Hyperbolic Manifolds - Lecture 20

Today:

1. Constructing more hyperbolic manifolds from known ones;
2. Fundamental domains;
3. A road map to Mostow rigidity;

Constructing more hyperbolic manifolds

Def. A group Γ is residually finite if $\forall\, \gamma \in \Gamma \setminus \{1\}$ there exists a finite quotient $\phi : \Gamma \twoheadrightarrow Q$ s.t. $\phi(\gamma) \neq 1$

$Q = \Gamma / \operatorname{Ker} \phi$, $\operatorname{Ker}\phi \triangleleft \Gamma$ f.i.

equivalently $\bigcap_{N \triangleleft \Gamma \text{ normal with finite index}} N = \{1\}$

Lemma (Malcev): $\Gamma < GL_n \mathbb{C}$ s.t. either

(a) Γ finitely generated

(b) $\Gamma < GL_n B$ where B is a f.g $\mathbb{Z}$-alg. $B \subset \mathbb{C}$ $B = \mathbb{Z}[\sqrt{2}]$

Then Γ is residually finite.

We know that complete closed hyperbolic manifolds are quotients $\mathbb{H}^n/\Gamma$ where $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n) = SO_0(n,1) < GL_{n+1}\mathbb{C}$ discrete and torsion free $\Rightarrow$ Malcev's Lemma applies to Γ.

$\Downarrow$

Inside Γ we find a lot of normal finite index subgroups $N \triangleleft \Gamma$

$\|$

$\Downarrow$ complete closed

Each N corresponds to a hyp. mfd $\mathbb{H}^n/N$ - - - -> $\mathbb{H}^n/\Gamma$ cpt $\Longleftarrow$ cpt.

finite covering

$N \triangleleft \Gamma$ f.i.

Fundamental domains

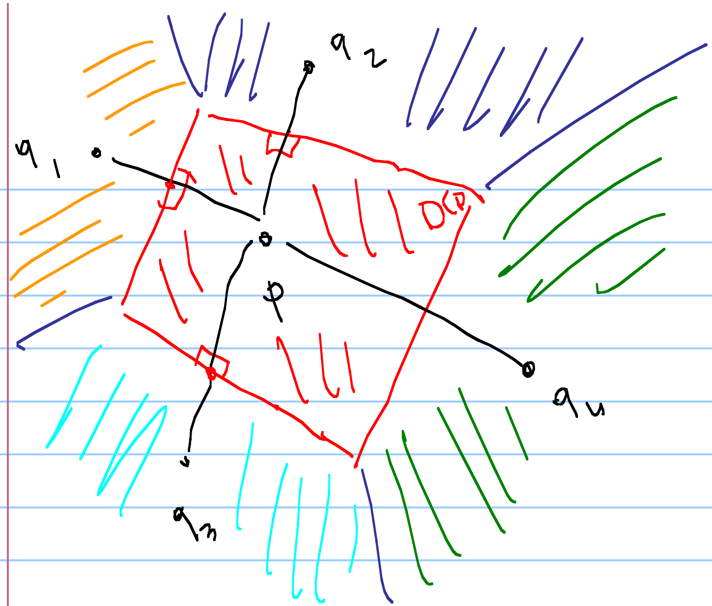
$\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ discrete, torsion free $\Longrightarrow$ $\mathbb{H}^n/\Gamma$ complete hyperbolic manifold

We now show that it is always possible to present $\mathbb{H}^n/\Gamma$ as a quotient of a convex polyhedron $P \subset \mathbb{H}^n$ with respect to isometric pairings of the faces.

Voronoi Tessellation: $S \subset \mathbb{H}^n$ discrete subset

$p \in S \longrightarrow D(p) := \{ x \in \mathbb{H}^n \mid d(p,x) \leq d(q,x) \ \forall q \in S \}$

$= \bigcap_{q \in S} \left(\mathcal{H}_{p<q} = \{ d(p,\bullet) \leq d(q,\bullet) \} \right)$



