HYPERBOLIC MANIFOLDS - EXERCISES

1. Exercises - November 20, 2020

Exercise 1.1 (Barycenters). We discuss different notions of barycenters in $\mathbb{H}^n$. Let $\mathcal{F} = \{x_1, \cdots, x_m\} \subset \mathbb{H}^n$ be a finite set of points.

(1) First notice that in $\mathbb{H}^n = \mathbb{I}_n \subset \mathbb{R}^{n,1}$ we can take $x := x_1 + \cdots + x_m$. Observe that x is always time-like. Use this to prove that every finite order isometry $f \in \mathrm{Isom}(\mathbb{H}^n)$ is elliptic.

(2) Let $P \subset \mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$ be a convex polygon with vertices $p_1, \cdots, p_m$ (that is, a convex region of $\mathbb{H}^2$ bounded by the closed concatenation of consecutive geodesic segments $p_1p_2, p_2p_3, \cdots, p_{m-1}p_m, p_mp_1$).

We call P *regular* if there every cyclic permutation of its vertices is realized by some isometry of $\mathbb{H}^2$. Show that the group of orientation preserving isometries of a regular polygon P is generated by an element which is a rotation of angle $2\pi/m$ around a point $p \in P$, the *center* of the polygon.

(3) Observe that the interior angles of a regular polygon are all equal and show that for every $0 \leq \alpha \leq \frac{m-1}{m}\pi$ there exists a unique (up to isometries) regular polygon P_α with interior angles all equal to α.

When we have a metric space there is a more geometric way of defining a notion of barycenter of a finite set of points: It is a point x in the space that minimizes the sum of the distances from all the points in the finite set. In general, barycenters do not have to exist or be unique. Luckily, in the hyperbolic space $\mathbb{H}^n$ the situation is very satisfactory:

(4) Recall that a function $g : \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$g\left(\frac{x+y}{2}\right) \leq \frac{g(x)+g(y)}{2}$$

for every $x < y$. It is strictly convex if the previous inequality is strict for every $x < y$. Show that a strictly convex function $g : \mathbb{R} \to \mathbb{R}$ that is also proper, that is $|g(x)| \to \infty$ as $|x| \to \infty$, has a unique minimum.

(5) (Convexity of distances) We say that a function $f : \mathbb{H}^n \to \mathbb{R}$ is *(strictly) convex* if for every geodesic $\gamma : \mathbb{R} \to \mathbb{H}^n$ the composition $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is (strictly) convex.

Let $x \in \mathbb{H}^n$ be any point. Consider the distance function $d_x : \mathbb{H}^n \to \mathbb{R}$ given by $d_x(y) = d(x, y)$. Show that d_x is strictly convex (recall that $\cosh(d(x, y)) = -\langle x, y\rangle_{(n,1)}$) and observe that it is proper, that is $d_x(y) \to \infty$ as $y \to \infty$.

Deduce that for every $x_1, \cdots, x_m \in \mathbb{H}^n$ the function $f := d_{x_1} + \cdots + d_{x_m}$ has a unique minimum in $\mathbb{H}^n$.

(6) Let us now pick a basepoint $x \in \mathbb{I}^n$ and point at infinity $[w] \in \mathbb{P}\mathcal{C}_+ = \partial\infty\mathbb{I}^n$. Observe that there exists a unique isotropic vector $w \in [w]$ with $\langle x, w \rangle = -1$. Let $d_w : \mathbb{I}^n \to \mathbb{R}$ denote the function $d_w(y) := \operatorname{arccosh}(-\langle w, y \rangle_{(n,1)})$. Show that d_w is strictly convex.

(7) Geometrically d_w has the following interpretation: Consider $v = w - x$. Observe that $v \perp x$ and $|v|_{(n,1)} = 1$. Let γ be the geodesic starting at x with velocity v. Observe that γ is the unique ray from x asymptotic to $[w]$. Show that d_w is the uniform limit on compact sets of $\mathbb{I}^n$ of the functions $d_{\gamma(t)}(\bullet) - t$. It is called the *Busemann function* corresponding to the ponint at infinity $[w]$ with basepoint x (or, equivalently, corresponding to the ray γ).

(8) Consider a finite collection of pairwise distinct points at infinity $[w_1], \cdots, [w_m] \in \partial_\infty\mathbb{I}^n$ and a basepoint $x \in \mathbb{I}^n$. Denote by d_{w_j} the corresponding Busemann functions. Show that, if $m \geq 3$, then $d_{w_1} + \cdots + d_{w_m}$ has a unique minimum in $\mathbb{H}^n$. What happens when $m = 1, 2$?

Exercise 1.2 (Geometry and dynamics of loxodromic motions). We study the action of loxodromic motions on the boundary at infinity.

