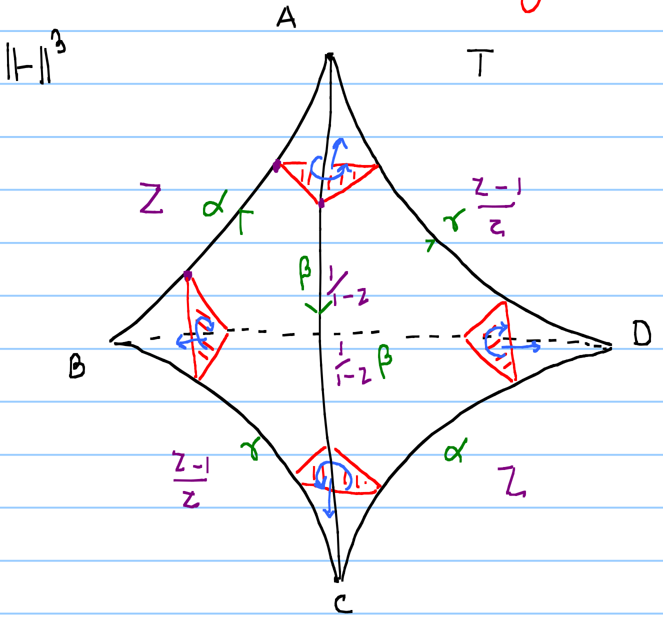
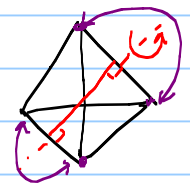
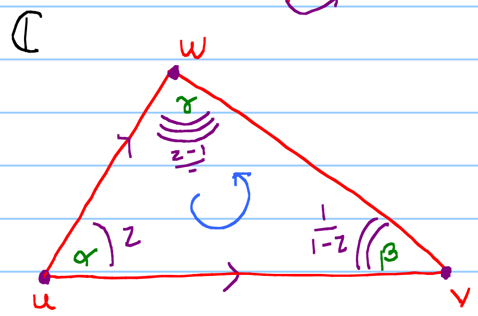
Hyperbolic Manifolds - Lecture 13

Note Title 15/12/2020

Today: • Volumes of ideal tetrahedra
• Geometric ideal triangulations in dim=3







$z = \frac{w-u}{v-u} \in \mathbb{C}$ $\arg(z) = \alpha$, $|z| =$ ratio length of sides

$\frac{u-v}{w-v} = \frac{1}{1-z}$

$\frac{v-w}{u-w} = \frac{z-1}{z}$

α, β, γ dihedral angles

Volumes

Theorem: $\mathrm{vol}\left(T(\alpha,\beta,\gamma)\right) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$

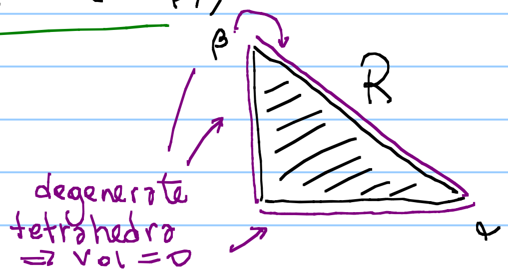
$$\Lambda(\theta) := -\int_0^\theta \log|2\sin(t)|\,dt \qquad \text{Lobachevsky function}$$

Corollary: There exists a unique ideal tetrahedron of maximal volume which corresponds to $\alpha = \beta = \gamma = \pi/3$.

Pf. $\alpha + \beta + \gamma = \pi \qquad \gamma = \pi - (\alpha + \beta)$

$\Rightarrow$ we study the function $\boxed{V(\alpha,\beta) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\pi - (\alpha+\beta))}$ defined over the simplex $R = \{0 \le \alpha, \beta, \pi - (\alpha+\beta) \le \pi\}$

