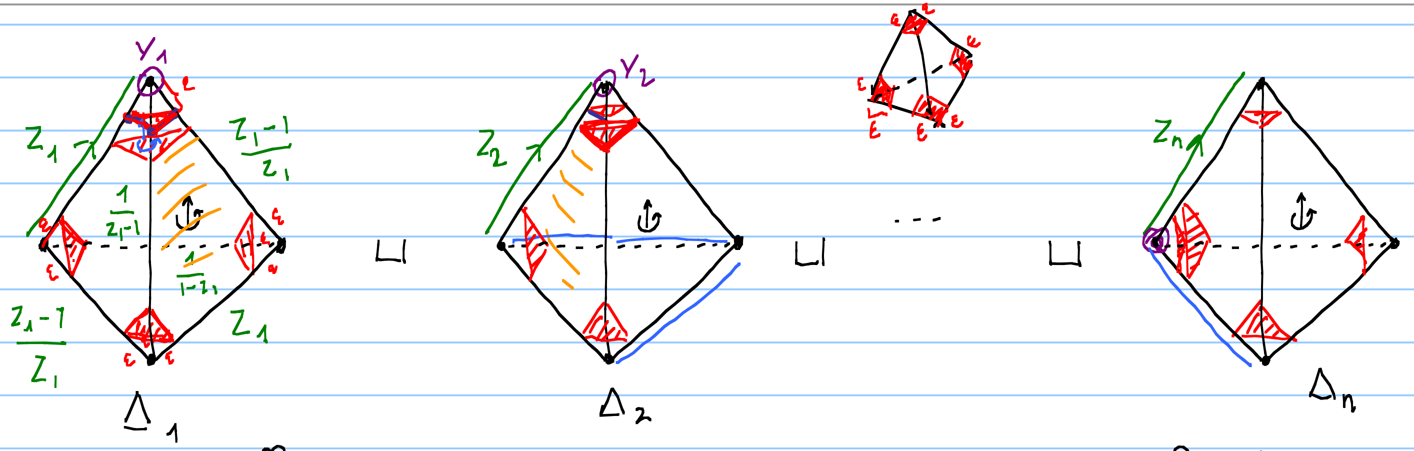
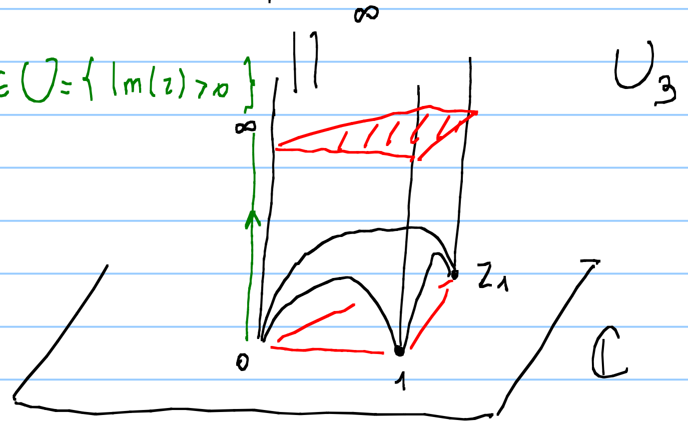
Hyperbolic Manifolds - Lecture 15

Note Title 12/01/2021

$X =$  $/$ face identifications

$z_j \in U = \{\mathrm{Im}(z) > 0\}$



Recall:

① Topology: $X - \text{vertices} \simeq \mathrm{int}(M) = M - \partial M$ with $\boxed{\partial M = \sqcup \text{ tori}}$

② Existence of a hyperbolic structure

$\Leftarrow$ { (i) no shear along edges
(ii) total dihedral angle around edges $= 2\pi$

