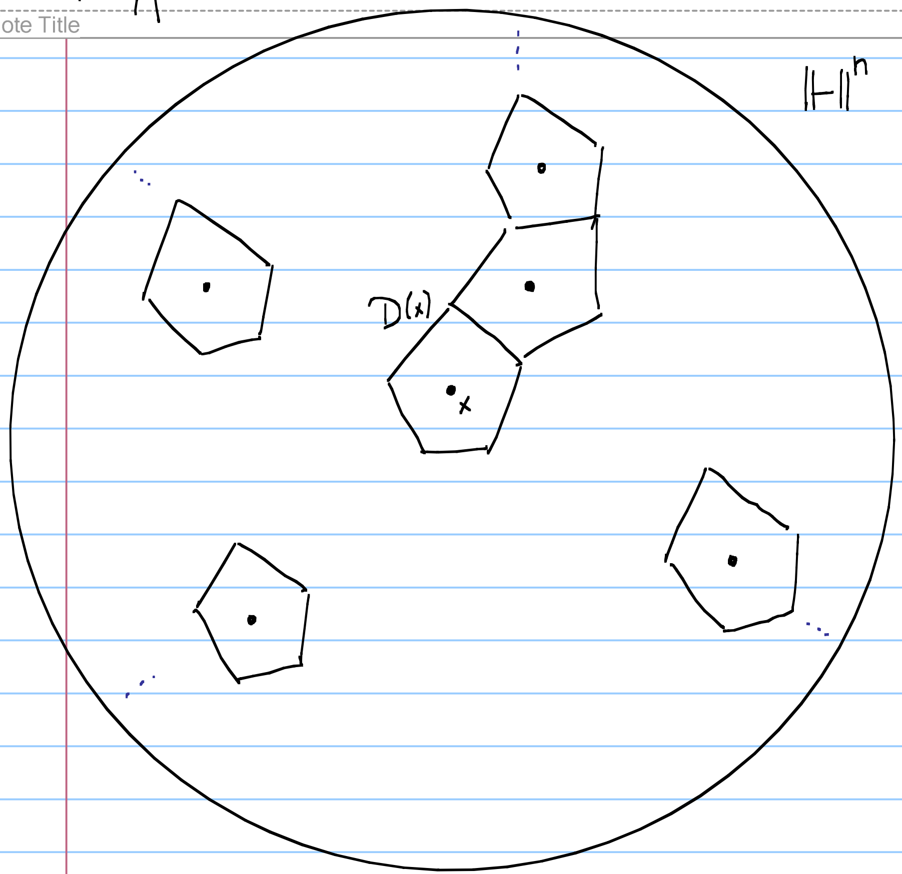
# Hyperbolic Manifolds - Lecture 26

Note Title 17/02/2021




$\mathrm{vol}(M = \mathbb{H}^n/\Gamma) \leq V_n \|M\|$ ($\geq$)

$z = \sum a_\sigma \sigma$ straight cycle representing $[M = \mathbb{H}^n/\Gamma]$

