Hyperbolic Manifolds - Lecture 23

Titolo nota 09/02/2021

$\Gamma_1 \overset{\rho}{\cong} \Gamma_2$

$f: \mathbb{H}^n \longrightarrow \mathbb{H}^n$ ρ-equiv QI

Last time: Defined an extension of f to $f: \partial\mathbb{H}^n \longrightarrow \partial\mathbb{H}^n$

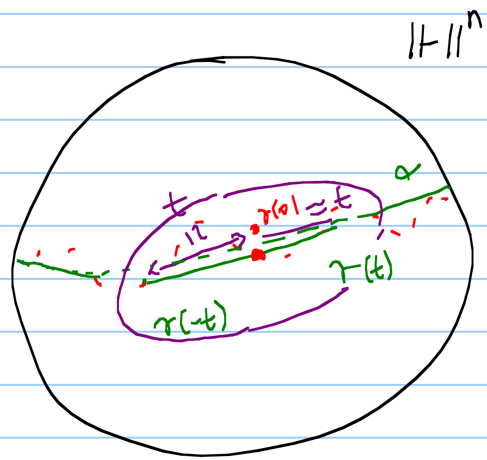
This is a consequence of stability of QG

Need to check: The extension f is ① well-def
② injective
③ continuous

End of Pf. of stability of QG for bi-infinite lines:

$\gamma : \mathbb{R} \longrightarrow \mathbb{H}^n$ L - QG

$t >> 0$ large time $\rightarrow$ Consider $[-t,t] \subset \mathbb{R}$

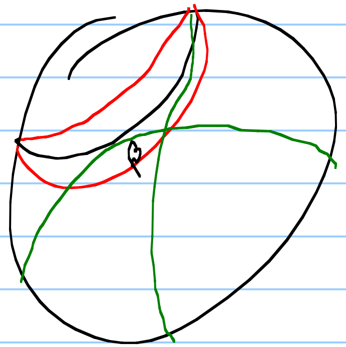
$\gamma : \mathbb{R} \longrightarrow$ 

By the finite case $[\gamma(-t), \gamma(t)] \subset N_{R_0}(\gamma[-t,t])$

$\Rightarrow$ all these geodesics pass unif. close to $\gamma(0)$

$\Rightarrow$ they converge up to subseq to a bi-infinite geo line α

with $\alpha \subset N_R(\gamma(\mathbb{R}))$ and $\gamma(\mathbb{R}) \subset N_R(\alpha)$



In order to conclude just observe that any two lines α, α' obtained as limits of subseq. of $[\gamma(-t), \gamma(t)]$ will have the property that $\alpha \subset N_{2R}(\alpha')$

$\alpha' \subset N_{2R}(\alpha)$

But the $2R$-neigh of any biinfinite line β only contains one biinfinite line (that is β)

$\Rightarrow \alpha = \alpha'$

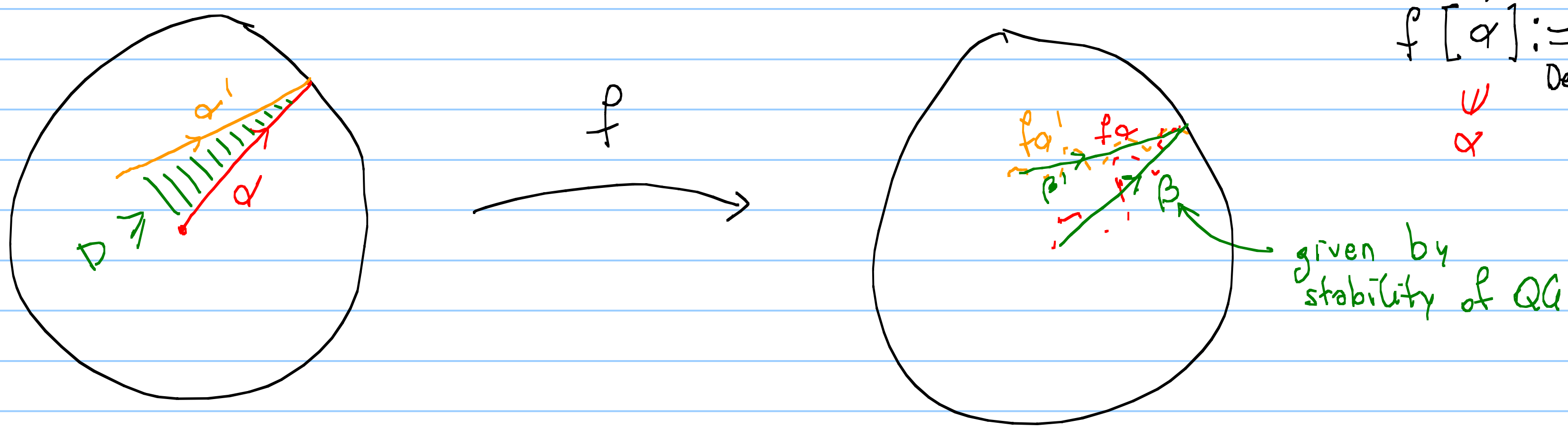
$\Rightarrow [\gamma(-t), \gamma(t)] \xrightarrow[t \to \infty]{} \alpha$. $\emptyset$