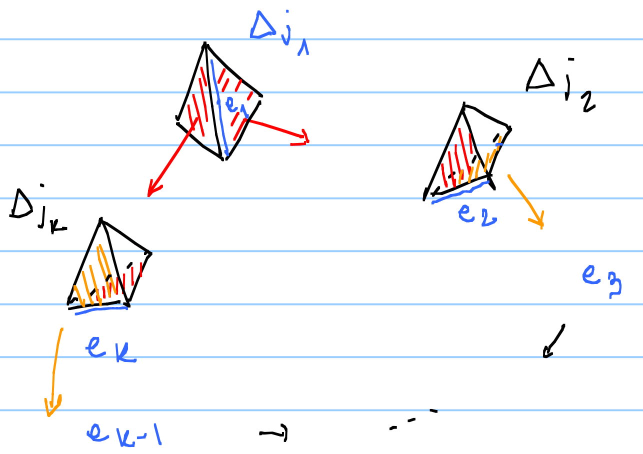
Consistency equations

$\forall$ edge $e \subset X$ $\quad \left| \prod_{\substack{\text{cycle of edges} \\ \text{incident in } e}} w_j \right| = 1$

$e = [\vec{e}_1, \ldots, \vec{e}_k]$

$\uparrow \qquad \uparrow$

$w_1 \;\cdots\; w_k$



③ If there is a solution, then there are many. They form a deformation space of $\underline{\dim = \text{number of vertices}(X)}$

$$\mathrm{Def}(X) = \{ (z_1, \ldots, z_n) \in \mathbb{U}^n \mid \text{consistency eqs hold} \}$$

number of edges of X = n = number of tetrahedra

$\partial M = \sqcup$ tori

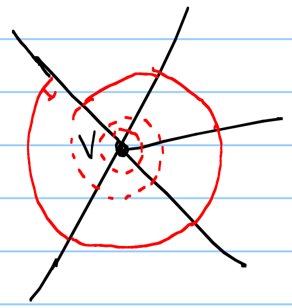
$\boxed{\text{if } \neq \emptyset}$ $\quad \dim_{\mathbb{C}} \mathrm{Def}(X) = \text{nb vertices}(X)$

Today $\longrightarrow$ Q: Are there complete solutions?

Q: What is the completion of an incomplete gluing?

Completeness of gluings

dim = 2



$\simeq S^1 \times [0,\infty)$

dim = 3

$v \in X$ ideal vertex, $N(v)$ = small neigh. of v

$\simeq L(v) \times (0,\infty) \simeq$  $\bullet\ v$

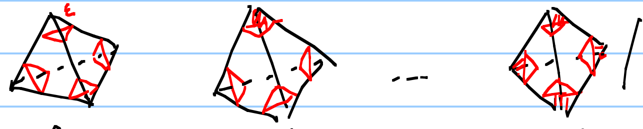
completeness of $N(v)$ is related to the behaviour of the horosections of the vertices of Δ_j that are incident to v

$$v = \{ v_1, \dots, v_r \}$$

$v_1 \in \Delta_{j_1}, \quad v_r \in \Delta_{j_r}$

this is called the <u>link</u> of the vertex → each comp of ∂M corresponds to an ideal vertex of X_M ↑ 3-mfd with ∂M

Proposition: TFAE

$X =$  / identifications

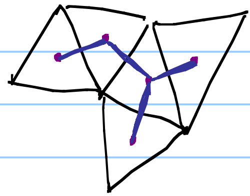
(1) $N(v)$ is <u>complete</u>

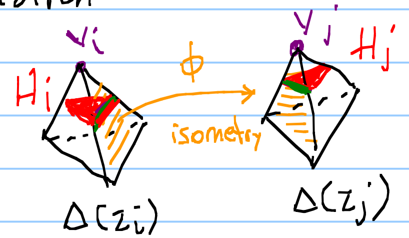
⇑

gives global geometric control on $N(v)$ → (2) It is possible to choose horosections $\{H_i\}$ of the $\{v_j\} = \{$vertices incident to $v\}$ that are <u>coherently</u> paired by the <u>geometric face pairings</u> of the ideal triangulation

⇑

just a "local" version of (2) → (3) Let G be the <u>dual graph</u> of the triangulation of <u>L</u> (<u>the link of v</u>). For each cycle of G we can coherently choose horosections corresp. to the vertices of the cycle so that they are paired by the geometric face pairings.





$$\boxed{\phi(H_i \cap \text{face}) = H_j \cap \phi(\text{face})}$$

just an alg description of cond (3)

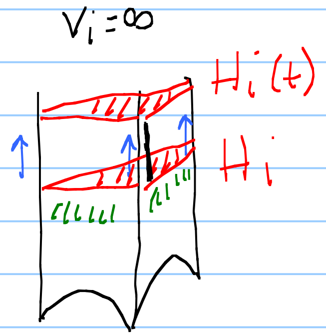
④ The total shear along each cycle is trivial.

