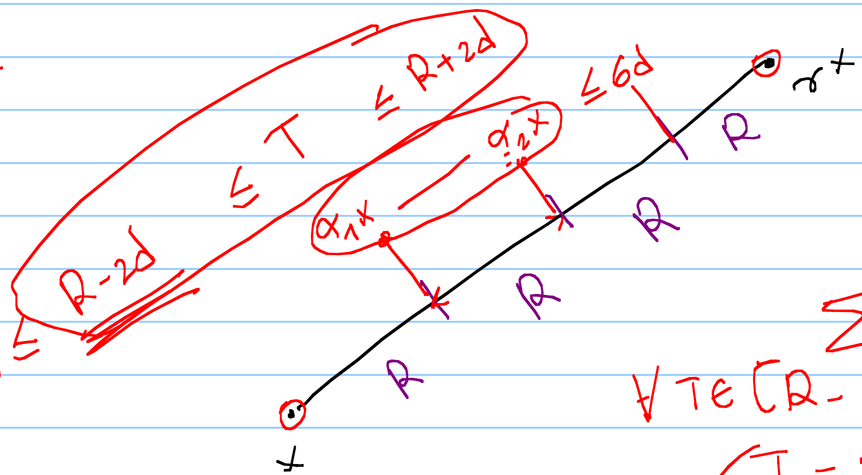
Hyperbolic Manifolds - Lecture 22

Note Title 03/02/2021

Extension of quasi-isometries



$\frac{1}{2}\sum_S d(a_i, a_{i+1})$

$\geq$

$\frac{1}{2}\sum d(a_i x, a_{i+1} x)$

$\geq$

$\sum d(a_i x, a_{i+1} x) - 2d$

$\forall T \in [R-2d, R+2d]$

$T - 2d \geq \frac{1}{2} T$

$R \geq 4d$

$\alpha \cdot \gamma$

$\Gamma_1 \curvearrowright (\Gamma_1, d_{S_1}) \qquad \gamma \xrightarrow{\alpha} \alpha \cdot \gamma \qquad \frac{1}{2} d(\alpha_1 x, \alpha_2 x) \leq$

$\downarrow \phi_1 \qquad \phi_1 \downarrow \qquad \downarrow \phi_1$

$\Gamma_1 \curvearrowright \mathbb{H}^n \qquad \gamma x \xrightarrow{\alpha} \alpha\gamma . x$

$\rho : \Gamma_1 \longrightarrow \Gamma_2$

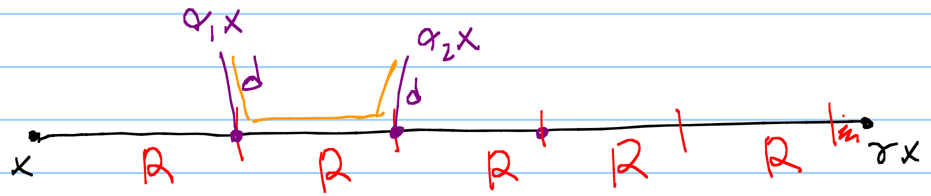
$\phi_1 \quad \Gamma_1$- equiv.

$\Gamma_1 \curvearrowright \mathbb{H}^n \xrightarrow{\phi_1^{-1}} \Gamma_1 \xrightarrow{\rho} \Gamma_2 \xrightarrow{\phi_2} \mathbb{H}^n \curvearrowleft \Gamma_2$

$\rho : \Gamma_1 \to \Gamma_2$

$\Gamma_2 \xrightarrow{\phi_2} \mathbb{H}^n \qquad \phi_2 \quad \Gamma_2$- equiv.