Boundary extension

$$\partial \mathbb{H}^n = \{\text{geo. rays}\}/\text{asymp. equiv}$$

$$f[\alpha] \underset{\text{Def}}{:=} [\beta]$$

α



Need to check:

① If $\alpha' \sim \alpha$, then $\beta \sim \beta'$

Pf. Since f is L-QI, we have $\begin{cases} f\alpha \subset N_{LD+L}(f\alpha') \\ f\alpha' \subset N_{LD+L}(f\alpha) \end{cases}$

$$d(f\alpha(t), f\alpha'(t)) \le L\, d(\alpha(t), \alpha'(t)) + L \le LD + L$$

$\Downarrow$ stability f is QI

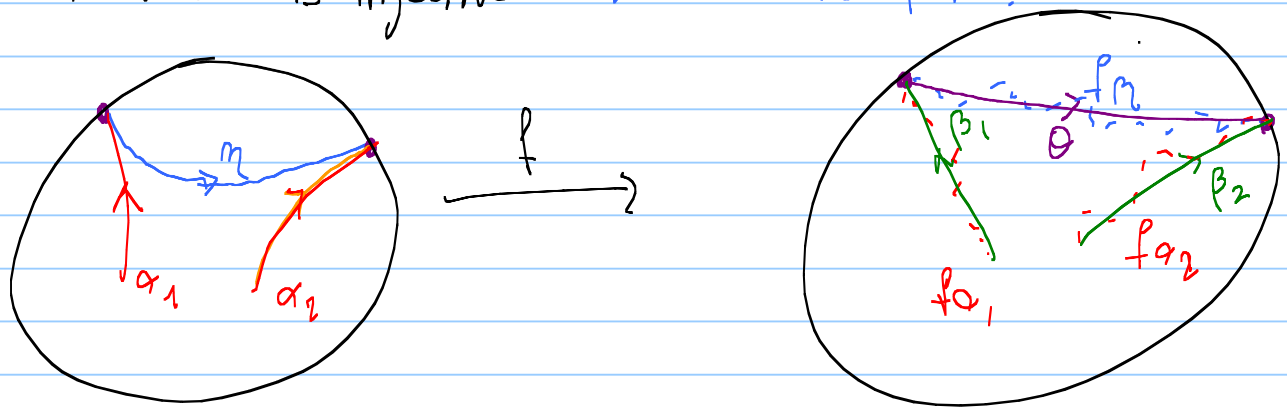
$$\begin{cases} \beta \subset N_R(f\alpha) \subset N_{R+LD+L}(f\alpha') \subset N_{\underbrace{2R+LD+L}_{R'}}(\beta') \\ \beta' \subset N_R(f\alpha') \end{cases} \Rightarrow \begin{cases} \beta \subset N_{R'}(\beta') \\ \beta' \subset N_{R'}(\beta) \end{cases}$$