Notice: Total shear is additive with respect to composition of cycles $\Rightarrow$ There is only a finite number of cycles (exactly $\underline{\underline{2}}$) to be checked.

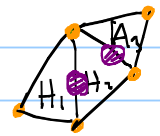
Pf. ② $\Rightarrow$ ① Suppose we have a coherent choice of horosections H_i = horosection of the vertex v_i. Then we can consider

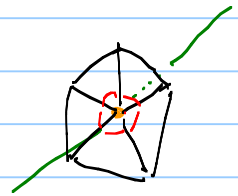
$$S_0 = \bigsqcup H_i \,/\, \text{face pairings} \subset \bigsqcup \Delta(z_j) \,/\, \text{face pairings}$$

Since the choice is coherent$^{(2)}$ $S_0 \subset N(v)$ is an emb. flat 2-torus.



Observe that if we have a coherent choice, then we have many: Just choose a height t and push each horosection H_i towards v_i at a distance t obtaining $H_i(t)$. The family $\{H_i(t)\}$ is also a coherent choice of horosections.





$\Rightarrow \forall\, t > 0$ we get $S_t = \sqcup H_i(t) \,/\,$ face pairings

$\Rightarrow N(v) = \bigsqcup_{t \geq 0} S_t$

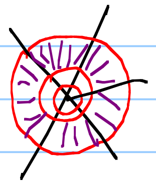
Notice that S_t and $S_{t'}$ are parallel and each path joining S_t to $S_{t'}$ in $N(v)$ has length at least $|t-t'|$.

$\Rightarrow$ get a neigh $S \times [0, \infty)$ where $S \times \{t\} \simeq S_t$

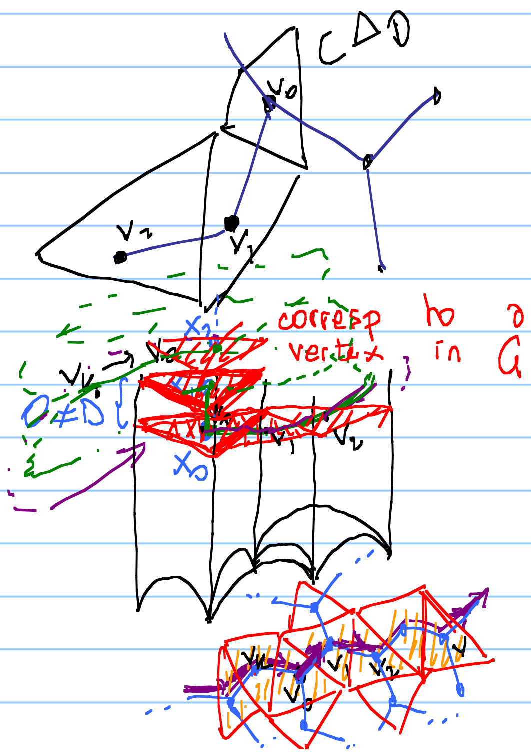
If we have a Cauchy sequence $\{x_n\} \subset$ neigh $\Rightarrow$ the sequence must be trapped between S_0 and some S_t but the region between S_0 and S_t is cpt so the Cauchy sequence converges.

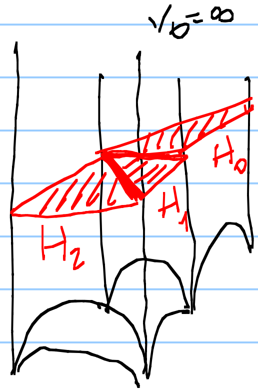
$x < S_t$ $\qquad$ $y > S_{t'}$

$d(x, y) \geq |t-t'|$



③ $\Rightarrow$ ② Fix $v_0 \in G$ a vertex and choose any horosection H_0 for the corresp vertex and tetrahedron



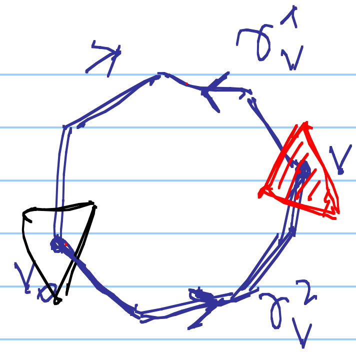


$\rightsquigarrow$ continue the process for any path in the graph.

