Hyperbolic Manifolds - Lecture 17

Note Title 19/01/2021

Discrete groups $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$

Some Topology

If there is no structure on X / If X is a mfd

$G \curvearrowright X \qquad G < \mathrm{Homeo}(X)\ ,\ \mathrm{Diff}(X)$

Def (Properly discontinuous): $G \curvearrowright X$ properly disc. if $\forall x \in X\ \exists x \in U_x$ neigh s.t. $\{ g \in G \mid g(U_x) \cap U_x \neq \emptyset \}$ is finite.

(Free) $G \curvearrowright X$ is free if every $g \in G \setminus \{1\}$ has no fixed point.

Hausdorff + connected (manifolds are reasonable)

Lemma: X is reasonable
$G \curvearrowright X$ free and properly discontinuous
Then $p : X \to X/G$ is a covering
$ \overset{\parallel}{Y}$

$\forall y \in Y = X/G \ \exists_{y \in} V_y \subset Y$ neigh.
s.t. $p^{-1}(V_y) = \coprod_{i \in I} U_i$ ← connected comp. of $p^{-1}(V_y)$

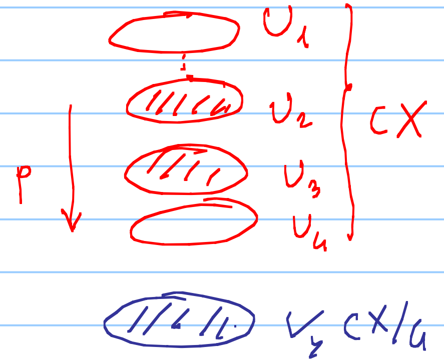
$\Rightarrow$ the restriction
$p|_{U_i} : U_i \to V_y$ is a homeo.

Hausdorff + path connected + locally contractible (manifolds are reasonable)

Fact: Every reasonable topological space Y can be written
as $Y = X/G$ where X is simply connected
and $G \curvearrowright X$ is free and properly discontinuous.

monodromy action ($G \simeq \pi_1(Y)$)

universal covering of Y



We are interested in the case $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n) \curvearrowright \mathbb{H}^n$

Lemma: $K \subset \mathbb{H}^n$ cpt. Then $I_K = \{ g \in \mathrm{Isom}^+(\mathbb{H}^n) \mid gK \cap K \neq \emptyset \}$ is cpt.

Pf. Think $\mathbb{H}^n = \mathbb{I}_n$ and $\mathrm{Isom}^+(\mathbb{H}^n) = SO_0(n,1) < GL(\mathbb{R}^{n,1})$.

$g_n \in I_K \Rightarrow \exists\ x_n \in K \subset \mathbb{I}_n$ s.t. $y_n = g_n x_n \in K \subset \mathbb{I}_n$

$\Rightarrow$ by cptness $x_n \to x \in K$

$y_n \to y \in K$

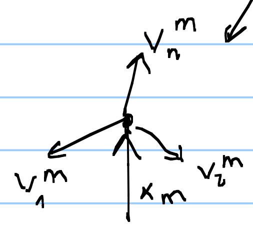
$v_j^m \to v_j \in x^\perp$

$g_m v_j^m \to w_j \in y^\perp$

$\Rightarrow g_m \to g : \begin{cases} x \to y \\ v_j \to w_j \end{cases} \in SO_0(n,1)$

$g \in I_K$ $\boxtimes$

p.o. o.n. basis of $x_m^\perp$



Γ closed, with the discrete topology

Ex: $1 \in \Gamma$ is not an accumulation point for Γ.

Corollary: $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$. Then $\Gamma \curvearrowright \mathbb{H}^n$ properly disc. $\Leftrightarrow$ Γ is discrete

Pf. ($\Rightarrow$) Suppose that Γ is not discrete

$\Rightarrow \gamma_n \to 1 \qquad \gamma_n \in \Gamma \setminus \{1\}$ pairwise distinct elements

$\Rightarrow \forall x \in \mathbb{H}^n$ we have $d(\gamma_n x, x) \longrightarrow 0$

$\Rightarrow \forall\ U_x$ neigh of x we have $\gamma_n U_x \cap U_x \neq \phi$ for ∞-many distinct γ_n $\Rightarrow$ $\Gamma \curvearrowright \mathbb{H}^n$ not prop. disc.

