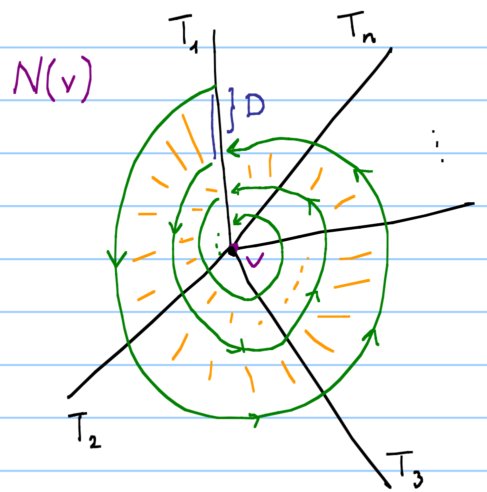
Hyperbolic Manifolds - Lecture 10

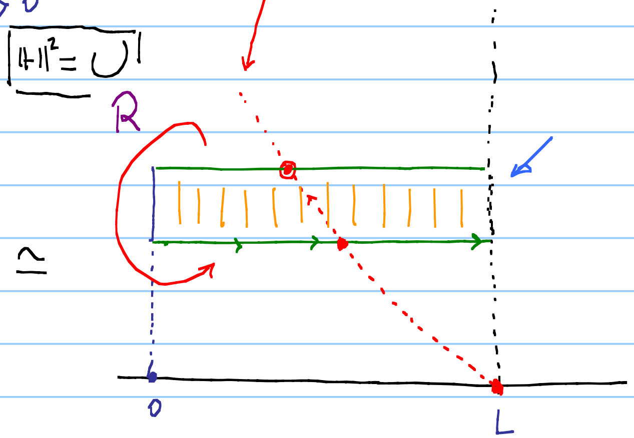
Review / Comments on completions:

$D = d_1 + \dots + d_n > 0$



$\|H\|^2 = U$

· homothety with center L and factor e^D



total length of green horocycle path $< \infty$

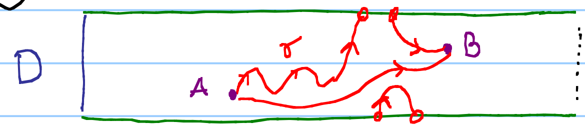
$N(v)$ incomplete vertex $D > 0$

① Want to compute the completion $\overline{N(v)}$ with respect to the path metric

② Can write $N(v) = R / \sim$

homothety

③ path metric on $R/\sim$

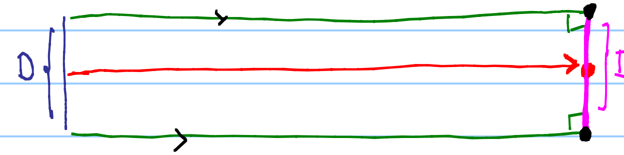


$d_R(A,B) = \inf\{\ell(\gamma) \mid \gamma \text{ joining } A \text{ and } B\}$

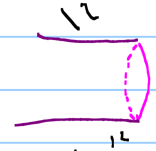
Remark: $d_{N(v)} \cong d_{\Sigma}|_{N(v)}$ have the same Cauchy sequences if the neigh is small enough.

$\rightarrow (R/\sim, d_R) \underset{\text{isometric}}{\simeq} (N(v), d_{N(v)})$

④ Completion of $R/\sim$

$\overline{N(v)} = \overline{R/\sim} =$  $/\sim$

⑤ In particular:



(i) $\overline{N(v)}$ is homeomorphic to $S^1 \times [0,1]$

$\Rightarrow \overline{\Sigma}$ is a surface with boundary

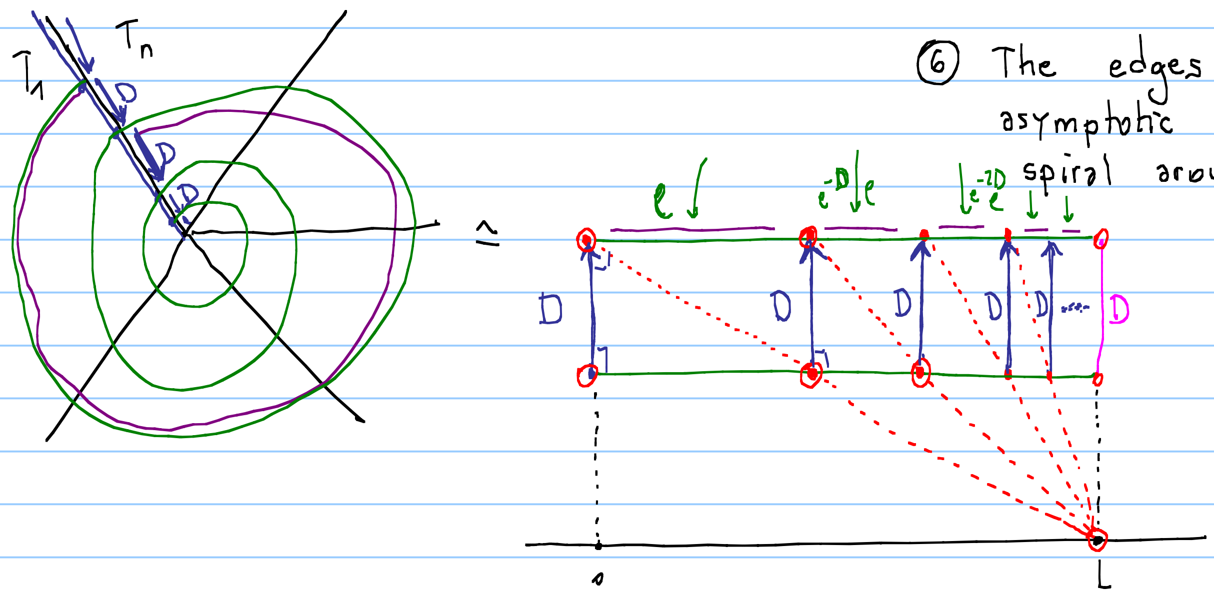
(ii) The Riemannian metric on $N(v) = R/\sim$ extends to a Riemannian metric on $\overline{N(v)} = \overline{R/\sim}$

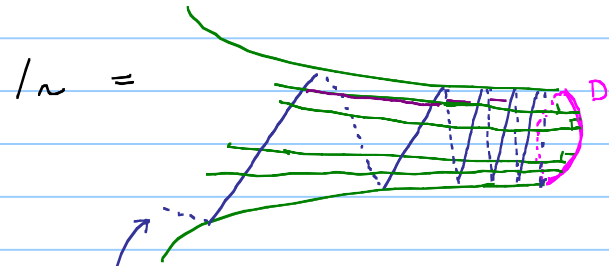
(iii) With respect to the Riem. metric, the boundary is totally geodesic

(iv) The natural distance on $\overline{R/\sim}$ is the path distance induced by the Riemannian metric

(v) The length of the boundary is $D = |d_1 + \dots + d_n|$

⑥ The edges of the triangles that are asymptotic to the incomplete vertex v spiral around the extra geodesic in the completion





edge asymp. to v spirals around pink geodesic boundary

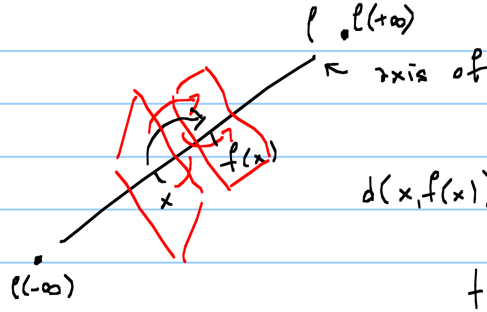
Exercises

① Barycenters and convexity of distances

② North - South dynamics and Ping-Pong

③ $PSL_2\mathbb{Z}$

North-south dynamics of loxodromic motions



$f \in \mathrm{Isom}(\mathbb{H}^n)$ loxodromic

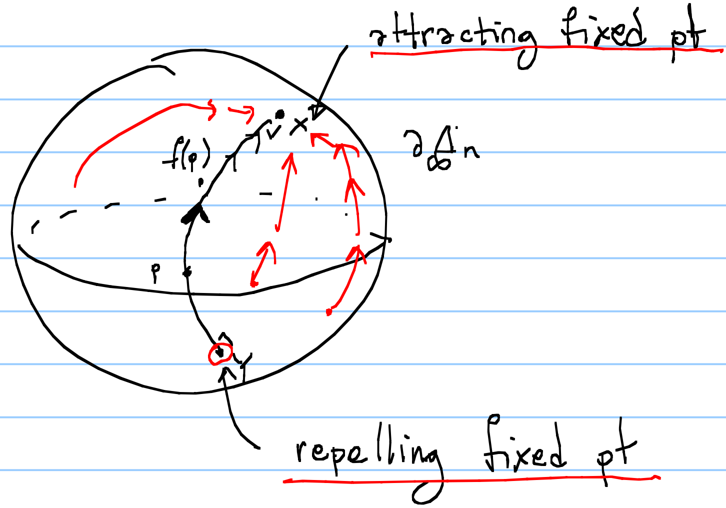
$d(x, f(x)) = \tau(f) > 0$

translation distance of f

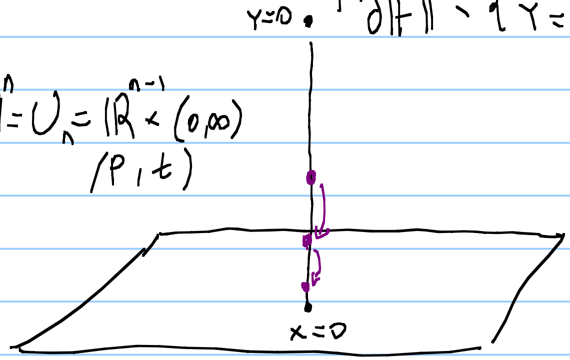
Want: Understand the dynamics of $f \curvearrowright \partial_\infty \mathbb{H}^n$

$\mathrm{Fix}(f) = \{x, y\}$

(1) Show that $f^n\big|_{\partial \mathbb{H}^n \setminus \{y = \text{repelling}\}} \xrightarrow{\text{unif. on cpt sets}} \text{const} \equiv x$



Pf. $\mathbb{H}^n = U_n = \mathbb{R}^{n-1} \times (0,\infty)$
(p, t)



Up to isometries we can assume that $x = 0$, $y = \infty$

By the normal form of loxodromic motions

$$f(p,t) = \underline{(\lambda A p, \lambda t)} \qquad \text{for some} \quad A \in O(n-1)$$

$\lambda > 0 \qquad \lambda \neq 0$

$\underline{\lambda < 1}$ because 0 is the attracting fixed pt

$$f\big|_{\partial_\infty \mathbb{H}^n = \partial_\infty U_n = \mathbb{R}^{n-1} \cup \{\infty\}} = \lambda A$$

If K is any cpt set in $\mathbb{R}^{n-1}$ and U is any neigh of 0
then for every n large enough $f^n K \subset U$

$\Rightarrow f^n \longrightarrow \text{const} \equiv 0$ unif. on cpt sets of $\mathbb{R}^{n-1}$ $\emptyset$

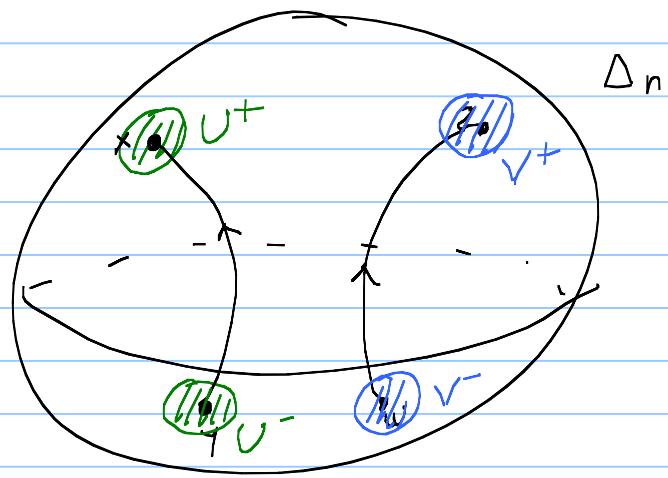
Similarly $f^{-n} \longrightarrow \text{const} \equiv \infty$ on $(\mathbb{R}^{n-1} \cup \{\infty\}) \setminus \{0\}$ $\emptyset$

(2) $f, g \in \mathrm{Isom}(\mathbb{H}^n)$ loxodromic

$\mathrm{Fix}(f) = \{x, y\} \cap \mathrm{Fix}(g) = \{w, z\} = \emptyset$
by point (1)

Observe that we can take n large enough so that

$$\begin{cases} f^m(\partial_\infty \mathbb{H}^{\overset{K}{\overset{\shortparallel}{n}}} - U^-) \subset U^+ \\ f^{-m}(\partial_\infty \mathbb{H}^n - U^+) \subset U^- \\ g^m(\partial_\infty \mathbb{H}^n - V^-) \subset V^+ \\ g^{-m}(\partial_\infty \mathbb{H}^n - V^+) \subset V^- \end{cases} \quad \forall\, m \geq n$$



Now we play ping-pong with f^m, g^m for m large enough

(3) Consider the subgroup G of $\mathrm{Isom}(\mathbb{H}^n)$ generated by f^m, g^m

Show that G is a non-abelian free group on two generators $G = \mathbb{F}_2$

In other words no composition of the form $\underbrace{f^{r_1} g^{s_1} f^{r_2} g^{s_2} \dots f^{r_k} g^{s_k}}_{= w}$

can be the identity.

let us assume that $m=1$ is enough to guarantee the separation properties of point (2)

Consider w as above, we want to check that $w \neq 1$

assume that $s_k > 0$ $\qquad w(V^+) = f^{r_1} g^{s_1} \dots f^{r_k} \underbrace{g^{s_k}(V^+)}_{\subset V^+}$

$$\subset f^{r_1} g^{s_1} \cdots \underbrace{f^{r_k}(V_+)}_{\subset\, U^+ \cup U^-}$$

$$\subset f^{r_1} g^{s_1} \cdots \underbrace{g^{s_{k-1}}(U^+ \cup U^-)}_{\subset\, V^+ \cup V^-}$$

$$\vdots$$

$$\subset f^{r_1}(V^+ \cup V^-) \subset U^+ \cup U^-$$

In conclusion, after ping-pong, we see that
$w(V^+) \subset U^+ \cup U^- \Rightarrow w \neq 1$ $\cancel{\emptyset}$

(c) Show that every element in $G \setminus \{1\}$ is loxodromic

Thm (Brouwer Fixed Point theorem): Every $f: \Delta_n \to \Delta_n$ continuous has a fixed point.

Want to apply to find fixed pts for $w \in G \setminus \{1\}$.

(We choose $U^\pm, V^\pm$ to be disks) $f^m(U^+) \subset U^+$
$f^{-m}(U^-) \subset U^-$

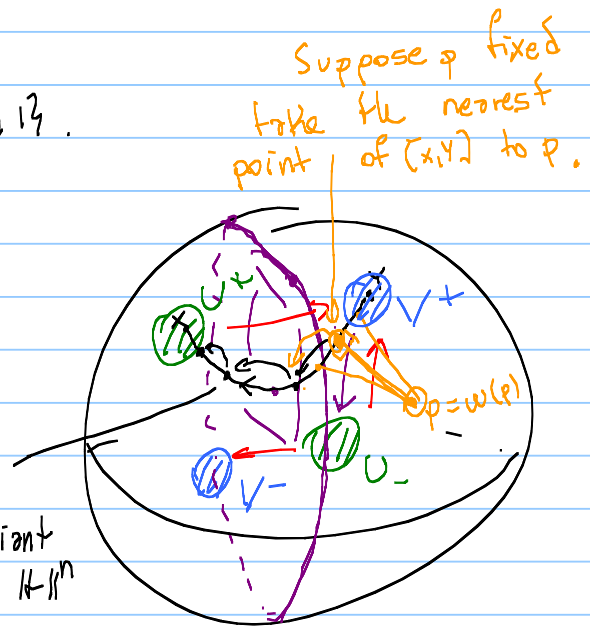
$$w = f^{r_1} g^{s_1} f^{r_2} g^{s_2} \cdots f^{r_k} g^{r_k}$$

Suppose $r_1 > 0$

$$w(U^+) = f^{r_1} g^{s_1} \cdots f^{r_k} g^{r_k}(U_+) \subset \cdots \subset U^+$$

(if $r_1 < 0$ we choose $w(U^-)$)

This is an invariant line in $\mathbb{H}^n$



$\Rightarrow$ By Brouwer, w has a fixed point in U^+

$\Rightarrow$ We do the same for $w^{-1} = g^{-s_k} f^{-r_k} \cdots g^{-s_1} f^{-r_1}$

suppose $-s_k > 0 \Rightarrow w^{-1}$ (for the same argument) has a fixed point in V^+

<u>OK</u> $\rightarrow$ If w translates point of $\ell = [x, y]$ by some positive amount then w must be loxodromic!

Suppose that this is not the case, that is $w|_{\ell=[x,y]} = Id$.

$\Rightarrow$ every hyperplane $H \perp \ell$ is invariant

$\Rightarrow$ look at trace $H \cap \partial_\infty \mathbb{H}^n$, it is also invariant under w

we want to obtain a contradiction by choosing H carefully

$w(\partial H) \subset \partial H$

$s_k < 0$

$$\omega = f^{r_1} g^{s_1} \cdots f^{r_k} \underbrace{g^{s_k}\left(\partial_\infty \mathbb{H}^n - V^+\right)}_{\subset V^-}$$

$$\subset \cdots \subset U^+$$

$\Rightarrow$ we only have to choose H_∞ that ∂H is not in $V^+ \cup U^+$

$\varnothing$