For every $v \in G$ we choose a path γ_v joining in G v_0 to v
$\Rightarrow$ using the above procedure, we can choose a horosection of v corresponding to γ_v, we denote it by $H(\gamma_v)$.

Claim: $H(\gamma_v)$ does not depend on the path γ_v chosen

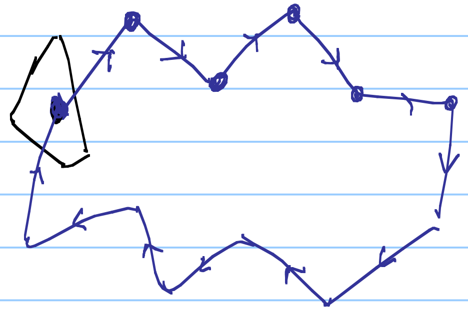
This is a consequence of condition 3

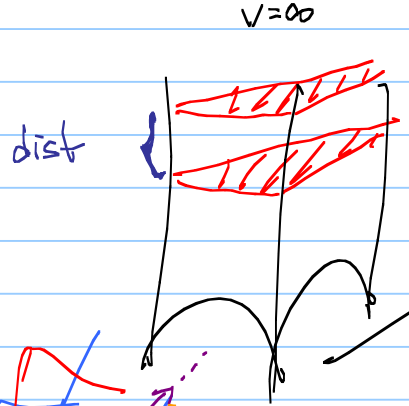


Since G is connected, we can coherently choose horosections for any of its vertices $\Rightarrow$ (2).

(3) $\Leftrightarrow$ (4) Condition 4 is just an algebraic restatement of 3

Consider a cycle γ of G, start with a vertex $v \in \gamma$ and a horosection H, we want to compute $H(\gamma)$ and compare it with H

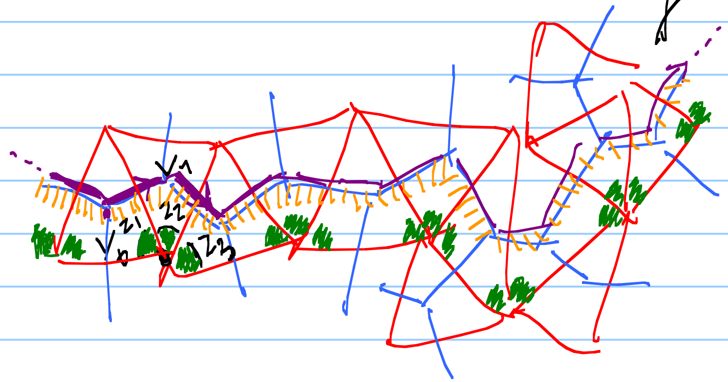




$\rightsquigarrow$ dist$(H, H(\gamma))$ = shear parameter associated to the cycle γ.

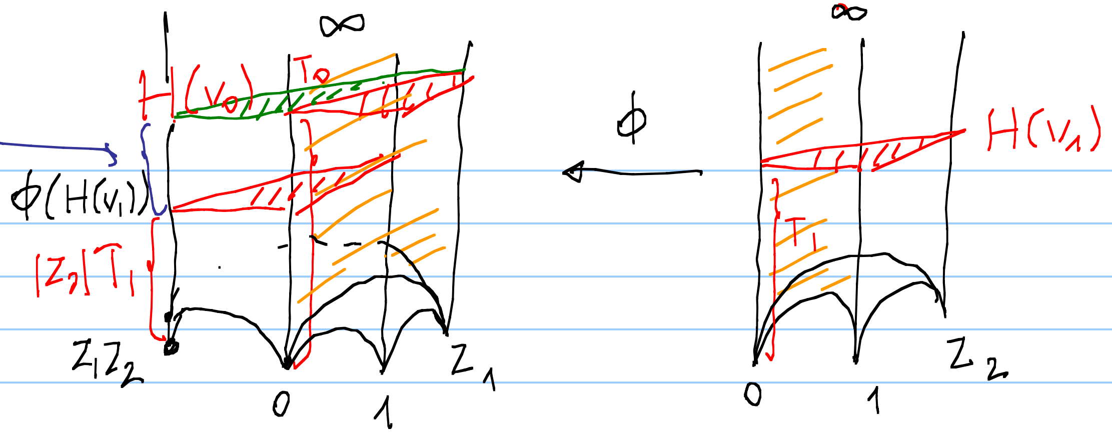
Start by choosing an arbitrary horosection for each vertex of G.

