mathbb{I}_n \iff \gamma_1^2(t) - \gamma_{n+1}(t)^2 = -1$

$\gamma(t)$ geodesic $\iff \|\gamma'(t)\| = 1 \quad \forall\, t \in \mathbb{R}$

$\gamma'(t) = \gamma_1(t)' e_1 + \gamma_{n+1}(t)' e_{n+1}$

$\|\gamma'(t)\|^2 = (\gamma_1(t)')^2 - (\gamma_{n+1}(t)')^2 = 1$

Lemma: $\gamma(t) = \sinh(t)\, e_1 + \cosh(t)\, e_{n+1}$

in particular γ is defined $\forall\, t \in \mathbb{R}$!

Lemma: The geodesic that starts at $x \in \mathbb{H}_n$ with unit speed $v \in T_x\mathbb{H}_n$ is given by $\gamma(t) = \cosh(t)\, x + \sinh(t)\, v$

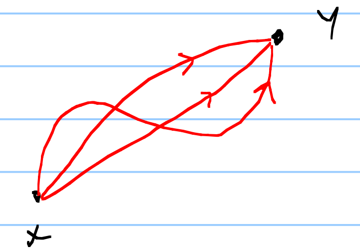
$$= \exp_x(tv)$$

exponential map

Metric structure of $\mathbb{I}_n$

$\mathbb{I}_n$ is a metric space with the path distance induced by the Riem. metric

$$d\left(\overset{\mathbb{I}_n}{\overset{\downarrow\quad\downarrow}{x\ ,\ y}}\right) := \inf\left\{ L(\gamma) \;\middle|\; \begin{array}{l} \gamma : I \longrightarrow \mathbb{I}_n \quad \text{smooth curve} \\ \text{that joins } x \text{ to } y \end{array} \right\}$$



Geodesics in $\mathbb{I}_n$ are those paths that locally minimize the distance between their points.

Hopf-Rinow: (M, g) Riem. mfd with path distance d.

Cor: $(\mathbb{I}_n, d)$ is a complete metric space

TFAE: ① (M, d) is a complete metric space]

② $B \subset M$ is cpt $\iff$ B is closed and totally bounded

③ Every geodesic in M can be extended to the whole $\mathbb{R}$

This holds true for $\mathbb{I}_n$! $\mapsto$ [④ There exists $x \in M$ s.t. $\exp_x$ is defined on all $T_x M$

Moreover, if M is complete, then $\forall$ $x, y \in M$ $x \neq y$ there exists a minimizing geodesic γ joining x to y with $L(\gamma) = d(x, y)$.

$\mathrm{Span}\{x, y\} \cap \mathbb{I}_n$

In $\mathbb{I}_n$ between $x, y \in \mathbb{I}_n$ $x \neq y$ there exists a unique geodesic so its length must be the distance between the two points

Lemma: The geodesic that starts at $x \in \mathbb{I}_n$ with unit speed $v \in T_x\mathbb{I}_n$ is given by $\gamma(t) = \cosh(t)\, x + \sinh(t)\, v$
$= \exp_x(tv)$

exponential map

Lemma: $\boxed{\cosh(d(x,y)) = -\langle x, y \rangle_{(n,1)}}$

Pf. geodesic between x and y $= \mathrm{Span}\{x, y\} \cap \mathbb{I}_n$
$= \mathrm{Span}\{x, v\} \cap \mathbb{I}_n$ $\quad v \in T_x\mathbb{I}_n$ $\quad v \in \mathrm{Span}\{x,y\} \cap x^\perp$, $\|v\| = 1$

$= \gamma(\mathbb{R})$ $\quad \gamma$ = geodesic starting at x with velocity v

$\gamma(t) = \cosh(t)\, x + \sinh(t)\, v$

At which $t \in \mathbb{R}$ we have $\gamma(t) = y$? Such a t is the length of γ between x and y and coincides with the distance between them

