LENGTH FUNCTIONS IN TEICHMÜLLER AND ANTI DE SITTER GEOMETRY

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Abstract. We establish a link between the behavior of length functions on Teichmüller space and the geometry of certain anti de Sitter 3-manifolds. As an application, we give new purely anti de Sitter proofs of results of Teichmüller theory such as (strict) convexity of length functions along shear paths and geometric bounds on their first and second variations along earthquakes. Along the way, we provide shear-bend coordinates for Mess' anti de Sitter 3-manifolds.

1. Introduction

The space $\mathcal{T}$ of hyperbolic metrics on a closed orientable surface Σ of genus $g \geqslant 2$ up to isotopy, known as *Teichmüller space*, is an object that appears ubiquitously as a space of parameters but also as a geometric object.

Comparing different hyperbolic metrics on Σ according to various measurements of distortion endows $\mathcal{T}$ with a wealth of geometry. An example is the *Lipschitz distortion* which corresponds to the so-called *Thurston's asymmetric metric*. Thurston proves in [19] that given hyperbolic metrics g_X, g_Y on Σ we have

$$\min_{f \text{ homotopic to Id}} \{\mathrm{Lip}(f) \mid f : (\Sigma, g_X) \to (\Sigma, g_Y)\} = \sup_{\gamma \in \pi_1(\Sigma) - \{1\}} \frac{L_Y(\gamma)}{L_X(\gamma)}$$

where $L_X(\gamma), L_Y(\gamma)$ is the length of the geodesic representatives of γ with respect to g_X, g_Y.

This phenomenon of expressing the measurement of distortion in terms of *length spectra* $L_Z(\bullet)$ is not exclusive of the Thurston metric, for example also the *Teichmüller* and *Weil-Petersson metrics* on $\mathcal{T}$ have this property.

It is therefore important to understand better how length functions behave on Teichmüller space. Often, this behavior is related to certain *geometric structures* on low dimensional manifolds. A celebrated example is the relation between *quasi-Fuchsian* hyperbolic 3-manifolds and Teichmüller geodesics discovered by Minsky [16].

Following an analogy between quasi-Fuchsian 3-manifolds and the so-called *Mess 3-manifolds*, in this article we bring together:

- 3-dimensional anti de Sitter geometry.
- Convexity of length functions along *shear paths* and *earthquakes.*

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In particular, we use the global scale geometry of Mess manifolds to give a proof of (strict) convexity of length functions. Using the same bridge, we also develop geometric bounds for the first and second variations on those functions along *earthquakes*. Our methods are inspired from ideas in 3-dimensional hyperbolic geometry.

1.1. Anti de sitter geometry.

Anti de Sitter geometry in dimension 3. is the geometry of $\mathbb{H}^{2,1} := \mathrm{PSL}_2(\mathbb{R})$ endowed with its natural pseudo-Riemannian metric of signature $(2,1)$. The link between Teichmüller theory and anti de Sitter 3-manifolds comes from the basic fact that the group of symmetries of this space is

$$\mathrm{Isom}_0(\mathrm{PSL}_2(\mathbb{R})) = \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$$

where $(A,B)\cdot X := AXB^{-1}$ and, at the same time, $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$. A vast literature explores various aspects of this relation starting with the seminal work of Mess [15] (for a survey on the topic and recent developments see [9]).

Mess representations. Let Σ be a closed orientable surface of genus $g \geqslant 2$ that we fix once and for all. We denote by $\Gamma := \pi_1(\Sigma)$ its fundamental group.

We realize the Teichmüller space $\mathcal{T}$ of hyperbolic metrics on Σ up to isotopy as a component of the representation space

$$\mathcal{T} \subset \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))/\mathrm{PSL}_2(\mathbb{R})$$

by associating to each hyperbolic structure X its holonomy representation $\rho_X : \Gamma \to \mathrm{PSL}_2(\mathbb{R})$.