Pick v_0 a first vertex in the cycle $\gamma \subset G$



dist = $\log |z_2| \frac{T_1}{T_0}$

this the first shear parameter



second edge $\Rightarrow$ second shear = $\log |z_3| \frac{T_2}{T_1}$

...

total shear = $\sum_{\text{cycle}} \log\left(|z_j| \frac{T_{j-1}}{T_j}\right) = \boxed{\sum \log |z_j|} + \cancel{\sum \log \frac{T_{j-1}}{T_j}}$

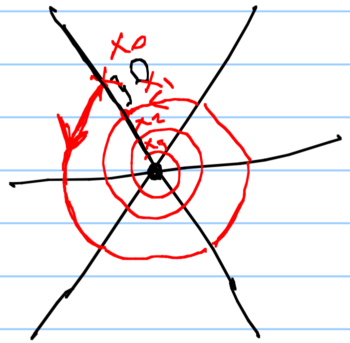
total shear

If the total shear along each cycle is 0 $\Rightarrow$ we can coherently choose horosections along the cycle.

This finishes the proof of ④ $\Rightarrow$ ③

Now, what remains is ① $\Rightarrow$ ④

In dim=2



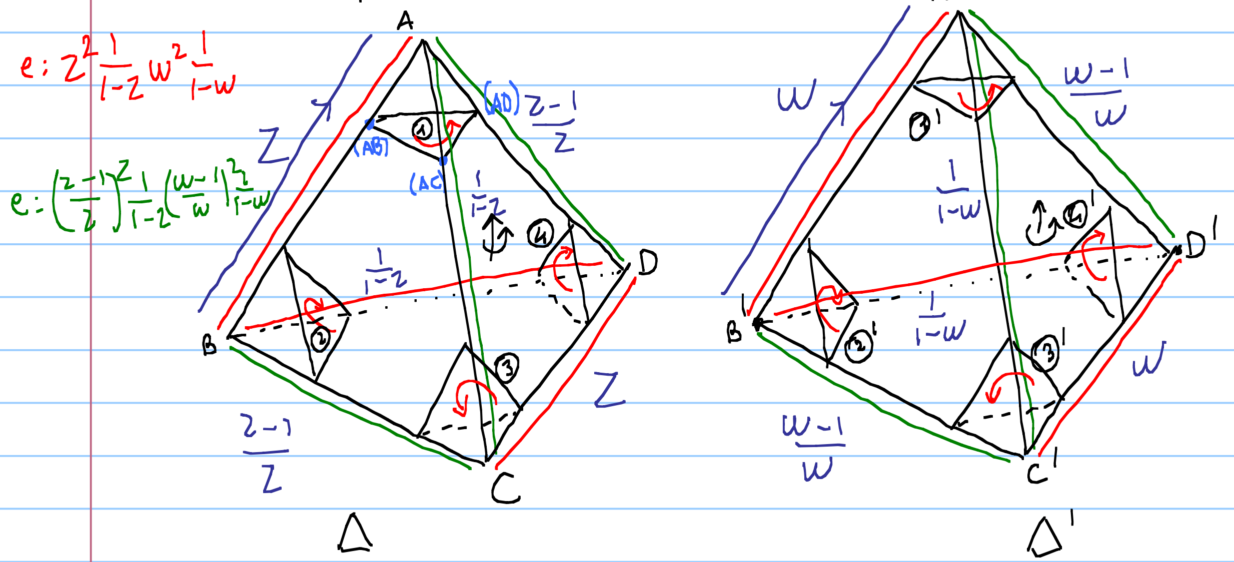
total length of the infinite red path is finite.

