Hyperbolic Manifolds - Lecture 24

Note Title 10/02/2021

$f : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ QI $\quad \rho$- equiv. $\quad \rho : \Gamma_1 \to \Gamma_2$

$f : \partial\mathbb{H}^n \longrightarrow \partial\mathbb{H}^n$ homeo. extending f. ρ- equiv.

If f sends vertices of reg. id. n-simpl to vertices of reg. id. n-simpl, then $f = \phi|_{\partial\mathbb{H}^n}$

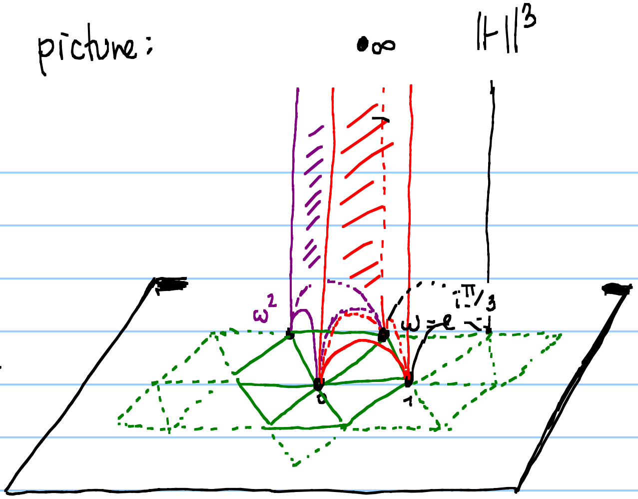
$\phi \in \mathrm{Isom}^+(\mathbb{H}^n)$

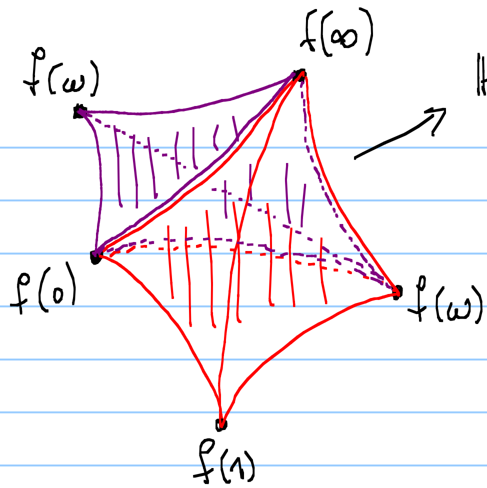
$n \geq 3$

[The reason why holds is that if $\Gamma_1 \simeq \Gamma_2$, then $\boxed{\mathrm{vol}(\mathbb{H}^n/\Gamma_1) = \mathrm{vol}(\mathbb{H}^n/\Gamma_2)}$.]

Heuristic picture:

$\mathbb{H}^3/\Gamma_1$ = figure-8 knot complement





$\mathbb{H}^3/\Gamma_2$ with $\Gamma_2 \cong \Gamma_1$

$\rightsquigarrow$ $\mathbb{H}^3 \to \mathbb{H}^3$

$\rightsquigarrow$ $\partial\mathbb{H}^3 \to \partial\mathbb{H}^3$

$\rho : \Gamma_1 \to \Gamma_2$ equiv.

$\Downarrow$

Can construct a map $F : \mathbb{H}^3 \to \mathbb{H}^3$ that extends $f : \partial\mathbb{H}^3 \to \partial\mathbb{H}^3$ in a simplicial way with respect to the tassellation provided by the figure-8 knot complement

If f does not send the vertices of the purple / red regular tetrahedra to regular ideal tetrahedra, then one of them has volume smaller than the volume of the regular ideal tetrahedron $\leftarrow v_3$

But then $\mathrm{vol}\left(\mathbb{H}^3/\Gamma_2\right) \leq \mathrm{vol}\left(F(\Delta)\right) + \mathrm{vol}\left(F(\Delta)\right) < 2V_3 = \mathrm{vol}\left(\mathbb{H}^3/\Gamma_1\right)$

and this would contradict $\underline{\mathrm{vol}\left(\mathbb{H}^3/\Gamma_1\right) = \mathrm{vol}\left(\mathbb{H}^3/\Gamma_2\right)}$.