Want to fix a scale $R > 0$ only depending on $\operatorname{diam} D(x) = d$ and work with this choice



$$R - 2d \le d(\sigma_1 x, \sigma_2 x) \le 2d + R$$

$$\left(\in [R-2d, R+2d] \right)$$

$$d(x, \gamma x) \ge \sum_i \left(d(\sigma_i x, \sigma_{i+1} x) - 2d \right) \ge \frac{1}{2} \sum d(\sigma_i x, \sigma_{i+1} x)$$

want to get rid of this

↑ can do this

if $\forall\ T \in [R-2d, R+2d]$ we have $T - 2d \geq \frac{1}{2} T$

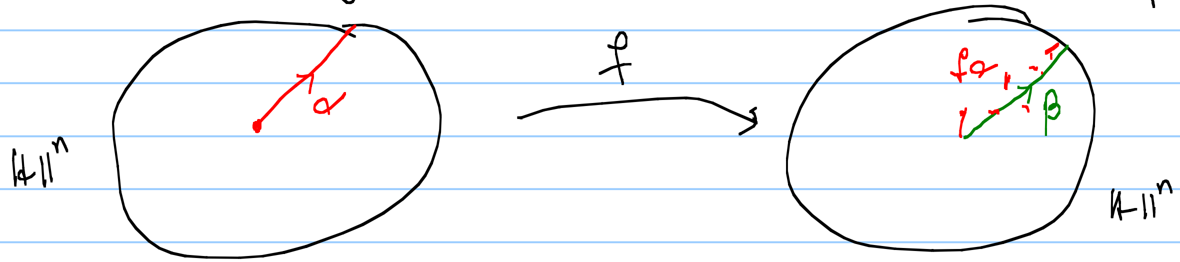
For this it is enough to choose $R = 6d$.

$f : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ quasi-isometry

Want: Extension $f : \partial\mathbb{H}^n \longrightarrow \partial\mathbb{H}^n$

f is QI $\Rightarrow$ Need to check: ① $[\alpha] \to [\beta]$ is well-def ② it is injective ③ it is continuous

Idea: Recall $\partial\mathbb{H}^n = \{$ geodesic rays $\alpha : [0,\infty) \longrightarrow \mathbb{H}^n \} /$ asymptotic equivalence



$f[\alpha] = [\beta]$

Def: A quasi-geodesic in $\mathbb{H}^n$ is a QI-embedding $\alpha : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{H}^n$

possibly $a = -\infty$, $b = +\infty$

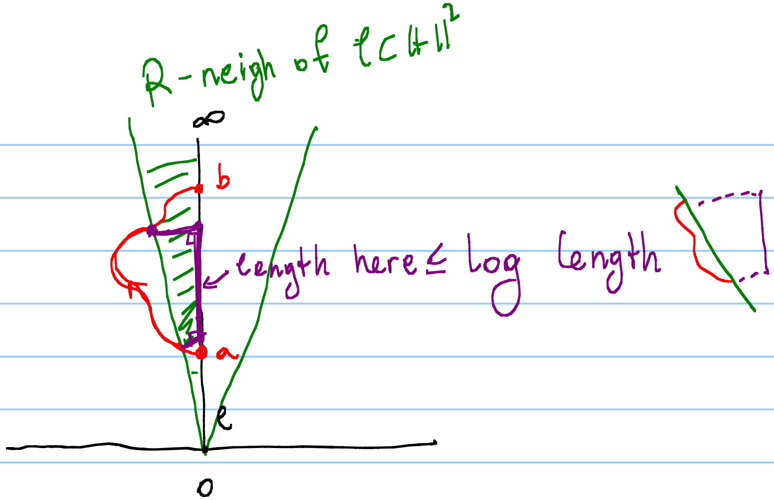
Rmk: $f : \mathbb{H}^n \to \mathbb{H}^n$ is a QI

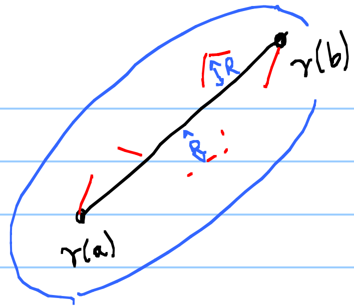
$\Rightarrow$ $f \circ \alpha : I \subset \mathbb{R} \to \mathbb{H}^n$ is a quasi geodesic for every geodesic $\alpha : I \to \mathbb{H}^n$

Proposition (Stability of quasi-geodesics): For every $L > 0$ there exists $R > 0$ s.t. the following holds: Let $\gamma : [a,b] \longrightarrow \mathbb{H}^n$ be a L-quasi geodesic

Then $\gamma[a,b] \subset N_R([\gamma(a), \gamma(b)])$ = R-neigh. of the geo $[\gamma(a), \gamma(b)]$