$$\Downarrow$$

$$\beta \sim \beta'$$

$\square$

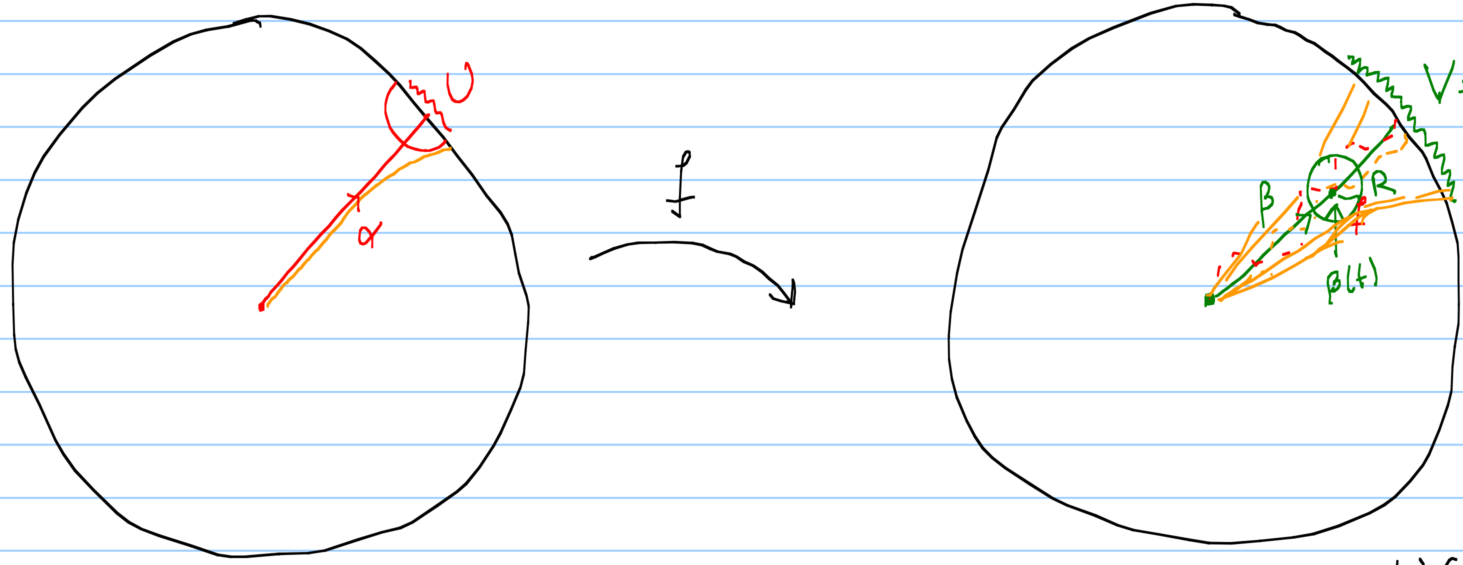
(2) $f: \partial \mathbb{H}^n \longrightarrow \partial \mathbb{H}^n$ is injective: Ex: Look at the picture!



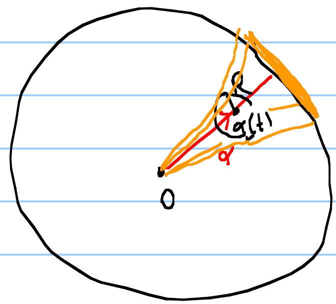
$\square$

③ $f: \partial \mathbb{H}^n \longrightarrow \partial \mathbb{H}^n$ continuous:

$fU \subset V$



A system of neigh of a geo ray:



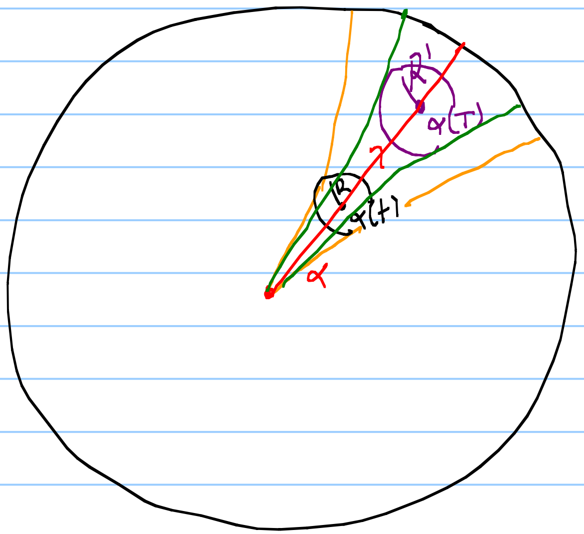
$\forall\, t > 0 \quad \forall\, R > 0$

$U_\alpha(t, R)$ = set of rays starting at $\alpha(0)$ and passing through the ball centered at $\alpha(t)$ of radius R

Claim: $\{U_\alpha(t,R)\}$ forms a system of open neigh of $[\alpha] \in \partial \mathbb{H}^n$.