Obs: V vanishes on the sides of R
$\Rightarrow$ V has a maximum inside R



by the regularity of V, at that point $\underline{dV=0}$

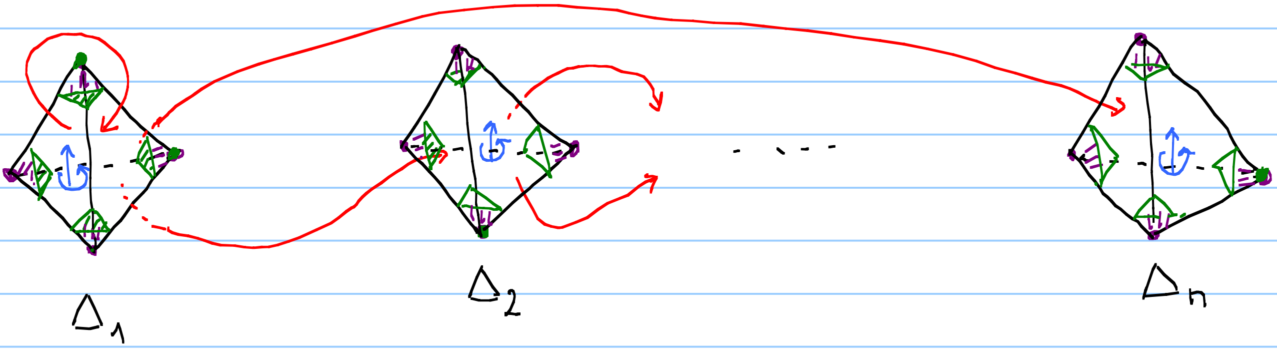
$$0=\frac{d}{d\alpha}V=\Lambda'(\alpha)-\Lambda'(\pi-(\alpha+\beta))=\log|2\sin(\alpha)|-\log|2\sin(\pi-(\alpha+\beta))|=0$$

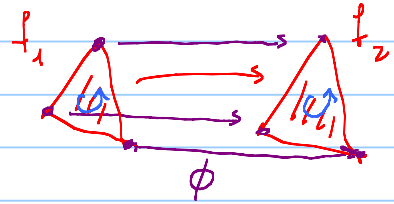
$$0=\frac{d}{d\beta}V=\Lambda'(\beta)-\Lambda'(\pi-(\alpha+\beta))=\log|2\sin(\beta)|-\log|2\sin(\pi-(\alpha+\beta))|$$

$$\rightsquigarrow \quad \alpha=\beta=\pi-(\alpha+\beta). \quad \boxtimes$$

Ideal triangulations in dim = 3

Take n copies of the standard oriented Euclidean 3-simplex Δ





The total number of faces is $4n$.

We choose a pairing of the $4n$ faces and for each pair (f_1, f_2) we choose an orientation reversing isometry $\phi : f_1 \longrightarrow f_2$.

Now we can form the space

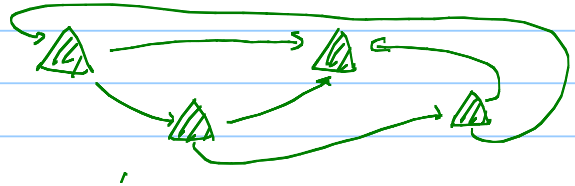
$$X := \left(\bigsqcup \Delta_i \right) \Big/ \bigsqcup_{\text{face pairs}} \phi$$

Differently from what happens in dim = 2, there is no reason to expect that X is a 3-mfd. In fact, generically it is not

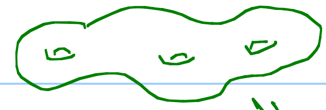
Theorem (Dunfield - Thurston): The proportion of 3-mfd gluings X among all possible gluings of n 3-simplices (as above) goes to 0 as $n \to \infty$!

The problem is that neigh of the vertices are in general not 3-balls!

$v \in$ vertex of X $\qquad$ $\mathrm{Link}(v, X) =$ gluing of triangles



$\Downarrow$ it is very likely that such a gluing gives a higher genus surface.



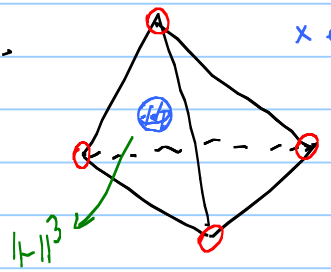
$\Downarrow$

$\mathrm{Neigh}(v, X) = \mathrm{Cone}(v, \mathrm{Link}) \neq$ 3-ball.

The situation is however not so bad: In fact X is a 3-mfd at each other pt.

