Hyperbolic Manifolds – Lecture 18

Titolo nota 20/01/2021

Arithmetic Constructions

$$\{ \text{complete oriented hyperbolic } n\text{-mfds} \} \xleftrightarrow{1:1} \{ \text{discrete torsion free subgroups of } \mathrm{Isom}^+(\mathbb{H}^n) \}$$

Basic idea: $\mathbb{H}^n = I_n = \{ x \in \mathbb{R}^{n,1} \mid \underbrace{x_1^2 + \dots + x_n^2 - x_{n+1}^2}_{\langle\,,\,\rangle_{(n,1)}} = -1 \ , \ x_{n+1} > 0 \}$

$$\mathrm{Isom}^+(\mathbb{H}^n) = \{ A \in SL_{n+1}\mathbb{R} \mid A \text{ preserves } \langle\,,\,\rangle_{(n,1)} \ , \ a_{n+1,n+1} > 0 \} = SO(n,1)$$

$a_{n+1,n+1} > 0 \Leftrightarrow A I_n \subset I_n$

finite index

$$\exists\ \Gamma(n) < \mathrm{Isom}^+(\mathbb{H}^n)_{\mathbb{Z}} = \{ A \in \mathrm{Isom}^+(\mathbb{H}^n) \cap SL_{n+1}\mathbb{Z} \} < \mathrm{Isom}^+(\mathbb{H}^n)$$

torsion free

$\mathbb{Z} \subset \mathbb{R}$ discrete $\Rightarrow$ discrete it has torsion, but it has also many torsion free finite index subgroups

$\mathrm{Ker}(\pi_n) \to SL_{n+1}\mathbb{Z} \xrightarrow{\pi_n} SL_{n+1}\mathbb{Z}/n\mathbb{Z}$

Example 1: Fix an integer $u>0$

Consider $B = x_1^2 + \dots + x_n^2 - u\, x_{n+1}^2$ quadratic form on $\mathbb{R}^{n+1}$

$\mathbb{I}_n(B) = \{ B(x,x) = -1,\ x_{n+1} > 0 \}$

$SO(B) = \{ A \in SL_{n+1}\mathbb{R} \mid A \text{ preserves } B,\ A\,\mathbb{I}_n(B) \subset \mathbb{I}_n(B) \}$

$\simeq SO(n,1)$ (conj with $\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \sqrt{u} \end{pmatrix}$)

$SO(B)_{\mathbb{Z}} = \{ A \in SO(B) \cap SL_{n+1}\mathbb{Z} \} < SO(B) \simeq SO(n,1)$

discrete ($\mathbb{Z} \subset \mathbb{R}$ ...)

Discussion for $n=3$

$B: \; x_1^2 + x_2^2 + x_3^2 - u x_4^2$

Def: B is admissible if there is no integer solution $x \neq 0$ to $B(x,x)=0$.

· By Meyer's Theorem, each quadratic form $a_1 x_1^2 + a_2 x_2^2 + \dots + a_n x_n^2 - a_{n+1} x_{n+1}^2$ with $a_j > 0$ and $n \geq 4$ has a non-trivial integer solution

Lemma: If $u=7$ then B is admissible ($n=3$)

Pf. Idea: look at $x_1^2 + x_2^2 + x_3^2 - 7x_4^2 \mod 8$ and find that $x_1^2 + \dots + x_4^2 = 0 \mod 8$
$\Rightarrow x_j \equiv 0 \mod 2$.

$\square$

$SO(B)_{\mathbb{Z}} < SO(B)$

Let's not worry about torsion because it always possible to kill it by passing to a finite index subgroup.

Recall: $SO(B)_{\mathbb{Z}} \curvearrowright \mathbb{I}_n(B)$ is properly disc.

Claim: $\mathbb{I}_n(B) / SO(B)_{\mathbb{Z}}$ is cpt.

$SO(B) \curvearrowright \mathbb{I}_n(B)$ transitively and with cpt stabilizers $\left(\mathrm{Stab}(x \in \mathbb{I}_n(B)) \simeq SO(n) \right)$

$\Rightarrow$ we can identify $\mathbb{I}_n(B)$ with $SO(B)/SO(n) \simeq \mathbb{I}_n(B)$

If we want to check that $SO(B)_{\mathbb{Z}} \backslash \mathbb{I}_n(B)$ cpt it is enough to check that $SO(B)_{\mathbb{Z}} \backslash SO(B)$ is cpt.

Rmk

$$\begin{array}{ccc} \boxed{SO(B)} & \subset & SL_{n+1}\mathbb{R} \\ \cup & & \cup \\ SO(B)_{\mathbb{Z}} & \subset & \boxed{SL_{n+1}\mathbb{Z}} \end{array}$$

Want to use Mahler here to check that the image is cpt.