Given $X, Y \in \mathcal{T}$ we can consider the corresponding *Mess representation*

$$\rho_{X,Y} = (\rho_X, \rho_Y) : \Gamma \to \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R}).$$

The group $\rho_{X,Y}(\Gamma)$ acts on $\mathbb{H}^{2,1}$ *convex cocompactly*, meaning that:

- There is an equivariant boundary map $\xi : \partial\Gamma \to \partial\mathrm{PSL}_2(\mathbb{R}) = \mathbb{P}\{A \in M_2(\mathbb{R}) \,|\mathrm{rank}(A) = 1\}$ whose image $\xi(\partial\Gamma) = \Lambda_{X,Y}$ has the property that for every $a, b, c \in \partial\Gamma$ the subspace $\mathbb{P}\{\mathrm{Span}\{\xi(a), \xi(b), \xi(c)\}\} \cap \mathrm{PSL}_2(\mathbb{R})$ is a *spacelike plane*, that is, it is isometric to $\mathbb{H}^2$.
- There is a canonical $\rho_{X,Y}(\Gamma)$-invariant properly convex open subset $\Omega_{X,Y} \subset \mathrm{PSL}_2(\mathbb{R})$ on which the action is properly discontinuous.
- We have $\partial\Omega_{X,Y} \cap \partial\mathrm{PSL}_2(\mathbb{R}) = \Lambda_{X,Y}$ and the group $\rho_{X,Y}(\Gamma)$ acts cocompactly on the convex hull $\mathcal{CH}_{X,Y} \subset \Omega_{X,Y}$ of $\Lambda_{X,Y}$.

In order to study the geometry of Mess representations, we will use *laminations* and *pleated surfaces* as we introduced in [14]. Let us briefly recall the construction.

Laminations and pleated surfaces. A geodesic lamination on a hyperbolic surface X is a $\rho_X(\Gamma)$-invariant closed subset $\lambda \subset \mathbb{H}^2$ that can be decomposed as a disjoint union of complete geodesics, the *leaves* of the lamination. The complementary regions $\mathbb{H}^2 - \lambda$ are ideal polygons, the *plaques* of the lamination. The lamination is called *maximal*, if all the plaques are ideal triangles. Conveniently, the data of a geodesic lamination can be encoded, by recording the endpoints of the leaves, as a Γ-invariant closed subset of the space of geodesics

$$\{(x,y)\in\partial\Gamma\times\partial\Gamma\mid x\neq y\}/(x,y)\sim(y,x).$$

This is the point of view that we adopt.

The boundary map $\xi:\partial\Gamma\to\Lambda_{X,Y}$ and the property of the curve $\Lambda_{X,Y}$ allow us to associate with every maximal lamination λ a *geometric realization*

$$\hat{\lambda}:=\bigcup_{(a,b)\in\lambda}[\xi(a),\xi(b)]\subset\mathcal{CH}_{X,Y}$$

and a *pleated set*

$$\hat{S}_\lambda:=\hat{\lambda}\cup\bigcup_{\Delta(a,b,c)\subset\mathbb{H}^2-\lambda}\Delta(\xi(a),\xi(b),\xi(c))\subset\mathcal{CH}_{X,Y}.$$

Here $[\xi(a),\xi(b)]$ denotes the *spacelike* geodesic with endpoints $\xi(a),\xi(b)$ while $\Delta(\xi(a),\xi(b),\xi(c))$ is the ideal *spacelike* triangle contained in the spacelike plane $\mathbb{P}\{\mathrm{Span}\{\xi(a),\xi(b),\xi(c)\}\}\cap\mathrm{PSL}_2(\mathbb{R})$ with vertices $\xi(a),\xi(b),\xi(c)$.