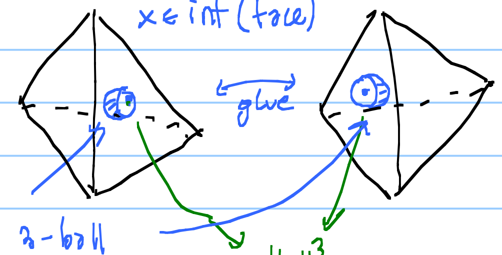
Lemma: $X -$ vertices $\underset{\text{homeo}}{\cong} \mathrm{int}(M)$ where M is a cpt oriented 3-mfd with $\partial M \neq \emptyset$ and $\mathrm{int}(M) = M - \partial M$.

Pf.



$x \in \mathrm{int}(\Delta) \rightsquigarrow$ never a problem

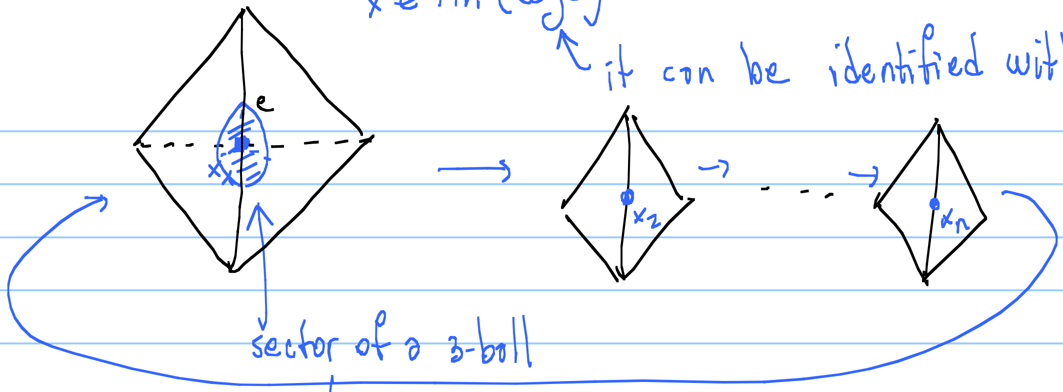
$x \in \mathrm{int}(\text{face}) \longrightarrow$ never a problem



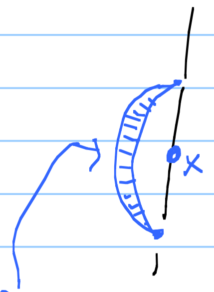
3-ball neigh of x in the quotient

$x \in \operatorname{int}(\text{edge})$

it can be identified with several other edges

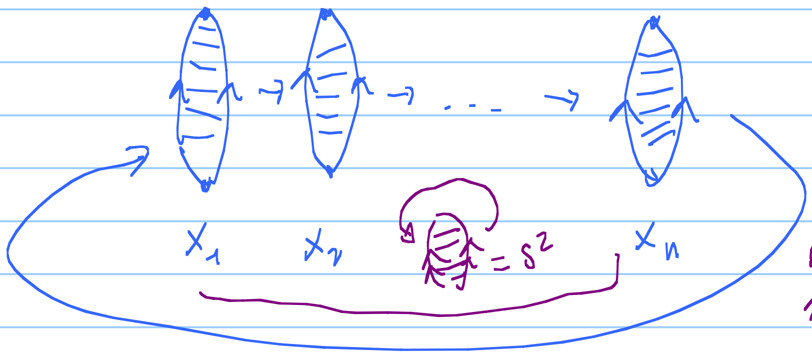


sector of a 3-ball



its contribution to the link of x in X is a sector of a sphere

$\Rightarrow$ the link of x in



$\simeq$ either S^2 or $\cancel{\mathbb{RP}^2}$

the orientation rules this out.

$\simeq \mathbb{RP}^2$

the link of x is homeo to S^2!

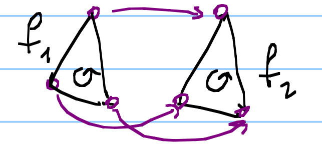
$\Rightarrow x$ is a 3-mfd. pt $\square$

Hyperbolic structure on ideal triangulations

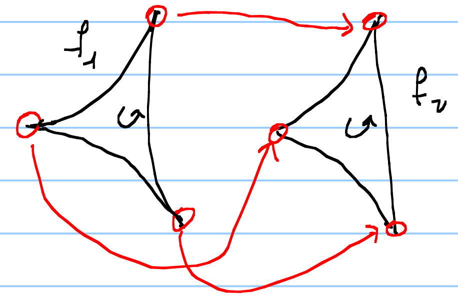
here we have some freedom, so we have to choose (α, β, γ) or the parameter $z \in \mathbb{C}$