Simplicial volume

In order to show that $\mathrm{vol}(\mathbb{H}^n/\Gamma_1) = \mathrm{vol}(\mathbb{H}^n/\Gamma_2)$ we will prove that the simplicial volumes of $\mathbb{H}^n/\Gamma_1$ and $\mathbb{H}^n/\Gamma_2$ agree and then we will prove the following proportionality principle:

Theorem (Gromov): let $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ be discrete, torsion free, cocompact, then we have

not trivial

$$\mathrm{vol}(\mathbb{H}^n/\Gamma) \overset{\geq}{\underset{\leq}{=}} V_n \cdot \| \mathbb{H}^n/\Gamma \|$$

simple

the volume of the regular ideal n-simplex

simplicial volume

X top space

$c \in C_n(X;\mathbb{R}) = \bigoplus \mathbb{R}\sigma$ real chains in X

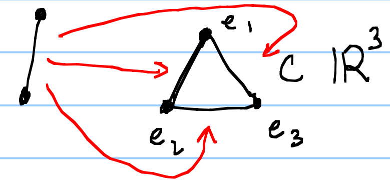
$\sigma : \Delta_n \to X$

continuous map from the standard Euclidean n-simplex

$c = \sum a_\sigma \cdot \sigma$, $a_\sigma \in \mathbb{R}$ ← finite linear comb.

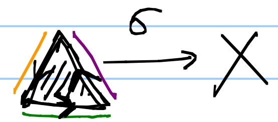
C_n has a norm defined by $|c| = \sum |a_\sigma|$

On C_n there is a boundary operator



$\partial : C_n(X;\mathbb{R}) \longrightarrow C_{n-1}(X,\mathbb{R})$

It is defined on basis elements $\sigma : \Delta_n \longrightarrow X$ by



$\partial\sigma = \sum \operatorname{sgn}(\tau) \cdot \sigma \circ \tau_j$

$\tau_j : \Delta_{n-1} \to \Delta_{n-1}$ j-th face of Δ_n

$\partial : C_n \to C_{n-1}$

n-th singular homology group of X with real coeff. is just

$$H_n(X;\mathbb{R}) := \frac{\mathrm{Ker}(\partial : C_n \to C_{n-1})}{\mathrm{Im}(\partial : C_{n+1} \to C_n)}$$

elements $c \in C_n$ s.t. $\partial c = 0$ are called cycles

elements c s.t. $c = \partial c'$ are called boundaries

$$= \frac{\text{Cycles}}{\text{Boundaries}}$$

There might be non-trivial classes for which $\|\cdot\|$ vanishes

Def: On $H_n(X;\mathbb{R})$ there is a natural semi-norm, called the Gromov norm, defined as follows:

$z = \sum a_\sigma \cdot \sigma$ $\qquad |z| : \sum |a_\sigma|$

$(v_1, \dots, v_n)$

on $\mathbb{R}^n$ $\quad |v|_{\ell_1} = \sum |v_j|$

$$\alpha \in H_n(X;\mathbb{R})$$

$$\|\alpha\| = \inf \{ \|z\| \mid z \in C_n(X,\mathbb{R}) \text{ is a cycle representing } \alpha \}$$

$$= \inf \{ \|z + \partial c\| \mid \begin{array}{l} z \in C_n(X,\mathbb{R}) \text{ is a fixed cycle rep. } \alpha \\ \text{and } c \in C_{n+1} \text{ varies among all possible chains} \end{array} \}$$

$$\cap$$

$$[0,\infty)$$

Ex: $\alpha, \beta \in H_n(X;\mathbb{R})$

① $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$

② $\lambda \in \mathbb{R} \setminus \{0\}$ $\quad \|\lambda \alpha\| = |\lambda| \, \|\alpha\|$