Write $\quad Y = \alpha x + \beta v \qquad \|x\|^2 = -1 \qquad \|v\| = 1$

$$\alpha = -\langle x, y\rangle_{(n,1)} \qquad \beta = \langle y, v\rangle$$

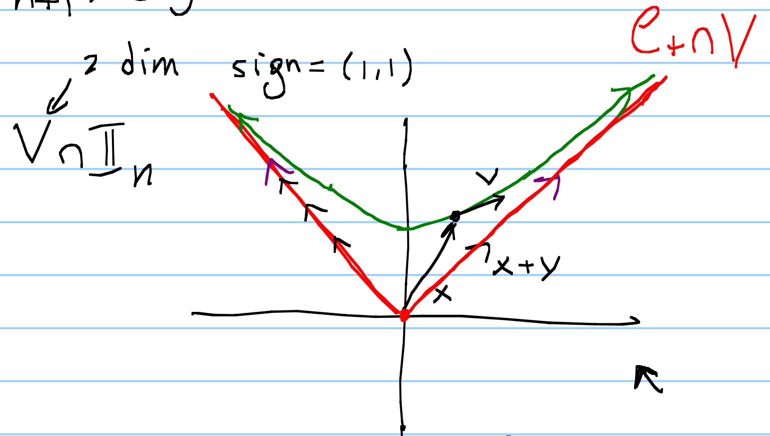
$$Y = \gamma(t) \iff \cosh(t) = -\langle x, Y\rangle_{(n,1)}.$$

$\boxtimes$

Boundary at infinity of $\mathbb{I}_n$

Obs: Every geodesic in $\mathbb{I}_n$ is asymptotic in the forward and backward directions to two different isotropic lines of

$$\mathcal{C}_+ = \{ w \in \mathbb{R}^{n,1} \mid \langle w, w \rangle_{(n,1)} = 0 \quad w_{n+1} > 0 \}$$



$V = \mathrm{Span}\{x, v\}$

Check: $x+v$ is the positive isotropic vector

Def (Boundary at infinity)

$$\partial_\infty \mathbb{I}_n = \mathbb{P}\mathcal{C}_+ = \mathcal{C}_+ / w \sim \lambda w \quad \lambda > 0$$

$\uparrow$

$$\|x+v\|^2_{(n,1)} = \|x\|^2_{(n,1)} + \|v\|^2_{(n,1)}$$

$$= -1 + 1 = 0$$

Rmk: Since $\mathrm{Isom}(\mathbb{I}_n) = O^+(n,1)$
we have $\mathrm{Isom}(\mathbb{I}_n) \curvearrowright \partial_\infty \mathbb{I}_n$
and this action "extends" the action on the interior

We now give a purely geometric interpretation of points in $\partial_\infty \mathbb{I}_n$

Notice that there is a map

$$\phi : \{ \alpha : [0,\infty) \longrightarrow \mathbb{I}_n \text{ half-lines} \} \longrightarrow\!\!\!\!\!\rightarrow \partial_\infty \mathbb{I}_n$$

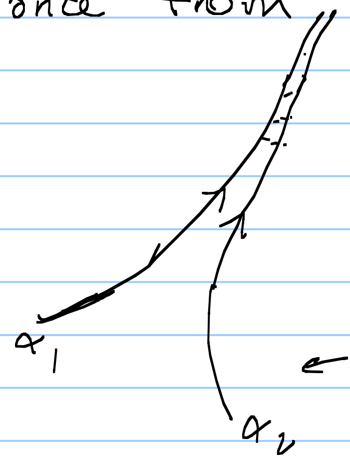
$$\alpha \longmapsto \text{forward isotropic direction } [x+v]$$

$x = \alpha(0)$

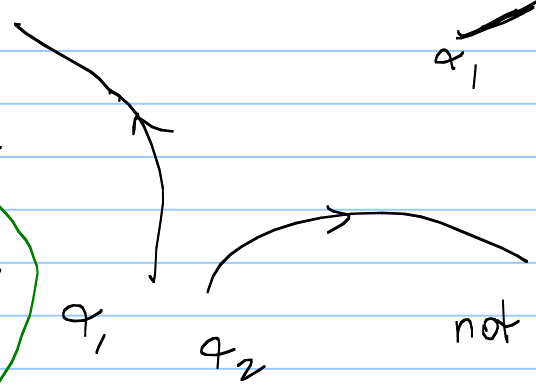
$v = \alpha'(0)$

Def (Asymptotic equivalence): $\alpha_1, \alpha_2 : [0,\infty) \longrightarrow \mathbb{H}_n$ half-lines. We say that α_1 is asymptotically equivalent to α_2 if they stay at a bounded distance from each other, i.e.

$$\sup_{t \geq 0} \{ d(\alpha_1(t), \alpha_2(t)) \} < \infty$$



Lemma: $\phi(\alpha_1) = \phi(\alpha_2) \Longleftrightarrow \alpha_1, \alpha_2$ are asymptotic equivalent



In particular $\underset{\displaystyle \mathbb{P}C_+}{\overset{\|}{}}\partial_\infty \mathbb{H}_n = \left\{ \begin{array}{l} \text{half lines up to} \\ \text{asymptotic equiv.} \end{array} \right\}$

Intrinsic!

