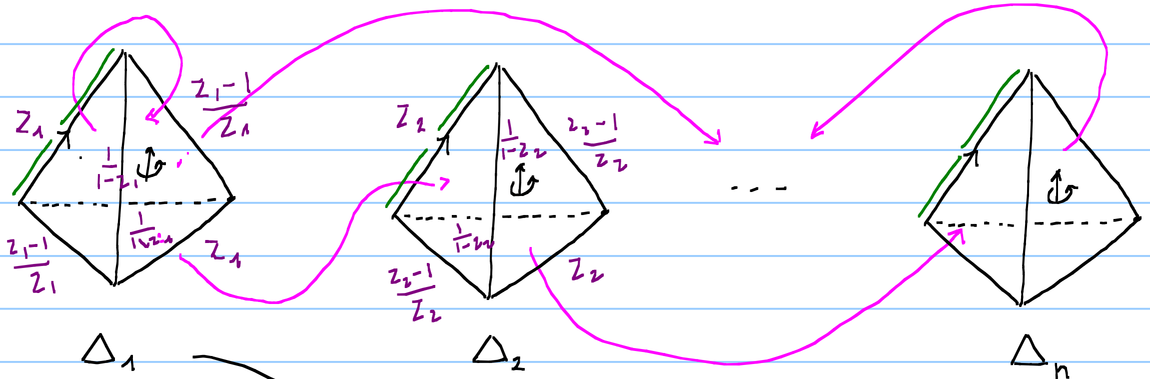
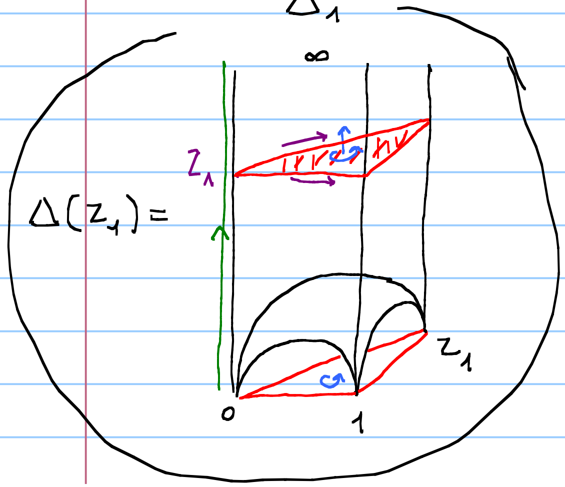
Note Title 16/12/2020

Hyperbolic Manifolds - Lecture 14





Geometric ideal triangulations in dim=3

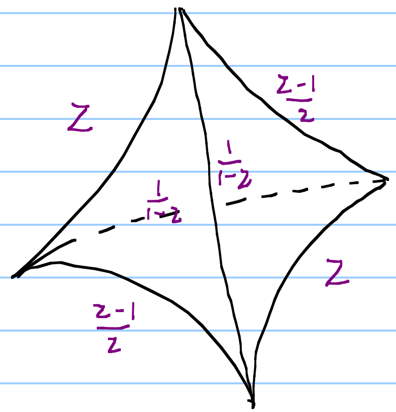
① $\Delta_1, \dots, \Delta_n$ oriented 3-simpl. with a paring on their faces and $\phi : f_1 \to f_2$ orientation reversing $\forall (f_1, f_2)$ face pair

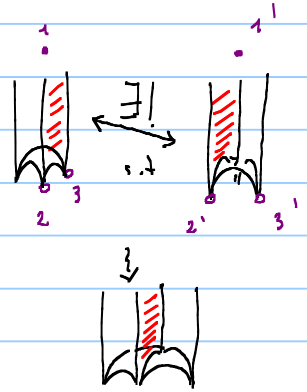
$$X = \bigsqcup \Delta_j \Big/ \bigsqcup_{\text{face pairs}} \phi$$

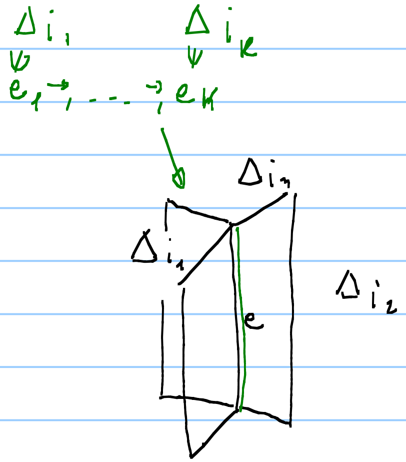
② X - vertices $= \text{int}(M) = M - \partial M$

