HYPERBOLIC MANIFOLDS

Introduction

In this class we will study hyperbolic manifolds, that is manifolds that are locally modeled on the hyperbolic spaces $\mathbb{H}^n$. These objects are interesting from several points of view and we will focus on the following two aspects:

(i) They appear everywhere in geometry and topology in dimensions 2 and 3.
(ii) They display one of the striking rigidity phenomena of locally symmetric spaces: Mostow rigidity.

Mostow rigidity. A major goal of the class will be the proof of the following theorem due to Mostow and Gromov-Thurston

Theorem (Mostow, Gromov-Thurston)**.** *Let M and N be closed orientable hyperbolic n-manifolds with $n \geq 3$. Let $f : M \to N$ be a continuous map of non-zero degree. Then*

$$\mathrm{vol}(M) \geq |\deg(f)| \,\mathrm{vol}(N)$$

with equality if and only if f is homotopic to a covering map.

This is a strengthening of the following result due to Mostow

Corollary (Mostow rigidity)**.** *Let M and N be closed orientable hyperbolic n-manifolds with $n \geq 3$. Suppose that M is homotopy equivalent to N, then M and N are isometric.*

Some structure. Another goal of the class is to discuss some general structure theory of finite volume hyperbolic manifolds.

We will see that they admit a canonical decomposition in thin pieces with a simple topology and thick parts. This is a consequence of the Margulis Lemma and is an important tool to study these objects.

A topological application is the following tamensess result

Theorem. *Every complete finite volume hyperbolic n-manifold M is diffeomorphic to the interior $\mathrm{int}(N) := N - \partial N$ of a compact manifold N with the property that ∂N carries a flat metric.*

Examples. Constructing finite volume hyperbolic n-manifolds is, in general, a non-trivial task. In the class we will spend some time discussing low dimensional ($n = 2, 3$) hyperbolic manifolds treating them as rich source of examples. However, we will not explore in any detail central topics surrounding them, such as Teichmüller theory or the Thurston's geometrization program.

When $n = 2$ or 3, there are explicit and elementary tools to construct finite volume hyperbolic manifolds: They are based on an the idea of assembling simple building blocks to create more complicated objects. This idea leads, for example, to the following

Theorem. *The space $\mathcal{T}(\Sigma_{g,b})$ of marked hyperbolic structures on a surface of finite type $\Sigma_{g,b}$ with $2g + b - 2 > 0$ and $b > 1$ is homeomorphic to $\mathbb{R}^d$ where $d = 6g + 2b - 6$.*

In the same spirit we will see that, assembling simple pieces, we can also create interesting examples also in dimension $n = 3$. For example:

Theorem (Riley,Thurston)**.** *The figure-8 knot complement has a unique complete hyperbolic metric of finite volume.*

Concerning higher dimensions $n \geq 4$, a very abstract recipe, namely, arithmetic constructions, ensures that there are finite volume and even closed hyperbolic n-manifolds for every $n \geq 4$.

Organization. The class is divided into five parts:

(1) Get acquainted of $\mathbb{H}^n$ and manifolds locally modeled on it.
(2) Construction of examples: $\mathbb{H}^2$ and hyperbolic surfaces.
(3) Margulis Lemma and thick-thin decomposition.
(4) Construction of examples: $\mathbb{H}^3$ and hyperbolic 3-manifolds.
(5) Mostow rigidity in the closed case.

Prereqisites. Some knowledge of the following would help:

i. Basic topology (manifolds, fundamental group, covering spaces)
ii. Basic differential geometry (Riemannian metric, connection, curvature).

We will try to keep the discussion as elementary as possible.

References

[1] R. Benedetti, C.Petronio, *Lectures on hyperbolic geometry.*
[2] B. Martelli, *An introduction to geometric topology.*