(1) (North-south dynamics) Let $f \in \operatorname{Isom}(\mathbb{H}^n)$ be a loxodromic motion with $\operatorname{Fix}(f) = \{x, y\}$. Show that, for one of the endponts, say x, f collapses $\partial_\infty\mathbb{H}^n - \{y\}$ to x, that is, the sequence f^n converges uniformly on compact subsets of $\partial_\infty\mathbb{H}^n - \{y\}$ to the constant map x. The point x is called the *attracting* fixed point of f. The other fixed point is called the *repelling* fixed point of f. Show that it is the attracting fixed point of f^{-1}.

(2) (Ping-Pong 1) Let $f, g \in \operatorname{Isom}(\mathbb{H}^n)$ be loxodromic motions with disjoint fixed point sets $\operatorname{Fix}(f) = \{u^+, u^-\}$ and $\operatorname{Fix}(g) = \{v^+, v^-\}$. Here u^+, v^+ are the attracting fixed points and u^-, v^- are the repelling ones. Consider four disjoint closed disks U^+, U^-, V^+, V^- surrounding respectively u^+, u^-, v^+, v^-. Show that there exists $N \in \mathbb{N}$ such that for every $n \geq N$ we have $f^n(\partial_\infty\mathbb{H} - U^-) \subset U^+$ and $f^{-n}(\partial_\infty\mathbb{H} - U^+) \subset U^-$ and similary $g^n(\partial_\infty\mathbb{H} - V^-) \subset V^+$ and $g^{-n}(\partial_\infty\mathbb{H} - V^+) \subset V^-$.

(3) (Ping-Pong 2) Consider the group generated by f^n, g^n, which we denote by $G := \langle f^n, g^n \rangle$. For simplicity, now assume that $n = 1$ is enough to guarantee the separation properties of the previous point. Show that if w is any non-trivial finite product of the form $w = f^{r_1} g^{s_1} f^{r_2} g^{s_2} \cdots g^{s_{k-1}} f^{r_k}$ with $r_j, s_j \neq 0$ for every $j \leq k-1$ then $w \neq 1$. Deduce that G is isomorphic to a free group $\mathbb{F}_2$ on two generators.

(4) Recall now the following fact from topology:

THEOREM 1.3 (Brouwer Fixed Point Theorem). *Any continuous map from a disk to itself has a fixed point.*

Use this to prove that every isometry in G is loxodromic.

(5) Show that there are loxodromic motions $f, g \in \mathrm{Isom}^+(\mathbb{H}^2)$ with disjoint fixed point sets such that the group they generate $G = \langle f, g \rangle$ contains ellliptic or parabolic motions.

EXERCISE 1.4 ($\mathrm{PSL}_2\mathbb{Z}$). Recall that $\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}_2\mathbb{R}$. In this exercise we investigate the subgroup $\mathrm{PSL}_2\mathbb{Z}$.

(1) (Fundamental domain) We work in the upper half space model $\mathbb{H}^2 = U$. Show that every point $x \in U$ can be moved by a fractional linear transformation in $\mathrm{PSL}_2\mathbb{Z}$ to a unique in the region
$$\mathcal{R} := \{z = x + iy \,|x \in [-1/2, 1/2], |z| \geq 1\}.$$
Compute also the area of $\mathcal{R}$. Which points in $\mathcal{R}$ are identified by acting with transformations in $\mathrm{PSL}_2\mathbb{Z}$?

(2) (Congruence subgroups) For every $n \in \mathbb{N}$ we have a homomorphism $\phi_n : \mathrm{PSL}_2\mathbb{Z} \to \mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$ sending a matrix A to $A \pmod n$. Show that ϕ_n is surjective. We denote by $\Gamma(n) < \mathrm{PSL}_2\mathbb{Z}$ the kernel of ϕ_n.

(3) Consider a set of representatives $\{A_1\Gamma(n), \cdots, A_m\Gamma(n)\}$ for the lateral classes of $\Gamma(n)$ in $\mathrm{PSL}_2\mathbb{Z}$. Show that every point $x \in U$ can be moved with a transformation in $\Gamma(n)$ to the region $\mathcal{R}_n = A_1\mathcal{R} \cup \cdots \cup A_m\mathcal{R}$. Compute the area of $\mathcal{R}_n$.

(4) Show that, if $n \geq 4$, then $\Gamma(n)$ has no elliptic elements.

(5) Recall that the *translation distance* $d(f)$ of a loxodromic motion $f \in \mathrm{Isom}(\mathbb{H}^n)$ is the amount of translation it induces on its axis. Show that $d(f \in \mathrm{PSL}_2\mathbb{R}) = 2\mathrm{arccosh}(\mathrm{tr}(f)/2)$.

(6) Denote by d_n the infimum of the translation distances of loxodromic motions in $\Gamma(n) < \mathrm{PSL}_2\mathbb{Z}$. Observe that it is realized by some $f_n \in \Gamma(n)$. Show that d_n diverges as $n \to \infty$.