$z$ is $\varepsilon$-efficient if

① $\mathrm{sgn}(a_\sigma) = \mathrm{sgn}(\sigma)$

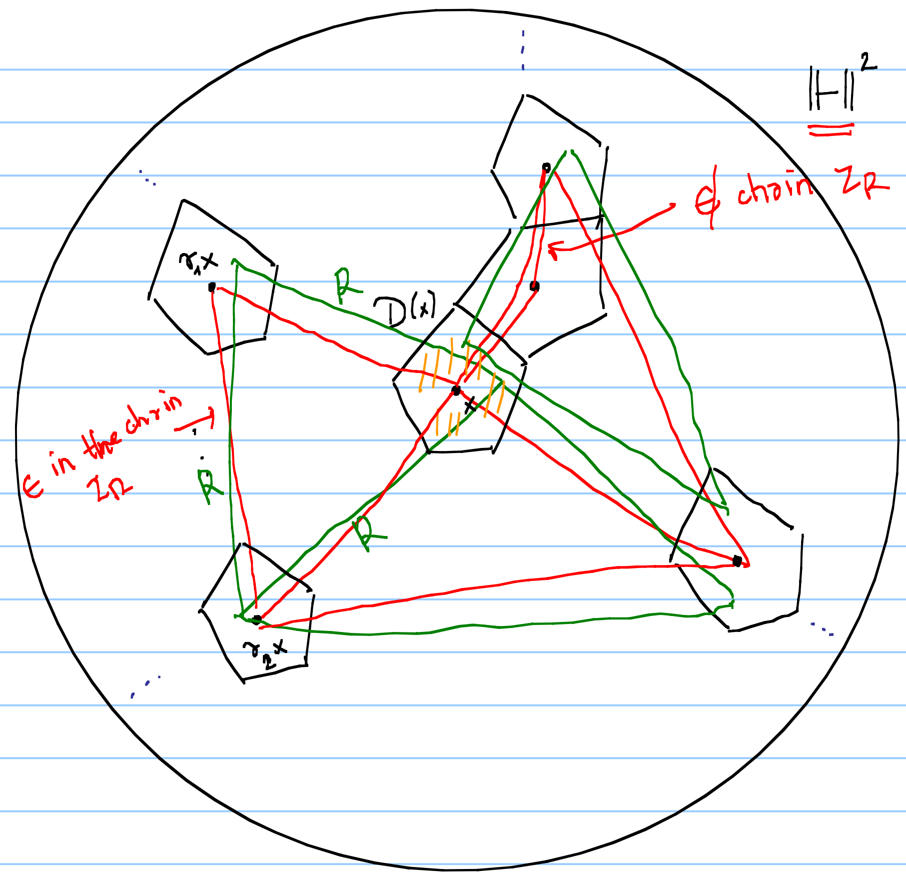
② $\mathrm{vol}(\sigma) \geq V_n - \varepsilon$ $\qquad \sum |a_\sigma|$ as small as possible

$$\mathrm{vol}(M) = \int_z \omega = \sum \underbrace{a_\sigma \,\mathrm{sgn}(\sigma)}_{>0} \underbrace{\mathrm{vol}(\sigma)}_{\text{as close to } V_n \text{ as possible}}$$

If $z$ is $\varepsilon$-efficient, the

$$\mathrm{vol}(M) = \sum a_\sigma \,\mathrm{sgn}(\sigma)\,\mathrm{vol}(\sigma)$$

$$\underset{①}{=} \sum |a_\sigma| \,\mathrm{vol}(\sigma) \underset{②}{\geq} |z|(V_n - \varepsilon) \geq (V_n - \varepsilon)\|M\|$$




$\Uparrow$
if $\varepsilon$ was arbitrary
$\Rightarrow$ we conclude
$\mathrm{vol}(M) \geq v_n \|M\|$.

Idea: Consider the following chains:

Fix $R > 0$ a large size
and $x \in \mathbb{H}^n$ a base point

$$\begin{array}{c} \mathbb{H}^n \\ \downarrow p \\ \mathbb{H}^n/\Gamma = M \end{array}$$

for every $\overset{1}{\overset{\shortparallel}{\gamma_0}}, \gamma_1, \ldots, \gamma_n$ define

$$\sigma(\gamma_0, \ldots, \gamma_n) = p \circ \hat{\sigma}(\gamma_0, \ldots, \gamma_n)$$

$\hat{\sigma}(\gamma_0, \ldots, \gamma_n)$ = straight $n$-simplex with vertices $\gamma_0 x, \ldots, \gamma_n x$

Define the chain

$$Z_R := \sum_{\underline{\gamma} \in \Gamma^n} a_R(\underbrace{1, \underline{\gamma}}_{(1, \gamma_1, \ldots, \gamma_n)}) \, \sigma(1, \underline{\gamma})$$

$\mathrm{vol}(\sigma(1,\underline{\gamma})) \geq v_n - \varepsilon$

where $a_R(1,\underline{r})$ = "number of regular finite n-simplices of sidelength $=R$ and vertices in $D(x)$, $D(r_1,x)$, ... $D(r_n x)$"

In order to count regular finite n-simplices of side length $=R$ we proceed as follows:

$$S(R) = \{ \Delta(x_0,\dots,x_n) \in \mathbb{H}^n \mid \text{~~oriented~~ regular finite } n\text{-simplices of side length} = R \}$$