$\Rightarrow$ we have a natural map $SO(B)_{\mathbb{Z}} \backslash SO(B) \hookrightarrow \boxed{SL_{n+1}\mathbb{Z} \backslash SL_{n+1}\mathbb{R}}$

$\cong$

this space parametrizes lattices in $\mathbb{R}^{n+1}$ (a lattice is a discrete subgroup $L < \mathbb{R}^{n+1}$, $L \simeq \mathbb{Z}^{n+1}$, think of $\mathbb{Z}^n \subset \mathbb{R}^{n+1}$)

$SL_{n+1}\mathbb{R} \curvearrowright$ lattices, transitively, the stabilizer of a pt is $SL_{n+1}\mathbb{Z}$

$\Rightarrow \{\text{lattices in } \mathbb{R}^{n+1}\} \simeq SL_{n+1}\mathbb{Z} \backslash SL_{n+1}\mathbb{R}$

Mahler Compactness Criterion: $C \subset \boxed{SL_n\mathbb{Z} / SL_n\mathbb{R}}$ has compact closure

$\Updownarrow$

every lattice Λ represented by C has bounded volume $\mathrm{vol}(\mathbb{R}^n/\Lambda) \leq V = V(C)$ and bounded systole, $\mathrm{sys}(\mathbb{R}^n/\Lambda) \geq \varepsilon = \varepsilon(C) > 0$

$$\mathrm{sys}(\mathbb{R}^n/\Lambda) = \inf_{x \in \Lambda \setminus \{0\}} \|x\|$$

$\Updownarrow$

$\cancel{\exists}$ sequence $g_n \in SL_n\mathbb{R}$, $v_n \in \mathbb{Z}^n \setminus \{0\}$ s.t.

$[g_n] \in C$, $g_n v_n \to 0$.

Let us check that $SO(B)_{\mathbb{Z}} \backslash SO(B)$ has cpt closure in $SL_n\mathbb{Z} \backslash SL_n\mathbb{R}$

(+ we also need to check that the image is closed)

Exercise.

Recall: B is admissible

Pf. $\exists\, g_n \in SO(B)$, $v_n \in \hat{\mathbb{Z}^n} \setminus \{0\}$ s.t. $g_n v_n \to 0$

$$B(g_n v_n, g_n v_n) = B(v_n, v_n) \in \mathbb{Z}$$

$\downarrow$

0 $\Rightarrow$ $B(v_n, v_n) = 0$ $\forall\, n$ large enough $\Rightarrow$ B is not admissible. $\cancel{\square}$

$\underline{SO(B)_{\mathbb{Z}}} < SO(B)$ discrete + compact quotient

$\Rightarrow$ we can now pass to a f.i. subgroup with the same properties:

$\Gamma(q_1) \cap \Gamma(q_2) \cap SO(B)_{\mathbb{Z}}$
discrete + torsion free
+ compact quotient
(because of finite index).

Pf. $X \in SO(B_{\mathbb{Z}})$ s.t. $X^p = I$

$\Rightarrow$ wlog we can assume that p is a prime number

Consider $SL_{n+1}\mathbb{Z} \xrightarrow{\pi_q} SL_{n+1}\mathbb{Z}/q\mathbb{Z}$

q prime number

$\Rightarrow$ pass to $\underbrace{\mathrm{Ker}\left(SO(B)_{\mathbb{Z}} \to SL_{n+1}\mathbb{Z}/q\mathbb{Z}\right)}_{\Gamma(q)''}$

$X \in \Gamma(q)$ $\quad X = I + M \quad M \equiv 0 \mod q$

$= I + q^k M_1 \quad$ with $q \nmid M_1$

$X^p = (I + q^k M_1)^p = I + pq^k M_1 + q^{2k} N$

$0 \quad \cancel{I} = X^p \equiv \cancel{I} + pq^k M_1 \quad (\mathrm{mod}\ q^{k+1})$

if $p \neq q \Rightarrow pq^k M_1 \not\equiv 0 \mod q^{k+1}$

if $p = q \Rightarrow$ Can consider instead $\Gamma(q_1) \cap \Gamma(q_2)$ for two different primes q_1, q_2. $\square$

Lemma (Selberg's Lemma): $\Gamma < GL_n\mathbb{C}$ f.g. subgroup. (or $\Gamma < GL_n B$ with $B \subset \mathbb{C}$ f.g. $\mathbb{Z}$-alg.) Then there exists a finite index normal subgroup $\Gamma_1 \lhd \Gamma$ which is torsion free.

Pf. Γ f.g. $\Rightarrow \gamma_1^{\pm 1}, \dots, \gamma_m^{\pm 1}$ finite generating set of matrices

$\Rightarrow B := \mathbb{Z}[\text{entries of} \ldots] < \mathbb{C}$ finitely generated $\mathbb{Z}$-alg.

