Hyperbolic Manifolds - Lecture 25

Note Title 16/02/2021

Proportionality principle

Theorem (Haagerup - Munkholm): Let v_n be the volume of the regular ideal n-simplex in $\mathbb{H}^n$. Let $\Delta \subset \mathbb{H}^n$ be any n-simplex.

Then $\mathrm{vol}(\Delta) \le v_n$ with equality if and only if Δ is a regular ideal n-simplex.

Theorem (Gromov-Thurston): Let $M = \mathbb{H}^n / \Gamma$ be a closed orientable hyperbolic n-mfd. Then

$$\mathrm{vol}(M) \overset{\le}{=} \underline{\underline{v_n}} \, \|M\| .$$

Riemannian volume simplicial volume

Corollary: If $f: \overset{\mathbb{H}^n/\Gamma_M}{\overset{\shortparallel}{M}} \longrightarrow \overset{\mathbb{H}^n/\Gamma_N}{\overset{\shortparallel}{N}}$ is a continuous function of non-zero degree between closed orientable hyperbolic n-mfds, then

$$\mathrm{vol}(M) \geqslant |\deg(f)|\,\mathrm{vol}(N)$$

$$\overset{\shortparallel}{v_n \|M\|} \geqslant |\deg(f)| \, \overset{\shortparallel}{v_n \|N\|}$$

$f: S^1 \longrightarrow S^1$

$f(z) = z^d$ has degree d

$f: M \longrightarrow N$ C^∞

$x \in N$ regular value for f meaning

$f^{-1}(x) = \{y_1, \dots, y_m\} \subset M$

$= \emptyset$

$df_{y_j}: T_{y_j}M \longrightarrow T_x N$ is an isom $\forall j$

$$\deg(f) = \sum_{j=1}^{m} \mathrm{sgn}(df_{y_j})$$

indep. of the choice of regular value

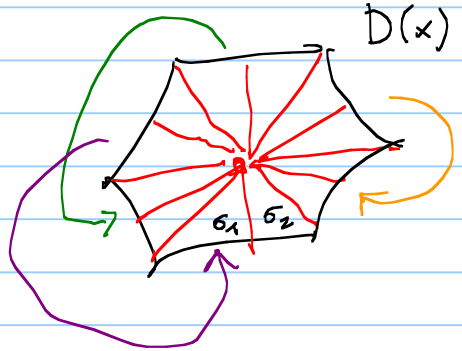
$$\begin{cases} +1 & \text{if } df_{y_j} \text{ is o.p.} \\ -1 & \text{if } df_{y_j} \text{ is o.r.} \end{cases}$$

Recall: $\|M\| = \|[M]\|$ $\qquad$ $M = \mathbb{H}^n/\Gamma$

$[M] \in H_n(M;\mathbb{R})$ fundamental class of M

$M = D(x) /$ identification of $\overset{\text{codim}=1}{\text{faces}}$ $\qquad$ $D(x) =$ Dirichlet polyhedron of $x \in \mathbb{H}^n$

$D(x) \underset{\text{break it into}}{\rightsquigarrow} \bigsqcup \overset{\text{n-simplices}}{\sigma}$ using barycentric subdivisions



$$[M] := [\textstyle\sum 1\cdot\sigma] \in H_n(M;\mathbb{R})$$

Notice: In general if M has a triangulation with K n-simpl.
Then $\|M\| \le K$ $\qquad$ $[M] = [\sum 1\cdot$ n-simpl. in the triang$]$

Fact: We can compute volumes of singular n-chains:

Consider ω the volume form of M

If $\sigma : \Delta_n \longrightarrow M$ $\mathcal{C}^1$- singular simplex

Then $$\int_\sigma \omega := \int_{\Delta_n} \sigma^* \omega \in \mathbb{R}$$

This extends by linearity to a map

$$\int \omega : \mathcal{C}_n^{1}(M; \mathbb{R}) = \bigoplus_{\sigma : \Delta_n \xrightarrow[\mathcal{C}^1]{} M} \mathbb{R}\sigma \longrightarrow \mathbb{R}$$

$$c = \sum a_\sigma \sigma \quad \Longrightarrow \quad \int_c \omega := \sum a_\sigma \int_\sigma \omega$$