($\Leftarrow$) U_x a neigh. of $x \in \mathbb{H}^n$ with cpt. closure $\overline{U_x}$

$$\{\gamma \in \Gamma \mid \gamma U_x \cap U_x \neq \phi\} \subset \{\gamma \in \Gamma \mid \gamma \overline{U_x} \cap \overline{U_x} \neq \phi\} = I_{\overline{U_x}} \cap \Gamma$$

cpt. + (closed) discrete $\Rightarrow$ finite.

$\square$

Lemma: $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ discrete. Then $\Gamma \curvearrowright \mathbb{H}^n$ free $\Leftrightarrow$ Γ is torsion free.

$\forall \gamma \in \Gamma \setminus 1 \ \forall n > 0$
$\gamma^n \neq 1$.

Pf. ($\Rightarrow$) Suppose $\gamma \in \Gamma \setminus 1$ is torsion, $\gamma^d = 1 \quad d > 1$
$\Rightarrow$ γ has a fixed pt. (the barycenter of $x, \gamma x, \gamma^2 x, \dots, \gamma^{d-1}x$).

see exercises.

($\Leftarrow$) Suppose that $\gamma \in \Gamma$ fixes $x \in \mathbb{H}^n$

$$\gamma \in \mathrm{Stab}_\Gamma(x) = I_{\{x\}} \cap \Gamma \Rightarrow \gamma \text{ is torsion.} \quad \emptyset$$

cpt. + discrete $\Rightarrow$ finite

$$\mathcal{A} = \{ V_y \xrightarrow{p|_{U_x}^{-1}} U_x \subset X \}_{y \in Y}$$

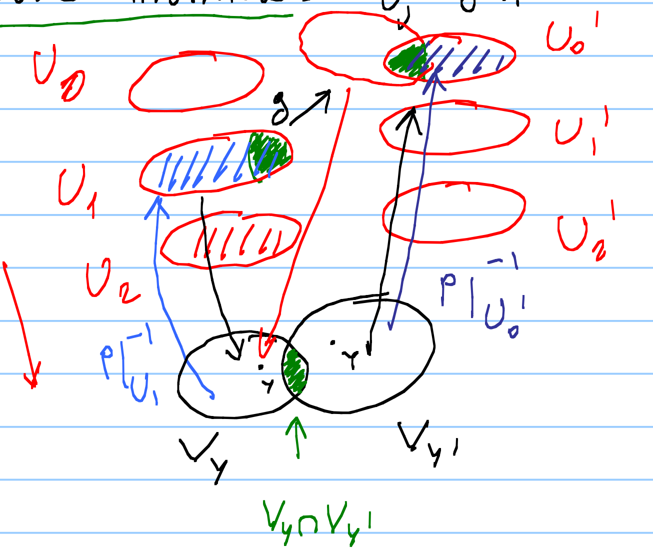
Notice that $Y = X/G$ has an atlas on X with change of charts in $G < \mathrm{Homeo}(X)$, $\mathrm{Diff}(X)$

In particular for $Y = \mathbb{H}^n/\Gamma$, $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$, Y is a hyperbolic manifold. $U_j = gU_1$ for some g

$$\bigsqcup U_j = \bigsqcup_{g \in G} gU_0$$

G permutes the components of $p^{-1}V$ in a transitive way

p



$$\bigsqcup U_i' = \bigsqcup_{g \in G} gU_0'$$

change of charts

$$(p|_{U_0'}^{-1})(p|_{U_1}) = g|_{U_1}$$

$V_y \cap V_{y'}$ is connected $\Rightarrow \exists!\ U_j$ s.t. $U_j \cap U_0' = p|_{U_0'}^{-1}(V_y \cap V_{y'})$

Is Y also complete? Yes, always.

Need:

Lemma: $p: M \to N$ local isometry between connected Riem. n-mfds. Then:

① If p is a covering, then M is complete $\Leftrightarrow$ N is complete. $\Uparrow$ Since $\mathbb{H}^n$ is complete and $\mathbb{H}^n \to \mathbb{H}^n/\Gamma$ is a covering, then $\mathbb{H}^n/\Gamma$ is compl.