Want to find U s.t. $f(U) \subset U_\beta(t,R)$

Obs:

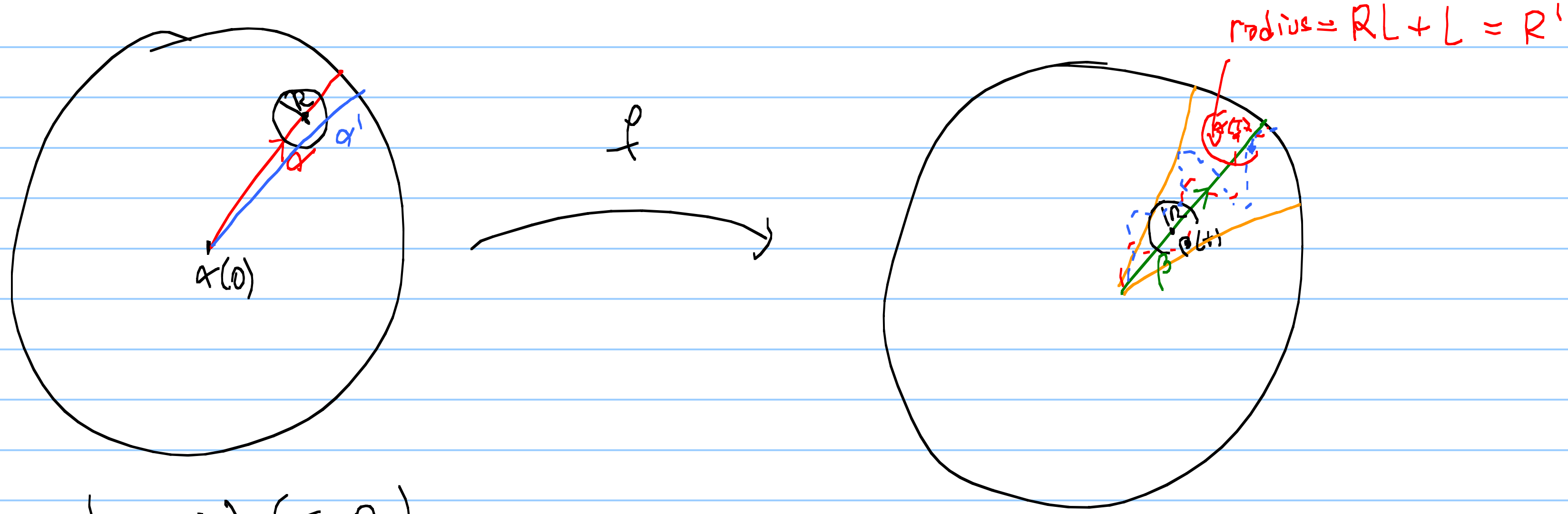


For fixed $R, R' > 0$ and $t > 0$

If $T > 0$ is much larger than $t > 0$

Then $U_\alpha(T, R') \subset U_\alpha(t, R)$

Now, back to the pf of continuity



Take $\alpha' \in U_\alpha(T, R)$

Since f is L-QI : $f\alpha'$ intersects $B(f\alpha(T), RL+L)$ $\Rightarrow$ by stability of QG we have that β' intersects