Def: Let M be a closed oriented n-mfd, the simplicial volume of M is the Gromov norm (denoted by $\|M\|$) of the fundamental class of M $[M] \in H_n(M, \mathbb{R})$

$$\|M\| = \|[M]\|$$

If M has a triangulation

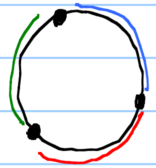
$M = \bigcup \sigma \implies \left[\sum 1 \cdot \sigma\right] = [M] \in H_n(M, \mathbb{R})$

σ n-dim simplex (parametrized $\sigma : \Delta_n \to M$)

Indep of the triangulation.

Example: $M = S^1$, $\|S^1\| = 0$

$[\text{blue} + \text{red} + \text{green}] \in H_1(S^1; \mathbb{R})$



$\leq |\text{blue} + \text{red} + \text{green}| = 3$

$\sigma_1 : I[0,1] \to S^1$

$\sigma_1 : [0,1] \longrightarrow$ 

$\|S^1\| \leq |\sigma_1| = 1$

$\sigma_2 : [0,1] \longrightarrow$ 

$[\frac{1}{2}\sigma_2] = [S^1]$ $\quad \|\frac{1}{2}\sigma_2\| = \frac{1}{2}$

$\vdots$

$\sigma_n : [0,1] \to$  $\left[\frac{1}{n}\sigma_n\right] = [S^1]$ $\qquad \left\|\frac{1}{n}\sigma_n\right\| = \frac{1}{n}$

$$\Downarrow$$

$$\|S^1\| = \inf\{\, |\text{cycles rep } [S^1]| \,\} = 0$$

Ex: Spheres S^n have vanishing simplicial volume.

General Properties of Gromov norms and simpl. volume

① $f : X \to Y$ continuous map, it induces a linear map

$$f_* : C_n(X;\mathbb{R}) \longrightarrow C_n(Y;\mathbb{R})$$

$$(\sigma : \Delta_n \to X) \longrightarrow (f \circ \sigma : \Delta_n \to Y)$$

$$f_*\left(\sum a_\sigma \sigma\right) = \sum a_\sigma f_*\sigma$$

It is Immediate to check that:

(i) $|f_* c| \leq |c|$

(ii) $\partial(f_*\sigma) = f_*(\partial\sigma)$

$\Rightarrow$ f_* induces a map $H_n(X;\mathbb{R}) \to H_n(Y;\mathbb{R})$

(i) + (ii) $\Rightarrow$ f_* is norm non-increasing

$\|\alpha\| \geq \|f_*\alpha\|$

Fact: $f, g : X \to Y$

② If f and g are homotopic, then $f_* = g_*$ on $H_n(X;\mathbb{R}) \to H_n(Y;\mathbb{R})$

As a consequence: If $f: X \to Y$ is a homotopy equivalence, then $f_* : H_n(X;\mathbb{R}) \longrightarrow H_n(Y;\mathbb{R})$ is an isometry

Pf. There exists $g: Y \longrightarrow X$ s.t. $gf : X \longrightarrow X$ is homotopic to Id_X and $fg : Y \to Y$ is homotopic to Id_Y

$$\begin{tikzcd} H_n(X;\mathbb{R}) \arrow[r, "f_*"] \arrow[rr, bend right] & H_n(Y;\mathbb{R}) \arrow[r, "g_*"] & H_n(X;\mathbb{R}) \end{tikzcd}$$

$(gf)_* = Id$ because $gf \simeq Id_X$

By ① $\|f_* \alpha\| \le \|\alpha\|$

$\|\alpha\| = \|g_* f_* \alpha\| \le \|f_* \alpha\|$

Notice: If $f : M \to N$ is a homotopy equivalence between closed orientable n-mfds, then $f_*[M] = \pm [N]$

In particular $\|M\| = \|[M]\| = \|[N]\| = \|N\|$

So M and N have the same simplicial volume.