Obs: $\mathrm{Isom}(\mathbb{H}^n) \curvearrowright S(R)$ in a transitive way with trivial stabilizer

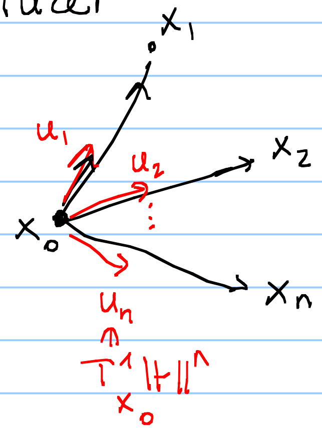
$\phi \Delta = \Delta$

$\phi(x_0) = x_0$

$d\phi(u_j) = u_j$

and $\{u_1, \dots, u_n\}$ is a basis of $T_{x_0}\mathbb{H}^n$

$\Rightarrow \phi = Id$




$\Downarrow$

Identify $S(R) = \mathrm{Isom}(\mathbb{H}^n)$

(choose a rep $\Delta^R \simeq \Delta(x_0^R, \dots, x_n^R) \in S(R)$)

$\phi \in \mathrm{Isom}(\mathbb{H}^n) \longrightarrow \phi \Delta^R$ )

We pick $\mu$ a <u>bi-invariant</u> measure on $\mathrm{Isom}(\mathbb{H}^n)$

(Explicitly we can define $\mu(B \subset \mathrm{Isom}(\mathbb{H}^n)) := \mathrm{vol}(\underline{\underline{Bx}})$

Fact: $\mu$ is bi-invariant and indep of $x \in \mathbb{H}^n$

$\mu(\phi B \psi) = \mu(B)$ )

$\gamma_0 = 1$

$$\boxed{\begin{aligned} a_R(1, \gamma_1, \ldots, \gamma_n) := {} & \mu\{\phi \in \mathrm{Isom}^+(\mathbb{H}^n) \mid \phi x_j^R \in D(\gamma_j x)\} \\ & - \mu\{\phi \in \mathrm{Isom}^-(\mathbb{H}^n) \mid \phi x_j^R \in D(\gamma_j x)\} \end{aligned}}$$

Lemma: Consider $z_R = \sum a_R(1,\underline{\gamma})\,\sigma(1,\underline{\gamma})$ as above then:

$$H_n(M;\mathbb{R}) \simeq \mathbb{R}$$
$$\alpha \longrightarrow \int_\alpha \omega$$

① $z_R$ is a cycle. In particular $[z_R] = \lambda_R [M]$ where $\lambda_R = \int_{z_R}\omega \Big/ \mathrm{vol}(M)$

② If $R$ is large enough, then $\mathrm{sgn}(a_R(1,\underline{\gamma})) = \mathrm{sgn}(\sigma(1,\underline{\gamma}))$
in particular $\lambda_R > 0$

③ If $R$ is large enough then $|z_R| = \mathrm{const}$.

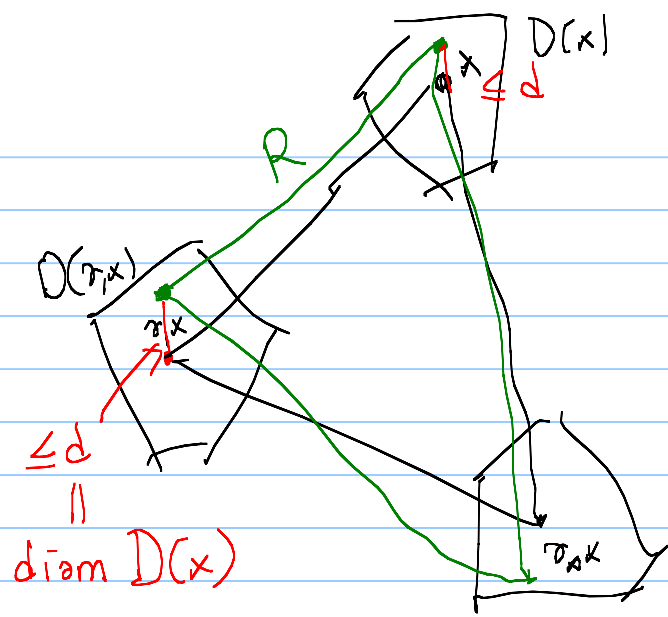
Lemma $\Rightarrow$ $z_R/\lambda_R$ for a fixed $\varepsilon > 0$ for $R$ large enough is a $\varepsilon$-efficient cycle.

