$\mathrm{SO}_0(2, n+1)$-MAXIMAL REPRESENTATIONS AND HYPERBOLIC SURFACES

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Abstract. We study maximal representations of surface groups $\rho : \pi_1(\Sigma) \to \mathrm{SO}_0(2, n+1)$ via the introduction of ρ-invariant pleated surfaces inside the pseudo-Riemannian space $\mathbb{H}^{2,n}$ associated to maximal geodesic laminations of Σ.

We prove that such pleated surfaces are always embedded, acausal, and possess an intrinsic pseudo-metric. We describe the hyperbolic structure of a pleated surface by constructing a shear cocycle from the cross ratio naturally associated to ρ. The process developed to this purpose applies to a wide class of cross ratios, including examples arising from Hitchin and Θ-positive representations in $\mathrm{SO}(p, q)$. We also show that the length spectrum of ρ dominates the ones of ρ-invariant pleated surfaces, with strict inequality exactly on curves that intersect the bending locus.

We observe that the canonical decomposition of a ρ-invariant pleated surface into leaves and plaques corresponds to a decomposition of the Guichard-Wienhard domain of discontinuity of ρ into standard fibered blocks, namely triangles and lines of photons. Conversely, we give a concrete construction of photon manifolds fibering over hyperbolic surfaces by gluing together triangles of photons.

The tools we develop allow to recover various results by Collier, Tholozan, and Toulisse on the (pseudo-Riemannian) geometry of ρ and on the correspondence between maximal representations and fibered photon manifolds through a constructive and geometric approach, bypassing the use of Higgs bundles.

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1. Introduction

The notion of maximal representations of the fundamental group Γ of a compact hyperbolic surface into a semi-simple Lie group of Hermitian type G was introduced by Burger, Iozzi, and Wienhard in their groundbreaking work [BIW10]. It provides a vast generalization of the notion of *Fuchsian representations*, namely discrete and faithful homomorphisms of Γ into $\mathrm{PSL}_2(\mathbb{R})$, which naturally arise as holonomies of complete hyperbolic structures on surfaces. As already observed in [BIW10], multiple dynamical and geometric properties of Fuchsian representations extend to this wider context: Every maximal representation $\rho : \Gamma \to G$ is faithful, its image $\rho(\Gamma)$ is a discrete subgroup of G acting freely and properly discontinuously on the Riemannian symmetric space associated to G, and the set of conjugacy classes of maximal representations constitutes a union of connected components of the character variety $\mathfrak{X}(\Gamma, G)$.

In recent years, a great variety of results have further investigated and strengthened the relations between Fuchsian representations and geometric structures that naturally arise from maximal representations, and this article is no exception. In our exposition we will consider maximal representations of the fundamental group Γ of a closed orientable surface Σ of genus $g \geqslant 2$ into the connected Lie group $G = \mathrm{SO}_0(2, n+1)$. Moreover, rather than investigating the properties of the action of Γ on the Riemannian symmetric space of $\mathrm{SO}_0(2, n+1)$, we will focus our attention on a class of pseudo-Riemannian and photon structures naturally associated to $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$, as previously done by Collier, Tholozan, and Toulisse in [CTT19].

The main aim of this paper is to provide a purely geometric approach to the study of $\mathrm{SO}_0(2, n+1)$-maximal representations, and establish a direct and explicit link with hyperbolic surfaces and classical Teichmüller theory. This gives a possible answer to the question addressed in [CTT19, Remark 4.13], and a suitable framework for generalizations to open surfaces. In particular, inspired by Thurston's and Mess' works in the study of hyperbolic 3-manifolds (see e.g. [Thu79, Chapter 8], or Canary, Epstein, and Green [CEG06, Chapter I.5] for a detailed exposition), and of constant curvature Lorentzian 3-manifolds (see [Mes07]), respectively, we will pursue this goal by introducing a notion of ρ-equivariant pleated surfaces inside $\mathbb{H}^{2,n}$, and we will investigate their topological, causal, and geometric properties.

We start by introducing the pseudo-Riemannian and photon spaces that we will be interested in. First, we recall that the group $\mathrm{SO}_0(2, n+1)$ is the identity component of the group of isometries of $\mathbb{R}^{2,n+1}$, which denotes the vector space $\mathbb{R}^{n+3}$ endowed with the quadratic form

$$\langle \bullet, \bullet \rangle_{(2,n+1)} := x_1^2 + x_2^2 - y_1^2 - \ldots - y_{n+1}^2.$$

There are multiple homogeneous spaces X naturally associated with $G = \mathrm{SO}_0(2, n+1)$, and each of them leads to a different class of (G, X)-structures in the sense of Thurston (see [Thu79, Chapter 3]). Here we will consider:

- The pseudo-Riemannian symmetric space $\mathbb{H}^{2,n}$ of negative lines of $\mathbb{R}^{2,n+1}$.
- The Photon space $\mathrm{Pho}^{2,n}$ of isotropic 2-planes of $\mathbb{R}^{2,n+1}$.

In both cases, every maximal representation $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ has a natural domain of discontinuity $\Omega_\rho(X) \subset X$, as a consequence of the work of Guichard and Wienhard [GW12] when $X = \mathrm{Pho}^{2,n}$, and of Danciger, Guéritaud, and Kassel [DGK17] when $X = \mathbb{H}^{2,n}$. Accordingly, any maximal representation ρ gives rise to:

- A pseudo-Riemannian manifold $M_\rho = \Omega_\rho(\mathbb{H}^{2,n})/\rho(\Gamma)$ of signature $(2, n)$.
- A closed photon manifold $E_\rho = \Omega_\rho(\mathrm{Pho}^{2,n})/\rho(\Gamma)$.

The geometries of these objects are strictly tied, as described by Collier, Tholozan, and Toulisse [CTT19]. Our work parallels in many aspects the article [CTT19] with a central difference: While in [CTT19] the geometric and topological information is extracted by relating the theory of Higgs bundles to the immersion data of equivariant maximal surfaces in $\mathbb{H}^{2,n}$, our techniques rely on the study of specific 1- and 2-dimensional subsets of the pseudo-Riemannian manifold M_ρ, namely *geodesic laminations* and *pleated surfaces*, in analogy with the tools originally developed by Thurston in his investigation of the structure of the ends of hyperbolic 3-manifolds (see Chapters 8 and 9 of [Thu79]).

A valuable feature of this approach, which is in many aspects explicit and constructive, is that it determines a concrete connection between maximal representations and hyperbolic structures on surfaces. Notice also that, if on the one hand the notion of equivariant pleated surfaces is well suited for generalizations to finite-type surfaces, the analytical techniques required for the study of Higgs bundles do not easily extend outside of the realm of closed orientable surface groups.

For convenience of the reader, we now summarize the main results of the paper. We will then provide a detailed description of each of them, together with the techniques developed for their proof, in the remainder of the introduction:

(a) For any maximal representation ρ and for any maximal geodesic lamination λ of Σ, there exists a $\rho(\Gamma)$-invariant, *acausal*, and properly embedded Lipschitz disk, the *pleated set* of λ,
$$\hat{S}_\lambda \cup \partial\hat{S}_\lambda \subset \mathbb{H}^{2,n} \cup \partial\mathbb{H}^{2,n}.$$
(see Theorem A). It is naturally decomposed as a union of spacelike geodesics and spacelike ideal triangles of $\mathbb{H}^{2,n}$ and is contained in the ρ-domain of discontinuity in $\mathbb{H}^{2,n}$. In particular, $S_\lambda = \hat{S}_\lambda/\rho(\Gamma)$ is a properly embedded subsurface of the pseudo-Riemannian manifold M_ρ. The decomposition of $\hat{S}_\lambda$ into lines and triangles corresponds to an analogue decompostion of the ρ-domain of discontinuity in $\mathrm{Pho}^{2,n}$ into *lines* and *triangles of photons* and we have a a natural fibration $E_\rho \to S_\lambda$ (see Proposition E).

(b) The pleated set $\hat{S}_\lambda$ has a natural intrinsic $\rho(\Gamma)$-invariant hyperbolic structure and a natural pseudo-metric induced by the pseudo-Riemannian metric of $\mathbb{H}^{2,n}$. The developing map $\hat{S}_\lambda \to \mathbb{H}^2$ is 1-Lipschitz with respect to the pseudo-metric on $\hat{S}_\lambda$ and the hyperbolic metric on $\mathbb{H}^2$ (see Theorem C). This implies that the length spectrum of the hyperbolic surface S_λ is *dominated* by the pseudo-Riemannian length spectrum of ρ, that is
$$L_\rho(\bullet) \geqslant L_{S_\lambda}(\bullet).$$
There is a simple characterization of those curves $\gamma \in \Gamma$ for which the strict inequality holds: They are exactly the curves that intersect essentially the *bending locus* of S_λ.

(c) The intrinsic hyperbolic structure on the pleated set $\hat{S}_\lambda$ is described by a *shear cocycle* σ^ρ_λ through Bonahon's shear parametrization of Teichmüller space (see [Bon96]). The definition of the cocycle σ^ρ_λ uniquely relies on the data of the lamination λ and of a Γ-invariant cross ratio on the Gromov boundary of Γ, naturally associated to the representation ρ. In fact, the construction applies in great generality, and associates to any *(strictly) positive* and *locally bounded* cross ratio β on $\partial\Gamma$, and to any maximal lamination λ, an intrinsic hyperbolic structure X_λ whose length spectrum coincides with the length spectrum of β on all measured laminations with support contained in λ.

(d) The set of hyperbolic surfaces S_λ arising as intrinsic hyperbolic structures on pleated surfaces lie on the boundary of the *dominated set* of ρ which is the subset of Teichmüller space defined by
$$\mathcal{P}_\rho := \{Z \in \mathcal{T} \,|\, L_Z(\bullet) \leqslant L_\rho(\bullet)\}.$$
The set $\mathcal{P}_\rho$ is convex for the Weil-Petersson metric and it is also convex in *shear coordinates*. Its interior $\mathrm{int}(\mathcal{P}_\rho)$ corresponds to those hyperbolic

surfaces Z that are *strictly dominated* by ρ, that is $L_\rho(\bullet) > cL_Z(\bullet)$ for some $c > 1$. Combining a geometric construction in $\mathbb{H}^{2,n}$ with the convexity of length functions along Weil-Petersson geodesics, we observe that if ρ is not Fuchsian, then $\mathrm{int}(\mathcal{P}_\rho)$ is always non-empty (Theorem D). This allows to recover part of the results described in [CTT19].

(e) We give an elementary construction of photon manifolds E that fiber over a closed hyperbolic surface S homeomorphic to Σ. The holonomies of these manifolds $\rho : \pi_1(E) \to \mathrm{SO}_0(2, n+1)$ are maximal and we have a natural ρ-equivariant pleated acausal embedding $\hat{S} \to \mathbb{H}^{2,n}$ of the universal cover $\hat{S} \to S$. The image is a pleated surface for some suitable lamination λ of S. The construction of $E \to S$ is completely analogous to the construction of a closed hyperbolic surface by first gluing together ideal hyperbolic triangles to form (incomplete) pair of pants, and then glue together the completions of the pair of pants. Here, instead of gluing ideal hyperbolic triangles, we will glue together triangles of photons forming (incomplete) fibered pairs of pants of photons, find suitable completions, and glue such completions to form closed manifolds (Theorem F and Proposition G).

We now describe more in detail each of the previous points.

Topology and acausality of pleated surfaces. Our discussion will heavily rely on the existence of equivariant boundary maps naturally associated to $\mathrm{SO}_0(2, n+1)$-maximal representations, which is guaranteed by the following result of Burger, Iozzi, Labourie, and Wienhard: We recall that the boundary at infinity $\partial\mathbb{H}^{2,n}$ of the pseudo-Riemannian symmetric space $\mathbb{H}^{2,n}$ identifies with the space of isotropic lines of $\mathbb{R}^{2,n+1}$.

Theorem ([BILW05, § 6], see also [CTT19, Theorem 2.5])**.** *If $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ is a maximal representation, then there exists a unique ρ-equivariant, continuous, and dynamics preserving embedding*

$$\xi : \partial\Gamma \to \partial\mathbb{H}^{2,n}$$

such that the image of ξ is an acausal *curve, meaning that for every triple of distinct points $a, b, c \in \partial\Gamma$, the subspace of $\mathbb{R}^{2,n+1}$ generated by the isotropic lines $\xi(a), \xi(b), \xi(c)$ has signature $(2, 1)$.*

Theorem 1 has a simple interpretation in terms of the geometry of $\mathbb{H}^{2,n}$: Every pair of distinct points $a, b \in \partial\Gamma$ is sent by ξ into the pair of endpoints of a unique spacelike geodesic of $\mathbb{H}^{2,n}$, and for every triple of distinct points $a, b, c \in \partial\Gamma$, the images $\xi(a), \xi(b), \xi(c)$ are the vertices of a unique ideal totally geodesic *spacelike* triangle in $\mathbb{H}^{2,n}$. In light of this phenomenon, the boundary map ξ allows us to naturally realize geodesic laminations on the surface Σ as ρ-invariant closed subsets of $\mathbb{H}^{2,n}$, and consequently in the pseudo-Riemannian manifold M_ρ.

To see this, we start by briefly recalling the notion of geodesic lamination, and the related terminology that will be used throughout our exposition. We will think of a geodesic ℓ in the universal cover $\widetilde{\Sigma}$ of Σ as an element of the space

$$\mathcal{G} := (\partial\Gamma \times \partial\Gamma - \Delta)/(x, y) \sim (y, x),$$

simply by identifying ℓ with the unordered pair of its endpoints. We say that two geodesics ℓ and ℓ' with endpoints a, b and a', b', respectively, are *crossing* if a' and b' lie in distinct connected components of $\partial\Gamma - \{a, b\}$ (recall that $\partial\Gamma$ is a topological circle). Two geodesics that are not crossing will be said to be *disjoint*. Within this framework, a geodesic lamination of Σ is a Γ-invariant closed subset $\lambda \subset \mathcal{G}(\widetilde{\Sigma})$ made of pairwise disjoint geodesics, and it is said to be *maximal* if every geodesic ℓ outside λ crosses some ℓ' in λ. The elements of a lamination will be also called its *leaves*, and the connected components P of $\widetilde{\Sigma} - \lambda$ will be called its *plaques*.

Let now ρ be a maximal representation, and let $\xi : \partial\Gamma \to \partial\mathbb{H}^{2,n}$ be its associated boundary map. For any leaf $\ell = [a, b]$ in λ, we can find a unique spacelike geodesic $\hat{\ell}$ in $\mathbb{H}^{2,n}$ with endpoints $\xi(a), \xi(b)$, and similarly for any plaque $P = \Delta(a, b, c)$, we have a unique spacelike ideal triangle $\hat{P}$ with endpoints $\xi(a), \xi(b), \xi(c) \in \partial\mathbb{H}^{2,n}$. We then define the *geometric realization of* λ *in* $\mathbb{H}^{2,n}$ to be

$$\hat{\lambda} := \bigcup_{\ell \text{ leaf of } \lambda} \hat{\ell},$$

and its associated *pleated set* as

$$\hat{S}_\lambda := \hat{\lambda} \cup \bigcup_{P \text{ plaque of } \lambda} \hat{P}.$$

Our first result establishes some structural properties about the topology and the causal features of these sets:

Theorem A. *Let $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ be a maximal representation. For every maximal lamination λ the pleated set $\hat{S}_\lambda \subset \mathbb{H}^{2,n}$ is an embedded Lipschitz disk which is also* acausal*, that is, every pair of points $x, y \in \hat{S}_\lambda$ is joined by a spacelike geodesic.*

The basic idea behind Theorem A is the following: A pair of geodesics $\hat{\ell}, \hat{\ell}'$ with endpoints on the limit curve $\Lambda_\rho = \xi(\partial\Gamma)$ form an acausal set $\hat{\ell} \cup \hat{\ell}'$ inside $\mathbb{H}^{2,n}$ if and only if the corresponding leaves ℓ, ℓ' of λ are disjoint.

This property immediately implies that the geometric realization $\hat{\lambda} \subset \mathbb{H}^{2,n}$ of any lamination λ is an acausal subset. In turn, working in the Poincaré model of $\mathbb{H}^{2,n}$ and using the fact that $\hat{\lambda}$ is acausal, we prove that adding the complementary triangles preserves the acausal property. By general properties of acausal subsets of $\mathbb{H}^{2,n}$, we deduce that $\hat{S}_\lambda \cup \Lambda_\rho \subset \mathbb{H}^{2,n} \cup \partial\mathbb{H}^{2,n}$ is a properly embedded Lipschitz disk.

The surface $S_\lambda = \hat{S}_\lambda/\rho(\Gamma) \subset M_\rho$ carries two natural geometries: It has an intrinsic *hyperbolic structure* and a *pseudo-metric* induced by the ambient space $\mathbb{H}^{2,n}$. We now focus our attention of the description of the former.

Cross ratio and shear cocycles. As in the case of pleated surfaces in hyperbolic 3-space $\mathbb{H}^3$, the hyperbolic structure on the pleated set $\hat{S}_\lambda$ can be recorded by a *shear cocycle* [Bon96].

In order to define it, we again rely on the properties of the boundary map: The acausality condition satisfied by ξ and the pseudo-Riemannian structure of the boundary $\partial\mathbb{H}^{2,n}$ uniquely determine a Γ-invariant *cross ratio* β^ρ on $\partial\Gamma$, satisfying the following properties:

- It is *(strictly) positive* on positively ordered quadruples in $\partial\Gamma$. This follows from the acausal properties of the boundary map ξ and implies, via general results of Martone and Zhang [MZ19], and Hamenstädt [Ham99], that β^ρ induces a length function L_ρ on the space of geodesic currents $\mathcal{C}$.

- It is *locally bounded*, meaning that there exists a hyperbolic structure X on Σ such that, for every compact subset K in the space of distinct 4-tuples in $\partial\Gamma$, we can find constants $C, \alpha > 0$ such that

$$|\log |\beta^\rho(a, b, c, d)|| \leqslant C \left|\log \left|\beta^X(a, b, c, d)\right|\right|^\alpha$$

  for every cyclically ordered 4-tuples $(a, b, c, d) \in K$, where β^X is the cross ratio on $\partial\Gamma$ determined by the structure $X \in \mathcal{T}$. This property is a consequence of the explicit definition of β^ρ and of the Hölder continuity of the limit map ξ.

Notice that examples of (strictly) positive and locally bounded cross ratios naturally arise also from other interesting contexts related to pseudo-Riemannian symmetric spaces $\mathbb{H}^{p,q}$ such as Hitchin representations in $\mathrm{SO}(p, p+1)$ or Θ-positive representations in $\mathrm{SO}(p, q)$ where similar pleated surface construction might be possible (see also Appendix A).

We also remark that positive cross ratios have been used by Martone and Zhang in [MZ19], by Labourie [Lab08], and Burger, Iozzi, Parreau, and Pozzetti [BIPP21] to study common features of Higher Teichmüller Theories.

Making use of the cross ratio β^ρ, we then describe the intrinsic hyperbolic structure of a pleated set S_λ through the data of a so-called *Hölder cocycle* σ^ρ_λ transverse to the maximal lamination λ, in the sense of [Bon96].

The notion of Hölder transverse cocycle has been introduced by Bonahon (see [Bon97b, Bon97a]), who for instance deployed them to provide a parametrization of Teichmüller space $\mathcal{T}$ of a closed orientable surface Σ in [Bon96], following ideas of Thurston [Thu98]. Heuristically speaking, if λ_Z is the geometric realization of λ on the hyperbolic surface Z, the shear cocycle σ^Z_λ records how the ideal triangles in $Z - \lambda_Z$ are glued together along the leaves of λ_Z. The space $\mathcal{H}(\lambda; \mathbb{R})$ of Hölder cocycles transverse to λ has a natural structure of vector space of dimension $3|\chi(\Sigma)|$, and the map that associates to any hyperbolic structure $Z \in \mathcal{T}$ its shear cocycle $\sigma^Z_\lambda \in \mathcal{H}(\lambda; \mathbb{R})$ embeds Teichmüller space as an open convex cone with finitely many faces inside $\mathcal{H}(\lambda; \mathbb{R})$. The resulting set of coordinates is usually referred to as *shear coordinates* with respect to the maximal lamination λ.

This point of view on Teichmüller space has proved to be fruitful also in the setting of Hitchin representations and, more generally, to analyze Anosov representations as witnessed by work of Bonahon and Dreyer [BD17], Alessandrini, Guichard, Rogozinnikov, and Wienhard [AGRW22], and Pfeil [Pfe21].

The underlying principle for the construction of a shear cocycle starting from a cross ratio is very elementary: The classical shear between two adjacent ideal triangles Δ and Δ' in the hyperbolic plane is an explicit function of the $\mathbb{R}\mathrm{P}^1$-cross ratio of the four ideal vertices of $\Delta \cup \Delta'$, and shears between triangles separated by finitely many leaves of λ can be expressed as a finite sum of shears between adjacent plaques. One can then define the ρ-shear between two adjacent plaques P, Q of λ simply by replacing the role of the $\mathbb{R}\mathrm{P}^1$-cross ratio with β^ρ. In fact with some additional (but elementary) work, this allows to introduce a natural notion of ρ-shear cocycle σ^ρ_λ for a large class of maximal laminations, namely laminations on Σ obtained by adding finitely many isolated leaves to pants decompositions (see Section 4.4).

The construction of the shear cocycle σ^ρ_λ that we describe relies only on the properties of the cross ratio β^ρ that we mentioned above, namely that β^ρ is strictly positive and locally bounded. Consequently, our techniques allow to deduce the following general statement:

Theorem B. *Let $\beta : \partial\Gamma^4 \to \mathbb{R}$ be a strictly positive locally bounded cross ratio. For every maximal lamination λ there exists a transverse Hölder cocycle $\sigma^\beta_\lambda \in \mathcal{H}(\lambda; \mathbb{R})$ with the following properties:*

(i) The cocycle σ^β_λ is the shear cocycle of a unique hyperbolic metric X_λ on Σ.
(ii) For every transverse measure μ on λ we have $L_{X_\lambda}(\mu) = L_\beta(\mu)$.
(iii) The map $\lambda \mapsto X_\lambda$ is continuous with respect to the Hausdorff topology on the space of maximal geodesic laminations.

The process to construct β-shear cocycles σ^β_λ for a generic maximal lamination is technically quite involved, and our strategy will heavily rely on multiple tools developed by Bonahon [Bon96] in his construction of shear coordinates for Teichmüller space, such as the notion of *divergence radius function* associated to the choice of a train track carrying λ (see also Bonahon and Dreyer [BD17, § 8.2]). However, if λ is a finitely leaved lamination and $\beta = \beta^\rho$ is the cross ratio associated to some maximal representation ρ, then the shear cocycle σ^ρ_λ has a simple interpretation in terms of horocycle foliations on the plaques of the pleated set S_λ, in direct analogy with Bonahon's original description of shear coordinates (see e.g. [Bon96, § 2]).

We call the cocycle σ^ρ_λ the *intrinsic shear cocycle* associated to λ and ρ.

Geometry of pleated surfaces. The other intrinsic geometric structure carried by the plated set $\hat{S}_\lambda$ is a ρ-invariant pseudo-metric: By Theorem A any two points $x, y \in \hat{S}_\lambda$ are connected by a unique spacelike geodesic segment $[x, y]$, we can define

$$d_{\mathbb{H}^{2,n}}(x, y) := \ell[x, y].$$

It is worth to mention that the function

$$d_{\mathbb{H}^{2,n}} : \hat{S}_\lambda \times \hat{S}_\lambda \to [0, \infty)$$

is not a distance in the traditional sense as it does not satisfy the triangle inequality nor its inverse (see also [GM21] and [CTT19]). However, it is continuous, it vanishes exactly on the diagonal, and its metric balls $B(x, r) = \{y \in \hat{S}_\lambda \,|\, d_{\mathbb{H}^{2,n}}(x, y) \leqslant r\}$ form a fundamental system of neighborhoods.

Nevertheless, the pseudo-distance $d_{\mathbb{H}^{2,n}}$ naturally relates to the hyperbolic structure X_λ associated to ρ and the maximal lamination λ. To see this, let us introduce the following notion: We say that a function $f : S_\lambda = \hat{S}_\lambda/\rho(\Gamma) \to X$ with values in a hyperbolic surface X is *K-Lipschitz* with respect to the intrinsic pseudo-metric if it lifts to a map $\hat{f} : \hat{S}_\lambda \to \mathbb{H}^2$ that satisfies

$$d_{\mathbb{H}^2}(\hat{f}(x), \hat{f}(y)) \leqslant K d_{\mathbb{H}^{2,n}}(x, y)$$

for any $x, y \in \hat{S}_\lambda$. Then we have:

Theorem C. *Let $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ be a maximal representation, and λ be a maximal lamination. If X_λ denotes the hyperbolic surface with* intrinsic shear cocycle *σ^ρ_λ, then the pleated surface $S_\lambda \subset M_\rho$ admits a unique developing homeomorphism $f : S_\lambda \to X_\lambda$ which is 1-Lipschitz with respect to the intrinsic pseudo-metric on S_λ. Furthermore, we have*

$$L_{X_\lambda}(\gamma) \leqslant L_\rho(\gamma)$$

for every $\gamma \in \Gamma$, where $L_{X_\lambda}, L_\rho : \Gamma \to (0, \infty)$ denote the length functions of the hyperbolic surface X_λ and the representation ρ, respectively, with strict inequality if and only if γ intersects the bending locus *of S_λ.*

The heuristic idea of Theorem C is the following: The pleated set $\hat{S}_\lambda$ has an intrinsic hyperbolic path metric (whose shear cocycle is exactly the intrinsic shear cocycle σ^ρ_λ). Using the fact that $\hat{S}_\lambda$ is an acausal subset that can be represented as a graph in the Poincaré model of $\mathbb{H}^{2,n}$, one can show that for every point $x, y \in \hat{S}_\lambda$ there exists a path $\alpha : I \to \hat{S}_\lambda$ joining them and with length bounded by the length of the spacelike geodesic $[x, y]$. This immediately implies that the path metric on $\hat{S}_\lambda$ is dominated by the intrinsic pseudo-metric.

We show that this picture is accurate in the case of finite leaved maximal laminations. The proof here is elementary and uses a cut-and-paste argument in the spirit of [CEG06, Theorem I.5.3.6]. In order to deduce the statement of Theorem C from the finite leaved case, we exploit continuity properties of pleated surfaces.

We conclude here our discussion on the existence of ρ-equivariant pleated surfaces and the study of their topology and geometry. In what follows, we deploy the results just described to extract information on the maximal representation ρ.

Length spectra of maximal representations. We now focus on the study of the set of pleated surfaces $\{X_\lambda\}_\lambda$ associated to a given maximal representation ρ, considered as a subset of Teichmüller space $\mathcal{T}$. As it turns out, it can be described as a subset of the boundary of a set that is *convex* with respect to multiple natural structures on $\mathcal{T}$. More precisely, given a maximal representation ρ let us define the *dominated set* of ρ as

$$\mathcal{P}_\rho := \{Z \in \mathcal{T} \,|\, L_Z(\gamma) \leqslant L_\rho(\gamma) \text{ for every } \gamma \in \Gamma\}.$$

We also define the companion $\mathcal{P}^{\mathrm{simple}}_\rho$ consisting of those hyperbolic surfaces whose *simple* length spectrum is dominated by the simple length spectrum of ρ.

By Theorem C, the set $\mathcal{P}_\rho$ is always non-empty as it contains all the hyperbolic structures of the pleated surfaces associated to ρ. Furthermore, it is convex with respect to the Weil-Petersson metric, by work of Wolpert [Wol87], and with respect to *shear paths*, by work of Bestvina, Bromberg, Fujiwara and Souto [BBFS13] and Théret [Thé14] generalizing a result of Kerckhoff [Ker83] (see also [MV] for a different approach).

We prove the following:

Theorem D. *Let $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ be a maximal representation. For every $Z \in \mathcal{T}$ define*

$$\kappa(Z) := \sup_{\gamma \in \Gamma - \{1\}} \frac{L_Z(\gamma)}{L_\rho(\gamma)}.$$

We have:

1. *$Z \in \mathrm{int}(\mathcal{P}_\rho)$ if and only if $\kappa(Z) < 1$.*
2. *If ρ is not Fuchsian, then $\mathrm{int}(\mathcal{P}_\rho) \neq \varnothing$.*
3. *If X_λ is the hyperbolic structure with shear cocycle σ^ρ_λ, then $X_\lambda \in \partial \mathcal{P}_\rho$.*

Furthermore,

4. *If $Z \notin \mathrm{int}(\mathcal{P}_\rho)$, then there exists $\mu \in \mathcal{ML}$ such that $\kappa(Z) = L_Z(\mu)/L_\rho(\mu)$.*
5. *$\mathcal{P}_\rho = \mathcal{P}^{\mathrm{simple}}_\rho$.*

As a consequence of properties (1) and (2), we obtain that if ρ is not Fuchsian, then there exist hyperbolic structures $Z \in \mathcal{T}$ whose length spectrum $L_Z(\bullet)$ is strictly dominated by the length spectrum of the maximal representation $L_\rho(\bullet)$. Thus, we recover the following:

Theorem (Collier, Tholozan, and Toulisse [CTT19])**.** *Let $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ be a maximal representation with $n \geqslant 1$. We have the following: Either ρ is Fuchsian or there exists a hyperbolic surface Z and a constant $c > 1$ such that $L_\rho(\bullet) > cL_Z(\bullet)$.*

In particular, the inequality $L_\rho(\bullet) > cL_Z(\bullet)$ immediately implies that the *entropy* of ρ, defined by

$$\delta_\rho := \limsup_{R\to\infty} \frac{\log |\{[\gamma] \in [\Gamma] \,|\, L_\rho(\gamma) \leqslant R\}|}{R},$$

where $[\Gamma]$ denotes the set of conjugacy classes of elements in Γ, is bounded by

$$\delta_\rho \leqslant 1/c \leqslant 1$$

and the equality $\delta_\rho = 1$ holds if and only if ρ is Fuchsian.

Let us briefly comment on properties (1), (2), and (3).

Property (1) characterizes interior points of $\mathcal{P}_\rho$ as those points Z whose length spectrum $L_Z(\bullet)$ is *strictly* dominated by the length spectrum $L_\rho(\bullet)$. The proof proceeds as follows. On the one hand, being strictly dominated is an open condition: For $Z \in \mathcal{T}$ and for every $K > 1$ there is a neighborhood U of Z such that every surface $Z' \in U$ is K-bilipschitz to Z and in particular $1/K \leqslant L_Z/L_{Z'} \leqslant K$. Therefore, if $\kappa(Z) < 1$ and $K < 1/\kappa(Z)$, then $\kappa(Z') < 1$. On the other hand, interior points are strictly dominated due to the strict convexity of length functions along Weil-Petersson geodesics.

The idea of (2) is the following: In order to prove that $\mathrm{int}(\mathcal{P}_\rho)$ is non empty, it is enough to show that $\mathcal{P}_\rho$ contains at least two distinct points X, Y. Indeed, by the strict convexity of length functions with respect to the Weil-Petersson metric (see Wolpert [Wol87] and [Wol06]), the midpoint $Z \in \mathcal{P}_\rho$ of the Weil-Petersson segment $[X, Y]$ is strictly dominated and, hence, by property (1), is an interior point.

If ρ is not Fuchsian, such pair of points $X, Y \in \mathcal{P}_\rho$ can be produced by taking two pleated surfaces S_α and S_β realizing simple closed curves α and β that intersect (completed to maximal laminations $\lambda_\alpha, \lambda_\beta$ by adding finitely many leaves spiraling around them). On the one hand, Theorem C tells us that $L_{S_\alpha}(\alpha) = L_\rho(\alpha)$ and $L_{S_\beta}(\beta) = L_\rho(\beta)$. On the other hand, as ρ is not Fuchsian, the bending locus of S_α, S_β is not empty. Since the bending loci are sublaminations of the maximal extensions $\lambda_\alpha, \lambda_\beta$, they contain the curves α, β respectively. Since α, β intersect essentially, the curve α intersects the bending locus of S_β and β intersects the bending locus of S_α, therefore $L_{S_\alpha}(\beta) < L_\rho(\beta)$ and $L_{S_\beta}(\alpha) < L_\rho(\alpha)$, again by Theorem C. In any case $S_\alpha \neq S_\beta$.

Property (3) follows from the fact that every measured lamination μ whose support does not intersect essentially the bending locus of S_λ realizes $L_{S_\lambda}(\mu) = L_\rho(\mu)$ which implies $\kappa(S_\lambda) = 1$ and, hence, by Property (1), $S_\lambda \in \partial\mathcal{P}_\rho$.

Lastly, let us also spend a couple of words on the simple length spectrum of ρ: It follows from (5) that the simple length spectrum alone completely determines the dominated set. This is an indication that there might be simple length spectrum rigidity for $\mathrm{SO}_0(2, n+1)$-maximal representations.

The proof of (5) depends on (4): In fact, on the one hand, we always have $\mathcal{P}_\rho \subset \mathcal{P}_\rho^{\mathrm{simple}}$, by definition. On the other hand, properties (1) and (4) imply together that $\partial\mathcal{P}_\rho \subset \partial\mathcal{P}_\rho^{\mathrm{simple}}$. As both sets are topological disks, by convexity with respect to Weil-Petersson geometry or with respect to shear paths, we can conclude that they are equal. The proof of (4) follows arguments of Thurston [Thu98] on the existence of a maximally stretched laminations.

Photon structures fibering over hyperbolic surfaces. We now describe the picture from the perspective of photon structures.

By work of Guichard and Wienhard [GW12], maximal representations $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ parametrize certain geometric structures, in the sense of Thurston, on appropriate closed manifolds. More precisely, every representation ρ determines a $(\mathrm{SO}_0(2, n+1), \mathrm{Pho}^{2,n})$-structure, called a photon structure, on a closed manifold E_ρ. By the Ehresmann-Thurston principle (see [Thu79, Chapter 3]), the topology of E_ρ does not change over each connected component of $\mathfrak{X}(\Gamma, \mathrm{SO}_0(2, n+1))$. However, different components can correspond to different topological types.

Using maximal surfaces in $\mathbb{H}^{2,n}$, Collier, Tholozan, and Toulisse showed in [CTT19] that E_ρ can always be seen as a *fibered photon bundle* over Σ with geometric fibers which are copies of $\mathrm{Pho}^{2,n-1}$ and, furthermore, its topology can be computed from some characteristic classes of ρ.

A fibered photon bundle over a surface $\pi : E \to S$ is an object that comes together with a developing map $\delta : \hat{E} \to \mathrm{Pho}^{2,n}$ and a natural associated map $f : \hat{S} \to \mathbb{H}^{2,n}$, where $\hat{S} \to S$ is the universal covering and $\hat{E} \to \hat{S}$ is the pull-back bundle on the universal covering, with the property that $\delta(\pi^{-1}(x)) = \mathrm{Pho}(f(x)^\perp)$. Here $f(x)^\perp \subset \mathbb{R}^{2,n+1}$ is the orthogonal subspace of the negative line $f(x) \in \mathbb{H}^{2,n}$. In particular, the fibered photon bundle $E \to S$ has an associated *underlying vector bundle* $V_E \to S$ where the fiber over x is the vector space $f(x)^\perp$.

In a similar spirit, using pleated surfaces, we show that every maximal lamination $\lambda \subset \Sigma$ induces a geometric decomposition of E_ρ into standard fibered blocks called *triangles of photons* which we briefly describe: Let $\Delta \subset \mathbb{H}^{2,n}$ be an ideal spacelike triangle with vertices $a, b, c \in \partial\mathbb{H}^{2,n}$. The standard triangle of photons $E(\Delta) \subset \mathrm{Pho}^{2,n}$ is the codimension 0 submanifold with boundary

$$E(\Delta) := \{V \in \mathrm{Pho}^{2,n} \,|\, V \perp x \text{ for some } x \in \Delta\}.$$

The boundary $\partial E(\Delta)$ is a union of *lines of photons* $\partial E(\Delta) = E(\ell_a) \cup E(\ell_b) \cup E(\ell_c)$ where ℓ_a, ℓ_b, ℓ_c are the boundary spacelike geodesics of Δ opposite to the ideal vertices a, b, c and

$$E(\ell) := \{V \in \mathrm{Pho}^{2,n} \,|\, V \perp x \text{ for some } x \in \ell\}$$

if $\ell \subset \mathbb{H}^{2,n}$ is a spacelike geodesic.

The triangle of photons $E(\Delta)$ naturally fibers over the ideal hyperbolic triangle Δ. We denote by $\pi : E(\Delta) \to \Delta$ the natural fibration. The fiber $\pi^{-1}(x)$ over the point $x \in \Delta$ is given by $\mathrm{Pho}(x^\perp) \simeq \mathrm{Pho}^{2,n-1}$.

We have:

Proposition E. *Let $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ be a maximal representation. Let $\hat{S}_\lambda \subset \mathbb{H}^{2,n}$ be the pleated set associated to the maximal lamination λ. Then the*

Guichard-Wienhard domain of discontinuity $\Omega_\rho \subset \mathrm{Pho}^{2,n}$ *naturally decomposes as*

$$\Omega_\rho = \bigsqcup_{\ell \subset \hat{\lambda}} E(\ell) \sqcup \bigsqcup_{\Delta \subset \hat{S}_\lambda - \hat{\lambda}} E(\Delta)$$

and we have an equivariant bundle projection $\Omega_\rho \to \hat{S}_\lambda$ *induced by the standard projections* $E(\Delta) \to \Delta$ *and* $E(\ell) \to \ell$.

Conversely, we are also able to describe a procedure to abstractly assemble triangles of photons and explicitly build fibered photon structures on a fiber bundle $E \to S$ with maximal holonomy $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$.

Our approach is completely analogous to the procedure that constructs a closed hyperbolic surface by gluing ideal triangles. First we construct *pair of pants of photons* $E_j \to S_j$ by gluing two copies of a standard triangle of photons $E(\Delta)$. As it happens for hyperbolic surfaces, if the holonomy around the boundary curves of S_j is *loxodromic* (with respect to a suitable notion of loxodromic), then $E_j \to S_j$ is the interior of a fibered photon structure with *totally geodesic boundary* $\bar{E}_j \to \bar{S}_j$.

For us, loxodromic means *bi-proximal*, a property that is equivalent to a suitable north-south dynamics on $\partial\mathbb{H}^{2,n}$: The set $\mathcal{L} \subset \mathrm{SO}_0(2, n+1)$ of loxodromic elements is an open subset with two connected components $\mathcal{L} = \mathcal{L}^+ \cup \mathcal{L}^-$ distinguished by the sign of the leading eigenvalue.

We remark that the condition of being loxodromic as well as the topology of the resulting bundles $\bar{E}_j \to \bar{S}_j$ can be read off the gluing maps without difficulties. We prove:

Theorem F. *The space of fibered photon structures with totally geodesc boundary on a bundle* $E \to S$ *over a pair of pants* S *admits a parametrization by*

$$\left\{ [\phi_a, \phi_b, \phi_c] \in \left(\prod_{j \in \{a,b,c\}} \mathrm{PStab}(\ell_j) \right) / \mathrm{PStab}(\Delta)^2 \; \middle| \rho_k = \phi_i \phi_j^{-1} \in \mathcal{L} \right\}$$

The topology of E *is determined by the first Stiefel-Whitney class* $w_1(V_E) \in H^1(S, \mathbb{Z}/2\mathbb{Z})$ *of the* underlying vector bundle $V_E \to S$. *The class* $w_1(V_E)$ *can be computed as follows: Let* $\gamma_a, \gamma_b, \gamma_c \subset S$ *be the peripheral curves corresponding to the vertices* a, b, c *respectively. Then*

$$w_1(V_E)[\gamma_k] = \begin{cases} 0 & \text{if } \rho_k \in \mathcal{L}^+, \\ 1 & \text{if } \rho_k \in \mathcal{L}^-, \end{cases}$$

Here $\mathrm{PStab}(\ell), \mathrm{PStab}(\Delta)$ are the stabilizers of the spacelike geodesic ℓ and the ideal spacelike triangle Δ that fix the endpoints of ℓ and the vertices of Δ in $\partial\mathbb{H}^{2,n}$, respectively. The space $\prod_{j \in \{a,b,c\}} \mathrm{PStab}(\ell_j)$ describes all possible gluing maps $E(\ell_j) \to E(\ell_j)$, but in a redundant way. This is due to the fact a pair of triples that differ by a pre- and post-composition by elements $\psi, \psi' \in \mathrm{PStab}(\Delta)$ give rise to isomorphic fibered photon structures.

As a second step, we take several pair of pants of photons with totally geodesic boundary $\bar{E}_j \to \bar{S}_j$ and glue them together. Again, some compatibility conditions must be fulfilled by the gluing maps in order to perform the gluing. As a result, we get a photon structure on a manifold E that naturally fibers over a hyperbolic surface S with geometric fibers and we also obtain a maximal geodesic lamination λ on S.

In analogy with [CTT19, Proposition 3.13], we have:

Proposition G. *The holonomy $\rho : \pi_1(E) \to \mathrm{SO}_0(2, n+1)$ of the fibered photon structure $E \to S$ descends to a maximal representation $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$. The hyperbolic surface S is the pleated surface that realizes λ in M_ρ.*

In combination with Theorems A and C, this provides an analogue of [CTT19, Corollary 4.12]. In fact, in order to prove Proposition G, we generalize results of [CTT19] about smooth spacelike surfaces to purely topological versions. This allows us to treat pleated surfaces.

The anti-de Sitter case. When $n = 1$ much of the above picture on pleated surfaces can be made explicit and quantitative. Due to the fact that $\mathrm{SO}_0(2, 2)$ is a 2-fold cover of $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$, maximal representations in $\mathrm{SO}_0(2, 2)$ naturally correspond to pairs of maximal representations in $\mathrm{PSL}_2(\mathbb{R})$, which, by Goldman's work [Gol80], are precisely the holonomies of hyperbolic structures on Σ. As such, this low dimensional case has a special connection with classical Teichmüller theory. This is highlighted by groundbreaking work of Mess [Mes07], who connected the study of globally hyperbolic maximal Cauchy compact (GHMC) anti-de Sitter 3-manifolds with maximal representations inside $\mathrm{SO}_0(2, 2)$, and gave a proof of Thurston's Earthquake Theorem based on the pseudo-Riemannian geometry of the manifold M_ρ. (Thurston's original approach is outlined in work of Kerckhoff [Ker83].) Since Mess' seminal paper, the study of GHMC anti-de Sitter 3-manifolds has propagated in multiple directions and has produced further connections with Teichmüller theory, as for example described in [ABB$^+$07], [BS10], [BS12], [BB09], [BBZ07], among other works. We refer to Bonsante and Seppi [BS20] for a detailed exposition of the current state-of-art and for further references.

We will explore this connection in a separate paper [MV] where we use pleated surfaces to obtain, among other results, an anti-de Sitter proof of (strict) convexity of length functions in shear coordinates for Teichmüller space (recovering the work of Bestvina, Bromberg, Fujiwara, and Suoto [BBFS13], and Théret [Thé14]) and a shear-bend parametrization of globally hyperbolic maximal Cauchy compact anti-de Sitter 3-manifolds.

Outline. This article is structured as follows:

In Section 2 we cover the background material that we need. More precisely: The geometry of the pseudo-hyperbolic space $\mathbb{H}^{2,n}$ and its boundary $\partial\mathbb{H}^{2,n}$, acausality, and the Poincaré model. The dynamical and geometric characterization of maximal representations $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$. Some classical Teichmüller theory: Geodesic laminations, geodesic currents, measured laminations, shear coordinates. Positive cross ratios and their Liouville currents.

In Section 3 we first discuss the geometric realizations $\hat{\lambda}$ of maximal laminations λ and the associated pleated sets $\hat{S}_\lambda \subset \mathbb{H}^{2,n}$. Then, we relate the acausal properties of the limit curve $\xi(\partial\Gamma) \subset \partial\mathbb{H}^{2,n}$ to the topology and the causal structure of $\hat{\lambda}$ and $\hat{S}_\lambda$ (see Propositions 3.5 and 3.6). The fact that the pleated set $\hat{S}_\lambda$ is acausal implies that it can be represented as a graph in the Poincaré model, and we show that the graph depends continuously on the lamination (see Proposition 3.7). Lastly, we analyze more in detail the locus where $\hat{S}_\lambda$ is folded and define the bending locus (see Proposition 3.9). The bending locus will control how the geometry of $\hat{S}_\lambda$ is distorted in $\mathbb{H}^{2,n}$. This will play a role in Sections 6 and 7.