$\boxed{\Rightarrow \quad \Gamma < GL_n(B)}$ Can assume that $\Gamma = GL_n B$

$\Rightarrow$ can consider $\ker\left(GL_n B \longrightarrow GL_n B/\mathfrak{m}\right) \cap \Gamma$

$\mathfrak{m}$ maximal ideal

$X^p = I$

$X = I + M \qquad M \equiv 0 \mod \mathfrak{m}$

$M \in \mathfrak{m}^k \setminus \mathfrak{m}^{k+1}$

Need $\bigcap \mathfrak{m}^k = 0$

this will be automatically a finite field.

E/F E finitely gen/F E field

$\Rightarrow E/F$ finite

$\cancel{I} = X^p = \cancel{I} + pM + O(\mathfrak{m}^{2k})$

$pM \equiv 0 \mod \mathfrak{m}^{k+1}$ $\equiv 0 \mod \mathfrak{m}^{2k}$

$E = F[u]$

$E = F[u_1, \dots, u_n]$

$= F(u_1)[u_2, \dots, u_n]$

$\dots$

Example 2

$\mathfrak{m} \cap \mathbb{Z}$ is a maximal ideal $\mathfrak{m} = (q, \dots)$

$B: \; x_1^2 + \dots + x_n^2 - \sqrt{2}\, x_{n+1}^2$

(In general we can also consider $B = x_1^2 + \dots + x_n^2 - u x_{n+1}^2$ with u an algebraic integer which is real and > 0

Def u is admissible if $u > 0$

All Galois conjugate of u are real and negative)

$p \neq q \Rightarrow p$ is invertible in $B/\mathfrak{m}$

$\Rightarrow ap = 1 + m$, $m \in \mathfrak{m}$

$$\underset{\mathfrak{m}^{k+1}}{\underset{\cap}{apM}} = \underset{\notin \mathfrak{m}^{k+1}}{M} + \underset{\in \mathfrak{m}^{k+1}}{\underset{\in\mathfrak{m}}{m}\,\underset{\in \mathfrak{m}^k}{M}}$$

If $\underline{x}$ is an integer sol. of $x_1^2 + \dots + x_n^2 - u\, x_{n+1}^2 = 0$

$\Rightarrow$ $\sigma_j : \mathbb{Q}(u) \longrightarrow \mathbb{C}$ all possible embeddings of $\mathbb{Q}(u)$ in the complex numbers $/\mathbb{Q}$

$\Gamma < GL_{n+1}(\mathbb{Q}(u))$

$\sigma(\underline{x})$ is a non trivial solution of

$$x_1^2 + \dots + x_n^2 - \underbrace{\sigma_j(u)}_{>0} x_{n+1}^2 = 0$$

this sum here is always >0

Notice: $\sqrt{2}$ is admissible

$SO(B) \cong SO(n,1)$

$SO(B)_{\mathbb{Z}[\sqrt{2}]} = \{ A \in SO(B) \cap SL_{n+1}(\mathbb{Z}[\sqrt{2}]) \}$

ring of integers of $\mathbb{Q}(\sqrt{2})$ $\leftarrow$ In general, take $SO(B)_{\mathbb{Z}[u]}$

$\mathbb{Z}[\sqrt{2}] \subset \mathbb{R}$ is not discrete.
However $SO(B)_{\mathbb{Z}[\sqrt{2}]}$ is discrete

pf. Consider $SO(B)_{\mathbb{Z}[u]}$

suppose we have $\gamma_n \to 1$

$\Rightarrow$ the entries of γ_n have norm ≤ 2 $\forall$ n large

Consider $\sigma_j(\gamma \in SO(B)_{\mathbb{Z}[u]}) \in SO(\sigma_j(B)) \simeq SO(n+1)$

$$\sigma_j(B) = x_1^2 + \cdots + x_n^2 \underbrace{- \sigma_j(u)}_{>0} x_{n+1}^2$$

in particular it is cpt

//

$\Leftarrow$

There is a uniform upper bound on the norm of the entries of $\sigma(\gamma)$.

$\Rightarrow \gamma_n, \sigma_j(\gamma_n)$ for n large

$a \in$ entry(γ)

a satisfies the polynomial $p(T) = (T-a)(T-\sigma_1(a)) \cdots (T-\sigma_d(a))$

$\mathbb{Z}[u]$

it has integer coeff.!
since the coeff. are symm. functions of
$a, \sigma_1(a), \ldots, \sigma_d(a)$
and all these numbers have unif. bdd. norm.
there are only finitely many possibilities for the polynomial $p(T)$

$\Downarrow$

There are only finitely many possible values for a

$\Downarrow$

the entries of a_n for n large vary in a finite set.

$\Downarrow$

This finishes the proof of discreteness. $\varnothing$