Pf.

① $z_R$ is a chain: If $a_R(1,\gamma) \neq 0$, then

$$d(x,\gamma_j x) \le R + 2d$$

by prop. disc. there is only a finite number of $\gamma_j$ for which this holds. $\boxtimes$

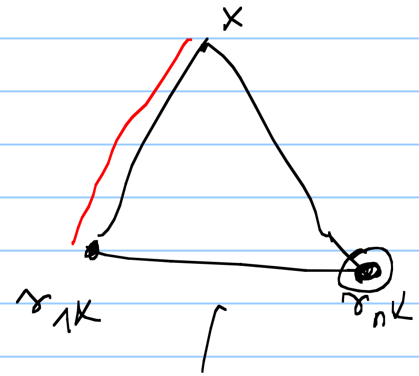



$z_R$ is a cycle: Consider a face of $\partial z_R$

say $\sigma(1,\gamma_1,\dots,\gamma_{n-1})$. The coeff of $\partial z_R$ along that face is

$$\mathrm{sgn}(\dots)\sum_{\gamma\in\Gamma} a_R(1,\gamma_1,\dots,\gamma_{n-1},\gamma)$$

$\|$

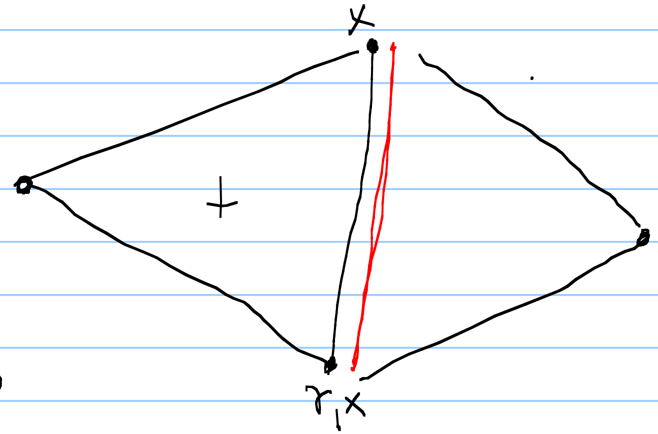


$$\sum_{\gamma\in\Gamma} \mu\left\{ \phi\in \mathrm{Isom}^+(\mathbb{H}^n) \;\middle|\; \begin{array}{l} \phi x_j^R \in D(\gamma_j x)^{j\le n-1} \\ \phi x_n^R \in D(\gamma x) \end{array} \right\}$$

$$-\,\mu\left\{ \phi\in \mathrm{Isom}^-(\mathbb{H}^n) \;\middle|\; \dots \right\}^{j\le n-1}$$

$$= \mu\left\{ \overbrace{\phi\in \mathrm{Isom}^+(\mathbb{H}^n) \;\middle|\; \phi x_j^R \in D(\gamma_j x)_{j\le n-1}}^{E_R^+} \right\}^{\gamma_0=1} \gamma x$$

$$-\,\mu\left\{ \underbrace{\phi\in \mathrm{Isom}^-(\mathbb{H}^n) \;\middle|\; \text{———}}_{E_R^-} \right\} = 0$$

