Hyperbolic Manifolds - Lecture 12

Note Title 09/12/2020

The hyperbolic 3-space $\mathbb{H}^3$

Summary:
- Isometries and $PSL_2\mathbb{C}$
- Ideal tetrahedra, classification and volume
- Discussion on (ideal) triangulations in dim = 3

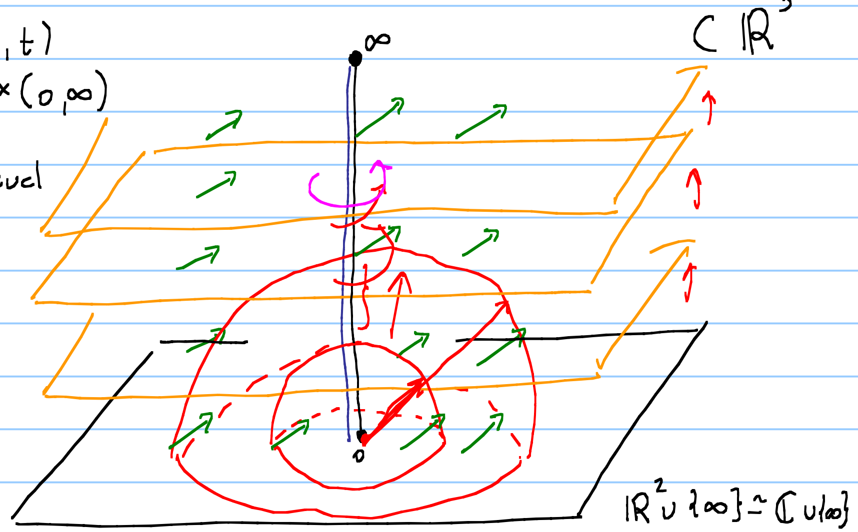
Isometries

$$\boxed{\text{Lemma 3: } \mathrm{Isom}^+(\mathbb{H}^3) \cong PSL_2\mathbb{C}}$$

(z,t)

$\mathbb{H}^3 = U_3 = \mathbb{C} \times (0,\infty)$

$g_U = \frac{1}{t^2} g_{Eucl}$

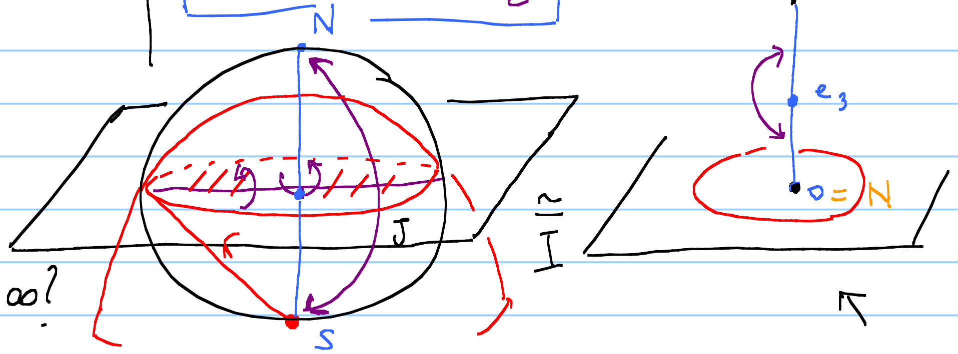


Looking at the normal forms we have in $Isom^+(U_3)$

	Action on U		Action on $\partial_\infty U = \mathbb{C}\cup\{\infty\} = \mathbb{CP}^1$
• Parabolic motions	$T_b:(z,t) \longrightarrow (z+b,t)$	$b\in\mathbb{C},\ b\neq 0$	$\tau_b:\ z \longrightarrow z+b$
• Loxodromic motions	$H_a(z,t) \longrightarrow (az,\lvert a\rvert t)$	$a\in\mathbb{C}^*,\ \lvert a\rvert\neq 1$	$h_a:\ z \longrightarrow az$
• Elliptic motions	$R_\theta:(z,t) \longrightarrow (e^{i\theta}z,t)$	$\theta\in(0,2\pi)$	$r_\theta:\ z \longrightarrow e^{i\theta}z$
• An inversion that exchanges 0 and ∞	$J:(z,t) \longrightarrow \left(-\dfrac{\bar z}{z\bar z+t^2},\ \dfrac{t}{z\bar z+t^2}\right)$		$\boxed{j:\ z \longrightarrow -\dfrac{1}{z}}$?