$B(f\alpha(T), R' + R_0)$

↑ stability const

and we know that $d(f\alpha(T), \beta) \leq R_0$

$\Rightarrow$ β' intersects $B(\beta(T'), R' + 2R_0)$

since $\beta(T')$ is close to $f\alpha(T)$ we have that

$T' = d(\beta(0), \beta(T'))$

$\simeq d(f\alpha(0), f\alpha(T))$

$\underset{QI}{\simeq} T$

In particular T' is very large $\not\emptyset$

$\beta' \subset U_\beta(T', R' + R_0)$

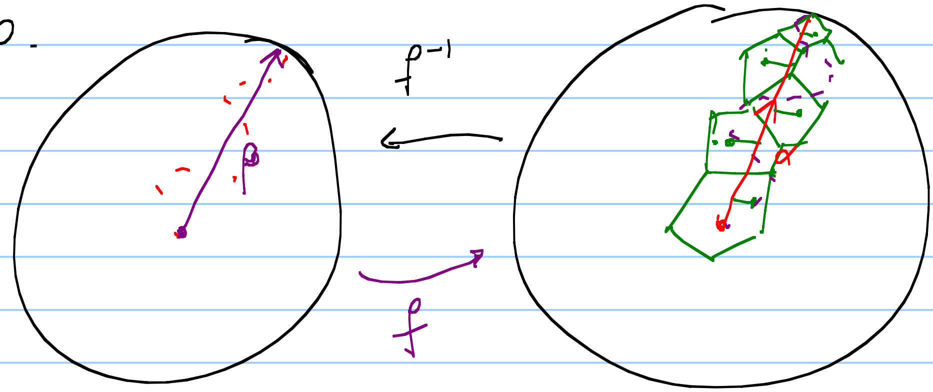
$\cap$

$U_\beta(t, R)$ if T, and hence T', is large enough.

Hw. cpt Hausdorff spaces

Now: $f : \partial \mathbb{H}^n \longrightarrow \partial \mathbb{H}^n$ is continuous + injective $\Rightarrow$ f homeo with the image

Ex: f is surjective $\Rightarrow$ f is a homeo.

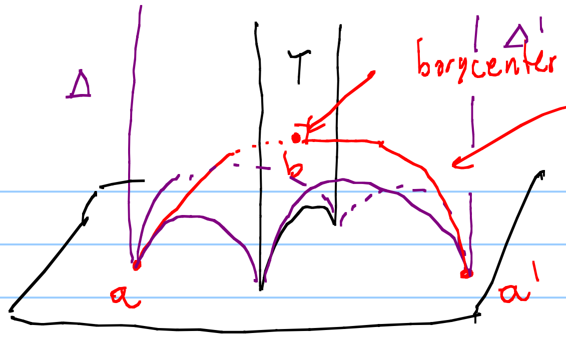


Lemma: Let $f: \partial\mathbb{H}^n \longrightarrow \partial\mathbb{H}^n$ be a homeomorphism. $n \geq 3$
Suppose that f sends the vertices of regular ideal n-simplices to the vertices of regular ideal simplices.
Then $f = \phi|_{\partial\mathbb{H}^n}$ where $\phi \in \mathrm{Isom}(\mathbb{H}^n)$

Notice: Any homeo of $\partial\mathbb{H}^2$ satisfies this property, but $\mathrm{Homeo}^+(S^1)$ is much larger than $\mathrm{Isom}^+(\mathbb{H}^2)$
$S^1 = \partial\mathbb{H}^2$, $\mathrm{Isom}^+(\mathbb{H}^2) = PSL_2\mathbb{R}$.

Pf. Need two properties

① Given a regular ideal $(n-1)$-simplex $T \subset \mathbb{H}^n$, there are exactly two regular ideal n-simplices with T as a face



Ex:

(i) Δ, Δ' are regular

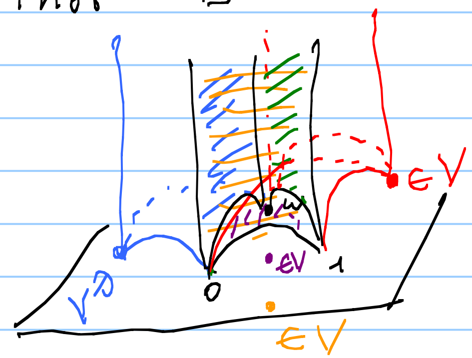
(ii) If Δ is regular and T is the face opposed to the vertex $a \in \Delta$, then the orthogonal proj of a to T is the barycenter of the face.