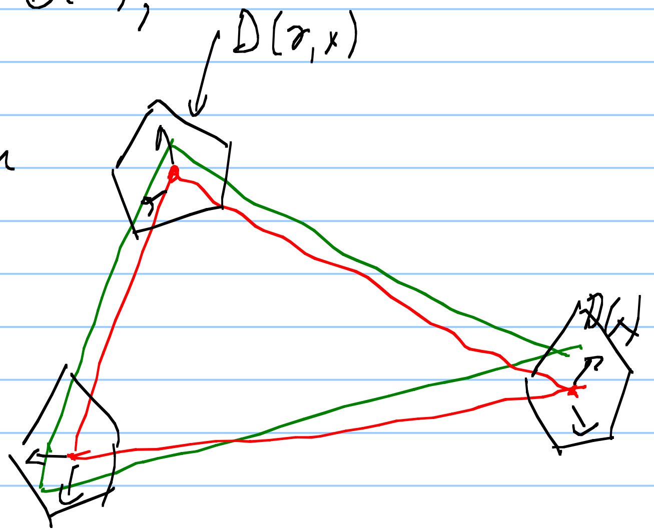



Obs: $E_R^+ \xrightarrow{\sim} E_R^{-1}$

Reflection along the face $x, \gamma_1 x, \dots, \gamma_{n-1}x$ gives a corresp. between $E_R^+$ and $E_R^-$
$\Rightarrow$ by bi-invariance of $\mu$, the two sets have the same measure. $\varnothing$

② If $R$ is much larger than the diameter of $D(x)$,

then perturbing the vertices $r_j x$ inside $D(r_j x)$ produces a $R$-simplex with the same orientation as $\sigma(1, r_1, \dots, r_n)$




$\Rightarrow$ If there are $\overset{\text{p.o.}}{\text{regular}}$ $R$-simpl. with vertices in $D(r_j x)$ $\Rightarrow$ there are no negative regular $R$-simpl. with vert. in $D(r_j x)$

$$\Rightarrow \quad \operatorname{sgn}(a_R(1, r_1, \dots, r_n)) = \operatorname{sgn}(\sigma(1, r_1, \dots, r_n))$$

from ② we know that one of the summands is 0 if $R$ is large enough

$$\mu\{+R\text{-simpl}\} - \mu\{-R\text{-simp}\}$$

③ $|Z_R| = \sum |a_R(1, \gamma_1, \dots, \gamma_n)|$

$$\gamma_0 = 1$$

$$= \sum_{(\gamma_1,\dots,\gamma_n)\in\Gamma^n} \mu\{\phi \in \mathrm{Isom}(\mathbb{H}^n) \mid \phi x_j^R \in D(\gamma_j \cdot x)\}$$

$$= \mu\{\phi \in \mathrm{Isom}(\mathbb{H}^n) \mid \phi x_0^R \in D(x)\} = \mathrm{vol}(D(x)) = \mathrm{vol}(M)$$

$\square$

$$\sum \mu(\cdot) = \mu(\bigcup \dots)$$

because the sets in the sum are essentially disjoint.

$\mu(A \cap B) = 0$

End of the proof of Mostow rigidity:

$\rho : \Gamma_1 \longrightarrow \Gamma_2$ isom. $\qquad \mathbb{H}^n/\Gamma_i$ closed orientable hyp.$^n$-mfd

$$\begin{array}{ccc} \Rightarrow \ f : \mathbb{H}^n & \longrightarrow & \mathbb{H}^n \\ \downarrow & & \downarrow \\ \bar{f} : \mathbb{H}^n/\Gamma_1 & \longrightarrow & \mathbb{H}^n/\Gamma_2 \end{array}$$