We now replace ① each Δ_j with an oriented ideal hyperbolic tetrahedron

② each $\phi : f_1 \to f_2$ with a hyperbolic isometry (completely determined by the behaviour of ϕ on the vertices).



$$\rightsquigarrow \quad X^{hyp} = \left(\bigsqcup \Delta_{j\,(z)}^{hyp} \right) \Big/ \left(\bigsqcup_{\text{face pairs}} \phi^{hyp} \right)$$



Notice that X^{hyp} is again locally modeled on $\mathbb{H}^3$ away from the edges

$\Rightarrow$ get a hyp metric / structure on $X^{hyp} -$ edges.

The edges are problematic at several levels:

(1) shearing

(1) dihedral angles

$e \in \text{edge}(X)$, e is represented by $e_1 \in \Delta_{i_1}$, $e_2 \in \Delta_{i_2}$, ..., $e_k \in \Delta_{i_k}$

The tetrahedra $\Delta_{i_1}, \dots, \Delta_{i_k}$ are arranged in a cycle around e

$\Delta_{i_1} \to \Delta_{i_2} \to \Delta_{i_3} \to \dots \to \Delta_{i_k} \to \Delta_{i_1}$

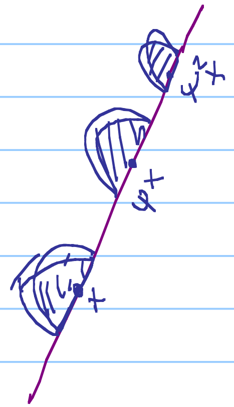
(perhaps there are some tetrahedra that appear multiple times)

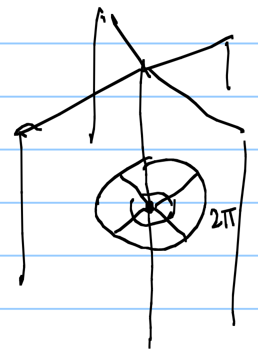
each such identification in X^{hyp} gives an isometry $\mathbb{R} \cong e_1^{hyp} \to e_2^{hyp} \to \dots \to e_k^{hyp} \to e_1^{hyp} \cong \mathbb{R}$

ψ an isometry of $\mathbb{R}$

It might be that φ is a non-trivial isometry $\varphi \neq Id$.

$\Longrightarrow$ If so several points of e_1^{hyp} are identified to each other!





Even if $\varphi = Id$ for each edge,

(2) In order to extend in a natural way the hyp.-structure over an edge the total dihedral angle around the edge must be 2π.

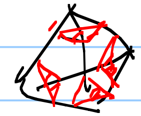
This gives topological restrictions:

(Recall that X - vertices $= \mathrm{int}(M) = M - \partial M$)

Lemma: If the sum of the dihedral angles around each edge is 2π then ∂M is a collection of tori and X has exactly n edges

equiv.

Pf. Suppose for simplicity that ∂M is connected



Idea: Consider Σ angles in the horospherical sections of the vertices $= 4\pi n$

$$= 2 \sum_{e \in \text{edges}(X)} \text{total dihedral angle around } e = 4\pi \cdot |\text{edges}|$$

$$\Rightarrow |\text{edges}| = n$$

Now let us make an Euler char computation

$$\chi(X) = |\text{vertices}| + \chi(M) - \chi(\partial M) = |\text{vertices}| - \tfrac{1}{2}\chi(\partial M)$$

$\parallel$

$|\text{vertices}| - |\text{edges}| + |\text{faces}| - |\text{tetrahedra}|$

$$\underbrace{|\text{vertices}| - \underset{n}{\underset{\parallel}{|\text{edges}|}} + \underset{2n}{\underset{\parallel}{|\text{faces}|}} - \underset{n}{\underset{\parallel}{|\text{tetrahedra}|}}}_{\substack{\parallel \\ 0}}$$

since $\chi(M) = \frac{1}{2}\chi(\partial M)$ $\forall$ cpt orientable 3-mfd.

$(\chi(\partial M) = 0)$

Poincaré-duality

$\Rightarrow \chi(\partial M) = 0 \Rightarrow \partial M$ is a torus. $\square$