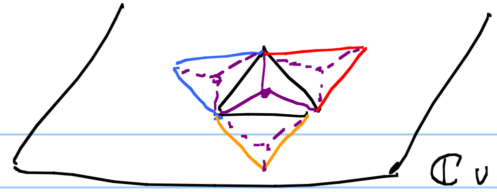
② Given a regular ideal n-simplex $\Delta \subset \mathbb{H}^n$, consider the subset $V \subset \partial\mathbb{H}^n$ obtained by consecutively reflecting Δ along its codim=1 faces, that is

G = group gen. by the reflections along the faces of Δ

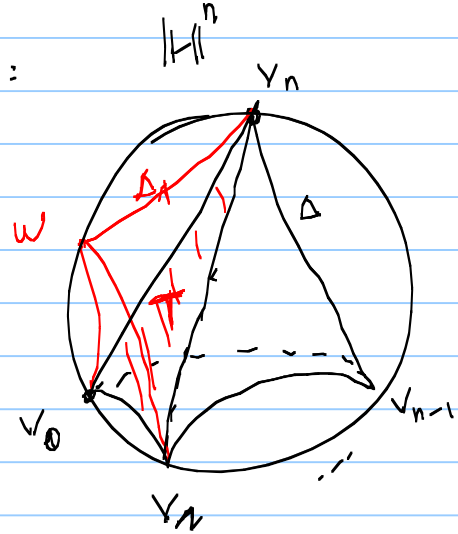
$V := G \cdot (\partial_\infty \Delta)$,

V is dense.

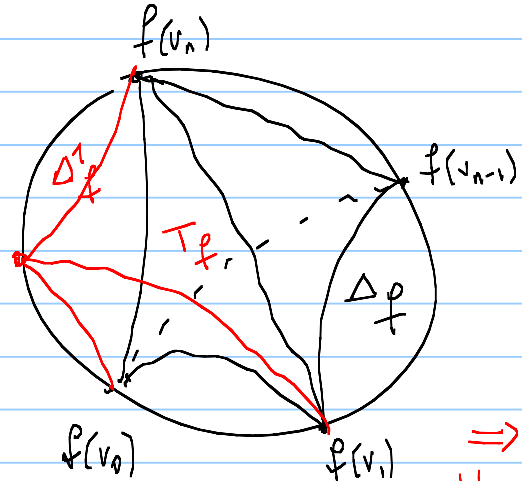




① + ② ⟹ Lemma:



$\xrightarrow{f}$



$\phi(w) = f(w)$

Property ① because ϕ is an isometry

f homeo ⟹ Δ'_f is the other regular ideal n-simplex with face T_f

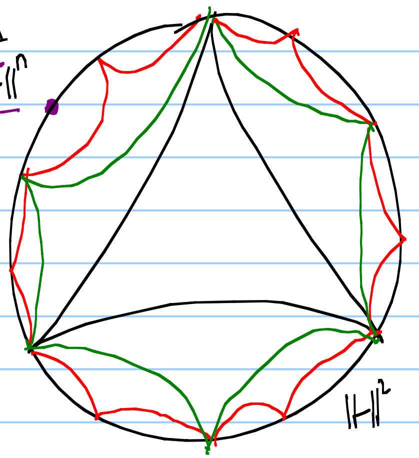
let $\phi \in \mathrm{Isom}(\mathbb{H}^n)$ an isometry that sends Δ to Δ_f (with the right order of vertices)

.... ⟹ ϕ and f agree on V

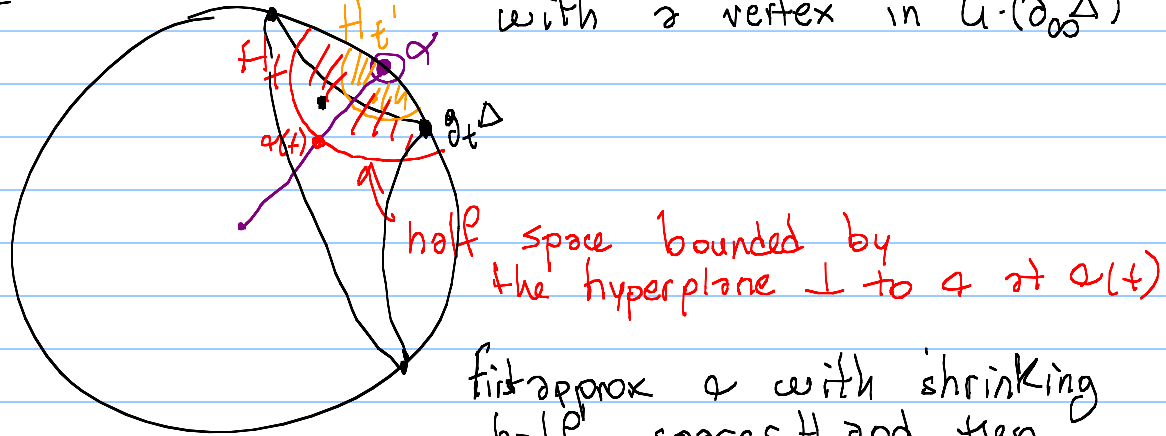
$\Rightarrow \quad \phi = f$ because V is dense and both are continuous $\blacksquare$