We have the following structural result:

Theorem (Theorems A, B, and C of [14])**.** *Let $\rho_{X,Y}$ be a Mess representation. Consider a maximal lamination $\lambda\subset\Sigma$. Let $\hat{S}_\lambda\subset\mathcal{CH}_{X,Y}\subset\Omega_{X,Y}$ be the corresponding* pleated set*. Then:*

1. *$\hat{S}_\lambda$ is an* acausal *Lipschitz disk with boundary $\Lambda_{X,Y}$. For every pair points $x,y\in\hat{S}_\lambda$ the geodesic $[x,y]$ joining them is spacelike. In particular $\hat{S}_\lambda$ has a pseudo-metric $d_{\mathbb{H}^{2,1}}(x,y):=\ell[x,y]$.*
2. *There is an* intrinsic hyperbolic structure *$Z_\lambda\in\mathcal{T}$ associated to λ with holonomy $\rho_\lambda:\Gamma\to\mathrm{PSL}_2(\mathbb{R})$. For every $\mu\in\mathcal{ML}_\lambda=\{\mu\in\mathcal{ML}\mid\mathrm{support}(\mu)\subset\lambda\}$ we have $L_{\rho_\lambda}(\mu)=L_\rho(\mu)$.*
3. *There exists a $(\rho_{X,Y}-\rho_\lambda)$-equivariant homeomorphism $\hat{f}:\hat{S}_\lambda\to\mathbb{H}^2$ which is 1-Lipschitz in the sense that $d_{\mathbb{H}^{2,1}}(x,y)\geqslant d_{\mathbb{H}^2}(\hat{f}(x),\hat{f}(y))$ and is totally geodesic on each leaf and plaque.*

Furthermore, we have

$$L_{\rho_\lambda}(\gamma)\leqslant L_{\rho_{X,Y}}(\gamma)$$

for every $\gamma\in\Gamma-\{1\}$, with strict inequality if and only if γ intersects the bending locus *of S_λ.*

Mess [15], inspired by work of Thurston (Chapter 8 of [18]), observes that $\partial^\pm\mathcal{CH}_{X,Y}$ is the pleated set $\hat{S}_{\lambda^\pm}$ of a lamination $\lambda^\pm$ and that measuring the total turning angle along paths $\alpha:I\to\partial^\pm\mathcal{CH}_{X,Y}$ endows $\lambda^\pm$ with a natural transverse measure, the *bending measure*. Then he shows that the surfaces

X, Y and $Z_{\lambda^+}, Z_{\lambda^-}$ are related by the following diagram

$$\begin{tikzcd}
& Z_{\lambda^+} \arrow[dl, "E^r_{\lambda^+}"'] \arrow[dr, "E^l_{\lambda^+}"] & \\
X & & Y \\
& Z_{\lambda^-} \arrow[ul, "E^r_{\lambda^-}"] \arrow[ur, "E^l_{\lambda^-}"'] &
\end{tikzcd}$$

where $E^l_{\lambda^+}, E^l_{\lambda^-}, E^r_{\lambda^+}, E^r_{\lambda^-}$ are the *left* and *right earthquakes* induced by the measured laminations λ^+, λ^-.

Recall that by work of Bonahon [6] and Thurston [19], for every maximal geodesic lamination λ of Σ, the Teichmüller space $\mathcal{T}$ can be realized as an open convex cone in a finite dimensional $\mathbb{R}$-vector space $\mathcal{H}(\lambda;\mathbb{R})$ via the so-called *shear coordinates* $\sigma_\lambda : \mathcal{T} \to \mathcal{H}(\lambda;\mathbb{R})$. Generalizing Mess, we prove:

Theorem 1. *Let $\rho_{X,Y}$ be a Mess representation. Consider a maximal lamination $\lambda \subset \Sigma$. Let $S_\lambda = \hat{S}_\lambda/\rho_{X,Y}(\Gamma)$ be the corresponding pleated surface. Then, in* shear coordinates $\mathcal{T} \subset \mathcal{H}(\lambda;\mathbb{R})$ *for λ we have:*

(1) The intrinsic hyperbolic structure Z_λ *of S_λ satisfies*

$$\sigma_\lambda(Z_\lambda) = \frac{\sigma_\lambda(X) + \sigma_\lambda(Y)}{2}.$$

(2) The intrinsic bending cocycle β_λ *of S_λ satisfies*

$$\beta_\lambda = \frac{\sigma_\lambda(X) - \sigma_\lambda(Y)}{2}.$$

Length functions. We now come to the main novelty of this article, namely, the anti de Sitter perspective on length functions in Teichmüller theory.

