Hyperbolic Manifolds - Lecture 2

Note Title 04/11/2020

References:

- An introduction to geometric topology, B. Martelli
  arXiv: 1610.02592v1
- Lectures on hyperbolic geometry, R. Benedetti and C. Petronio

Summary:

1. hyperboloid model of $\mathbb{H}^n$, group of isometries
2. totally geodesic k-subspaces
3. geodesics and Riemannian distance

The hyperbolic n-space $\mathbb{H}^n$: Hyperboloid model

$\mathbb{H}^n \subset$ standard Lorentzian $(n+1)$-space $= \mathbb{R}^{n,1}$

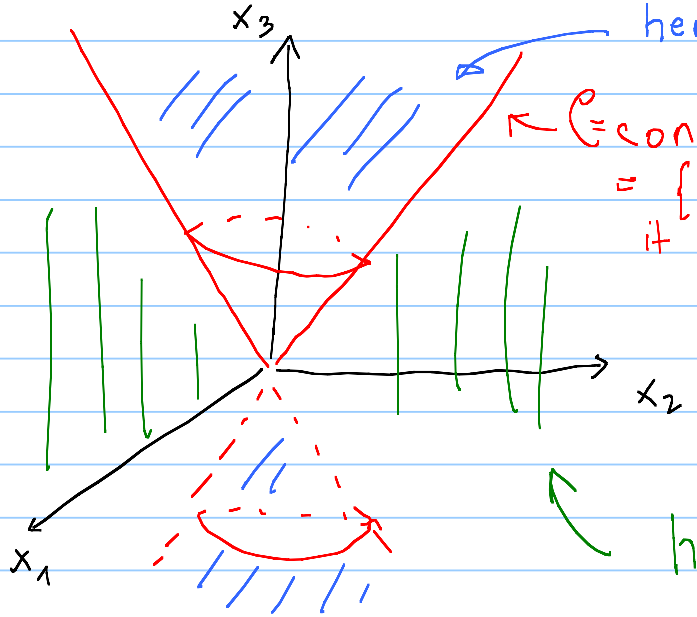
$\mathbb{R}^{n,1} = \mathbb{R}^{n+1}$ endowed with the quadratic form of signature $(n,1)$

$$\langle x, y \rangle_{(n,1)} := x_1 y_1 + x_2 y_2 + \cdots + x_n y_n - x_{n+1} y_{n+1}$$

$x, y \in \mathbb{R}^{n+1}$

$$= {}^t x \, J \, y \quad \text{where} \quad J = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$$

Picture for $n=2$ $\qquad \mathbb{R}^{2,1} = \mathbb{R}^3, \langle\ ,\ \rangle_{(2,1)}$



here each vector x has $\langle x,x\rangle_{(2,1)} < 0$

time-like vectors

$\mathcal{C}$ = cone of isotropic vectors

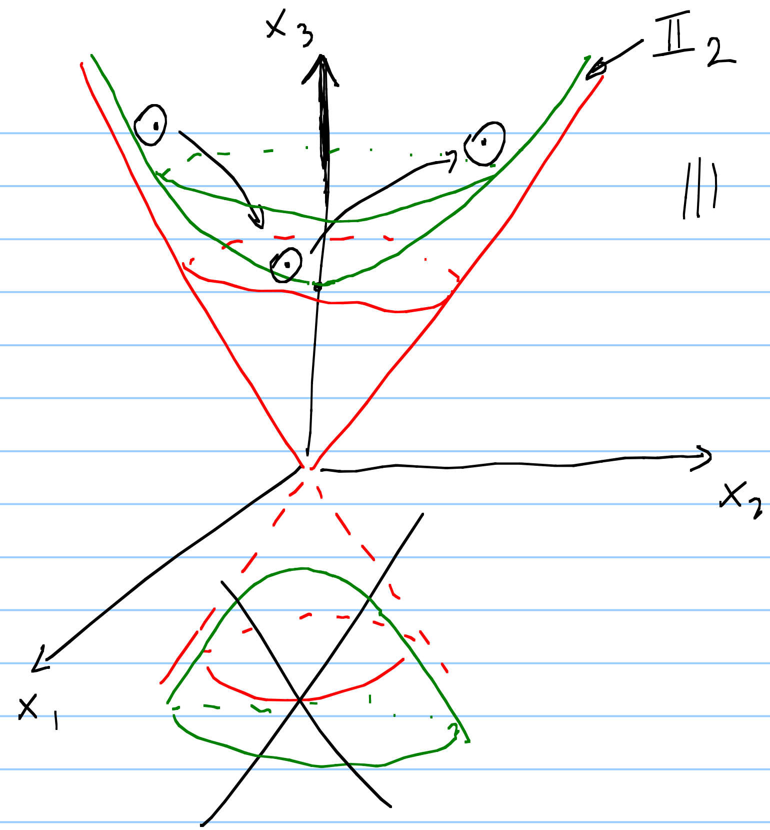
$= \{ x \in \mathbb{R}^3 \mid \langle x,x\rangle_{(2,1)} = 0 \}$

