Hyperbolic Manifolds - Lecture 21

Note Title 02/02/2021

Mostow Rigidity

necessary, if $n=2$, this fails

Prasad-Mostow: It holds for finite volume mfds as well. $\mathbb{H}^n/\Gamma_i$ cpt.

Theorem (Mostow Rigidity): Let $n \geqslant 3$.

Consider $\Gamma_1, \Gamma_2 < \mathrm{Isom}^+(\mathbb{H}^n)$ discrete, torsion free, cocompact.

Suppose that $\rho : \Gamma_1 \longrightarrow \Gamma_2$ is an isomorphism, then

there exists $\phi \in \mathrm{Isom}^+(\mathbb{H}^n)$ s.t. $\rho(\gamma) = \phi \gamma \phi^{-1}$ $\forall \gamma \in \Gamma$

Equivalently, $\phi : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ is a ρ-equivariant isometry. In particular

ϕ descends to an isometry $\mathbb{H}^n/\Gamma_1 \xrightarrow{\sim} \mathbb{H}^n/\Gamma_2$.

Step 1 of Pf: Find a ρ-equivariant quasi-isometry $\phi: \mathbb{H}^n \longrightarrow \mathbb{H}^n$

Def: $\phi: \mathbb{H}^n \longrightarrow \mathbb{H}^n$ is a quasi-isometry (QI) if there exists $L > 0$ s.t.

① $L\,d(x,y) + L \geq d(\phi(x), \phi(y)) \geq \frac{1}{L} d(x,y) - L$ (QI-embedding)

② $\forall\, y \in \mathbb{H}^n \;\exists\, x \in \mathbb{H}^n$ s.t. $d(\phi(x), y) \leq L$ (coarsely surjective)

Def: Let Γ be a finitely generated group with a finite generating set $S = \{s_1, \dots, s_n\}$ (every $\gamma \in \Gamma$ can be written as a word $\gamma = s_{j_1}^{\pm 1} s_{j_2}^{\pm 1} \cdots s_{j_k}^{\pm 1}$); S induces a word metric d_S on Γ that makes (Γ, d_S) a metric space; d_S is defined as follows: $d_S(\alpha, \beta) =$ minimal length of a word in the generators $s_1^{\pm 1}, \dots, s_n^{\pm 1}$ that represents $\alpha^{-1}\beta$.

$d(1, s_1) = 1$
$d(1, s_2^{-1}) = 1$
$d(s_1 s_2^{-1}, 1) \leq 2$
$\vdots$

(Milnor-Schwarz lemma)

Lemma: $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ discrete, torsion free and cocompact ($\mathbb{H}^n/\Gamma$ is cpt). Then

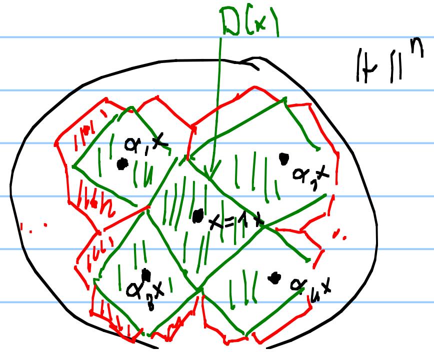
① Γ is finitely generated,

② Fix a generating set S and a pt $x \in \mathbb{H}^n$ and consider the orbit map

$$f : (\Gamma, d_S) \longrightarrow \mathbb{H}^n, \quad \alpha \longmapsto \alpha x$$

then f is a QI.