Let us first recall the following: For every element $\gamma \in \Gamma - \{1\}$ the isometry $\rho_{X,Y}(\gamma)$ has a unique pair of invariant spacelike lines: The *axis* $\ell \subset \mathcal{CH}_{X,Y}$ on which it acts by translations by $L_\rho(\gamma) = (L_X(\gamma) + L_Y(\gamma))/2$ and the dual axis $\ell^* \subset \mathbb{H}^{2,1} - \Omega_{X,Y}$ on which it acts by translations by $\theta_\rho(\gamma) = (L_X(\gamma) - L_Y(\gamma))/2$.

We prove:

Theorem 2. *Let $\rho_{X,Y}$ be a Mess representation. Let $\gamma \in \Gamma - \{1\}$ be a non-trivial element, denote by $\ell \subset \mathcal{CH}_{X,Y}$ the axis of $\rho_{X,Y}(\gamma)$. Let $\lambda \subset \Sigma$ be a maximal lamination, let $Z_\lambda \in \mathcal{T}$ be the intrinsic hyperbolic structure on $\hat{S}_\lambda/\rho_{X,Y}(\Gamma)$ where $\hat{S}_\lambda \subset \mathcal{CH}_{X,Y}$ is the pleated set associated with λ.*

(a) Let δ be the maximal timelike distance of ℓ from $\hat{S}_\lambda$. Then:

$$\cosh(L_{Z_\lambda}(\gamma)) \leqslant \cos(\delta)^2 \cosh(L_\rho(\gamma)) + \sin(\delta)^2 \cosh(\theta_\rho(\gamma)).$$

(b) Let $\delta^\pm$ be the maximal timelike distance of ℓ from $\lambda^\pm$. Then:

$$\cosh\big(i(\lambda^\pm, \gamma)\big) \leqslant \sin\big(\delta^\pm\big)^2 \cosh(L_\rho(\gamma)) + \cos\big(\delta^\pm\big)^2 \cosh(\theta_\rho(\gamma)),$$

and

$$i(\lambda^{\pm}, \gamma) \geqslant \cos\big(\delta^{\pm}\big)^2 \theta_\rho(\gamma).$$

Here $i(\bullet, \bullet)$ *is the* geometric intersection *form.*

When combined, the previous results (Theorem 1 and Theorem 2) give a purely anti de Sitter proof of (strict) convexity of length functions in shear coordinates, recovering simultaneously results of Bestvina, Bromberg, Fujiwara, and Souto [4], and Théret [17]:

Theorem 3. *Let* $\lambda \subset \Sigma$ *be a maximal lamination. The following holds:*

- *(a) Let* $\gamma \in \Gamma - \{1\}$ *be a non-trivial loop. The length function* $L_\gamma : \mathcal{T} \subset \mathcal{H}(\lambda; \mathbb{R}) \to (0, \infty)$ *is convex. Moreover, convexity is strict if* γ *intersects essentially every leaf of* λ.
- *(b) Let* $\gamma \in \mathcal{ML}$ *be a measured lamination. The length function* $L_\gamma : \mathcal{T} \subset \mathcal{H}(\lambda; \mathbb{R}) \to (0, \infty)$ *is convex. Furthermore, convexity is strict if the support of* γ *intersects transversely each leaf of* λ.

Note that (b) does not imply (a): In (a) the loop γ does not necessarily represent a simple curve.