Stokes: $c \in C^1_{n+1}(M, \mathbb{R}) \Rightarrow \int_{\partial c} \overset{\text{volume form}}{\omega} = \int_c d\omega$

$\Rightarrow$ We get $\int \omega : H_n(M; \mathbb{R}) \longrightarrow \mathbb{R}$ linear

$$\begin{array}{rcl} \overset{\rotatebox{90}{$\in$}}{\alpha} & \longrightarrow & \int_z \omega \quad \leftarrow \text{a } C^1\text{-cycle representing } \alpha \\ z \to z + \partial c & \longrightarrow & \int_z \omega + \int_{\partial c} \omega \\ & & \qquad\qquad \| \\ & & \qquad\qquad \int_c d\omega = 0 \end{array}$$

In particular: $$\int_{[M]} \omega = \sum 1 \cdot \int_\sigma \omega = \sum_{\sigma \text{ in bar. sub. of } D(x)} \mathrm{vol}(\sigma) = \mathrm{vol}(D(x)) = \mathrm{vol}(M)$$

$$[M] = \Big[\sum_{\sigma \text{ in the barycentric sub. of } D(x)} 1 \cdot \sigma\Big]$$

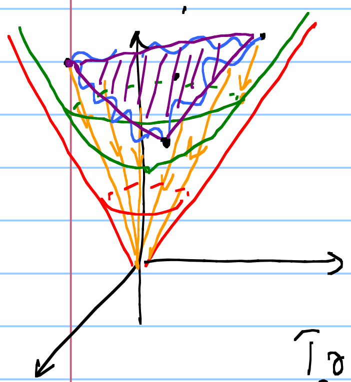
Thm: $\int_\bullet \omega : H_n(M;\mathbb{R}) \rightsquigarrow \mathbb{R}$ is a linear isomorphism.

Notice: If $z = \sum a_\sigma \sigma$ is a C^1 cycle rep. $[M]$, then $\boxed{\sum a_\sigma \int_\sigma \omega = \mathrm{vol}(M)}$

This suggests the following: If we want $\sum |a_\sigma|$ to be small, it is convenient to make $\int_\sigma \omega$ large.

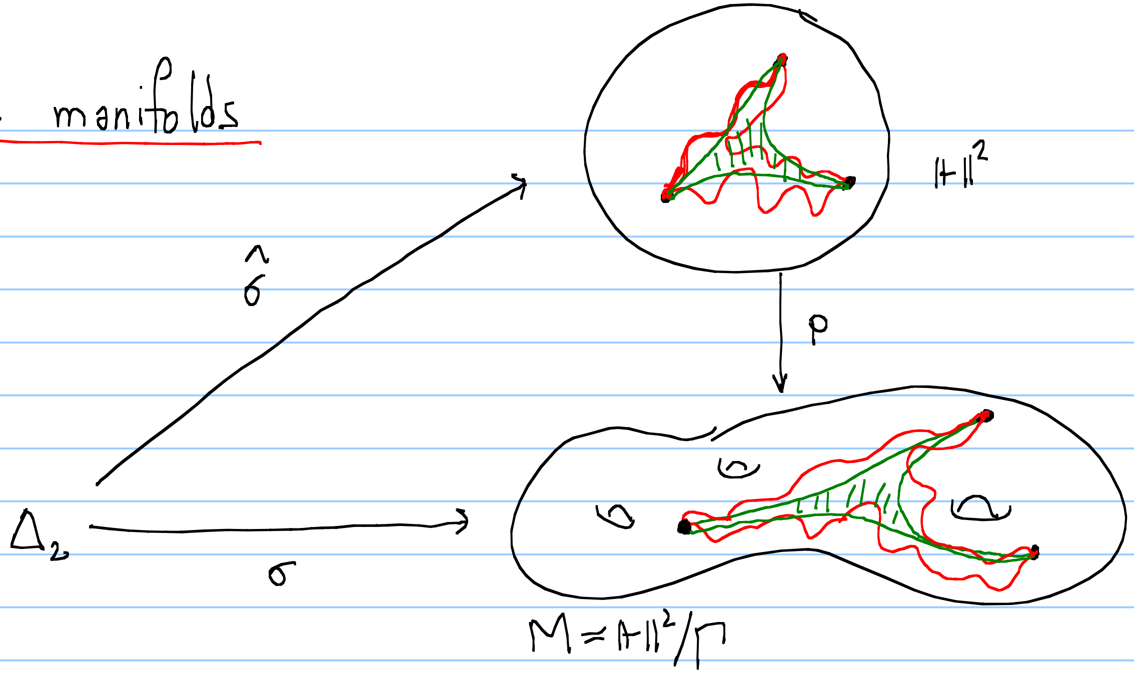
Straightening in hyperbolic manifolds

A canonical way to replace
$\hat{\sigma} : \Delta_m \longrightarrow \mathbb{H}^n = \mathbb{I}_n \subset \mathbb{R}^{n,1}$
with a straigth simplex