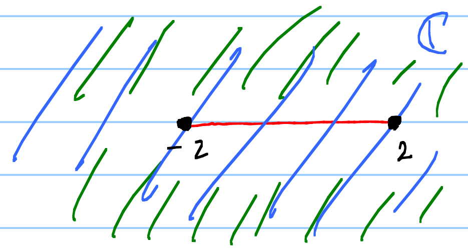
Lemma 1: The transformations τ_b, h_a, r_θ, j generate $PSL_2\mathbb{C}$

Q: Is there an isom of $\mathbb{H}^3$ that acts like $-\frac{1}{z}$ at ∞?



Lemma 2: The transformations $T_b, H_\alpha, R_\theta, J$ generate $\mathrm{Isom}^+(\mathbb{H}^3)$

Lemma 1 + Lemma 2 $\Longrightarrow$ Lemma 3.

$\operatorname{tr}(A) \in$ 

We have the following matrix criterion:

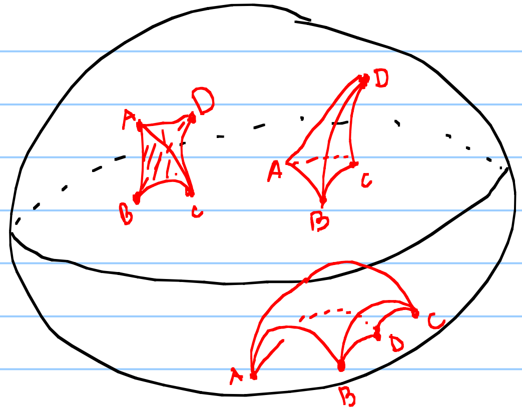
Lemma: $A \in PSL_2\mathbb{C} \setminus \{Id\}$. Then A is

- parabolic $\iff$ $\operatorname{tr} A \overset{\in \mathbb{C}}{=} \pm 2$ $\iff$ A conj. to $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ $\quad \underline{(z,t) \longrightarrow (z+b, t)}$
- loxodromic $\iff$ $\operatorname{tr} A \notin [-2,2]$ $\iff$ A conj. to $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ $\quad \underline{(z,t) \longrightarrow (\lambda^2 z, |\lambda|^2 t)}$
- elliptic $\iff$ $\operatorname{tr} A \in (-2,2)$ $\iff$ A conj. to $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ $\quad (z,t) \longrightarrow (e^{2i\theta} z, t)$

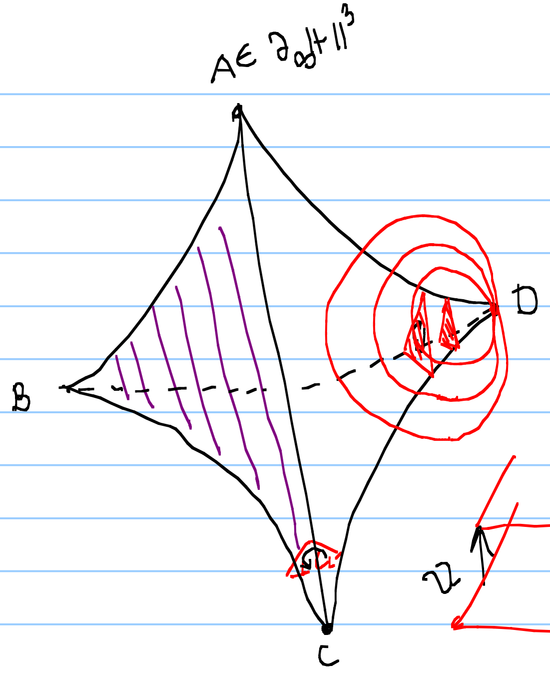
Pf. Use the Jordan normal form. $\square$

Ideal tetrahedra

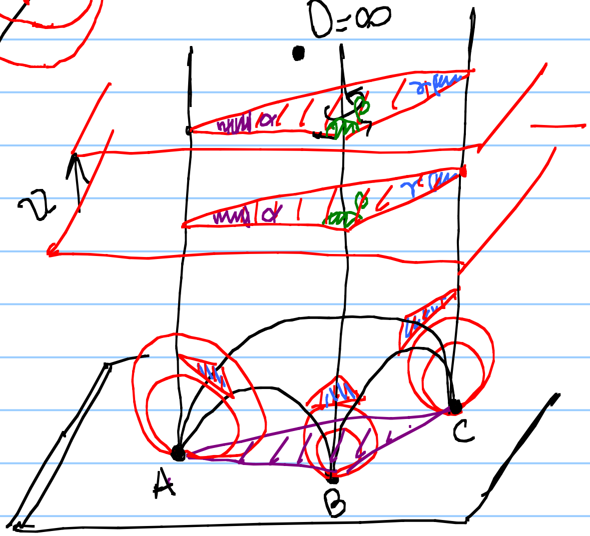
Any quadruple $A, B, C, D \in \mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$ span a tetrahedron in $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$