③ $f: M \longrightarrow N$ a degree $= d \neq 0$ map between closed orientable n-mfds, then $\|M\| \geq |d| \|N\|$ and equality holds when f is a covering.

As a consequence, n-mfds with self-maps of degree ≥ 2 have vanishing simplicial volume.

Later: If M is hyperbolic then M has no self map of degree ≥ 2 because $\|M\| > 0$.

discrete, torsion free, cocompact.

Lemma: If $\Gamma_1 \simeq \Gamma_2$, then $\mathbb{H}^n/\Gamma_1$ is homotopy equiv. to $\mathbb{H}^n/\Gamma_2$

$\mathbb{H}^n/\Gamma_1 \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{H}^n/\Gamma_2$

In particular $\| \mathbb{H}^n/\Gamma_1 \| = \| \mathbb{H}^n/\Gamma_2 \|$.

Pf. Take $x \in \mathbb{H}^n$ and consider its fundamental polyhedron $D(x)$

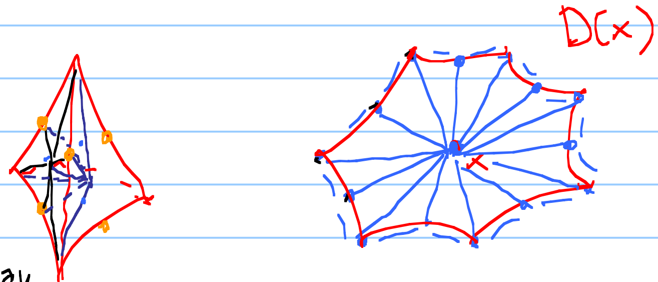
Rmk: The bar. sub. provides a rep. of $[\mathbb{H}^n/\Gamma_1]$.

Take a barycentric subdivision of $D(x)$

since $\mathbb{H}^n/\Gamma_1$ is cpt.

This divides $D(x)$ into simplices.

$D(x)$ is a finite polyhedron



Let V be the set of vertices

Define $F: V \longrightarrow \mathbb{H}^n$

to be any ρ-equiv. map

Extend F over the simplices in a simplicial way.

F now defines a map $\mathbb{H}^n \longrightarrow \mathbb{H}^n$ (by extending $F: D(x) \longrightarrow \mathbb{H}^n$ by equivariance)

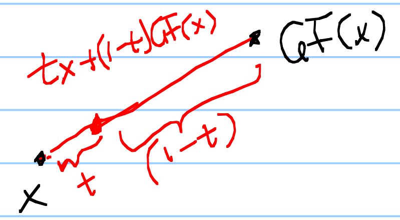
and also $f: \mathbb{H}^n/\Gamma_1 \to \mathbb{H}^n/\Gamma_2$

Similarly we construct $G: \mathbb{H}^n \longrightarrow \mathbb{H}^n$ $\quad \rho^{-1}: \Gamma_2 \to \Gamma_1$ equivariant.

$$G: \mathbb{H}^n/\Gamma_2 \longrightarrow \mathbb{H}^n/\Gamma_1$$

Claim: F and G are homotopy inverses

that is $GF \simeq Id$ (Γ_1-equiv.), but finding a Γ_1-equiv. homotopy is simple: Just take $(x,t) \in \mathbb{H}^n \times [0,1] \longrightarrow tx + (1-t)\,GF(x)$



By construction this homotopy is Γ_1-equiv. $\varnothing$

F is ρ-equiv

G is ρ^{-1}-equiv.

$$GF(\gamma x) = G(\rho(\gamma)F(x))$$
$$= \rho^{-1}(\rho(\gamma))GF(x)$$

$$= \gamma\, GF(x)$$

$$\begin{tikzcd} x \arrow[r, dash] \arrow[d, "\gamma"] & GF(x) \\ \gamma x \arrow[r, dash] & \gamma GF(x). \end{tikzcd}$$