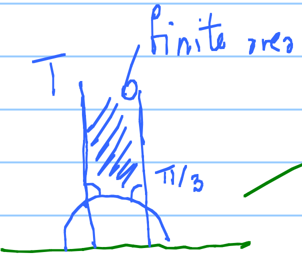
② If M is complete, then p is a covering.

So we proved the following:

Proposition: $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ discrete and torsion free.
Then $\mathbb{H}^n/\Gamma$ is a complete hyperbolic manifold and $p: \mathbb{H}^n \to \mathbb{H}^n/\Gamma$ is a covering.

Problem: How do we produce Γ for which $\mathbb{H}^n/\Gamma$ has finite volume?

$\mathbb{H}^2 \cong U_2$



Ex: $\Gamma = \langle \gamma \rangle$

$\Gamma \curvearrowright \mathbb{H}^n$ is free and prop. disc $\Leftrightarrow$ γ is lox or parabolic.

Ex: $\mathrm{Isom}(\mathbb{H}^2) = PSL_2\mathbb{R}$

$PSL_2\mathbb{Z} < PSL_2\mathbb{R}$

discrete, but it has torsion $\tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\tau^4 = 1$. However

$\Gamma(n) = \mathrm{Ker}\left(PSL_2\mathbb{Z} \to PSL_2 \mathbb{Z}/n\mathbb{Z}\right)$

$n >> 0$

has no elliptic elements $\Rightarrow$ it acts freely on $\mathbb{H}^2$.

$\Gamma(n) < PSL_2\mathbb{Z}$ finite index $\Rightarrow$ it has a fund. domain consisting of finitely many copies of T

$\Rightarrow$ $\mathbb{H}^2/\Gamma(n)$ is a complete hyperbolic surface with finite area

nb of copies $= [PSL_2\mathbb{Z} : \Gamma(n)] < \infty$

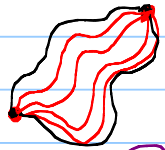
$$\mathrm{area}\left(\mathbb{H}^2/\Gamma(n)\right) = \mathrm{area}(T)\,[PSL_2\mathbb{Z} : \Gamma(n)].$$

$\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ discrete, torsion free $\rightsquigarrow$ $\mathbb{H}^n/\Gamma$

In fact: Every complete hyperbolic n-mfd (M,g) is of this form $(M,g) \simeq \mathbb{H}^n/\Gamma$

$\mathbb{H}^n/\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$

$(M,g) \Rightarrow$ Can write M as $M = X/G$ $\cong$ where X is simply conn. and $G \curvearrowright X$ is free and properly disc., $p: X \to M$ is a covering projection



(X, p^*g) is a Riem. mfd locally isometric to M.

$\Rightarrow$ it is locally isometric to $\mathbb{H}^n$

[Furthermore $G \curvearrowright X$ by isometries]

Theorem: If X is simply connected, complete Riem. n-mfd locally isometric to $\mathbb{H}^n$, then X is isometric to $\mathbb{H}^n$.

$\Rightarrow$ there is a bijection $\{$ complete hyp. n-mfds $\} \leftrightarrow \{ \mathbb{H}^n/\Gamma$ with $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ discrete, torsion free $\}$

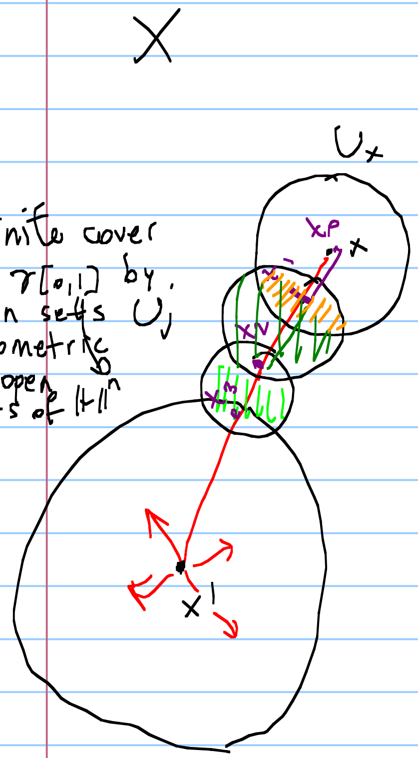
Pf. Step 1: X locally isometric to $\mathbb{H}^n$ + simply connected $\Rightarrow$ construct a local isometry $p: X \to \mathbb{H}^n$

The construction of p follows the same lines of the principle of analytic continuation.