$=$ Voronoi cell of p

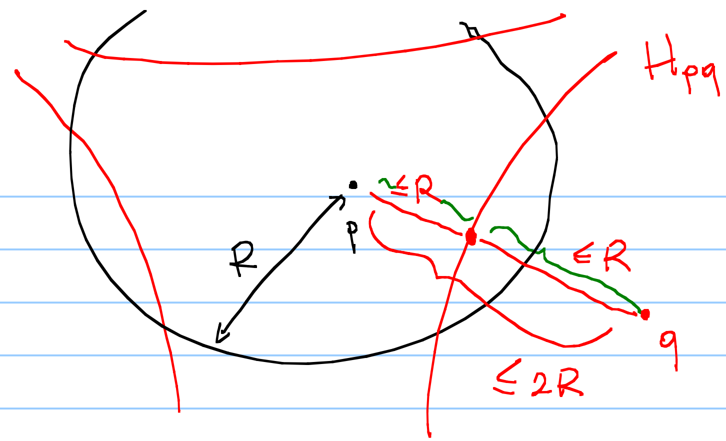
this is a half space bounded by the hyperplane $H_{pq} =$
$= \{ d(p,\cdot) = d(q,\cdot) \}$

Lemma: $D(p)$ is a polyhedron

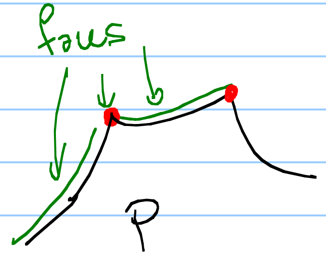
$=$ a locally finite intersection of half spaces

in a small neigh around every point there can be only finitely many boundary hyperplanes

Pf. We only have to check that the intersection is loc. finite.



Since S is discrete there are only finitely many pts in $B(p, 2R) \cap S$. $\square$



A face of a polyhedron P is $\partial H \cap P$ if H is a half space with $\operatorname{int}(P) \cap H = \emptyset$ and $P \cap H \neq \emptyset$

Consider $p \neq p'$ $\quad p, p' \in S$ and the corresponding cells $D(p), D(p')$

Lemma: ① $D(p) \cap D(p') = \underbrace{D(p) \cap H_{pp'}} = D(p') \cap H_{pp'}$

this is a face of $D(p)$

$= \partial \mathcal{H}_{p' < p} \cap D(p)$

similarly

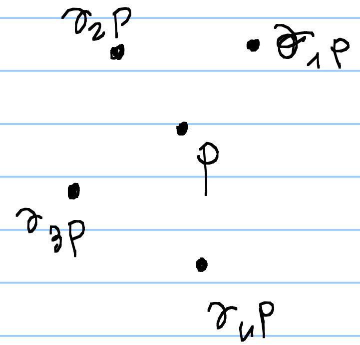
② $\bigcup_{p \in S} D(p) = \mathbb{R}^n$

any pt $x \in \mathbb{R}^n$ has a nearest neighbour $p_x \in S \iff x \in D(p_x)$
because S is discrete.

We apply the above construction to $\Gamma \curvearrowright \mathbb{H}^n$ with Γ discrete:

Pick $p \in \mathbb{H}^n$, denote by $\Gamma p = \{\gamma p \mid \gamma \in \Gamma\}$ the orbit of p

since Γ is discrete, Γp is discrete as well.