M cpt. oriented 3-mfd with boundary

Choose an (oriented) edge E_j for each Δ_j







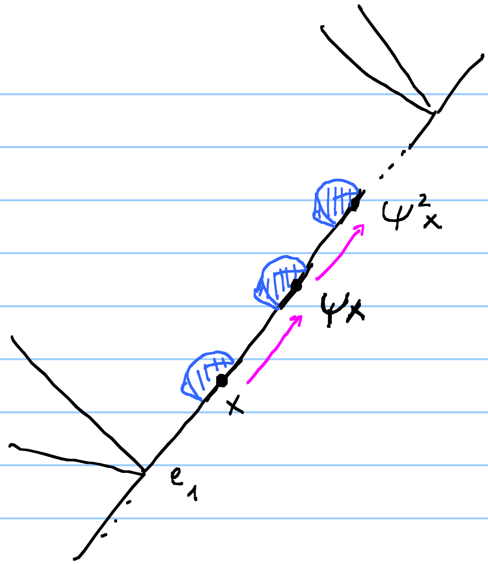
③ Choose $z_j \in U$ and replace Δ_j with $\Delta(z_j)$ = ideal hyperbolic tetrahedron with complex angle z_j at e_j.

Then replace $\phi : f_1 \longrightarrow f_2$ with the unique hyperbolic isometry with the same behaviour on the vertices

④ $$X^{hyp} = \bigsqcup \Delta(z_j) \Big/ \bigsqcup_{\text{face pairs}} \phi^{hyp}$$

X^{hyp} has a hyperbolic metric away from the edges.

Problems: (1) Shearing: $\Delta_{i_1} \dots \Delta_{i_k}$ cycle of tetrahedra around the edge e, with Δ_{i_j} incident at e on e_j



$\Rightarrow$ cycle of isometric identifications

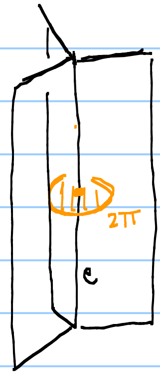
$\psi: e_1 \longrightarrow e_2 \longrightarrow \dots \longrightarrow e_k \longrightarrow e_1$

We could have $\psi \neq Id$ in which case infinitely many points on e_1 are identified

In particular: • $X^{hyp} \neq X$ − vertices $= int(M) = M - \partial M$

• no natural extension of the hyperbolic structure

(b) dihedral angles : If the hyperbolic metric has a natural extension, then the total dihedral angle around each edge is 2π.



Consequences: We must have that ∂M consists of tori.

[From now on we only work with gluings for which M has toroidal boundary.]

Consistency equations

Proposition: e edge $\Delta_{i_1}, \dots, \Delta_{i_k}$ cycle of tetrahedra around e with incident edges $e_1, \dots, e_k$ and associated complex parameters $w_1, \dots, w_k \in U \subset \mathbb{C}$

⚠ some tetrahedra might appear several times

$w_j \in \left\{ z_{i_j}, \frac{1}{1-z_{i_j}}, \frac{z_{i_j}-1}{z_{i_j}} \right\}$

TFAE: (1) $\forall$ e edge we have no shearing and the total dihedral angle is 2π

$\Updownarrow$

(2) $\forall$ e edge we have $|w_1 \dots w_k| = 1$ and $\arg(w_1) + \dots + \arg(w_k) = 2\pi$

$\Downarrow$ $\Uparrow$

$\boxed{\text{Snappy}}$ $\rightarrow$ (3) $\forall$ edge we have $w_1 \dots w_k = 1$?

equiv: $\log(w_1) + \dots + \log(w_k) = 2\pi i$

Notice: $\log$ has a well-def branch on U

Moreover, if the conditions are satisfied, then the hyperbolic metric naturally extends over the edges.

Pf. $(1) \Leftrightarrow (2)$

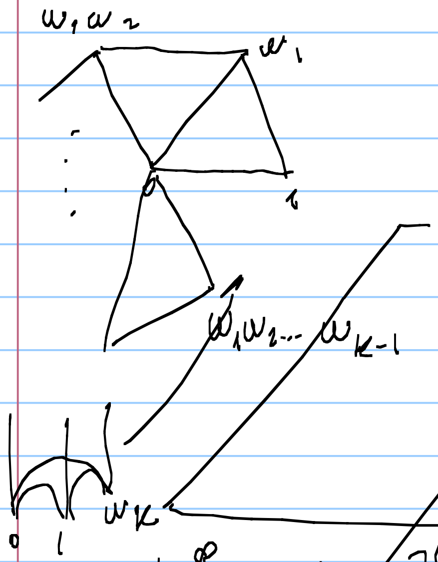
there is no shearing
$\Updownarrow$ $|w_1 \cdots w_k| = 1$