Idea:



Since $\mathbb{H}^n/\Gamma$ is cpt
$D(x)$ is cpt.
$\Rightarrow$ it has finite diameter
$\Rightarrow$ the inclusion

of the orbit

$(\Gamma x, d_{\mathbb{H}^n}) \subset \mathbb{H}^n$ QI

Claim: Γ is generated by the first generation of translates of $D(x)$ and $(\Gamma, d_S) \longrightarrow (\Gamma x, d_{\mathbb{H}^n})$ is a QI

Pf. ① Let $d = \operatorname{diam} D(x)$

Consider $S = \{ \gamma \in \Gamma \mid d(x, \gamma x) \leq 4d \}$

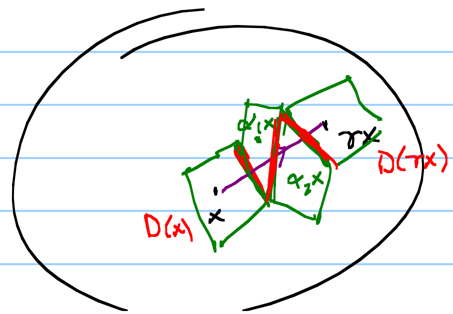
$= \{ \gamma_1, \dots, \gamma_m \}$

proper discontinuity

Then S generates Γ, in fact consider any $\gamma \in \Gamma$

If $d(x, \gamma x) \leq 4d \Rightarrow \gamma \in S$

If $d(x, \gamma x) > 4d \Rightarrow$ consider a geodesic joining x to γx, it intersects finitely



many translates of $D(x)$, the first one being $D(x)$ and the last one $D(\gamma x)$

$D(\alpha_1 x)$, $D(\alpha_2 x)$, ..., $D(\alpha_k x)$

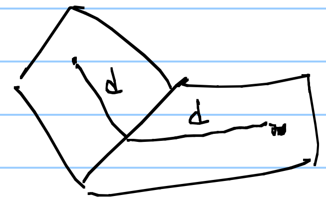
$\alpha_1 = 1$, $\alpha_k = \gamma$

$D(\alpha_{j-1}x) \cap D(\alpha_j x) \neq \emptyset$

$D(\alpha x) = \alpha D(x)$

$\Downarrow$

$d(\alpha_{j-1}x, \alpha_j x) \leq 2d$



$\Downarrow$

$d(x, \alpha_j^{-1}\alpha_{j-1}x) \leq 2d$

$\Downarrow$

$\alpha_j^{-1}\alpha_{j-1} \in S$

$\Rightarrow$ $\gamma = \alpha_k$

$= (\alpha_k\alpha_{k-1}^{-1})(\alpha_{k-1}\alpha_{k-2}^{-1}) \cdots (\alpha_2\alpha_1^{-1})$

$\alpha_k\alpha_{k-1}^{-1} \in S$, $\alpha_{k-1}\alpha_{k-2}^{-1} \in S$, $\alpha_2\alpha_1^{-1} \in S$

$\Rightarrow$ γ is a product of elements of S.

(2) Let S be the set of generators produced by the pf of (1)

We show that $(\Gamma, d_S) \longrightarrow (\Gamma x, d_{\mathbb{H}^n})$ is a QI

$\hookrightarrow \mathbb{H}^n$ QI

If S_1, S_2 are different generating sets, then $(\Gamma, d_{S_1}) \xrightarrow[Id]{} (\Gamma, d_{S_2})$

this is a QI. (Pf. Later).

Lipschitz property of the orbit map:

$$d_{\mathbb{H}^n}(\alpha x, \beta x) = d_{\mathbb{H}^n}(x, \alpha^{-1}\beta x)$$

$$\alpha^{-1}\beta = \gamma_{j_1}^{\pm 1} \gamma_{j_2}^{\pm 1} \cdots \gamma_{j_k}^{\pm 1} \qquad k = d_S(\alpha, \beta)$$

$$= d_{\mathbb{H}^n}(x, \gamma_{j_1}^{\pm 1} \cdots \gamma_{j_k}^{\pm 1})$$

$$\leq d(x, \gamma_{j_1}^{\pm 1} \cdots \gamma_{k-1}^{\pm 1} x) + d(\gamma_{j_1}^{\pm 1} \cdots \gamma_{j_k}^{\pm 1} x, \gamma_{j_1}^{\pm 1} \cdots \gamma_{j_{k-1}}^{\pm 1} x)$$