Pf. $\alpha_j(t) = \cosh(t)\, x_j + \sinh(t)\, v_j$ $\qquad x_j \in \mathbb{I}_n$

$v_j \in T_{x_j}\mathbb{I}_n \quad \|v_j\| = 1$

$\phi(\alpha_j) = [\underbrace{x_j + v_j}_{= w_j}] \Rightarrow v_j = w_j - x_j$

$$\cosh(d(\alpha_1(t), \alpha_2(t))) = -\langle \alpha_1(t), \alpha_2(t)\rangle_{(n,1)}$$

$$= -\langle \tfrac{1}{2}\sinh(t)\, w_1 + e^{-t} x_1, \tfrac{1}{2}\sinh(t)\, w_2 + e^{-t} x_2 \rangle$$

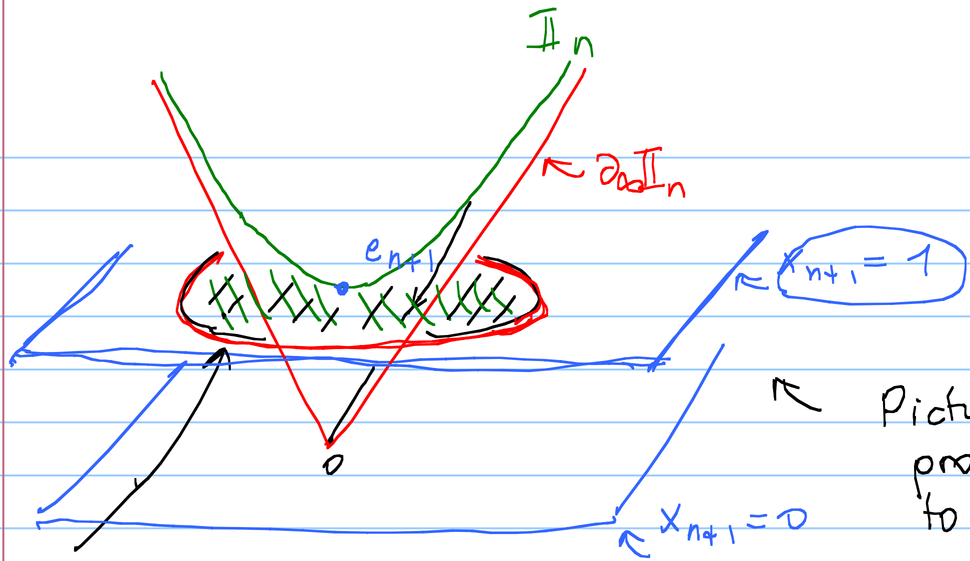
$$= -\tfrac{1}{4}\sinh(t)^2 \langle w_1, w_2\rangle + O(1)$$

In particular $\sup_{t>0}\{\cosh(d(\alpha_1(t), \alpha_2(t)))\} < \infty \iff \langle w_1, w_2 \rangle = 0$

$\iff w_1 = \lambda w_2$ for some $\lambda > 0$

$\iff [w_1] = [w_2]$

$\emptyset$



Now: Want to give a topology to $\mathbb{H}_n \cup \partial_\infty \mathbb{H}_n$ s.t. it makes sense to say that $\alpha(t) \xrightarrow[t\to\infty]{} \omega$.

Picture for the stereographic projection π from the origin $o \in \mathbb{R}^{n,1}$ to the screen $\{x_{n+1} = 1\}$

$\pi(\mathbb{H}_n \cup \partial_\infty \mathbb{H}_n)$ here has a natural topology, and we will use this one.
(equivalently this is the topology coming from the inclusion of $\mathbb{H}_n \cup \partial_\infty \mathbb{H}_n$ in projective space $\mathbb{P}(\mathbb{R}^{n,1})$)

$\boxed{\text{Klein model of } \mathbb{H}^n}$

$$\pi : \mathbb{H}_n \cup \mathcal{C}_+ \subset \mathbb{R}^{n,1} \longrightarrow \mathbb{P}(\mathbb{R}^{n,1}) = \mathbb{R}^{n,1} / v \sim \lambda v$$
$[x_1, \dots, x_{n+1}]$

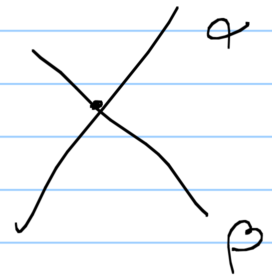
$\pi|\mathbb{H}_n$ is injective.

$$\pi(\mathcal{C}_+) = \underline{\{x_1^2 + x_2^2 + \dots + x_n^2 - x_{n+1}^2 = 0\}} \qquad \pi(\mathbb{H}_n) = \underline{\{x_1^2 + \dots + x_n^2 - x_{n+1}^2 < 0\}}$$

Lemma: $\underset{\substack{\psi \\ A}}{\mathrm{Isom}(\mathbb{I}_n)} \longrightarrow \mathrm{Aut}(\partial_\infty \mathbb{I}_n)$ is an injective homomorphism.