$[\gamma(a), \gamma(b)] \subset N_R(\gamma[a,b])$

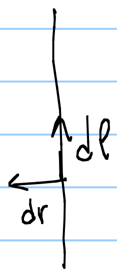




Idea: $\mathbb{H}^2$

In fact if we work in normal coord around ℓ the metric of $\mathbb{H}^2$ can be written as

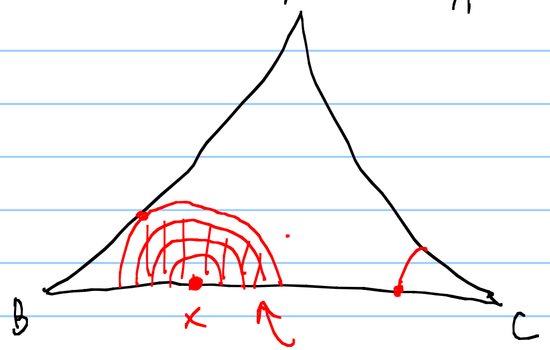
$$ds^2_{\mathbb{H}^2} = \cosh(r)^2 d\ell^2 + dr^2$$



Lemma: There exists $\delta > 0$ s.t. the following holds: For every piecewise geodesic path $\gamma : [a,b] \longrightarrow \mathbb{H}^n$, then
$\uparrow$ finite interval

$$d(x, \gamma[a,b]) \leq \delta \log_2 \ell(\gamma) + 1$$
$\forall \quad x \in [\gamma(a), \gamma(b)]$

The reason for this is δ-hyperbolicity, namely the following property



$\subset \mathbb{H}^n$

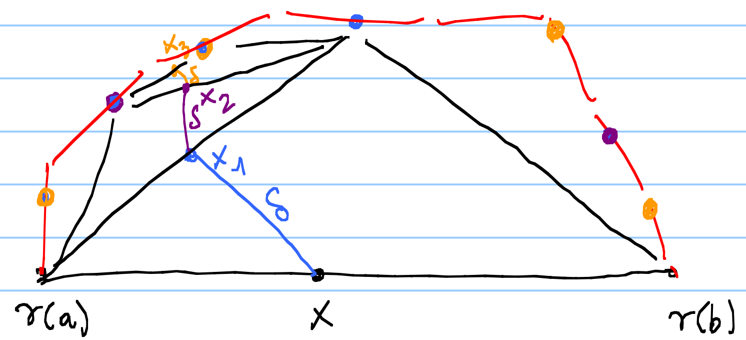
the area of this half disk $\leq$ area $(ABC) \leq \pi$

e^{r}, r = radius of the disk

$\Downarrow$

Conclude: Each $x \in [BC]$ is $\underbrace{\text{uniformly close}}_{= \delta \simeq \log(\pi)}$ to $[AB] \cup [AC]$

Pf lemma:



Divide γ into 2^n segments of length ≤ 1

$n \simeq \log_2 \ell(\gamma)$

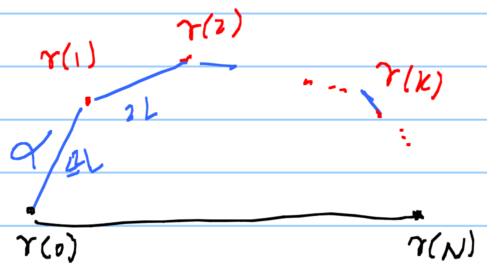
Pick $x \in [\gamma(a), \gamma(b)]$

look at triangles as in the picture

$\Rightarrow d(x, \gamma[a,b]) \simeq \delta n$ $\boxtimes$

Pf of stability of quasi geodesics

Ex: α is a L'-QG for some L' only dep on L.