In Sections 4 and 5 we explain how to attach a natural Hölder cocycle $\sigma^\beta_\lambda \in \mathcal{H}(\lambda; \mathbb{R})$ to every positive and locally bounded cross ratio β and every maximal lamination λ (see Theorem 4.1). Section 4 mainly focuses on the study finite leaved laminations, setting that can be treated with elementary techniques (see Propositions 4.9 and 4.11). In Section 5 we extend the construction to the case of a general maximal lamination. The procedure here is analytic: We define the shear cocycle as a limit of elementary finite approximations. The process needed to establish the convergence of finite approximations is quite delicate as it depends on the geometry of λ on a fine scale. In the end, we show that the shear cocycle σ^β_λ is contained in the closure of Teichmüller space $\mathcal{T} \subset \mathcal{H}(\lambda;\mathbb{R})$ and it is in $\mathcal{T}$ if β is strictly positive.

In Section 6 we formally define pleated surfaces and study their intrinsic geometric properties. Our analysis here is based on a precise understanding of the case of finite leaved maximal laminations and on continuity properties of pleated surfaces. The pleated set $\hat{S}_\lambda$ has an intrinsic length space structure that makes it locally isometric to $\mathbb{H}^2$ and the local isometry $\hat{f} : \hat{S}_\lambda \to \mathbb{H}^2$ is 1-Lipschitz with respect to the intrinsic pseudo-metric (see Proposition 6.5). We check that the shear cocycle of the intrinsic path metric on $\hat{S}_\lambda$ coincides with σ^ρ_λ (Proposition 6.6). In both cases, the proofs are elementary. The general case (Proposition 6.7) follows from the finite leaved case by continuity arguments. As a consequence, we derive a precise comparison between the length spectrum of S_λ and the length spectrum of ρ (Proposition 6.8).

In Section 7 we link the geometry of the maximal representation $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ to the geometry of the dominated set $\mathcal{P}_\rho \subset \mathcal{T}$ consisting of those hyperbolic surfaces Z whose length spectrum $L_Z(\bullet)$ is strictly dominated by $L_\rho(\bullet)$. Such a subset is non-empty, as it contains all pleated surfaces associated to ρ, and is convex with respect to the Weil-Petersson metric. We describe the structure of interior points and of boundary points (see Lemma 7.4 and Proposition 7.7) and show that if the representation ρ is not Fuchsian then the interior of $\mathcal{P}_\rho$ is never empty (Proposition 7.5).

In Section 8 we discuss the point of view of fibered photon structures. We introduce triangles and lines of photons $E(\Delta)$ and $E(\ell)$. We show that given a maximal lamination λ the Guichard-Wienhard domain of discontinuity admits a fibration $\pi : \Omega_\rho \to \hat{S}_\lambda$ where $\pi^{-1}(\ell) = E(\ell)$ and $\pi^{-1}(\Delta) = E(\Delta)$ for every leaf $\ell \subset \hat{\lambda}$ and plaque $\Delta \subset \hat{S}_\lambda - \hat{\lambda}$ (see Proposition 8.5). In the opposite direction, we construct fibered photon structures by gluing together triangles of photons along lines of photons. We discuss in detail pants of photons which are basic gluings of two triangles of photons. Provided that the holonomy along the boundary curves is loxodromic such pants of photons are the interior of fibered photon structures with totally geodesic boundary (see Lemma 8.9). We completely classify those (see Theorem 8.10). Gluing pants of photons with totally geodesic boundary one obtains fibered photon structures over closed hyperbolic surfaces $E \to S$. The holonomy $E \to S$ is always maximal (see Lemma 8.11).

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2. Preliminaries

In this section we review some basic facts that we will need in our exposition.

We start by discussing the geometry and causal structure of the pseudo-Riemannian space $\mathbb{H}^{2,n}$ and its boundary $\partial\mathbb{H}^{2,n}$. In particular, we discuss the Poincaré model of $\mathbb{H}^{2,n}$ (see Proposition 2.4) which is a useful device to examine the structure of acausal sets (see Lemmas 2.5 and 2.6).

Then, we introduce maximal representations $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ and describe the acausal and dynamical properties of their associated limit curve $\xi : \partial\Gamma \to \partial\mathbb{H}^{2,n}$ (see Theorem 2.9). This is the starting point of our constructions in Section 3.

Afterwards, we recall some background material from classical Teichmüller theory and introduce geodesic laminations, geodesic currents, and shear coordinates. Geodesic laminations are the objects that provide us a direct link between maximal representations and hyperbolic surfaces. We explain how to associate to a maximal lamination λ a pleated set $\hat{S}_\lambda \subset \mathbb{H}^{2,n}$ in Sections 3. We investigate the topology and causal properties of $\hat{S}_\lambda$ in Section 4 and we describe its geometries in Sections 5, and 6.

Geodesic currents and Teichmüller geometry are the main tools that we will use to analyze the length spectrum of maximal representations in Section 7.

We end the section by describing (positive) cross ratios on $\partial\Gamma$ and their associated Liouville currents (see Theorem 2.21). As we will see in Section 4, every maximal representation ρ has a natural strictly positive cross ratio β^ρ induced by the boundary map. As a consequence, its length spectrum can be represented by a Liouville current $\mathscr{L}_\rho$. Our use of cross ratios will be twofold: In Sections 4 and 5, we use β^ρ to encode the shear cocycle of a pleated surface. In Section 7, we use the Liouville current $\mathscr{L}_\rho$ to study the structure of the set of pleated surfaces.

2.1. The pseudo-Riemannian space $\mathbb{H}^{2,n}$.

We first introduce the linear and projective models of $\mathbb{H}^{2,n}$ and $\partial\mathbb{H}^{2,n}$: Let $\mathbb{R}^{2,n+1}$ denote the vector space $\mathbb{R}^{n+3}$ endowed with the quadratic form

$$\langle x, y\rangle_{(2,n+1)} := x_1y_1 + x_2y_2 - x_3y_3 - \ldots - x_{n+3}y_{n+3}$$

of signature $(2, n+1)$. Consider the hyperboloid

$$\widehat{\mathbb{H}}^{2,n} := \{x \in \mathbb{R}^{2,n} \mid \langle x, x\rangle_{(2,n+1)} = -1\}.$$

The restriction of the quadratic form $\langle \bullet, \bullet\rangle_{(2,n+1)}$ to each tangent space

$$T_x\widehat{\mathbb{H}}^{2,n} = x^\perp$$

has signature $(2, n)$ and, therefore, endows $\widehat{\mathbb{H}}^{2,n}$ with a pseudo-Riemannian structure of the same signature. The group $\mathrm{SO}_0(2, n+1)$ acts transitively and by orientation preserving isometries on $\widehat{\mathbb{H}}^{2,n}$. However, the action is not proper as point stabilizers are not compact.

Tangent vectors $v \in T_x\widehat{\mathbb{H}}^{2,n}$ split into three types:

$$v \text{ is } \begin{cases} \text{spacelike} & \text{if } \langle v, v\rangle > 0, \\ \text{lightlike} & \text{if } \langle v, v\rangle = 0, \\ \text{timelike} & \text{if } \langle v, v\rangle < 0. \end{cases}.$$

Similarly, we call a curve $\alpha : I \to \widehat{\mathbb{H}}^{2,n}$ spacelike, lightlike, or timelike if $\dot{\alpha}$ is always spacelike, lightlike, or timelike.

Geodesics in the linear model $\widehat{\mathbb{H}}^{2,n}$ are easy to describe: Let $x \in \widehat{\mathbb{H}}^{2,n}$ be a point and $v \in T_x\widehat{\mathbb{H}}^{2,n}$ a tangent vector. Let $\gamma : \mathbb{R} \to \widehat{\mathbb{H}}^{2,n}$ be the geodesic starting at x with velocity v. Then

$$\gamma(t) = \begin{cases} \cosh(t)x + \sinh(t)v & \text{if } \langle v, v\rangle = 1, \\ x + tv & \text{if } \langle v, v\rangle = 0, \\ \cos(t)x + \sin(t)v & \text{if } \langle v, v\rangle = -1. \end{cases} .$$

The pseudo-Riemannian space $\mathbb{H}^{2,n}$ is the quotient

$$\mathbb{H}^{2,n} := \widehat{\mathbb{H}}^{2,n}/(x \sim -x)$$

and can be realized as an open subset of the projective space $\mathbb{RP}^{n+2}$. The projection $\mathbb{R}^{2,n+1} - \{0\} \to \mathbb{RP}^{n+2}$ induces the 2-to-1 covering projection $\widehat{\mathbb{H}}^{2,n} \to \mathbb{H}^{2,n}$. In the projective model, the geodesic starting at x with velocity v is just the intersection of the projective line corresponding to the 2-plane $\mathrm{Span}\{x, v\}$ with $\mathbb{H}^{2,n}$. Given two points $x, y \in \mathbb{H}^{2,n}$, they are always connected by a geodesic, namely, the projective line corresponding to $\mathrm{Span}\{x, y\}$. The type of the line can be determined using the following simple criterion:

Lemma 2.1 ([GM21, Proposition 3.2])**.** *Two distict points $x, y \in \mathbb{H}^{2,n}$ are joined by:*

- *A spacelike geodesic if and only if $|\langle x, y\rangle_{(2,n+1)}| > 1$.*
- *A lightlike geodesic if and only if $|\langle x, y\rangle_{(2,n+1)}| = 1$.*
- *A timelike geodesic if and only if $|\langle x, y\rangle_{(2,n+1)}| < 1$.*

Similar to what happens for geodesics, the intersection of a linear space

$$\mathbb{P}\left(V \subset \mathbb{R}^{2,n+1}\right) \subset \mathbb{RP}^{n+2}$$

with $\mathbb{H}^{2,n}$ also gives a totally geodesic subspace. In particular, every 3-dimensional subspace V on which the restriction of the quadratic form has signature $(2, 1)$ provides a totally geodesic subspace of $\mathbb{H}^{2,n}$ isometric to $\mathbb{H}^2$.

The space $\mathbb{H}^{2,n}$ has a natural boundary at infinity which can be described in the projective model as the projectivization of the cone of isotropic vectors $\partial\mathbb{H}^{2,n} = \mathbb{P}(C)$, where

$$C := \{x \in \mathbb{R}^{2,n} \mid \langle x, x\rangle_{(2,n+1)} = 0\}.$$

In the linear model $\widehat{\mathbb{H}}^{2,n}$, the boundary at infinity is $\partial\widehat{\mathbb{H}}^{2,n}$, a two-fold covering of $\partial\mathbb{H}^{2,n}$, which can be described as $C/x \sim \lambda^2 x$. We can topologize

$$\widehat{\mathbb{H}}^{2,n} \cup \partial\widehat{\mathbb{H}}^{2,n}$$

by simultaneously embedding them in the sphere of rays $\mathbb{R}^{2,n+1} - \{0\}/x \sim \lambda^2 x$.

Similar to Lemma 2.1, we have

Lemma 2.2. *Two distict points $x, y \in \partial\mathbb{H}^{2,n}$ are joined by:*

- *A spacelike geodesic if and only if $\langle x, y\rangle_{(2,n+1)} \neq 0$.*
- *A lightlike geodesic if and only if $\langle x, y\rangle_{(2,n+1)} = 0$.*

Centered at each point $a \in \partial\mathbb{H}^{2,n}$ we have a family of *horoballs*

$$O = \mathbb{P}\{x \in \widehat{\mathbb{H}}^{2,n} \mid 0 < |\langle x, v\rangle_{2,n+1}| < c\} \subset \mathbb{H}^{2,n}$$

where $v \in \mathbb{R}^{2,n}$ is a representative of a and $c > 0$ is a positive constant. The boundary $\partial O \subset \mathbb{H}^{2,n}$ is a *horosphere* centered at a.

The terminology is justified by the fact that every space-like 2-plane H of $\mathbb{H}^{2,n}$ whose boundary at infinity contains $a \in \partial\mathbb{H}^{2,n}$ intersects O and ∂O in a (usual) horoball or horocycle based at $a \in \partial H$.

For more material on the geometry of $\mathbb{H}^{2,n}$, we also refer to Section 2 of [GM21] and Section 3 of [CTT19].

2.2. **Acausal sets and Poincaré model.** Certain subsets of $\widehat{\mathbb{H}}^{2,n}$ display some features that make them similar to metric spaces and, in particular, their geometry can be compared to the one of subsets of $\mathbb{H}^2$.

Definition 2.3 (Acausal)**.** Let X be a subset of $\widehat{\mathbb{H}}^{2,n} \cup \partial\widehat{\mathbb{H}}^{2,n}$. We say that X is *acausal* if any pair of distinct points in X is joined by a spacelike geodesic.

Acausal subsets $X \subset \widehat{\mathbb{H}}^{2,n}$ are naturally endowed with a pseudo-distance, denoted by $d_{\widehat{\mathbb{H}}^{2,n}}$: The distance between two points $x, y \in X$ is the length of the unique spacelike segment $[x, y]$ joining them and is computed by

$$\cosh\left(d_{\widehat{\mathbb{H}}^{2,n}}(x, y) = \ell[x, y]\right) = -\langle x, y\rangle.$$

Using this formula, it is simple to check that $d_{\widehat{\mathbb{H}}^{2,n}} : X \times X \to [0, \infty)$ satisfies the following two properties:

- It is continuous.
- It vanishes precisely on the diagonal $\Delta_X \subset X \times X$.

However, it does not satisfy the triangle inequality. The two properties together imply that, for every $x \in X$, the sets $B_r(x) := \left\{y \in X \,\middle|\, d_{\widehat{\mathbb{H}}^{2,n}}(x, y) < r\right\}$ form a fundamental system of neighbourhoods of x.

We now introduce a useful way of representing acausal subsets which we will use extensively later on in our computations. This is the Poincaré model $\mathbb{H}^{2,n}$, as described in [CTT19]: Let $E \subset \mathbb{R}^{2,n+1}$ be a spacelike 2-plane. Let $E^\perp$ its orthogonal with respect to $\langle \bullet, \bullet\rangle_{2,n+1}$. Consider the Euclidean disk

$$\mathbb{D}^2 := \{u \in E \mid \langle u, u\rangle_{2,n+1} < 1\}$$

and the round sphere

$$\mathbb{S}^n := \{v \in E^\perp \mid \langle v, v\rangle_{2,n+1} = -1\}.$$

Define the maps

$$\begin{array}{rccc} \Psi : & \mathbb{D}^2 \times \mathbb{S}^n & \longrightarrow & \widehat{\mathbb{H}}^{2,n} \\ & (u, v) & \longmapsto & \frac{2}{1-\|u\|^2} u + \frac{1+\|u\|^2}{1-\|u\|^2} v. \end{array}$$

and

$$\begin{array}{rccc} \partial\Psi : & \partial\mathbb{D}^2 \times \mathbb{S}^n & \to & \partial\widehat{\mathbb{H}}^{2,n} \\ & (u, v) & \longmapsto & u + v. \end{array}$$

We have

Proposition 2.4 ([CTT19, Proposition 3.5])**.** *For any spacelike 2-plane $E \subset \mathbb{R}^{2,n+1}$, we have the following*

(a) The map $\Psi = \Psi_E$ is a diffeomorphism.

(b) *The pull-back pseudo-Riemannian metric can be written as*

$$\Psi^* g_{\widehat{\mathbb{H}}^{2,n}} = \left(\frac{2}{1-\|u\|^2}\right)^2 |\mathrm{d}u|^2 - \left(\frac{1+\|u\|^2}{1-\|u\|^2}\right)^2 g_{\mathbb{S}^n}.$$

(c) *The map $\partial\Psi$ is a diffeomorphism and extends continuously Ψ.*

The Poincaré model is especially useful when dealing with acausal subsets for the following reasons: Firstly, acausal subsets $X \subset \widehat{\mathbb{H}}^{2,n}$ can always be written as graphs of functions $g : U \subset \mathbb{D}^2 \to \mathbb{S}^n$ that are 1-Lipschitz with respect to the hemispherical metric

$$g_{\mathbb{S}^2} := \left(\frac{2}{1+\|u\|^2}\right)^2 |\mathrm{d}u|^2$$

on $\mathbb{D}^2$ and the spherical metric on $\mathbb{S}^n$. Secondly, the graph map $u : U \subset \mathbb{D}^2 \to X \subset \widehat{\mathbb{H}}^{2,n}$ is 1-Lipschitz with respect to the hyperbolic metric

$$g_{\mathbb{H}^2} := \left(\frac{2}{1-\|u\|^2}\right)^2 |\mathrm{d}u|^2$$

on $\mathbb{D}^2$ and the intrinsic pseudo-metric on $X \subset \widehat{\mathbb{H}}^{2,n}$. In both cases, for us, compactness properties of 1-Lipschitz maps will translate in compactness properties of acausal subsets.

We start with the following lemma:

Lemma 2.5. *Let $E \subset \mathbb{R}^{2,n+1}$ be a spacelike 2-plane. Let $\Psi : \mathbb{D}^2 \times \mathbb{S}^n \to \widehat{\mathbb{H}}^{2,n}$ be the corresponding Poincaré model. Consider the points $x := \Psi(u,v)$ and $x' = \Psi(u',v')$. We have the following:*

(1) *x, x' are joined by a spacelike segment if and only if they are contained in the same connected component of $\widehat{\mathbb{H}}^{2,n} - x^\perp$ and*

$$d_{\mathbb{S}^n}(v,v') < d_{\mathbb{S}^2}(u,u')$$

where we endow $\mathbb{D}^2$ with the hemispherical distance.

(2) *x, x' are joined by a lightlike segment if and only if*

$$d_{\mathbb{S}^n}(v,v') = d_{\mathbb{S}^2}(u,u')$$

where we endow $\mathbb{D}^2$ with the hemispherical distance.

(3) *If x, x' are joined by a spacelike geodesic, then*

$$d_{\mathbb{H}^2}(u,u') \geqslant d_{\widehat{\mathbb{H}}^{2,n}}(x,x'),$$

where we endow $\mathbb{D}^2$ with the hyperbolic distance.

Proof. We prove each property separately.

Property (1). The spherical distance between two points $v, v' \in \mathbb{S}^n$ is computed by

$$\cos\big(d_{\mathbb{S}^n}(v,v')\big) = \langle v, v'\rangle_{\mathbb{S}^n} = -\langle v, v'\rangle_{2,n+1}.$$

Similarly, the hemispherical distance between $u, u' \in \mathbb{D}^2$ is given by

$$\cos\big(d_{\mathbb{S}^2}(u,u')\big) = \frac{2}{1+\|u\|^2}\frac{2}{1+\|u'\|^2}\langle u, u'\rangle_{2,n+1} + \frac{1-\|u\|^2}{1+\|u\|^2}\frac{1-\|u'\|^2}{1+\|u'\|^2}.$$

By Lemma 2.1, x, x' are joined by a spacelike geodesic if and only

$$\begin{aligned} -1 &> \langle x, x' \rangle = \langle \Psi(u,v), \Psi(u',v') \rangle \\ &= \frac{2}{1-\|u\|^2}\frac{2}{1-\|u'\|^2}\langle u, u' \rangle_{2,n+1} + \frac{1+\|u\|^2}{1-\|u\|^2}\frac{1+\|u'\|^2}{1-\|u'\|^2}\langle v, v \rangle_{2,n+1} \end{aligned}$$

which can be rewritten as

$$\frac{1+\|u\|^2}{1-\|u\|^2}\frac{1+\|u'\|^2}{1-\|u'\|^2}\left(\cos\left(d_{\mathbb{S}^2}(u,u')\right) - \cos\left(d_{\mathbb{S}^n}(v,v')\right)\right) < 0.$$

The latter is equivalent to

$$d_{\mathbb{S}^n}(v, v') < d_{\mathbb{S}^2}(u, u'),$$

as desired.

Property (2). By Lemma 2.1, x, x' are joined by a lightlike geodesic if and only if $\langle x, x' \rangle = -1$. The coclusion follows from a computation which is exactly analogue to the one of Property (1).

Property (3). If x, x' are joined by a spacelike geodesic, then $\langle x, x' \rangle < -1$, and their distance in $\widehat{\mathbb{H}}^{2,n}$ is computed by

$$\begin{aligned} \cosh\left(d_{\widehat{\mathbb{H}}^{2,n}}(x,x')\right) &= -\langle \Psi(u,v), \Psi(u',v') \rangle \\ &= -\frac{2}{1-\|u\|^2}\frac{2}{1-\|u'\|^2}\langle u, u' \rangle + \frac{1+\|u\|^2}{1-\|u\|^2}\frac{1+\|u'\|^2}{1-\|u'\|^2}\langle v, v' \rangle. \end{aligned}$$

The hyperbolic distance in $\mathbb{D}^2$ is computed by

$$\cosh\left(d_{\mathbb{H}^2}(u,u')\right) = -\frac{2}{1-\|u\|^2}\frac{2}{1-\|u'\|^2}\langle u, u' \rangle + \frac{1+\|u\|^2}{1-\|u\|^2}\frac{1+\|u'\|^2}{1-\|u'\|^2}.$$

Since $v, v' \in \mathbb{S}^n$, we have $|\langle v, v' \rangle| \leqslant 1$. Therefore, we have

$$\cosh\left(d_{\widehat{\mathbb{H}}^{2,n}}(x,x')\right) \leqslant \cosh\left(d_{\mathbb{H}^2}(u,u')\right)$$

with equality if and only if $v = v'$. $\square$

We can now prove that acausal subsets can be described as graphs of 1-Lipschitz functions. This is the content of the next lemma:

Lemma 2.6. *Let $E \subset \mathbb{R}^{2,n+1}$ be a spacelike 2-plane. Let $\Psi : \mathbb{D}^2 \times \mathbb{S}^n \to \widehat{\mathbb{H}}^{2,n}$ be the corresponding Poincaré model and let $\pi : \widehat{\mathbb{H}}^{2,n} \to \mathbb{D}^2$ be the projection to the first factor. Let $X \subset \widehat{\mathbb{H}}^{2,n}$ be an acausal subset. Then*

(1) The projection $\pi : X \to \mathbb{D}^2$ is injective. In particular, we can write X as the graph of a function $g : \pi(X) \subset \mathbb{D}^2 \to \mathbb{S}^n$.

(2) The function g is strictly 1-Lipschitz with respect to the hemispherical metric on $\mathbb{D}^2$ and the standard spherical metric on $\mathbb{S}^n$, that is,

$$d_{\mathbb{S}^n}(g(u), g(u')) < d_{\mathbb{S}^2}(u, u')$$

for any distinct $u, u' \in \pi(X)$.

(3) Viceversa, the graph of any strictly 1-Lipschitz function $g : U \subset \mathbb{D}^2 \to \mathbb{S}^n$ defined on a connected subset $U \subset \mathbb{D}^2$ is an acausal subset.

Proof. **Property (1).** By Property (3) of Lemma 2.5, the restriction of the projection π on X is injective. In particular, for any $u \in \pi(X) \subseteq \mathbb{D}^2$, there exists a unique $g(u) \in \mathbb{S}^n$ such that $\Psi(u, g(u)) \in X$.
Property (2). As X is acausal, for every $u, u' \in \mathbb{D}^2$ the points $\Psi(u, g(u))$ and $\Psi(u', g(u'))$ are joined by a spacelike segment. The conclusion follows from Property (1) of Lemma 2.5.
Property (3) follows from Property (1) of Lemma 2.5 once we observe the following: Consider a point $u \in U$. Let $x := \Psi(u, g(u))$. As g is strictly 1-Lipschitz, by Property (2) of Lemma 2.5, its graph cannot intersect $x^\perp$. Therefore, by connectedness of U, the graph is contained in the same component of x. □

We now restrict our attention to some special acausal subsets, namely spacelike geodesics and spacelike planes and prove a couple of topological properties that will be useful later on:

Lemma 2.7. *Let $E \subset \mathbb{R}^{2,n+1}$ be a spacelike 2-plane. Let $\Psi : \mathbb{D}^2 \times \mathbb{S}^n \to \widehat{\mathbb{H}}^{2,n}$ be the corresponding Poincaré model and let $\pi : \widehat{\mathbb{H}}^{2,n} \to \mathbb{D}^2$ be the projection to the first factor. Then*

1. *If $H \subset \widehat{\mathbb{H}}^{2,n}$ is a spacelike plane, then the restriction of $\pi : H \to \mathbb{D}^2$ is a diffeomorphism and extends continuously to $\partial H \to \partial \mathbb{D}^2$.*
2. *If $\ell \subset \widehat{\mathbb{H}}^{2,n}$ is a pacelike geodesic, then $\pi(\ell)$ is a smooth properly embedded curve. Either it is a diameter or it intersects every diameter at most once.*

Proof. We prove the properties in order.
Property (1). Since H is transverse to the fibers $\Psi(\{x\} \times \mathbb{S}^n)$ of π, the restriction $\pi : H \to \mathbb{D}^2$ is a local diffeomorphism. As the pseudo-distance $d_{\widehat{\mathbb{H}}^{2,n}}$ restricts to the hyperbolic distance on H, by Property (3) of Lemma 2.5, we also have that π is distance non-decreasing when we put on $\mathbb{D}^2$ the hyperbolic metric. In particular, π is proper and injective. Thogether, the two facts imply that $\pi : H \to \mathbb{D}^2$ is a diffeomorphism.
Property (2). By the previous point, $\pi : \ell \to \mathbb{D}^2$ is a smooth proper embedding. We claim that the projection $\pi(\ell)$ is either a diameter of $\mathbb{D}^2$, or it intersects every diameter of $\mathbb{D}^2$ at most once. In order to see this, parametrize ℓ as $\ell(t) = e^t a + e^{-t} b$ and write $a = u_a + v_a$ and $b = u_b + v_b$ with $u_a, u_b \in \partial \mathbb{D}^2$ and $v_a, v_b \in \mathbb{S}^n$. The projection of $\ell(t)$ to $\mathbb{D}^2$ is a curve $u(t)$ satisfying

$$e^t u_a + e^{-t} u_b = \frac{2}{1 - |u|^2} u.$$

In particular, $u(t)$ intersects the line $pu^1 + qu^2 = 0$ if and only if $p(e^t u_a^1 + e^{-t} u_b^2) + q(e^t u_a^1 + e^{-t} u_b^2) = 0$ and this has at most one solution unless $\pi(\ell) = \{pu^1 + qu^2 = 0\}$. □

2.3. **Maximal representations.** We now introduce maximal representations in $\mathrm{SO}_0(2, n+1)$ and a couple of geometric objects that are naturally associated to them.

The first geometric object which one can attach to every representation $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ is a flat vector bundle $V_\rho \to \Sigma$. The total space V_ρ is defined as follows:

$$V_\rho := \hat{\Sigma} \times \mathbb{R}^{2,n+1} / (x, v) \sim (\gamma x, \rho(\gamma) v).$$

Here $\hat{\Sigma}$ is the universal covering of Σ and γ acts on it as a deck transformation. The bundle projection $V_\rho \to \Sigma$ is just the one induced by the the universal covering projection $\hat{\Sigma} \to \Sigma$.

The vector bundle $V_\rho \to \Sigma$ has an associated cohomological invariant $T(\rho) \in \mathbb{Z}$, called the *Toledo invariant* (see [BIW10]). The number $T(\rho)$ always satisfies a Milnor-Wood inequality $|T(\rho)| \leqslant 2|\chi(\Sigma)|$.

Definition 2.8 (Maximal Representaion)**.** A representation ρ is called *maximal* if it satisfies $|T(\rho)| = 2|\chi(\Sigma)|$.

Thank to the work of Burger, Iozzi, Labourie and Wienhard [BILW05], we can equivalently describe maximal representations in terms of *boundary maps*. Here we will mostly adopt this more geometric perspective.

Theorem 2.9 (Burger, Iozzi, Labourie and Wienhard [BILW05])**.** *A representation $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ is maximal if and only if there exists a ρ-equivariant Hölder continuous embedding*

$$\xi : \partial\Gamma \to \partial\mathbb{H}^{2,n}$$

such that $\Lambda_\rho := \xi(\partial\Gamma)$ is an acausal *curve, meaning that for every triple of distinct points $u, v, w \in \partial\Gamma$, the subspace of $\mathbb{R}^{2,n+1}$ spanned by the lines $\xi(u), \xi(v), \xi(w)$ has signature $(2, 1)$.*

The second geometric object that we associate to a maximal representation ρ is a pseudo-Riemannian manifold M_ρ locally modeled on $\mathbb{H}^{2,n}$: Even though the group $\rho(\Gamma)$ does not act properly discontinuously on the whole $\mathbb{H}^{2,n}$, it admits a nice domain of discontinuity. In fact, maximal representations in $\mathrm{SO}_0(2, n+1)$ are convex cocompact in the sense of [DGK18] and [DGK17]. This means that in the projective model $\mathbb{H}^{2,n} \subset \mathbb{P}(\mathbb{R}^{2,n+1})$ they preserve a properly convex open subset $\Omega_\rho \subset \mathbb{H}^{2,n}$ with $\mathcal{C}^1$-boundary $\partial\Omega_\rho$ containing the limit curve $\Lambda_\rho \subset \partial\Omega_\rho$ and that the action is cocompact on the convex hull $\mathcal{CH}(\Lambda_\rho) \cap \Omega_\rho$ of the limit curve.

As the representation acts by projective transformations on Ω_ρ, it preserves the natural Hilbert metric on the convex domain and, hence, the action on Ω_ρ is properly discontinuous. Furthermore, as every $\rho(\gamma)$ acts by isometries on Ω_ρ and has an attracting fixed point on $\Lambda_\rho \subset \partial\Omega_\rho$, it follows that $\rho(\gamma)$ cannot have fixed points in Ω_ρ, so that the action is also free. In conclusion, since the action is free and properly discontinuous, we can associate to ρ the pseudo-Riemannian manifold $M_\rho := \Omega_\rho/\rho(\Gamma)$. The quotient

$$\mathcal{CC}(M_\rho) := (\mathcal{CH}(\Lambda_\rho) \cap \Omega_\rho)/\rho(\Gamma)$$

is the convex core of M.

Notice that Ω_ρ is by no means unique, however, the subset $\mathcal{CH}(\Lambda_\rho) \subset \Omega_\rho$ does not depend on the choice of the domain. Therefore, the geometry of the convex core $\mathcal{CC}(M_\rho)$ is also independent of the choice of Ω_ρ.

Let us observe that, as

$$\Omega_\rho \cup \partial\Omega_\rho \subset \mathbb{H}^{2,n} \cup \partial\mathbb{H}^{2,n}$$

is simply connected, we can lift it to $\widehat{\mathbb{H}}^{2,n} \cup \partial\widehat{\mathbb{H}}^{2,n}$.

2.4. **Hyperbolic surfaces and Teichmüller space.** When $n = 0$, Goldman [Gol80] has shown that maximal representations in $\mathrm{SO}_0(2,1)$ correspond exactly to holonomies of hyperbolic structures on Σ.

We will denote by $\mathcal{T}$ the classical Teichmüller space that parametrizes such hyperbolic structures on Σ up to isotopy. We recall that our goal is to relate the geometry of maximal representations to the one of hyperbolic surfaces.

We end this section by collecting some facts from classical Teichmüller theory that will be needed later on starting from geodesic laminations which are one of our main tools.

2.4.1. *Geodesic laminations.* We start with some familiar properties of the hyperbolic plane: Every geodesic on $\mathbb{H}^2$ can be uniquely identified by its pair of endpoints on $\partial\mathbb{H}^2 = S^1$.

Definition 2.10 (Space of Geodesics)**.** The space of unoriented geodesics of $\mathbb{H}^2$ is

$$\mathcal{G} := (\partial\Gamma \times \partial\Gamma - \Delta)/(x, y) \sim (y, x).$$

Given two geodesics $\ell, \ell' \subset \mathbb{H}^2$ we can also describe their relative position by looking at the configuration of their endpoints at infinity. More precisely:

Definition 2.11 (Crossing and Disjoint)**.** Let $a, b, a', b' \in S^1$ be four points on a circle such that $a \neq b$ and $a' \neq b'$. We say that the pairs (a, b) and (a', b') are *crossing* if a', b' are contained in distinct components of $S^1 - \{a, b\}$ and *disjoint* otherwise.

We now fix once and for all a reference hyperbolic structure on Σ and identify $\partial\mathbb{H}^2$ with $\partial\Gamma$.

Definition 2.12 (Geodesic Lamination)**.** A geodesic lamination λ on Σ is a closed Γ-invariant subset $\hat{\lambda} \subset \mathcal{G}$ such that every pair of geodesics $\ell, \ell' \in \lambda$, the *leaves* of the geodesic lamination, is disjoint.

A lamination is *maximal* if $\mathbb{H}^2 - \hat{\lambda}$ consists of ideal triangles. Each connected component of $\mathbb{H}^2 - \hat{\lambda}$ is called a *plaque*.

We denote by $\mathcal{GL}$ the space of geodesic laminations on Σ. As geodesic laminations are subsets of the spaces of closed subsets of $\mathcal{G}$, the space $\mathcal{GL}$ is naturally endowed with Chabauty (or Hausdorff) topology. It is a standard fact (see Proposition I.4.1.7 in [CEG06]) that $\mathcal{GL}$ is compact with respect to this topology.

For more material on laminations on surfaces we refer to Chapter I.4 of [CEG06].

2.4.2. *Geodesic currents and measured laminations.* Geodesic currents were introduced by Bonahon [Bon88]. They are defined as follows:

Definition 2.13 (Geodesic Currents)**.** A *geodesic current* is a Γ-invariant locally finite Borel measure μ on $\mathcal{G}$. We denote by $\mathcal{C}$ the space of geodesic currents.

The space $\mathcal{C}$ has a natural structure of a cone and is endowed with a natural weak-$\star$ topology. Furthermore, it is endowed with a natural continuous symmetric bilinear form

$$i(\bullet, \bullet) : \mathcal{C} \times \mathcal{C} \to [0, \infty),$$

called the *intersection form*. We briefly recall its definition, as we will use it later on: Let $\mathcal{J} \subset \mathcal{G} \times \mathcal{G}$ be the space of pair of geodesics (ℓ, ℓ') intersecting transversely $\ell \pitchfork \ell'$. The group Γ acts properly discontinuously on $\mathcal{J}$. Given two geodesic currents

$\alpha, \beta \in \mathcal{C}$, they induce a Γ-invariant measure $\alpha \times \beta$ on $\mathcal{J}$ and, hence, a well defined measure $\alpha \times \beta$ on the quotient $\mathcal{J}/\Gamma$ which it is possible to show to be always finite. The intersection between α, β is defined as

$$i(\alpha, \beta) := (\alpha \times \beta)(\mathcal{J}/\Gamma).$$

Definition 2.14 (Measured Lamination)**.** A *measured lamination* on Σ is a geodesic current $\mu \in \mathcal{C}$ such that $i(\mu, \mu) = 0$.

It is a standard fact, that, with this definition, the support of a measured lamination is a geodesic lamination of Σ (see Proposition 17 of Bonahon [Bon88]). We denote by $\mathcal{ML}$ the space of measured laminations on Σ.

Bonahon shows that the following natural objects associated to Σ embed canonically in $\mathcal{C}$:

- The space $\mathcal{S}$ of free homotopy class of closed curves of Σ.
- The space $\mathcal{T}$ of isotopy classes of hyperbolic metrics on Σ.

We will make no distinction between a point in these spaces and its image in the space of currents $\mathcal{C}$. Bonahon also proves that, with respect to the intersection form $i(\bullet, \bullet)$ we have the following relations:

- If $\alpha, \beta \in \mathcal{S}$, then $i(\alpha, \beta)$ is the intersection number between α, β.
- If $\alpha \in \mathcal{S}$ and $X \in \mathcal{T}$, then $i(X, \alpha) = \ell_X(\alpha)$ is the length of α on X.

In particular, the intersection form provides a continuous extension of the length function $\ell_X(\bullet) : \mathcal{S} \to (0, \infty)$ to a continuous positive function on the space of geodesic currents as $i(X, \bullet) : \mathcal{C} \to (0, \infty)$. For more details on such properties, we refer to Bonahon [Bon88].

2.4.3. *Shear coordinates.* Let λ be a maximal lamination of Σ. Following Bonahon [Bon96], we have the following definition:

Definition 2.15 (Hölder Cocycle)**.** A *Hölder cocycle* transverse to λ is a real-valued function on the set of pairs of distinct plaques of $\hat{\lambda}$ that satisfies:

1. Symmetry: For every pair of distinct plaques P, Q, we have $\sigma(P, Q) = \sigma(Q, P)$.
2. Additivity: For every pair of distinct plaques P, Q, and for every plaque R that separates P from Q, we have $\sigma(P, Q) = \sigma(P, R) + \sigma(R, Q)$.
3. Invariance: For every pair of distinct plaques P, Q and for every $\gamma \in \Gamma$, we have $\sigma(P, Q) = \sigma(\gamma P, \gamma Q)$.

We denote by $\mathcal{H}(\lambda; \mathbb{R})$ the space of Hölder cocycles transverse to λ. It is a real vector space of dimension $3|\chi(\Sigma)|$.

Hölder cocycles are a useful device that allow to encode among other things the following data:

- Every hyperbolic metric $X \in \mathcal{T}$ has an associated shear cocycle $\sigma_\lambda^X \in \mathcal{H}(\lambda, \mathbb{R})$ that describes the relative position of the plaques of $X - \lambda$.
- Every measured lamination $\mu \in \mathcal{ML}$ with support contained in λ has an associated transverse cocycle $\mu \in \mathcal{H}(\lambda, \mathbb{R})$ and a length functional $L_\mu : \mathcal{H}(\lambda; \mathbb{R}) \to \mathbb{R}$ whose evaluation on shear cocycles σ_λ^X coming from hyperbolic metrics $X \in \mathcal{T}$ equals $L_X(\mu)$.

We refer to Bonahon [Bon96] for the details of the construction.

The space of Hölder cocycles transverse to λ is naturally endowed with a symplectic form $\omega_\lambda(\bullet,\bullet)$, called the *Thurston's symplectic form*, which essentially generalizes the notion of intersection between geodesic currents to transverse Hölder distributions, in the sense of [RS75]. The form ω_λ can be also described concretely in terms of the classical algebraic intersection between 1-chains on a surface, interpretation that will be recalled in Section 4.5.1 from the work of Bonahon [Bon96].

The Thurston symplectic form was deployed by Bonahon [Bon96] to relate the notion of shear cocycle σ_λ^X associated to a hyperbolic structure $X \in \mathcal{T}$ with the notion of hyperbolic length for measured laminations. Concretely, we have that for every $X \in \mathcal{T}$ and $\mu \in \mathcal{ML}$ with support contained in λ, the following relation holds:

$$\omega(\sigma_\lambda^X, \mu) = L_X(\mu)$$

(see in particular [Bon96, Theorem E]).

The Thurston symplectic form is particularly relevant in the study of shear cocycles because it provides a complete characterization of the set of transverse Hölder cocycles that can be realized as shears of hyperbolic metrics. Inspired by ideas of Thurston [Thu98], Bonahon proved the following parametrization result:

Theorem 2.16 (Bonahon [Bon96, Theorems A, B])**.** *For any maximal geodesic lamination λ of Σ the map $\mathcal{T} \to \mathcal{H}(\lambda,\mathbb{R})$ sending*

$$X \in \mathcal{T} \to \sigma_\lambda^X \in \mathcal{H}(\lambda,\mathbb{R})$$

is a real analytic diffeomorphism. The image of the map is the open convex cone

$$C := \{\sigma \in \mathcal{H}(\lambda,\mathbb{R}) \mid \omega(\mu,\sigma) > 0 \text{ for every } \mu \in \mathcal{ML} \text{ with } \mathrm{supp}(\mu) \subset \lambda\}$$

where $\omega(\bullet,\bullet)$ is the Thurston's symplectic form on $\mathcal{H}(\lambda,\mathbb{R})$.

The resulting set of coordinates for Teichmüller space are called *shear coordinates* relative to λ.

Let us mention that, Bonahon and Sözen [SB01] proved that the pulll-back of the Thurston's symplectic form ω via the above diffeomorphism is (a multiple of) the Weil-Petersson symplectic form on Teichmüller space. In Section 7, we will use the Weil-Petersson geometry of Teichmüller to study the length spectrum of maximal representations $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$.

When dealing with different spaces of Hölder cocycles $\mathcal{H}(\lambda;\mathbb{R})$ relative to nearby laminations $\lambda \in \mathcal{GL}$, it is useful to identify all such spaces with the space of real weights $\mathcal{W}(\tau;\mathbb{R})$ of a suitable train track τ carrying all the laminations considered. This is particularly convenient when studying continuity properties of maps $\lambda \in \mathcal{GL} \to \sigma_\lambda \in \mathcal{H}(\lambda;\mathbb{R})$ as we will need later on.

Thus, we now briefly introduce train tracks and systems of real weights.

2.4.4. *Train tracks.* We recall the necessary terminology (see e.g. [PH92, Bon97b, Bon97a, BD17]). We define a *branch* inside Σ to be a homeomorphism $\varphi : [0,1] \times [0,1] \to B$ (which we abusively identify with its image B). We refer to: (the images of) the curves $t \mapsto \varphi(t,\cdot)$ as the *ties* of the branch B, to $\partial_v B := \varphi(\{0,1\} \times [0,1])$ and $\partial_h B := \varphi([0,1] \times \{0,1\})$ as its *vertical* and *horizontal* boundaries, respectively, and to the images of the points in $\{0,1\} \times \{0,1\}$ through the map φ as its *vertices*.

We then define a (trivalent) *train track* τ as a closed subset of Σ that can be decomposed into the union of a finite number of branches $(B_i)_i$ satisfying the following conditions:

i) every connected component of the intersection $B_i \cap B_j$ between two distinct branches coincides a connected component of $\partial_v B_i$, it is contained in a component of $\partial_v B_j$, and it contains exactly one vertex of B_j (up to exchanging the roles of i and j);

ii) for every i, each vertex of B_i is contained in the vertical boundary of some branch B_j, with $i \neq j$;

iii) no complementary region of the interior of τ is homeomorphic to a disc that intersects 0, 1 or 2 distinct components of the vertical boundaries of the branches $(B_i)_i$.

Any tie of a branch B_i of τ that is not strictly contained inside a connected component of the vertical of some (possibly different) branch B_j will be simply called a *tie* of the train track τ. The *horizontal boundary* $\partial_h \tau$ of τ is defined as the union of the horizontal boundaries of its branches, and the closure of $\partial \tau - \partial_h \tau$ is called the *vertical boundary* $\partial_v \tau$ of τ. The ties of τ that contain a component of $\partial_v \tau$ are called *switches*. A switch coincides with a connected component of the vertical boundary of some branch B_i in τ, and strictly contains two components of the vertical boundary of some branches B_j, B_k of τ (possibly two of the three branches B_i, B_j, B_k coincide). Moreover, every switch contains exactly one connected component c of the vertical boundary of τ.

If $\tilde{\tau}$ is the preimage of τ in the universal cover of Σ, then a branch of $\tilde{\tau}$ is simply the lift of some branch of τ. Similarly we define the ties, the switches, the vertical and horizontal boundary of $\tilde{\tau}$ and of its branches. We say that a train track τ carries a lamination λ if λ is contained in the interior of τ and every tie of τ is transverse to the leaves of λ.

Train tracks come naturally together with a vector space of real weights as we now describe.

2.4.5. *Systems of real weights.* Given τ a trivalent train track of Σ, a *system of real weights* $\eta = (\eta_i)_i$ of τ is a real-valued function on the set of branches $(B_i)_i$ of τ that satisfies a natural linear constraint for every switch of τ (compare with [Bon97a], or [Bon96, Section 3]): for any switch s, let B_i^s, B_j^s, B_k^s be the branches of τ adjacent to s, and assume that s coincides with a connected component of the vertical boundary of the branch B_i^s. If $\eta_i^s, \eta_j^s, \eta_k^s$ denote the weights associated by η with B_i^s, B_j^s, B_k^s, respectively, then we require η to satisfy $\eta_i^s = \eta_j^s + \eta_k^s$, for any switch s of τ.

We denote by $\mathcal{W}(\tau; \mathbb{R})$ the space of systems of real weights of τ. Observe that $\mathcal{W}(\tau; \mathbb{R})$ is naturally endowed with a real vector space structure, and its dimension is completely determined by the topology of τ (see e. g. [Bon97b, Theorem 15]). In particular, if τ carries a maximal lamination, which will be the only case we will be interested in, then $\mathcal{W}(\tau; \mathbb{R}) \cong \mathbb{R}^{-3\chi(S)}$. Moreover, for any maximal lamination λ' carried by τ, there exists a natural isomorphism $\mathcal{H}(\lambda'; \mathbb{R}) \cong \mathcal{W}(\tau; \mathbb{R})$, which can be described as follows: let α be a Hölder cocycle transverse to λ', and let B_i be a branch of τ. Select arbitrarily a lift $\widetilde{B}_i$ of B_i to the universal cover $\widetilde{\Sigma}$, and select a tie k_i of $\widetilde{B}_i$ disjoint from its vertical boundary. Since τ carries λ', there exist two distinct plaques P_i', Q_i' of λ' whose interior contain the endpoints of k_i. Then we define the real weight of α associated with B_i to be $\alpha_i := \alpha(P_i', Q_i') \in \mathbb{R}$. By the properties of Hölder cocycles (see Definition 2.15), it is easy to check that the weight α_i does not depend on the choice of the lift of k_i, and the weights $(\alpha_i)_i$ satisfy the switch conditions described above. The corresponding map $\mathcal{H}(\lambda;\mathbb{R}) \to \mathcal{W}(\tau;\mathbb{R})$ is a linear isomorphism, as shown in [Bon97b, Theorem 11].