$\Rightarrow$ produce a Cauchy sequence x_n
with $x_n \rightarrow$ ideal vertex

Same principle works in dim=3.

$\square$

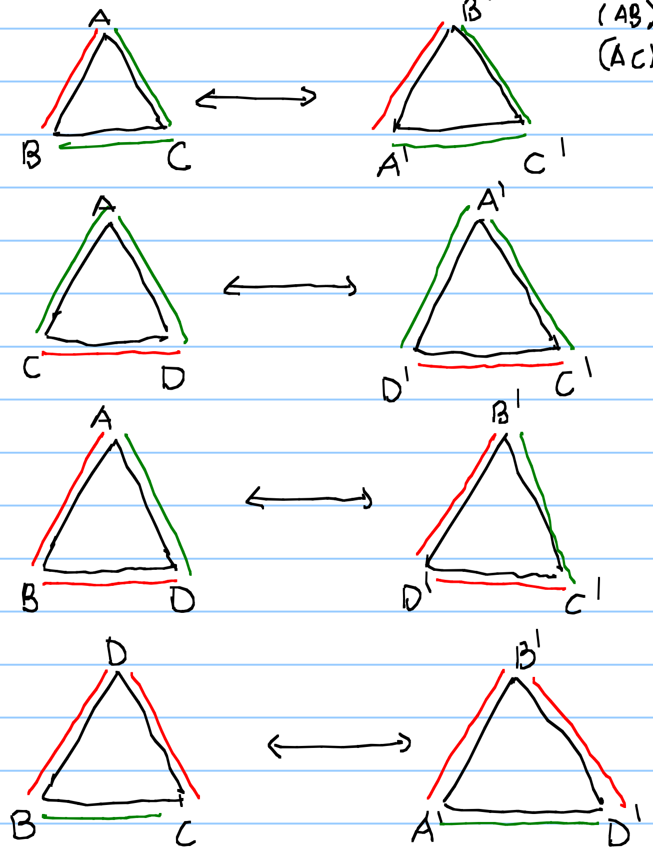
An example:



$X = \Delta \cup \Delta' /$ identifications

⤳ In X we have a red edge e and a green edge e

Face identifications



$(AB) \to (B'A')$
$(AC) \to (B'C')$

(1) $X \overset{\text{vertices}}{\simeq} \text{int}(M) = M - \partial M$ ✓ (In fact $X \setminus$ vertices $\simeq$ figure-8 knot)

M cpt with $\partial M = \cup$ tori

Lemma: This is the case $\Leftrightarrow$ number of edges of X = number of tetrahedra.

χ - computation.

(2) Consistency equations

e : $z^2 w^2 / (1-z)(1-w) = 1$

e : $\dfrac{(1-z)(1-w)}{z^2 w^2} = 1$

$\updownarrow$ one depends on the other (and this is not a surprise)

$\mathrm{Def}(X) = \{ (z,w) \in U^2 \mid z^2 w^2 / (1-z)(1-w) = 1 \}$

$U = \{ \mathrm{Im} > 0 \} \subset \mathbb{C}$

$\mathrm{Def}(X) \neq \emptyset$

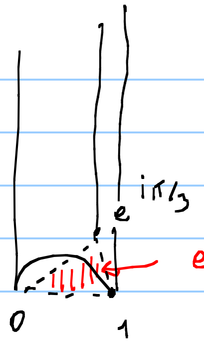
$\dfrac{z^2}{1-z} = \dfrac{w^2}{1-w} = \pm 1$ $\Leftarrow$ z, w solve $T^2 - T + 1 = 0$

$T = e^{i\pi/3}$ is a solution

$\boxed{z = w = e^{i\pi/3}} \in \mathrm{Def}(X).$

③ Completeness

If $z = e^{i\pi/3}$ then $\Delta(z) =$



$K = \{1, (12)(34), (13)(24), (14)(23)\}$

$\cap$

A_4
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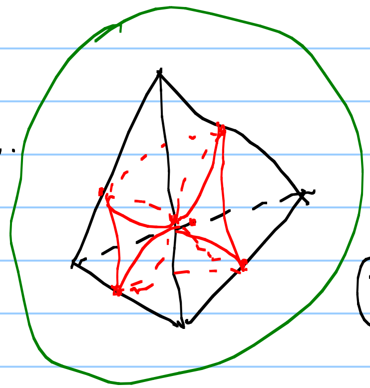
$\Delta(z)$ is the most symmetric $\left(\mathrm{Isom}^+(\Delta(z)) \cong \mathrm{Symmetries}^+(\text{abstract tetrahedron, or a regular Euclid. tetr.}) \right)$ tetrahedron and it is the one that maximizes the volume.