① Case $[a,b]$ finite

$\gamma : [a,b] \longrightarrow \mathbb{H}^n$ $\quad L$-QG $\qquad [a,b] = [0, N]$, $N \in \mathbb{N}$

We discretize γ as $\gamma(0), \gamma(1), \gamma(2) \dots \gamma(N)$

Since γ is QG $\quad d(\gamma(i), \gamma(i+1)) \leq 2L$

Connect all these consecutive pts by geo. and get a piecewise α with the property

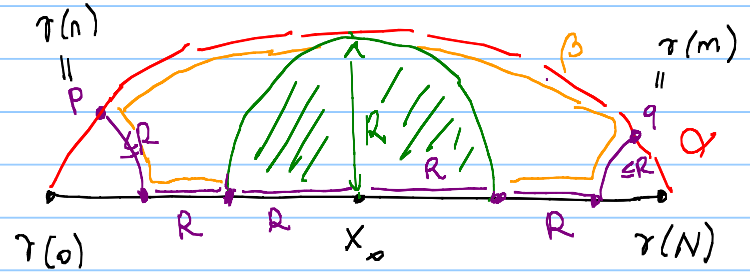
$$\gamma[0,N] \subset N_{2L}(\alpha)$$

$$\alpha \subset N_{2L}(\gamma[0,N])$$

Pf of $[\gamma(0), \gamma(N)] \subset N_{R_0}(\alpha)$ for some unif. R_0 only dep on L

$$R = \max_{x \in [\gamma(0), \gamma(N)]} d(x, \alpha) = d(x_0, \alpha)$$

Now consider the configuration as in the picture



by the Lemma: $R \le S \log_2 \ell(\beta) + 1$

$\le 4R + \ell(\alpha|_{[p,q]})$

up to a tiny error we can choose p, q to be $p = \gamma(n)$, $\gamma(m) = q$ for some $n, m \in [0, N] \cap \mathbb{N}$

α piecewise geo $\rightarrow$ $\| \quad \sum_{n \le j \le m-1} d(\gamma(j), \gamma(j+1))$

$$\overset{\cdot}{\leq} 2L|m-n|$$

$$\leq 2L\left(L\, d(\gamma(n),\gamma(m)) + L\right)$$

γ is a L-QG

$\iff$

$$\frac{1}{L}|t-s| - L \leq d(\gamma(t),\gamma(s)) \leq L|t-s| + L$$

$$\leq 2L^2 6R + 2L^2$$

c only depends on L

In conclusion: $R \leq 8 \log_2 (cR + c) + 1$

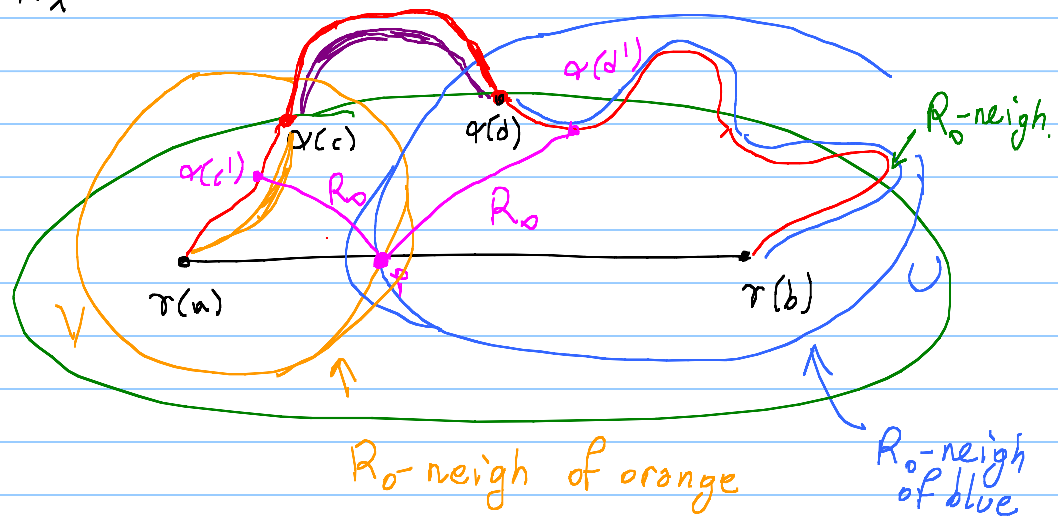
$\Rightarrow$ R cannot be too large, that is, $R \leq R_0$ only depending on L.

We now show that $\gamma[a,b] \subset N_{R_1}([\gamma(a),\gamma(b)])$ with R_1 only dep on L.