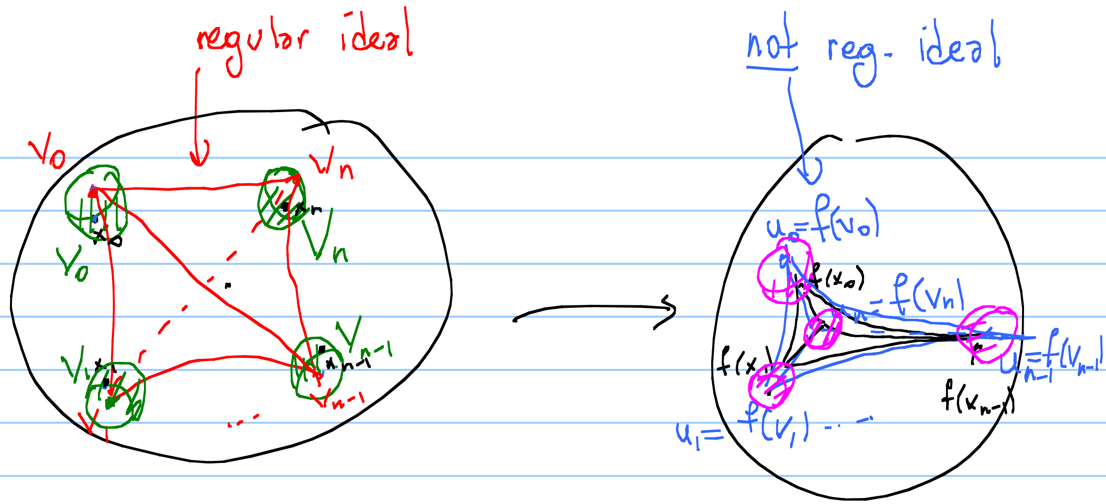
$\rho$-equiv. QI

we can assume that $f$ is a $\rho$-equiv. homotopy equiv.

$\Rightarrow f : \partial\mathbb{H}^n \longrightarrow \partial\mathbb{H}^n$ boundary extension $\rho$-equiv., homeo.

Need to check that $f$ sends the vertices of reg. ideal $n$-simpl. to the vertices of reg. ideal $n$-simpl.

Suppose this is not the case




vol $\le V_n - \delta$

for some $\delta > 0$

$\Rightarrow$ there are neigh. $V_j$ of the vertices $v_j \in \partial \mathbb{H}^n$

s.t. $\forall \; x_j \in V_j$ we have $\boxed{vol\left( \Delta(f(x_0), \dots, f(x_n)) \right) \le V_n - \delta/2}$

Take the almost regular ideal representation of $[\mathbb{H}^n/\Gamma_1]$ given by

$$\frac{1}{\lambda_R} z_R = \sum_{\underline{\gamma}\in\Gamma_1^n} \frac{1}{\lambda_R} a_R(1,\underline{\gamma})\, \sigma(1,\underline{\gamma})$$

Consider $f_* \frac{z_R}{\lambda_R} = \sum_{\underline{\gamma}\in\Gamma_1^n} \frac{1}{\lambda_R} a_R(1,\underline{\gamma})\, f_*\sigma(1,\underline{\gamma})$, since $f : \mathbb{H}^n/\Gamma_1 \to \mathbb{H}^n/\Gamma_2$ is a homotopy equivalence, we have $f_* z_R/\lambda_R$ represents $[\mathbb{H}^n/\Gamma_2]$ and so does $\mathrm{str}(f_* \frac{z_R}{\lambda_R}) = \frac{1}{\lambda_R} \sum a_R(1,\underline{\gamma})\, \mathrm{str}(f_*\sigma(1,\underline{\gamma}))$

$$\mathrm{str}(f_*\sigma(1,\underline{\gamma})) = \sigma(\rho(\underline{\gamma}))$$

$f(\gamma x) = \rho(\gamma) f(x)$

straight simplex with vertices $\rho(\gamma_i) f(x)$.

$$\Rightarrow \mathrm{vol}(\mathbb{H}^n/\Gamma_2) = \int_{\mathrm{str}(f_* \frac{z_R}{\lambda_R})} \omega = \frac{1}{\lambda_R} \sum_{\underline{\gamma}\in\Gamma^n} a_R(1,\underline{\gamma})\, \mathrm{sgn}(\sigma(\rho(\underline{\gamma})))\, \mathrm{vol}(\sigma(\rho(\underline{\gamma})))$$

We divide $Z_R$ into a good part and a bad part $Z_R = Z_R^{bad} + Z_R^{good}$

where $Z_R^{bad}$ is the part for which $\sigma(\rho(\underline{\gamma}))$ has a translate

with vertices in $U_j$ and in particular satisfy $vol(\sigma(\rho(\underline{\gamma}))) \leq V_n - \delta/2$

$|Z_R| = \text{Const}$

**Claim:** $|Z_R^{bad}| \geq \alpha > 0$ with $\alpha$ indep of $R$

$$|Z_R^{bad}| = \sum_{\underline{\gamma} \in \Gamma^n_{bad}} |a_R(1,\underline{\gamma})| = \sum \mu \{ \phi \in \mathrm{Isom}(\mathbb{H}^n) \mid \phi x_j^R \in D(\gamma_j x) \text{ and there exists } \gamma \in \Gamma \text{ s.t. } \gamma\phi x_j^R \in V_j \}$$

$$\geq \mu\{ \phi \in \mathrm{Isom}(\mathbb{H}^n) \mid \phi x_j^R \in V_j \} \geq \alpha \overset{?}{>} 0 \quad \underline{\text{Ex.}}$$