$\longleftarrow \quad \Delta_m : \hat{\sigma}$

Take convex comb. of the vertices of $\hat{\sigma}$ inside $\mathbb{R}^{n,1}$ and then project everything back to the hyperboloid



$$\begin{tikzcd} & \mathbb{H}^2 \arrow[d, "p"] \\ \Delta_2 \arrow[ur, "\hat{\sigma}"] \arrow[r, "\sigma"'] & M = \mathbb{H}^2/\Gamma \end{tikzcd}$$

$\sigma = p \circ \hat{\sigma}$

Keep the vertices of $\hat{\sigma}$ fixed and straighten the rest

$$\hat{\sigma} : \underset{\text{Conv hull of } e_0, \dots, e_n \text{ of } \mathbb{R}^{n+1}}{\underset{\shortparallel}{\Delta_m}} \longrightarrow \mathbb{I}_n$$

$$\operatorname{str}(\hat{\sigma}) := \left(t_0 e_0 + \dots + t_n e_n \longrightarrow \underset{\overset{\cup\!\!\shortmid}{\mathbb{I}_n}}{} \frac{t_0 \hat{\sigma}(e_0) + \dots + t_n \hat{\sigma}(e_n)}{\left(- \| t_0 \hat{\sigma}(e_0) + \dots + t_n \hat{\sigma}(e_n) \|^2_{n,1} \right)^{1/2}} \right)$$

This gives us $\overset{\text{by linearity}}{\vee}$ $\operatorname{str} : C_m(M;\mathbb{R}) \longrightarrow C^1_m(M;\mathbb{R})$

$$\operatorname{str} : \underset{p \circ \hat{\sigma}}{\underset{\shortparallel}{\sigma}} \longrightarrow p \circ \operatorname{str}(\hat{\sigma})$$

Properties: ① $\operatorname{str}(\partial \sigma) = \partial \operatorname{str}(\sigma)$

② σ and $\operatorname{str}(\sigma)$ are canonically homotopic $\underset{?}{\text{rel}}$ vertices

Sanity check:
If $\sigma = p \circ \hat{\sigma}_1$
$= p \circ \hat{\sigma}_2$

Then $\hat{\sigma}_2 = \gamma \circ \hat{\sigma}_1$
so $\operatorname{str}(\hat{\sigma}_2) = \gamma \circ \operatorname{str}(\hat{\sigma}_1)$
$\Downarrow$
the procedure is well-def

↑

take the homotopy $t\,\hat{\sigma}(x) + (1-t)\,\mathrm{str}(\hat{\sigma})(x)$

① + ② $\Longrightarrow$ str defines an isomorphism

$$\mathrm{str}: H_n(M;\mathbb{R}) \xrightarrow{\sim} H_n(M;\mathbb{R})$$

<u>Obs:</u> (i) $\mathrm{str}: C_n(M;\mathbb{R}) \longrightarrow \mathcal{C}^1(M;\mathbb{R})$ is <u>norm non-increasing</u>

$$c = \underline{\sum a_\sigma \sigma} \longrightarrow \underset{\mathrm{str}(c)}{\overset{\textstyle\sum a_\sigma \,\mathrm{str}(\sigma)}{\parallel}}$$

$$|\mathrm{str}(c)| \leq |c|$$

As a consequence the Gromov norm of any class α can be computed just using straigth representatives