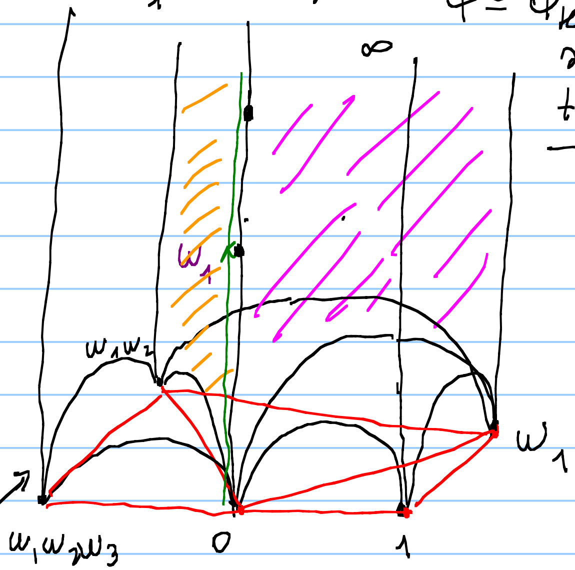
$\Delta_1 = \Delta(w_1)$

$\psi = \phi_k \phi_{k-1} \cdots \phi_2 (\phi_1 = Id)$ acts on e_1 as $\underline{t \to |w_1 \cdots w_k| t}$

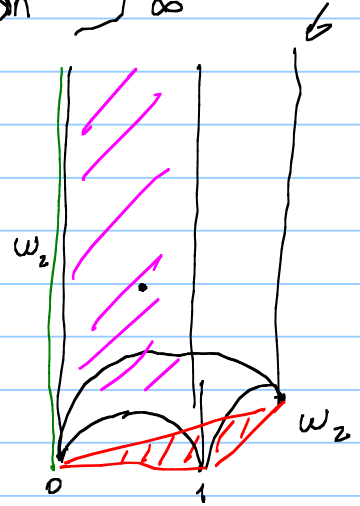
ϕ_2 acts on e_1 by $t \to |w_2| t$
ϕ_3 acts on e_1 by $t \to |w_3| t$
$\vdots$
ϕ_k acts on e_1 by $t \to |w_k| t$

$\Delta_2 = \Delta(w_2)$



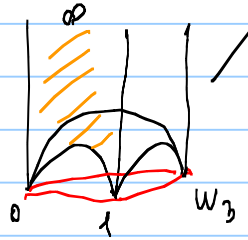


$\exists!$ isom ϕ_2 that sends the pink face to the other pink face
$0 \to 0$
$1 \to w_1$
$\infty \to \infty$



$\phi_2: (z,t) \to (az, |a| t)$

$1 \to w_1 \Rightarrow a = w_1$
$\Rightarrow \phi(w_2, 0) = (w_1 w_2, 0)$



$\exists!$ ϕ_3: $0 \to 0$, $\infty \to \infty$, $1 \to w_1 w_2$

....

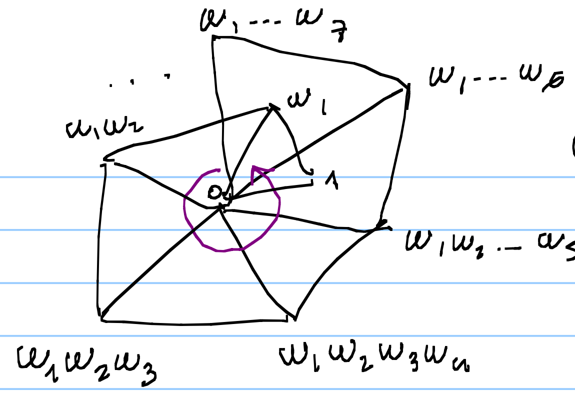
$\phi(z,t) = (bz, |b| t)$
$b = w_1 w_2$
$\Rightarrow w_3 \to w_1 w_2 w_3$

ϕ_k: $0 \to 0$, $\infty \to \infty$, $1 \to w_1 \cdots w_k$

(1) $\Leftrightarrow$ (2)

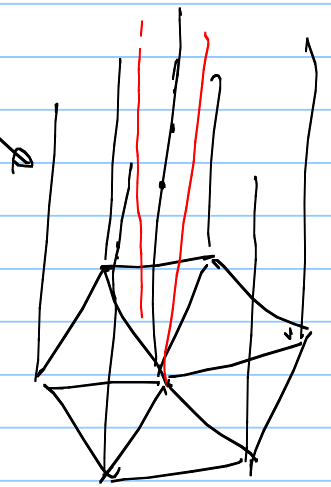
No shearing $\Leftrightarrow |w_1 \dots w_k| = 1$