it separates $\mathbb{R}^3$ into two parts

light-like vectors

space-like vectors

here $\langle x,x\rangle_{(2,1)} > 0$



$\mathbb{I}_n = \{ x \in \mathbb{R}^{n+1} \mid \underline{\langle x, x \rangle = -1} \atop x_{n+1} > 0 \}$

hyperboloid

Lemma: $\mathbb{I}_n$ is a submanifold of $\mathbb{R}^{n+1}$.

pf. It is the preimage of the regular value -1 of the smooth function
$f: x \in \mathbb{R}^{n+1} \longrightarrow \langle x, x \rangle_{(2,1)}$

$df_x(v) = 2\langle x, v \rangle$

(conclude by the Implicit Function Thm) $\square$

$T_x \mathbb{I}_n = \operatorname{Ker} df_x = \operatorname{Ker} 2\langle x, \cdot \rangle = x^{\perp_{(n,1)}}$

Observe that since $\langle x, x \rangle = -1$ for $x \in \mathbb{I}_n$ we have that $\langle \, , \, \rangle_{(n,1)} \big|_{x^\perp = T_x \mathbb{I}_n}$ is positive definite

$v \perp x \Rightarrow \langle v, v \rangle_{(n,1)} > 0$

As a consequence $\mathbb{I}_n$ has a natural Riemannian metric g given by the restriction of $\langle \cdot, \cdot \rangle_{(n,1)}$ to the tangent spaces

$$g_x = \langle \, , \, \rangle_{(n,1)} \big|_{T_x \mathbb{I}_n}$$

Def: $(\mathbb{I}_n, g)$ is the hyperboloid model of $\mathbb{H}^n$.

Isometries of $\mathbb{I}_n$

We will see that $(\mathbb{I}_n, g)$ is as symmetric as it can possibly be.

Def (Isometry): $f : (M, g) \longrightarrow (M', g')$ smooth map between Riem. n-mfds is an isometry if $\forall x \in M$ $df_x : (T_x M, g_x) \longrightarrow (T_{f(x)} M', g'_{f(x)})$ is an Isometry and also f is a diffeomorphism.

We now construct many isometries of $\mathbb{I}_n$.

$${}^tAJA = J \qquad J = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$$

$$O^+(n,1) := \left\{ A \in GL_{n+1}\mathbb{R} \;\middle|\; \begin{array}{l} ① \ \forall x, y \in \mathbb{R}^{n+1} \\ \quad \underline{\langle Ax, Ay \rangle}_{(n,1)} = \underline{\langle x, y \rangle}_{(n,1)} \\ ② \ a_{n+1,n+1} > 0 \longleftarrow Ae_{n+1} \subset \{x_{n+1} > 0\} \end{array} \right.$$

Clearly every $A \in O^+(n,1)$ satisfies $\underline{A\,\mathbb{I}_n = \mathbb{I}_n}$

by ① it has $\|Ax\|^2_{(n,1)} = \|x\|^2_{(n,1)} = -1$, $\Uparrow$ $\mathbb{I}_n$

by ② A does not switch the two components of $\{ \|x\|^2_{(n,1)} = -1 \}$

moreover, since A is linear, it coincides with it's differential $dA_x = A$, so A is an isometry of $\mathbb{I}_n$.

Lemma: $O^+(n,1) < \mathrm{Isom}(\mathbb{I}_n, g)$ $\square$

We now show that $O^+(n,1)$ acts <u>transitively</u> on <u>base frames</u> of $\mathbb{I}_n$

Def (Base frame): A base frame is a pair $(x \in \mathbb{I}_n, (v_1, \dots, v_n)$ a orthonormal basis of $T_x\mathbb{I}_n)$

$\|v_j\| = 1 \quad \langle v_i, v_j \rangle = 0$ if $i \neq j$

Lemma: $O^+(n,1)$ acts transitively on base frames

Pf. We choose our favourite base frame $e_{n+1} = (0,\dots,0,1) \in \mathbb{I}_n$

$T_{e_{n+1}}\mathbb{I}_n = \{x_{n+1} = 0\}$

$(\underset{\underset{(1,0,\dots,0)}{\shortparallel}}{e_1}, \dots, \underset{\underset{(0,\dots,1,0)}{\shortparallel}}{e_n})$ orthonormal

Pick another base frame $(x \in \mathbb{I}^n, (\underline{v_1}, \dots, \underline{v_n})$ basis of $T_x\mathbb{I}_n)$ $\quad T_x\mathbb{I}_n = x^\perp$

A: $e_{n+1} \longrightarrow x$

$e_j \longrightarrow v_j \quad \forall\, j \le n$

$A = (\underbrace{v_1 | \dots | v_n | x}_{\text{the columns form a orthonormal basis for } \mathbb{R}^{n,1}}) \in \underline{O^+(n,1)}$.

$\square$

Corollary: $\mathbb{I}_n$ has constant curvature!

Pf. Obs: If $A : \mathbb{I}_n \longrightarrow \mathbb{I}_n$ is an isometry and $P \subset T_x\mathbb{I}_n$ is a 2-plane

then $\sec(P) = \sec(\underbrace{AP \subset T_{Ax}\mathbb{I}_n}_{\text{2-plane}})$.

By the above lemma $\mathrm{Isom}(\mathbb{I}_n, g)$ acts transitively on pairs $(x \in \mathbb{I}_n,\ P \subset T_x\mathbb{I}_n \text{ 2-plane})$. $\boxtimes$

Q: Is $O^+(n,1)$ the group of isometries of $(\mathbb{I}_n, g)$?

A: Yes, in order to prove it we need the following

Lemma (Isometries are determined by their first order):

$f_1, f_2 : (M, g) \longrightarrow (M', g')$ isometries, M connected

Suppose that $\exists\ x \in M$ s.t. $f_1(x) = f_2(x)$, $(df_1)_x = (df_2)_x$

Then $f_1 = f_2$.

Corollary: $O^+(n,1) = \mathrm{Isom}(\mathbb{I}_n, g)$

Pf. $f \in \mathrm{Isom}(\mathbb{I}_n, g)$

$(x \in \mathbb{I}_n, (v_1, \dots, v_n))$ = base frame of $\mathbb{I}_n$

$(f(x) \in \mathbb{I}_n, (df(v_1), \dots, df(v_n)))$ = another base frame

$\Rightarrow$ Find $A \in O^+(n,1)$ that sends the first base frame to the second one

$\Rightarrow$ by the previous lemma A and f agree everywhere. $\square$

Pf of lemma: We first show that if $f_1(x)=f_2(x)$ and $(df_1)_x = (df_2)_x$ then f_1 and f_2 agree on a neigh. of x and then use a connectedness argument to show that they agree everywhere.

Recall that $\forall v \in T_x M$ there exists a unique geodesic γ_v with $\gamma_v(0)=x$ and $\gamma_v'(0)=v$ (also denoted by $\gamma_v(t) = \exp_x(tv)$)

exponential map

$\exp_x : T_x M \longrightarrow M$

Step 1: Observe that f_1, f_2, being isometries, send geo. to geo. In particular

$$f_j \circ \gamma_v(t) = \gamma_{(df_j)_x(v)}(t)$$

geodesic in M' with initial value $\underline{\underline{f_j(x)}}$ and velocity $\underline{\underline{(df_j)_x(v)}}$

So f_1 and f_2 agree on each geodesic starting from x !
Since geodesics starting from x cover an open neigh of x
($\exp_x : T_xM \longrightarrow M$ is a diff. in a neigh. of $0_x \in T_xM$) we conclude that f_1 and f_2 agree on that neigh.

Step 2: The locus $\{ x \in M \mid f_1(x) = f_2(x) \text{ and } (df_1)_x = (df_2)_x \}$ is closed non-empty (by assumption) and open (by Step 1). So, by connectedness of M it coincides with M. $\square$

K- Subspaces

Just like $\mathbb{R}^n$ has many k-dim linear subspaces isometric to $\mathbb{R}^k$, also $\mathbb{I}_n$ has many k-dim "linear" subspaces isometric to $\mathbb{I}_k$.

Def (k-Subspace) A k-subspace of $\mathbb{I}_n$ is a subspace of $\mathbb{I}_n$ obtained as non-empty intersection $V \cap \mathbb{I}_n$ where $V \subset \mathbb{R}^{n,1}$ is a $(k+1)$-dim linear subspace.

Ex: $V \cap \mathbb{I}_n = \phi \iff \langle \cdot , \rangle_{(n,1)} \big|_V$ is semi positive definite.

Lemma: A k-subspace $V \cap \mathbb{I}_n$ is isometric to $\mathbb{I}_k$

Pf. $\langle \ \rangle_{(n,1)} \big|_V$ has signature $(k,1) \Rightarrow (V, \langle \ \rangle_{(n,1)} \big|_V) \simeq \mathbb{R}^{k,1}$

$$V \cap \mathbb{I}_n \longrightarrow \mathbb{I}_k \quad \square$$

Rmk: There are many properties of subspaces analogue to those of subspaces of $\mathbb{R}^n$: (1) Intersections of subspaces, if nonempty, is again a subspace

(2) Given a family of subspaces, there is a smallest subspace containing all of them.

(3) $\mathrm{Isom}(\mathbb{H}_n)$ acts transitively on k-subspaces.

Geodesics and hyperplanes

Def (Totally Geodesic): $X \subset Y$ submanifold of a Riem. mfd Y, we say that X is totally geodesic if every geodesic in X (with respect to the induced Riem. structure) is also a geodesic in Y.

Lemma: Suppose that $X = \mathrm{fix}(\phi)$ with $\phi: Y \to Y$ isometry.

$\Downarrow$ If $\gamma: J \to Y$ is a geo with $\gamma(0) \in X$ and $\gamma'(0) \in T_x X$ then $\gamma \subset X$.

Then X is totally geodesic.

Pf. Take $\gamma: I \to Y$ a geodesic with $\underset{x}{\overset{\shortparallel}{\gamma(0) \in X}}$ and $\underset{v}{\overset{\shortparallel}{\gamma'(0) \in T_x X}}$

Then also $\phi \circ \gamma: I \to Y$ is a geodesic!

but it has $\phi\gamma(0) = \phi(x) = x \in X$

and also $d\phi_x \gamma'(0) = \gamma'(0)$

by uniqueness of geodesics $\phi \circ \gamma = \gamma \implies \gamma \subset \mathrm{fix}(\phi) = X$ $\boxtimes$

Lemma: Hyperplanes ((n-1)- subspaces of $\mathbb{I}_n$) are totally geodesic.

Pf. Enough to consider $H = \text{Span}(e_2, \dots, e_{n+1})$ (because $\text{Isom}(\mathbb{I}_n)$ acts transitively on hyperplanes and preserves the property of being totally geodesic)

Define $\phi : \mathbb{I}_n \longrightarrow \mathbb{I}_n$ by

$$\phi(x_1, x_2, \dots, x_{n+1}) = (-x_1, x_2, \dots, x_{n+1}) \in \text{Isom}(\mathbb{I}_n)$$

$\text{Fix}(\phi) = H$ conclude using the above lemma. $\boxtimes$

Corollary: k- subspaces are totally geodesics.

Pf. $\underbrace{\text{1-subspaces}}_{\text{tot geo}} \subset \underbrace{\text{2-subspaces}}_{\text{tot geo}} \subset \dots \subset \underbrace{\text{(n-1)-subspaces}}_{\text{tot geo}} \subset \underbrace{\mathbb{I}_n}_{\text{tot geo.}}$

Ex: If $X \subset Y$ tot. geo and $Y \subset Z$ tot. geo then $X \subset Z$ tot. geo $\boxtimes$

Corollary: 1-subspaces are the support of geodesics in $\mathbb{I}_n$.

Corollary: Between two points $x, y \in \mathbb{I}_n$, $x \neq y$ there is a unique geodesic corresponding to the 1-subspace generated by x and y.