The space of real weights $\mathcal{W}(\tau;\mathbb{R})$ provides us a way to compare shear cocycles associated with distinct maximal geodesic laminations that are close with respect to the Hausdorff topology. Indeed, if $(\lambda_m)_m$ is a sequence of maximal geodesic laminations that converges to λ, and λ is carried by a train track τ, then for m sufficiently large τ carries λ_m. In particular, we have isomorphisms

$$\mathcal{H}(\lambda_m;\mathbb{R}) \cong \mathcal{W}(\tau;\mathbb{R}) \cong \mathcal{H}(\lambda;\mathbb{R}).$$

2.5. **Cross ratios.** The last tool that we need are cross-ratios.

Our use of cross ratios will be twofold: On the one hand, we will use them to abstractly define the shear cocycles of our pleated surfaces (the basic computation will be exploited in Remark 4.6). On the other hand, they will also help us in the study of the length spectrum of a maximal representation ρ as they provide a natural Liouville current $\mathscr{L}_\rho$ such that $i(\mathscr{L}_\rho, \bullet)$ extends continuously the length spectrum $L_\rho(\bullet)$ from the space of closed geodesics $\mathcal{S}$ to the space of geodesic currents $\mathcal{C}$.

Let us remark that cross ratios are also objects of interests in their own and have been widely used to study maximal and Hitchin representations [Lab08, MZ19, BIPP21], and questions about length spectrum rigidity of negatively curved manifolds [Ota90, Led95, Ham99, Ham97].

We now introduce these objects formally. Observe that the Gromov boundary $\partial\Gamma$ admits a natural Hölder structure. To see this, recall that the choice of a Fuchsian representation $\hat{\rho} : \Gamma \to \mathrm{PSL}_2(\mathbb{R})$ determines a unique $\hat{\rho}$-equivariant homeomorphism $\phi_{\hat{\rho}} : \partial\Gamma \to \partial\mathbb{H}^2$. Different choices of Fuchsian representations $\hat{\rho}, \hat{\rho}'$ provide homeomorphisms $\phi_{\hat{\rho}}, \phi_{\hat{\rho}'}$ that differ by post-composition with a quasi-symmetric homeomorphism of $\partial\mathbb{H}^2 \cong \mathbb{RP}^1$. Since quasi-symmetric homeomorphisms are bi-Hölder continuous with respect to any choice of a Riemannian distance on $\partial\mathbb{H}^2$, the notion of Hölder continuous functions $f : \partial\Gamma \to \mathbb{R}$ is independent of the choice of the Fuchsian representation $\hat{\rho}$, and therefore intrinsic of the Γ-space $\partial\Gamma$.

Definition 2.17 (Cross Ratio)**.** Let $\partial\Gamma^{(4)}$ denote the space of 4-tuples $(u, v, w, z) \in (\partial\Gamma)^4$ satisfying $u \neq z$ and $v \neq w$. A *cross ratio* is a Hölder continuous function $\beta : \partial\Gamma^{(4)} \to \mathbb{R}$ that satisfies the following properties:

i) β is Γ-invariant with respect to the diagonal action of Γ on $\partial\Gamma^{(4)}$, i.e. $\gamma \cdot (u, v, w, z) = (\gamma u, \gamma v, \gamma w, \gamma z)$;

ii) For every $u, v, w, z, x \in \partial\Gamma$ we have

$$\begin{aligned} \beta(u,v,w,z) &= 0 \quad \Leftrightarrow u = w \text{ or } v = z, \\ \beta(u,u,w,z) &= \beta(u,v,w,w) = 1, \\ \beta(u,v,w,z) &= \beta(w,z,u,v), \\ \beta(u,v,w,z) &= \beta(u,v,x,z)\beta(u,v,w,x), \\ |\beta(u,v,w,z)| &= |\beta(u,w,v,z)\beta(u,z,w,v)|, \end{aligned} \tag{1}$$

whenever the 4-tuples appearing above belong to $\partial\Gamma^{(4)}$.

Remark 2.18. Observe that the second and fourth relations in (1) imply that for any 4-tuple of pairwise distinct points $u, v, w, z \in \partial\Gamma$ we have

$$\beta(u,v,z,w) = \beta(u,v,w,z)^{-1}. \tag{2}$$

In turn, relation (2) and the third symmetry in (1) imply that

$$\beta(u, v, w, z) = \beta(v, u, z, w). \tag{3}$$

We alert the reader of the existence of multiple non-equivalent definitions of cross ratios in the literature. For the reader's convenience, we summarize the relations between Definition 2.17 and other notions in the literature in Appendix A.

We now recall the notion of positive cross ratios from [Ham99] (see also [MZ19]):

Definition 2.19 (Positive Cross Ratio)**.** A cross ratio $\beta : \partial\Gamma^{(4)} \to \mathbb{R}$ is said to be *positive* if for every 4-tuple of pairwise distinct cyclically ordered points $x, y, w, z \in \partial\Gamma$ it satisfies $\beta(x, y, z, w) \geqslant 1$. We say that β is *strictly positive* if for every 4-tuple $x, y, w, z \in \partial\Gamma$ as above we have $\beta(x, y, z, w) > 1$.

A positive cross ratio has a natural notion of length functions associated to any non-trivial element $\gamma \in \Gamma$. We briefly recall its definition:

Definition 2.20 (Period of a Cross Ratio)**.** Let $\beta : \partial\Gamma^{(4)} \to \mathbb{R}$ be a cross ratio. For any $\gamma \in \Gamma - \{e\}$ we define the β-period of γ to be

$$L_\beta(\gamma) := \log \left|\beta(\gamma^+, \gamma^-, x, \gamma x)\right|,$$

for some $x \in \partial\Gamma - \{\gamma^+, \gamma^-\}$, where γ^+ and γ^- denote the attracting and repelling fixed points of γ in $\partial\Gamma$.

It is simple to deduce from the symmetries of cross ratio (see (1), (2), (3)) that the quantity $L_\beta(\gamma)$ does not depend on the choice of $x \in \partial\Gamma - \{\gamma^+, \gamma^-\}$, and it satisfies $L_\beta(\gamma) = L_\beta(\gamma^{-1}) = L_\beta(\delta\gamma\delta^{-1})$ for any $\gamma, \delta \in \Gamma$, with $\gamma \neq e$.

As observed by Hamenstädt [Ham97] (see also Martone-Zhang [MZ19]), any positive cross ratio β uniquely determines a geodesic current compatible with its period functions, as described by the following result:

Theorem 2.21 ([Ham97, Lemma 1.10], [MZ19, Appendix A])**.** *Every positive cross ratio $\beta : \partial\Gamma^{(4)} \to \mathbb{R}$ is represented by a geodesic current $\mathscr{L}_\beta \in \mathcal{C}$, that is, for every $\gamma \in \Gamma - \{e\}$ we have*

$$L_\beta(\gamma) = i(\mathscr{L}_\beta, \gamma),$$

where $L_\beta(\gamma)$ denotes the β-period of γ.

The geodesic current $\mathscr{L}_\beta$ will be called the *Liouville current* of β, in analogy with the terminology introduced by Bonahon [Bon88, § 2] in the case of hyperbolic structures on closed surfaces. By Theorem 2.21, the non-negative function

$$\begin{array}{rccc} L_\beta : & \mathcal{C} & \longrightarrow & \mathbb{R} \\ & c & \longmapsto & i(\mathscr{L}_\beta, c) \end{array}$$

naturally extends the β-period functions to the entire space of geodesic currents. Moreover we have:

Lemma 2.22. *If β is a strictly positive cross ratio, then $L_\beta(c) > 0$ for any non-trivial geodesic current $c \in \mathcal{C}$.*

Proof. The first part of the assertion follows immediately from the definition of the intersection form on geodesic currents (see Section 2.4.2). Consider now a non-trivial geodesic current $c \in \mathcal{C}$, and select a leaf ℓ' in the support of c, namely a point inside $\mathcal{G} = \mathcal{G}(\widetilde{\Sigma})$ for which all neighborhoods have positive c-measure. We now choose a geodesic $\ell \in \mathcal{G}$ that crosses ℓ'. We can find small intervals I, J and

I', J' inside $\partial\Gamma$ around the endpoints of ℓ and ℓ', respectively, so that every pair of geodesics $\ell \in I\tilde{\times}J := (I \times J)/\sim$ and $\ell' \in I'\tilde{\times}J' := (I' \times J')/\sim$ is crossing (here $\sim$ denotes the equivalence relation $(x, y) \sim (y, x)$ on $(\partial\Gamma \times \partial\Gamma)/\sim$, compare with Section 2.4.1). Accordingly with the notation introduced in Section 2.4.2, we have

$$i(\mathscr{L}_\beta, c) = (\mathscr{L}_\beta \times c)(\mathcal{J}/\Gamma),$$

where $\mathcal{J}$ denotes the set of pairs of crossing geodesics in $\mathcal{G} \times \mathcal{G}$. Since Γ acts freely and properly discontinuously on $\mathcal{J}$, up to choosing smaller intervals I, J, I', J', we can assume that the Borel-measurable set

$$\mathcal{K} := (I\tilde{\times}J) \times (I'\tilde{\times}J') \subset \mathcal{J}$$

projects injectively inside $\mathcal{J}/\Gamma$. In particular we have

$$\begin{aligned}(\mathscr{L}_\beta \times c)(\mathcal{J}/\Gamma) &\geqslant (\mathscr{L}_\beta \times c)(\mathcal{K}) \\ &= (\mathscr{L}_\beta)(I\tilde{\times}J) \cdot c(I'\tilde{\times}J').\end{aligned}$$

By construction $c(I'\tilde{\times}J') > 0$, so it is enough to show that $\mathscr{L}_\beta(I\tilde{\times}J) > 0$. This is in fact a direct consequence of the definition of the Liouville current $\mathscr{L}_\beta$, and the fact that β is strictly positive: indeed the measure $\mathscr{L}_\beta$ satisfies

$$\mathscr{L}_\beta([a, b]\tilde{\times}[c, d]) = \log \beta(a, b, c, d)$$

for any pair of disjoint intervals $[a, b], [c, d]$ in $\partial\Gamma$, where a, b, c, d are cyclically ordered (compare with [Ham97, Lemma 1.10], [MZ19, Appendix A]). Therefore, being β strictly positive, we immediately conclude that $\mathscr{L}_\beta(I\tilde{\times}J) > 0$, and consequently $i(\mathscr{L}_\beta, c) > 0$, as desired. ☐

3. Laminations and pleated sets

In order to understand the geometry of maximal representations $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ and relate it to the geometry of Teichmüller space $\mathcal{T}$, we study certain 1- and 2-dimensional $\rho(\Gamma)$-invariant objects contained in the pseudo-Riemannian symmetric space $\mathbb{H}^{2,n}$, namely *geodesic laminations* and *pleated sets*. Recall that every maximal representation $\rho : \Gamma \to \mathrm{SO}_0(2, n + 1)$ has an associated acausal circle $\xi : \partial\Gamma \to \Lambda_\rho \subset \partial\mathbb{H}^{2,n}$.

Definition 3.1 (Λ-Lamination)**.** Let $\Lambda \subset \partial\mathbb{H}^{2,n}$ be an acausal curve. A Λ-lamination is a closed subset $\lambda \subset (\Lambda \times \Lambda - \Delta)/(x, y) \sim (y, x)$ such that every pair of points $(a, b), (a', b') \in \lambda$ gives disjoint pairs on Λ.

As Λ is an acausal curve, every $(a, b) \in \Lambda \times \Lambda - \Delta$ represents a spacelike line $[a, b] \subset \mathbb{H}^{2,n}$. We define the geometric realization of a Λ-lamination λ in $\mathbb{H}^{2,n}$ to be $\hat{\lambda} = \cup_{(a,b)\in\lambda}[a, b]$.

Notice that the geometric realization $\hat{\lambda}$ of λ is a closed subset of $\mathbb{H}^{2,n}$ contained in the convex hull $\mathcal{CH}(\Lambda) \subset \mathbb{H}^{2,n}$ of the acausal curve $\Lambda \subset \partial\mathbb{H}^{2,n}$.

Definition 3.2 (ρ-Lamination)**.** Let $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ be a maximal representation with associated acausal curve $\Lambda_\rho \subset \partial\mathbb{H}^{2,n}$. A ρ-lamination is a $\rho(\Gamma)$-invariant Λ_ρ-lamination.

As a first step, we establish the existence of geodesic laminations and discuss their causal structure and topological features: As it turns out, they are always acausal subsets (see Proposition 3.5).

The main idea for acausality of laminations is the following: Consider two geodesics $\ell, \ell' \subset \mathbb{H}^{2,n}$ with distinct endpoints on an acausal curve $\Lambda \subset \partial\mathbb{H}^{2,n}$. The property of being acausal for the subset $\ell \cup \ell'$ is equivalent to the condition that $\partial\ell, \partial\ell'$ are not crossing. This is the content of Lemma 3.4. Acausality of the geometric realization $\hat{\lambda}$ of any Λ-lamination λ follows immediately.

When the Λ-lamination λ is maximal, we can add to its geometric realization $\hat{\lambda}$ the geometric realizations of the complementary triangles. Together, they form the pleated set $\hat{S}_\lambda$ of λ.

Definition 3.3 (Pleated Set)**.** Let $\Lambda \subset \partial\mathbb{H}^{2,n}$ be an acausal curve. Let λ be a maximal Λ-lamination. The pleated set associated to λ is the set consisting of $\hat{\lambda}$ and all spacelike triangles Δ bounded by leaves of $\hat{\lambda}$.

As it happens for geodesic laminations, also pleated sets are acausal subsets of $\mathbb{H}^{2,n}$ (see Proposition 3.6). In particular, they are always nicely embedded Lipschitz subsurfaces of $\mathbb{H}^{2,n}$. Later, in the next sections, we will study the intrinsic geometry of such surfaces $\hat{S}_\lambda$: They are always endowed with an intrinsic hyperbolic metric and with an intrinsic pseudo-metric.

The proof of acausality of $\hat{S}_\lambda$ relies on the acausality of $\hat{\lambda}$ and on purely topological arguments: We show that in every Poincaré model $\mathbb{D}^2 \times \mathbb{S}^n$ of $\widehat{\mathbb{H}}^{2,n}$, the restriction of the induced projection $\pi : \widehat{\mathbb{H}}^{2,n} \to \mathbb{D}^2$ to the pleated set $\hat{S}_\lambda$ is bijective. We prove that by analyzing $\mathbb{D}^2 - \pi(\hat{\lambda})$. With little work this is quickly seen to imply that $\hat{S}_\lambda$ is acausal and, hence, it is the graph of a function $g_\lambda : \mathbb{D}^2 \to \mathbb{S}^n$ which is 1-Lipschitz with respect to the spherical metrics.

This brings us to the second part of the section where we show that the map $\lambda \in \mathcal{GL} \to g_\lambda \in \mathrm{Lip}_1(\mathbb{D}^2, \mathbb{S}^n)$ is continuous with respect to the natural topologies of the two spaces (see Proposition 3.7).

In the last part, we conclude by studying the locus where the pleated set $\hat{S}_\lambda$ is folded. This is the bending locus of $\hat{S}_\lambda$. It does not necessarily agree with $\hat{\lambda}$, but we show that it is always a sublamination of it (see Proposition 3.9). The bending locus will be used to characterize those curves that are strictly shorter in the intrinsic hyperbolic metric of $S_\lambda = \hat{S}_\lambda/\rho(\Gamma)$ than in the manifold M_ρ.

3.1. **Crossing geodesics and acausality.**

The following is the main computation of the section: It shows that the topological property of being crossing or disjoint is equivalent to a geometric property in the pseudo-Riemannian space $\mathbb{H}^{2,n}$.

Lemma 3.4. *Let $a, b, a', b' \in \Lambda$ be four distinct points on an acausal circle $\Lambda \subset \partial\mathbb{H}^{2,n}$ such that the geodesics $[c, d]$ with $c, d \in \{a, b, a', b'\}$ are all spacelike. Then:*

(i) The pairs (a, b) and (a', b') are disjoint *if and only if the geodesics $\ell = [a, b], \ell' = [a', b']$ are disjoint and the subset $\ell \cup \ell' \subset \mathbb{H}^{2,n}$ is acausal.*

(ii) The pairs (a, b) and (a', b') are crossing *if and only if there is a timelike geodesic which is orthogonal to both geodesics $\ell = [a, b]$ and $\ell' = [a', b']$.*

Proof. We work in the quadric model $\widehat{\mathbb{H}}^{2,n}$: Lift a, b, a', b' to representatives in the isotropic cone of $\langle \bullet, \bullet \rangle$. We can parametrize ℓ by $\ell(t) = (e^t a + e^{-t} b)/\sqrt{-2\langle a, b\rangle}$ and

ℓ' by $\ell'(s) = (e^s a' + e^{-s} b')/\sqrt{-2\langle a', b'\rangle}$. With these parametrizations we have:

$$\begin{aligned}
&-2\sqrt{\langle a,b\rangle\langle a',b'\rangle}\cdot\langle \ell(t),\ell'(s)\rangle\\
&= -\langle e^t a + e^{-t} b, e^s a' + e^{-s} b'\rangle\\
&= -e^{t+s}\langle a,a'\rangle - e^{t-s}\langle a,b'\rangle - e^{-t+s}\langle b,a'\rangle - e^{-t-s}\langle b,b'\rangle.
\end{aligned}$$

As the geodesics $[a,a'],[a,b'],[b,a'],[b,b']$ are all spacelike, we have that the four products $\langle a,a'\rangle,\langle a,b'\rangle,\langle b,a'\rangle,\langle b,b'\rangle$ are all negative and, therefore, the function $(t,s)\to -\langle \ell(t),\ell'(s)\rangle$ is proper. As a consequence, it has a global minimum $m>0$. A small computation shows that the function has a unique critical point (t_0,s_0), which must coincide with the point of minimum, where it assumes the value

$$\begin{aligned}
m &= -\langle \ell(t_0),\ell'(s_0)\rangle\\
&= \sqrt{\frac{\langle a,a'\rangle\langle b,b'\rangle}{\langle a,b\rangle\langle a',b'\rangle}} + \sqrt{\frac{\langle a,b'\rangle\langle b,a'\rangle}{\langle a,b\rangle\langle a',b'\rangle}}
\end{aligned}$$

Let $e_1,\cdots e_{n+3}$ be the canonical basis of $\mathbb{R}^{2,n+1}$. Up to isometries and rescaling, we can assume that $a=-e_2+e_3, a'=e_1+e_3, b=e_2+e_3, b'=\alpha e_1+\beta e_2+\gamma e_3+u$ with u in the linear span of $e_4,\cdots,e_{n+2}$ and $\alpha^2+\beta^2=1$. Furthermore, by acausality, we have $\gamma>\alpha,\beta$ and $\beta+\gamma>0$. We identify $\widehat{\mathbb{H}}^{2,n}$ with $\mathbb{D}^2\times\mathbb{S}^n$ via the chart Ψ_E induced by the orthogonal decomposition $\mathbb{R}^{2,n+1}=E\oplus F$ where $E=\mathrm{Span}\{e_1,e_2\}$ and $F=\mathrm{Span}\{e_3,\cdots,e_{n+3}\}$. Under such identification we have

$$\pi_{\mathbb{D}^2}(a)=(0,-1),\pi_{\mathbb{D}^2}(b)=(0,1),\pi_{\mathbb{D}^2}(a')=(0,1),\pi_{\mathbb{D}^2}(b')=(\alpha,\beta)\in\partial\mathbb{D}^2,$$

and the above expression becomes

$$\sqrt{\frac{\langle a,a'\rangle\langle b,b'\rangle}{\langle a,b\rangle\langle a',b'\rangle}} + \sqrt{\frac{\langle a,b'\rangle\langle b,a'\rangle}{\langle a,b\rangle\langle a',b'\rangle}} = \sqrt{\frac{\gamma-\beta}{2(\gamma-\alpha)}}+\sqrt{\frac{\beta+\gamma}{2(\gamma-\alpha)}}.$$

Notice that $\pi_{\mathbb{D}^2}$ induces a homeomorphism between Λ and $\partial\mathbb{D}^2$ so that the cyclic order of a,b,a',b' on Λ is the same as the cyclic order of their projections to $\partial\mathbb{D}^2$.

A little algebraic manipulation shows that this quantity is $\leqslant 1$ if and only if

$$\sqrt{\gamma^2-\beta^2}\leqslant -\alpha.$$

Also notice that, as b' is isotropic, we always have $0=|b'|^2=\alpha^2+\beta^2-\gamma^2-|u|^2$ with $|u|^2>0$ and, hence, $\sqrt{\gamma^2-\beta^2}\leqslant|\alpha|$.

Thus, we have the following two cases: If $\alpha>0$, that is, if the pairs (a,b) and (a',b') are disjoint, then the inequality $\sqrt{\gamma^2-\beta^2}<-\alpha$ is never satisfied and, hence, $m>1$. Therefore, $\ell(t),\ell'(s)$ are always distinct and the geodesic segments $[\ell(t),\ell'(s)]$ are always spacelike which implies that the subset $\ell\cup\ell'$ is acausal.

If $\alpha<0$, that is, if the pairs (a,b) and (a',b') are crossing, then the inequality $\sqrt{\gamma^2-\beta^2}\leqslant-\alpha$ is always satisfied and, hence $m\leqslant 1$. In this case, the geodesic segment $[\ell(t_0),\ell'(s_0)]$ is timelike and, as $\langle\ell(t_0),\ell'(s_0)\rangle$ realizes the minimum of $\langle\ell(t),\ell'(s)\rangle$, it is also orthogonal to ℓ,ℓ' at $\ell(t_0)$ and $\ell'(s_0)$. □

As a consequence of Lemma 3.4, we immediately get:

Proposition 3.5. *Let $\Lambda\subset\partial\mathbb{H}^{2,n}$ be an acausal curve. Let $\hat{\lambda}$ be the geometric realization of a Λ-lamination λ. Then $\hat{\lambda}$ is a proper acausal subset.*

3.2. **Pleated sets.** We now study the topology and causal structure of a pleated set associated to a maximal lamination.

A priori, a pleated set can be a very complicated topological subspace of $\mathbb{H}^{2,n}$. We now show that, instead, Proposition 3.5 forces a good topological behaviour:

Proposition 3.6. *Let $\Lambda \subset \partial\mathbb{H}^{2,n}$ be an acausal curve. Let λ be a maximal. Let $\hat{S}_\lambda \subset \mathbb{H}^{2,n}$ be the associated pleated set. Then $\hat{S}_\lambda$ is a topological Lipschitz acausal subsurface.*

Proof. We lift Λ to an acausal curve $\hat{\Lambda} \subset \partial\widehat{\mathbb{H}}^{2,n}$.

The proof strategy is as follows: We show that in every Poincaré model $\mathbb{D}^2 \times \mathbb{S}^n$ of $\widehat{\mathbb{H}}^{2,n}$ the pleated set $\hat{S}_\lambda$ is the graph of a function $g : \mathbb{D}^2 \to \mathbb{S}^n$. We deduce that $\hat{S}_\lambda$ is achronal which implies that the function g is 1-Lipschitz with respect to the spherical metrics on $\mathbb{D}^2$ and $\mathbb{S}^n$. Therefore, being a graph of a Lipschitz function, the pleates set $\hat{S}_\lambda$ is a Lipschitz subsurface. Acausality will follow from the fact that $\hat{S}_\lambda$ does not contain lightlike segments.

Let

$$\Psi : (\mathbb{D}^2 \cup \partial\mathbb{D}^2) \times \mathbb{S}^n \to \widehat{\mathbb{H}}^{2,n} \cup \partial\widehat{\mathbb{H}}^{2,n}$$

be the Poincaré model associated to a splitting $\mathbb{R}^{2,n+1} = E \oplus E^\perp$ with E spacelike 2-plane. Let $\pi : \widehat{\mathbb{H}}^{2,n} \cup \partial\widehat{\mathbb{H}}^{2,n} \to \mathbb{D}^2 \cup \partial\mathbb{D}^2$ be the projection to the first factor.

Recall that, as $\hat{\Lambda} \subset \partial\widehat{\mathbb{H}}^{2,n}$ is an acausal circle, the projection π restricts to a homeomorphism $\pi : \hat{\Lambda} \to \partial\mathbb{D}^2$.

Consider the restriction of π to $\hat{\lambda}$. By Proposition 3.5 and Lemma 2.6, $\pi : \hat{\lambda} \to \mathbb{D}^2$ is injective. It is also proper as π is a fibration with compact fibers. Hence, it is a homeomorphism onto the image $\pi(\hat{\lambda})$. Notice that, by Lemma 2.7, every $\pi(\ell)$ is a smooth proper arc joining the projections of the endpoints of ℓ on $\hat{\Lambda}$.

We now show that the connected components of $\mathbb{D}^2 - \pi(\hat{\lambda})$ correspond to the triangles associated to $\hat{\lambda}$. This comes from the fact that both triangles and connected components can be characterized in terms of cyclic order of the endpoints of the leaves of $\hat{\lambda}$ and $\pi(\hat{\lambda})$ and π induces a homeomorphism between $\hat{\Lambda}$ and $\partial\mathbb{D}^2$.

First notice that π maps triangles to connected components: Let $\Delta = \Delta(a, b, c)$ be a triangle bounded by the leaves $[a, b], [b, c], [c, a]$ of $\hat{\lambda}$. Since Δ is contained in a totally geodesic spacelike plane, by Lemma 2.7, the restriction of π to Δ is a homeomorphism onto the image. The image $\pi(\mathrm{int}(\Delta))$ must be disjoint from the other leaves $\pi(\ell)$ of $\pi(\hat{\lambda})$, otherwise $\pi(\ell)$ would intersect one of the sides $\pi(\partial\Delta) = \pi[a, b] \cup \pi[b, c] \cup \pi[c, a]$. Thus $\pi(\mathrm{int}(\Delta))$ is a connected component of $\mathbb{D}^2 - \pi(\hat{\lambda})$.

Then we show that every connected component of $\mathbb{D}^2 - \pi(\hat{\lambda})$ arises as a projection of a triangle.

Let $U \subset \mathbb{D}^2 - \pi(\hat{\lambda})$ be a connected component. The boundary ∂U is contained in $\pi(\hat{\lambda})$, but there is more structure. In fact:

Claim 1. The boundary ∂U consists of a union of projections of leaves $\pi(\ell)$.

Proof of the claim. Suppose that for a leaf ℓ of $\hat{\lambda}$ we have $\pi(\ell) \cap \partial U \neq \varnothing$. We show that $\pi(\ell) \cap \partial U$ is open and closed in $\pi(\ell)$ and, hence, by connectedness, we have $\pi(\ell) \subset \partial U$. It is clear that $\pi(\ell) \cap \partial U$ is closed as ∂U is closed in $\mathbb{D}^2$. Let us show that it is open. Notice that $\pi(\ell)$ divides $\mathbb{D}^2$ in two half spaces $\mathbb{D}^2 - \pi(\ell) = A \cup A'$ and U is contained in one of them, say $U \subset A$. We claim that there exists a small neighborhood B of x such that $B \cap A \subset \mathbb{D}^2 - \pi(\hat{\lambda})$. If this is the case, then $B \cap \pi(\ell)$

contains an open segment around x entirely contained in ∂U proving that $\partial U \cap \pi(\ell)$ is open in $\pi(\ell)$.

Suppose that we cannot find such a neighborhood B. Then there exists a sequence of distinct leaves ℓ_n and points $x_n \in \pi(\ell_n)$ such that $x_n \in A$ and $x_n \to x$. The sequence of leaves ℓ_n converges to ℓ in the Hausdorff topology of $\mathbb{H}^{2,n} \cup \partial\mathbb{H}^{2,n}$ and, hence, their projections $\pi(\ell_n)$ converge to $\pi(\ell)$ in the Hausdorff topology of $\mathbb{D}^2 \cup \partial\mathbb{D}^2$. As a consequence, if y is any point in U, for n large enough, $\pi(\ell_n)$ separates it from $\pi(\ell)$ as it will be contained in a small neighborhood of the closure of $\pi(\ell)$. This contradicts the fact that $x \in \partial U$. □

Using the fact that λ is maximal, we show that

Claim 2. The boundary ∂U consists of the projection of three leaves of the form $\pi[a, b], \pi[b, c], \pi[c, a]$.

Proof of the claim. If ℓ, ℓ' are leaves with disjoint endpoints of λ, then there is a leaf ℓ^* whose endpoints separate the endpoints of ℓ from the endpoints of ℓ'. As a consequence, $\pi(\ell^*)$ must separates $\pi(\ell)$ from $\pi(\ell')$ which means that $\pi(\ell), \pi(\ell')$ cannot be the boundary of a single connected component of $\mathbb{D}^2 - \pi(\hat{\lambda})$. We deduce that every pair of boundary leaves of U share exactly one endpoint. This is possible only if ∂U has at most three leaves. On the other hand, given two leaves $\pi(\ell), \pi(\ell')$, in every connected component of $\mathbb{D}^2 - (\pi(\ell) \cup \pi(\ell'))$ there is another leaf of $\pi(\hat{\lambda})$. Therefore, ∂U must have at least three leaves so that ∂U corresponds to a triangle bounded by leaves of $\hat{\lambda}$. □

In conclusion, we have that the projection π restricted to the pleated set $\hat{S}_\lambda$ is injective and has image $\mathbb{D}^2$.

We can now show that $\hat{S}_\lambda$ is a topological achronal subsurface.

Claim 3. $\hat{S}_\lambda$ is a topological achronal subsurface.

Proof of the claim. Suppose there there are two points on $\hat{S}_\lambda$ connected by a timelike geodesic τ. Consider suitable local coordinates $\Psi : \mathbb{D}^2 \times \mathbb{S}^n \to \widehat{\mathbb{H}}^{2,n}$ adapted to a timelike sphere T containing τ and an orthogonal spacelike plane H, that is $\Psi(\mathbb{D}^2 \times \{v\}) = H$ and $\Psi(\{0\} \times \mathbb{S}^n) = T$. As the projection $\pi : \widehat{\mathbb{H}}^{2,n} \to \mathbb{D}^2$ collapses τ to a point, its restriction to $\hat{S}_\lambda$ cannot be injective and this contradicts the previous claim. As a consequence $\hat{S}_\lambda$ is achronal and the function $g : \mathbb{D}^2 \to \mathbb{S}^n$ describing it as a graph is 1-Lipschitz by Lemma 2.6. In particular $\hat{S}_\lambda$ is a topological Lipschitz subsurface of $\mathbb{H}^{2,n}$. □

Finally, we have to show that $\hat{S}_\lambda$ is acausal.

Claim 4. $\hat{S}_\lambda$ is acausal.

Proof of the claim. Again, we work in local coordinates adapted to a timelike sphere T and an orthogonal spacelike plane H, so that we can write $\hat{S}_\lambda$ as the graph of a 1-Lipschtiz function $g : \mathbb{D}^2 \to \mathbb{S}^n$. By Lemma 2.5, the points $p = (x, g(x))$ and $q = (y, g(y))$ on $\hat{S}_\lambda$ are connected by a lightlike geodesic if and only if $d_{\mathbb{S}^n}(g(x), g(y)) = d_{\mathbb{S}^2}(x, y)$. As g is 1-Lipschitz, this means that $d_{\mathbb{S}^n}(g(t), g(s)) = d_{\mathbb{D}^2}(t, s)$ for every t, s on the geodesic arc $[x, y]$ (in the hemispherical metric). Therefore the lightlike geodesic $[p, q]$ is contained in $\hat{S}_\lambda$. In particular, either $[p, q]$ is contained in a leaf of $\hat{\lambda}$, or $[p, q]$ meets two different leaves, or it meets the interior of a complementary triangle. However a leaf, a pair of distinct leaves, and a complementary region are all acausal subsets. Therefore, all these cases are not possible and we conclude that $\hat{S}_\lambda$ must be acausal. □

This finishes the proof. □

In Sections 4, 5, and 6, we will prove that every pleated set $\hat{S}_\lambda \subset \mathbb{H}^{2,n}$ associated to a maximal representation $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ and a maximal ρ-lamination λ has a natural associated $\rho(\Gamma)$-invariant hyperbolic structure $\mathbb{H}^2$ and admits a developing map $f : \hat{S} \to \mathbb{H}^2$ which is 1-Lipschitz with respect to the intrinsic pseudo-metric on $\hat{S}_\lambda$ and the hyperbolic metric.

The data of the pleated set $S_\lambda = \hat{S}_\lambda/\rho(\Gamma)$ together with the intrinsic pseudo-Riemannian metric, the intrinsic hyperbolic structure, and the 1-Lipschitz developing map $f : \hat{S}_\lambda \to \mathbb{H}^2$ is what we will call a *pleated surface*.

3.3. **Continuity of pleated sets.** We now discuss continuity properties of pleated sets associated to a maximal representation $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ and maximal laminations $\lambda \in \mathcal{GL}$.

First notice that if $\Omega_\rho \subset \mathbb{H}^{2,n}$ is a properly convex open $\rho(\Gamma)$-invariant subset, then Ω_ρ admits a lift to the 2-fold cover $\widehat{\mathbb{H}}^{2,n}$. In particular, we can identify a pleated set in $\mathbb{H}^{2,n}$, which is always contained in $\mathcal{CH}(\Lambda_\rho) \subset \Omega_\rho$, with its lift in $\widehat{\mathbb{H}}^{2,n}$.

We will deal with two topologies: On the one hand, as pleated sets $\hat{S}_\lambda$ are closed subset of $\widehat{\mathbb{H}}^{2,n}$, they are endowed with a natural Chabauty topology. On the other hand, if we fix a Poincaré model $\Psi : \mathbb{D}^2 \times \mathbb{S}^n \to \widehat{\mathbb{H}}^{2,n}$ each $\hat{S}_\lambda$ can be written as a graph of a 1-Lipschitz function $g_\lambda : \mathbb{D}^2 \to \mathbb{S}^n$ (by Proposition 3.6 and Lemma 2.6) and, therefore, we can also endow pleated sets with the topology of uniform convergence of the functions g_λ. Notice that covergence with respect to this topology implies Chabauty convergence.

Proposition 3.7. *Let $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ be a maximal representation. For every Poincaré model $\Psi : \mathbb{D}^2 \times \mathbb{S}^n \to \widehat{\mathbb{H}}^{2,n}$, the map*

$$\lambda \in \mathcal{GL} \to g_\lambda \in \mathrm{Lip}_1(\mathbb{D}^2, \mathbb{S}^n)$$

is continuous with respect to the Chabauty topology on $\mathcal{GL}$ and the uniform convergence on compact subsets on $\mathrm{Lip}_1(\mathbb{D}^2, \mathbb{S}^n)$.

Proof. Let λ_m be a sequence of maximal laminations converging to a maximal lamination λ in the Chabauty topology.

Let $\hat{S}_m, \hat{S}$ be the corresponding pleated sets and let $g_m, g : \mathbb{D}^2 \to \mathbb{S}^n$ be the corresponding 1-Lipschitz maps. We want to show that $g_m \to g$. Notice that, being 1-Lipschitz, the maps g_n converge uniformly on compact sets to a 1-Lipschitz function $g' : \mathbb{D}^2 \to \mathbb{S}^n$ up to subsequences. If we show that $g' = g$, then the convergence $g_n \to g$ would follow.

We now argue that

Claim. Each $x \in \hat{S}$ is the limit of a sequence $x_m \in \hat{S}_m$.

This will be enough to conclude: In fact, suppose that this is the case. Pick $x \in \hat{S}$ and find $x_m \in \hat{S}_m$ as in the claim. Represent x_m as $(y_m, g_m(y_m))$ and x as $(y, g(y))$. By assumption $x_m \to x$ so that $y_m \to y$ and $g_m(y_m) \to g(y)$. However, by uniform convergence $g_m \to g'$, we have $g_m(y_m) \to g'(y)$ and, hence, $g'(y) = g(y)$.

Proof of the claim. In the proof of the claim we distinguish whether x belongs to a leaf or to a plaque, but the arguments are very similar.

Consider a point x on a leaf $\hat{\ell}$ of $\hat{\lambda}$. As λ_m converges in the Chabauty topology to λ, the leaf $\ell \subset \lambda$ is the limit of a sequence of leaves $\ell_m \subset \lambda_m$ and, hence, also the the geometric realization $\hat{\ell}$ is the Chabauty limit of the sequence of geometric realizations $\hat{\ell}_m$. Therefore, $x \in \hat{\ell}$ is the limit of a sequence of points $x_m \in \hat{\ell}_m$.

Consider a point x on a plaque $\hat{\Delta}$ of $\hat{S} - \hat{\lambda}$. As λ_m converges in the Chabauty topology to λ, the plaque Δ is the limit of a sequence of plaques Δ_m of λ_m and, hence, also the the geometric realization $\hat{\Delta}$ is the Chabauty limit of the sequence of geometric realizations $\hat{\Delta}_m$. Therefore $x \in \hat{\Delta}$ is the limit of a sequence of points $x_m \in \hat{\Delta}_m$. □

This finishes the proof of the proposition. □

3.4. **Bending locus.** A pleated set $\hat{S}_\lambda$ associated to a maximal ρ-lamination λ is not necessarily bent along all the leaves of $\hat{\lambda}$.

Definition 3.8 (Bending Locus)**.** Let $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ be a maximal representation. Let λ be a maximal ρ-lamination with geometric realization $\hat{\lambda}$. Let $\hat{S}_\lambda$ be the corresponding pleated set. A point $x \in \ell \subset \hat{\lambda}$ is in the bending locus if there is no geodesic (necessarily spacelike) segment τ entirely contained in $\hat{S}_\lambda$ and such that $\mathrm{int}(\tau) \cap \ell = x$.

We now prove the following:

Proposition 3.9. *The bending locus is a sublamination of* $\hat{\lambda}$.

Proof. We show that if $x \in \ell \subset \hat{\lambda}$ is not in the bending locus, then a neighborhood of ℓ is contained in a spacelike plane and, therefore, its intersection with $\hat{\lambda}$ is not in the bending locus. This implies that the bending locus is closed and consists of a disjoint union of the leaves of $\hat{\lambda}$.

Before staritng the proof, we recall a general structural result (see Theorem I.4.2.8 in [CEG06]) for ρ-laminations: Every ρ-lamination λ decomposes as a disjoint union of a finite number of ρ-sublaminations λ_j that are minimal and a finite number of orbits $\rho(\Gamma)\ell$ of isolated leaves that are asymptotic in both directions to leaves of the minimal components λ_j. A minimal component has the property that every leaf is dense, that is, for every $\ell \subset \lambda_j$ there exists a sequence of distinct elements $\gamma_n \in \Gamma$ such that $\ell_n = \rho(\gamma_n)\ell \to \ell$.

Suppose that $x \in \ell \subset \hat{\lambda}$ is not contained in the bending locus. Let $\tau \subset \hat{S}_\lambda$ be a spacelike segment transverse to ℓ at x and entirely contained in $\hat{S}_\lambda$. According to the above decompostion, we have that either ℓ is isolated or it is contained in a minimal component of λ.

If ℓ is isolated, then $\ell = \Delta \cap \Delta'$ for two different components of $\hat{S}_\lambda - \hat{\lambda}$ and τ intersects both Δ, Δ'. In this case, we immediately conclude that the two triangles Δ, Δ' must be contained in the same spacelike plane. In particular, the whole line ℓ is not contained in the bending locus.

If ℓ is contained in a minimal component of λ, then ℓ is dense in such component. In this case, there are infinitely many pairwise distinct segments τ_n entirely contained in $\hat{S}_\lambda$ and intersecting transversely ℓ: In fact, we can find a sequence of pairwise distinct translates ℓ_n of ℓ such that $\ell_n \to \ell$. If the endpoints of ℓ_n are sufficiently close to the endpoints of ℓ, then ℓ_n must intersect τ transversely, and every transverse intersection $\tau \cap \ell_n$ can be translated back to $\tau_n \cap \ell$ using the isometry that brings ℓ_n to ℓ.

Consider now two distinct spacelike segments τ, τ' entirely contained in $\hat{S}_\lambda$ and intersecting transversely ℓ. Every geodesic $\ell' \subset \hat{\lambda}$ with endpoints sufficiently close to those of ℓ must intersect both τ and τ'. In particular, if Δ' and Δ'' are triangles of $\hat{S}_\lambda$, with edges ℓ', ℓ'' sufficiently close to ℓ, then τ, τ' intersect both $\mathrm{int}(\Delta')$ and $\mathrm{int}(\Delta'')$. This implies that Δ', Δ'' lie on the same spacelike plane. Hence, all triangles which have an edge sufficiently close to ℓ lie on the same spacelike plane. By density of triangles in $\hat{S}_\lambda$, we conclude that a neighborhood of ℓ in $\hat{S}_\lambda$ lies on a spacelike plane H and, therefore, every point $x \in \ell$ does not lie in the bending locus. $\square$

4. Hyperbolic structures on pleated sets I

Let $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ be a maximal representation, and let λ be a maximal geodesic lamination with associated pleated set $\hat{S}_\lambda \subset \mathbb{H}^{2,n}$. After investigating the causal and topological properties of $\hat{S}_\lambda$, we now turn our attention on its geometric structure. Being pleated sets obtained as unions of spacelike geodesics and ideal triangles, a natural question that arises is whether the metrics on each totally geodesic region "patch nicely together", determining an intrinsic hyperbolic metric on Σ.

Inspired by the work of Bonahon [Bon96] in the context of hyperbolic surfaces, we now intend to answer to this question by recording the relative position of the hyperbolic triangles that make up the pleated set $\hat{S}_\lambda$ into a shear cocycle $\sigma^\rho_\lambda \in \mathcal{H}(\lambda; \mathbb{R})$ transverse to the lamination λ (see Section 2.4.3). Making use of Bonahon's characterization of hyperbolic shear cocycles in terms of lengths of measured laminations (see Theorem 2.16), this will determine, for every maximal representation ρ and for any maximal lamination λ, the intrinsic hyperbolic structure $X^\rho_\lambda \in \mathcal{T}$ of the pleated set S_λ. The construction of the shear cocycles σ^ρ_λ and the investigation of their properties are going to be the main subject of the current and next sections.

In fact, the process that we will outline applies in a wider generality than the one specifically needed for the study of pleated sets in $\mathbb{H}^{2,n}$. Indeed, the definition of the cocycle σ^ρ_λ will rely only on certain analytic properties of the cross ratio β^ρ naturally associated to the representation ρ, namely on its *positivity* (see Definition 2.19) and *local boundedness* (see Definition 4.2). Examples of cross ratios satisfying these properties occur frequently in the literature about Higher Teichmüller Theories: This is for instance the case for Hitchin representations in $\mathrm{SO}_0(p, p+1)$ or Θ-positive representations in $\mathrm{SO}_0(p, q)$ (see e.g. Beyrer and Pozzetti [BP21] and Appendix A).

We can now describe our main result in this context:

Theorem 4.1. *Let $\beta : \partial\Gamma^{(4)} \to \mathbb{R}$ be a positive and locally bounded cross ratio. Then for every maximal lamination λ, the β-shear cocycle σ^β_λ belongs to the closure of the cone $C(\lambda) \subset \mathcal{H}(\lambda; \mathbb{R})$, that is*

$$\omega_\lambda(\sigma^\beta_\lambda, \mu) \geqslant 0$$

for every measured lamination μ with $\mathrm{supp}\, \mu \subseteq \lambda$. Moreover, if the cross ratio β is strictly positive, then $\omega_\lambda(\sigma^\beta_\lambda, \mu) > 0$ for every non-trivial measured lamination μ as above, and consequently there exists a unique hyperbolic structure $Y = Y^\beta_\lambda \in \mathcal{T}$ such that $\sigma^\beta_\lambda = \sigma^Y_\lambda \in \mathcal{H}(\lambda; \mathbb{R})$.

The proof of Theorem 4.1 will be concluded in Section 5.5. The current and next sections (namely Section 4 and 5) focus on the development of the required ingredients.

We start by briefly introducing the cross ratios β^ρ associated to maximal representations ρ and investigating their properties.

4.1. **Cross ratios of maximal representations.** In what follows we describe a cross ratio β^ρ on $\partial\Gamma$ naturally associated to $\rho : \partial\Gamma \to \mathrm{SO}_0(2, n+1)$ and its limit map ξ. To this purpose, we start by defining a sign function on the set of 4-tuples of points in $\partial\Gamma$ as follows. Given ϕ some fixed homeomorphism between $\partial\Gamma$ and $\mathbb{R}\mathrm{P}^1$, we set

$$\mathrm{Sgn}(u, v, w, z) := \mathrm{sgn}\left(\frac{\phi(u) - \phi(w)}{\phi(u) - \phi(z)}\frac{\phi(v) - \phi(z)}{\phi(v) - \phi(w)}\right),$$

for any $(u, v, w, z) \in \partial\Gamma^{(4)}$, where $\mathrm{sgn}(t) = +1$ if $t > 0$, $\mathrm{sgn}(0) = 0$, and $\mathrm{sgn}(t) = +1$ if $t < 0$. It is simple to check that the function Sgn is independent of the choice of the homeomorphism ϕ, and that $\mathrm{Sgn}(u, v, w, z) = 0$ if and only if $u = w$ or $v = z$.

For any maximal representation $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ with associated acausal limit map $\xi : \partial\Gamma \to \partial\mathbb{H}^{2,n}$ we then define

$$\beta^\rho(u, v, w, z) := \mathrm{Sgn}(u, v, w, z)\left(\frac{\langle\xi(u), \xi(w)\rangle}{\langle\xi(u), \xi(z)\rangle}\frac{\langle\xi(v), \xi(z)\rangle}{\langle\xi(v), \xi(w)\rangle}\right)^{1/2}, \tag{4}$$

for any $(u, v, w, z) \in \partial\Gamma^{(4)}$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product $\langle\cdot,\cdot\rangle_{(2,n+1)}$, and we are implicitly selecting representatives in $\mathbb{R}^{2,n+1}$ of the equivalence classes $\xi(y)$, for $y \in \{u, v, w, z\}$. By Theorem 2.9, the scalar products involved in the definition above are all non-zero. Moreover, the quantity appearing under the square root does not depend on the chosen lightlike representatives of the equivalence classes $\xi(u), \xi(v), \xi(w), \xi(z)$, and it is always non-negative. Since ρ preserves the scalar product $\langle\cdot,\cdot\rangle_{(2,n+1)}$ and the diagonal action of Γ on $\partial\Gamma^{(4)}$ preserves Sgn, the function β^ρ is Γ-invariant. Finally, the Hölder continuity of β^ρ at any point $(u, v, w, z) \in \partial\Gamma$ is a direct consequence of the Hölder continuity of the limit map ξ, which is guaranteed by Theorem 2.9). It is straightforward to check that maps β^ρ associated to $\mathrm{SO}_0(2, n+1)$-maximal representations as in (4) satisfy the symmetries listed in (1).

We now turn our attention to the notion of locally bounded cross ratio. To this purpose, we need to introduce some notation. We select an identification of the universal cover $\widetilde{\Sigma}$ with $\mathbb{H}^2$ through the choice of a hyperbolic structure $X \in \mathcal{T}$. Moreover, given ℓ an oriented geodesic of $\mathbb{H}^2$, we denote by ℓ^+ and ℓ^- the positive and negative endpoints of ℓ in $\partial\mathbb{H}^2 \cong \partial\Gamma$, respectively. If ℓ and h are two disjoint oriented geodesics in $\mathbb{H}^2$, then we say that ℓ and h are *coherently oriented* if their endpoints satisfy $\ell^+ \leqslant h^+ < h^- \leqslant \ell^- < \ell^+$ with respect to some cyclic order on $\partial\Gamma$. With this notation, we now define:

Definition 4.2 (Locally Bounded)**.** A cross ratio $\beta : \partial\Gamma^{(4)} \to \mathbb{R}$ is said to be *locally bounded* if there exists a (and consequently for any) hyperbolic structure $X \in \mathcal{T}$ such that, for any constant $D > 0$ we can find $C, \alpha > 0$ such that

$$\left|\log\beta(h^+, \ell^+, \ell^-, h^-)\right| \leqslant C\left|\log\beta^X(h^+, \ell^+, \ell^-, h^-)\right|^\alpha$$

for any pair of coherently oriented geodesics ℓ, h in $\widetilde{\Sigma} \cong \mathbb{H}^2$ satisfying $0 < d_{\mathbb{H}^2}(\ell, h) \leqslant D$.

The term $\log \beta^X(h^+,\ell^+,\ell^-,h^-)$ appearing in the definition above has a precise geometrical interpretation in terms of 2-dimensional hyperbolic geometry, as described by the following lemma:

Lemma 4.3. *Let $X \in \mathcal{T}$. For any pair of coherently oriented disjoint geodesics ℓ, h in $\mathbb{H}^2 \cong \widetilde{\Sigma}$, the value $\beta^X(\ell^+,h^+,h^-,\ell^-)$ is strictly positive and satisfies*

$$\log \beta^X(h^+,\ell^+,\ell^-,h^-) = 2\log\cosh \frac{d_{\mathbb{H}^2}(\ell,h)}{2}.$$

We now prove positivity and local boundedness of the cross ratios β^ρ, properties that will be crucial for the construction of shear cocycles developed in Section 5.

Lemma 4.4. *For every maximal representation $\rho : \Gamma \to \mathrm{SO}_0(2,n+1)$, the cross ratio β^ρ is strictly positive and satisfies relation* (17)*.*

Proof. Let (u,v,w,z) be a 4-tuple of distinct and cyclically ordered points in $\partial\Gamma$. Up to the action of $\mathrm{SO}_0(2,n+1)$, we can assume that $\xi(u) = e_2 + e_3$, $\xi(v) = -e_1 + e_3$, $\xi(w) = -e_2 + e_3$. Moreover, since u,v,w,z are cyclically ordered, $\xi(z)$ can be expressed as $\xi(z) = \cos\vartheta\, e_1 + \sin\vartheta\, e_2 + x$, where $\vartheta \in (-\pi/2,\pi/2)$, and x is some timelike vector of norm -1 orthogonal to the spacelike plane spanned by e_1 and e_2. Being $\xi(\partial\Gamma)$ a spacelike curve in $\mathbb{H}^{2,n}$, we see that ϑ and x must satisfy

$$-1 \leqslant \langle e_3, x\rangle < -|\sin\vartheta|. \tag{5}$$

Observe that $\mathrm{Sgn}(u,v,w,z) = +1$. Therefore, by definition of β^ρ and the normalization selected, we have

$$\beta^\rho(u,v,w,z) = \left(\frac{2(\langle e_3,x\rangle - \cos\vartheta)}{\langle e_3,x\rangle + \sin\vartheta}\right)^{1/2}.$$

From this identity, it is immediate to see that $\beta^\rho(u,v,w,z) > 1$ if and only if $\langle e_3, v\rangle - \sin\vartheta < 2\cos\vartheta$. This inequality is always satisfied: The left-hand side is negative by (5), while the right-hand side is positive since $\vartheta \in (-\pi/2,\pi/2)$. ☐

Lemma 4.5. *For any maximal representation $\rho : \Gamma \to \mathrm{SO}_0(2,n+1)$, the positive cross ratio β^ρ is locally bounded.*

Proof. Fix $D > 0$, and let $A = A_D$ be the subset of $\partial\Gamma^{(4)}$ given by

$$\{(h^+,\ell^+,\ell^-,h^-) \mid \ell, h \text{ coherently oriented}, \ 0 < d_{\mathbb{H}^2}(\ell,h) \leqslant D\}.$$

If $F : A \to \mathbb{R}$ denotes the function

$$F(u,v,w,z) := \frac{|\log \beta^\rho(u,v,w,z)|}{|\log \beta^X(u,v,w,z)|^\alpha},$$

then the statement is equivalent to F being bounded (observe that F is well defined on A by Lemma 4.3). Since F is invariant with respect to the diagonal action of Γ, it induces a continuous function on the quotient space A/Γ. We introduce a convenient exhaustion by compact subsets of A/Γ. For any $d \in (0,D]$, let A_d be the subset of A given by

$$\{(h^+,\ell^+,\ell^-,h^-) \mid \ell, h \text{ coherently oriented}, \ d \leqslant d_{\mathbb{H}^2}(\ell,h) \leqslant D\}.$$

Then it is immediate to see that the fundamental group Γ acts cocompactly on A_d for every $d > 0$. In particular F admits an upper bound on A_d for any $d > 0$.

Assume now that F is not bounded over A. Then there exist sequences of coherently oriented geodesics ℓ_n, h_n in $\mathbb{H}^2$ such that $F(h_n^+,\ell_n^+,\ell_n^-,h_n^-)$ tends to $+\infty$

as n goes to ∞, and $d_{\mathbb{H}^2}(\ell_n, h_n) \leqslant D$. Since A_d/Γ is compact for every $d > 0$, and the 4-tuples $(h_n^+, \ell_n^+, \ell_n^-, h_n^-)$ are escaping every compact subset of A/Γ, we must have that $d_{\mathbb{H}^2}(\ell_n, h_n)$ tends to 0 as n goes to ∞. Up to the action of Γ, we can assume that all the geodesics ℓ_n intersect a fixed fundamental domain for Σ in $\mathbb{H}^2$, and therefore that $(\ell_n)_n$ converges to some geodesic ℓ, up to subsequence. Since $d_{\mathbb{H}^2}(\ell_n, h_n)$ tends to 0, the geodesics h_n also converge up to subsequence to some h, with $d_{\mathbb{H}^2}(\ell, h) = 0$. In particular, we have two possibilities: Either $\ell = h$, or ℓ and h are asymptotic to each other. In both cases we have that $\ell^\pm \neq h^\mp \in \mathbb{RP}^1$. Moveover, up to choosing a different identification between $\widetilde{\Sigma}$ and $\mathbb{H}^2$, we can assume that the points $\ell^\pm$ are $h^\pm$ are different from $\infty \in \mathbb{RP}^1$.

By the properties of cross ratios and their continuity we observe that both terms $\beta^\rho(h_n^+, \ell_n^+, \ell_n^-, h_n^-), \beta^X(h_n^+, \ell_n^+, \ell_n^-, h_n^-)$ converge to 1 as $n \to \infty$. The rest of the proof will be dedicated to the study of the order of convergence to 0 of the logarithm of these terms, which will lead to a contradiction with F being unbounded.

We start from the term involving β^X: By the symmetries of the standard hyperbolic cross ratio, we observe

$$\begin{aligned}\beta^X(h_n^+, \ell_n^+, \ell_n^-, h_n^-) &= 1 - \beta^X(h_n^+, \ell_n^-, \ell_n^+, h_n^-) \\ &= 1 - \frac{h_n^+ - \ell_n^+}{h_n^+ - h_n^-}\frac{\ell_n^- - h_n^-}{\ell_n^- - \ell_n^+}\end{aligned}$$

Notice that the denominator $(h_n^+ - h_n^-)(\ell_n^- - \ell_n^+)$ converges to $(h^+ - h^-)(\ell^- - \ell^+)$, which is different from 0. On the other hand, since $d_{\mathbb{H}^2}(\ell, h) = 0$, the factor $(h_n^+ - \ell_n^+)(\ell_n^- - h_n^-)$ is infinitesimal. Therefore we deduce that

$$\lim_{n\to\infty} \frac{\left|\log \beta^X(h_n^+, \ell_n^+, \ell_n^-, h_n^-)\right|}{\left|(h_n^+ - \ell_n^+)(\ell_n^- - h_n^-)\right|} = \frac{1}{|(h^+ - h^-)(\ell^- - \ell^+)|} =: M, \tag{6}$$

where M is strictly positive.

Let now $\xi : \partial\Gamma \to \mathrm{SO}_0(2, n+1)$ denote the limit map associated to the representation ρ. In order to study the behavior of the cross ratios $\beta^\rho(h_n^+, \ell_n^+, \ell_n^-, h_n^-)$, it will be convenient to introduce representatives of the projective classes $\xi(\ell_n^\pm), \xi(h_n^\pm)$, by selecting some affine hyperplane V in $\mathbb{R}^{2,n+1}$ intersecting the projective classes $\xi(\ell^\pm), \xi(h^\pm)$ and pick representatives belonging to V. We will continue to denote with abuse these representatives by $\xi(\ell_n^\pm), \xi(h_n^\pm)$. Consider now

$$\beta^\rho(h_n^+, \ell_n^+, \ell_n^-, h_n^-)^2 = \frac{\langle \xi(h_n^+), \xi(\ell_n^-)\rangle}{\langle \xi(h_n^+), \xi(h_n^-)\rangle}\frac{\langle \xi(\ell_n^+), \xi(h_n^-)\rangle}{\langle \xi(\ell_n^+), \xi(\ell_n^-)\rangle}.$$

Since $\xi(\ell_n^\pm), \xi(h_n^\pm) \to \xi(\ell^\pm), \xi(h^\pm)$, respectively, the above quantity converges to 1. A simple algebraic manipulation shows that

$$\begin{aligned}\beta^\rho(h_n^+, \ell_n^+, \ell_n^-, h_n^-)^2 - 1 = \;&\frac{\langle \xi(h_n^+), \xi(\ell_n^-) - \xi(h_n^-)\rangle\langle \xi(\ell_n^+) - \xi(h_n^+), \xi(h_n^-)\rangle}{\langle \xi(h_n^+), \xi(h_n^-)\rangle\langle \xi(\ell_n^+), \xi(\ell_n^-)\rangle} + \\ &- \frac{\langle \xi(\ell_n^+) - \xi(h_n^+), \xi(\ell_n^-) - \xi(h_n^-)\rangle\langle \xi(h_n^+), \xi(h_n^-)\rangle}{\langle \xi(h_n^+), \xi(h_n^-)\rangle\langle \xi(\ell_n^+), \xi(\ell_n^-)\rangle}\end{aligned}$$

Let now $L > 0$ be a positive constant such that $|\langle u, v\rangle| \leqslant L\|u\|_0\|v\|_0$, for some fixed Euclidean norm $\|\cdot\|_0$ on $\mathbb{R}^{2,n+1}$. We deduce that

$$\left|\beta^\rho(h_n^+, \ell_n^+, \ell_n^-, h_n^-)^2 - 1\right| \leqslant 2L\|\xi(h_n^+)\|_0\|\xi(h_n^-)\|_0 \frac{\|\xi(\ell_n^+) - \xi(h_n^+)\|_0\|\xi(\ell_n^-) - \xi(h_n^-)\|_0}{\left|\langle \xi(h_n^+), \xi(h_n^-)\rangle\langle \xi(\ell_n^+), \xi(\ell_n^-)\rangle\right|}.$$

Since $\xi(\ell_n^\pm), \xi(h_n^\pm)$ are converging to $\xi(\ell^\pm), \xi(h^\pm)$, and $\xi(\ell^\pm) \neq \xi(h^\mp)$, we can find a constant $M' > 0$ such that

$$\left|\beta^\rho(h_n^+, \ell_n^+, \ell_n^-, h_n^-)^2 - 1\right| \leqslant M' \|\xi(\ell_n^+) - \xi(h_n^+)\|_0 \|\xi(\ell_n^-) - \xi(h_n^-)\|_0.$$

Moreover, being ξ a Hölder continuous function with exponent α, we conclude that for n sufficiently large

$$\begin{aligned} \left|\log \beta^\rho(h_n^+, \ell_n^+, \ell_n^-, h_n^-)\right| &= \frac{1}{2} \log\bigl(1 + (\beta^\rho(h_n^+, \ell_n^+, \ell_n^-, h_n^-)^2 - 1)\bigr) \\ &\leqslant \frac{M'}{2} \|\xi(\ell_n^+) - \xi(h_n^+)\|_0 \|\xi(\ell_n^-) - \xi(h_n^-)\|_0 \\ &\leqslant M'' \left|(h_n^+ - \ell_n^+)(\ell_n^- - h_n^-)\right|^\alpha, \end{aligned}$$

for some constant $M'' > 0$ (here we are considering ξ as a Hölder function from a neighborhood of $\ell^\pm, h^\pm$ in $\mathbb{R} \subset \mathbb{R}\mathrm{P}^1 = \partial\mathbb{H}^2$ to $\mathbb{H}^{2,n} \subset V$ with their Euclidean metrics). Finally, combining this inequality with relation (6), we obtain that

$$\limsup_{n\to\infty} \frac{|\log \beta^\rho(u,v,w,z)|}{\left|\log \beta^X(u,v,w,z)\right|^\alpha} \leqslant \frac{M''}{M},$$

which contradicts the fact that $F(h_n^+, \ell_n^+, \ell_n^-, h_n^-)$ diverges. We conclude that F is bounded on A, and therefore that there exists a constant $C > 0$ satisfying the requirements. □

We therefore conclude that the cross ratios β^ρ associated to maximal representations $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ satisfy the hypotheses of Theorem 4.1.

4.2. **Outline of the construction.** We now move to the definition of shear cocycles associated to positive locally bounded cross ratios and maximal laminations. Throughout the rest of Section 4, β will always denote a cross ratio on $\partial\Gamma$, and λ a maximal geodesic lamination on Σ.

Consider two distinct plaques P, Q of λ. The shear $\sigma_\lambda^\beta(P, Q)$ between the distinct plaques P, Q of λ will be defined following a careful approximation argument which depends on the fine properties of maximal geodesic laminations in hyperbolic surfaces. In order to describe the first steps of our construction, let us introduce some notation: We say that a plaque R (or a leaf ℓ) of λ separates P from Q if P and Q are contained in distinct connected components of $\widetilde{\Sigma} - R$ (or $\widetilde{\Sigma} - \ell$). We denote by $\mathcal{P}_{PQ}$ the set of plaques of λ that separates P from Q.

In the remainder of the current section we will proceed as follows:

- **§ 4.3:** We start by recalling a simple process – already described by Bonahon in [Bon96] – that, starting from a finite subset of plaques $\mathcal{P} \subseteq \mathcal{P}_{PQ}$, produces a finite lamination $\lambda_\mathcal{P}$ of $\widetilde{\Sigma}$ containing all the leaves of λ that lie in the boundary of some plaque in $\mathcal{P}$. We introduce the elementary β-shear between two adjacent complementary regions of $\lambda_\mathcal{P}$, which naturally generalizes the classical definition in hyperbolic geometry. We then define the *finite β-shear with respect to $\mathcal{P}$*, denoted by $\sigma_\mathcal{P}^\beta(P, Q)$, as the sum of the elementary shears between all adjacent complementary regions of $\lambda_\mathcal{P}$.
- **§ 4.4:** In this section we focus our attention on the notion of β-shears for finite leaved maximal laminations. We observe how the relations satisfied by finite β-shears (from Section 4.3) allow to define β-shear cocycles associated to finite leaved maximal lamination in a fairly elementary and natural way (see in particular Proposition 4.9).

§ 4.5: Lastly, we investigate the connections between β-shear cocycles associated to a finite leaved lamination, and the β-periods of its closed leaves (see Proposition 4.11). The bridge between these notions is provided by the *Thurston symplectic form*, through which hyperbolic shear cocycles are fully characterized (see Theorem 2.16).

4.3. **Finite shears between plaques.** We start by introducing the notion of finite β-shear between plaques of a maximal geodesic lamination.

Let X be a hyperbolic structure on Σ. Given any finite subset $\mathcal{P}$ of $\mathcal{P}_{PQ}$, we select an X-geodesic path k in $(\widetilde{\Sigma}, \widetilde{X})$ joining two points in the interior of (the $\widetilde{X}$-geodesic realizations of) P and Q, and we index the plaques $P_1, \ldots, P_n$ in $\mathcal{P}$ according to their order along k, moving from P to Q. It is immediate to check that the ordering is independent of the choice of the arc k. We set also $P_0 := P$ and $P_{n+1} := Q$. For every i, let ℓ_i^P and ℓ_i^Q be the boundary leaves of P_i that face P and Q, respectively. If S_i denotes the (possibly empty) region of $\widetilde{\Sigma}$ delimited by ℓ_i^Q and ℓ_{i+1}^P, for every $i \in \{0, \ldots, n\}$, we define d_i to be the geodesic that joins the negative endpoints of the leaves ℓ_i^Q and ℓ_{i+1}^P, as we orient them as boundary of the strip S_i (if ℓ_i^Q and ℓ_{i+1}^P share one or two endpoints, then we take $d_i = \ell_i^Q$). For every $\mathcal{P} \subset \mathcal{P}_{PQ}$ as above, let now $\lambda_{\mathcal{P}}$ be the (finite) geodesic lamination of $\widetilde{\Sigma}$ given by

$$\lambda_{\mathcal{P}} := \{\ell_0^Q, d_0, \ell_1^P, \ell_1^Q, d_1, \ldots, \ell_n^P, \ell_n^Q, d_n, \ell_{n+1}^P\},$$

where the leaves are listed as we move from P to Q. The complementary set of the lamination $\lambda_{\mathcal{P}}$ in $\widetilde{\Sigma}$ consists of two half-planes containing the plaques P and Q, and a finite number of *spikes*, i. e. regions bounded by two distinct asymptotic geodesics, that separate P from Q.

Consider now two adjacent complementary regions R, R' of $\lambda_{\mathcal{P}}$. We denote by ℓ the leaf of $\lambda_{\mathcal{P}}$ shared by R and R', and we select arbitrarily an orientation on ℓ. Let u_l (u_r resp.) be the ideal vertex in $(R \cup R') \cap \partial\Gamma$ that lies on the left (right resp.) of the geodesic ℓ. If one of the regions R, R' on the sides of ℓ coincides with a half-plane containing P or Q, then we select u_l or u_r to be the vertex of the plaque P or Q different from ℓ^+ and ℓ^-. Then we set

$$\sigma^\beta(R, R') := \log\left|\beta(\ell^+, \ell^-, u_l, u_r)\right|, \tag{7}$$

and we define the *finite β-shear between P and Q* relative to $\mathcal{P}$ to be

$$\sigma^\beta_{\mathcal{P}}(P, Q) := \sum_{i=0}^{m} \sigma^\beta(R_i, R_{i+1}),$$

where $R_0, R_1, \ldots, R_{m+1}$ are the complementary regions of $\lambda_{\mathcal{P}}$ as we move from P to Q. By the symmetry (3) of the cross ratio β, each term $\sigma^\beta(R_i, R_{i+1})$ does not depend on the choice of the orientation of the leaf separating R_i and R_{i+1}, and $\sigma^\beta(R_i, R_{i+1}) = \sigma^\beta(R_{i+1}, R_i)$ for every i. Notice also that $\sigma^\beta_{\mathcal{P}}(P, Q) = \sigma^\beta_{\mathcal{P}}(Q, P)$.

Remark 4.6. The definition of the cross ratio β^ρ provided in Section 4.1 is designed so that the shear $\sigma^\rho(T, T')$ between two adjacent ideal triangles (or spikes) coincides with the classical shear between their spacelike realizations $\hat{T}$ and $\hat{T}'$ inside $\mathbb{H}^{2,n}$ (i.e. if T has ideal vertices $a, b, c \in \partial\Gamma$, then $\hat{T}$ is the spacelike triangle with ideal vertices $\xi(a), \xi(b), \xi(c) \in \partial\mathbb{H}^{2,n}$), measured with respect to the induced hyperbolic path metric on $\hat{T} \cup \hat{T}'$.

In order to justify this assertion we need to introduce some notation. As we did previously, we denote by ℓ the geodesic shared by R and R' together with an arbitrary choice of orientation, and by u_l and u_r the ideal vertices in $(R \cup R') \cap \partial\Gamma$ that lie on the left and on the right of ℓ, respectively. Since $\xi : \partial\Gamma \to \mathbb{H}^{2,n}$ is a spacelike curve by Theorem 2.9, there exist unique spacelike planes H_l and H_r in $\mathbb{H}^{2,n}$ whose boundary at infinity contain the triples $\xi(\ell^+), \xi(\ell^-), \xi(u_l)$ and $\xi(\ell^+), \xi(\ell^-), \xi(u_r)$, respectively. If $\hat{R}$ and $\hat{R}'$ denote the regions of H_l and H_r delimited by the spacelike geodesics corresponding to the boundary leaves of R and R', respectively, then the set $\hat{R} \cup \hat{R}'$ possesses a natural hyperbolic metric with geodesic boundary induced by the hyperbolic distances on H_l and H_r.

Let now $\tilde{\xi}(\ell^\pm)$, $\tilde{\xi}(u_l)$, $\tilde{\xi}(u_r)$ be representatives of the projective classes $\xi(\ell^\pm)$, $\xi(u_l)$, $\xi(u_r)$, respectively, so that all their pairwise scalar products are negative (this is possible again by Theorem 2.9). The vectors $\tilde{\xi}(\ell^+)$ and $\tilde{\ell}(\ell^-)$ generate a 2-plane V in $\mathbb{R}^{2,n+1}$ of signature $(1,1)$. Moreover, the orthogonal projection of a vector $w \in \mathbb{R}^{2,n+1}$ onto V can be expressed as

$$p(w) = \frac{\langle w, \tilde{\xi}(u_-)\rangle}{\langle \tilde{\xi}(u_+), \tilde{\xi}(u_-)\rangle} \tilde{\xi}(u_+) + \frac{\langle w, \tilde{\xi}(u_+)\rangle}{\langle \tilde{\xi}(u_+), \tilde{\xi}(u_-)\rangle} \tilde{\xi}(u_-).$$

From here a simple computation shows that $\log|\beta^\rho(\ell^+, \ell^-, u_l, u_r)|$ coincides with the signed distance between the projective classes of $p(\tilde{\xi}(u_l))$ and $p(\tilde{\xi}(u_r))$ along the oriented spacelike geodesic $[\xi(\ell^-), \xi(\ell^+)]$, which can be parametrized by

$$\ell(t) = \frac{1}{\sqrt{-2\langle \tilde{\xi}(u_+), \tilde{\xi}(u_-)\rangle}} (e^t \tilde{\xi}(u_+) + e^{-t} \tilde{\xi}(u_-)).$$

On the other hand, the projection $p(\xi(u_l))$ can be characterized in terms of the hyperbolic metric of $\hat{R} \cup \hat{R}'$ as the unique point of the line $\ell = [\xi(\ell^-), \xi(\ell^+)]$ that is joined to the ideal vertex $\xi(u_l)$ by a geodesic ray in H_l orthogonal to ℓ, and similarly for $p(\xi(u_r))$ and H_r. Since the classical hyperbolic shear between two ideal triangles (or spikes) that share a boundary geodesic h coincides with the signed distance between the projection of their ideal vertices different from $h^\pm$, we deduce that $\log|\beta^\rho(u_+, u_-, u_l, u_r)|$ coincides with the classical notion of shear between the plaques $\hat{R}, \hat{R}'$.

We now highlight a few properties satisfied by finite β-shears. Since the proofs of these relations are elementary and only rely on the symmetries of the cross ratio β, we postpone them to Appendix B. In what follows, we fix a maximal geodesic lamination λ, and we denote by λ_c the sublamination of λ consisting of the lifts of all simple closed geodesics contained in λ. Notice that λ_c is non-empty for any finite leaved maximal lamination λ.

4.3.1. *Shear between plaques sharing a vertex.* Let P and Q be two distinct plaques of λ that share an ideal vertex $w \in \partial\Gamma$. We label the vertices of P and Q that are different from w as u_P, v_P and u_Q, v_Q, respectively, so that the leaves $[w, v_P]$ and $[w, v_Q]$ separate the interior of the plaque P from the interior of the plaque Q. Then we have:

Lemma 4.7. *For every finite subset $\mathcal{P} \subset \mathcal{P}_{PQ}$*

$$\sigma^\beta_{\mathcal{P}}(P, Q) = \log|\beta(w, v_P, u_P, v_Q)\beta(w, v_Q, v_P, u_Q)|.$$

In particular the shear between P and Q is independent of the selected family of plaques $\mathcal{P} \subset \mathcal{P}_{PQ}$.

4.3.2. *Shear between plaques asymptotic to a closed leaf.* Consider now two plaques P and Q of λ that are separated by exactly one component ℓ of λ_c. Select arbitrarily an orientation of ℓ, and assume that the plaque P has a vertex equal to ℓ^+ and that lies on the left of ℓ. Similarly, assume that Q lies on the right of ℓ and that one of its vertices is equal to ℓ^-. We denote by x_P, y_P and x_Q, y_Q the vertices of P and Q different from ℓ^+ and ℓ^-, respectively, so that $[y_P, \ell^+]$ and $[y_Q, \ell^-]$ are the boundary components of P and Q closest to ℓ. Then we have:

Lemma 4.8. *For every finite subset $\mathcal{P} \subset \mathcal{P}_{PQ}$*

$$\sigma^\beta_{\mathcal{P}}(P, Q) = \log \left|\beta(\ell^+, y_P, x_P, \ell^-)\beta(\ell^-, \ell^+, y_Q, y_P)\beta(\ell^-, y_Q, x_Q, \ell^+)\right|.$$

In particular the shear between P and Q is independent of the selected family of plaques $\mathcal{P} \subset \mathcal{P}_{PQ}$.

4.4. **Shear cocycles: Finite case.** We now focus on the construction of β-shear cocycles σ^β_λ associated to *finite leaved* maximal geodesic laminations, and the investigation of their properties. Thanks to the relations described in Lemmas 4.7 and 4.8, it is possible to carry out the analysis of shear cocycles with respect to finite leaved laminations in a fairly elementary way, without any subtle approximation argument.

Even though not generic, the convenience of examining the finite leaved case separately is twofold. On the one hand, it serves as a guideline and motivation for the analysis in the general case. On the other, the naturality of β-shear cocycles for finite leaved laminations, combined with the continuous dependence from Proposition 5.10, shows that the approximation process described in Section 5.2 produces cocycles that are independent of their construction (see in particular Section 5.1 and Lemma 5.7).

Until the end of the current section, λ will denote a finite leaved maximal lamination of Σ. Recall that every leaf of a lamination of this form projects in Σ either onto a simple closed geodesic, or onto a simple bi-infinite geodesic, and in the latter case each of its ends accumulates onto a (possibly common) simple closed geodesic.

We start by outlining the definition of the β-shear cocycle relative to λ. Consider two plaques P and Q of λ, and denote by ℓ_P and ℓ_Q the boundary leaves of P and Q, respectively, that separate the interior of P from the interior of Q. Notice that the geodesics ℓ_P, ℓ_Q, lying in the boundary of a plaque of λ, project onto simple bi-infinite geodesics in Σ. We then choose arbitrarily an oriented geodesic segment k starting at a point in the interior of P and reaching a point in the interior of Q. By compactness, there exist only finitely many leaves of λ that intersect k and that project onto simple closed geodesics of Σ. We label them as $\ell_1, \dots, \ell_n$, following the order in which we meet them moving along the segment k, and we orient each ℓ_i from right to left with respect to k. For any i, we now select plaques P_i and Q_i that lie on the left and on the right of ℓ_i, respectively, and that have ℓ_i^+ or ℓ_i^- as one of their vertices (if P has a vertex equal to ℓ_1^+ or ℓ_1^-, then we choose $P_1 = P$, and similarly for Q, Q_n, and ℓ_n). Since Q_i and P_{i+1} are not separated by any lift of simple closed leaves, the set of plaques $\mathcal{P}_{Q_i P_{i+1}}$ is finite for any $i = 1, \dots, n-1$. For the same reason we see that the sets $\mathcal{P}_{PP_1}$ and $\mathcal{P}_{Q_n Q}$ are finite. Finally, we set

$$\mathcal{P} := \mathcal{P}_{PP_1} \cup \mathcal{P}_{Q_1 P_2} \cup \dots \cup \mathcal{P}_{Q_{n-1} P_n} \cup \mathcal{P}_{Q_n Q},$$

and we define the *β-shear between P and Q* to be

$$\sigma^\beta_\lambda(P, Q) := \sigma^\beta_{\mathcal{P}}(P, Q).$$

It is not difficult to show that the quantity $\sigma^\beta_\lambda(P, Q)$ is independent of the collection of plaques $\mathcal{P}$ selected following the aforementioned procedure. To see this, it is in fact enough to check that $\sigma^\beta_{\mathcal{P}}(P, Q) = \sigma^\beta_{\mathcal{P}'}(P, Q)$ for any finite extension $\mathcal{P}' \supset \mathcal{P}$ obtained as above. Notice that any such $\mathcal{P}' \subset \mathcal{P}_{PQ}$ is of the form

$$\mathcal{P}' = \mathcal{P}_{PP'_1} \cup \mathcal{P}_{Q'_1 P'_2} \cup \cdots \cup \mathcal{P}_{Q'_{n-1} P'_n} \cup \mathcal{P}_{Q'_n Q},$$

where P'_i and Q'_i are plaques that lie on the left and on the right of ℓ_i, respectively. In fact, it is not restrictive to assume that $P'_i \neq P_i$ and $Q_i \neq Q'_i$, in which case both P'_i and Q'_i separate P_i from Q_i. Observe also that both pairs P_i, P'_i and Q_i, Q'_i share one of the endpoints of ℓ_i as a vertex, and exactly one of the following hold: Either the plaques P_i and Q_i (and consequently P'_i and Q'_i) have a common vertex, equal to ℓ_i^+ or ℓ_i^-, or P_i and Q_i do not share any vertex. By Lemma 4.7 in the former case, and Lemma 4.8 in the latter, we have

$$\sigma^\beta(P_i, Q_i) = \sigma^\beta(P_i, P'_i) + \sigma^\beta(P'_i, Q'_i) + \sigma^\beta(Q'_i, Q_i)$$

for every $i = 1, \ldots, n$, which implies the equality between the finite β-shears computed with respect to the set of plaques $\mathcal{P}$ and $\mathcal{P}'$.

To prove that σ^β_λ is indeed a transverse Hölder cocycle (see Definition 2.15), one can proceed with a process analogous to the one described in the proof of Lemma 5.9, without the need of any convergence argument. Based on this, we omit a proof of this assertion, and postpone it to the general case. We can summarize the above discussion in the following statement:

Proposition 4.9. *Let β be a cross ratio. Then for every finite leaved maximal lamination λ, the map $(P, Q) \mapsto \sigma^\beta_\lambda(P, Q)$ defines a Hölder cocycle $\sigma^\beta_\lambda \in \mathcal{H}(\lambda; \mathbb{R})$ naturally associated to β and λ.*

4.5. **Shears and length functions: Finite case.** We conclude our analysis of β-shear cocycles associated to finite leaved maximal laminations examining their relations with the periods of the cross ratio β (see in particular Proposition 4.11). As already observed in the work of Bonahon [Bon96], length functions provide a complete characterization of the set of transverse Hölder cocycles in $\mathcal{H}(\lambda; \mathbb{R})$ that arise as shear cocycles of hyperbolic structures on a closed surface with respect to the maximal lamination λ (see in particular Theorem 2.16). The connection between β-shears and β-periods rely on the properties of the *Thurston symplectic form* on $\mathcal{H}(\lambda; \mathbb{R})$, a skew-symmetric non-degenerate bilinear form, whose definition is recalled in Section 4.5.1.

The combination of the analysis in the finite leaved case (developed in this section) together with the continuity results of β-shear cocycles (described in Section 5.3 below) will eventually allow us to relate β-shear cocycles of strictly positive and locally bounded cross ratios to hyperbolic structures on Σ, as described in Theorem 4.1.

4.5.1. *Thurston symplectic form.* We start by briefly recalling the definition of the Thurston symplectic form ω_λ on the space of transverse Hölder cocycles $\mathcal{H}(\lambda; \mathbb{R})$ to a maximal geodesic lamination λ. As described by Bonahon in [Bon96, § 3], the symplectic form ω_λ can be intrinsically characterized in terms of the intersection form on the space of geometric currents supported on λ, in the sense of [RS75]. However, for our purposes it will be more convenient to have an elementary (but less intrinsic) description of the Thurston symplectic form, through the choice of a train track τ carrying λ and its induced isomorphism $\mathcal{H}(\lambda;\mathbb{R}) \cong \mathcal{W}(\tau;\mathbb{R})$ (see Section 5.3).

In the following we briefly introduce the necessary notation. Given any switch s of the train track τ, we denote by B_s the unique branch of τ whose vertical boundary contains s, and we select arbitrarily lifts $\widetilde{B}_s$ and $\tilde{s} \subset \widetilde{B}_s$ of B_s and s to the universal cover of Σ. The switch $\tilde{s}$ is then adjacent to two other distinct branches $\widetilde{B}_l^s$ and $\widetilde{B}_r^s$ of $\tilde{\tau}$, with $\widetilde{B}_l^s$ and $\widetilde{B}_r^s$ lying on the left and on the right of $\tilde{s}$, respectively, if seen from $\widetilde{B}_s$ with respect to the orientation of $\widetilde{\Sigma}$. For any transverse Hölder cocycle $\alpha \in \mathcal{H}(\lambda;\mathbb{R})$, we then denote by α_l^s and α_r^s the real weights associated by α to the branches $\widetilde{B}_l^s, \widetilde{B}_r^s$. Finally, the *Thurston symplectic form* ω_λ applied to $\alpha, \beta \in \mathcal{H}(\lambda;\mathbb{R})$ can be expressed as

$$\omega_\lambda(\alpha,\beta) = \frac{1}{2}\sum_s (\alpha_r^s\,\beta_l^s - \alpha_l^s\,\beta_r^s), \tag{8}$$

where the sum is taken over all switches s of τ.

As recalled in Section 2.4.3, the Thurston symplectic form provides a characterization of the set of transverse Hölder cocycles that can be obtained as shear cocycles of hyperbolic structures on Σ (see Theorem 2.16, or [Bon96, Theorem 20]). In addition, it is worth to mention that the Thurston symplectic form is also intimately related to the geometry of Teichmüller space, and in particular to its Weil-Petersson symplectic structure, as observed in [SB01]. We refer to [Bon96, § 3] (see also [Pap86], [PH92, § 3.2], [SB01]) for a more detailed description of the Thurston symplectic form and its properties.

4.5.2. *Lengths.* The relation between β-shear cocycles with respect to finite leaved laminations and β-periods relies on elementary arguments. The main ingredients are the combinatorial description of the Thurston symplectic form from relation (8), and the following statement:

Lemma 4.10. *Let λ be a finite leaved maximal lamination, and let γ be a non-trivial element of Γ whose axis $\tilde{\gamma}$ projects onto a closed leaf of λ. Consider a plaque P of λ that has one of the endpoints $\gamma^\pm$ of $\tilde{\gamma}$ as a vertex, and assume that it lies on the left of $\tilde{\gamma}$. Then*

$$\sigma_\lambda^\beta(P,\gamma P) = \pm L_\beta(\gamma),$$

with positive sign if P has γ^+ as one of its vertices, and with negative sign otherwise.

The proof of Lemma 4.10 relies only on the symmetries of the cross ratio β, and we postpone it to Appendix B.

We conclude the current section with the following result, which will play an important role in the proof of Theorem 4.1, described in Section 5:

Proposition 4.11. *Let β be a positive cross ratio. Then for every finite leaved maximal lamination λ and for weighted multicurve μ with* $\operatorname{supp}\mu \subseteq \lambda$*, we have*

$$L_\beta(\mu) = \omega_\lambda(\sigma_\lambda^\beta, \mu),$$

where ω_λ denotes the Thurston symplectic form on the space of transverse Hölder cocycles $\mathcal{H}(\lambda;\mathbb{R})$.

Proof. It is sufficient to consider the case in which μ consists of a single simple closed curve with weight 1. Let γ be an element of Γ whose axis $\tilde{\gamma}$ projects onto a simple closed leaf of λ. We orient $\tilde{\gamma}$ so that it points towards the attracting fixed point $\gamma^+ \in \partial\Gamma$ and moves away from the repelling fixed point γ^-. We will denote by $\tilde{\gamma}^{-1}$ the axis of γ endowed with the opposite orientation, and by $|\gamma| \in \mathcal{C}$ the associated geodesic current.

We now select a train track carrying λ. Being λ a finite leaved lamination, there exist bi-infinite leaves of λ that spiral around γ both from its left and its right (with respect to the orientation of γ and of the surface Σ). It is not restrictive to assume that λ is carried by a train track τ obtained from a tubular neighborhood of γ by adding two branches on its sides, and then properly extended away from γ. (Such a train track can be obtained by taking a sufficiently small metric neighborhood of λ with respect to some hyperbolic structure X, and possibly by a small deformation to guarantee the trivalence of every switch.)

In order to provide an explicit expression for the evaluation of the Thurston symplectic form $\omega_\lambda(\sigma^\beta_\lambda, |\gamma|)$ with respect to the train track τ, we need to introduce some notation. Let k be a tie of the train track τ that intersects γ. We select arbitrarily a lift $\tilde{k}$ of k in $\tilde{\Sigma}$ that crosses $\tilde{\gamma}$, and we denote by P and Q the plaques of λ that contain the endpoints of $\tilde{k}$, so that P lies on the left of $\tilde{\gamma}$ and Q on its right. Both plaques have an ideal vertex equal to $\tilde{\gamma}^+$ or $\tilde{\gamma}^-$. We now introduce the following sign convention: We say that the left sign of τ with respect to γ, denoted by $\mathrm{sgn}_l(\tau, \gamma)$, is equal to $+1$ if the plaque P lying on the left of $\tilde{\gamma}$ has γ^+ as one of its ideal vertices, and we set it equal to -1 otherwise. On the other hand, we define the right sign of τ with respect to γ to be $\mathrm{sgn}_r(\tau, \gamma) = +1$ if the plaque Q lying on the right of $\tilde{\gamma}$ has γ^- as one of its vertices, and -1 otherwise. It is not difficult to see that the sign functions $\mathrm{sgn}_l(\tau, \gamma)$ and $\mathrm{sgn}_r(\tau, \gamma)$ depend only on the train track τ and the choice of the orientation of the curve γ: For instance, we can alternatively define $\mathrm{sgn}_l(\tau, \gamma)$ to be $+1$ if the branch of τ that enters in the tubular neighborhood of γ from its left follows the orientation of γ and -1 otherwise; a similar description holds for $\mathrm{sgn}_r(\tau, \gamma)$.

There are only finitely many possible configurations for the switches and branches of τ that intersected s. By applying relation (8) to each possible configuration, we obtain the expression:

$$\omega_\lambda(\sigma^\beta_\lambda, |\gamma|) = \frac{1}{2}\left(\mathrm{sgn}_l(\tau, \gamma)\, \sigma^\beta_\lambda(P, \gamma P) - \mathrm{sgn}_r(\tau, \gamma)\, \sigma^\beta_\lambda(Q, \gamma Q)\right).$$

Notice that by definition $\mathrm{sgn}_r(\tau, \gamma) = -\mathrm{sgn}_l(\tau, \gamma^{-1})$. Moreover, for any plaque R of λ we have

$$\sigma^\beta_\lambda(R, \gamma R) = \sigma^\beta_\lambda(\gamma^{-1} R, R) = \sigma^\beta_\lambda(R, \gamma^{-1} R),$$

since σ^β_λ is a transverse Hölder cocycle. In particular, the term $\omega_\lambda(\sigma^\beta_\lambda, |\gamma|)$ can be equivalently expressed as

$$\omega_\lambda(\sigma^\beta_\lambda, |\gamma|) = \frac{1}{2}\left(\mathrm{sgn}_l(\tau, \gamma)\, \sigma^\beta_\lambda(P, \gamma P) + \mathrm{sgn}_l(\tau, \gamma^{-1})\, \sigma^\beta_\lambda(Q, \gamma^{-1} Q)\right).$$

Since P lies on the left of $\tilde{\gamma}$ and Q lies on the left of $\tilde{\gamma}^{-1}$, we can now apply Lemma 4.10 to both terms appearing above, obtaining

$$\omega(\sigma^\beta_\lambda, |\gamma|) = \frac{1}{2}\left(L_\beta(\gamma) + L_\beta(\gamma^{-1})\right) = L_\beta(\gamma),$$

which proves the desired identity. ☐

5. Hyperbolic structures on pleated sets II

This section is dedicated to the definition and the study of shear cocycles associated to positive and locally bounded cross ratios and general maximal geodesic laminations. We generalize the phenomena observed in Section 4.5 for finite leaved maximal laminations, and we provide a proof of Theorem 4.1 in full generality.

The construction of the β-shear cocycle $\sigma^\beta_\lambda \in \mathcal{H}(\lambda;\mathbb{R})$ for a general maximal lamination λ will require several auxiliary choices and a fine analysis of the convergence of finite β-shears. Nevertheless, we will observe that the resulting β-shear cocycles satisfy a series of natural properties:

(a) For every *finite leaved* maximal lamination λ, the transverse Hölder cocycle $\sigma^\beta_\lambda \in \mathcal{H}(\lambda;\mathbb{R})$, obtained through the general process described in Section 5.2.3, coincides with the β-shear cocycle introduced in Section 4.4 (see in particular Proposition 4.9).

(b) The map

$$\mathcal{GL} \ni \lambda \longmapsto \sigma^\beta_\lambda \in \mathcal{W}(\tau;\mathbb{R})$$

is continuous with respect to the Hausdorff topology on the space of maximal geodesic laminations.

Consequently, since maximal finite leaved laminations are dense in the entire set of maximal geodesic laminations (see e.g. [CEG06, Theorem I.4.2.19]) and since the β-shear cocycles σ^β_λ constructed in Section 4.4 do not require any auxiliary choice, we can conclude that the transverse Hölder cocycle σ^β_λ only depends on λ and β, even in the case of a general maximal geodesic lamination.

Outline of the construction. The β-shear $\sigma^\beta_\lambda(P,Q)$ between the plaques P,Q will be defined as a limit of certain finite β-shears $\sigma^\beta_{\mathcal{P}_n}(P,Q)$ associated to a suitably chosen exhaustion $\mathcal{P}_n$ of the set of plaques $\mathcal{P}_{PQ}$ separating P from Q. The choice of $\mathcal{P}_n$ depends on the geometry of λ on a fine scale. More precisely, in order to select it, we will use a divergence radius function associated to the choice of a hyperbolic structure X, a train track τ that carries λ, and a geodesic arc k joining P to Q.

We emphasize however that, as previously observed, the continuity properties of the construction (Proposition 5.10) and the naturality in the case of finite leaved laminations (see Section 4.4) make the cocycle $\sigma^\beta_\lambda(P,Q)$ independent of all the auxiliary choices required for its definition. The rest of the section is structured as follows:

§ 5.1: We dedicate this section to the description of *divergence radius functions*, which were originally introduced by Bonahon in [Bon96] to study the convergence of the shearing maps between hyperbolic surfaces.

§ 5.2: In this section we give the general definition of the β-shear $\sigma^\beta_\lambda(P,Q)$: We deploy divergence radius functions to carefully select an exhausting sequence of finite sets of plaques $(\mathcal{P}_n)_n$ inside $\mathcal{P}_{PQ}$, whose associated finite shears converges.

§ 5.3: In this section we prove that β-shear cocycles σ^β_λ satisfy a suitable notion of continuity with respect to the maximal lamination λ (endowing the space of maximal geodesic laminations with the Chabauty topology).

§ 5.4: We then study the relations between β-shear cocycles and β-periods associated to a positive and locally bounded cross ratio β, generalizing what observed in Section 4.5 for finite leaved laminations (see in particular Proposition 5.11).

§ 5.5: We conclude our analysis with the proof of Theorem 4.1, combining the results from the previous sections with Bonahon's shear parametrization (see Theorem 2.16)

5.1. **Divergence radius functions.** In order to define the β-shear between two plaques P and Q in the case of a general maximal lamination, we will need to determine an exhaustion $(\mathcal{P}_n)_n$ of the set of plaques separating P from Q whose associated sequence of finite shears $(\sigma^{\beta}_{\mathcal{P}_n}(P, Q))_n$ converges (compare with Section 4.3). This part of our analysis requires some care, because of the (not particularly strong) control between finite β-shears associated to different collections of plaques (see Lemma 5.5). In particular, we make use of the so-called *divergence radius function* $r : \mathcal{P}_{PQ} \to \mathbb{N}$, associated to the choice of a trivalent train track carrying the lamination λ (see Section 2.4.4 for the related terminology), a hyperbolic structure X on Σ, and a (X-)geodesic path joining P to Q (see Bonahon-Dreyer [BD17], and Bonahon [Bon96, § 1]). Any such function depends on the choice of:

- A hyperbolic structure X on Σ.
- A (trivalent) train track τ inside Σ.
- A maximal geodesic lamination λ (which will be identified with its X-geodesic realization in the universal cover of (Σ, X)) carried by τ.
- Two distinct plaques P and Q of λ.
- A geodesic segment k that joins a point in the interior of P to a point in the interior of Q.

Once we fix these data, the associated *divergence radius function*

$$r = r_{X,\tau,\lambda,k} : \mathcal{P}_{PQ} \longrightarrow \mathbb{N}$$

associates to every plaque R that separates P from Q a natural number $r(R)$, which roughly measures the length of the geodesic arc $R \cap k$ in terms of the combinatorics of the fixed train track τ and the boundary leaves of R that intersect k.

In order to be more precise, we need to introduce some notation. For any plaque $R \in \mathcal{P}_{PQ}$, let ℓ_R, ℓ'_R be the boundary leaves of R that intersect the arc k. If the geodesic segment $R \cap k$ is not entirely contained in the lift $\tilde{\tau}$ of the train track τ to the universal cover of Σ, then we set $r(R) = 0$. If this does not occur, then the intersection points between k and the boundary leaves ℓ_R, ℓ'_R lie in a common branch $\widetilde{B}_0$ of $\tilde{\tau}$. We now orient ℓ_R, ℓ'_R so that they share their negative endpoint, and we denote by

$$\ldots, \widetilde{B}_2, \widetilde{B}_{-1}, \widetilde{B}_0, \widetilde{B}_1, \widetilde{B}_2, \ldots$$

the branches of $\tilde{\tau}$ that ℓ_R passes through, indexed in consecutive order according to the orientation of ℓ_R. We then define $r(R) := n + 1$, where n is the largest natural number such that ℓ'_R passes through the branches $\widetilde{B}_m$ for every integer $m \in \{-n, -n+1, \ldots, n-1, n\}$. Then we have:

Lemma 5.1 (see [BD17, Lemma 5.3])**.** *The divergence radius function $r : \mathcal{P}_{PQ} \to \mathbb{N}$ satisfies the following properties:*

(1) there exist constants $A, M > 0$ such that

$$A^{-1} \, e^{-M^{-1} \, r(R)} \leqslant L_{\tilde{X}}(k \cap R) \leqslant A \, e^{-M \, r(R)}$$

for every $R \in \mathcal{P}_{PQ}$;

(2) *there exists $N \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$ the preimage $r^{-1}(n)$ contains at most N plaques.*

Divergence radius functions were first introduced by Bonahon in [Bon96, § 1] to study the convergence of shear maps with respect to maximal geodesic laminations (see also Bonahon [Bon97b, Bon97a], and Bonahon and Dreyer [BD17]). In our exposition, these functions will be useful to select exhaustions $(\mathcal{P}_n)_n$ of $\mathcal{P}_{PQ}$ by finite nested subsets whose associated finite β-shear $\sigma^{\beta}_{\mathcal{P}_n}(P,Q)$ converge. However, in certain steps of our analysis (see in particular Lemma 5.7 and Proposition 5.10), it will useful to have a better understanding of the dependence of the functions r and of the corresponding constants A, M, N with respect to the choices of the lamination λ carried by τ, and the transverse path k. We summarize the necessary refinements of Lemma 5.1 in the following statements. Fixed a hyperbolic structure X on Σ, a train track τ, a maximal lamination λ carried by τ, and two plaques P and Q of λ, we have:

Lemma 5.2. *For any geodesic arc k joining the interiors of P and Q, there exist constants $A, M, N > 0$ and a open neighborhood U of λ in the space of maximal laminations (endowed with the Chabauty topology) such that the following properties hold:*

- *Every maximal lamination $\lambda' \in U$ is carried by τ.*
- *The X-geodesic path k is transverse to (the X-geodesic realization of) every $\lambda' \in U$.*
- *For every $\lambda' \in U$ there exist distinct plaques P', Q' of λ' in X that contain the endpoints of k.*
- *For any maximal lamination $\lambda' \in U$ and plaques P', Q' as above, the associated divergence radius function $r' = r_{X,\tau,\lambda',k} : \mathcal{P}_{P'Q'} \to \mathbb{N}$ satisfies properties (1) and (2) in Lemma 5.1 with constants $A, M, N > 0$ (which in particular are uniform in $\lambda' \in U$).*

Lemma 5.3. *For any choice of X-geodesic paths k and k' with endpoints lying in (the geodesic realizations of) the plaques P, Q, the associated divergence radius functions $r = r_k, r' = r_{k'} : \mathcal{P}_{PQ} \to \mathbb{N}$ provided by Lemma 5.1 are coarsely equivalent, i. e. there exist constants $H, K > 0$ such that*

$$H^{-1}\, r'(R) - K \leqslant r(R) \leqslant H\, r'(R) + K$$

for every plaque $R \in \mathcal{P}_{PQ}$.

We postpone the proofs of Lemmas 5.1, 5.2, and 5.3 to Appendix C.

5.2. **Shear cocycles: General case.** We now focus our attention on the construction of β-shear cocycles relative to a general maximal lamination λ. For the remainder of Section 5, we will assume the cross ratio $\beta : \partial\Gamma^{(4)} \to \mathbb{R}$ to be *locally bounded* (see Definitions 2.17, 4.2). Furthermore, we fix once and for all a hyperbolic structure X, and a train track τ carrying λ.

We start our analysis with two elementary Lemmas: The first (Lemma 5.4) describes how the shear between two plaques changes under the operation of *diagonal exchange* in the region separating P from Q. The second (Lemma 5.5) provides a bound between finite β-shears computed with respect to two finite families of plaques $\mathcal{P}, \mathcal{P}' \subset \mathcal{P}_{PQ}$ with $\mathcal{P} \subseteq \mathcal{P}'$. The bound described by Lemma 5.5 will be essential for the study of the approximation process needed to define σ^{β}_{λ}.

5.2.1. *Change of shear under diagonal exchange.* Let P, Q be two plaques of λ that share no ideal vertex. We denote by ℓ_P (resp. ℓ_Q) the boundary leaf of P (resp. Q) that separates the interior of P from the interior of Q, and by S the region of $\widetilde{\Sigma}$ bounded by ℓ_P and ℓ_Q. Given a coherent orientation of ℓ_P and ℓ_Q, we define d and d' to be the crossing geodesics $[\ell_P^+, \ell_Q^-]$ and $[\ell_P^-, \ell_Q^+]$, respectively. Finally, let R, T (resp. R', T') denote the complementary regions of d (resp. d') inside S.

To simplify the notation, we set

$$\sigma_d^\beta(P, Q) := \sigma^\beta(P, R) + \sigma^\beta(R, T) + \sigma^\beta(T, Q),$$
$$\sigma_{d'}^\beta(P, Q) := \sigma^\beta(P, R') + \sigma^\beta(R', T') + \sigma^\beta(T', Q).$$

Then we have:

Lemma 5.4. *The following relation holds:*

$$\left|\sigma_d^\beta(P, Q) - \sigma_{d'}^\beta(P, Q)\right| = 2\left|\log \beta(\ell_P^+, \ell_Q^+, \ell_Q^-, \ell_P^-)\right|.$$

As for Lemmas 4.7 and 4.8, the proof of Lemma 5.4 is an elementary consequence of the symmetries satisfied by the cross ratio β, and it will be described in Appendix B.

5.2.2. *Enlarging the finite set of plaques.* The next goal is to determine the behavior of the finite shear $\sigma_{\mathcal{P}}^\beta$ as we enlarge the finite family of plaques $\mathcal{P} \subset \mathcal{P}_{PQ}$. The statement that follows will play an essential role in the approximation process to determine $\sigma_\lambda^\beta(P, Q)$. Recall that, since β is a locally bounded cross ratio (see Definition 4.2), for $D = L_{\tilde{X}}(k) > 0$ (the length of k in $(\widetilde{\Sigma}, \widetilde{X})$), we can find constants $C, \alpha > 0$ (depending on the fixed hyperbolic structure X, the cross ratio β, and $L_{\tilde{X}}(k)$) such that

$$\left|\log \beta(h^+, \ell^+, \ell^-, h^-)\right| \leqslant C\left|\log \beta^X(h^+, \ell^+, \ell^-, h^-)\right|^\alpha \tag{9}$$

for every pair of coherently oriented geodesics ℓ, h in $(\widetilde{\Sigma}, \widetilde{X})$ such that $0 < d_{\tilde{X}}(\ell, h) \leqslant L_{\tilde{X}}(k)$. We then have:

Lemma 5.5. *For any pair of finite subsets $\mathcal{P}, \mathcal{P}'$ of $\mathcal{P}_{PQ}$ satisfying $\mathcal{P} \subseteq \mathcal{P}'$, we have*

$$\left|\sigma_{\mathcal{P}}^\beta(P, Q) - \sigma_{\mathcal{P}'}^\beta(P, Q)\right| \leqslant 2C\left|\mathcal{P}' - \mathcal{P}\right| \left(\sum_{d \subset k - \bigcup \mathcal{P}} L_{\tilde{X}}(d)^\alpha\right),$$

where $|\mathcal{P}' - \mathcal{P}|$ denotes the cardinality of the set $\mathcal{P}' - \mathcal{P}$, d varies among the (countable) set of connected components of $k - \bigcup \mathcal{P}$, and C, α are the constants associated with X, β, $L_{\tilde{X}}(k)$ as above.

Proof. We first consider the case in which $\mathcal{P}' = \mathcal{P} \cup \{R\}$. If

$$P = P_0, P_1, \ldots, P_n, P_{n+1} = Q$$

are the plaques of $\mathcal{P}$, indexed as we encounter them along the arc k from P to Q, then the plaque R will lie inside one of the components of $\widetilde{\Sigma} - \bigcup \mathcal{P}$ that separate P_i from P_{i+1}, for some i. We will denote by S such a region.

The laminations $\lambda_{\mathcal{P}}$ and $\lambda_{\mathcal{P}'}$ differ by a sequence of elementary moves, each of which either adds leaves to the lamination, or performs a diagonal exchange inside S. By Lemma 4.7, the shear between P and Q computed through the intermediate laminations λ and λ' does not change when λ' is obtained from λ by introducing new leaves. Therefore, it is sufficient to compute the change of the shear cocycle that occurs when a diagonal exchange is performed.

Let σ^β_λ and $\sigma^\beta_{\lambda'}$ be the shears associated with the plaques P and Q through the finite laminations λ and λ', respectively, which differ by a diagonal exchange in the region bounded by the leaves ℓ and h. We select orientations on ℓ and h so that they are coherently oriented. By Lemma 5.4 we have

$$\left|\sigma^\beta_\lambda(P,Q) - \sigma^\beta_{\lambda'}(P,Q)\right| = 2\left|\log \beta(h^+,\ell^+,\ell^-,h^-)\right|$$

Combining this equality with relation (9) and Lemma 4.3, we deduce that

$$\left|\sigma^\beta_\lambda(P,Q) - \sigma^\beta_{\lambda'}(P,Q)\right| \leqslant 2C\, d_{\tilde{X}}(\ell,h)^\alpha \leqslant 2C\, L_{\tilde{X}}(k \cap S)^\alpha,$$

where the last inequality holds since $k \cap S$ is a path that connects points lying on the leaves ℓ and h. When $\mathcal{P}$ and $\mathcal{P}'$ differ by a single plaque R, then by adding leaves and performing exactly one flip, we can move from the lamination $\lambda_{\mathcal{P}}$ to $\lambda_{\mathcal{P}'}$. If $\mathcal{P}'$ is obtained by adding to $\mathcal{P}$ n_S plaques lying inside the same region S, then it is simple to check that $\lambda_{\mathcal{P}}$ and $\lambda_{\mathcal{P}'}$ differ by a suitable sequence of moves, exactly n_S of which are diagonal exchanges. The difference in the shears $\sigma^\beta_{\mathcal{P}}(P,Q)$ and $\sigma^\beta_{\mathcal{P}'}(P,Q)$ can then be bounded by $2C\, n_S\, L_{\tilde{X}}(k \cap S)^\alpha$, by the same argument outlined above. The desired statement follows by applying this process in any complementary region S of $\widetilde{\Sigma} - \bigcup \mathcal{P}$, and noticing that $n_S \leqslant |\mathcal{P}' - \mathcal{P}|$ for any S. □

Remark 5.6. Notice that the argument described above makes use of the local boundedness of β only on pairs of leaves of the lamination λ. In particular, the machinery described in this section in fact applies to cross ratios β that are λ-*locally bounded*, i. e. that locally satisfy the control

$$\left|\log \beta(h^+,\ell^+,\ell^-,h^-)\right| \leqslant C\left|\log \beta^X(h^+,\ell^+,\ell^-,h^-)\right|^\alpha$$

for any pair of coherently oriented distinct leaves ℓ, h of the lamination λ.

5.2.3. *Constructing β-shear cocycles.* We are now ready to describe the approximation process for the β-shear cocycle σ^β_λ in the case of a general maximal lamination. Throughout the current section, we denote by P and Q two distinct plaques of some fixed maximal lamination λ, and by X an auxiliary hyperbolic structure on Σ.

We start our construction by selecting a well behaved exhaustion $(\mathcal{P}_n)_n$ by nested finite subsets of $\mathcal{P}_{PQ}$ through the notion of divergence radius function. Concretely, let k be a X-geodesic segment joining points in the interior of the plaques P and Q, and let $r = r_{X,\tau,\lambda,k} : \mathcal{P}_{PQ} \to \mathbb{N}$ be the corresponding divergence radius function, defined as in Section 5.1. Then, for every $n \in \mathbb{N}$ we set

$$\mathcal{P}_n := \{R \in \mathcal{P}_{PQ} \mid r(R) \leqslant n\}.$$

Notice that by Lemma 5.1 the cardinality of $\mathcal{P}_{n+1} - \mathcal{P}_n$ is bounded above by a constant $N > 0$ independent of n, and the union $\bigcup_n \mathcal{P}_n$ is equal to $\mathcal{P}_{PQ}$.

We are now ready to prove the first technical step of our construction:

Lemma 5.7. *The series*

$$\sum_n \left|\sigma^\beta_{\mathcal{P}_n}(P,Q) - \sigma^\beta_{\mathcal{P}_{n+1}}(P,Q)\right|$$

converges, and in particular the limit

$$\sigma^\beta_\lambda(P,Q) := \lim_{n\to\infty} \sigma^\beta_{\mathcal{P}_n}(P,Q)$$

is finite. Moreover, the quantity $\sigma^\beta_\lambda(P,Q) \in \mathbb{R}$ is independent of the choice of the geodesic arc k selected to construct the divergence radius function $r = r_{X,\tau,\lambda,k}$ and the set of plaques $(\mathcal{P}_n)_n$.

Proof. For simplicity, let $\sigma_n(P,Q)$ denote the quantity $\sigma^\beta_{\mathcal{P}_n}(P,Q)$. By Lemma 5.5 we have

$$\begin{aligned}|\sigma_{n+1}(P,Q) - \sigma_n(P,Q)| &\leqslant 2C|\mathcal{P}_{n+1} - \mathcal{P}_n| \left(\sum_{d\subset k - \bigcup \mathcal{P}_n} L_{\tilde{X}}(d)^\alpha\right) \\ &\leqslant 2CN \left(\sum_{d\subset k - \bigcup \mathcal{P}_n} L_{\tilde{X}}(d)^\alpha\right).\end{aligned}$$

It is not restrictive to assume $\alpha < 1$, in which case we have

$$\sum_{d\subset k - \bigcup \mathcal{P}_n} L_{\tilde{X}}(d)^\alpha \leqslant \sum_{R\in\mathcal{P}_{PQ}: r(R)\geqslant n+1} L_{\tilde{X}}(k\cap R)^\alpha.$$

Combining this estimate with the properties of the divergence radius function r described Lemma 5.1, we obtain

$$\begin{aligned}\sum_{d\subset k - \bigcup \mathcal{P}_n} L_{\tilde{X}}(d)^\alpha &\leqslant \sum_{R\in\mathcal{P}_{PQ}: r(R)\geqslant n+1} A^\alpha e^{-\alpha M r(R)} \\ &\leqslant A^\alpha N \sum_{j>n} e^{-\alpha M j} \\ &\leqslant \frac{A^\alpha N}{1-e^{-\alpha M}} e^{-\alpha M(n+1)},\end{aligned} \tag{10}$$

where $A, M, N > 0$ are the constants provided by Lemma 5.1. Therefore we deduce

$$\sum_{n\in\mathbb{N}} |\sigma_{n+1}(P,Q) - \sigma_n(P,Q)| \leqslant \frac{2CA^\alpha N^2}{1-e^{-\alpha M}} \sum_{n\in\mathbb{N}} e^{-\alpha M(n+1)} < +\infty,$$

which concludes the proof of the first part of the statement.

Let $(\mathcal{P}'_n)_n$ be the sequence of plaques associated with a different choice of geodesic segment k', and hence divergence radius function r' as in Lemma 5.1. By Lemma 5.3, there exist two natural numbers l, m such that

$$\mathcal{P}_n \subseteq \mathcal{P}'_{ln+m}, \quad \mathcal{P}'_n \subseteq \mathcal{P}_{ln+m}$$

for every $n \in \mathbb{N}$. Moreover, by property (2) of Lemma 5.1, there exists a constant $N > 0$ such that the cardinality of the sets $\mathcal{P}_{l(ln+m)+m} - \mathcal{P}_n$ is bounded above by $N(l^2-1)n + lm + m)$ for every $n \in \mathbb{N}$. The same function of n in particular provides an upper bound of the cardinality of the set $\mathcal{P}'_{ln+m} - \mathcal{P}_n \subseteq \mathcal{P}_{l(ln+m)+m} - \mathcal{P}_n$. Applying Lemma 5.5 and relation (10), we deduce

$$\begin{aligned}|\sigma^\beta_{\mathcal{P}_n}(P,Q) - \sigma^\beta_{\mathcal{P}'_{ln+m}}(P,Q)| &\leqslant 2CN(l^2-1)n + lm + m) \left(\sum_{d\subset k - \bigcup \mathcal{P}_n} L_{\tilde{X}}(d)^\alpha\right) \\ &\leqslant \frac{2A^\alpha C N^2}{1-e^{-\alpha M}} (l^2-1)n + lm + m) e^{-\alpha M(n+1)}.\end{aligned}$$

This proves in particular that the difference between $\sigma^{\beta}_{\mathcal{P}_n}(P,Q)$ and $\sigma^{\beta}_{\mathcal{P}'_{ln+m}}(P,Q)$ tends to 0 as $n \to \infty$, and therefore we conclude

$$\lim_{n\to\infty} \sigma^{\beta}_{\mathcal{P}_n}(P,Q) = \lim_{n\to\infty} \sigma^{\beta}_{\mathcal{P}'_n}(P,Q),$$

as desired. □

Remark 5.8. Fix a locally bounded cross ratio β, a hyperbolic structure X, a train track τ carrying λ, and a geodesic arc k transverse to λ. The estimates appearing in the proof of Lemma 5.7 show that, given two distinct plaques P and Q of λ, there exist constants $C' = C'(C, \alpha, A, M, N), M' = M'(\alpha, M) > 0$ such that for every $n \in \mathbb{N}$

$$\left|\sigma^{\beta}_{\lambda}(P,Q) - \sigma^{\beta}_{\mathcal{P}_n}(P,Q)\right| \leqslant C' e^{-M'n},$$

where the constants $A, M, N > 0$, provided by Lemma 5.1, depend only the structure X, the train track τ carrying λ, and the path k and $\alpha, C > 0$ are provided by the local boundedness of β (see Definition 4.2), with the choice of $D = L_{\tilde{X}}(k)$.

By Lemma 5.2 we can then find a neighborhood U of λ in the space of maximal geodesic laminations such that, for every $\lambda' \in U$, the finite β-shears $\sigma^{\beta}_{\mathcal{P}'_n}(P',Q')$ associated with the arc k and the corresponding divergence radius function $r' = r_{X,\tau,\lambda',k} : \mathcal{P}_{P'Q'} \to \mathbb{N}$ converge to $\sigma^{\beta}_{\lambda'}(P',Q')$ and satisfy

$$\left|\sigma^{\beta}_{\lambda'}(P',Q') - \sigma^{\beta}_{\mathcal{P}'_n}(P',Q')\right| \leqslant C' e^{-M'n},$$

with uniform constants $C', M' > 0$ with respect to $\lambda' \in U$ (compare with the notation of Lemma 5.2). For future reference (see in particular Proposition 5.10), we notice that the constants $C', M' > 0$ also satisfy

$$\sum_{d \subset k - \bigcup \mathcal{P}_n} L_{\tilde{X}}(d)^{\alpha} \leqslant \frac{C' e^{-M'n}}{2CN}. \tag{11}$$

(Compare with relation (10).)

We finally define the *β-shear relative to λ between the plaques P and Q* to be

$$\sigma^{\beta}_{\lambda}(P,Q) := \lim_{n\to\infty} \sigma^{\beta}_{\mathcal{P}_n}(P,Q),$$

where $(\mathcal{P}_n)_n$ is the exhausting sequence of $\mathcal{P}_{PQ}$ associated with the divergence radius function $r = r_{X,\tau,\lambda,k} : \mathcal{P}_{PQ} \to \mathbb{N}$, for some choice of a X-geodesic path k joining P and Q. By Lemma 5.7, the value $\sigma^{\beta}_{\lambda}(P,Q)$ is independent of the choice of k. We are now ready to conclude the construction of β-shear cocycles:

Proposition 5.9. *The map $(P,Q) \mapsto \sigma^{\beta}_{\lambda}(P,Q)$, constructed following the process described above, is a Hölder cocycle trasverse to λ.*

Proof. All the properties are simple consequences of the definition of finite β-shears from Section 4.3, and of the independence of the quantity σ^{β}_{λ} from the selected geodesic path k and the associated divergence radius $r : \mathcal{P}_{PQ} \to \mathbb{N}$, as established by Lemma 5.7.

To prove property (1) from Definition 2.15, it suffices to select the same path k (and hence same divergence radius function r) to approximate both $\sigma^{\beta}_{\lambda}(P,Q)$ and $\sigma^{\beta}_{\lambda}(Q,P)$. Indeed, by the symmetries of the cross ratio β we have $\sigma^{\beta}_{\mathcal{P}_n}(P,Q) = \sigma^{\beta}_{\mathcal{P}_n}(Q,P)$ for every $n \in \mathbb{N}$.

To see property (2), let k be a path connecting the plaques P and Q, with associated function $r : \mathcal{P}_{PQ} \to \mathbb{R}$, and let $R \in \mathcal{P}_{PQ}$. We select a subarc k' of k that connects P to R, and we set $k'' = \overline{k - k'}$. Observe that the restriction of $r = r_k$ to the set of plaques $\mathcal{P}_{PR}$ coincides with the divergence radius function r' associated to k'. The same holds for the restriction of r to $\mathcal{P}_{RQ}$ and the path k''. Therefore, the divergence radius functions r' and r'' associated to k' and k'' determine sequences of finite collections of plaques $(\mathcal{P}'_n)_n$ and $(\mathcal{P}''_n)_n$, respectively, satisfying

$$\lim_{n\to\infty} \sigma^{\beta}_{\mathcal{P}'_n}(P, R) = \sigma^{\beta}_{\lambda}(P, R), \qquad \lim_{n\to\infty} \sigma^{\beta}_{\mathcal{P}''_n}(R, Q) = \sigma^{\beta}_{\lambda}(R, Q).$$

Moreover, if $(\mathcal{P}_n)_n$ denotes the exhaustion of $\mathcal{P}_{PQ}$ associated to k and r, then by construction

$$\mathcal{P}_n = \mathcal{P}'_n \cup \{R\} \cup \mathcal{P}''_n$$

for every $n \geqslant r(R)$. Moreover the finite β-shears satisfy

$$\sigma^{\beta}_{\mathcal{P}_n}(P, Q) = \sigma^{\beta}_{\mathcal{P}'_n}(P, R) + \sigma^{\beta}_{\mathcal{P}''_n}(R, Q)$$

again for every $n \geqslant r(R)$. By taking the limit as $n \to \infty$, we obtain the additivity property described in property (2) of Definition 2.15.

Finally, to show property (3), let $\gamma \in \Gamma$ and select $\gamma(k)$ as a path joining the interiors of the plaques γP to γQ. The associated divergence radius function coincides with $r \circ \gamma^{-1} : \mathcal{P}_{\gamma P \, \gamma Q} \to \mathbb{N}$, where $r : \mathcal{P}_{PQ} \to \mathbb{N}$ is the divergence radius function of k. If $(\mathcal{P}_n)_n$ denotes the sequence of finite family of plaques associated with k and r, then $\gamma(k)$ and $r \circ \gamma^{-1}$ have associated sequence $(\gamma\mathcal{P}_n)_n$. Moreover, being β Γ-invariant, we have

$$\sigma^{\beta}_{\mathcal{P}_n}(P, Q) = \sigma^{\beta}_{\gamma\mathcal{P}_n}(\gamma P, \gamma Q)$$

for every $n \in \mathbb{N}$. The identity $\sigma^{\beta}_{\lambda}(P, Q) = \sigma^{\beta}_{\lambda}(\gamma P, \gamma Q)$ then follows by taking the limit as $n \to \infty$. $\square$

5.3. **Continuity of shear cocycles.** We now study the continuity properties of the map

$$\mathcal{GL} \ni \lambda \longmapsto \sigma^{\beta}_{\lambda} \in \mathcal{H}(\lambda; \mathbb{R}).$$

As recalled in Section 2.4.5, the choice of a train track τ that carries a maximal lamination λ determines natural identifications between its associated system of real weights $\mathcal{W}(\tau; \mathbb{R})$ and the space of Hölder cocycles $\mathcal{H}(\lambda'; \mathbb{R})$ transverse to any lamination λ' carried by τ. In particular, there exists a sufficiently small neighborhood U of λ inside $\mathcal{GL}$ for which the map

$$U \ni \lambda' \longmapsto \sigma^{\beta}_{\lambda'} \in \mathcal{W}(\tau; \mathbb{R})$$

is well defined. Within this framework, it makes sense to ask ourselves whether the map $\lambda' \mapsto \sigma^{\beta}_{\lambda'}$ is continuous. The next statement answers affirmatively to this question:

Proposition 5.10. *Let $(\lambda_m)_m$ be a sequence of maximal geodesic laminations converging to λ in the Chabauty topology. Given τ a train track that carries λ, we identify $\mathcal{H}(\lambda; \mathbb{R})$ and $\mathcal{H}(\lambda_m; \mathbb{R})$ with $\mathcal{W}(\tau; \mathbb{R})$, the space of real weights of τ (for m sufficiently large). Then*

$$\lim_{m\to\infty} \sigma^{\beta}_{\lambda_m} = \sigma^{\beta}_{\lambda} \in \mathcal{W}(\tau; \mathbb{R}).$$

Proof. If k is a tie of the lift of the train track τ in $(\widetilde{\Sigma}, \widetilde{X})$, then the endpoints of k lie in the interior of two plaques P, Q of λ. Moreover, since λ_m converges to λ in the Chabauty topology, there exists a $m_0 \in \mathbb{N}$ such that for every $m > m_0$ the endpoints of k lie in the interior of two plaques $P^{(m)}, Q^{(m)}$ of λ_m. Then the statement is equivalent to show that, for any k as above

$$\lim_{m \to \infty} \sigma^{\beta}_{\lambda_m}(P^{(m)}, Q^{(m)}) = \sigma^{\beta}_{\lambda}(P, Q).$$

Let $\mathcal{P}_k$ (resp. $\mathcal{P}_k^{(m)}$) denote the set of plaques of λ (resp. λ_m) that separate P from Q (resp. $P^{(m)}$ from $Q^{(m)}$). If k is the geodesic arc joining the endpoints of k, then Lemma 5.1 provide us functions

$$r : \mathcal{P}_k \to \mathbb{N}, \quad r_m : \mathcal{P}_k^{(m)} \to \mathbb{N}$$

satisfying properties (1), (2) with respect to constants $A, M, N > 0$ that are *independent of* m, and defined in terms of the same train track τ and arc k (see in particular Remark 5.8). To simplify the notation, for every $n \in \mathbb{N}$ and $m > m_0$ we set

$$\begin{aligned} \mathcal{P}_n &:= \{R \in \mathcal{P}_k \mid r(R) \leqslant n\}, \\ \mathcal{P}_n^{(m)} &:= \{R \in \mathcal{P}_k^{(m)} \mid r_m(R) \leqslant n\}, \end{aligned}$$

and

$$\begin{aligned} \sigma &:= \sigma^{\beta}_{\lambda}(P, Q), & \sigma^{(m)} &:= \sigma^{\beta}_{\lambda_m}(P^{(m)}, Q^{(m)}), \\ \sigma_n &:= \sigma^{\beta}_{\mathcal{P}_n}(P, Q), & \sigma_n^{(m)} &:= \sigma^{\beta}_{\mathcal{P}_n^{(m)}}(P^{(m)}, Q^{(m)}). \end{aligned}$$

Let now $N, C', M' > 0$ be positive constants satisfying the requirements of Lemma 5.1 and Remark 5.8. We will prove the desired assertion by showing that

$$\limsup_{m \to \infty} |\sigma^{(m)} - \sigma| \leqslant (2 + n) C' e^{-M'n} \tag{12}$$

for every $n \in \mathbb{N}$. Since the left-hand side of the inequality is independent of n, and the right-hand side converges to 0 as $n \to \infty$, the assertion will follow.

We will divide the proof of relation (12) into smaller steps. In order to describe them, we need to introduce some notation. For any $R \in \mathcal{P}_n$, we choose arbitrarily a point x_R in the interior of R. Since $\lambda_m \to \lambda$, we can find a $m_1 > 0$ sufficiently such that, for any $m > m_1$, there exists a unique plaque $R^{(m)} \in \mathcal{P}_k^{(m)}$ whose interior contains x_R. Being $\mathcal{P}_n$ finite, up to selecting a larger m_1 we can assume that this holds for every plaque $R \in \mathcal{P}_n$. We then introduce the sets

$$\mathcal{Q}_n^{(m)} := \{R^{(m)} \in \mathcal{P}_k^{(m)} \mid R \in \mathcal{P}_n\},$$

for $m > m_1$. Finally, we set

$$\hat{\sigma}_n^{(m)} := \sigma^{\beta}_{\mathcal{Q}_n^{(m)}}(P^{(m)}, Q^{(m)}).$$

Step 1. For every $n \in \mathbb{N}$ there exists a $m_2 \geqslant m_1$ such that $\mathcal{Q}_n^{(m)} \subseteq \mathcal{P}_n^{(m)}$ for all $m > m_2$. Moreover

$$\left|\hat{\sigma}_n^{(m)} - \sigma_n^{(m)}\right| \leqslant 2CNn \left(\sum_{d \subset k - \bigcup \mathcal{Q}_n^{(m)}} L_{\tilde{X}}(d)^{\alpha} \right),$$

where $C, \alpha, N > 0$ are the constants appearing in Lemmas 5.1 and 5.5 (see also Remark 5.8).

Proof of Step 1. We will show that $\mathcal{Q}_n^{(m)} \subseteq \mathcal{P}_n^{(m)}$ for every m sufficiently large. The second part of the assertion will follow by applying Lemma 5.5 and noticing that

$$\left|\mathcal{P}_n^{(m)} - \mathcal{Q}_n^{(m)}\right| \leqslant \left|\mathcal{P}_n^{(m)}\right| \leqslant Nn$$

by Lemma 5.1.

Let $R \in \mathcal{P}_n$, and denote by $\ell_R, h_R \subset \lambda$ the boundary leaves of R that cross the tie k. Similarly, let $\ell_R^{(m)}, h_R^{(m)} \subset \lambda_m$ be the boundary leaves of $R^{(m)} \in \mathcal{Q}_n^{(m)}$ that cross k. It is enough to prove that $\lim_{m\to\infty} r_m(R^{(m)}) = n$ for every $R \in \mathcal{P}_n$.

Since the laminations λ_m converge to λ and the plaques $R^{(m)}$ contain a fixed point x_R in the interior of the plaque R, the leaves $\ell_R^{(m)}, h_R^{(m)}$ converge in the Hausdorff topology to ℓ_R, h_R as $m \to \infty$ (up to relabeling). Recalling the definition of the divergence radius functions r, r_m from Section 5.1, the condition $r(R) = n$ is equivalent to say that the leaves ℓ_R, h_R cross $n+1$ common branches of $\tilde{\tau}$ in both directions (as we count starting from the branch containing the tie k) before taking different paths at some switch of $\tilde{\tau}$. Since the boundary leaves of $R^{(m)}$ that meet k converge to the boundary leaves ℓ_R and h_R, we can find a sufficiently large $m_2 \geqslant m_1$ such that $\ell_R^{(m)}$ passes through the same $n+1$ branches of $\tilde{\tau}$ as ℓ_R in both directions, and similarly for $h_R^{(m)}$, for all $m \geqslant m_2$. This implies in particular that $r_m(R^{(m)}) = n$. Since $\mathcal{P}_n$ is finite, up to enlarging m_2 we can assume that this holds for every $R \in \mathcal{P}_n$, as desired. □

Step 2. For every $n \in \mathbb{N}$ we have

$$\lim_{m\to\infty} \sum_{d \subset k - \bigcup \mathcal{Q}_n^{(m)}} L_{\tilde{X}}(d)^\alpha = \sum_{d \subset k - \bigcup \mathcal{P}_n} L_{\tilde{X}}(d)^\alpha \leqslant \frac{C' e^{-M'n}}{2CN}.$$

Proof of Step 2. As observed in the proof of the previous step, the boundary leaves of $R^{(m)}$ converge to the boundary leaves of R with respect to the Chabauty topology for every $R \in \mathcal{P}_n$. In particular, each subarc $k \cap R$ of k is equal to the limit of the subarcs $(k \cap R^{(m)})_m$. Since $\mathcal{P}_n$ is a finite collection of plaques, the set

$$\{d \mid d \text{ connected component of } k - \bigcup \mathcal{Q}_n^{(m)}\}$$

is finite, and the length of each of its components converges to the length of the corresponding component of $k - \bigcup \mathcal{P}_n$. This implies the equality appearing in the statement. The upper bound of the limit follows from Remark 5.8, and more specifically relation (11). □

Step 3. For every $n \in \mathbb{N}$ we have $\lim_{m\to\infty} \hat{\sigma}_n^{(m)} = \sigma_n$.

Proof of Step 3. We denote as above by ℓ_R, h_R (resp. $\ell_R^{(m)}, h_R^{(m)}$) the leaves of $R \in \mathcal{P}_n$ (resp. $R^{(m)} \in \mathcal{Q}_n^{(m)}$) that cross k, and we orient them from right to left as we follow the geodesic arc k, moving from P to Q (resp. $P^{(m)}$ and $Q^{(m)}$). By what observed above we have

$$\lim_{m\to\infty} (\ell_R^{(m)})^\pm = \ell_R^\pm, \quad \lim_{m\to\infty} (h_R^{(m)})^\pm = h_R^\pm \in \partial\Gamma.$$

By definition, the quantity $\sigma_n = \sigma^{\beta}_{\mathcal{P}_n}(P, Q)$ is a finite sum of elementary shears, defined as in relation (7), where the points at infinity u_+, u_-, u_l, u_r belong to the set $\{\ell^{\pm}_R, h^{\pm}_R \mid R \in \mathcal{P}_n\}$, and similarly for $\sigma^{(m)}_n$ and the set $\{(\ell^{(m)}_R)^{\pm}, (h^{(m)}_R)^{\pm} \mid R \in \mathcal{P}_n\}$. From the construction it follows that the finite laminations $\lambda_{\mathcal{Q}^{(m)}_n}$, defined as in Section 4.3, converge to the lamination $\lambda_{\mathcal{P}_n}$ as $m \to \infty$. By the continuity of the cross ratio β, it is now immediate to see that the finite sum of shears $\sigma^{(m)}_n$ converge to σ_n as $m \to \infty$. □

Step 4. For every $n \in \mathbb{N}$ and $m > m_0$

$$\begin{aligned} |\sigma - \sigma_n| &\leqslant C' e^{-M' n}, \\ |\sigma^{(m)} - \sigma^{(m)}_n| &\leqslant C' e^{-M' n}. \end{aligned}$$

Proof of Step 4. This is an immediate consequence of Remark 5.8 and the definition of $\sigma, \sigma_n, \sigma^{(m)}, \sigma^{(m)}_n$. □

We now have all the ingredients to conclude our argument. First we observe that

$$|\sigma^{(m)} - \sigma| \leqslant |\sigma^{(m)} - \sigma^{(m)}_n| + |\sigma^{(m)}_n - \hat{\sigma}^{(m)}_n| + |\hat{\sigma}^{(m)}_n - \sigma_n| + |\sigma_n - \sigma|.$$

By Steps 1 and 2 we have

$$\limsup_{m \to \infty} |\sigma^{(m)}_n - \hat{\sigma}^{(m)}_n| \leqslant n\, C' e^{-M' n}.$$

Relation (12) now follows by combining the above inequalities with Steps 3 and 4:

$$\begin{aligned} \limsup_{m \to \infty} |\sigma^{(m)} - \sigma| &\leqslant 2C' e^{-M' n} + \limsup_{m \to \infty} |\sigma^{(m)}_n - \hat{\sigma}^{(m)}_n| \\ &\leqslant (2 + n) C' e^{-M' n}. \end{aligned}$$

This concludes the proof of Proposition 5.10. □

As mentioned in the introduction of the section, continuity and the fact that σ^{β}_{λ} coincides with the one defined in the previous section for maximal finite leaved laminations implies that σ^{λ}_{β} only depends on β, λ and not on all the auxiliary choices made to define it.

5.4. **Shears and length functions: General case.** Now that we have established the continuous dependence of β-shear cocycles σ^{β}_{λ} in the maximal lamination λ, we can easily generalize the relation between shear cocycles and β-periods observed in Section 4.5 for finite leaved laminations to any maximal geodesic lamination. More precisely:

Proposition 5.11. *Let β be a positive and locally bounded cross ratio. Then for every maximal lamination λ and for every measured lamination μ with* $\operatorname{supp} \mu \subseteq \lambda$, *we have*

$$L_{\beta}(\mu) = \omega_{\lambda}(\sigma^{\beta}_{\lambda}, \mu),$$

where ω_{λ} denotes the Thurston symplectic form on the space of transverse Hölder cocycles $\mathcal{H}(\lambda; \mathbb{R})$, and L_{β} is the length function introduced in Section 2.5.

Proof. When λ is a finite leaved maximal lamination, then the statement is equivalent to Proposition 4.11. Consider now a general maximal lamination λ and a measured lamination μ with support contained in λ. Without loss of generality we can assume that μ is minimal, so it can be approximated in $\mathcal{ML}$ by a sequence of weighted simple closed curves $(a_n \gamma_n)_n$. Moreover, following the procedure described by Canary, Epstein, and Green in [CEG06, Theorem I.4.2.14], for every n we can extend the curve γ_n to a finite leaved lamination λ_n so that, up to subsequence, λ_n converges in the Chabauty topology to λ.

Select now a train track τ that carries λ. Since the laminations λ_n are converging to λ in the Chabauty topology, the train track τ carries λ_n for n sufficiently large. In particular, we can identify the spaces of transverse Hölder cocycles $\mathcal{H}(\lambda; \mathbb{R})$ and $\mathcal{H}(\lambda_n; \mathbb{R})$ with the space of real weights $\mathcal{W}(\tau; \mathbb{R})$. Notice that the isomorphisms $\mathcal{H}(\lambda_n; \mathbb{R}) \cong \mathcal{W}(\tau; \mathbb{R}) \cong \mathcal{H}(\lambda; \mathbb{R})$ are linear symplectomorphisms with respect to the associated Thurston symplectic forms and the algebraic intersection pairing ω_τ on $\mathcal{W}(\tau; \mathbb{R})$, in light of the description provided in Section 4.5.1. By Lemma 4.11 we have

$$a_n L_\beta(\gamma_n) = \omega_\tau(\sigma^\beta_{\lambda_n}, a_n \gamma_n) \tag{13}$$

for every n sufficiently large (we are identifying with abuse the cocycles $\sigma^\beta_{\lambda_n}, a_n \gamma_n \in \mathcal{H}(\lambda_n; \mathbb{R})$ with their image inside $\mathcal{W}(\tau; \mathbb{R})$). Now, by Theorem 2.21 the left-hand side $a_n L_\beta(\gamma_n)$ is continuous in $a_n \gamma_n$, and hence converges to $L_\beta(\mu)$, while the right-hand side converges to $\omega_\lambda(\sigma^\beta_\lambda, \mu)$ by Proposition 5.10. By taking the limit as $n \to \infty$ of relation (13) we obtain the statement for the lamination λ and the minimal measured lamination μ. □

5.5. **The proof of Theorem 4.1.** We finally have all the elements to prove the main result of the section:

Theorem 4.1. *Let $\beta : \partial\Gamma^{(4)} \to \mathbb{R}$ be a positive and locally bounded cross ratio. Then for every maximal lamination λ, the β-shear cocycle σ^β_λ belongs to the closure of the cone $C(\lambda) \subset \mathcal{H}(\lambda; \mathbb{R})$, that is*

$$\omega_\lambda(\sigma^\beta_\lambda, \mu) \geqslant 0$$

for every measured lamination μ with $\operatorname{supp} \mu \subseteq \lambda$. Moreover, if the cross ratio β is strictly positive, then $\omega_\lambda(\sigma^\beta_\lambda, \mu) > 0$ for every non-trivial measured lamination μ as above, and consequently there exists a unique hyperbolic structure $Y = Y^\beta_\lambda \in \mathcal{T}(\Sigma)$ such that $\sigma^\beta_\lambda = \sigma^Y_\lambda \in \mathcal{H}(\lambda; \mathbb{R})$.

Proof. By Theorem 2.21, every positive cross ratio has an associated Liouville current $\mathscr{L}_\beta$, and the corresponding β-length $L_\beta = i(\mathscr{L}_\beta, \bullet)$ is a non-negative function on the space of geodesic currents. Hence the first part of the assertion follows directly from Proposition 5.11.

As observed in Lemma 2.22, if the cross ratio β is strictly positive, then $L_\beta(c) > 0$ for any non-trivial geodesic current c. Therefore, combining Proposition 5.11 with Theorem 2.16, we deduce that for every maximal geodesic lamination λ there exists a unique hyperbolic structure Y such that $\sigma^\beta_\lambda = \sigma^Y_\lambda$, as desired. □

Remark 5.12. We point out to the reader that the work of Burger, Iozzi, Parreau, and Pozzetti on geodesic currents (see in particular [BIPP21, Theorems 1.3, 1.7, Corollary 1.9]) can be deployed to investigate in detail the set of measured laminations μ with trivial β-length, by examining the geometric decomposition of the Liouville current $\mathscr{L}_\beta$. This in turn determines the set of maximal geodesic laminations λ for which the associated β-shear cocycle lies in the boundary $\partial C(\lambda)$ of the shear parametrization from Theorem 2.16.

6. Geometry of pleated surfaces

In this section we prove the main structural result about the geometry of pleated surfaces, that is, Theorem C.

Our strategy is as follows: We first analyze explicitly the case of finite leaved maximal laminations and prove the existence of a 1-Lipschitz developing map, as given in Theorem C, in that setting. The proofs here are completely elementary (see Propositions 6.5 and 6.6). Then, we exploit continuity properties of pleated surfaces to deduce the existence of a 1-Lipschitz developing map in the general case (see Proposition 6.7). The comparison of the length spectra (see Proposition 6.8) will be a consequence of the existence of 1-Lipschitz developing map together with some properties the bending locus.

Let us remark that for the proof of the strict domination theorem (Theorem 7.6) the elementary case of finite leaved maximal laminations is sufficient.

In general, it is always possible to define on the pleated set $\hat{S}_\lambda$ a natural *length space* structure: Recall that a Lipschitz function is differentiable almost everywhere.

Definition 6.1 (Regular Path)**.** A *(weakly) regular path* is a Lipschitz map $\gamma : I = [a, b] \to \mathbb{H}^{2,n}$ such that $\dot{\gamma}$ is spacelike (or lightlike) almost everywhere. The length of a weakly regular path is

$$\ell(\gamma) := \int_I \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} dt.$$

The Lipschitz property implies that the length $\ell(\gamma)$ is always finite.

If λ is a maximal lamination and $\hat{S}_\lambda \subset \mathbb{H}^{2,n}$ is the associated pleated set we say that a path $\gamma : I \to \hat{S}_\lambda$ is (weakly) regular if it is (weakly) regular as a path in $\mathbb{H}^{2,n}$.

A regular path $\gamma : I \to \hat{S}_\lambda$ is transverse to $\hat{\lambda}$ if $\gamma^{-1}\hat{\lambda}$ has Lebesgue measure zero.

It is not difficult to check, using the representation of $\hat{S}_\lambda$ as the graph of a (strictly) 1-Lipschitz function $g_\lambda : \mathbb{D}^2 \to \mathbb{S}^n$ in a Poincaré model $\mathbb{D}^2 \times \mathbb{S}^n$ of $\widehat{\mathbb{H}}^{2,n}$ that every pair of points $x, y \in \hat{S}_\lambda$ can be joined by a weakly regular path. In fact, the graph of any Lipschitz path $\alpha : I \to \mathbb{D}^2$ joining the projections $\pi(x), \pi(y)$ would work. When λ is a finite leaved maximal lamination, it is possible to join any two points $x, y \in \hat{S}_\lambda$ with a regular path that intersects the the lamination in countably many points (hence, transversely).

Definition 6.2 (Pleated Surface)**.** Let $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ be a maximal representation. A *pleated surface* for ρ realizing the maximal lamination λ consists of the following data:

(1) The pleated set $S_\lambda = \hat{S}_\lambda / \rho(\Gamma)$.
(2) The hyperbolic surface $X_\lambda \in \mathcal{T}$ with shear cocycle equal to the intrinsic shear cocycle $\sigma_\lambda^{X_\lambda} = \sigma_\lambda^\rho \in \mathcal{H}(\lambda; \mathbb{R})$.
(3) A homeomorphism $f : S_\lambda \to X_\lambda$ that is totally geodesic on every leaf of λ and each plaque of $S_\lambda - \lambda$ and is 1-Lipschitz with respect to the intrinsic pseudo-metric (see Section 2.2) and the hyperbolic metric.

We call a homeomorphism $f : S_\lambda \to X_\lambda$ with the the properties of (3) a *1-Lipschitz developing map.*

In a more precise way, 1-Lipschitz developing maps of pleated surfaces are totally geodesic on the leaves of the bending locus and on each of the complementary components as we prove in the next lemma. Later on, we will see that developing maps are strictly contracting (in a suitable sense) in directions transverse to the bending locus.

Lemma 6.3. *Let $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ be a maximal representation. Let S_λ be a pleated surface realizing a maximal lamination λ and let $f : S_\lambda \to X_\lambda$ be a 1-Lipschitz developing map. Then f is totally geodesic on the complement of the bending locus.*

Proof. Consider a component W of the complement of the bending locus of S_λ. In this case, the restriction of the pseudo-metric to W is a hyperbolic metric. As the restriction of $f : S_\lambda \to X_\lambda$ to $f : W \to f(W)$ is a 1-Lipschitz map between hyperbolic surfaces of the same area, we conclude that $f : W \to f(W)$ is an isometry (see e.g. Thurston [Thu98]). $\square$

Lemma 6.4. *Let $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ be a maximal representation. Let S_λ be a pleated surface realizing a maximal lamination λ and let $f : S_\lambda \to X_\lambda$ be a 1-Lipschitz developing map. Then f sends regular paths $\gamma : I \to S_\lambda$ to a Lipschitz (hence rectifiable) paths $f\gamma : I \to X_\lambda$.*

Proof. Lift f to the universal covering $\hat{f} : \hat{S}_\lambda \subset \widehat{\mathbb{H}}^{2,n} \to \mathbb{H}^2$. We show that $\hat{f}$ maps regular paths to rectifiable paths. In order to do so, it is convenient to work in a Poincaré model $\mathbb{D}^2 \times \mathbb{S}^n$ of $\widehat{\mathbb{H}}^{2,n}$ and represent $\hat{S}_\lambda$ as a graph of a 1-Lipschitz function $g : \mathbb{D}^2 \to \mathbb{S}^n$. Let us denote by $u : \mathbb{D}^2 \to \hat{S}_\lambda$ the graph map $u(x) = (x, g(x))$ and by $h : \mathbb{D}^2 \to \mathbb{H}^2$ the composition $h = fu$. By Lemma 2.5 the map u is 1-Lipschitz with respect to the hyperbolic metric on $\mathbb{H}^2$ and the pseudo-metric on $\hat{S}_\lambda$. As the developing map $\hat{f}$ is 1-Lipschitz with respect to the intrinsic path metric on $\hat{S}_\lambda$ and the hyperbolic metric on $\mathbb{H}^2$, we conclude that h is 1-Lipschitz with respect to the hyperbolic metric on both source and target. Let $\gamma : I \to \hat{S}_\lambda$ be a regular path. We can write it as $\gamma = u\alpha$ where $\alpha : I \to \mathbb{D}^2$ is the Lipschitz path obtained by composing γ with the Lipschitz projection $\widehat{\mathbb{H}}^{2,n} \to \mathbb{D}^2$. As $f\gamma = h\alpha$, we deduce that $f\gamma$ is a Lipschitz path. $\square$

6.1. **Finite leaved maximal laminations.** We prove the main results for finite leaved laminations:

Proposition 6.5. *Let $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ be a maximal representation. Let λ be a finite leaved maximal lamination of Σ. Let $\hat{S}_\lambda$ be the associated pleated set. Then there is an equivariant homeomorphism $f : \hat{S}_\lambda \to \mathbb{H}^2$ with the following properties:*

(i) It is totally geodesic on every leaf and plaque.

(ii) It is 1-Lipschitz, that is, $d_{\mathbb{H}^{2,n}}(x, y) \geqslant d_{\mathbb{H}^2}(f(x), f(y))$ for every $x, y \in \hat{S}_\lambda$.

Proof. Let $\mu = \gamma_1 \sqcup \cdots \sqcup \gamma_k \subset \lambda$ denote the collection of the closed leaves of λ. Each leaf ℓ of $\lambda - \mu$ is an isolated leaf and is adjacent to two distinct triangular components P, P', that is, $\ell = P \cap P'$. In particular, each component of $S_j \subset S - \mu$ has an intrinsic incomplete hyperbolic metric. The abstract completion of S_j is a hyperbolic surface with totally geodesic boundary $\bar{S}_j$.

We lift S_j and $\bar{S}_j$ to their universal covering $\hat{S}_j \subset \hat{S}_\lambda \subset \widehat{\mathbb{H}}^{2,n}$ and $\bar{U}_j \subset \mathbb{H}^2$.

Claim. We have

$$d_{\mathbb{H}^{2,n}}(x, y) \geqslant d_{\bar{U}_j}(x, y)$$

for every $x, y \in \hat{S}_j$.

Proof of claim. Consider points $x, y \in \hat{S}_j$. Let H be a spacelike hyperplane containing x, y and let τ be a timelike sphere orthogonal to H at x. The choice of H, τ provides a global chart $\Psi : \mathbb{D}^2 \times \mathbb{S}^n \to \widehat{\mathbb{H}}^{2,n}$ such that $\Psi(\mathbb{D}^2 \times \{v\}) = H$ and $\Psi(0, v) = x$. Let $\ell \subset \hat{\lambda}$ be a leaf.

By Lemma 2.7, the projection of ℓ to $\mathbb{D}^2$ intersects (transversely) every diameter of $\mathbb{D}^2$ at most once. As a consequence, if we represent $\hat{S}_\lambda$ as a graph of a 1-Lipschitz function $g : \mathbb{D}^2 \to \mathbb{S}^n$, the geodesic segment $[x, y] \subset \mathbb{D}^2$ lifts on a curve γ joining $x, y \in \hat{S}_j$ that stays on $\hat{S}_j$, otherwise, $[x, y]$ would cross the projection of a some leaf $\ell \subset \partial \hat{S}_j \subset \hat{\mu}$ at least twice.

Using the explicit expression of the metric given by Proposition 2.4, we conclude that $\ell(\gamma) \leqslant \ell(\alpha)$ with equality if and only if $[x, y] \subset \hat{S}_j$. In fact, let us parametrize $[x, y] \subset \mathbb{D}^2$ as $\alpha : I \to \mathbb{D}^2$ and write $\gamma(t) = (\alpha(t), g(\alpha(t)))$. We have

$$\begin{aligned} \ell(\gamma) &= \int_I \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle} dt \\ &= \int_I \sqrt{|\dot{\alpha}|^2_{\mathbb{H}^2} - \frac{4}{(1 - |\alpha|^2)^2} |\dot{g\alpha}|^2} dt \\ &\leqslant \int_I |\dot{\alpha}|_{\mathbb{H}^2} dt = \ell_{\mathbb{H}^2}(\alpha). \end{aligned}$$

The conclusion follows from $d_{\bar{U}_j}(x, y) \leqslant \ell(\gamma)$ and $\ell_{\mathbb{H}^2}(\alpha) = d_{\mathbb{H}^{2,n}}(x, y)$. $\square$

Thus, the inclusion $\hat{S}_j \subset \bar{U}_j$ is uniformly continuous, in the sense that

$$d_{\bar{U}_j}(f(x), f(y)) \leqslant d_{\mathbb{H}^{2,n}}(x, y).$$

Being uniformly continuous, it extends to a map $\hat{f}_j$ from the closure of $\hat{S}_j$ in $\hat{S}$, obtained from $\hat{S}_j$ by adding the lifts of the leaves of μ adjacent to S_j, to the completion $\bar{U}_j$. By construction and continuity, the map $\hat{f}$ is $\pi_1(S_j)$-equivariant.

Claim. The extension $\hat{f}_j$ of the inclusion $\hat{S}_j \subset \bar{U}_j$ satisfies the following properties:

- It is a homeomorphism.
- It maps each of the leaves $\ell \subset \partial \hat{S}_j$ to the geodesic boundary of $\bar{U}_j$ in a totally geodesic way.
- We have $\ell(\hat{f}_j(\gamma)) = \ell(\gamma)$ for every regular path $\gamma : I \to \hat{S}_j \cup \partial \hat{S}_j$ that intersects $\hat{\lambda}$ in countably many points.

Proof of claim. Let ℓ be a boundary leaf of $\partial \hat{S}_j$ and let $[x, y] \subset \ell$ be a finite subsegment. Notice that there exists a sequence of leaves $\ell_n \subset \hat{S}_j$ that converges to ℓ. Thus, we can approximate $[x, y]$ with a sequence of segments $[x_n, y_n] \subset \ell_n$ for which $d_{\bar{U}_j}(\hat{f}_j(x_n), \hat{f}_j(y_n)) = d_{\mathbb{H}^{2,n}}(x_n, y_n)$. By continuity, we conclude that $d_{\bar{U}_j}(\hat{f}_j(x), \hat{f}_j(y)) = d_{\mathbb{H}^{2,n}}(x, y)$. Thus, $\hat{f}_j$ maps each boundary leaf $\ell \subset \partial \hat{S}_j$ to a boundary leaf of $\partial \bar{U}_j$ in a totally geodesic way.

If $\gamma : I \to \hat{S}_j$ is a regular path that intersects $\hat{\lambda}$ in countably many points then $\ell(\gamma) = \ell(\gamma - \hat{\lambda})$. As $\hat{f}_j$ is totally geodesic on each component of $\hat{S}_j - \hat{\lambda}$, it follows that $\hat{f}_j \gamma$ is rectifiable and $\ell(\hat{f}_j \gamma) = \ell(\gamma)$. $\square$

Let $\gamma \subset \mu$ be a leaf adjacent to the components S_i, S_j (possibly the same component), denote by $\alpha_i \subset \partial \bar{S}_i, \alpha_j \subset \partial \bar{S}_j$ the boundary components corresponding to γ. There is a unique way to glue the completions $\bar{S}_i, \bar{S}_j$ along α_i, α_j so that the identifications with γ agree. Thus, after gluing all the completions $\bar{S}_j$ along their boundary components as prescribed by the leaves of μ, we get a hyperbolic surface

$$X_\lambda = \bigsqcup_{S_k \text{ component of } S_\lambda - \mu} \bar{S}_k \,/\alpha_i \sim \alpha_j$$

and a homeomorphism $f : S_\lambda \to X_\lambda$ which is isometric on each of the leaves of λ and plaques of $S_\lambda - \lambda$.

Lift f to a map $\hat{f} : \hat{S}_\lambda \to \mathbb{H}^2$.

Claim. The map $\hat{f}$ sends regular paths that intersect $\hat{\mu}$ in countably many points to rectifiable paths of the same length and we have

$$d_{\mathbb{H}^{2,n}}(x, y) \geqslant d_{\mathbb{H}^2}(\hat{f}(x), \hat{f}(y))$$

for every $x, y \in \hat{S}_\lambda$.

Proof of the claim. The proof is similar to the one of the first claim.

By continuity and density, it is enough to restrict our attention to $x, y \in \hat{S} - \hat{\lambda}$. As in the first claim, let H be a spacelike hyperplane containing x, y and let τ be a timelike sphere orthogonal to H at x. This choice corresponds to a global chart $\Psi : \mathbb{D}^2 \times \mathbb{S}^n \to \mathbb{H}^{2,n}$ such that $\Psi(\mathbb{D}^2 \times \{v\}) = H$ and $\Psi(0, v) = x$.

Let $\ell \subset \hat{\lambda}$ be a leaf. By Lemma 2.7, the projection of ℓ to $\mathbb{D}^2$ intersects (transversely) every diameter of $\mathbb{D}^2$ at most once. As a consequence, the geodesic segment $[x, y] \subset \mathbb{D}^2$ intersects the projections of the leaves of $\hat{\mu}$ in finitely many points $p_1, \cdots, p_k \in [x, y]$. For simplicity, set also $p_0 := x$ and $p_{k+1} := y$. Let α_j, α be the lifts as a graph of $[p_j, p_{j+1}], [x, y]$ to $\hat{S}$. The path $\hat{f}(\alpha) \subset \mathbb{H}^2$ is the composition of the paths $\hat{f}(\alpha_j)$. By the above discussion, as α_j is entirely contained in (the closure of) a component $\hat{S}_i$, each $\hat{f}(\alpha_j)$ is a rectifiable path of length $\ell(\hat{f}(\alpha_j)) = \ell(\alpha_j)$. Therefore, $\hat{f}(\alpha)$ is also a rectifiable path of length $\ell(\hat{f}(\alpha)) = \ell(\alpha)$ and joins $\hat{f}(x)$ to $\hat{f}(y)$. Thus,

$$d_{\mathbb{H}^2}(\hat{f}(x), \hat{f}(y)) \leqslant \ell(\hat{f}(\alpha)) = \ell(\alpha) = d_{\mathbb{H}^{2,n}}(x, y).$$

This concludes the proof of the claim. □ □

The only missing piece in the finite leaved setting is the equivalence of the intrinsic hyperbolic structure X_λ and the one given by the intrinsic shear cocycle σ^ρ_λ.

Proposition 6.6. *Let λ be a finite maximal lamination, and let S be the pleated surface of M realizing λ. Then the ρ-shear cocycle σ^ρ_λ coincides with the shear coordinates of the intrinsic hyperbolic metric $X = X_\lambda$ described in Proposition 6.5. In other words, we have $\sigma^\rho_\lambda = \sigma^X_\lambda$.*

Proof. By additivity of the shear cocycles $\sigma^\rho_\lambda, \sigma^X_\lambda \in \mathcal{H}(\lambda; \mathbb{R})$, it is enough to show that $\sigma^\rho_\lambda(P, Q) = \sigma^X_\lambda(P, Q)$ when P and Q are separated by at most one component $\tilde{\gamma}$ of λ_c, the set of leaves of λ that project onto simple closed geodesics.

If no component of λ_c separates P from Q, then there exists a finite collection of plaques $P = P_0, P_1, \ldots, P_n, P_{n+1} = Q$ such that P_i and P_{i+1} are adjacent for every i. Again by additivity, it is sufficient to check that $\sigma^X_\lambda(P_i, P_{i+1}) = \sigma^\rho_\lambda(P_i, P_{i+1})$, and this follows from what we observed in Remark 4.6.

Therefore it is enough to consider the case in which P and Q are separated by exactly one component of λ_c. By what we just proved, we can further reduce the discussion to the case in which both plaques P and Q have exactly one ideal vertex equal to one of the endpoints $\tilde{\gamma}^{\pm}$ of $\tilde{\gamma}$. Up to relabeling the plaques and change orientation of $\tilde{\gamma}$, we can assume that P lies on the left of $\tilde{\gamma}$ and has one vertex equal to $\tilde{\gamma}^+$. We denote by x_P, y_P the vertices of P different from $\tilde{\gamma}^{\pm}$, so that $[y_P, \tilde{\gamma}^+]$ is the boundary component of P that is closest to $\tilde{\gamma}$. If z_Q denote the vertex of Q that coincides with one of the endpoints of $\tilde{\gamma}$, then we label the other vertices of Q as x_Q, y_Q, so that $[y_Q, z_Q]$ is the boundary component of Q that is closest to $\tilde{\gamma}$.

As usual, we denote by $\mathcal{P}_{PQ}$ the set of plaques of λ that separate P from Q. By Lemmas 4.7 and 4.8, for any finite collection $\mathcal{P} \subset \mathcal{P}_{PQ}$ we have

$$\begin{aligned} \sigma^{\rho}_{\mathcal{P}}(P,Q) &= \log\left|\beta^{\rho}(\tilde{\gamma}^+, y_P, x_P, y_Q)\beta^{\rho}(\tilde{\gamma}^+, y_Q, y_P, x_Q)\right|, \\ \sigma^{X}_{\mathcal{P}}(P,Q) &= \log\left|\beta^{X}(\tilde{\gamma}^+, y_P, x_P, y_Q)\beta^{X}(\tilde{\gamma}^+, y_Q, y_P, x_Q)\right|, \end{aligned} \tag{14}$$

if $z_Q = \tilde{\gamma}^+$, and

$$\begin{aligned} \sigma^{\rho}_{\mathcal{P}}(P,Q) &= \log\left|\beta^{\rho}(\tilde{\gamma}^+, y_P, x_P, \tilde{\gamma}^-)\beta^{\rho}(\tilde{\gamma}^+, \tilde{\gamma}^-, y_P, y_Q)\beta^{\rho}(y_Q, \tilde{\gamma}^-, \tilde{\gamma}^+, x_Q)\right|, \\ \sigma^{X}_{\mathcal{P}}(P,Q) &= \log\left|\beta^{X}(\tilde{\gamma}^+, y_P, x_P, \tilde{\gamma}^-)\beta^{X}(\tilde{\gamma}^+, \tilde{\gamma}^-, y_P, y_Q)\beta^{X}(y_Q, \tilde{\gamma}^-, \tilde{\gamma}^+, x_Q)\right|, \end{aligned} \tag{15}$$

if $z_Q = \tilde{\gamma}^-$. In particular $\sigma^{\rho}_{\lambda}(P,Q) = \sigma^{\rho}_{\mathcal{P}}(P,Q)$ and $\sigma^{X}_{\lambda}(P,Q) = \sigma^{X}_{\mathcal{P}}(P,Q)$ are independent of the choice of $\mathcal{P} \subset \mathcal{P}_{PQ}$.

Select now an identification between the universal cover of Σ and $\mathbb{H}^2$ compatible with the intrinsic hyperbolic structure $X = X_\lambda \in \mathcal{T}$. Then the classical shear $\sigma^X_\lambda(P,Q)$ can be characterized as follows (see e. g. [Bon96]):

Fact. *Let $\hat{\ell} = \hat{\ell}(s)$ be a unit speed parametrization of the geodesic $[\tilde{\gamma}^+, \tilde{\gamma}^-]$ pointing towards $\tilde{\gamma}^+$, and denote by $\hat{v}_P, \hat{v}_Q \in \mathbb{H}^2 \cong_X \widetilde{\Sigma}$ the projections of the ideal vertices x_P, x_Q onto the spacelike geodesics $[\tilde{\gamma}^+, y_P], [z_Q, y_P]$, respectively. In addition we set $\hat{\ell}(s_P)$ (resp. $\hat{\ell}(s_Q)$) to be the intersection point between $\hat{\ell}$ and the horocycle of $\mathbb{H}^2$ based at $\tilde{\gamma}^+$ (resp. z_Q) that passes through $\hat{v}_P$ (resp. $\hat{v}_Q$). Then $s_Q - s_P = \sigma^X_\lambda(P,Q)$.*

On the other hand, we will prove that a similar description holds for the shear σ^{ρ}_{λ}:

Claim. *Let $\ell = \ell(t)$ be a unit speed parametrization of the space-like geodesic $[\xi(\tilde{\gamma}^+), \xi(\tilde{\gamma}^-)]$ pointing towards $\xi(\tilde{\gamma}^+)$, and denote by $v_P, v_Q \in \mathbb{H}^{2,n}$ the projections of the ideal vertices $\xi(x_P), \xi(x_Q)$ onto the space-like geodesics $[\xi(\tilde{\gamma}^+), \xi(y_P)], [\xi(z_Q), \xi(y_P)]$, respectively. In addition we set $\ell(t_P)$ (resp. $\ell(t_Q)$) to be the intersection point between ℓ and the horosphere of $\mathbb{H}^{2,n}$ based at $\xi(\tilde{\gamma}^+)$ (resp. $\xi(z_Q)$) that passes through v_P (resp. v_Q). Then $t_Q - t_P = \sigma^{\rho}_{\lambda}(P,Q)$.*

Assuming that such characterization holds true, we can finally prove the statement. Let $\hat{S}$ denote the lift of the pleated surface S realizing λ to $\mathbb{H}^{2,n}$ (compare with the proof of Proposition 6.5). Since any horosphere O based at $\xi(\tilde{\gamma}^+) \in \mathrm{Ein}^{1,n}$ (or $\xi(z_Q)$) intersects every plaque that has an ideal vertex equal to $\xi(\tilde{\gamma}^+)$ (or $\xi(z_Q)$) into a horocycle, it follows that the curve $\partial O \cap \hat{S}$ is a horocycle based at $\tilde{\gamma}^+$ (or z_Q) with respect to the intrinsic hyperbolic metric of $\hat{S}$. On the other hand, the points v_P and v_Q are uniquely determined by the intrinsic hyperbolic structures of the plaques P and Q, so the horocycles $\partial O \cap \hat{S}$ pass through the point of $\hat{S}$ corresponding to $\hat{v}_P$ (or $\hat{v}_Q$). This implies that $s_Q - s_P = t_Q - t_P$, and therefore $\sigma^X_\lambda(P,Q) = \sigma^{\rho}_{\lambda}(P,Q)$, which was what we were left to prove.

Proof of the claim. In order to simplify the notation, we will implicitly (and with abuse of notation) identify $\hat{S}$ with one of its lifts to $\widehat{\mathbb{H}}^{2,n}$, and similarly for the boundary map ξ associated with the representation ρ. The projections $v_P, v_Q \in \widehat{\mathbb{H}}^{2,n}$ satisfy

$$v_P = \sqrt{-\frac{\langle \tilde\xi(y_P), \tilde\xi(\tilde\gamma^+)\rangle}{2\langle \tilde\xi(x_P), \tilde\xi(\tilde\gamma^+)\rangle\langle \tilde\xi(x_P), \tilde\xi(y_P)\rangle}} \left(\frac{\langle \tilde\xi(x_P), \tilde\xi(\tilde\gamma^+)\rangle}{\langle \tilde\xi(y_P), \tilde\xi(\tilde\gamma^+)\rangle} \tilde\xi(y_P) + \frac{\langle \tilde\xi(x_P), \tilde\xi(y_P)\rangle}{\langle \tilde\xi(y_P), \tilde\xi(\tilde\gamma^+)\rangle} \tilde\xi(\tilde\gamma^+) \right),$$

$$v_Q = \sqrt{-\frac{\langle \tilde\xi(y_Q), \tilde\xi(z_Q)\rangle}{2\langle \tilde\xi(x_Q), \tilde\xi(z_Q)\rangle\langle \tilde\xi(x_Q), \tilde\xi(y_Q)\rangle}} \left(\frac{\langle \tilde\xi(x_Q), \tilde\xi(z_Q)\rangle}{\langle \tilde\xi(y_Q), \tilde\xi(z_Q)\rangle} \tilde\xi(y_Q) + \frac{\langle \tilde\xi(x_Q), \tilde\xi(y_Q)\rangle}{\langle \tilde\xi(y_Q), \tilde\xi(z_Q)\rangle} \tilde\xi(z_Q) \right),$$

where $\tilde\xi(x)$ denotes a representative of the projective class $\xi(x) \in \mathrm{Ein}^{1,n} \subset \mathbb{R}\mathrm{P}^{n+2}$ (compare with Remark 4.6). Consider now the parametrization of the leaf $[\xi(\tilde\gamma^+), \xi(\tilde\gamma^-)]$ given by

$$\ell(t) = \frac{1}{\sqrt{-2\langle \tilde\xi(\tilde\gamma^+), \tilde\xi(\tilde\gamma^-)\rangle}} (e^t \tilde\xi(\tilde\gamma^+) + e^{-t} \tilde\xi(\tilde\gamma^-))$$

The horosphere based at $\xi(\tilde\gamma^+)$ that passes through v_P intersects the spacelike geodesic $[\xi(\tilde\gamma^+), \xi(\tilde\gamma^-)]$ at $\ell(t_P)$, where

$$e^{t_P} = \sqrt{\frac{\langle \tilde\xi(\tilde\gamma^+), \tilde\xi(\tilde\gamma^-)\rangle\langle \tilde\xi(x_P), \tilde\xi(y_P)\rangle}{\langle \tilde\xi(x_P), \tilde\xi(\tilde\gamma^+)\rangle\langle \tilde\xi(y_P), \tilde\xi(\tilde\gamma^+)\rangle}}.$$

Similarly, the horosphere based at $\xi(z_Q)$ that passes through v_Q intersects the spacelike geodesic $[\xi(\tilde\gamma^+), \xi(\tilde\gamma^-)]$ at $\ell(t_Q)$, where

$$e^{\pm t_Q} = \sqrt{\frac{\langle \tilde\xi(\tilde\gamma^+), \tilde\xi(\tilde\gamma^-)\rangle\langle \tilde\xi(x_Q), \tilde\xi(y_Q)\rangle}{\langle \tilde\xi(x_Q), \tilde\xi(\tilde\gamma^\pm)\rangle\langle \tilde\xi(y_Q), \tilde\xi(\tilde\gamma^\pm)\rangle}}$$

if $z_Q = \tilde\gamma^\pm$, respectively. In particular we have

$$\begin{aligned} t_Q - t_P &= \frac{1}{2} \log \frac{\langle \tilde\xi(x_Q), \tilde\xi(y_Q)\rangle\langle \tilde\xi(x_P), \tilde\xi(\tilde\gamma^+)\rangle\langle \tilde\xi(y_P), \tilde\xi(\tilde\gamma^+)\rangle}{\langle \tilde\xi(x_Q), \tilde\xi(\tilde\gamma^+)\rangle\langle \tilde\xi(y_Q), \tilde\xi(\tilde\gamma^+)\rangle\langle \tilde\xi(x_P), \tilde\xi(y_P)\rangle} \\ &= \log \left|\beta^\rho(\tilde\gamma^+, y_P, x_P, y_Q)\beta^\rho(\tilde\gamma^+, y_Q, y_P, x_Q)\right| \\ &= \sigma^\rho_{\mathcal{P}}(P, Q) \end{aligned}$$

if $z_Q = \tilde\gamma^+$, and

$$\begin{aligned} t_Q - t_P &= \frac{1}{2} \log \frac{\langle \tilde\xi(x_Q), \tilde\xi(\tilde\gamma^-)\rangle\langle \tilde\xi(y_Q), \tilde\xi(\tilde\gamma^-)\rangle}{\langle \tilde\xi(\tilde\gamma^+), \tilde\xi(\tilde\gamma^-)\rangle\langle \tilde\xi(x_Q), \tilde\xi(y_Q)\rangle} \frac{\langle \tilde\xi(x_P), \tilde\xi(\tilde\gamma^+)\rangle\langle \tilde\xi(y_P), \tilde\xi(\tilde\gamma^+)\rangle}{\langle \tilde\xi(\tilde\gamma^+), \tilde\xi(\tilde\gamma^-)\rangle\langle \tilde\xi(x_P), \tilde\xi(y_P)\rangle} \\ &= \log \left|\beta^\rho(\tilde\gamma^+, y_P, x_P, \tilde\gamma^-)\beta^\rho(\tilde\gamma^+, \tilde\gamma^-, y_P, y_Q)\beta^\rho(y_Q, \tilde\gamma^-, \tilde\gamma^+, x_Q)\right| \\ &= \sigma^\rho_{\mathcal{P}}(P, Q) \end{aligned}$$

if $z_Q = \tilde\gamma^-$. □

This concludes the proof of Proposition 6.6. □

6.2. **General maximal laminations.** We now extend the result from finite leaved laminations to the general case using the continuity of the construction.

Proposition 6.7. *Let $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ be a maximal representation. Let λ be a maximal lamination with associated pleated set $S_\lambda = \hat S_\lambda/\rho(\Gamma)$. Let $X_\lambda \in \mathcal{T}$ be the hyperbolic surface whose shear coordinates with respect to the lamination λ agree with the intrinsic shear cocycle $\sigma^\rho_\lambda \in \mathcal{H}(\lambda; \mathbb{R})$. Then there exists a (unique)* *developing map $f : S_\lambda \to X_\lambda$ which is 1-Lipschitz with respect to the intrinsic pseudo-metric on S_λ and the hyperbolic metric on X_λ.*

Proof. Let λ_m be a sequence of finite leaved maximal laminations that converges to λ in the Hausdorff topology. By Propositions 3.6 and 6.5 for every m we can find a pleated set $S_m = \hat{S}_m/\rho(\Gamma)$, a hyperbolic surface $X_m \in \mathcal{T}$ with shear cocycle $\sigma_m = \sigma^\rho_{\lambda_m} \in \mathcal{H}(\lambda_m; \mathbb{R})$ and a developing map $f_m : S_m \to X_m$ which is 1-Lipschitz with respect to the pseudo-metric and the hyperbolic metric.

Let $S = \hat{S}/\rho(\Gamma)$ be the pleated set associated to λ.

By Proposition 5.10 the cocycles $\sigma_m = \sigma^\rho_{\lambda_m}$ converge to the cocycle $\sigma = \sigma^\rho_\lambda$ naturally associated with the lamination λ.

The cocycle σ represents a hyperbolic surface $X \in \mathcal{T}$, that is $\sigma^\rho_\lambda = \sigma^X_\lambda$ and the convergence $\sigma^\rho_{\lambda_m} \to \sigma^\rho_\lambda$ implies $X_m \to X$ in $\mathcal{T}$.

In order to obtain convergence of developing maps, we will work in the Poincaré model $\Psi : \mathbb{D}^2 \times \mathbb{S}^n \to \widehat{\mathbb{H}}^{2,n}$ associated to the choice of an orthogonal splitting $\mathbb{R}^{2,n+1} = E \oplus F$ where E is a $(2,0)$-plane. We write $\hat{S}_m, \hat{S}$ as graphs of 1-Lipschitz functions $g_m, g : \mathbb{D}^2 \to \mathbb{S}^n$, that is, they are the images of the functions u_m, u defined by

$$\begin{array}{rccc} u_m, u : & \mathbb{D}^2 & \longrightarrow & \widehat{S}_m, \widehat{S} \subset \widehat{\mathbb{H}}^{2,n} \\ & x & \longmapsto & \Psi(x, g_m(x)), \Psi(x, g(x)). \end{array}$$

By Proposition 3.7, we have $g_m \to g$ and $u_m \to u$ uniformly on compact subsets of $\mathbb{D}^2$. Furthermore, by property (3) of Lemma 2.5, the maps u_m, u are 1-Lipschitz with respect to the hyperbolic distance of $\mathbb{D}^2$ and the pseudo-distance $d_{\mathbb{H}^{2,n}}$, that is, they satisfy

$$d_{\mathbb{H}^{2,n}}(u(x), u(y)), d_{\mathbb{H}^{2,n}}(u_m(x), u_m(y)) \leqslant d_{\mathbb{H}^2}(x, y) \tag{16}$$

for every $x, y \in \mathbb{D}^2$.

Let $\hat{f}_m : \widehat{S}_m \to \widehat{X}_m$ denote the lifts of the developing maps f_m to the universal covers. Fix $x_0 \in \mathbb{D}^2$ so that $u(x_0)$ lies in the interior of a plaque of $\widehat{S}$. This implies in particular that $u_m(x_0) \notin \hat{\lambda}_m$ for m sufficiently large. Choose now identifications $\widehat{X}_m \simeq \mathbb{H}^2$ so that the sequence $(\hat{f}_m u_m(x_0))_m$ converges to some $y_0 \in \mathbb{H}^2$.

By Proposition 3.6 and relation (16), the maps $h_m := \hat{f}_m u_m : \mathbb{D}^2 \to \mathbb{H}^2$ are 1-Lipschitz with respect to the hyperbolic metrics of both domain and codomain, and $h_m(x_0) \to y_0$ as m goes to ∞. By Ascoli-Arzelà, up to subsequences, we have that h_m converges uniformly on compact sets to a 1-Lipschitz map $h : \mathbb{D}^2 \to \mathbb{H}^2$ with $h(x_0) = y_0$. Finally, we set $\hat{f} := h\pi : \widehat{S} \to \mathbb{H}^2$, where $\pi : \mathbb{H}^{2,n} \to \mathbb{D}^2$ is the projection determined by the map Ψ.

Notice that each $\hat{f}_m$ is (ρ, ρ_{X_m})-equivariant where ρ_{X_m} is the holonomy associated to the chosen identification $\hat{X}_m \simeq \mathbb{H}^2$. The sequence of holonomies ρ_{X_m} converges to ρ_X, a representative of the holonomy of the hyperbolic surface X: As $X_n \to X$ in Teichmüller space $\mathcal{T}$, we only have to check that the sequence is precompact. This follows from the fact that

$$\rho_{X_m}(\gamma) h_m(x_0) = h_m \pi(\rho(\gamma) u_m(x_0)) \to h\pi(\rho(\gamma) u(x_0)).$$

As a consequence, we deduce that $\hat{f}$ is (ρ, ρ_X)-equivariant: Take $x \in \widehat{S}$ and select $x_m \in \widehat{S}_m$ that converge to x. Then

$$\begin{aligned}\hat{f}(\rho(\gamma)x) &= h\pi(\rho(\gamma) \lim_{m\to\infty} x_m) \\ &= \lim_{m\to\infty} h_m \pi(\rho(\gamma)x_m) \\ &= \lim_{m\to\infty} \rho_{X_m}(\gamma) h_m \pi(x_m) \\ &= \rho_X(\gamma)\hat{f}(x),\end{aligned}$$

where in the second equality we used the uniform convergence of the maps h_m.

We now show that $\hat{f}$ is the lift of a 1-Lipschitz developing map $f : S \to X$. In order to do so, we have to prove that:

- $\hat{f}$ is injective.
- $\hat{f}$ is totally geodesic on each leaf of $\hat{\lambda}$, each plaque of $\widehat{S} - \hat{\lambda}$, and on the complement of the bending locus.

This will be enough to conclude the proof.

Let now P be a plaque of $\widehat{S}$, and consider a sequence of plaques $P_m \subset \widehat{S}_m$ that converge to P. By hypothesis $\hat{f}_m = h_m \pi$ is an isometric embedding on P_m, and therefore the same holds for the restriction of $\hat{f} = h\pi$ on P. In the same way we see that $\hat{f}(\ell)$ is a parametrized geodesic for every leaf ℓ of $\hat{\lambda}$.

Since distinct plaques of $\widehat{S}_m$ are sent by $\hat{f}_m$ into ideal triangles of $\mathbb{H}^2$ with disjoint interiors, the same property is verified by $\hat{f}$ and the plaques of $\widehat{S}$. In particular the map h restricts to a homeomorphism between $\mathbb{D}^2 \backslash \pi(\hat{\lambda})$ and $\mathbb{H}^2 \backslash \hat{f}(\hat{\lambda})$. In addition, if P, Q, R are plaques of $\widehat{S}$ and R separates P from Q, then $\hat{f}(R)$ separates $\hat{f}(P)$ from $\hat{f}(Q)$. From here it is simple to see that $\hat{f}$ is in fact globally injective.

□

6.3. **Length estimates.** The length spectrum of a pleated surface is dominated by the length spectrum of the maximal representation:

Proposition 6.8. *Let $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ be a maximal representation. Consider a pleated surface S_λ associated to ρ and a maximal lamination λ, together with a 1-Lipschitz developing map $f : S_\lambda \to X_\lambda$. Then, for every $\gamma \in \Gamma$ we have*

$$L_{X_\lambda}(\gamma) \leqslant L_\rho(\gamma),$$

where the strict inequality holds if and only if γ intersects essentially the bending locus of S_λ.

Proof. We proceed as in Proposition 3.38 of [CTT19].

If γ does not intersect essentially the bending locus then the invariant geodesic ℓ of $\rho(\gamma)$ is contained in $\hat{S}_\lambda$. By Lemma 6.3, the 1-Lipschitz developing map $\hat{f}$: $\hat{S}_\lambda \to \mathbb{H}^2$ is an isometry on the complement of the bending locus. Therefore, we have $L_\rho(\gamma) = L_{X_\lambda}(\gamma)$.

Assume that γ intersects essentially the bending locus of S_λ.

Let ℓ be the axis of $\rho(\gamma)$. We first observe that ℓ is not contained in $\hat{S}_\lambda$: If this was the case, then, as γ intersects the bending locus essentially, the geodesic ℓ must also intersects some bending line $\ell' \subset \hat{\lambda}$. However, this would contradict the fact that ℓ' is a bending line by the definition of bending locus. Therefore ℓ is not contained in $\hat{S}_\lambda$.

As ℓ is not contained in $\hat{S}_\lambda$, it can be connected to $\hat{S}_\lambda$ by a timelike geodesic and we can take the timelike geodesic $[x,p]$ of maximal length $\ell[x,p] = T > 0$ joining a point $x \in \ell$ to a point $p \in \hat{S}_\lambda$. We parametrize ℓ as $\ell(t) = \cosh(t)x + \sinh(t)w$ with w a spacelike vector orthogonal to x and we write $p := \cos(T)x + \sin(T)v$ with v timelike and orthogonal to x, w (as $[x,p]$ maximizes the timelike distance between ℓ and $\hat{S}_\lambda$). The isometry $\rho(\gamma)$ acts on ℓ by translating points by $L = L_\rho(\gamma)$ and acts on $\mathrm{Span}\{x,v\}^\perp$ by an isometry A. Thus, we have $\rho(\gamma)p = \cos(T)(\cosh(L)x + \sinh(L)w) + \sin(T)Av$.

$$\begin{aligned}\cosh\left(d_{\mathbb{H}^{2,n}}(p, \rho(\gamma)^n p)\right) &= -\langle p, \rho(\gamma)^n p\rangle \\ &= \cos(T)^2 \cosh(nL) - \sin(T)^2 \langle v, A^n v\rangle.\end{aligned}$$

Since the developing map $\hat{f} : \hat{S}_\lambda \to \mathbb{H}^2$ is 1-Lipschitz and equivariant, we get

$$d_{\mathbb{H}^2}(f(p), \rho_{X_\lambda}(\gamma)^n f(p)) \leqslant d_{\mathbb{H}^{2,n}}(p, \rho(\gamma)^n p).$$

Furthermore, for a hyperbolic isometry $\rho_{X_\lambda}(\gamma)^n$, we have that the minimal displacement coincides with the translation length so that

$$nL_{X_\lambda}(\gamma) = L_{X_\lambda}(\gamma^n) \leqslant d_{\mathbb{H}^2}(f(p), \rho_{X_\lambda}(\gamma)^n f(p)).$$

Putting together the previous inequalities we get

$$\cosh\left(nL_{X_\lambda}(\gamma)\right)) \leqslant \cos(T)^2 \cosh(nL) - \sin(T)^2 \langle v, A^n v\rangle.$$

Since the spectral radius of A is strictly smaller than e^L, we can choose n sufficiently large so that $|\langle v, A^n v\rangle| < \cosh(nL)$ (see for example [CTT19, Corollary 2.6] and [BPS19]). For this value of n we get

$$\begin{aligned}\cosh\left(nL_{X_\lambda}(\gamma)\right)) &\leqslant \cos(T)^2 \cosh(nL) - \sin(T)^2 \langle v, A^n v\rangle \\ &< \cos(T)^2 \cosh(nL) + \sin(T)^2 \cosh(nL) = \cosh(nL).\end{aligned}$$

Which implies $L_{X_\lambda}(\gamma) < L$. □

7. Teichmüller geometry and length spectra

In this section we relate the geometry of maximal representations to the geometry of Teichmüller space and use Teichmüller geometry to study the length spectrum of maximal representations. Our main goal is the proof of Theorem D from the introduction.

7.1. **Outline.**

Let us briefly describe the picture.

By Theorem C and Proposition 6.8, we know that the length spectrum $L_\rho(\bullet)$ of a maximal representation $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ dominates the length spectrum $L_{S_\lambda}(\bullet)$ of all the pleated surfaces S_λ. Furthermore, we have characterized those curves $\gamma \in \Gamma$ for which the strict inequality $L_{S_\lambda}(\gamma) < L_\rho(\gamma)$ holds. They are precisely the ones that do not intersect essentially the bending locus of S_λ.

Thus, we can consider the dominated set of ρ which is the following space

Definition 7.1 (Dominated Set)**.** The *dominated set* of the maximal representation ρ is the subset of Teichmüller space $\mathcal{T}$ defined by

$$\mathcal{P}_\rho := \{Z \in \mathcal{T} \mid L_Z(\bullet) \leqslant L_\rho(\bullet)\}$$

where L_Z, L_ρ are the length spectra of Z, ρ.

Similarly, the *simply dominated set* is

$$\mathcal{P}_\rho^{\mathrm{simple}} := \left\{ Z \in \mathcal{T} \;\middle|\; L_Z^{\mathrm{simple}}(\bullet) \leqslant L_\rho^{\mathrm{simple}}(\bullet) \right\}$$

where $L_Z^{\mathrm{simple}} \leqslant L_\rho^{\mathrm{simple}}$ are the simple length spectra of Z, ρ.

Clearly $\mathcal{P}_\rho \subset \mathcal{P}_\rho^{\mathrm{simple}}$.

Let us stress the fact that the set $\mathcal{P}_\rho$ is non-empty as it contains the hyperbolic structures X_λ of all pleated surfaces S_λ associated to maximal laminations λ, but it always has more structure: By work of Bestvina, Bromberg, Fujiwara, and Souto [BBFS13], and Théret [Thé14] on convexity of length functions in shear coordinates (see also [MV] for a different approach), the dominated set $\mathcal{P}_\rho$ is convex with respect to shear paths. By results of Wolpert [Wol87], [Wol06] on covexity of length functions along Weil-Petersson geodesics, it is also convex with respect to the Weil-Petersson metric.

We will analyze more carefully the structure of the dominated set. In order to do so, let us introduce the following useful auxiliary function which is an analogue of the Thurston's distortion function [Thu98] for hyperbolic surfaces:

Definition 7.2 (Maximal Length Distotion)**.** The maximal length distortion $\kappa : \mathcal{T} \to (0, \infty)$ is the function defined by

$$\kappa(Z) := \sup_{c \in \mathcal{C} - \{0\}} \frac{L_\rho(c)}{L_Z(c)}.$$

Similarly, we also define

$$\kappa^{\mathrm{simple}}(Z) := \sup_{\mu \in \mathcal{ML} - \{0\}} \frac{L_\rho(\mu)}{L_Z(\mu)}.$$

As both L_ρ, L_Z are continuous homogeneous positive functions on the space of geodesic currents $\mathcal{C}$ their ratio κ descends to a continuous positive function on the projectivization $\mathbb{P}\mathcal{C}$. Since the the projectivization $\mathbb{P}\mathcal{C}$ is compact, the supremum $\kappa(Z)$ is a maximum $\kappa(Z) = L_\rho(\bar{c})/L_Z(\bar{c})$, which is achieved at some current $\bar{c} \in \mathcal{C}$.

In the first part of the section, we use the maximal length distortion to characterize interior points $Z \in \mathrm{int}(\mathcal{P}_\rho)$ as those points for which $\kappa(Z) < 1$ (see Lemma 7.4). In other words, those are exactly the points that are strictly dominated by ρ. Thus, Theorem 7.6 is equivalent to $\mathrm{int}(\mathcal{P}_\rho) \neq \varnothing$ in this setting.

Using strict convexity of length functions along Weil-Petersson one shows that $\mathrm{int}(\mathcal{P}_\rho) \neq \varnothing$ provided that $\mathcal{P}_\rho$ contains at least two distinct points. If ρ is not Fuchsian, such points can be produced by considering pleated surfaces associated to maximal extensions of two intersecting simple closed curves α, β. This is the content of Proposition 7.5.

For convenience of the reader, we recall the definition of Fuchsian representation

Definition 7.3 (Fuchsian)**.** A maximal representation $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ is Fuchsian if it preserves a spacelike plane $H \subset \mathbb{H}^{2,n}$.

In the second part of the section, we consider points on the boundary $Z \in \partial\mathcal{P}_\rho$ and exterior points $Z \in \mathcal{T} - \mathcal{P}_\rho$. An immediate observation is that the pleated surfaces S_λ all lie on $\partial\mathcal{P}_\rho$ as thy have $\kappa(S_\lambda) = 1$ since we have $L_{S_\lambda}(\mu) = L_\rho(\mu)$ for every measured lamination $\mu \in \mathcal{ML}$ whose support is contained in λ. We show that for every Z outside $\mathrm{int}(\mathcal{P}_\rho)$, the maximum $\kappa(Z) \geqslant 1$ is realized by some measured lamination (see Proposition 7.7). The proof of this fact follows arguments of Thurston [Thu98] on the existence of maximally stretched laminations between two hyperbolic surfaces.

As a consequence we deduce that $\mathcal{P}_\rho$ coincides with the simply dominated set $\mathcal{P}_\rho^{\mathrm{simple}}$ (see Corollary 7.8). In fact, on the one hand, we have $\mathcal{P}_\rho \subset \mathcal{P}_\rho^{\mathrm{simple}}$ directly from the definition. On the other hand, from the above discussion we get $\partial\mathcal{P}_\rho \subset \partial\mathcal{P}_\rho^{\mathrm{simple}}$. As both subsets are topological disks, we conclude that equality holds.

7.2. **Structure of the dominated set.** We start our analysis of the dominated set by characterizing interior points.

Lemma 7.4. *A point $Z \in \mathcal{P}_\rho$ lies in the interior* $\mathrm{int}(\mathcal{P}_\rho)$ *if and only if we have* $\kappa(Z) < 1$.

Proof. Suppose that $\kappa(Z) < 1$. There exists a small neighbourhood U of $Z \in \mathcal{T}$ such that for every $X \in U$ we have $L_X(\bullet)/K < L_Z(\bullet) < KL_X(\bullet)$ with $K = 1/\kappa$. Thus, for every surface in $X \in U$ we have $L_X/L_\rho \leqslant KL_Z/L_\rho \leqslant K\kappa = 1$, that is, $X \in \mathcal{P}_\rho$.

Viceversa, if $Z \in \mathrm{int}(\mathcal{P}_\rho)$, then Z is the midpoint of a WP geodesic $[Z, Z']$ entirely contained in $\mathrm{int}(\mathcal{P}_\rho)$. Let $c \in \mathcal{C}$ be a geodesic current such that $\kappa = \frac{L_Z(c)}{L_\rho(c)}$. By strict convexity of length functions along Weil-Petersson geodesics (see Wolpert [Wol06]), we have $L_Z(c) < (L_{Z'}(c) + L_{Z''}(c))/2 \leqslant L_\rho(c)$. Therefore $\kappa < 1$. □

We remark that exactly the same argument also shows that a point $Z \in \mathcal{P}_\rho^{\mathrm{simple}}$ lies in the interior $\mathrm{int}(\mathcal{P}_\rho^{\mathrm{simple}})$ if and only if we have $\kappa^{\mathrm{simple}}(Z) < 1$.

We now show that $\mathrm{int}(\mathcal{P}_\rho)$ is never empty when ρ is not Fuchsian.

Proposition 7.5. *If ρ is not fuchsian then* $\mathrm{int}(\mathcal{P}_\rho) \neq \varnothing$.

Proof. We prove the statement in two steps: First we show that if $\mathcal{P}_\rho$ contains two distinct points then $\mathrm{int}(\mathcal{P}_\rho) \neq \varnothing$. Then we show that if ρ is not fuchsian then $\mathcal{P}_\rho$ contains at least two points.

The first step only uses the Weil-Petersson geometry of Teichmüller space: Let $X, Y \in \mathcal{P}_\rho$ be distinct points. Let Z be their Weil-Petersson midpoint. We show that Z is an interior point: By Lemma 7.4 this is equivalent to $\kappa(Z) = \sup_{\gamma\in\Gamma}\{L_Z(\gamma)/L_\rho(\gamma)\} < 1$. Let $c \in \mathcal{C}$ be a geodesic current that achieves $\kappa = L_Z(c)/L_\rho(c)$. By results of Wolpert, the length of a geodesic current is strictly convex along a Weil-Petersson geodesic. Hence $L_Z(c) < (L_X(c) + L_Y(c))/2 \leqslant L_\rho(c)$. Therefore $\kappa(Z) < 1$.

The second step, instead, relies on the pseudo-Riemannian geometry of $\mathbb{H}^{2,n}$. Let α and β be intersecting essential simple closed curves. Extend α, β to two finite leaved maximal laminations μ, ν of Σ by adding ideal triangulations of their complementary regions $\Sigma - \alpha, \Sigma - \beta$. Let $S_\lambda, S_\mu \subset M$ be the pleated surfaces realizing λ, μ for ρ. Denote by X_λ, X_μ their intrinsic hyperbolic structures. Note that $L_{X_\lambda}(\alpha) = L_\rho(\alpha)$ and $L_{X_\mu}(\beta) = L_\rho(\beta)$.

Since ρ is not fuchsian, the bending loci of S_λ and S_μ are both non-empty and, by Proposition 3.9, they are sublaminations of λ and μ. By construction, any non-trivial sublamination of λ, μ contains α and β as every leaf of $\lambda - \alpha, \mu - \beta$ spirals around α, β. Therefore, the bending locus of S_λ, S_μ contains α, β. As α, β are intersecting, we conclude, by Proposition 6.8, that $L_{X_\lambda}(\beta) < L_\rho(\beta)$ and $L_{X_\mu}(\alpha) < L_\rho(\alpha)$. Hence X_λ, X_μ are different hyperbolic surfaces. □

From Lemma 7.4 and Proposition 7.5 we deduce the following result of Collier, Tholozan, and Toulisse [CTT19]

Theorem 7.6. *Let ρ be a maximal representation of a surface group into $\mathrm{SO}_0(2, n+1)$. Then either ρ is fuchsian or there exists $\kappa > 1$ and a fuchsian representation σ such that $L_\rho \geqslant \kappa L_\sigma$*

7.3. **Simple length spectrum.** We now analyze $\kappa(Z)$ for points outside $\mathcal{P}_\rho$.

Proposition 7.7. *For every $Z \in \mathcal{T} - \mathrm{int}(P_\rho)$, the maximum $\kappa(Z)$ is achieved by some measured lamination $\lambda \in \mathcal{ML}$.*

Proof. Let us first consider $Z \in \mathcal{T} - \mathcal{P}_\rho$. Following an argument of Thurston [Thu98], we show that

Claim. $\kappa(Z) = \kappa^{\mathrm{simple}}(Z) := \sup_{\gamma \text{ simple}} L_Z(\gamma)/L_\rho(\gamma)$.

Once we know that $\kappa(Z)$ can be computed by restricting to simple closed curves, it immediately follows that the maximum is achieved at a measured lamination $\lambda \in \mathcal{ML}$.

Proof of the claim. In order to prove the claim, we show that if γ is not simple and we have $L_Z(\gamma)/L_\rho(\gamma) > 1$, then there is a shorter curve α such that $L_Z(\alpha)/L_\rho(\alpha) \geqslant L_Z(\gamma)/L_\rho(\gamma)$: As γ is not simple, it describes an immersed figure 8 in Z. Let $P \to Z$ be the covering corresponding to the immersed figure 8. The surface P is a pair of pants with boundary curves $\alpha_1, \alpha_2, \alpha_3$. Let S be a pleated surface realizing the curves α_j for the maximal representation given by the restriction of ρ to the subgroup corresponding to $\pi_1(P)$. We have $L_\rho(\alpha_j) = L_S(\alpha_j)$ for $j \leqslant 3$ by construction and $L_\rho(\gamma) \geqslant L_S(\gamma)$ by the Lipschitz properties of pleated surfaces. Furthermore, as S is a hyperbolic surface pair of pants, we have $L_S(\gamma) \geqslant L_S(\alpha_j)$ for every $j \geqslant 3$.

As a consequence, we get

$$L_Z(\gamma)/L_\rho(\gamma) \leqslant L_P(\gamma)/L_S(\gamma).$$

By Lemma 3.4 of [Thu98], we have that

$$L_P(\gamma)/L_S(\gamma) \leqslant \max_{j \leqslant 3}\{L_P(\alpha_j)/L_S(\alpha_j)\})$$

which yields the conclusion. ☐

Lastly, we take care of boundary points $Z \in \partial\mathcal{P}_\rho$: Let Z_n be a sequence of points outside P_ρ converging to Z. By the previous steps, we can associate to each of them a measured lamination $\lambda_n \in \mathcal{ML}$ such that $L_{Z_n}(\lambda_n)/L_\rho(\lambda_n) = \kappa(Z_n) > 1$. Up to subsequence and rescaling, we can assume that the sequence of measured laminations λ_n converges to some $\lambda \in \mathcal{ML}$. By continuity of length functions, we have $L_{Z_n}(\lambda_n)/L_\rho(\lambda_n) \to L_Z(\lambda)/L_\rho(\lambda) \geqslant 1$. As $Z \in \mathcal{P}_\rho$, we also have the opposite inequality so we conclude that equality holds and $L_Z(\lambda)/L_\rho(\lambda) = 1 = \kappa(Z)$. ☐

From Proposition 7.7, we deduce the following

Corollary 7.8. *We have $\mathcal{P}_\rho = \mathcal{P}_\rho^{\mathrm{simple}}$.*

Proof. Observe that, directly from the definitions, we always have $\mathcal{P}_\rho \subset \mathcal{P}_\rho^{\mathrm{simple}}$.

Also notice that both sets are topological convex disks with non-empty interior.

If we knew that $\partial\mathcal{P}_\rho \subset \partial\mathcal{P}_\rho^{\mathrm{simple}}$, then the claim would follow from a topological argument based on the following:

Claim. Let $D, D' \subset \mathbb{R}^n$ be topological n-disks such that $D \subset D'$ and $\partial D \subset \partial D'$. Then $D = D'$.

Proof of the claim. Consider the map $\iota_* : H_n(D, \partial D) \to H_n(D', \partial D')$ induced by the proper inclusion $\iota : (D, \partial D) \to (D', \partial D')$. We now show that ι is degree one, that is, ι_* is an isomorphism. By well-known consequences, we deduce that ι is surjective which implies the claim.

The computation of the degree can be done as follows: Let $\star \in \mathrm{int}(D) \subset \mathrm{int}(D')$ be any interior point. As $D - \star, D' - \star$ deformation retract to $\partial D, \partial D'$, we have that the degree n relative homology groups are isomorphic to the local homology groups $H_n(D, \partial D) = H_n(D, D - \star)$ and $H_n(D', \partial D') = H_n(D', D' - \star)$. By the excision theorem, if $U \subset \mathrm{int}(D)$ is a small ball around $\star$, then $H(D, D - \star) = H_n(U, U - \star)$ and $H(D', D' - \star) = H_n(U, U - \star)$. As ι restricts to the identity $U \to U$, we conclude that ι_* is an isomorphism.

□

Hence, it is sufficient to show that $\partial \mathcal{P}_\rho \subset \partial \mathcal{P}_\rho^{\mathrm{simple}}$. Consider $Z \in \partial \mathcal{P}_\rho$, by Lemma 7.4, we have $\kappa(Z) = 1$. Furthermore, by Proposition 7.7, the maximum is realized by a measured lamination $\lambda \in \mathcal{ML}$. Therefore $\kappa^{\mathrm{simple}}(Z) = 1$ and, hence, $Z \in \partial \mathcal{P}_\rho^{\mathrm{simple}}$, as interior points of $\mathcal{P}_\rho^{\mathrm{simple}}$ are the ones for which $\kappa^{\mathrm{simple}}(Z) < 1$. □

8. Fibered photon structures

As shown by Guichard and Wienhard [GW12], maximal representations $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ parametrize deformations of photon structures, a class of geometric structures in the sense of Thurston (see Chapter 3 of [Thu79]), on certain closed manifolds E.

Definition 8.1 (Photon Structure)**.** A *photon* of $\mathbb{R}^{2,n+1}$ is an isotropic 2-plane. We denote by $\mathrm{Pho}^{2,n}$ the space of photons in $\mathbb{R}^{2,n+1}$. The group $\mathrm{SO}_0(2, n+1)$ acts transitively on the homogeneous space $\mathrm{Pho}^{2,n}$ with non-compact stabilizer. We call a $(\mathrm{SO}_0(2, n+1), \mathrm{Pho}^{2,n})$-structure on a manifold M a *photon structure*.

The construction of Guichard and Wienhard is the following: The maximal representation ρ has a natural domain of discontinuity $\Omega_\rho \subset \mathrm{Pho}^{2,n}$ obtained by removing from the space of photons, the closed subset

$$K_\rho := \{F \in \mathrm{Pho}^{2,n} \,|\, \ell \subset F \text{ for some isotropic line } \ell \in \Lambda_\rho\}.$$

The group $\rho(\Gamma)$ acts properly discontinuously, freely, and cocompactly on Ω_ρ so that the quotient $E_\rho := \Omega_\rho/\rho(\Gamma)$ is a closed manifold endowed with a photon structure. By the Ehresmann-Thurston principle [Thu79], the topology of E_ρ does not change as we vary ρ continuously.

Collier, Tholozan, and Toulisse [CTT19] have shown that the manifold E_ρ has a natural description as $\mathrm{O}(n)/\mathrm{O}(n-2)$-bundle $E_\rho \to \Sigma$ in a way compatible with the geometric structure, that, is, in such a way that the fibers are also geometric.

Definition 8.2 (Fibered Photon Structure)**.** Let $E \to S$ be a fiber bundle over the surface S such that the total space E has a photon structure. Let $\hat{S}$ be the universal covering of S and let $\pi : \hat{E} \to \hat{S}$ be the pull-back bundle. Let $\delta : \hat{E} \to \mathrm{Pho}^{2,n}$ be the developing map. We say that the photon structure is fibered if $\delta(\pi^{-1}(x)) = \mathrm{Pho}(f(x)^\perp)$ for some $f(x) \in \mathbb{H}^{2,n}$. In this case, there is an associated map $\hat{\Sigma} \to \mathbb{H}^{2,n}$ defined by $x \to f(x)$.

In this section we consider the point of view of fibered photon structures $E \to \Sigma$ associated to maximal representations. We use pleated surfaces to give a geometric decomposition of $E \to S$, namely triangles and lines of photons which we now introduce.

Definition 8.3 (Triangles and Lines of Photons)**.** For every ideal spacelike triangle $\Delta \subset \mathbb{H}^{2,n}$ and spacelike geodesic $\ell \subset \mathbb{H}^{2,n}$ we define a *triangle of photons* $E(\Delta) \subset \mathrm{Pho}^{2,n}$ and a *line of photons* $E(\ell) \subset \mathrm{Pho}^{2,n}$ consisting of those photons V that are orthogonal to some point $x \in \Delta$ and $x \in \ell$ respectively. Triangles and lines of photons $E(\Delta), E(\ell)$ are naturally fiber bundles over Δ, ℓ where the fiber over the point $x \in \Delta$ is the space $\mathrm{Pho}(x^\perp)$.

Triangles of photons $E(\Delta)$ are codimension 0 submanifolds of $\mathrm{Pho}^{2,n}$ with boundary. The boundary $\partial E(\Delta)$ consists of three components which are smooth submanifolds of $\mathrm{Pho}^{2,n}$. Each boundary component is a line of photons. Notice that lines of photons carry an action of the subgroup $\mathrm{SO}(1,1) \times \mathrm{SO}(1,n)$ which is compatible with the fibration $E(\ell) \to \ell$. Furthermore, they have natural ideal boundaries:

Definition 8.4 (Ideal Boundary)**.** Triangles and lines of photons have both an ideal boundary. Boundary components correspond to isotropic lines and have have all the following form: For every isotropic line $[a] \in \partial\mathbb{H}^{2,n}$, we consider the subspace

$$E(a) := \mathrm{Pho}(a^\perp) = \{F \in \mathrm{Pho}^{2,n} \,|a \subset F\}.$$

If $\ell \subset \mathbb{H}^{2,n}$ is a spacelike geodesic with endpoints at infinity $a, b \in \partial\mathbb{H}^{2,n}$, then ideal boundary of $E(\ell)$ is $E(a) \cup E(b)$. The subspace $E(a) \cup E(\ell) \cup E(b)$ is the closure of $E(\ell)$ in $\mathrm{Pho}^{2,n}$. Similarly, if $\Delta \subset \mathbb{H}^{2,n}$ is a spacelike ideal triangle with vertices $a, b, c \in \partial\mathbb{H}^{2,n}$, then ideal boundary of $E(\Delta)$ is $E(a) \cup E(b) \cup E(c)$. The subspace $E(\Delta) \cup E(a) \cup E(b) \cup E(c)$ is the closure of $E(\Delta)$ in $\mathrm{Pho}^{2,n}$.

After having proved the geometric decomposition, we will explain, conversely, how to explicitly construct abstractly photon structures that fiber over hyperbolic surfaces by assembling together triangles of photons. The process is completely analogous to the construction of hyperbolic surfaces by gluing ideal triangles. The holonomy of such photon structures will correspond to maximal representations $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$, the hyperbolic surface S, which is the base of the fibering, corresponds to a pleated surface for ρ, and the gluing parameters of the triangles of photons correspond to the bending of the pleated surface.

The goal of the section is to develop this picture in more details.

8.1. **A geometric decomposition.** We have the following is geometric decomposition of the Guichard-Wienhard domain of discontinuity $\Omega_\rho \subset \mathrm{Pho}^{2,n}$:

Proposition 8.5. *Let $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ be a maximal representation with Guichard-Wienhard domain of discontinuity $\Omega_\rho \subset \mathrm{Pho}^{2,n}$. Let $\hat\Sigma$ be the universal covering of Σ. Let $f : \hat\Sigma \to \mathbb{H}^{2,n}$ be a ρ-equivariant embedding with acausal image $\hat S = f(\hat\Sigma)$. Then we have:*

- *The closure of $\hat S$ in $\mathbb{H}^{2,n} \cup \partial\mathbb{H}^{2,n}$ is $\hat S \cup \Lambda_\rho$.*
- *Ω_ρ is foliated by $\mathrm{Pho}(f(x)^\perp)$ and the map $\Omega_\rho \to \hat\Sigma$ that associates to a point $y \in \Omega_\rho$ the unique leaf $\mathrm{Pho}(f(x)^\perp)$ that contains it is an equivariant fibration.*

In [CTT19] these properties are proved for smooth equivariant spacelike embeddings (see Lemma 3.23, Lemma 4.11, and Theorem 5.3 of [CTT19]). Here we slightly generalize their results in a purely topological setting which is necessary when we deal with pleated surfaces.

Proof. Let us first prove the first point.

We lift f to an acausal embedding $\hat{f} : \hat{\Sigma} \to \widehat{\mathbb{H}}^{2,n}$. We will work in different Poincaré models of $\widehat{\mathbb{H}}^{2,n}$, for now we fix an arbitrary one $\psi : \mathbb{D}^2 \times \mathbb{S}^n \to \widehat{\mathbb{H}}^{2,n}$ and denote by $\pi : \widehat{\mathbb{H}}^{2,n} \to \mathbb{D}^2$ the associated projection.

As $\hat{S} = \hat{f}(\hat{\Sigma})$ is acausal, by Lemma 2.6, it can be represented as the graph of a 1-Lipschitz function $g : \pi(\hat{S}) \subset \mathbb{D}^2 \to \mathbb{S}^n$. The domain $\pi(\hat{S}) \subset \mathbb{D}^2$ is a simply connected open subset by invariance of domain as the map $\pi\hat{f} : \hat{\Sigma} \to \mathbb{D}^2$ is injective. Let D denote the projection $\pi(\hat{S})$ and let $\bar{D}$ be its closure in $\mathbb{D}^2 \cup \partial\mathbb{D}^2$. As g is 1-Lipschitz, it continuously extends to a 1-Lipschitz function $\bar{g} : D \subset \mathbb{D}^2 \cup \partial\mathbb{D}^2 \to \mathbb{S}^n$. We deduce that the closure $\hat{S} \cup \partial\hat{S}$ of $\hat{S}$ in $\widehat{\mathbb{H}}^{2,n} \cup \partial\widehat{\mathbb{H}}^{2,n}$ is the graph of $\bar{g}$.

We first show that

Claim. We have $D = \mathbb{D}^2$.

Proof of the claim. As $\rho(\Gamma)$ acts cocompactly on $\hat{S}$, we can find a compact fundamental domain $F \subset \hat{S}$. Let $U \subset \hat{S}$ be an open neighborhood of F in $\hat{S}$. By compactness, there exists $r > 0$ such that $d_{\mathbb{H}^{2,n}}(x, y) \geqslant r$ for every $x \in F$ and $y \in \partial U$. As $\rho(\Gamma)$ preserves $\hat{S}$ and its pseudo metric, we deduce that every point $x \in \hat{S}$ has an open neighborhood $U_x \subset \hat{S}$ such that $d_{\mathbb{H}^{2,n}}(x, \partial U_x) \geqslant r$.

Recall that $\mathbb{D}^2$ is endowed with a hyperbolic metric. As $\hat{S}$ is acausal, by Lemma 2.5, we have that $d_{\mathbb{H}^2}(\pi(x), \pi(y)) \geqslant d_{\mathbb{H}^{2,n}}(x, y)$ for every $x, y \in \hat{S}$. In particular, for every $x \in \hat{S}$ we have

$$d_{\mathbb{H}^2}(\pi(x), \pi(\partial U_x)) \geqslant d_{\mathbb{H}^{2,n}}(x, \partial U_x) \geqslant r.$$

Since $\pi\hat{f} : \hat{\Sigma} \to \mathbb{D}^2$ is an injective map, by invariance of domain, it is also open. Therefore, for every $x \in \hat{S}$ the set $\pi(U_x)$ is an open neighborhood of $\pi(x)$. Furthermore, by the above discussion, it contains the hyperbolic metric ball of radius r centered at $\pi(x)$. We are now ready to conclude: The projection $D = \pi(\hat{S})$ is a subset of $\mathbb{D}^2$ with the property that its hyperbolic r-neighborhood $N_r(D)$ is still contained in D. This is only possible if $D = \mathbb{D}^2$. □

Using the dynamical properties of ρ we now show that $\partial\hat{S} = \hat{\Lambda}_\rho$.

First, recall that for every $\gamma \in \Gamma$ the element $\rho(\gamma)$ preserves a spacelike geodesic $[a, b]$, on which it acts by translations by $L > 0$, and its orthogonal subspace $\mathrm{Span}\{a, b\}^\perp$, on which it acts with (generalized) largest eigenvalue ν with $|\nu| < e^L$ (see [BPS19] or Corollary 2.6 of [CTT19]). Up to replacing γ with γ^{-1}, we assume that $\rho(\gamma)a = e^L a, \rho(\gamma)b = e^{-L}b$ where $L > 1$ and e^L is larger than $|\nu|$.

Fix $\gamma \in \Gamma$ with invariant axis $[a, b]$. Every $x \in \widehat{\mathbb{H}}^{2,n}$ can be written as $x = \alpha a + \beta b + u$ with $\alpha, \beta \in \mathbb{R}$ and $u \in V = \mathrm{Span}\{a, b\}^\perp$.

Claim. There exists $x \in \hat{S}$ that can be written as $x = \alpha a + \beta b + u$ with either $\alpha \neq 0$ or $\beta \neq 0$.

Proof of the claim. Suppose that this is not the case, then $\hat{S} \subset V$. Let $e \in \mathrm{Span}\{a, b\}$ be a spacelike vector. As V has signature $(1, n)$, there exists $e' \in V$ spacelike. Consider the Poincaré model $\psi : \mathbb{D}^2 \times \mathbb{S}^n \to \widehat{\mathbb{H}}^{2,n}$ associated to the orthogonal splitting $\mathbb{R}^{2,n+1} = E \oplus E^\perp$ where $E = \mathrm{Span}\{e, e'\}$. If a point $x = \psi(u, v)$ lies on V then

$$\begin{aligned}0 &= \langle \psi(u,v), e\rangle \\ &= \langle \frac{2}{1-|u|^2}u + \frac{1+|u|^2}{1-|u|^2}v, e\rangle \\ &= \frac{2}{1-|u|^2}\langle u, e\rangle.\end{aligned}$$

Therefore the projection of V to $\mathbb{D}^2$ lies on the line $\langle \bullet, e\rangle = 0$ of E. Since $\hat{S}$ is acausal, the projection $\pi_E(\hat{S})$ is an open subset of $\mathbb{D}^2$, therefore, there is a point $x \in \hat{S}$ which is not contained in V. □

Suppose that there is a point $x = \alpha a + \beta b + u \in \hat{S}$ with $\alpha \neq 0$. Then a lies in $\partial\hat{S}$: By $\rho(\Gamma)$-invariance, we have $\rho(\gamma)^m x = \alpha e^{mL} a + \beta e^{-mL} b + \rho(\gamma)^n u \in \hat{S}$. As the largest (generalized) eigenvalue of the restriction of $\rho(\gamma)$ to $V = \mathrm{Span}\{a,b\}^\perp$ is smaller than e^L, the sequence $[\alpha e^{mL} a + \beta e^{-mL} b + \rho(\gamma)^n u]$ converges to $[a]$ in the sphere of rays $\mathbb{R}^{2,n+1} - \{0\}/y \sim \lambda^2 y$. Thus $[a] \in \partial\hat{S}$.

Similarly, if there is a point $x = \alpha a + \beta b + u \in \hat{S}$ with $\beta \neq 0$, then b lies in $\partial\hat{S}$.

By $\rho(\Gamma)$-invariance, the orbit $\rho(\Gamma)a$ is contained in $\Lambda_\rho \cap \partial\hat{S}$ and moreover, it is dense in Λ_ρ. Therefore $\Lambda_\rho \subset \partial\hat{S}$. As Λ_ρ and $\partial\hat{S}$ are both graphs of functions $\partial\mathbb{D}^2 \to \mathbb{S}^n$, we conclude that $\Lambda_\rho = \partial\hat{S}$.

This concludes the proof of the first point.

For the second point we need the following three properties:

(1) For every $x \in \hat{\Sigma}$, the space $\mathrm{Pho}(f(x^\perp))$ is contained in Ω_ρ.
(2) If $p \in \Omega_\rho$, then $p \in f(x^\perp)$ for some $x \in \hat{\Sigma}$.
(3) If $x, y \in \hat{\Sigma}$ are distinct points, then $\mathrm{Pho}(f(x^\perp)), \mathrm{Pho}(f(y^\perp))$ are disjoint.

Together, the properties imply that Ω_ρ is foliated by $\mathrm{Pho}(f(x)^\perp)$ for $x \in \hat{\Sigma}$ and is equipped with an equivariant map $\Omega_\rho \to \hat{\Sigma}$.

We now prove the properties.

Property (1). The first property follows from the following fact

Claim. $\mathbb{P}(f(x)^\perp) \cap \Lambda_\rho = \varnothing$ for every $x \in \hat{\Sigma}$.

Proof of the claim. If $a \in \mathbb{P}(f(x)^\perp) \cap \partial\mathbb{H}^{2,n}$, then the 2-plane $\mathrm{Span}\{a, f(x)\}$ is lightlike, that is, $a, f(x)$ are joined by a lightlike geodesic. Let $\mathbb{D}^2 \times \mathbb{S}^n$ be a Poincaré model where $a, f(x)$ correspond respectively to $(p, \bar{g}(p))$ and $(o, \bar{g}(o))$ where $p \in \partial\mathbb{D}^2$ and $o \in \mathbb{D}^2$ is the center. By Lemma 2.5, since $[a, f(x)]$ is lightlike, we have $d_{\mathbb{S}^n}(\bar{g}(o), \bar{g}(p)) = d_{\mathbb{S}^2}(o, p)$. As $\bar{g}$ is 1-Lipschitz we must have $d_{\mathbb{S}^n}(\bar{g}(o), \bar{g}(t)) = d_{\mathbb{S}^2}(o, t)$ for every t on the radial segment $[o, p] \subset \mathbb{D}^2$ which is a minimal geodesic for the hemispherical metric on $\mathbb{D}^2$. However, by Lemma 2.5, this implies that $(o, g(o))$ and $(t, g(t))$ are connected by a lightlike geodesic. This contradicts the fact that $\hat{S}$, the graph of g, is acausal. □

Recall that $\Omega_\rho \subset \mathrm{Pho}^{2,n}$ is the complement of the set

$$K_\rho := \{F \in \mathrm{Pho}^{2,n} \,|\, \ell \subset F \text{ for some isotropic line } \ell \in \Lambda_\rho\}.$$

If $\mathrm{Pho}(f(x)^\perp) \cap K \neq \varnothing$, then there is a photon F orthogonal to $f(x)$ containing an isotropic line $a \in \Lambda_\rho$. In particular, $a \subset f(x)^\perp$ which cannot happen by the claim.

Property (3). The last property follows from the fact that $f(x), f(y)$ are joined by a spacelike segment: Suppose that there is a photon F that is simultaneously orthogonal to $f(x)$ and $f(y)$. Then it is orthogonal to the 2-plane $\mathrm{Span}\{f(x), f(y)\}$

which has signature $(1,1)$ as $f(x), f(y)$ are joined by a spacelike segment. However, the orthogonal of such plane, having signature $(1,n)$, cannot contain photons.

Property (2). The second property follows from the fact that every timelike sphere intersects $f(\hat{\Sigma})$ exactly once and $\Lambda_\rho = \partial f(\hat{\Sigma})$.

Let $F \in \mathrm{Pho}^{2,n}$ be a photon. The orthogonal $F^\perp$ is non-positive definite and can be approximated by negative definite $(n+1)$-planes F_n. Each such plane intersects $f(\hat{\Sigma})$ exactly once in a point $f(x_n)$. Thus $F^\perp$ either intersects $f(\hat{\Sigma})$ in some point $f(x)$ or it intersects Λ_ρ. In the first case, $F \subset f(x)^\perp$, that is $F \in \mathrm{Pho}(f(x)^\perp)$, and, moreover, by Property (3), $F^\perp$ intersects $f(\hat{\Sigma})$ in exactly the point $f(x)$. In the second case, $a \subset F^\perp$ for some isotropic line $a \in \Lambda_\rho$ which implies $a \subset F$ and, hence, $F \in K_\rho$.

Note that, as a by-product of the proof, we can describe explicitly the projection $\Omega \to \hat{S}$ as $F \to \mathbb{P}(F^\perp) \cap \hat{S}$. □

8.2. **Gluing triangles of photons.** We start with a simple computation: Let Δ be an ideal spacelike triangle and let ℓ be a spacelike geodesic. Let $f \in \mathrm{SO}_0(2,n+1)$ be an isometry such that $f(E(\Delta)) = E(\Delta)$ or $f(E(\ell)) = E(\ell)$. As f induces an orientation preserving homeomorphism of $\mathrm{Pho}^{2,n}$, we have that f extends to a homeomorphism of the closures of $E(\Delta)$ and $E(\ell)$. In particular, f must permute the ideal vertices of $E(\Delta)$ or $E(\ell)$.

We denote by $\mathrm{PStab}_{\mathrm{SO}_0(2,n+1)}(E(\Delta))$ and $\mathrm{PStab}_{\mathrm{SO}_0(2,n+1)}(E(\ell))$ the elements that stabilize $E(\Delta)$ and $E(\ell)$ without permuting the ideal vertices. Observe that if $f(E(u)) = E(u)$ for some isotropic line $u \in \partial\mathbb{H}^{2,n}$, then $f(u) = u$ in $\partial\mathbb{H}^{2,n}$. As a consequence, we have the following

Lemma 8.6. *We have*

- $\mathrm{PStab}_{\mathrm{SO}_0(2,n+1)}(E(\Delta)) = \mathrm{PStab}_{\mathrm{SO}_0(2,n+1)}(\Delta)$.
- $\mathrm{PStab}_{\mathrm{SO}_0(2,n+1)}(E(\ell)) = \mathrm{PStab}_{\mathrm{SO}_0(2,n+1)}(\ell)$.
- $\mathrm{Stab}_{\mathrm{SO}_0(2,n+1)}(E(a)) = \mathrm{Stab}_{\mathrm{SO}_0(2,n+1)}(a)$.

We now fix once and for all an ideal spacelike triangle $\Delta \subset \mathbb{H}^{2,n}$ with vertices $a, b, c \in \partial\mathbb{H}^{2,n}$. Denote by ℓ_a, ℓ_b, ℓ_c the edges opposite to a, b, c respectively.

Consider $H := \mathrm{Span}\{a, b, c\} \subset \mathbb{R}^{2,n+1}$ and denote by $F = H^\perp$ the orthogonal complement. Let $\iota_F \in \mathrm{O}(2, n+1)$ be an isometric involution that restricts to an orthogonal reflection on F and to $-\mathrm{I}$ on H. We have:

- $\mathrm{PStab}_{\mathrm{SO}_0(2,n+1)}(\Delta) = \mathrm{SO}(F) \sqcup \iota_F \mathrm{SO}(F)$.
- $\mathrm{PStab}_{\mathrm{SO}_0(2,n+1)}(\ell_u) = \mathrm{SO}_0(L_u) \times \mathrm{SO}_0(L_u^\perp) \sqcup \iota_F \mathrm{SO}_0(L_u) \times \mathrm{SO}_0(L_u^\perp)$.

Let $r_a, r_b, r_c \in \mathrm{O}(2, n+1)$ be the isometries of $\mathbb{R}^{2,n+1}$ that on H coincide with the orthogonal reflections along the planes $L_a = \mathrm{Span}\{b, c\}, L_b = \mathrm{Span}\{c, a\}, L_c = \mathrm{Span}\{a, b\}$ and restrict to the identity on F.

Consider the standard triangle of photons $E(\Delta) \to \Delta$. We endow the boundary components $\partial E(\Delta)$ with the induced orientation. We now construct fibered photon structures by gluing together two copies of $E(\Delta)$ along suitably boundary identifications. We now describe which gluing maps $\phi : \partial E(\Delta) \to \partial E(\Delta)$ are admissible: For every edge ℓ_u of Δ, we start by choosing an isometry $\phi_u^B : \ell_u \to \ell_u$. Then, we choose $\beta_u \in \mathrm{PStab}(E(\ell_u))$ covering $\phi_u^B : \ell_u \to \ell_u$ and form the orientation reversing isometry $\phi_u := \beta_u r_u$. Observer that ϕ_u coincides with β_u on $E(\ell_u)$.

The choices of the three gluing maps of the base $\phi_a^B, \phi_b^B, \phi_c^B$ determine an (incomplete) hyperbolic surface

$$S = \Delta \cup \Delta / \phi_a^B \cup \phi_b^B \cup \phi_c^B,$$

a photon structure

$$E = E(\Delta) \cup E(\Delta)/\phi_a \cup \phi_b \cup \phi_c,$$

where a $(\mathrm{SO}_0(2, n+1), \mathrm{Pho}^{2,n})$-local chart around a point $x \in E(\ell_u)$ is obtained by juxtaposing $E(\Delta)$ and $\phi_u(E(\Delta))$.

We also have a natural fiber bundle projection

$$E \to S$$

with geometric fibers.

Observe that, if we choose elements ψ, ψ' in $\mathrm{PStab}(E(\Delta))$ and we change the gluing maps ϕ_u with $\psi\phi_u\psi'$ for all $u \in \{a, b, c\}$, then, if we denote by E' the new photon structure, there is an isomorphism of photon structures $E \to E'$ covering the identity $S = S'$ induced by $\psi \cup \psi'$. Thus, the space of parameters for the gluing maps is

$$\mathrm{PStab}_{\mathrm{SO}_0(2,n+1)}(\ell_a) \times \mathrm{PStab}_{\mathrm{SO}_0(2,n+1)}(\ell_b) \times \mathrm{PStab}_{\mathrm{SO}_0(2,n+1)}(\ell_c)$$

modulo the action by left and right multiplications of

$$\mathrm{PStab}(\Delta) \times \mathrm{PStab}(\Delta).$$

The completion of the hyperbolic surface S is a hyperbolic pair of pants with totally geodesic boundary $\bar{S}$ and we can easily read off the length of the boundary components from the gluing maps ϕ_u^B. We now study the completion of E. In particular, we give conditions on the gluing maps ϕ_u under which the fibered photon structure E admits a completion $\bar{E}$, which is a fibered photon structure with photon boundary that naturally fibers over $\bar{S}$.

First, let us compute the holonomies ρ_a, ρ_b, ρ_c around the vertices a, b, c of S corresponding to the vertices a, b, c of Δ: A direct computation from the definition of the photon structure on E shows that

$$\rho_v = \phi_u \phi_w^{-1}$$

where the u, v, w are in cyclic order.

Definition 8.7 (Loxodromic Isometry)**.** An isometry $\rho \in \mathrm{SO}(2, n+1)$ is loxodromic if it admits an invariant spacelike line $\ell = [a, b]$ such that the following holds:

- $\rho^n \to a$ uniformly on $\mathbb{H}^{2,n} - \mathbb{P}(b^\perp)$.
- $\rho^{-n} \to b$ uniformly on $\mathbb{H}^{2,n} - \mathbb{P}(a^\perp)$.

We say that a and b are the attracting and repelling fixed points of ϕ.

Note that a loxodromic element ϕ with attracting and repelling fixed points a, b has north-south dynamics on $\mathrm{Pho}^{2,n}$ in the following sense:

- $\rho^n \to E(a)$ uniformly on $\mathrm{Pho}^{2,n} - E(b)$.
- $\rho^{-n} \to E(b)$ uniformly on $\mathrm{Pho}^{2,n} - E(a)$.

In particular, it acts properly discontinuously and freely on $\mathrm{Pho}^{2,n} - (E(b) \cup E(a))$.

Definition 8.8 (Fibered Photon Structure with Geodesic Boundary)**.** Let $\bar{S}$ be an orientable compact surface with boundary and let $\bar{E} \to \bar{S}$ be a fiber bundle over it.

A *half space* of $\mathrm{Pho}^{2,n}$ is a subspace of the form

$$E(W) = \{V \in \mathrm{Pho}^{2,n} \,|V \perp x \text{ for some } x \in W\}$$

where $W \subset H$ is a half plane in a spacelike plane $H \subset \mathbb{H}^{2,n}$.

A photon structure with *totally geodesic boundary* on $\bar{E}$ is a maximal atlas of charts with values in a half space of $\mathrm{Pho}^{2,n}$ with change of charts that are restrictions of transformations in $\mathrm{SO}_0(2, n+1)$.

The photon structure is fibered if the following holds: Let $\hat{S} \to \bar{S}$ be the universal covering and let $\hat{E} \to \hat{S}$ be the pull-back bundle. Let $\delta : \hat{E} \to \mathrm{Pho}^{2,n}$ be the developing map. Then $\delta(\pi^{-1}(x)) = \mathrm{Pho}(f(x)^{\perp})$ for some $f(x) \in \mathbb{H}^{2,n}$.

The map $f : \hat{S} \to \mathbb{H}^{2,n}$ is the associated map of the fibered photon structure.

We have the following:

Lemma 8.9. *Let a, b, c be the vertices of S. Suppose that the holonomies $\rho_v := (\phi_u \phi_w^{-1}) \in \mathrm{Stab}(v)$ for $\{v, u, w\} = \{a, b, c\}$ and u, v, w cyclically ordered are all loxodromic. Denote the invariant lines by $\ell(\rho_v)$. Then there is a completion $E \subset \bar{E}$ which is a fibered photon structure with totally geodesic boundary where the boundary component adjacent to v is $E(\ell(\rho_v))/\rho_v$.*

Proof. Let $\ell(\rho_v)$ be the invariant spacelike line of ρ_v. Notice that, as ρ_v is loxodromic and leaves invariant $E(v)$, we have $\ell(\rho_v) = [v, t]$. Let $E(\ell(\rho_v))$ be the corresponding ρ_v-invariant line of photons. Note that the action $\rho_v \curvearrowright E(\ell(\rho_v))$ is properly discontinuous and free. Let

$$E(\ell(\rho_v))/\rho_v \to \ell(\rho_v)/\rho_v$$

be the corresponding quotient bundle.

We now give local charts for

$$E \cup (E(\ell(\rho_v))/\rho_v) \to S \cup (\ell(\rho_v)/\rho_v).$$

Let $\hat{E} \to \hat{S}$ be the pull-back bundle to the universal covering $\hat{S} \to S$. Choose a lift v of the vertex. Let $\hat{S}_v$ be the fan of triangles with an ideal vertex in v. Let $\hat{E}_v$ be the fan of all triangles of photons of $\hat{E}$ with ideal vertex $E(v)$.

Claim. The restriction of the associated map $f : \hat{S} \to \mathbb{H}^{2,n}$ to $\hat{S}_v$ is an acausal embedding.

Proof of the claim. The proof is a simpler version of the one of Proposition 3.6.

We lift $f : \hat{S} \to \mathbb{H}^{2,n}$ to the two fold cover $\widehat{\mathbb{H}}^{2,n} \to \mathbb{H}^{2,n}$ and work in a Poincaré model $\mathbb{D}^2 \times \mathbb{S}^n$ of $\widehat{\mathbb{H}}^{2,n}$. Observe that $\hat{S}_v = \bigcup_{j \in \mathbb{Z}} \Delta_j$ where $\Delta_j = \Delta(v, u_{j-1}, u_j)$. Notice that the cyclic order of the vertices of $\hat{S}_v$ on $\partial \hat{S}$ is

$$v < \cdots < u_{j-1} < u_j < u_{j+1} < \cdots < v.$$

Let $\pi : \widehat{\mathbb{H}}^{2,n} \cup \partial\widehat{\mathbb{H}}^{2,n} \to \mathbb{D}^2 \cup \partial\mathbb{D}^2$ be the natural projection. Consider two consecutive triangles $\Delta_j = \Delta(v, u_{j-1}, u_j), \Delta_{j+1} = \Delta(v, u_j, u_{j+1})$ intersecting along the geodesic $[v, u_j]$. By construction, $f(\Delta_j \cup \Delta_{j+1})$ is acausal so the projections $\pi(v), \pi(u_{j-1}), \pi(u_j), \pi(u_{j+1})$ of the vertices v, u_{j-1}, u_j, u_{j+1} to $\partial\mathbb{D}^2$ appear in this exact cyclic order on $\partial\mathbb{D}^2$. We deduce that that the projections of the vertices $\pi(u_j)$ appear in the same order

$$\pi(v) < \cdots < \pi(u_{j-1}) < \pi(u_j) < \pi(u_{j+1}) < \cdots < \pi(v)$$

as they appear on $\partial\hat{S}$. As a consequence, the restriction of f to the union $\lambda = \bigcup_{j\in\mathbb{Z}} \partial\Delta_j$ of the sides of the triangles Δ_j is an acausal embedding: For every $\ell, \ell' \subset \lambda$ we have that the endpoints of ℓ, ℓ' are in disjoint position.

