Hyperbolic Manifolds - Lecture 11

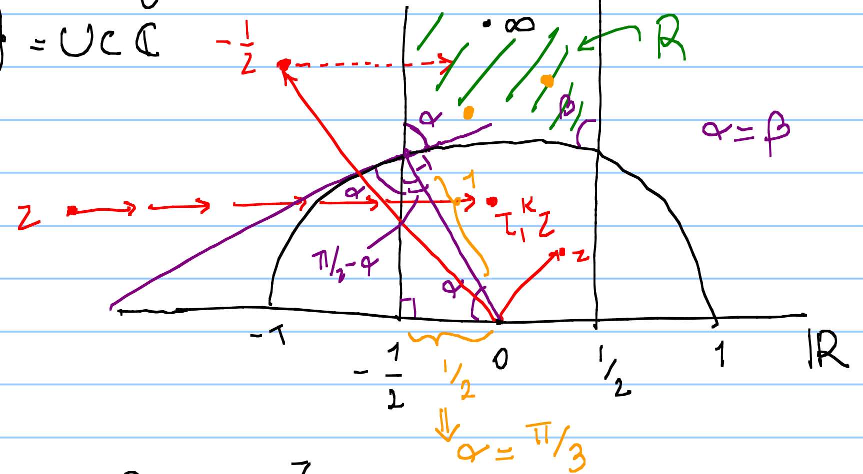
Exercises

(1.4) $PSL_2\mathbb{Z} < PSL_2\mathbb{R} = \mathrm{Isom}^+(U)$ (with $U = \mathbb{H}^2$)

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2\mathbb{R}$ acts on U as

$$z \longrightarrow \frac{az+b}{cz+d}$$

$\{\mathrm{Im} > 0\} = U \subset \mathbb{C} \subset \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$



(1) Show that every pt $z \in U$ can be moved into $\mathcal{R} = \{ z = x + iy \mid x \in [-\frac{1}{2}, \frac{1}{2}], |z| \geq 1 \}$

Pf. $PSL_2\mathbb{Z}$ contains $\tau_1: \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $z \longrightarrow z+1$. Using $\tau_1^{\pm 1}$ several times we can make sure that $\mathrm{Re}(\tau_1^k z) \in [-\frac{1}{2}, \frac{1}{2}]$.

Now: we choose a translate of z inside the vertical strip s.t. the Im part is the largest. Observe that $PSL_2\mathbb{Z}$ contains $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z \longrightarrow -\frac{1}{z}$

and $\mathrm{Im}\left(-\frac{1}{z}\right) = \mathrm{Im}\left(-\frac{\bar{z}}{|z|^2}\right) = \frac{\mathrm{Im}(z)}{|z|^2}$

in particular, if $|z|<1$, then $\mathrm{Im}\left(-\frac{1}{z}\right) > \mathrm{Im}(z)$

$\Rightarrow$ there exists a translate of z in R with larger imaginary part. $\square$

A useful formula: $\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\mathrm{Im}(z)}{|cz+d|^2}$ ($\Rightarrow$ there is a translate with largest Im).

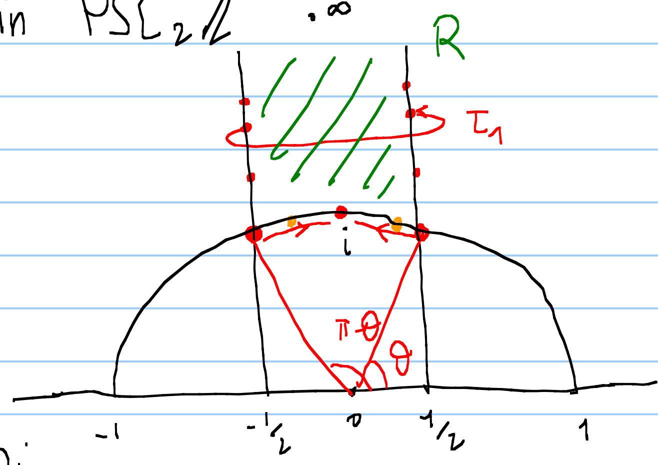
(1') Compute Area(R)

R is a triangle $\Rightarrow$ Area$(R) = \pi - (\alpha + \beta + \gamma)$ where α, β, γ are the angles

(with $\alpha = \pi/3$, $\beta = \pi/3$, $\gamma = 0$)

$= \pi/3$

(1'') Compute which points of $\mathcal{R}$ are identified by motions in $PSL_2\mathbb{Z}$



Some immediate identifications are:

- vertical sides identified by $\tau_1(z) = z+1$
- since S^1 is invariant under $z \to -\frac{1}{z}$ there (might) be identifications on the horizontal side

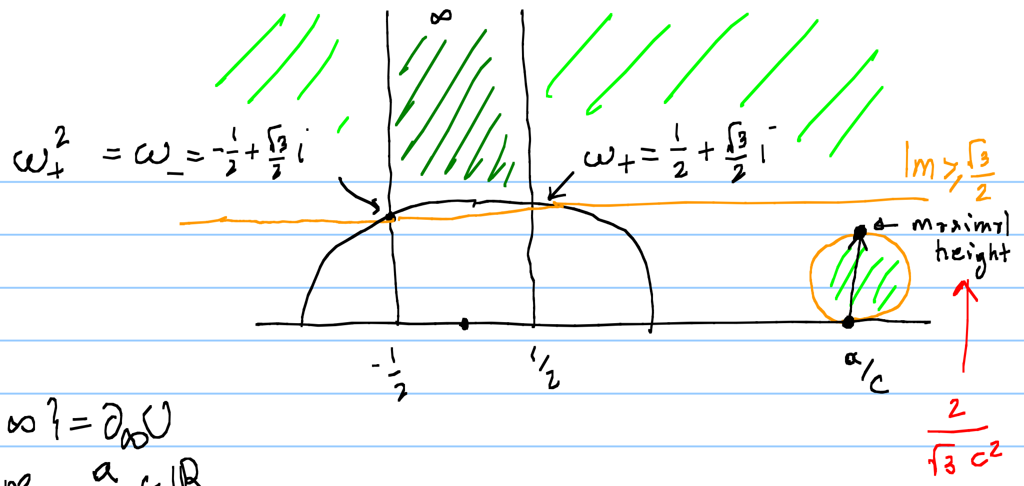
$z = e^{i\theta}$ $\quad -\frac{1}{z} = e^{\pi i} e^{-i\theta} = e^{(\pi-\theta)i}$

What else? $\quad$ $\mathcal{R} \circlearrowleft$ $z \to \frac{az+b}{cz+d}$ $\circlearrowleft \mathcal{R}$? $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2\mathbb{Z}$

Case ①: $c=0 \Rightarrow 1 = ad - bc = ad \Rightarrow a = b = \pm 1 \Rightarrow z \to z \pm b$ $\quad \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix} z \to z \pm b$

None of these identify two pts in $\mathcal{R}$ unless $b = \pm 1$ (and in this case the identifications are on the vertical sides)

Case ② $|c| \geq 2$



$$\infty \rightsquigarrow \frac{a\infty + b}{c\infty + d} = \frac{a}{c}$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \{ \mathrm{Im} = \frac{\sqrt{3}}{2} \}$ = a circle tangent to $\mathbb{R} \cup \{\infty\} = \partial_\infty U$ at the center of the horosphere $\frac{a}{c} \in \mathbb{R}$

useful formula

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) \overset{b}{=} \frac{\mathrm{Im}(z)}{|cz+d|^2} = \frac{\sqrt{3}/2}{|(cx+d)^2 + \frac{3}{4}c^2|} \Rightarrow \text{ it is largest for } cx+d=0$$

$\uparrow$ $\mathrm{Im}(z) = \frac{\sqrt{3}}{2}$

$z = x + i\sqrt{3}/2$

$$\Rightarrow = \frac{2}{\sqrt{3}\, c^2}$$

When $\frac{2}{\sqrt{3} c^2} < \frac{\sqrt{3}}{2}$?

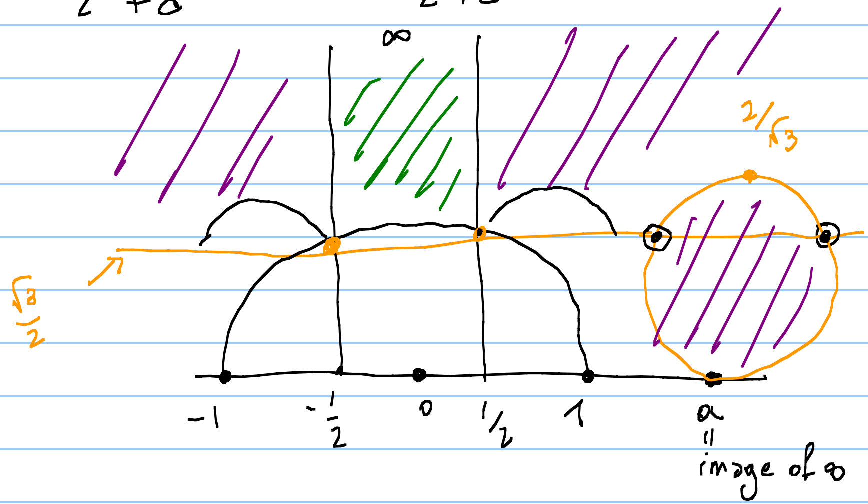
$\Leftrightarrow c^2 > \frac{4}{3} \Leftrightarrow |c| \geq 2$.

$\Rightarrow$ In case ② we do not get any identification between pts in $\mathcal{R}$.

Case ③ $c = \pm 1$, let us assume $c=1$

$$\Rightarrow \frac{az+b}{cz+d} = \frac{az+b}{z+d} = a + \frac{bc-ad}{z+d} = a - \frac{1}{z+d}$$

$bc - ad = -\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -1$



Case ③.1. If $|a| \geq 1$, then the purple disk is disjoint from $\mathcal{R}$ (see the picture)

$\Rightarrow$ no other identifications (except $\omega_+ \simeq \omega_-$)

Case ③.2

$c=1$
$a=0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ 1 & d \end{pmatrix}$$

$\det = 1 \Rightarrow b = -1$

$$= \begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix} : z \longrightarrow -\frac{1}{z+d}$$

$d=0$ already know $z \to -\frac{1}{z}$

$d=1$

$$-\frac{1}{z+1}$$

$\infty \to 0$

$-1 \to \infty$

$1 \to -\frac{1}{2}$

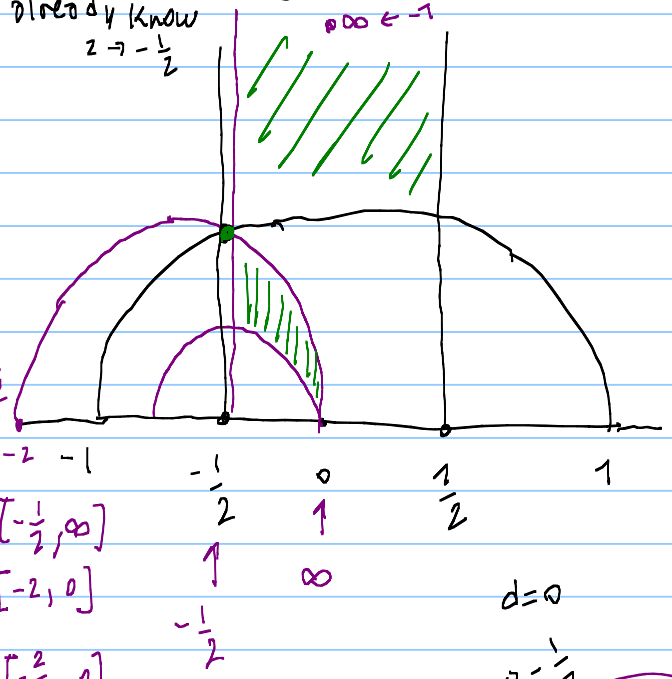
$\frac{1}{2} \to -\frac{2}{3}$

$-\frac{1}{3} \to -2$

$[-1,1] \to [-\frac{1}{2},\infty]$

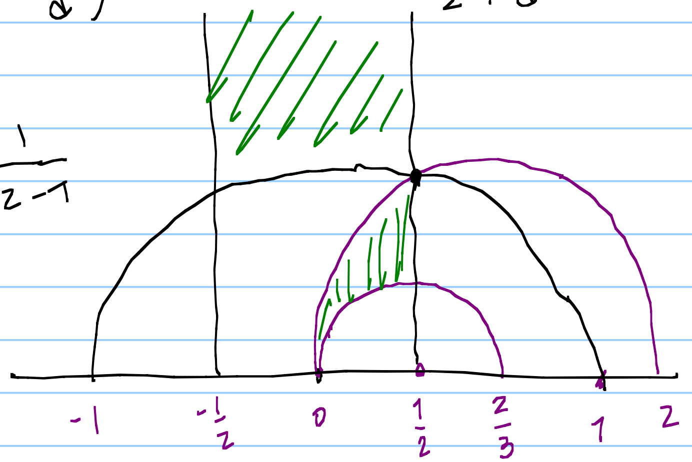
$[-\frac{1}{2},\infty] \to [-2,0]$

$[\frac{1}{2},\infty] \to [-\frac{2}{3},0]$



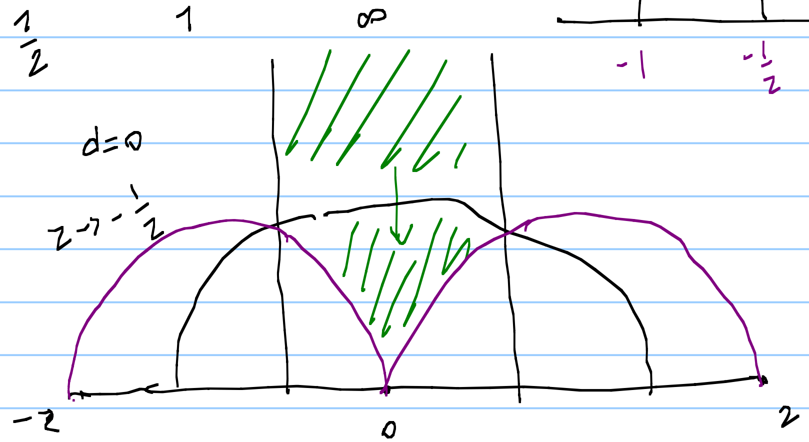
$d=-1$

$$z \to -\frac{1}{z-1}$$



$d=0$

$z \to -\frac{1}{z}$

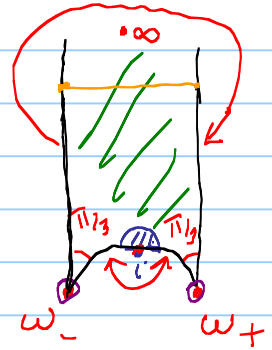


$d = \pm 1$

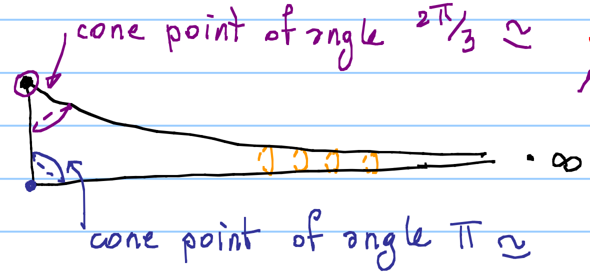
$\Rightarrow$ no new identifications

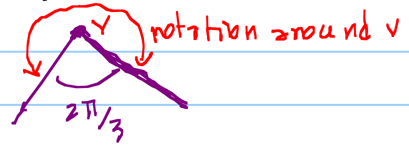
Claim: $|d| > 1 \Rightarrow$ the image of R is disjoint from $R \Rightarrow$ no new identifications.

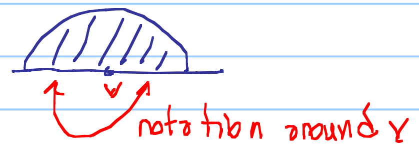
Thus R is "almost" a fundamental domain for $PSL_2\mathbb{Z} \curvearrowright U$ and we computed the identifications on the sides



$\Rightarrow U / PSL_2\mathbb{Z} \simeq$

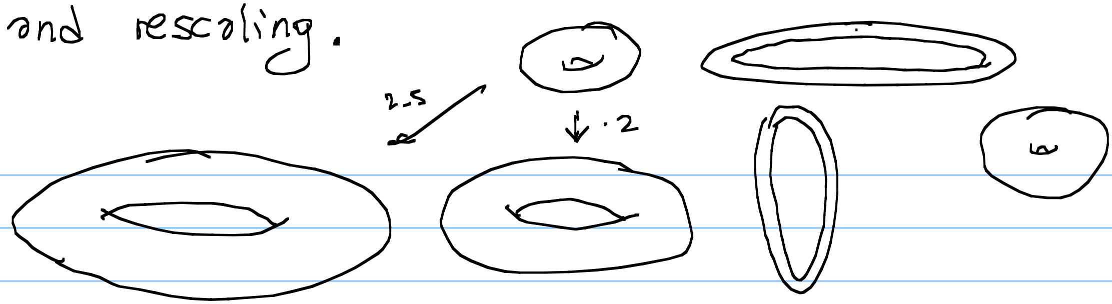






↑ space of orbits

↑ This is called the modular curve, it can be identified with the moduli space of flat tori up to isometries and rescaling.



(2) Consider $PSL_2\mathbb{Z} \xrightarrow{\phi_n} PSL_2\mathbb{Z}/n\mathbb{Z} = \left\{ \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \;\middle|\; \begin{matrix} \bar{a},\bar{b},\bar{c},\bar{d} \in \mathbb{Z}/n\mathbb{Z} \\ \bar{a}\bar{d} - \bar{b}\bar{c} = 1 \pmod{n} \end{matrix} \right\}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \pmod{n}$ ↖ finite

$\ker \Phi_n = \Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2\mathbb{Z} \;\middle|\; \begin{matrix} a \equiv d \equiv 1 \pmod{n} \\ b \equiv c \equiv 0 \pmod{n} \end{matrix} \right\} < PSL_2\mathbb{Z}$

($\Gamma(n)$ called <u>principal congruence subgroup</u>)

Show that ϕ_n is surjective ($\Rightarrow [PSL_2\mathbb{Z} : \Gamma(n)] = |PSL_2\mathbb{Z}/n\mathbb{Z}|$?)

Pick $\begin{pmatrix} \bar a & \bar b \\ \bar c & \bar d \end{pmatrix} \in PSL_2 \mathbb{Z}/n\mathbb{Z}$ want to lift it to $PSL_2\mathbb{Z}$

can lift to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, but it might be that $\det = ad - bc \neq 1$

Notice that there is a set of matrices that are easily liftable

$$\begin{pmatrix} 1 & \bar b \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in PSL_2\mathbb{Z}$$

$$\begin{pmatrix} 1 & 0 \\ \bar b & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \bar 1 \\ \overline{-1} & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Recall that $\forall X \in M_{2\times 2}(\mathbb{Z})$ we can find $A, B \in PSL_2\mathbb{Z}$ (compositions of elementary matrices of the type [the three above]) s.t. $AXB = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$

Apply this to $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \bar{A} \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \bar{B} = \begin{pmatrix} \bar{u} & 0 \\ 0 & \bar{v} \end{pmatrix} \pmod n$

$\Rightarrow uv \equiv 1 \pmod n$

$\Rightarrow$ it is enough to lift $\begin{pmatrix} \bar{u} & 0 \\ 0 & \bar{v} \end{pmatrix}$ to $PSL_2\mathbb{Z}$

$\Rightarrow uv = 1 + rn \Rightarrow \begin{pmatrix} u & r \\ n & v \end{pmatrix} \in PSL_2\mathbb{Z}$

$\Rightarrow$ can obtain $\begin{pmatrix} \bar{u} & \bar{r} \\ 0 & \bar{v} \end{pmatrix}$

$\Rightarrow$ by applying some $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ we can get $\begin{pmatrix} \bar{u} & 0 \\ 0 & \bar{v} \end{pmatrix}$. $\square$

(4) Show that $\Gamma(n)$ does not contain elliptics for $n \geq 4$

$$X = \begin{pmatrix} 1+\alpha n & \beta n \\ \gamma n & 1+\delta n \end{pmatrix}$$

Know that X is elliptic $\iff$ $|\mathrm{tr}\, X| < 2$

$|\mathrm{tr}\, X| = |2 + (\alpha+\delta)n|$

if $\alpha + \delta = 0 \implies |\mathrm{tr}\, X| = 2$ OK

If $\alpha+\delta \neq 0 \implies |\alpha+\delta| \geq 1$

$|2+(\alpha+\delta)n| \geq |(\alpha+\delta)n| - 2$ triang ineq.

$\geq n - 2$

If $n \geq 4$ ≥ 2.

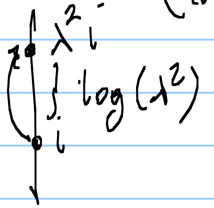
(5) Loxodromics in $\Gamma(n)$

X is loxodromic $\iff |\mathrm{tr} X| > 2$

In this case X is conj in $PSL_2\mathbb{R}$ to $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $\lambda \neq 1$

Such a matrix has the same transl. dist as X (because they are conj) (and the same trace)

$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} z = \lambda^2 z$ $\qquad d\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right) = \log \lambda^2 = 2 \log \lambda$



$\Rightarrow$ $\cosh(d(X)/2) = \cosh(\log \lambda) = |\mathrm{tr}(X)|/2$

(6) Show that the minimal transl dist of lox. in $\Gamma(n)$ $\uparrow \infty$ as $n \to \infty$ with a very explicit rate

$$\cosh\left(d\left(\underset{\begin{pmatrix} 1+\alpha n & \beta n \\ \gamma n & 1+\delta n \end{pmatrix}}{\overset{\parallel}{X \in \Gamma(n)}} \right)/2 \right) = |tr(X)|/2 = \frac{|2+(\alpha+\delta)n|}{2} \geq \frac{n-2}{2}$$

since X is lox $|\alpha+\delta| \neq 0$ (otherwise X is parabolic)

$\Rightarrow \min_{X \in \Gamma(n)} d(X)$ diverges as $2\operatorname{arccosh}\left(\frac{n-2}{2}\right)$. ⊠