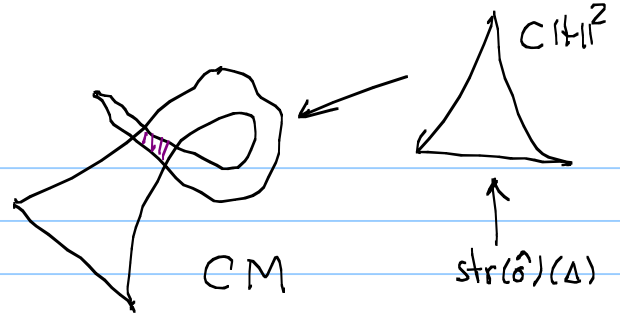
↑ $|\alpha| = \inf\left\{ |z| \;\middle|\; \begin{array}{l} z \text{ cycle} \\ \text{rep. } \alpha \end{array} \right\}$

(ii) $$\underset{\parallel}{\int_{\mathrm{str}(\sigma)}} \omega = \boxed{\pm}\, \mathrm{vol}\big(\mathrm{str}(\hat{\sigma})(\Delta)\big)$$

↑ $\sigma = p \circ \hat{\sigma}$

± depends on whether str(σ) preserves the orientation or not.

$$\begin{gathered} \| \\ \int_{\Delta_n} str(\sigma)^* \omega \end{gathered}$$



Lemma: $\mathrm{vol}(M) \leq v_n \|M\|$

Pf. ω = volume form of M

$Z = \sum a_\sigma \, str(\sigma)$ any straight cycle rep $[M]$

We have

$$\mathrm{vol}(M) = \int_Z \omega = \sum a_\sigma \int_{str(\sigma)} \omega = \sum a_\sigma \, \mathrm{sgn}(str(\sigma)) \, \mathrm{vol}(str(\hat{\sigma})(\Delta))$$

$$\leq \sum |a_\sigma| \cdot \underbrace{\mathrm{vol}(str(\hat{\sigma})(\Delta))}_{\underset{HM}{\leq} V_n} \leq v_n |Z|$$

Taking the inf over straight cycle finishes the proof. $\emptyset$

Opposite inequality

$$\mathrm{vol}(M) = \underline{\sum_\sigma a_\sigma \,\mathrm{sgn}(\sigma)\, \overset{\le V_n}{\overset{\downarrow}{\mathrm{vol}(\hat{\sigma})}}} \qquad \overset{\text{finite}}{\overset{\downarrow}{\sum a_\sigma \sigma}} \text{ is a straight cycle rep. } [M]$$

Want: $(\underline{\sum |a_\sigma|})$ as small as possible. It is efficient for a cycle to have

Want to find efficient cycles that satisfy $\longrightarrow$

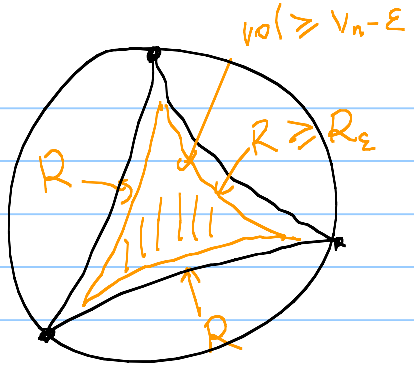
$$\begin{cases} ① \quad \mathrm{sgn}(a_\sigma) = \mathrm{sgn}(\sigma) \\ ② \quad \mathrm{vol}(\hat{\sigma}) \text{ very close to } V_n \text{ for all } \sigma \end{cases}$$

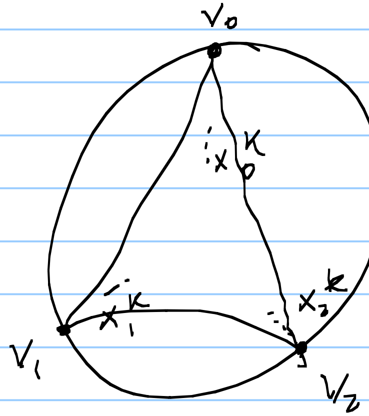
Efficient cycles

Fix $\varepsilon > 0$.