Pf of property (2)

Idea: $G\Delta$ covers $\mathbb{H}^n$



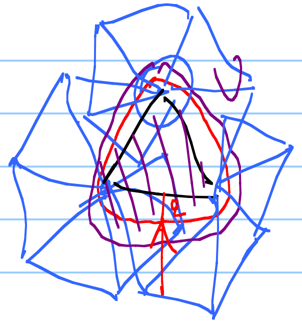
If $G\Delta$ covers $\mathbb{H}^n$, then we can approx each pt on $\partial\mathbb{H}^n$ with a vertex in $G\cdot(\partial_\infty\Delta)$



first approx α with shrinking half spaces H_t and then use $G\Delta = \mathbb{H}^n$ to find $g_t\Delta$ which intersects H_t some vertex in

$\Rightarrow \partial_\infty(g_t\Delta)$ approx α.

$\mathbb{R}^n$



You can easily cover an open neigh of Δ by reflecting Δ along its faces U

$N_\varepsilon(\Delta) \subset U$

$S \subset \mathbb{R}^n$ $\qquad S = G\Delta \Rightarrow S = \mathbb{R}^n$

S is open

S is also closed

$< \varepsilon$

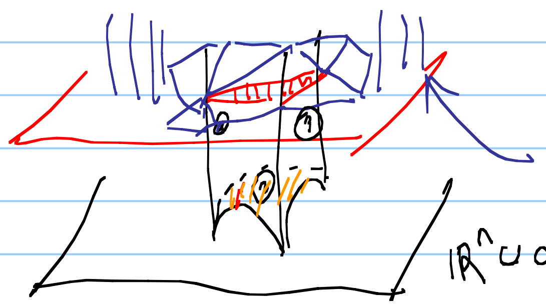
$g_1 x_n \quad g_N x_N \quad y$

$y \in N_\varepsilon(g_N \Delta)$

$\Downarrow$

$y \in G\Delta$

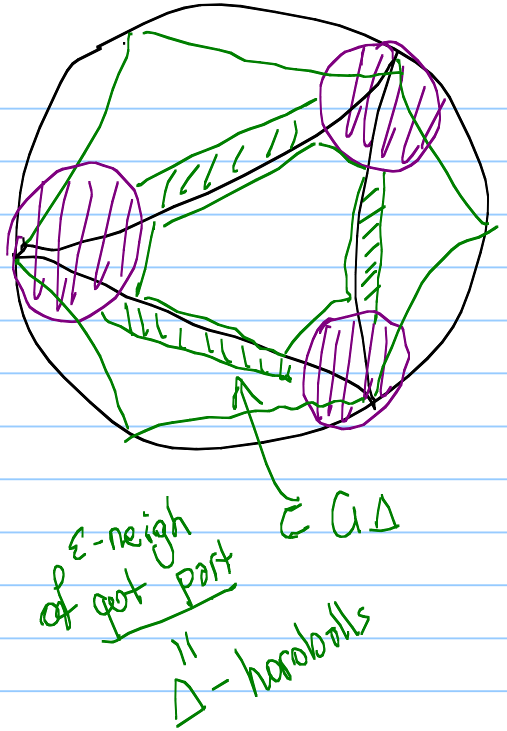
Enough to show that $N_\varepsilon(\Delta) \subset G\Delta$



$\mathbb{H}^n = U_n$

$\mathbb{R}^n \cup \infty = \partial U_n$

the Euclidean picture shows that the blue region is contained in $G\Delta$



The horoballs are in $G\Delta$

$\Rightarrow \exists\, \varepsilon > 0$ s.t. $N_\varepsilon(\Delta) \subset G\Delta$.

$\Rightarrow$ by connectedness $G\Delta = \mathbb{H}^n$. $\square$