Hyperbolic Manifolds - Lecture 1

Titolo nota 03/11/2020

Info:

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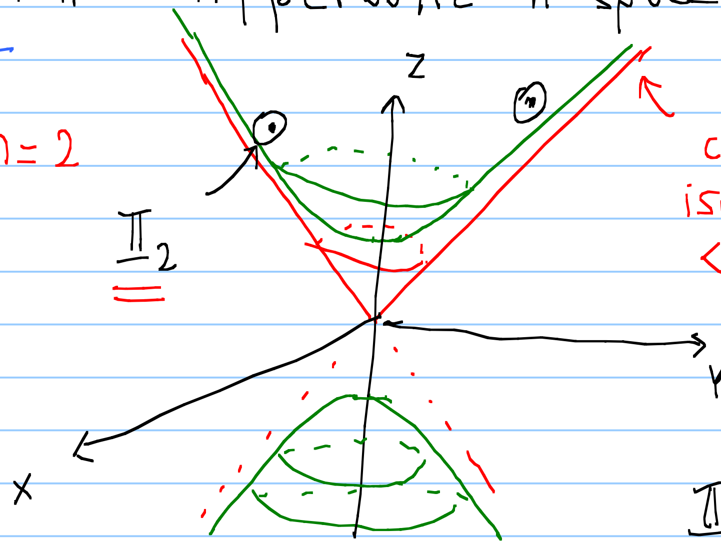
- webpage: mathi.uni-heidelberg.de/~gviaggi/hypgeo2020.
- schedule: Tue 11:15 – 12:50
  Wed 16:15 – 17:50
- notes from the lecture uploaded to webpage.
- exercise sessions: Sporadic. After we have seen some theory, we will pause and discuss exercises
- prerequisites: some basic topology and differential geometry

Introduction

$\mathbb{H}^n$ = hyperbolic n-space, Riemannian n-mfd of const curvature -1, simply connected, complete

$n=2$



cone of isotropic vectors $\langle x, x\rangle_{(2,1)} = 0$

$\mathbb{R}^3$, $\langle X, Y\rangle_{(2,1)} = X_1Y_1 + X_2Y_2 - X_3Y_3$

$\mathbb{R}^3 \quad \mathbb{R}^3$

$$= {}^tX \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} Y$$

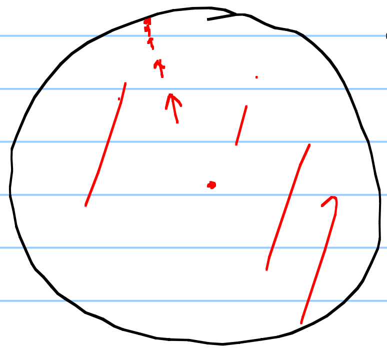
signature $(2,1)$

$$\mathbb{H}_2 = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} \langle x, x\rangle = -1 \\ x_3 > 0 \end{array} \right\}$$

hyperboloid model of hyperbolic 2-space

In the first lectures we will familiarize with this object and its geometry of <u>locally symmetric space</u>

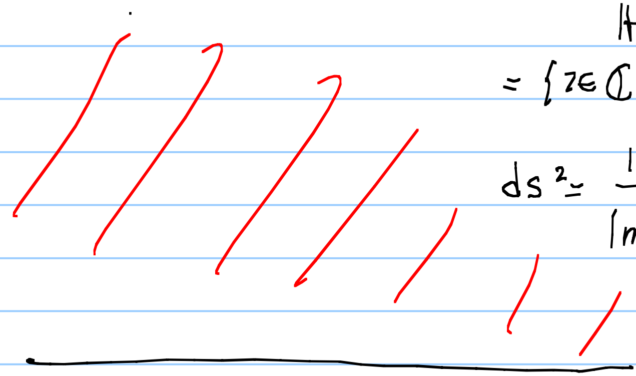
Other shapes of $\mathbb{H}^2 = \mathbb{I}_2$