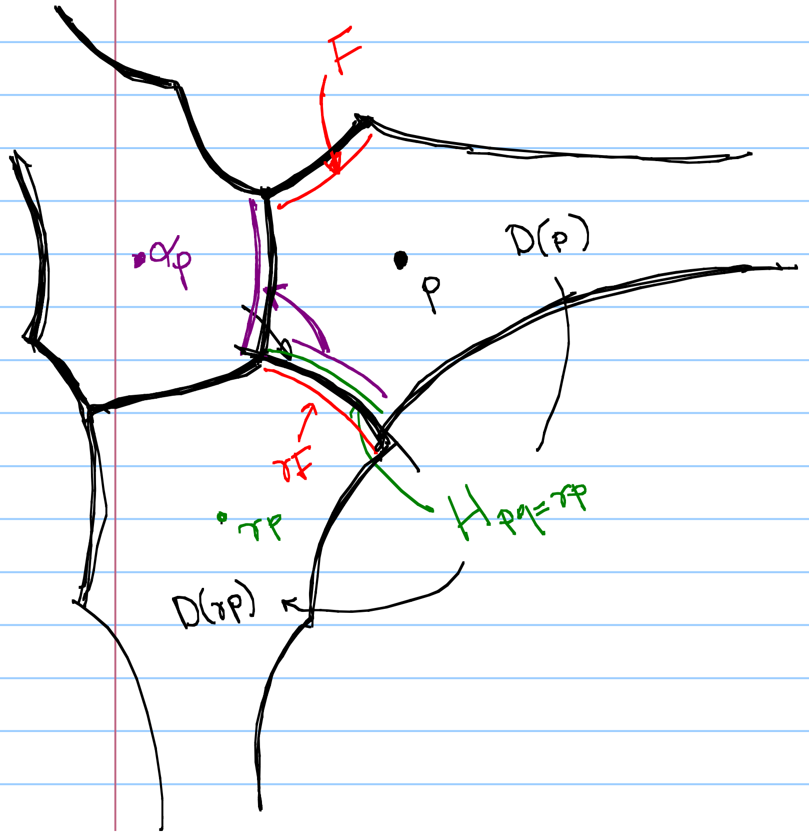
$\Rightarrow$ we can construct the Voronoi tesselation of $\mathbb{H}^n$ relative to $\underbrace{S = \Gamma p}_{\Gamma\text{-inv}}$. $\Rightarrow$ $\mathbb{H}^n = \bigsqcup_{\gamma \in \Gamma} D(\gamma p)$

$\Downarrow$

$D(\gamma p)$ is an isometric copy of $D(p)$, in fact $D(\gamma p) = \gamma D(p)$.

Notice that there are no two points $x, y \in D(p)$ that are identified by an isometry in Γ (exercise)

$\Rightarrow$ $\operatorname{int}(D(p)) \hookrightarrow \mathbb{H}^n/\Gamma$ isometric embedding, the complement is the projection of $\partial D(p)$ (which is a loc. finite collection of pieces of hyperplanes)



$\Rightarrow$ $\mathbb{H}^n/\Gamma = D(p) \big/$ identifications of the faces induced by Γ

$D(p)$ = Dirichlet fundamental domain of Γ relative to $p \in \mathbb{H}^n$.

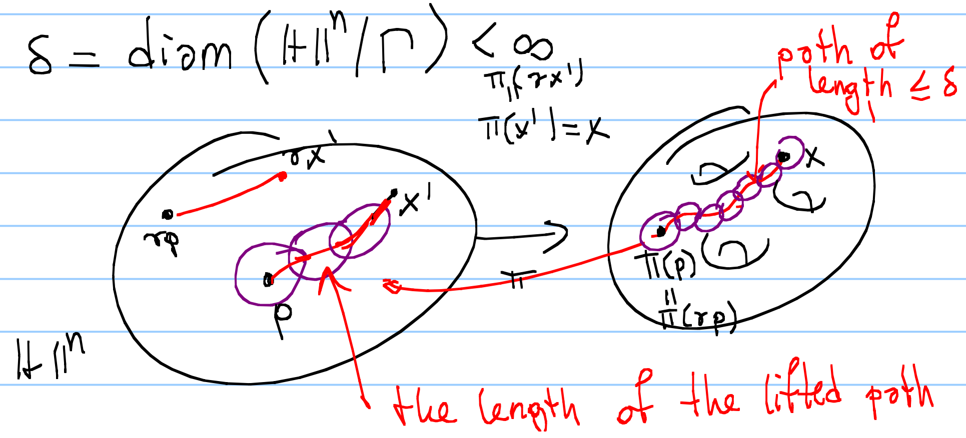
Lemma: We have the following:

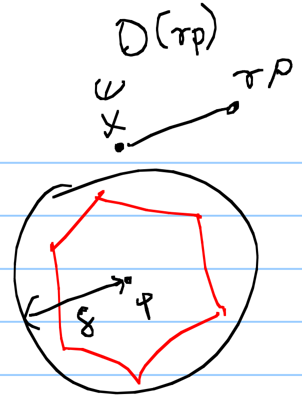
① $\mathrm{vol}(\mathbb{H}^n/\Gamma) = \mathrm{vol}(D(p))$

② $\mathbb{H}^n/\Gamma$ is cpt $\Longleftrightarrow$ $D(p)$ is cpt.