$\underline{\gamma} \in \Gamma^n_{bad} \Leftrightarrow$ a translate $\sigma(\rho(\underline{\gamma}))$ has vertices in $U_0, U_1, \dots, U_n$

$\Uparrow$

$\sigma(\underline{v})$ has vertizes in $V_0, \dots, V_n$ $\qquad$ $f(V_j) \subset U_j$

$\phi$ bar of $\Delta^R$

$\shortparallel$

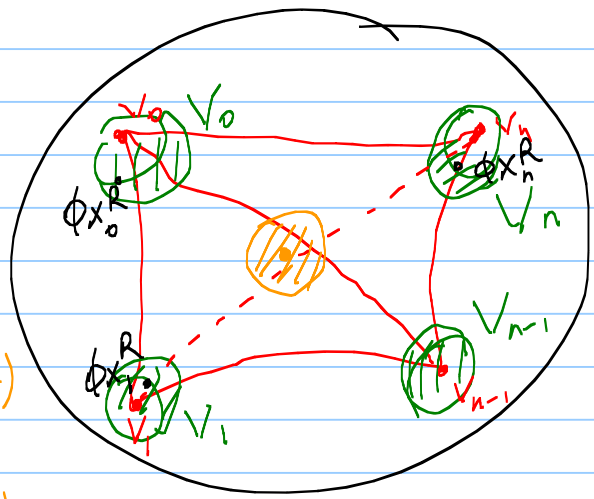
$\Delta(x_0^R, \dots, x_n^R)$

$\cap$

some ball

around the bar. of $\Delta(v_0, \dots, v_n)$

$\Rightarrow$ cpt closure



Now:

$$\mathrm{vol}\left(\mathbb{H}^n/\Gamma_2\right) \overset{\int_{\mathrm{str}\left(\rho_*\frac{z_R}{\lambda_R}\right)}\omega}{=} \frac{1}{\lambda_R}\sum a_R(1,\gamma)\ \mathrm{vol}\left(\sigma(\rho(\gamma))\right)$$

$$= \frac{1}{\lambda_R}\sum_{\gamma\in\Gamma^n_{bad}} a_R(1,\gamma)\underbrace{\mathrm{vol}(\ldots)}_{\leq V_n-\delta} + \frac{1}{\lambda_R}\sum_{\gamma\in\Gamma^n_{good}} a_R(1,\gamma)\underbrace{\mathrm{vol}(\ldots)}_{\leq V_n}$$

$$(V_n-\delta)\sum_{bad}|a_R|\frac{1}{\lambda_R} \qquad V_n\sum_{good}|a_R|\frac{1}{\lambda_R}$$

$$\leq \underbrace{\frac{1}{\lambda_R}V_n\sum_{\gamma\in\Gamma^n}|a_R(1,\gamma)|}_{\| \ V_n\frac{|z_R|}{\lambda_R} \to \|\mathbb{H}^n/\Gamma_1\|} - \delta\sum_{\gamma\in\Gamma^n_{bad}}\frac{1}{\lambda_R}|a_R(1,\gamma)|$$

$$\wedge \quad \text{claim}$$

$$\delta\alpha$$

Passing to the liminf as $R \to \infty$ we get

$$\mathrm{vol}(\mathbb{H}^n/\Gamma_2) \le \underbrace{v_n \, \| \mathbb{H}^n/\Gamma_1 \|}_{= \, \mathrm{vol}(\mathbb{H}^n/\Gamma_1)} - \delta\alpha \le \mathrm{vol}(\mathbb{H}^n/\Gamma_1) - \delta\alpha$$

$$\mathrm{vol}(\mathbb{H}^n/\Gamma_1) = \mathrm{vol}(\mathbb{H}^n/\Gamma_2)$$

by the proportionality principle.

$\square$