$$\Delta = \{ z \in \mathbb{C} \mid |z| < 1 \}$$

$$ds^2 = \frac{4}{(1-|z|^2)^2}\, g_{Eucl}$$

Poincaré disk model of $\mathbb{H}^2$


$\mathbb{R}$

$\mathbb{H}^2 = U$

$= \{ z \in \mathbb{C} \mid Im(z) > 0 \}$

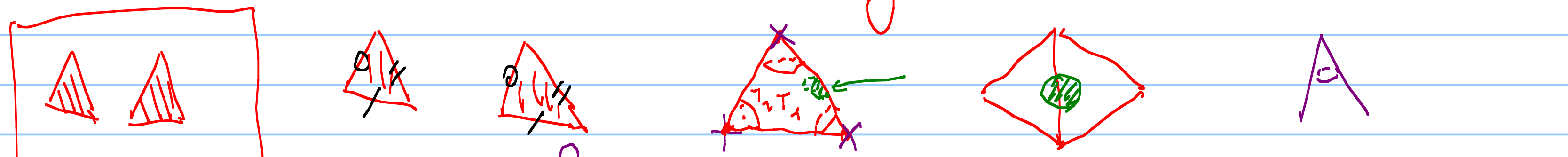
$$ds^2 = \frac{1}{Im(z)^2}\, g_{Eucl.}$$

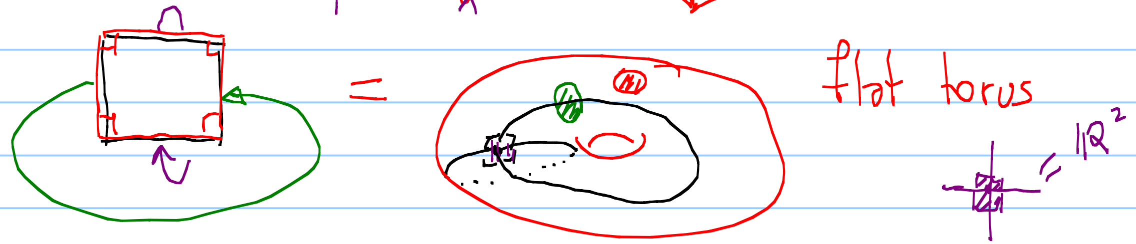
$\mathbb{H}^2 = \mathbb{I}_2 = \Delta = U$

they are all isometric to each other!

Other hyperbolic n-mfds can be thought as obtained by cutting from $\mathbb{H}^n$ some pieces and the reassembling them together to form a Riemannian mfd.

Example (Euclidean): How to build a 2-dim object that locally looks like $\mathbb{R}^2$?
Take a piece of paper, a pair of scissors, and use some glue!



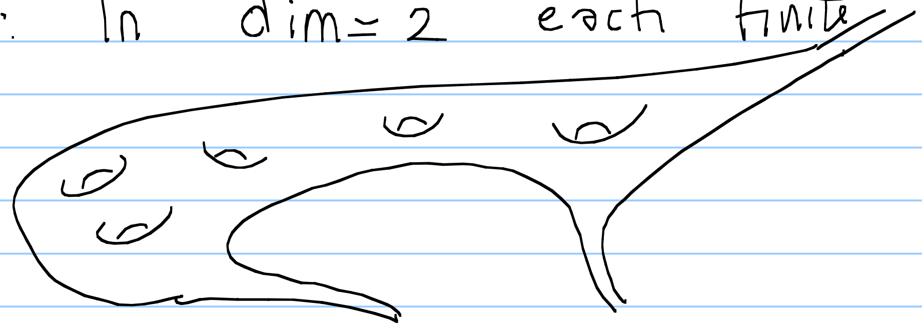


The same idea of cut and glue applies to an $\mathbb{H}^2$- piece of paper!

A major goal of the class is to construct finite volume or even closed hyperbolic manifolds.

We will explore the cut and glue techniques in dim = 2 and 3 (this is wher the cut and glue construction is most powerful and flexible).