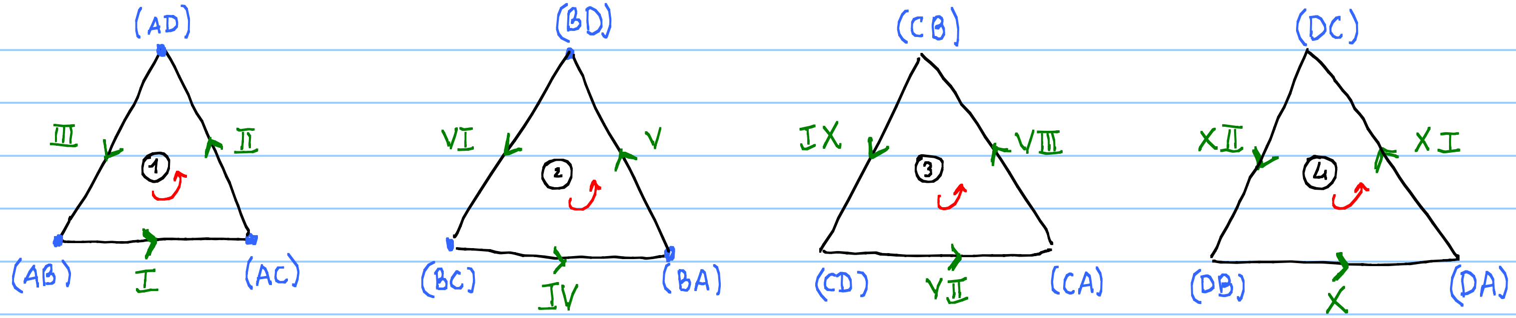
There is a very symm. choice of horosections:



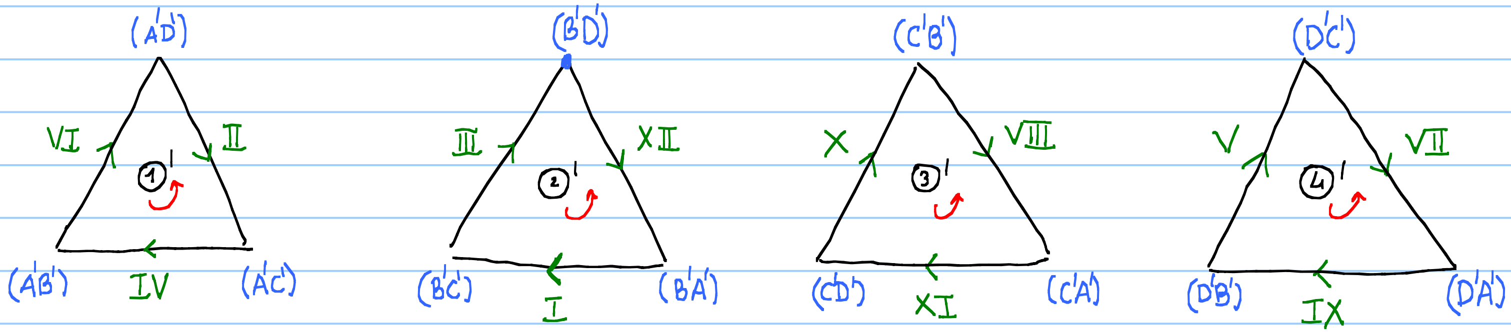
(Exercise)

The face pairings respect this choice

$\Rightarrow$ ② holds $\Rightarrow$ the structure $(z,w) = (e^{i\pi/3}, e^{i\pi/3})$ is complete.