Step 2: X complete + $p: X \to \mathbb{H}^n$ local isometry $\Rightarrow$ p is a covering.

p must be the trivial covering (i.e. a homeomorphism) because $\mathbb{H}^n$, being simply connected, has no non-trivial covering.

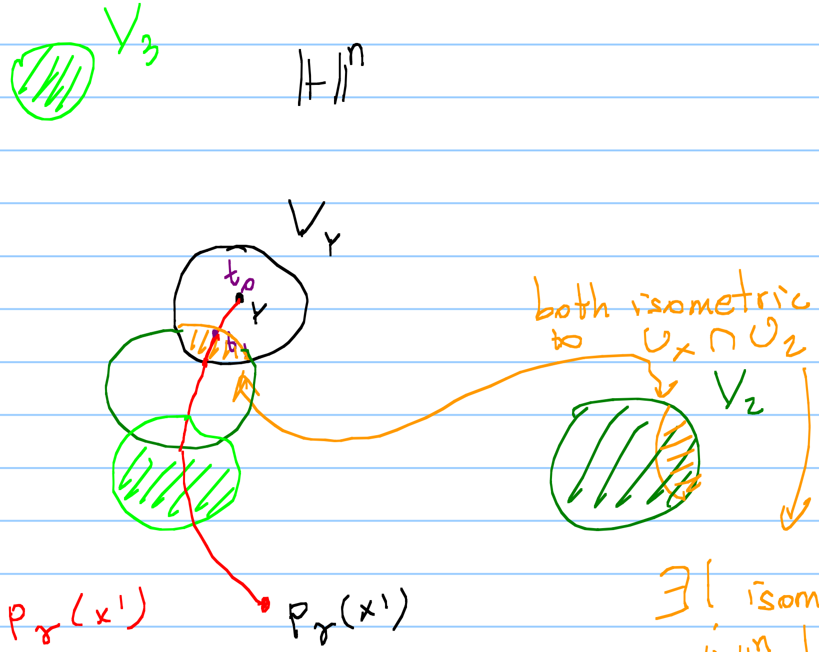
Pf of step 1 (sketch)



finite cover of $\gamma[0,1]$ by open sets U_j isometric to open sets of $\mathbb{H}^n$

$p : U_x \to V_y$

$\gamma[t_1,t_2] \subset U_2 \longrightarrow V_2$



Pick $\gamma : [0,1] \to X$ a path joining x to x' $\Rightarrow$ want to define $p_\gamma(x')$

$\Rightarrow$ Find a subdivision $0 = t_0 < t_1 < \dots < t_{k-1} < t_k = 1$ s.t. $\gamma[t_{j-1}, t_j] \subset$ one of the sets U_j

$\exists!$ isom of $\mathbb{H}^n$ that attaches V_2' to $p(U_x \cap U_2)$

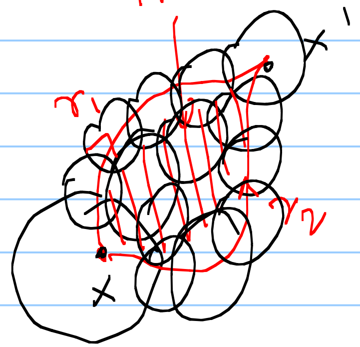
<u>Claim:</u> $p_\gamma(x')$ does not depend on the choices made

because X is simply connected

$\Downarrow$

p extends to a local isometry on X

$H : [0,1]^2 \to X$ homotopy between γ_1, γ_2



$\Downarrow$

same process as before extends $p : U_x \to V_y$ over H. $\boxtimes$