Pf. ② ($\Longleftarrow$) clear: The covering projection $\mathbb{H}^n \xrightarrow{\pi} \mathbb{H}^n/\Gamma$ is surjective when restricted to $D(p)$.

($\Longrightarrow$) Let $\delta = \mathrm{diam}(\mathbb{H}^n/\Gamma) < \infty$





$\leq \delta$ as well

$\Rightarrow$ by the definition of $D(p)$ we must have $D(p) \subset B(p, \delta)$

(every point in $\mathbb{H}^n$ is δ-close to a point in the orbit of p) $\boxtimes$

A road map to Mostow Rigidity

Theorem (Mostow): Suppose that $\begin{cases} \boxed{n \geq 3} \\ \Gamma_1, \Gamma_2 < \mathrm{Isom}^+(\mathbb{H}^n) \text{ discrete torsion free} \\ \mathbb{H}^n/\Gamma_j \text{ is a closed manifold for } j=1,2 \\ \rho : \underset{\pi_1(\mathbb{H}^n/\Gamma_1)}{\underset{\shortparallel}{\Gamma_1}} \longrightarrow \underset{\pi_1(\mathbb{H}^n/\Gamma_2)}{\underset{\shortparallel}{\Gamma_2}} \text{ is an isomorphism} \end{cases}$

Then there exists $\phi \in \mathrm{Isom}^+(\mathbb{H}^n)$ s.t. $\rho(\gamma) = \phi\gamma\phi^{-1}$ $\forall \gamma \in \Gamma_1$

In particular ϕ descends to an isometry between the quotient manifolds $\mathbb{H}^n/\Gamma_1 \xrightarrow[\phi]{} \mathbb{H}^n/\Gamma_2$.

$\Gamma_2 = \phi\Gamma_1\phi^{-1}$ the groups are conjugate

A road map: ① We want to produce a ρ-equivariant isometry $\mathbb{H}^n \to \mathbb{H}^n$, and we will start by constructing a ρ-equivariant quasi-isometry F

② Quasi-isometries $F : \mathbb{H}^n \to \mathbb{H}^n$ extend to the boundary $F = \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ (ρ-equivariant) ; the extension is a homeomorphism of the boundary.

We want to show that $F = \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ is the restriction of an isometry ϕ of $\mathbb{H}^n$ to $\partial\mathbb{H}^n \Rightarrow$ the isometry ϕ will be automatically ρ-equivariant.

We will rely on the following criterion:

If $F : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ sends the vertices of every regular ideal simplex to the vertices of a regular ideal simplex then F is the restriction of an isometry.

regular ideal simplices are exactly those simplices with maximal volume.

③ In order to check that F sends simplices of maximal volume to simplices of maximal volume we will introduce a volume invariant for Γ_1, Γ_2, which is called simplicial volume which will be universally proportional to $\mathrm{vol}(\mathbb{H}^n/\Gamma_i)$ (Proportionality principle).

Using this invariant we will be able to prove property .

As preparation for ①:

Def $f : (X, d_X) \longrightarrow (Y, d_Y)$ a map between metric spaces.

* f is a <u>quasi-isometric embedding</u> (QI emb) if there exists a const $C > 0$ s.t.

$$C\, d_X(p,q) + C \geq d_Y(f(p), f(q)) \geq \frac{1}{C} d_X(p,q) - C$$

* f is <u>coarsely surjective</u> if there exists $C > 0$ s.t. $\forall\, y \in Y\ \exists\, x \in X$ s.t.

$$d_Y(f(x), y) \leq C$$

f is a <u>quasi-isometry</u> if it is a coarsely surj. QI emb.

In this case we say that X is quasi isometric to Y

Ex: Show that if X is QI to Y, then Y is QI to X.

$\Rightarrow$ QI is an equivalence relation between metric spaces.

$$* \in K \qquad \text{QI}$$

example

$$\begin{tikzcd} \mathbb{Z} \arrow[r, shift left, "\text{inclusion}"] & \mathbb{R} \arrow[l, shift left] \end{tikzcd} \qquad \text{QI}$$

$$[x] \longleftarrow x$$

$\parallel$

integral part of x