We will only consider those tetrahedra that have all vertices at ∞. These are called ideal tetrahedra



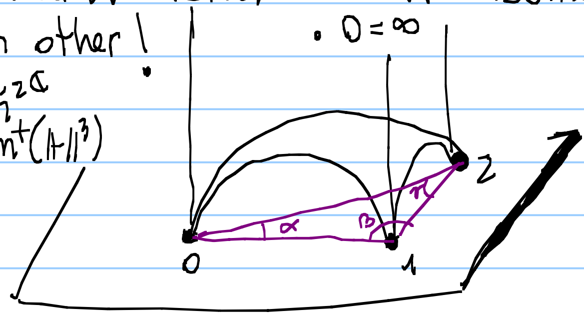
Note: • Each face of an ideal tetrahedron is an ideal triangle! Recall that ideal triangles are all isometric to each other.



• In a gluing of ideal tetrahedra we do not have to worry about what happens at the vertices.

• Not all ideal tetrahedra are isometric to each other!

Up to $\mathrm{Isom}^+(\mathbb{H}^3) \cong PSL_2\mathbb{C}$ $\longrightarrow$



In order to classify ideal tetrahedra we look at small horospherical sections around the vertices. All these sections are Euclidean triangles homothetic to each other and we record the similarity classes of these triangles

equivalently, we only record the angles of the triangles $\Delta(\alpha,\beta,\gamma)$

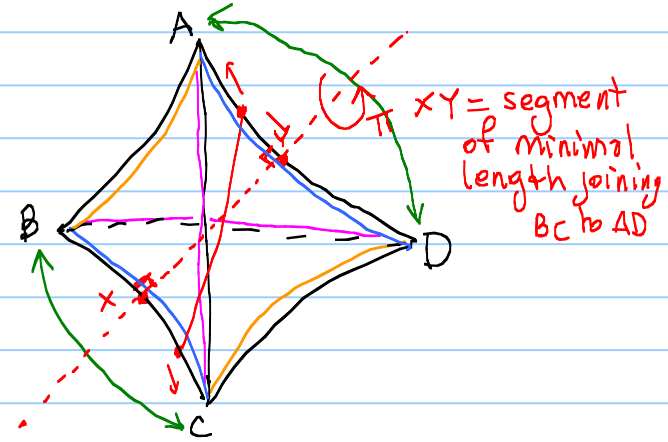
Notice that α,β,γ are the dihedral angles of the corresponding edges (because horospheres centered at the vertex are $\perp$ to the edges)

Obs: Given a vertex D and the angles α,β,γ of any small horosection we can completely reconstruct the tetrahedron $ABCD$ (up to isometries).

Want to parametrize similarity classes of oriented Euclidean triangles.

Before going on let us obeserve that every ideal tetrahedron has <u>many</u> useful simmetries

Observation: For every pair of opposite edges of $T(ABCD)$ there exists an orientation preserving isometry that exchanges the endpoints of the edges as in the picture.



The π-rotation around XY exchanges $A \leftrightarrow D$, $B \leftrightarrow C$

($\Rightarrow$ it leaves $T(ABCD)$ invariant)

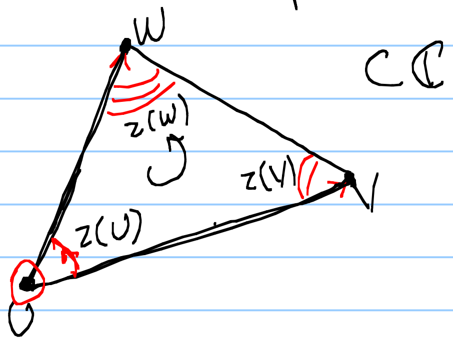
$\Rightarrow$ We have isometries that realize the permutations $(AB)(CD)$, $(AD)(BC)$, $(AC)(BD)$

in particular $\mathrm{Isom}^+(T(ABCD))$ acts transitively on the vertices!

$\Rightarrow$ There is a well-defined oriented similarity class of Euclidean triangles associated to every oriented ideal tetrahedron

(it does not depend on the choice of a vertex or the small horosection).

How do we parametrize oriented similarity classes of Euclidean triangles?