Fact: There exists $R_\varepsilon > 0$ s.t. the volume of a finite regular n-simplex Δ with side length $= R \geq R_\varepsilon$ is at least $\mathrm{vol}(\Delta) \geq v_n - \varepsilon$

More generally: If $\Delta(v_0, \dots, v_n)$ is a regular ideal n-simplex and $x_j^k \longrightarrow v_j$ then $\mathrm{vol}(\Delta(x_0^k, \dots, x_n^k)) \longrightarrow v_n$



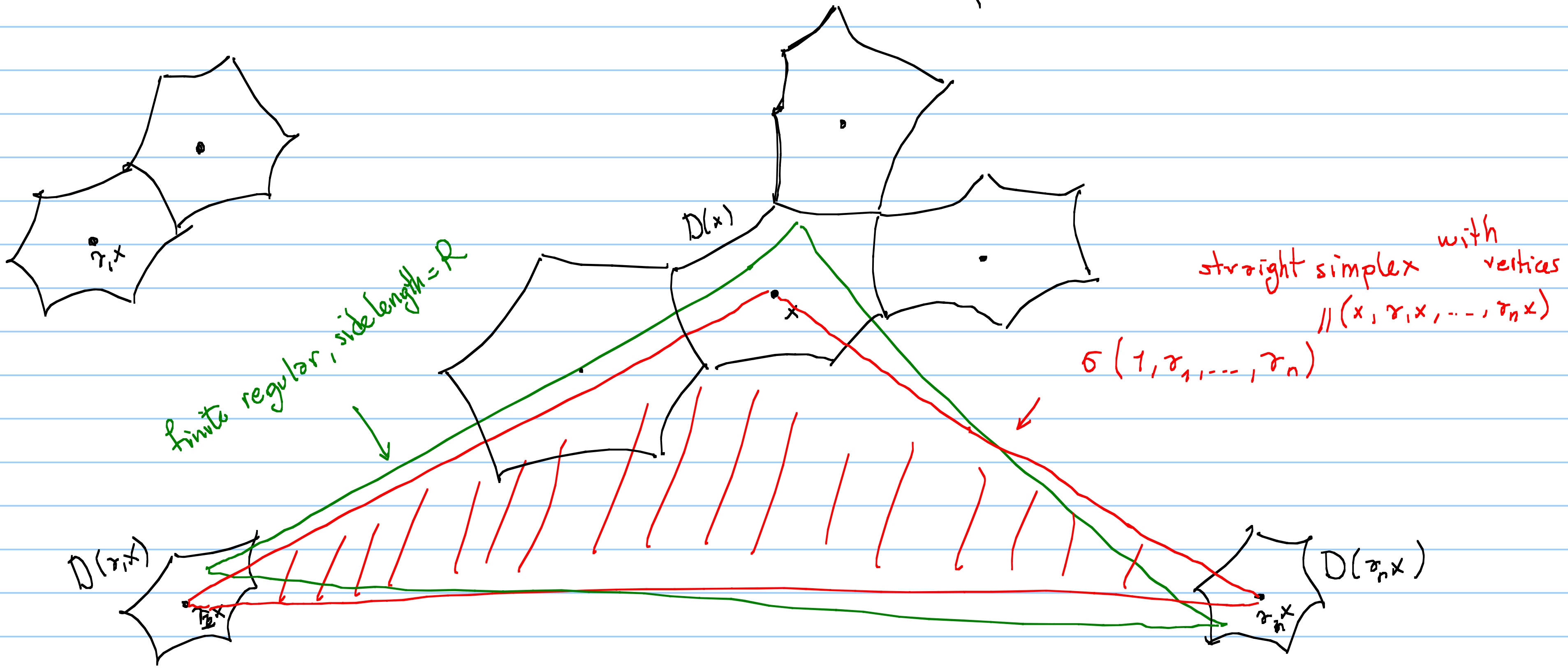


Idea:

$x \in \mathbb{H}^n$ $\quad D(x)$

$$z = \sum a_{\underline{\gamma}}\, \sigma(\underline{\gamma})$$

$\underline{\gamma} = (1,\gamma_1,\dots,\gamma_n)$



we want to include $\sigma(1,\gamma_1,\dots,\gamma_n)$ in the sum if there is a finite regular simplex with large side length and vertices in $D(x), D(\gamma_1 x), \dots, D(\gamma_n x)$

straight simplex with vertices in $x, \gamma_1 x, \dots, \gamma_n x$

Claim: $z_R = \sum a_{\underline{\gamma}}^R \, \sigma(\underline{\gamma})$ is an "efficient cycle"

$a_{\underline{\gamma}}^R =$ "number of finite regular n-simpl. with side length $= R$ and vertices in $D(x), D(\gamma_1 x), \dots, D(\gamma_n x)$