Total dihedral angle $= 2\pi \Leftrightarrow \sum \arg(w_j) = 2\pi$

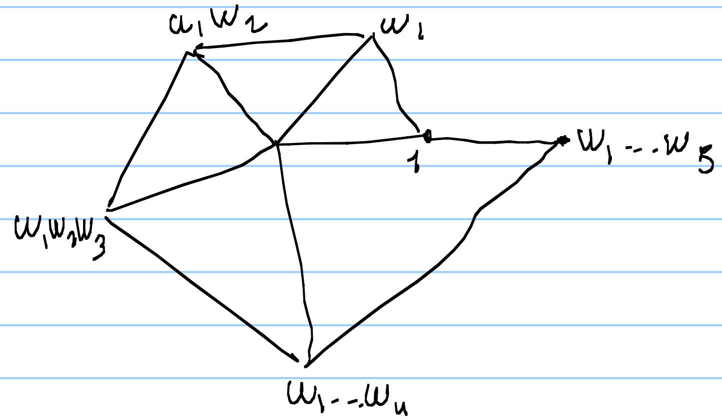


$\mathbb{C}$

a neigh of the edge in the quotient



If total angle $= 2\pi$



If $|w_1 \dots w_k| = 1$

(3) ⇒ (2) Know that $w_1 \cdots w_k = 1$ $\forall$ edges

Need to check that $\sum \arg(w_j) = 2\pi$

$m(e) \in \mathbb{N}$ $m(e) \geq 1$

$$w_1 \cdots w_d = 1 \implies \sum_{\text{edge}=e} \arg(w_j) = 2\pi\, m(e)$$

$$2\pi\, n = \sum_{\text{edges}} \sum_{e} \arg(w_j) \overset{?}{=} 2\pi \sum_{e \in \text{edge}} m(e)$$

$|\text{edges}| = |\text{tetrahedra}|$

$\Uparrow$ χ-char comp.

Last time: The number of edges $= n$ if $\boxed{M \text{ has toroidal boundary}}$

$\implies m(e) = 1$ $\square$

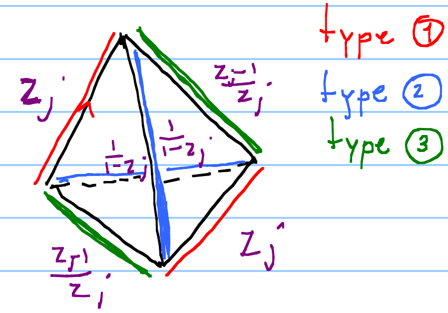
$$\chi(X) = |\text{vertices}| - |\text{edges}| + 2n - n$$

$$\chi(X) = |\text{vertices}| - \tfrac{1}{2}\chi(\partial M)$$

$= 0$ for ∂M = Torus

Solutions to the consistency equations: Parameter space

An explicit expression for the consistency equation for the edge e



$$1 = w_1 \cdots w_k = \left[\prod_{j=1}^{n} z_j^{A_{j,1}} \left(\frac{1}{1-z_j} \right)^{A_{j,2}} \left(\frac{z_j - 1}{z_j} \right)^{A_{j,3}} \right]$$

$\rightarrow A_{j,i}$ = number of edges of type ⓘ in the tetrahedron Δ_j that are incident to e $\in \{0,1,2\}$

$$\Leftrightarrow \quad 2\pi i = \sum_{j=1}^{n} A_{j,1} \log(z_j) + A_{j,2} \log\left(\frac{1}{1-z_j}\right) + A_{j,3} \log\left(\frac{z_j - 1}{z_j}\right) = G_e(z_1, \dots, z_n)$$

$n = |\text{edges}|$

$G : \mathbb{C}^n \longrightarrow \mathbb{C}^n \qquad G = \left(G_e^{(z)}\right)_{e \in \text{edges}}$

Lemma: $\mathrm{Im}(G) \subset$ linear subspace of $\dim = n - |\text{vertices}|$

Proposition: dG is surjective at each point $(z_1, \dots, z_n) \in \mathbb{C}^n$

Lemma + Proposition $\Rightarrow$ $G^{-1}(2\pi i, \dots, 2\pi i) = \begin{cases} \phi \\ \text{or} \quad \text{it is a } \dim_{\mathbb{C}} = |\text{vertices}| \text{ submanifold of } \mathbb{C}^n. \end{cases}$