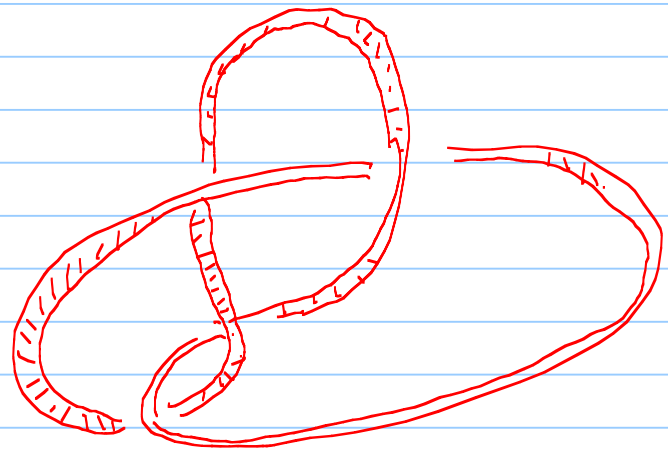
For example: In dim = 2 each finite type surface with $\chi < 0$

(like this one: ) can be given plenty of hyperbolic metrics!

This leads to Teichmüller theory which studies the deformation spaces of hyperbolic structures on surfaces and their natural geometries. It is a very active and modern research area.

In dim $= 3$ we have the following example which can be obtained with cut and glue construction (Thurston, Riley, Jørgensen)

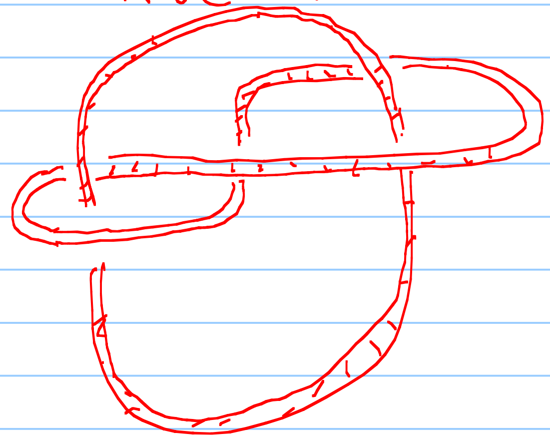
figure 8 knot:

$K =$  $\subset \mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$

Thurston: $S^3 \setminus K$ admits a complete finite volume hyperbolic structure.

We will prove this in the class!

The same is true for

$W =$ 

Whitehead link

$S^3 \setminus W$ has a complete finite vol. hyp. structure!

In fact many 3-mfds admit hyperbolic structures as discovered by Thurston. The study of the geometry of hyperbolic 3-mfds is again an active research area.

In higher dim ≥ 4 the cut and glue construction is much more difficult to handle and also less powerful.

Nontheless closed or finite volume hyperbolic n-mfds exist for each $n \geq 2$!

In order to construct them we will use <u>Arithmetic constructions</u>.

Idea: $\underset{\text{complete hyp. mfd}}{M} = \overset{\text{univ. covering}}{\tilde{M}} / \pi_1 M \curvearrowright \tilde{M}$

deck group action is by isometries, it is free (1) and it is properly discontinuous (2)

$$\tilde{M} \;\|\; \mathbb{H}^n / \pi_1 M < \mathrm{Isom}(\mathbb{H}^n)$$

Viceversa: Suppose that we have $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ which satisfies:

(1) torsion free $\alpha^n \neq 1$

(2) discrete (with respect to the topology of $\mathrm{Isom}(\mathbb{H}^n)$)