triangle ineq

$$= d_{\mathbb{H}^n}(\gamma_{j_k}^{\pm 1} x, x) \leq 4d$$

$$\leq \cdots \quad \cdots + 4d$$

$$\leq 4d \cdot k = 4d \; d_S(\alpha, \beta)$$

$$d_S(1, \alpha^{-1}\beta) \qquad d_{\mathbb{H}^n}(x, \alpha^{-1}\beta x)$$

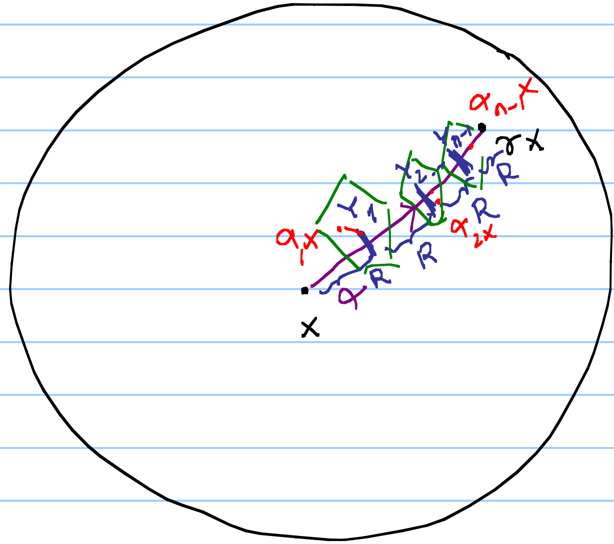
$$\alpha^{-1}\beta = \gamma$$

The other inequality: Want $\frac{1}{L} d_S(\alpha,\beta) - L \le d_{\mathbb{H}^n}(\alpha x, \beta x)$.

Enough to consider $(\alpha,\beta) = (1,\gamma)$

Consider the geodesic joining x to γx

Sample α at regular intervals of size R.



$\alpha : [0, nR] \longrightarrow \mathbb{H}^n$

$\alpha(nR) = \gamma x$

$\alpha(0) = x$

$y_j = \alpha(jR)$

Each y_j is d-close to a point $\alpha_j x$ in the orbit of x

$$d(x,\gamma x) = \text{length of } \alpha = nR$$

$$= \sum_{j\le n} d(\gamma_{j-1},\gamma_j)$$

$$\geq \sum_{j\le n} d(\alpha_{j-1}x,\alpha_j x) - 2d$$

If R is large enough

$$\geq \frac{1}{2}\sum d(\alpha_{j-1}x,\alpha_j x)$$

$$\| $$

$$d(x,\alpha_j^{-1}\alpha_{j-1}x)$$

$$\geq c\sum d_S(\alpha_j^{-1}\alpha_{j-1},1)$$

$$\geq c\, d_S(\gamma,1)$$

triangle inequality

because $\gamma = (\alpha_n\alpha_{n-1}^{-1})\dots(\alpha_2\alpha_1^{-1})$

$$d(\gamma_{j-1},\gamma_j) \geq d(\alpha_{j-1}x,\alpha_j x) - 2d$$

$$6d \geq R+2d \geq d(\alpha_{j-1}x,\alpha_j x) \geq R-2d \geq d$$

$$\Downarrow$$

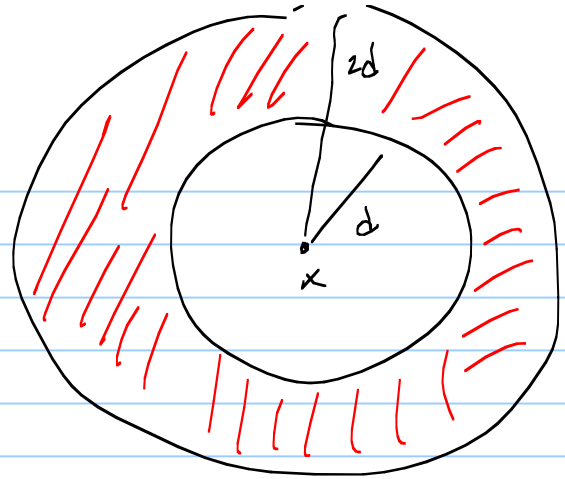
$$d(\alpha_j^{-1}\alpha_{j-1}x, x) \in [d, 6d]$$