We immediately deduce that $f(\mathrm{int}(\Delta_j)) \cap f(\mathrm{int}(\Delta_i)) = \varnothing$ for all $j \neq i$ which says that f is an embedding: We already know that is is the case when $|i-j| = 1$. Assume $|i-j| > 1$. Note that πf is an embedding on both Δ_j, Δ_i and the images coincide with the topological disks bounded by the closures in $\mathbb{D}^2 \cup \partial\mathbb{D}^2$ of $\pi f(\partial\Delta_j), \pi f(\partial\Delta_i)$. Those curves are disjoint and not nested. The conclusion follows.

Checking that $f(\hat{S}_v)$ is acausal is simple: Suppose that τ is a lightlike geodesic contained in $f(\hat{S}_v)$ then τ has a subsegment entirely contained in one of the triangles $f(\Delta_j)$ but such triangle is spacelike, so it cannot contain any lightlike segment. $\square$

A consequence of the claim is that the restriction of the developing map $\delta : \hat{E} \to \mathrm{Pho}^{2,n}$ to $\hat{E}_v$ is an embedding.

Notice that the images $\delta(\hat{E}_v)$ are ρ_v-invariant and contained in $\mathrm{Pho}^{2,n} - E(\ell(\rho_v))$: In fact, if $E(\ell(\rho_v))$ intersects the image of one of the triangles of photons $E(\Delta')$ in $\hat{E}_v$ under the developing map, then we have $\rho_v\delta(E(\Delta')) \cap \rho_v\delta(E(\Delta') \neq \varnothing$ as both $\delta(E(\Delta'))$ and $E(\ell(\rho_v))$ have a vertex in $E(v)$. But ρ_v moves every triangle $\delta(E(\Delta'))$ off itself.

Furthermore, by the loxodromic assumption on ρ_v, the ρ_v-orbit of every triangle in $\delta(\hat{E}_v)$ accumulates to $E(\ell(\rho_v))$ either in the forward or backward direction and to $E(v)$ or $E(t)$ in the opposite one. We deduce that the union $\delta(\hat{E}_v) \cup E(\ell(\rho_v)) \subset \mathrm{Pho}^{2,n}$ is a ρ_v-invariant submanifold with boundary $E(\ell(\rho_v))$. This provides local charts for $E \cup (E(\ell(\rho_v))/\rho_v)$ at points in $E(\ell(\rho_v))/\rho_v$. $\square$

We now describe the topology of the completion: We associate to the fibered photon structure $\bar{E} \to \bar{S}$ the vector bundle $\bar{V} \to \bar{S}$ where the fiber over the point $x \in \bar{S}$ is the vector space $x^{\perp} \subset \mathbb{R}^{2,n+1}$. Vector spaces over $\hat{S}$ are classified by the first Stiefel-Whitney class $w_1(\bar{V}) \in H^1(\bar{S}, \mathbb{Z}/2\mathbb{Z})$ that can be computed as follows: Let $[\gamma] \in H_1(\bar{S}, \mathbb{Z}/2\mathbb{Z})$ be represented by the loop $\gamma : S^1 \to \hat{S}$. Consider the pull-back bundle $\gamma^*\bar{V} \to S^1$, we define $w_1(\bar{V})[\alpha] \in \mathbb{Z}/2\mathbb{Z}$ to be 0 if α^*V is orientable and 1 if it is non-orientable. This defines a homomorphism $w_1(\bar{V}) : H_1(\bar{S}, \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ representing the first Stiefel-Whitney class.

If $\gamma_a, \gamma_b, \gamma_c$ are the boundary curves of $\bar{S}$ corresponding to the vertices a, b, c, then, in order to compute $w_1(\bar{V})$, it is enough to know two among $w_1(\bar{V})[\gamma_a]$, $w_1(\bar{V})[\gamma_b]$, $w_1(\bar{V})[\gamma_c]$ as two classes among $[\gamma_a]$, $[\gamma_b]$, $[\gamma_c]$ generate $H_1(\bar{S}, \mathbb{Z}/2\mathbb{Z})$.

Let us compute $w_1(\bar{V})[\gamma_a]$: Let $\rho_a = \phi_c\phi_b^{-1}$ be the holonomy around γ_a. Recall that ρ_a is a loxodromic element.

Loxodromic elements in $\mathrm{SO}_0(2, n+1)$ are divided into two connected components $\mathcal{L}^+, \mathcal{L}^-$: Let $\rho \in \mathrm{SO}_0(2, n+1)$ be a loxodromic isometry with invariant line $\ell(\rho) = [a, b]$. Consider the action of ρ on the 2-plane $L := \mathrm{Span}\{a, b\}$. Then ρ can either preserve or exchange the connected components of $L \cap \widehat{\mathbb{H}}^{2,n}$. This feature distinguishes the two connected components of $\mathrm{SO}(L)$ and the two connected components of loxodromic elements in $\mathrm{SO}_0(2, n+1)$. In the first case, when $\rho \in \mathrm{SO}_0(L)$, the bundle $V(\ell)/\rho$ is the trivial bundle over ℓ/ρ. In the second case, when $\rho \notin \mathrm{SO}_0(L)$, the bundle $V(\ell)/\rho \to \ell/\rho$ is the unique non-orientable bundle over ℓ/ρ.

The computation of the connected component of the loxodromic isometry $\rho_a = \phi_c\phi_b^{-1}$ is simple: The connected components $\mathcal{L}^+, \mathcal{L}^-$ are distinguished by the sign of the extremal eigenvalues. We already know that one of the eigenvectors is the isotropic line a, so we can read off the connected component from $\rho_a(a)$. If the sign is positive, then $w_1(\bar{V})[\gamma_a] = 0$. Otherwise, $w_1(\bar{V})[\gamma_a] = 1$.

Theorem 8.10. *The space of fibered photon structures with totally geodesc boundary on a bundle $E \to S$ over a hyperbolic pair of pants S admits a parametrization by*

$$\left\{ [\phi_a, \phi_b, \phi_c] \in \left(\prod_{j\in\{a,b,c\}} \mathrm{PStab}(\ell_j) \right) / \mathrm{PStab}(\Delta)^2 \ \middle|\rho_k = \phi_i\phi_j^{-1} \in \mathcal{L} \right\}$$

The topology of E is determined by the first Stiefel-Whitney class $w_1(V) \in H^1(S, \mathbb{Z}/2\mathbb{Z})$ of the underlying vector bundle $V \to S$. The class $w_1(V)$ can be computed as follows

$$w_1(V)[\gamma_k] = \begin{cases} 0 & \textit{if } \rho_k \in \mathcal{L}^+, \\ 1 & \textit{if } \rho_k \in \mathcal{L}^-, \end{cases}$$

8.3. **Gluings pants of photons.** Let $\bar{E}_j \to \bar{S}_j$ be $2g-2$ oriented pants of photons with totally geodesic boundary. We label by $\gamma_{a_j}, \gamma_{b_j}, \gamma_{c_j}$ the boundary components of $\bar{S}_j$ and by $E(\gamma_{a_j}), E(\gamma_{b_j}), E(\gamma_{c_j})$ the corresponding boundary components of $\bar{E}_j$.

For every oriented pants of photons $\bar{E}_j \to \bar{S}_j$, we fix a developing map $\delta_j : \hat{E}_j \to \mathrm{Pho}^{2,n}$ and corresponding holonomy, and we denote by $\rho_{a_j}, \rho_{b_j}, \rho_{c_j}$ the holonomies of the boundary curves $\gamma_{a_j}, \gamma_{b_j}, \gamma_{c_j}$ and by $\ell(\rho_{a_j}), \ell(\rho_{b_j}), \ell(\rho_{c_j})$ their invariant spacelike geodesics.

Let τ be an orientation reversing pairing of the boundary components of the fibered pairs of pants $\bar{E}_j \to \bar{S}_j$.

If we want to perform a geometric gluing of the blocks $\bar{E}_j \to \bar{S}_j$ that implements the pairing τ, some compatibility conditions must be fulfilled: Every time we have an identification of a boundary component of $\bar{S}_i$ labeled by v_i with some boundary component of $\bar{S}_j$ labeled by v_j, the holonomies ρ_{v_i}, ρ_{v_j} must be conjugate in $\mathrm{SO}_0(2, n+1)$.

If this happens we choose for every pair of boundary components $E(\gamma_{v_i}), E(\gamma_{v_j})$ that are paired by τ an arbitrary initial orientation reversing identification $c_{v_jv_i} : E(\gamma_{v_i}) \to E(\gamma_{v_j})$ as bundles over $\gamma_{v_i}, \gamma_{v_j}$ which is induced by an element $c_{v_jv_i} \in \mathrm{SO}_0(2, n+1)$ such that $\rho_{v_j} = c_{v_jv_i}\rho_{v_i}c_{v_jv_i}^{-1}$.

All other admissible gluing maps will be of the form $\tau_{v_jv_i} := \beta_{v_j}c_{v_jv_i}\beta_{v_i}$ where $\beta_{v_i} \in \mathrm{PStab}(\ell(\rho_{v_i}))$ and $\beta_{v_j} \in \mathrm{PStab}(\ell(\rho_{v_j}))$ are isometries that commute with ρ_{v_i} and ρ_{v_j} respectively. The restriction $\tau^B_{v_jv_i} : \ell(\rho_{v_i}) \to \ell(\rho_{v_j})$ induces an orientation reversing isometry between the boundary components $\gamma_{v_i}, \gamma_{v_j}$.

Thus we can form: A hyperbolic structure over a closed surface

$$S := \bigcup_{j\leqslant 2g-2} \bar{S}_j \Bigg/ \bigcup_{j\leqslant 2g-2} \tau^B_{v_jv_i} \ ,$$

a photon structure over a closed manifold

$$E := \bigcup_{j \leqslant 2g-2} \bar{E}_j \Big/ \bigcup_{j \leqslant 2g-2} \tau_{v_j v_i} \ ,$$

where a $(\mathrm{SO}_0(2, n+1), \mathrm{Pho}^{2,n})$-local chart around a point $x \in E(\gamma_{v_j}) = E(\gamma_{v_i})$ is obtained by juxtaposing $\delta_j(\hat{E}_j)$ and $\tau_{v_j v_i}\delta_i(\hat{E}_i)$.

As before, since gluing and fibering are compatible, we also get a fiber bundle projection

$$E \to S$$

with geometric fibers.

The following, which is analogous to Proposition 3.13 of [CTT19], shows that the holonomy $\rho : \pi_1(S) \to \mathrm{SO}_0(2, n+1)$ of $E \to S$ is maximal.

Lemma 8.11. *Let $\rho : \Gamma \to \mathrm{SO}_0(2, n+1)$ be a representation. Suppose that there exists a ρ-equivariant local acausal embedding $f : \hat{\Sigma} \to \mathbb{H}^{2,n}$, meaning that every point $x \in \hat{\Sigma}$ has a neighborhood U such that $f|_U : U \to \mathbb{H}^{2,n}$ is an embedding with acausal image. Then ρ is maximal.*

Proof. We can lift f to a local acausal embedding $f : \hat{\Sigma} \to \widehat{\mathbb{H}}^{2,n}$. By assumption, every point $x \in \hat{\Sigma}$ has a neighborhood U such that $f|_U$ is an embedding with acausal image. In particular, by Lemma 2.6, we can represent $f(U)$ in a Poincaré model $\mathbb{D}^2 \times \mathbb{S}^n$ as a graph of a strictly 1-Lipschitz function $g : \pi(U) \subset \mathbb{D}^2 \to \mathbb{S}^n$.

As Γ acts cocompactly on $\hat{\Sigma}$ and f is ρ-equivariant, we can choose the neighborhoods U in a uniform way. In order to do so we proceed as follows: We endow $\hat{\Sigma}$ with a Γ-invariant hyperbolic metric obtained by pulling back a hyperbolic metric on Σ. We cover Σ with the projections of the neighborhoods U and find $r > 0$ a Lebesgue number for the covering which is smaller than the injectivity radius of Σ. With these choices the restriction of f to $B(x, r)$ is an embedding with acausal image for every $x \in \hat{\Sigma}$.

Observe that we can choose a continuous $\rho(\Gamma)$-invariant family of orthogonal splittings $P_x \oplus N_x$ of $\mathbb{R}^{2,n+1}$ where P_x is a spacelike 2-plane: Let $\mathrm{Gr}_{(2,0)}(\mathbb{R}^{2,n})$ be the Grassmannian of spacelike 2-planes in $\mathbb{R}^{2,n}$. Topologically, we have an identification

$$\mathrm{Gr}_{(2,0)}(\mathbb{R}^{2,n}) = \mathrm{SO}(2, n)/\mathrm{SO}(2) \times \mathrm{SO}(n),$$

where the right hand side is the symmetric space of $\mathrm{SO}(2, n)$, in particular the Grassmannian is contractible. Let $G \to \widehat{\mathbb{H}}^{2,n}$ be the bundle with fiber $\mathrm{Gr}_{(2,0)}(x^\perp)$ over the point $x \in \widehat{\mathbb{H}}^{2,n}$ and let $f^*G \to \hat{\Sigma}$ be the corresponding pull-back bundle. As the fiber is contractible, we can always find a Γ-invariant global section. Such a section corresponds to the desired continuous family of orthogonal splittings.

Consider the plane bundle

$$P \to \hat{\Sigma}$$

whose fiber over $x \in \hat{\Sigma}$ is the spacelike plane P_x and let $P^1 \subset P$ the corresponding unit circle bundle whose fiber over x is $\mathbb{S}^1_x \subset P_x$.

We now define a ρ-equivariant isomorphism between P^1 and the unit tangent bundle $T^1\hat{\Sigma}$: As a concrete model of $T^1\hat{\Sigma}$, we exploit the Γ-invariant hyperbolic metric obtained by pulling back a hyperbolic metric on Σ, and, using the exponential map, we identify $T^1\hat{\Sigma} \to \hat{\Sigma}$ with

$$B^1 := \{(x,y) \in \hat{\Sigma} \times \hat{\Sigma} \,|\, d(x,y) = r\} \to \hat{\Sigma}$$

where the bundle projection is the projection to the first factor and the fiber over x is the unit circle B^1_x around x in $\hat{\Sigma}$. We will show that B^1 is equivariantly isomorphic to P^1.

For every $y \in B^1_x$ let $\xi_x(y)$ be the endpoint at infinity of the spacelike geodesic ray issueing from $f(x)$ and passing through $f(y)$. Explicitly, if $t_x(y) \in T^1\widehat{\mathbb{H}}^{2,n}$ is the direction of the ray, then

$$\xi_x(y) = [f(x) + t_x(y)] \in \partial\widehat{\mathbb{H}}^{2,n}.$$

Notice that $\xi_x(B^1_x) \subset \partial\widehat{\mathbb{H}}^{2,n}$ is a loop freely homotpic to a generator of the fundamental group: This can be seen in the Poincaré disk model $\widehat{\mathbb{H}}^{2,n} \simeq \mathbb{D}^2 \times \mathbb{S}^n$ associated to the splitting $\mathbb{R}^{2,n+1} = P_x \oplus N_x$ where the circle $f(B^1_x)$ is the graph of a topological circle around the origin and $\xi_x(B^1_x)$ is obtained by projecting radially such circle to $\partial\mathbb{D}^2$ and then mapping it to $\partial\widehat{\mathbb{H}}^{2,n}$ via the graph map.

We now exhibit an explicit degree one map $\phi_x : B^1_x \to \mathbb{S}^1_x$. If $[a] \in \partial\widehat{\mathbb{H}}^{2,n}$ is an isotropic ray, then, we can represent it uniquely as $u_x([a]) + v_x([a])$ with $u_x([a]) \in P_x$ and $v_x([a]) \in N_x$ vectors of norm 1 and -1 respectively. Explicitely,

$$u_x([a]), v_x([a]) := \pi_{P_x}(a)/\sqrt{\langle \pi_{P_x}(a), \pi_{P_x}(a)\rangle}, \pi_{N_x}(a)/\sqrt{-\langle \pi_{N_x}(a), \pi_{N_x}(a)\rangle}$$

where $\pi_{P_x}, \pi_{N_x} : \mathbb{R}^{2,n+1} \to P_x, N_x$ are the orthogonal projections and a is any representative of $[a]$ in the isotropic cone. We denote by $\mathbb{S}^1_x, \mathbb{S}^n_x$ the unit spheres of P_x, N_x and by $u_x, v_x : \partial\widehat{\mathbb{H}}^{2,n} \to \mathbb{S}^1_x, \mathbb{S}^n_x$ the two projections.

Observe that u_x satisfies

$$u_{\gamma x}(\rho(\gamma)[a]) = \rho(\gamma)u_x([a]).$$

In fact, $\rho(\gamma)$ maps isometrically $P_x \oplus N_x$ to $P_{\gamma x} \oplus N_{\gamma x}$.

Consider now $y \in B^1_x$. Split $\xi_x(y) = f(x) + t_x(y)$ as $u_x(\xi_x(y)) + v_x(\xi_x(y))$. We define

$$\phi_x(y) := u_x(\xi_x(y)).$$

The map $\Phi : B^1 \to P^1$ defined by

$$\Phi(x,y) := (f(x), \phi_x(y))$$

is a ρ-equivariant continuous bundle map which covers the identity of $\hat{\Sigma}$ and is degree one on every fiber. From this, we deduce that the (flat) circle bundles B^1/Γ and $P^1/\rho(\Gamma)$ have the same (maximal) Euler number and, hence, ρ is maximal. □

8.4. **Topology of the gluing.** We finish the discussion with a computation of the topology of the gluing: In order to do so, we have to compute the first and second Stiefel-Whitney classes of the vector bundle $V \to S$ naturally associated to $E \to S$.

Let G be the dual graph associated to the gluing.

The homology group $H_1(S,\mathbb{Z})$ is generated by $[\gamma_{a_j}], [\gamma_{b_j}], [\gamma_{c_j}]$ for $j \leqslant 2g-2$ and by the simple cycles of G. Let $\kappa \subset G$ be a simple cycle. Let $u_1, \cdots u_r$ be the vertices appearing along κ. Every edge $u_{j+1}u_j$ corresponds to an identification $\tau_{u_{j+1}u_j} : E(\gamma_{u_j}) \to E(\gamma_{u_{j+1}})$. Then, the holonomy around the cycle is given by

$$\rho_\kappa = \tau_{u_1u_r} \cdots \tau_{u_2u_1}.$$

As the representation ρ is maximal, the holonomy ρ_κ is loxodromic. The value $w_1(V)[\kappa]$ is 0 if ρ_κ belongs to $\mathcal{L}^+$ and 1 if ρ_κ belongs to $\mathcal{L}^-$.

The second Stiefel-Whitney class $w_2(V)$ can be computed as follows: Choose for every pair of identified boundary components $E(\gamma_{v_i}), E(\gamma_{v_j})$ a pair of non-vanishing sections $\sigma_{v_i}, \sigma_{v_j}$ of the underlying vector bundles $V \to \gamma_{v_i}, V \to \gamma_{v_j}$ that are identified under the gluing map $\tau_{v_i v_j}$. The Stiefel-Whitney number $w_2(V)[S]$, that uniquely determines $w_2(V)$, can be computed as the sum of the relative Stiefel-Whitney numbers $w_2(V_j, \sigma_j)[S_j, \partial S_j]$ which are the mod 2 reductions of the relative Euler numbers $e(V_j, \sigma_j)$.

Appendix A. Other cross ratios

There are multiple non-equivalent definitions of cross ratios in the literature.

For the reader's convenience, we summarize the relations between Definition 2.17 and the notions of cross ratios studied by Ledrappier [Led95], Hamenstädt [Ham97, Ham99], and Labourie [Lab08]. If $\beta = \beta(u, v, w, z)$ satisfies Definition 2.17, then:

- The function $(u, v, w, z) \mapsto \log|\beta(u, v, w, z)|$ is a Ledrappier's cross ratio (compare with [Led95, Définition 1.f], [MZ19, Definition 2.4]).
- The function $(u, v, w, z) \mapsto |\beta(u, v, w, z)|$ is a Hamenstädt's cross ratio (compare with [Ham97, Ham99]).
- If β further satisfies

$$\beta(u, v, w, z) = 1 \Leftrightarrow u = v \text{ or } w = z, \tag{17}$$

  then the function $B(u, v, w, z) := \beta(u, w, v, z)$ is a Labourie's cross ratio (see [Lab08, Definition 3.2.1]).

For the sake of completeness, even if we do not investigate in detail the properties of such cross ratios in this paper, we briefly discuss other examples of positive and locally bounded cross ratios from the literature strictly related to representations in $\mathrm{SO}(p, q)$ and pseudo-hyperbolic spaces $\mathbb{H}^{p,q}$. They come from respectively:

- Hitchin representations $\rho : \Gamma \to \mathrm{SO}(p, p+1)$.
- More generally, Θ-positive Anosov representations $\rho : \Gamma \to \mathrm{SO}(p, q)$.

In both cases (strict) positivity comes from transversality of the boundary maps (as explained in [BP21]) and local boundedness comes from their Hölder regularity (following the same strategy of Lemma 4.5).

As studied by Martone and Zhang in [MZ19] there are other natural classes of positive cross ratios arising from the study of Anosov representations. The ones that we mentioned above close to the setting of our interest and can have a more direct link with similar pleated surface constructions in $\mathbb{H}^{p,q}$.

Appendix B. Shears and symmetries of cross ratios

The current appendix is dedicated to the proofs of the relations satisfied by cross ratios and their associated shears, which were deployed throughout Section 5. We start by proving the following elementary relation:

Lemma B.1. *Let β be a cross ratio. Then for every 6-tuple of pairwise distinct points $a, b, c, d, e, x \in \partial\Gamma$ we have*

$$|\beta(a, b, c, d)\beta(a, d, b, e)| = |\beta(a, b, c, x)\beta(a, x, b, d)\beta(a, d, x, e)|$$

Proof. It is sufficient to apply the symmetries of the cross ratio β in (1) as follows

$$\begin{aligned} |\beta(a,b,c,d)\beta(a,d,b,e)| &= |\beta(a,b,c,x)\beta(a,b,x,d)\beta(a,d,b,e)| \\ &= |\beta(a,b,c,x)\beta(a,b,x,d)\beta(a,d,b,x)\beta(a,d,x,e)| \\ &= |\beta(a,b,c,x)\beta(a,x,b,d)\beta(a,d,x,e)|, \end{aligned}$$

where we used in the order twice the fourth relation and once the fifth relation from (1). By applying log to both members we obtain relation (18). □

Making use of the relation described in Lemma B.1, we can now provide a proof of the properties satisfied by finite β-shears and described by Lemmas 4.7 and 4.8:

Proof of Lemma 4.7. Let S denote the (closure of the) connected component of $\widetilde{\Sigma}\backslash\{P,Q\}$ that separates P from Q. Observe that the right-hand side of the statement can be expressed as $\sigma^\beta(P,S)+\sigma^\beta(S,Q)$. Consider now any geodesic g lying in the interior of S with endpoints w and x, and denote by R and R' the complementary regions of g inside S adjacent to P and Q, respectively. We claim that the following equality holds:

$$\sigma^\beta(P,S)+\sigma^\beta(S,Q)=\sigma^\beta(P,R)+\sigma^\beta(R,R')+\sigma^\beta(R',Q). \tag{18}$$

This is in fact a simple consequence of Lemma B.1. To see this, observe that, by definition of the finite shear σ^β, the left-hand side coincides with

$$\log|\beta(w,v_P,u_P,v_Q)\beta(w,v_Q,v_P,u_Q)|,$$

while the right-hand side is equal to

$$\log|\beta(w,v_P,u_P,x)\beta(w,x,v_P,v_Q)\beta(w,v_Q,x,u_Q)|.$$

Therefore relation (18) follows from Lemma B.1 applied to the 6-tuple $a=w$, $b=v_P$, $c=u_P$, $d=v_Q$, $e=u_Q$, and $x=x$.

The relation appearing in the statement can now be deduced simply by applying relation (18) enough times: at the k-th step we introduce inside the region S a leaf ℓ_k lying in the boundary of some plaque in $\mathcal{P}$, obtaining a finite lamination $\lambda_k=\lambda_{k-1}\cup\{\ell_k\}$. Relation (18) then allows us to split the sum of the shears between the complementary regions of λ_{k-1} as the sum of the shear of the complementary regions of λ_k. In a finite number of steps we obtain that $\sigma^\beta_{\mathcal{P}}(P,Q)$ coincides with $\sigma^\beta(P,S)+\sigma^\beta(S,Q)$, as desired (observe that, in the notation of §4.3, the lamination $\tilde{\lambda}_{\mathcal{P}}$ does not contain any geodesic of type d_i, since every spike has ideal vertex equal to w under our assumptions). □

Proof of Lemma 4.8. Among all the elements of $\mathcal{P}$ that lie on the left (resp. on the right) of g, we denote by P' (resp. Q') the plaque that is closest to g. Let x'_P, y'_P (resp. x'_Q, y'_Q) be the vertices of P' (resp. Q') different from g^+ (resp. g^-), so that $[y'_p, g^+]$ (resp. $[y'_Q, g^-]$) is the boundary component of P' (resp. Q') closest to g.

By following the process outlined in §4.3, we see that the shear $\sigma_{\mathcal{P}}(P,Q)$ satisfies

$$\sigma^\beta_{\mathcal{P}}(P,Q)=\sigma^\beta_\lambda(P,R_P)+\sigma^\beta(R_P,R_Q)+\sigma^\beta_\lambda(R_Q,Q),$$

where R_P and R_Q denote the plaques of $\tilde{\lambda}_{\mathcal{P}}$ with vertices g^+,g^-,y'_P and g^+,g^-,y'_Q, respectively. By Lemma 4.7, the shear $\sigma^\beta_\lambda(P,R_P)$ is independent of the set of plaques that separate P and R_P inside $\mathcal{P}$, since P and R_P share the ideal vertex g^+. The exact same argument applies for $\sigma_\lambda^\beta(R_Q, Q)$. Furthermore we have

$$\sigma_\lambda^\beta(P, R_P) = \log\left|\beta(g^+, y_P, x_P, y'_P)\beta(g^+, y'_P, y_P, g^-)\right|,$$
$$\sigma_\lambda^\beta(R_Q, Q) = \log\left|\beta(g^-, y_Q, x_Q, y'_Q)\beta(g^-, y'_Q, y_Q, g^+)\right|.$$

On the other hand, the plaques R_P and R_Q share the boundary component $[g^+, g^-]$ and their shear satisfies

$$\sigma^\beta(R_P, R_Q) = \log\left|\beta(g^+, g^-, y'_P, y'_Q)\right|.$$

By applying Lemma B.1 to the 6-tuple $a = g^+$, $b = y_P$, $c = x_P$, $d = g^-$, $e = y'_Q$, and $x = y'_P$, we obtain

$$\sigma_\lambda^\beta(P, R_P) + \sigma_\lambda^\beta(R_P, R_Q) = \log\left|\beta(g^+, y_P, x_P, g^-)\beta(g^+, g^-, y_P, y'_Q)\right|.$$

Combining this identity with the expression for $\sigma_\lambda^\beta(R_Q, Q)$ we deduce

$$\begin{aligned}
&\sigma_\lambda^\beta(P, R_P) + \sigma_\lambda^\beta(R_P, R_Q) + \sigma_\lambda^\beta(R_Q, Q) \\
&\quad = \log\left|\beta(g^+, y_P, x_P, g^-)\beta(g^+, g^-, y_P, y'_Q)\beta(g^-, y_Q, x_Q, y'_Q)\beta(g^-, y'_Q, y_Q, g^+)\right| \\
&\quad = \log\left|\beta(g^+, y_P, x_P, g^-)\beta(g^-, g^+, y'_Q, y_P)\beta(g^-, y_Q, x_Q, y'_Q)\beta(g^-, y'_Q, y_Q, g^+)\right| \\
&\quad = \log\left|\beta(g^+, y_P, x_P, g^-)\beta(g^-, g^+, y_Q, y_P)\beta(g^-, y_Q, x_Q, g^+)\right|
\end{aligned}$$

where in the second equality we applied relation (3), and in the last line we applied again Lemma B.1 to the 6-tuple $a = g^+$, $b = y_P$, $c = x_P$, $d = g^-$, $e = y'_Q$, and $x = y'_P$. This concludes the proof of the statement. □

We now provide a proof of Lemma 5.4, which again follows easily from the symmetries of cross ratios:

Proof of Lemma 5.4. Let u denote the vertex of P that is not an endpoint of ℓ_P, and by v the vertex of Q that is not an endpoint of ℓ_Q. Then the left-hand side of the equation can be expressed as

$$\left|\log\left|\frac{\beta(\ell_P^+, \ell_P^-, u, \ell_Q^-)\beta(\ell_P^+, \ell_Q^-, \ell_P^-, \ell_Q^+)\beta(\ell_Q^+, \ell_Q^-, \ell_P^+, v)}{\beta(\ell_P^+, \ell_P^-, u, \ell_Q^+)\beta(\ell_Q^+, \ell_P^-, \ell_P^+, \ell_Q^-)\beta(\ell_Q^+, \ell_Q^-, \ell_P^-, v)}\right|\right|$$

Applying the third symmetry in (1), we obtain the identities

$$\left|\beta(\ell_P^+, \ell_P^-, u, \ell_Q^-)\right| = \left|\beta(\ell_P^+, \ell_P^-, u, \ell_Q^+)\beta(\ell_P^+, \ell_P^-, \ell_Q^+, \ell_Q^-)\right|,$$
$$\left|\beta(\ell_Q^+, \ell_Q^-, \ell_P^-, v)\right| = \left|\beta(\ell_Q^+, \ell_Q^-, \ell_P^-, \ell_P^+)\beta(\ell_Q^+, \ell_Q^-, \ell_P^+, v)\right|.$$

By replacing these terms in the expression above we obtain

$$\begin{aligned}
\left|\sigma_d^\beta(P, Q) - \sigma_{d'}^\beta(P, Q)\right| &= \left|\log\left|\frac{\beta(\ell_P^+, \ell_P^-, \ell_Q^+, \ell_Q^-)\beta(\ell_P^+, \ell_Q^-, \ell_P^-, \ell_Q^+)}{\beta(\ell_Q^+, \ell_P^-, \ell_P^+, \ell_Q^-)\beta(\ell_Q^+, \ell_Q^-, \ell_P^-, \ell_P^+)}\right|\right| \\
&= \left|\log\left|\beta(\ell_P^+, \ell_P^-, \ell_Q^+, \ell_Q^-)^2\beta(\ell_P^+, \ell_Q^-, \ell_P^-, \ell_Q^+)^2\right|\right|
\end{aligned}$$

where in the last equality we made use of (2) and (3). The desired expression then follows by applying the fourth relation in (1) and (2). (Notice that $\beta(g^+, h^+, h^-, g^-) > 1$ for any pair of coherently oriented geodesics g, h that share no endpoint.) □

We are now left with the proof of Lemma 4.10, which directly relates β-periods and β-shears:

Proof of Lemma 4.10. Let $x, y, \gamma^{\pm} \in \partial\Gamma$ be the vertices of P in counterclockwise order along $\partial\Gamma$. By Lemma 4.7, we have

$$\sigma^{\beta}_{\lambda}(P, \gamma P) = \log \left|\beta(\gamma^{\pm}, x, y, \gamma y)\beta(\gamma^{\pm}, \gamma y, x, \gamma x)\right|.$$

The proof of the relation appearing in the statement now reduces to a careful applications of the symmetries of the cross ratio β (see in particular (1), (2)). In what follows, we express the chain of equalities that leads to the proof, reporting on the right the relations that are applying (the symbol $(1.n)$ refers to the n-th symmetry of β appearing in (1)):

$$\begin{aligned}
& |\beta(\gamma^{\pm}, x, y, \gamma y)\beta(\gamma^{\pm}, \gamma y, x, \gamma x)| \\
\text{(1.4)} \qquad & = \left|\beta(\gamma^{\pm}, x, \gamma x, \gamma y)\beta(\gamma^{\pm}, x, y, \gamma x)\beta(\gamma^{\pm}, \gamma y, x, \gamma x)\right| \\
\text{(1.5)} \qquad & = \left|\beta(\gamma^{\pm}, \gamma x, x, \gamma y)\beta(\gamma^{\pm}, x, y, \gamma x)\right| \\
(\Gamma\text{-inv.}) \qquad & = \left|\beta(\gamma^{\pm}, x, \gamma^{-1}x, y)\beta(\gamma^{\pm}, x, y, \gamma x)\right| \\
\text{(1.4)} \qquad & = \left|\beta(\gamma^{\pm}, x, \gamma^{-1}x, \gamma x)\right| \\
\text{(1.4)} \qquad & = \left|\beta(\gamma^{\pm}, x, \gamma^{\mp}, \gamma x)\beta(\gamma^{\pm}, \gamma x, x, \gamma^{\mp})\right| \\
\text{(1.5)} \qquad & = \left|\beta(\gamma^{\pm}, \gamma^{\mp}, x, \gamma x)\right| \\
\text{(2)} \qquad & = \left|\beta(\gamma^{+}, \gamma^{-}, x, \gamma x)\right|^{\pm 1}.
\end{aligned}$$

Taking the logarithm of this relation we obtain the identity $\sigma^{\beta}(P, \gamma P) = \pm L_{\beta}(\gamma)$, as desired. □

Appendix C. On divergence radius functions

In our construction of β-shear cocycles, we made use of a series of technical properties satisfied by divergence radius functions, described in Lemmas 5.1, 5.2, and 5.3. We remark that the statement of Lemma 5.1 already appeared in the work of Bonahon and Dreyer [BD17]. The underlying strategy of proof is essentially the same as the one described by Bonahon in [Bon96, Lemmas 3,5]. However, since the work [Bon96] uses a definition of divergence radius function that is weaker than the one we introduced in Section 5.1, we describe how to adapt the argument accordingly. The strategy of proof is in fact particularly useful to understand the dependence of the constants, as asserted in Lemma 5.2.

We start by fixing some hyperbolic metric on Σ and a train track τ that carries a maximal lamination λ. Furthermore, we introduce the following terminology: If B is a branch of τ, we define the *width* of B (with respect to the chosen metric) to be the distance between the components of the horizontal boundary of $\tilde{b}$, for some lift $\tilde{B}$ of B in $\widetilde{\Sigma}$. Similarly, the *length* of B is defined as the distance between the components of the vertical boundary of $\tilde{B}$, for some lift $\tilde{B}$ of B.

Proof of Lemma 5.1. We start by selecting suitable constants $M, A_0, \theta > 0$, which depends exclusively on the train track τ and the fixed hyperbolic structure X:

- We select $M < 1$ so that every branch of the train track τ has length within M and M^{-1}.

- We let $A_0 > 1$ be such that every component of the vertical boundary of τ (compare with the terminology introduced in Section 2.4.4) has endpoints at distance $\geqslant A_0^{-1}$, and such that every branch of τ has width $\leqslant A_0$.
- We choose $\theta \in (0, \pi/2)$ a lower bound for the intersection angle between the geodesic arc k and the leaves of the lamination λ.

Consider now the following situation: Let ℓ and ℓ' be two distinct asymptotic geodesics in $(\widetilde{\Sigma}, \widetilde{X})$, and let u be their common endpoint in $\partial\Gamma$. Consider a geodesic segment k' joining a point $p \in \ell$ to a point of ℓ', and assume that the angles between k' and ℓ, ℓ' satisfy

$$\theta \leqslant \left|\angle(k', \ell)\right|, \left|\angle(k', \ell)\right| \leqslant \pi - \theta.$$

Finally, select a parametrization by arc length of the geodesic $\ell = \ell(t)$ such that $\ell(t)$ tends to u as $t \to -\infty$ and $\ell(0) = p$, and assume that there exists some positive $t > 0$ for which $\ell(t)$ satisfies

$$A_0^{-1} \leqslant d_{\tilde{X}}(\ell(t), \ell') \leqslant A_0.$$

A simple computation in the upper half plane model of $\mathbb{H}^2$ then shows that there exists a constant $A > 0$, which depends only on A_0 and θ, such that

$$A^{-1} e^{-t} \leqslant L_{\tilde{X}}(k') \leqslant A e^{-t}. \tag{19}$$

We now have all the technical ingredients for the proof the desired statement: First recall the definition of the divergence radius function $r : \mathcal{P}_{PQ} \to \mathbb{N}$ outlined in Section 5.1, select any plaque $R \in \mathcal{P}_{PQ}$, and denote by $s = s_R$ the switch of the lift of the train track τ that separates the branches $\widetilde{B}_{r(R)-1}$ and $\widetilde{B}_{r(R)}$ (see Section 5.1 for the necessary terminology). By definition of the divergence radius function r, the boundary leaves ℓ_R and ℓ'_R of R that separate P from Q fellow travel along the branches $\widetilde{B}_n$ for all $n < r(R)$, and then take different turns at the switch s. Indeed, while the leaf ℓ_R crosses s to then enter in the branch $\widetilde{B}_{r(R)}$, the leaf ℓ'_R passes through the unique branch of $\tilde{\tau}$ adjacent to s and different from $\widetilde{B}_{r(R)-1}$ and $\widetilde{B}_{r(R)}$.

Now, if γ_R denotes the subsegment of ℓ_R that joins $k \cap R$ to the switch s of the train track $\tilde{\tau}$, then by the choice of M we have

$$M\, r(R) \leqslant L_{\tilde{X}}(\gamma_R) \leqslant M^{-1}\, r(R) \tag{20}$$

whenever $r(R) > 1$. Moreover, if we travel along the geodesic ℓ_R at distance $\ell(\gamma_R)$ towards the positive direction of ℓ_R, the geodesics ℓ_R and ℓ'_R are at distance $d(s \cap \ell_R, s \cap \ell'_R) \in (A_0^{-1}, A_0)$ by our initial choices. (Notice that the switch s contains exactly one connected component of the vertical boundary of $\tilde{\tau}$, whose endpoints are at distance between A_0^{-1} and A_0.) We then are in right setting to apply relation (19) to $k' = k \cap R$, $\ell = \ell_R$, $\ell' = \ell'_R$ and $t = L_{\tilde{X}}(\gamma_R)$: consequently we conclude that

$$A^{-1}\, e^{-L_{\tilde{X}}(\gamma_R)} \leqslant L_{\tilde{X}}(k \cap R) \leqslant A\, e^{-L_{\tilde{X}}(\gamma_R)}$$

Combining this comparison with relation (20), we obtain the control appearing in property (1) of Lemma 5.1 for all $r(R) > 1$. Now, up to enlarging the multiplicative constant $A > 0$ to obtain a bound from above of the diameter of every complementary region of τ in X, we can then make sure that *(1)* holds for every $R \in \mathcal{P}_{P,Q}$.

The proof of the second bound appearing in (2) is a simple generalization of [Bon96, Lemma 4]: in his work Bonahon showed that, if k_0 is a geodesic arc transverse to $\tilde{\lambda}$ that projects onto an *embedded* arc in Σ, then the number of plaques $R \in \mathcal{P}_{PQ}$ satisfying $r_{k_0}(R) = n$ is bounded above by an explicit function $N_0 = N_0(\Sigma)$ that depends only on the topology of Σ. For a general geodesic arc k, we can argue as follows: there exists a natural number m such that the arc k can be subdivided into m subsegments $(k_i)_i$ with disjoint interiors and such that every k_i projects onto an embedded geodesic arc in Σ. Then the cardinality of $r_k^{-1}(n)$ is bounded above by $N := mN_0(\Sigma)$. Observe also that, if ε_0 is equal to the injectivity radius of X, then $m \leqslant \ell(k)/\varepsilon_0$. □

From the proof provided above, and in particular from the definition of the constants $A, M, N > 0$, Lemma 5.2 easily follows:

Proof of Lemma 5.2. We fix a hyperbolic structure X on Σ, and we select a train track τ that carries λ and a X-geodesic ark k joining the interiors of the plaques P and Q. We denote by $M, A_0, A, \theta > 0$ the constants introduced in the proof of Lemma 5.1. Up to selecting a smaller $\theta > 0$, we can find a small neighborhood U of λ inside $\mathcal{GL}$ satisfying the following conditions:

- Every $\lambda' \in U$ is carried by τ.
- For every $\lambda' \in U$, the geodesic segment k is transverse to λ' and $\theta > 0$ is a uniform lower bound of the intersection angle between k and λ'.
- the endpoints of k lie in the interior of two distinct plaques P', Q' of λ', for every $\lambda' \in U$.

The constants $M, A_0 > 0$ depends only on the train track τ (and the hyperbolic structure X), and A is determined by A_0 and θ. In particular, A and M satisfy relations (19), (20) for any divergence radius function $r' = r_{X,\tau,\lambda',k} : \mathcal{P}_{P'Q'} \to \mathbb{N}$ associated to a lamination $\lambda' \in U$ and the path k. Relations (19), (20) in turn imply property (1) for all such divergence radius functions r'. Finally, it is immediate from the explicit description of the constant $N > 0$ satisfying property (2) provided in the proof of Lemma 5.1 that we can assume N to be uniform in $\lambda' \in U$. □

The only technical statement left to prove is Lemma 5.3. For its proof, we will make use of an elementary lemma of planar hyperbolic geometry. In order to recall its statement, we need to introduce some notation.

If $X \in \mathcal{T}$ is a hyperbolic structure and $(\widetilde{\Sigma}, \widetilde{X}) \cong \mathbb{H}^2$ is a fixed identification between the universal cover of Σ and the hyperbolic plane determined by X, then we select d_∞ a fixed Riemannian distance on $\partial\Gamma \cong \partial\mathbb{H}^2$. The choice of the metric d_∞ determines a distance (which we will continue to denote with abuse by d_∞) on the space of oriented geodesics of $\widetilde{\Sigma}$, by setting

$$d_\infty(g, h) := d_\infty(g^+, h^+) + d_\infty(g^-, h^-)$$

for any pair of oriented geodesics g and h. Then we have:

Lemma C.1. *Let λ be a maximal geodesic lamination on Σ, and let P and Q be two distinct plaques of λ. For any geodesic segment k joining two points in the interior of P and Q, respectively, we can find a constant $C = C(k) > 0$ such that, for every plaque $R \in \mathcal{P}_{PQ}$*

$$C^{-1}\, d_\infty(\ell_R, h_R) \leqslant L_{\tilde{X}}(k \cap R) \leqslant C\, d_\infty(\ell_R, h_R),$$

where ℓ_R, ℓ'_R denote the boundary leaves of R that separate P from Q.

We are now ready to prove Lemma 5.3:

Proof of Lemma 5.3. By property *(1)* of Lemma 5.1, there exist positive constants $A, A', M, M' > 0$ such that

$$A^{-1} e^{-M^{-1} r(R)} \leqslant L_{\tilde{X}}(k \cap R) \leqslant A e^{-M r(R)}$$
$$(A')^{-1} e^{-(M')^{-1} r'(R)} \leqslant L_{\tilde{X}}(k' \cap R) \leqslant A' e^{-M' r'(R)}$$

for every $R \in \mathcal{P}_{PQ}$. On the other hand, by Lemma C.1, there exist constants $S, T > 0$ such that for every $R \in \mathcal{P}_{PQ}$ we have

$$W^{-1} d_\infty(\ell_R, h_R) \leqslant L_{\tilde{X}}(k \cap R) \leqslant W\, d_\infty(\ell_R, h_R),$$
$$(W')^{-1} d_\infty(\ell_R, h_R) \leqslant L_{\tilde{X}}(k' \cap R) \leqslant W'\, d_\infty(\ell_R, h_R).$$

By combining the inequalities above, we obtain

$$\begin{aligned} e^{M r(R)} &\leqslant \frac{A}{L_{\tilde{X}}(k \cap R)} \\ &\leqslant \frac{AW}{d_\infty(\ell_R, h_R)} \\ &\leqslant \frac{AWW'}{L_{\tilde{X}}(k' \cap R)} \\ &\leqslant AA'WW' e^{M' r'(R)}, \end{aligned}$$

which implies the upper bound appearing in the statement with suitable choices of $H, K > 0$. By exchanging the roles of r and r' in the argument above we determine the existence of the lower bound. □

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