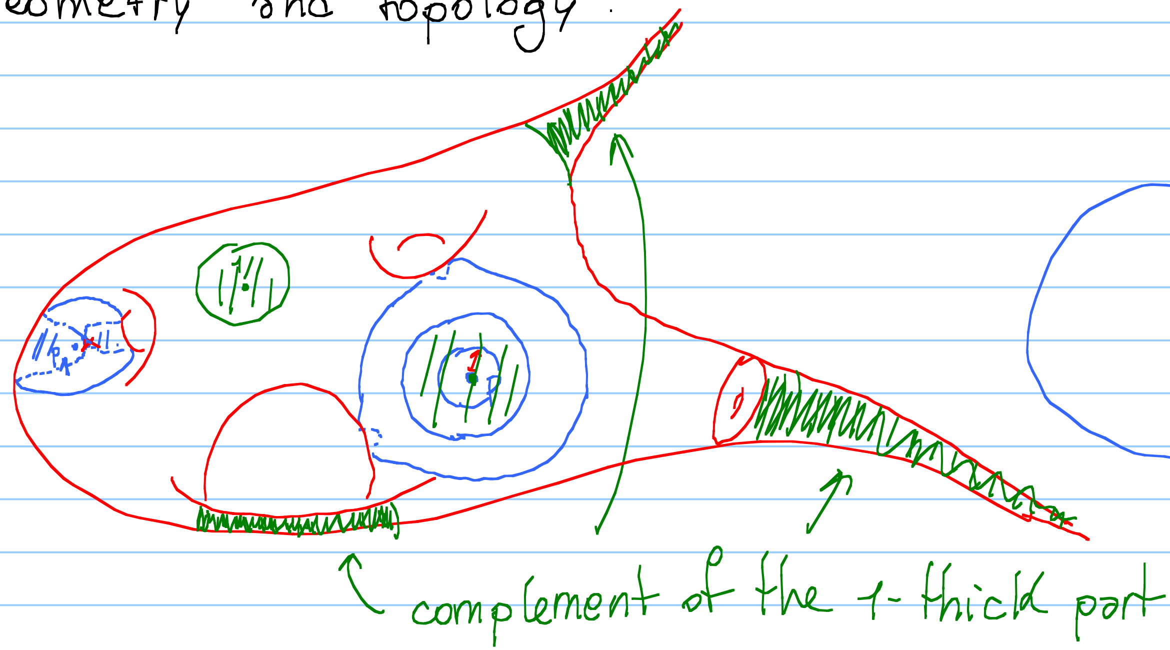
Then $\mathbb{H}^n/\Gamma$ is a complete hyperbolic n-mfd.

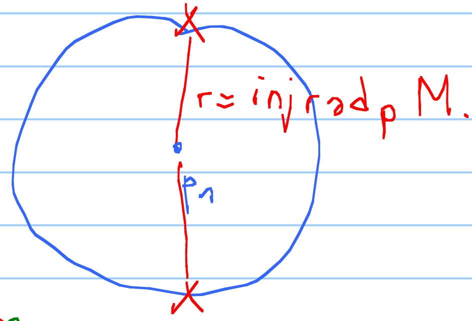
We will produce finite volume hyperbolic n-mfds with $n \geq 2$ by finding suitable torsion free discrete subgroups of $\mathrm{Isom}(\mathbb{H}^n)$. This is what the arithmetic constructions do for us.

Now that we know that finite vol. hyp. mfds exist, and are abundant, we want to try to understand better their geometry and topology.