$$\Downarrow$$

there are only finitely many elements with this property

$\exists\, c>0$ s.t. $d(\alpha_j^{-1}\alpha_{j-1}x, x) \geq c\, d_S(\alpha_j^{-1}\alpha_{j-1}, 1)$ $\Longleftarrow$

$\forall\, \alpha_j, \alpha_{j-1}$ s.t. $d(\alpha_j^{-1}\alpha_{j-1}x, x) \in [d, 2d]$



$\square$

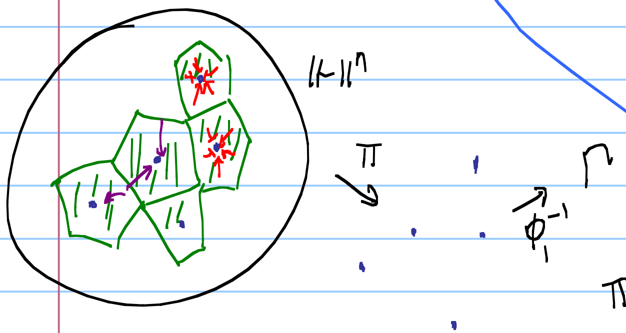
Now, back to the equivariant quasi isometry:

Lemma: S_1, S_2 gen. sets for $\Gamma \Rightarrow \mathrm{Id}: (\Gamma, d_{S_1}) \to (\Gamma, d_{S_2})$ is a QI

$$
\begin{tikzcd}
(\mathbb{H}^n, d_{\mathbb{H}^n}) \arrow[r, bend right, "\text{QI (Lemma)}"'] \arrow[rrrd, bend right] & (\Gamma_1, d_{S_1}) \arrow[l, "\phi_1"'] \arrow[r, "\rho"] & (\Gamma_2, d_{\rho(S_1)}) \arrow[d, dashed, "\text{Need QI}"] \arrow[d, phantom, "\mathrm{Id}\,\|"'] & \\
 & & (\Gamma_2, d_{S_2}) \arrow[r, "\phi_2", "\text{QI (by Lemma)}"'] & \mathbb{H}^n
\end{tikzcd}
$$

Isometry because ρ is an isom

$\phi_1 : \Gamma \to \Gamma x$ is a bijection



π selects for each point the center of a fundamental cell containing it.

Pf of Lemma:

S_1, S_2 every $\sigma \in S_2$ can be written as a word in the generators of S_1 of length at most $c > 0$ for some large const. c, similarly for $\beta \in S_1$ with generators in S_2.

$\Rightarrow d_{S_1}(\sigma, 1) = n$

$\Rightarrow \sigma = \underbrace{s_{j_1}^{\pm 1}}_{} \cdots \underbrace{s_{j_n}^{\pm 1}}_{} = \text{----}$

length of this word is at most nc

can be written as a word of length $\leq c$ in the gen. S_2

$\Rightarrow \quad d_{S_2}(\gamma, 1) \le c\, d_{S_1}(\gamma, 1)$

Switching the roles of S_1 and S_2

$\Rightarrow \quad d_{S_1}(\gamma, 1) \le c\, d_{S_2}(\gamma, 1)$ $\square$

$\Rightarrow \quad Id : (\Gamma, d_{S_1}) \to (\Gamma, d_{S_2})$ is (bijective and) c-bilipschitz.

Now : Step 2: We start with an equivariant QI $f : \mathbb{H}^n \to \mathbb{H}^n$ and we extend it to a map $\partial\mathbb{H}^n \to \partial\mathbb{H}^n$

$\partial\mathbb{H}^n$ = equivalence classes of geo rays

Idea: