Hyperbolic Manifolds - Lecture 19

Note Title 26/01/2021

Arithmetic constructions, part 2

Example 2

Recall: $B = x_1^2 + \ldots + x_n^2 - \sqrt{2}\, x_{n+1}^2$ $\quad \mathbb{Z}[\sqrt{2}]$

$SO(B) \simeq SO(n,1) \curvearrowright \mathbb{I}_B \subset \mathbb{R}^{n+1}$

$SO(B)_{\mathbb{Z}[\sqrt{2}]} = SO(B) \cap SL_{n+1}(\mathbb{Z}[\sqrt{2}])$

Rmk: Since B is admissible, $SO(B)_{\mathbb{Z}[\sqrt{2}]} \subset SO(B)$ is discrete

$\downarrow$

$B: x_1^2 + \ldots + x_n^2 - u\, x_{n+1}^2$ $\quad u$ an alg. integer

admissibility: $u \in \mathbb{R}$ $\quad u > 0$

All Galois conj. of u are real and negative

Def (Admissible III): $F/\mathbb{Q}$ a number field (a finite extension of $\mathbb{Q}$)

Denote by $\sigma_1, \dots, \sigma_d : F \longrightarrow \mathbb{C}$ all possible embeddings of F into $\mathbb{C}$ as a field

F is totally real if $\sigma_j(F) \subset \mathbb{R}$

Let B be a non-degenerate quadratic form on $\mathbb{R}^{n+1}$ with coeff. in $\sigma_1(F)$

$B : \sum a_{ij} x_i x_j \qquad a_{ij} \in \sigma_1(F)$

B is admissible if it has signature $(n,1)$ and all other conjugates of B have signature $(n+1, 0)$

$j > 1 \qquad \sigma_j B : \sum \sigma_j(\sigma_1^{-1} a_{ij}) x_{ij}$

$$\sigma_1 B$$
$$\shortparallel$$
$$u = \sqrt{2} \qquad B = x_1^2 + \dots + x_n^2 - \sqrt{2} x_{n+1}^2$$
$$\sigma_2 B = x_1^2 + \dots + x_n^2 + \sqrt{2} x_{n+1}^2$$

Last time:

Lemma: If B is admissible, then $SO(B) \cap \underbrace{SL_{n+1}(\mathcal{O}_F)}_{} < SO(B)$ is discrete.

$SO(B)_{\mathcal{O}_F} = SO(B)\cap SL_{n+1}(\mathcal{O}_F)$

$\mathcal{O}_F$: ring of integers of F

$F/\mathbb{Q}$ finite extension

$\alpha \in F \Rightarrow \alpha$ satisfies a polynomial with rational coeff. $p \in \mathbb{Q}[T]$ s.t. $p(\alpha) = 0$

$1, \alpha, \alpha^2, \alpha^3, \dots$ ← there are many $\mathbb{Q}$-linear relations between them,

$\mathcal{O}_F = \{ \alpha \in F \mid \alpha \text{ is algebraic over } \mathbb{Z} \}$

α is algebraic over $\mathbb{Z}$ $\Leftrightarrow$ α satisfies a monic polynomial with integer coeff.

$\mathcal{O}_F < F$ is a subring.

e.g. $\sqrt{2}$: $T^2 - 2$

Pf. $\gamma \in SO(B)_{\mathcal{O}_F}$

α = an entry of γ

$\sigma_1(\alpha), \dots, \sigma_d(\alpha)$

$\sigma_1(\alpha)$ = an entry of $\sigma_1(\gamma), \dots, \sigma_d(\alpha)$

$\sigma_1(\gamma) \in SO(\sigma_1 B) = SO(n,1)$

$\sigma_d(\alpha) \in SO(\sigma_d B)$; $SO(\sigma_2 B), \dots, SO(\sigma_d B) = SO(n+1)$ in particular they are all cpt.

$\Rightarrow$ the entries of $\sigma_j(\gamma)$ for $j \neq 1$

vary in a cpt set.

$\Rightarrow \quad |\sigma_j(\alpha)| \leq K_j < \infty \quad j \neq 1$

Discreteness $\Leftrightarrow \nexists\ \gamma_n \overset{\in \Gamma \setminus \{1\}}{\longrightarrow} \mathbb{1}$

$\gamma \in \underline{U(\mathbb{1}, R)} = \{$ matrices in SL_{n+1} where the entries are $\leq R\}$

$\gamma_n \in U(\mathbb{1}, R) \quad \forall\ n$ large

$\Rightarrow$ all the entries of γ_n are bounded by R

$\alpha \in \gamma_n$ entry $\Rightarrow \underline{|\sigma_1(\alpha)| < R}$, $\underline{|\sigma_j(\alpha)| \leq K_j}$

$\Rightarrow$ If $\underline{\alpha \in \mathcal{O}_F}$ satisfies the polynomial with $\underline{\text{integer coeff}}$ $p(T) = \prod (T - \sigma_j(\alpha))$

the coeff. of $p(T)$ are sym.

functions of the roots $\sigma_j(a)$

$\Rightarrow$ they are unif. bounded

$\Rightarrow$ Since the coeff are integers, there only finitely many poss. for $p(T)$ $\Rightarrow$ there are only finitely many possibilities for a $\square$

last time: Torsion is never a problem, because of Selberg's Lemma

The only property that we have to check is cocompactness:

$$\mathbb{H}_B / SO(B)_{\mathcal{O}_F} \text{ is cpt.}$$

let us consider only

$B: x_1^2 + \dots + x_n^2 - \sqrt{2}\, x_{n+1}^2$

$$\mathbb{H}_B / SO(B)_{\mathbb{Z}[\sqrt{2}]} \text{ cpt.} \Leftarrow SO(B) / SO(B)_{\mathbb{Z}[\sqrt{2}]} \text{ cpt.}$$

$$\mathbb{I}_B \cong SO(B)/SO(n) = \underset{SO(B)}{\mathrm{stab}}(pt)$$

How do we check that $SO(B)/SO(B)_{\mathbb{Z}[\sqrt{2}]}$ is cpt? We want to apply the Mahler Compactness criterion!

Last time: $SO(x_1^2+x_2^2+x_3^2-7x_4^2) \hookrightarrow SL_4\mathbb{R}/SL_4\mathbb{Z}$

$$\overline{SO(\ldots)_{\mathbb{Z}}}$$

$$\parallel$$

$$SO(\ldots)\cap SL_4\mathbb{Z}$$

the image of this emb. is closed, and it has cpt closure by Mahler.

Ideally: $SO(B)/SO(B)_{\mathbb{Z}[\sqrt{2}]} \hookrightarrow SL_N(\mathbb{R})/SL_N(\mathbb{Z})$ with closed image
with cpt closure.

The tool for finding such an embeddis is a trick called Weil restriction of scalars:

$\mathbb{Z}[\sqrt{2}] \subset \mathbb{R}$ is not discrete

$$\begin{array}{c} \mathbb{Z}[\sqrt{2}] \\ \parallel \\ \mathbb{Z} \oplus \mathbb{Z}\sqrt{2} \\ \cap \\ \mathbb{R} \oplus \mathbb{R} \end{array}$$

$\leftarrow$ this is discrete!

Think of B as a quadratic form on

$SO(B)_{\mathbb{Z}[\sqrt{2}]}$

$$B : \underbrace{\mathbb{Z}[\sqrt{2}]^{n+1} \oplus \mathbb{Z}[\sqrt{2}]^{n+1}} \longrightarrow \mathbb{Z}[\sqrt{2}]$$

$$\begin{array}{cc} \parallel & \parallel \\ (\mathbb{Z} \oplus \mathbb{Z}\sqrt{2})^{n+1} & (\mathbb{Z} \oplus \mathbb{Z}\sqrt{2})^{n+1} \\ \cap & \cap \end{array}$$

Strategy: $\mathbb{Z}[\sqrt{2}]^{n+1} = L$

On L we have a quadratic form B. $L \subset \mathbb{R}^{n+1}$ is not a lattice, but we can embed L as a lattice in $\mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$ via the two emb. of $\mathbb{Z}[\sqrt{2}]$ in $\mathbb{R}$

and we can extend the quadratic form B to one Q on $\mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$ via the same emb.

$\Rightarrow$ the orthogonal group

$(\mathbb{R}\oplus\mathbb{R})^{n+1}$ $(\mathbb{R}\oplus\mathbb{R})^{n+1}$

$SO(Q) \simeq SO(\sigma_1 B)\times SO(\sigma_2 B)$

$SO(Q)/SO(Q)_{\mathbb{Z}} \twoheadrightarrow SO(\sigma_1 B)/SO(B)_{\mathbb{Z}[\sqrt{2}]}$

Let $\sigma_1, \sigma_2 : \mathbb{Z}[\sqrt{2}] \longrightarrow \mathbb{R}$ be the two embeddings of $\mathbb{Z}[\sqrt{2}]$ in $\mathbb{R}$

$a+\sqrt{2}b \longrightarrow a \pm \sqrt{2} b$

Need to check that image in $SL_N\mathbb{R}/SL_N\mathbb{Z}$ is closed (ex) and has cpt closure (application of Mahler

Using these embeddings we get embeddings

$\sigma : \mathbb{Z}[\sqrt{2}]^{n+1} \longrightarrow \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$

$v \longrightarrow (\sigma_1(v), \sigma_2(v))$

Notice that $\sigma(\mathbb{Z}[\sqrt{2}]^{n+1})$ lies in $\mathbb{R}^{n+1}\oplus\mathbb{R}^{n+1} \simeq \mathbb{R}^{2(n+1)}$ as $\mathbb{Z}^{2(n+1)} \subset \mathbb{R}^{2(n+1)}$

$\Rightarrow$ can define a "quadratic form" Q on

$\mathbb{R}^{n+1}\oplus\mathbb{R}^{n+1}$ by $Q((x_1,y_1),(x_2,y_2)) = (\sigma_1 B(x_1,y_1), \sigma_2 B(x_2,y_2)) \in \mathbb{R}^2$

$Q : \mathbb{R}^{2(n+1)} \times \mathbb{R}^{2(n+1)} \longrightarrow \mathbb{R}^2$

$$SO(Q) = \{ M \mid Q(M\underline{x}, M\underline{y}) = Q(\underline{x}, \underline{y}) \} < SL_{N = 2(n+1)}(\mathbb{R})$$

$$SO(Q) \simeq SO(\sigma_1 B) \times SO(\sigma_2 B) \xrightarrow{\pi_1} SO(\sigma_1 B)$$

Ex: $M : \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1} \circlearrowleft$ that preserves Q

$\Rightarrow$ M preserves $\mathbb{R}^{n+1} \oplus 0$, $0 \oplus \mathbb{R}^{n+1}$ and act on them as isometries $\sigma_1 B$, $\sigma_2 B$

$\Rightarrow$ Get $SO(Q) \to SO(\sigma_1 B) \times SO(\sigma_2 B)$

$$SO(Q)_{\mathbb{Z}} = SO(Q) \cap SL_N \mathbb{Z} \xrightarrow{\pi_1} SO(\sigma_1 B)_{\mathbb{Z}[\sqrt{2}]} < SO(\sigma_1 B)$$

$\Rightarrow$ we get a surjective map $SO(Q) / SO(Q)_{\mathbb{Z}} \to \underset{\overset{\parallel}{B}}{SO(\sigma_1 B) / SO(B)_{\mathbb{Z}[\sqrt{2}]}}$

$\Rightarrow$ in order to prove cptness of it is enough to show that $SO(Q)/SO(Q)_{\mathbb{Z}}$ is cpt.

Again: $SO(Q)/SO(Q)_{\mathbb{Z}} \longleftrightarrow SL_N\mathbb{R}/SL_N\mathbb{Z}$

with closed image (exercise) and cpt closure (by Mahler).

Example 3

In general we have the following arithmetic way of constructing closed hyperbolic n-mfds:

Proposition: Let $F/\mathbb{Q}$ be a totally real number field

Let $\mathcal{O}_F$ be the ring of integers of F

Let B be an admissible quadratic form on $\mathbb{R}^{n+1}$ with coeff in F

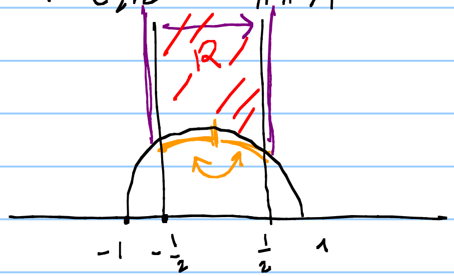
Then $\Gamma = SO(B)_{\mathcal{O}_F} < SO(B) \simeq SO(n,1)$ is discrete and $\mathbb{H}_B/\Gamma$ is cpt.

An example that works in any dim $\quad B: x_1^2 + \dots + x_n^2 - \sqrt{2}\, x_{n+1}^2$.

Fundamental domains

Given a complete hyp. manifold presented as $M = \mathbb{H}^n / \Gamma$ we would like to express it as $M = P/\sim$ where $P \subset \mathbb{H}^n$ is some polyhedron and $\sim$ is an isometric identification of its faces.

Ex: $\Gamma = PSL_2\mathbb{Z}$ $\quad \mathbb{H}^2/\Gamma = R/\sim$



Voronoi Tessellations:

$S \subset \mathbb{H}^n$ a discrete set

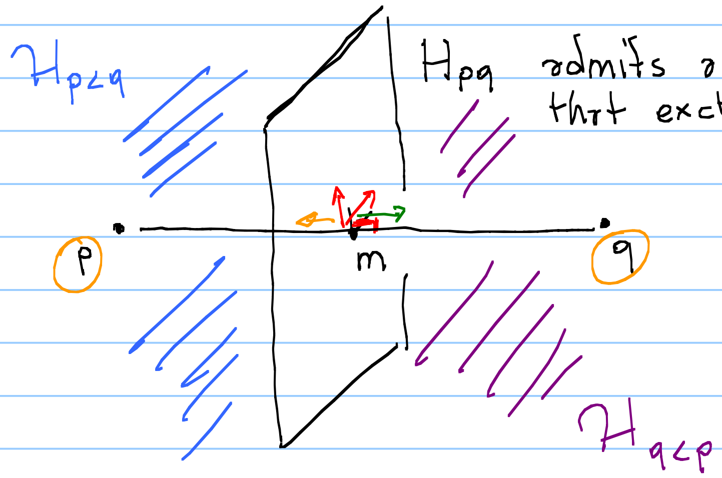
$p \in S$

$$D(p) := \{ x \in \mathbb{H}^n \mid d(x,p) \leq d(x,q) \ \forall q \in S \}$$

$$= \bigcap_{q \in S} \mathcal{H}_{p<q} = \{ d(p,\cdot) \leq d(q,\cdot) \}$$

$= \text{Voronoi cell}$

Rmk: $\mathcal{H}_{p<q}$ ($p,q \in S$) = a half space bounded by the hyperplane $H_{pq} = \{ d(p,\cdot) = d(q,\cdot) \}$



A polyhedron is a locally finite intersection of half spaces

around every pt there are only finitely many half spaces that pass nearby.

Lemma: $D(p)$ is a polyhedron.