M

$\varepsilon = 1$



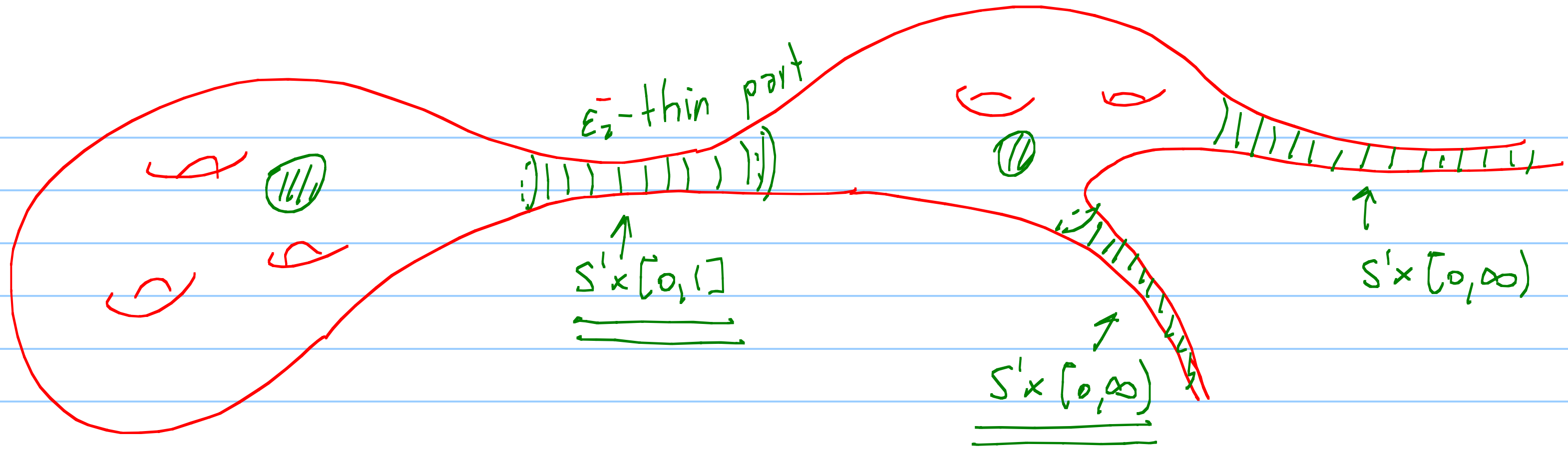


We begin studying M around a point $p \in M$

At a small scale, depending on the point p, we just see a small ball of $\mathbb{H}^n$. Then we start inflating the ball will intersect itself (the mfd has finite vol...). The radius at which this happens is called the injectivity radius of M at p. It governs the scale at which M looks like $\mathbb{H}^n$ around p.

Fix a scale $\varepsilon > 0$. We can always decompose M into a ε-thick part, where each point has a ball around it of radius $= \varepsilon$ which looks like a ball of the same radius in $\mathbb{H}^n$

The Margulis Lemma will tell us that there exist a universal scale $\varepsilon_n > 0$, only depending on the $\dim = n$, such that the complement of the ε_n-thick part (called ε_n-thin part) of a hyperbolic n-mfd has a very simple geometry and topology!



Using the thick-thin decomposition it is possible to relate the "topological complexity" of a finite vol. hyp. mfd M to its volume, meaning that it is always possible to construct a simplicial complex K_M which is homotopy equivalent to M and has a number of simplices uniformly proportional to the volume of M.

[In a certain sense, the volume of a finite volume hyperbolic n-mfd, captures the topological complexity of the manifold.]

This leads us to the last part of the class where we will see how volumes are at the base of one of the striking rigidity phenomena of locally symmetric spaces

Mostow, Gromov-Thurston: M, N closed orientable hyperbolic n-mfds $\boxed{n \geq 3}$

$f : M \to N$ continuous map with $\deg(f) \neq 0$

Then

$$(*) \quad \mathrm{vol}(M) \geq |\deg(f)| \,\mathrm{vol}(N)$$

In the class we will prove this one! — and if equality holds in $(*)$ then f is homotopic to a Riemannian covering.

Rmks: • This is really a phenomenon of $\dim \geq 3$, because on a closed orientable surface there are plenty on non-equivalent hyp. structures and still $\mathrm{vol}(\Sigma) = 2\pi|\chi(\Sigma)|$

$$(\Sigma, \sigma) \overset{\mathrm{Id}}{=} (\Sigma, \sigma') \qquad \mathrm{vol} = \mathrm{vol}$$

• Also notice that if $\boxed{f : M \to N}$ is a homotopy equivalence with homotopic inverse $\boxed{g : N \to M}$

$$fg \simeq \mathrm{Id}_N \qquad gf \simeq \mathrm{Id}_M$$

we have $|\deg(f)| = |\deg(g)| = 1$

and also
$$\mathrm{vol}(M) \geq |\deg(f)|\,\mathrm{vol}(N) = \mathrm{vol}(N)$$

$$\mathrm{vol}(N) \geq |\deg(g)|\,\mathrm{vol}(M) = \mathrm{vol}(M)$$

So that $\mathrm{vol}(M) = \mathrm{vol}(N)$ and we have equality in the Thm.

$\Rightarrow$ f is homotopic to a Riem. cov., but a Riem covering of $|\deg = 1$ is an isometry! $\Longrightarrow$ M and N are isometric!

Corollary: If M, N closed, orientable, n-mfds $n \geq 3$ and $M \simeq N$ homotopy equiv Then M is isometric to N.

- Since hyp. mfds are aspherical, $f: M \to N$ is a homotopy equiv $\Leftrightarrow$ $f_*: \pi_1 M \longrightarrow \pi_1 N$ is an isomorphism.

Corollary: M, N closed, orientable, n-mfds, $n \geq 3$ If $\pi_1 M \simeq \pi_1 N$ Then they are isometric.