Pf. suppose that $A|_{\partial_\infty \mathbb{I}_n} = \mathrm{Id}$.

Then A leaves invariant each geodesic in $\mathbb{I}_n$.
Now take $p \in \mathbb{I}_n$



$p = \alpha \cap \beta$ with α, β geodesics

$Ap = A(\alpha \cap \beta) = (A\alpha) \cap (A\beta) = \alpha \cap \beta = p$.

$\Rightarrow A = \mathrm{Id}$. $\square$

Fixed points of isometries

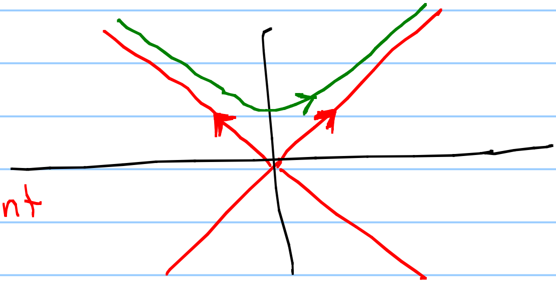
Lemma: Every $A \in \mathrm{Isom}(\mathbb{H}_n)$ has a fixed point in $\mathbb{H}_n \cup \partial_\infty \mathbb{H}_n$

Pf. This can be deduced by topological means from the Brouwer Fixed Point Thm.

Here is an elementary proof using linear algebra: We proceed by induction:

$n=1$ $\quad \mathbb{H}_1 \subset \mathbb{R}^{1,1}$ $\quad O^+(n,1)$

Either they preserve the isotropic directions
or they act on $\mathbb{H}_1 = \mathbb{R}$ with a fixed point
(think of isometries of $\mathbb{R}$).



In general $A \in O^+(n,1)$

A has an invariant 2-plane V (by the Jordan normal form)

Let's look at the signature of V:

① If it is $(1,1)$, then we are in the base case again.

② If it is $(1,0)$, with a 1-dim radical, then A must preserve the isotropic line.

③ If it is $(2,0)$, then $V^\perp$ is preserved by A, so A has a fixed point in $V^\perp \cap (\mathbb{I}_n \cup \partial_\infty \mathbb{I}_n)$ by the inductive hypothesis.

$\square$

Elliptic, Loxodromic and Parabolic isometries

Def (Elliptic): $A \in \mathrm{Isom}(\mathbb{I}_n)$ is elliptic if A fixes a pt in $\mathbb{I}_n$.

Non-elliptic isometries cannot fix too many boundary points

Lemma: If $A \in \mathrm{Isom}(\mathbb{I}_n)$ fixes three distinct boundary points then A is elliptic

Pf. $A[w_j] = [w_j]$ for $[w_1], [w_2], [w_3] \in \partial \mathbb{I}_n$

$\Updownarrow$

At the level of $\mathbb{R}^{n,1}$ we have $A w_j = \lambda_j w_j$.

Notice now that $\overset{\overset{0}{\#}}{\langle w_i, w_j \rangle} = \langle A w_i, A w_j \rangle = \lambda_i \lambda_j \overset{\overset{0}{\#}}{\langle w_i, w_j \rangle}$ $i \neq j$

$\Rightarrow \lambda_i \lambda_j = 1$ $\qquad \lambda_1 \lambda_2 = \lambda_2 \lambda_3 = \lambda_3 \lambda_1 = 1$ which together with $\lambda_j > 0$ gives $\lambda_1 = \lambda_2 = \lambda_3 = 1$, so that $A w_j = w_j$

$\Rightarrow$ Define $x = \dfrac{w_1 + w_2 + w_3}{3}$

$\Rightarrow$ x is invariant under A and corresponds to a fixed point in $\mathbb{I}_n$.

Def: A non-elliptic isometry $A \in \mathrm{Isom}(\mathbb{I}_n)$ can be either

- parabolic if it fixes exactly one boundary point.
- loxodromic if it fixes exactly two distinct boundary points.