Suppose that $[c,d] \subset [a,b]$ is a maximal interval on which $\alpha[c,d]$ is outside $N_{R_0}([\gamma(a),\gamma(b)])$

We know that $[\gamma(a),\gamma(b)] \subset N_{R_0}(\alpha)$

$\Rightarrow$ picture



$[\gamma(a),\gamma(b)] \subset U \cup V$

$\Rightarrow \exists\ p \in U \cap V \cap [\gamma(a),\gamma(b)]$

$c' < c < d < d'$

$d(p, \alpha(c')), d(p, \alpha(d')) \leq R_0$

$$\frac{1}{L}|c'-d'| - L \leq d(\alpha(c'), \alpha(d')) \leq 2R_0$$

α is a L-QG

$$\Rightarrow \quad |c-d| \leq |c'-d'| \leq L(2R_0 + L)$$

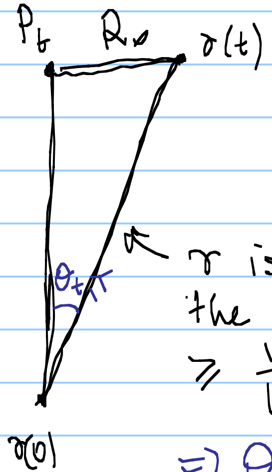
$$\Rightarrow \quad \alpha[c,d] \subset N_{R_1}([\gamma(a), \gamma(b)])$$

$R_1 = R_0 + L^2(2R_0 + L) + L$ $\square$

② Case $[a,b]=[0,\infty]$

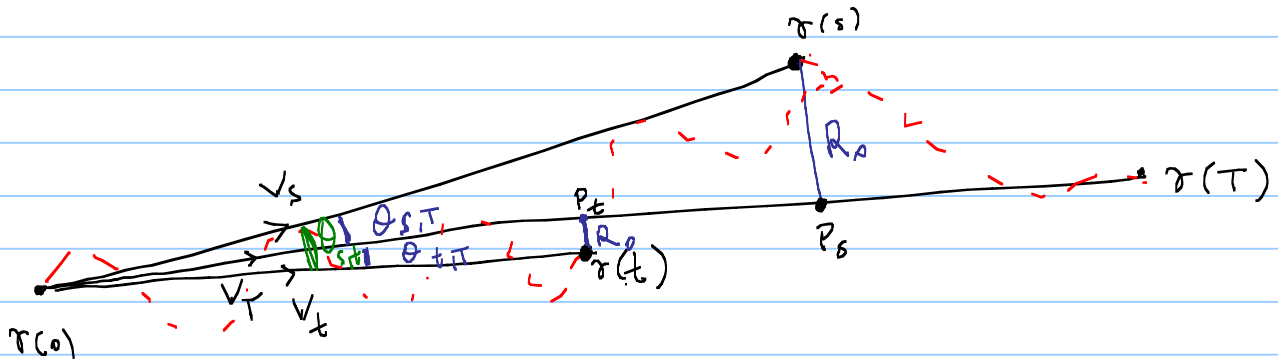
We take a limit of the finite case.

$t < s < T$ very large



γ is QG
the length
$\geq \frac{1}{L}t - L \approx t$

$\Rightarrow \theta_{t,T}$ must be small (exercise in $\mathbb{H}^2$)



$\gamma[0,T] \subset N_{R_0}([\gamma(0),\gamma(T)])$

$\Rightarrow \theta_{s,T}, \theta_{t,T}$ are very small

$\Rightarrow \theta_{s,t}$ is very small as well

$\Rightarrow$ The sequence of unit vectors $\{V_t \in T^1_{\gamma(0)}\mathbb{H}^n\}$ parallel to $[\gamma(0), \gamma(t)]$ is a Cauchy sequence (for the spherical metric = angles)

$\Rightarrow$ V_t converges

$\Rightarrow$ $[\gamma(0), \gamma(t)]$ converges to a geodesic ray β

Since $\gamma[0,t] \subset N_{R_0}([\gamma(0), \gamma(t)])$

and $[\gamma(0), \gamma(t)] \subset N_{R_0}(\gamma[0,T])$

$\Rightarrow \gamma[0,\infty] \subset N_{R_2}(\beta)$

$\beta \subset N_{R_2}(\gamma[0,\infty])$ $\boxtimes$

③ Case $[a,b) = [-\infty, \infty]$