(AD)
III
II
1
(AB)
I
(AC)
(BD)
VI
V
2
(BC)
IV
(BA)
(CB)
IX
VIII
3
(CD)
VII
(CA)
(DC)
XII
XI
4
(DB)
X
(DA)

I → (ABC)


(A'D')
VI
II
1'
(A'B')
IV
(A'C')
(B'D')
III
XII
2'
(B'C')
I
(B'A')
(C'B')
X
VIII
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(C'D')
XI
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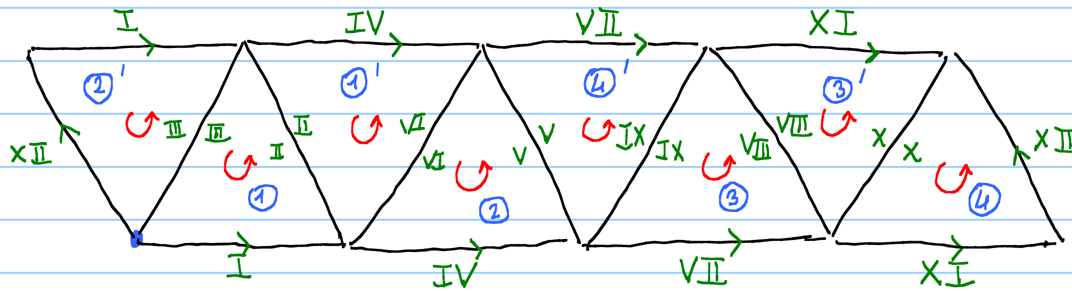
(D'C')
V
VII
4'
(D'B')
IX
(D'A')


I
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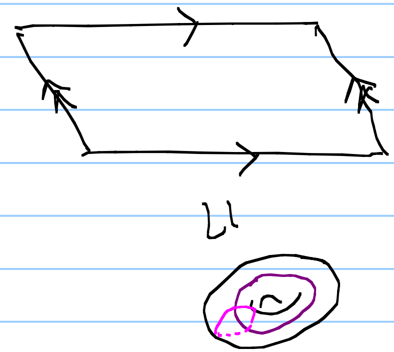

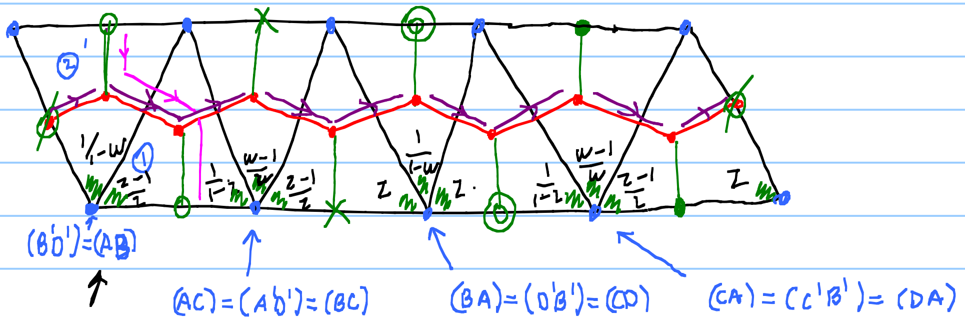

cycle : $\left| \left(\frac{z-1}{z}\right)^3 \frac{1}{(1-z)^2} z^3 \left(\frac{w-1}{w}\right)^2 \frac{1}{(1-w)^2} \right| = 1$

$\left(\frac{z-1}{z}\right)^2 \frac{1}{(1-z)^2} z^4 \left(\frac{w-1}{w}\right)^2 \frac{1}{(1-w)^2} = 1$

$\frac{z-1}{w^2} = 1$

$z^2/w^2 = 1$

Lemma: the vanishing of all total shears along cycles

$|w_1 \dots w_k| = 1$

is equiv to

$w_1 \dots w_k = 1$

$\forall$ each cycle.

consistency $\left\{ z^2 w^2 / (w-1)(z-1) = 1 \right.$

completeness $\left. \frac{z-1}{w^2} = 1 \right\}$ $\Downarrow$ $\left\{ \begin{array}{l} z^2 w^2 / (w-1)(z-1) = 1 \\ z^2 = w^2 \end{array} \right.$

$\leadsto$ See that there exists only one solution! $(z,w) = (e^{i\pi/3}, e^{i\pi/3})$

This is also not a surprise (Mostow Rigididity).