We associate to each vertex a complex number

$$u \to \left[\frac{w-u}{v-u} = z(u)\right)$$

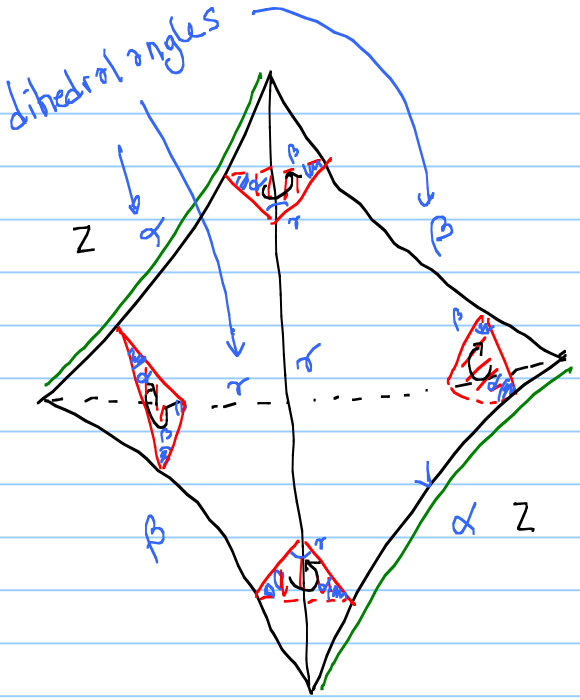
$$v \to \left[\frac{u-v}{w-v} = z(v)\right)$$

$$w \longrightarrow \left[\frac{v-w}{u-w} = z(w) \right.$$

There are some relations between the three parameters

$$\left. \begin{aligned} z(u)\, z(v)\, z(w) &= -1 \\ 1 - z(u) + z(u)\, z(v) &= 0 \end{aligned} \right\}$$

$\Rightarrow$ Similarity classes of oriented Euclidean triangles are parametrized by $\{ (z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 z_2 z_3 = -1 \,,\; 1 - z_1 + z_2 z_1 = 0 \} \simeq \mathbb{C}$ - dim = 1.

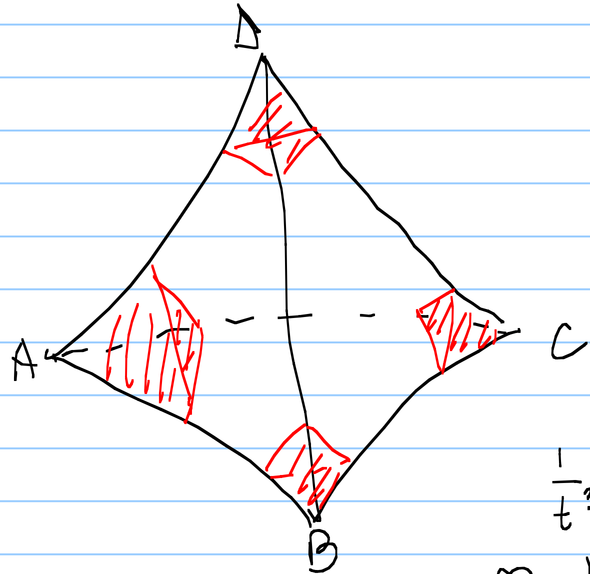


$\alpha+\beta+\gamma=\pi$

$\Rightarrow$ get an oriented similarity class

Volumes of ideal tetrahedra

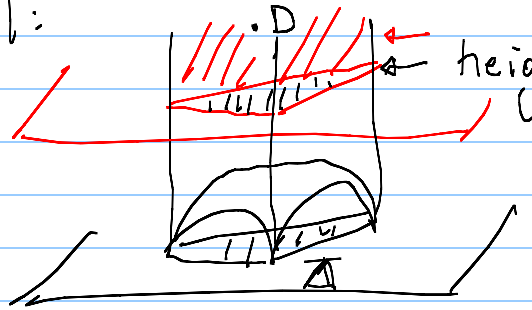
Lemma: $\operatorname{vol}(T(ABCD)) < \infty$



Pf. After chopping off neigh. of ideal vertices we get something cpt which has certainly finite vol.

Enough to show that the red pieces have finite vol:

$\{t=T\}$
$\cap$
$\mathbb{C}\times(0,\infty)$



$$\operatorname{volume}(\operatorname{red}(D)) = \int_\Delta \int_T^\infty d\mathrm{vol}_{g_U}$$

$$d\mathrm{vol}_{g_U} = \frac{1}{t^3}\,dx\,dy\,dt$$

$$= \left[\left(\int_T^\infty \frac{1}{t^3}\,dt\right)\right] \operatorname{Area}(\Delta) < \infty \qquad \square$$

The same pf. shows that ideal n-simplices in $\mathbb{H}^n$ have finite vol.

Dihedral angles

$\alpha+\beta+\gamma=\pi$

Theorem: $\mathrm{vol}(T(\alpha,\beta,\gamma)) = \Lambda(\alpha)+\Lambda(\beta)+\Lambda(\gamma)$

where $\Lambda(\theta) = -\int_0^\theta \log|2\sin(t)|\,dt$ is the Lobachevsky function. $\varnothing$