In the case of *earthquakes*, they allow us to get the following infinitesimal geometric bounds. We should mention that these bounds can also be deduced from work of Kerckhoff [13] and Wolpert [21] respectively.

Theorem 4. *Let* $\lambda \in \mathcal{ML}$ *be a measured lamination. Let* $E_\lambda : [a, b] \to \mathcal{T}$ *be an earthquake path driven by* λ. *Set* $L_\gamma(t) := \ell_\gamma(E_\lambda(t))$. *Then:*

- *(i) For every* $\gamma \in \Gamma - \{1\}$ *we have:*

$$\left|\dot{L}_\gamma\right| \leqslant i(\gamma, \lambda).$$

- *(ii) For every* $\gamma \in \Gamma - \{1\}$ *we have:*

$$\ddot{L}_\gamma \geqslant \frac{1}{\sinh(L_\gamma)} \left|\dot{L}_\gamma\right| \left(i(\gamma, \lambda) - \left|\dot{L}_\gamma\right|\right).$$

Anti de Sitter proofs. We now briefly discuss the main new ideas and ingredients that go into the anti de Sitter proofs.

Theorem 2. The idea is that as we move a closed geodesic $\gamma \subset M_{X,Y}$ orthogonally along timelike directions, the length shrinks. Heuristically speaking: Every closed geodesic $\gamma \subset M_{X,Y}$ is the core of an (immersed) anti de Sitter annulus $A_\gamma \subset M_{X,Y}$ whose intrinsic metric has the form $\mathrm{d}s^2 = -\mathrm{d}t^2 + \sin(t)^2 \mathrm{d}\ell^2$. Hence, the length of $\gamma(s) = (0, s)$ (in (t, ℓ) coordinates) contracts as we move it away from the core $\{t = 0\}$ along orthogonal timelike directions. In the proof of the theorem we make precise some aspects of this picture. In particular, we understand how various avatars of A_γ intersect the pleated surfaces $\hat{S}_\lambda/\rho_{X,Y}(\Gamma)$ and $\partial^{\pm}\mathcal{CC}(M_{X,Y}) = \partial^{\pm}\mathcal{CH}_{X,Y}/\rho_{X,Y}(\Gamma)$.

Theorem 3. (Strict) convexity is equivalent to the (strict) inequality

$$L_\gamma\left(\frac{X+Y}{2}\right) \leqslant \frac{L_\gamma(X)+L_\gamma(Y)}{2}$$

for every $X, Y \in \mathcal{T} \subset \mathcal{H}(\lambda; \mathbb{R})$. We note that the right hand side is $L_\rho(\gamma)$ for $\rho_{X,Y}$ and the left hand side is, by Theorem 1, $L_{Z_\lambda}(\gamma)$ where Z_λ is the hyperbolic structure on the pleated surface $\hat{S}_\lambda/\rho(\gamma)$ associated with λ and ρ. The inequality is then a consequence of part (a) of Theorem 2. The inequality is not strict exactly when $\delta = 0$ which happens if and only if $\ell \subset \hat{S}_\lambda$. This is possible if and only if γ does not intersect the *bending locus.*

The proof for laminations requires a significantly more refined argument based on the following heuristic principle: Every time ℓ passes at timelike distance $\delta > 0$ from $\hat{S}_\lambda$ it creates a gap of size $\kappa > 0$ between $L_{Z_\lambda}(\gamma)$ and $L_\rho(\gamma)$.

Theorem 4. The idea is to analyze the geometry of the representations $\rho_t := \rho_{Z_{-t},Z_t}$ where $Z_t = E^l_\lambda(t)$ as $t \to 0$. Notice that, by Theorem 1, the bending lamination on $\partial^+\mathcal{CH}_{Z_{-t},Z_t}$ is $\lambda^+_t = t\lambda$ and the hyperbolic structure is constant $Z^+_t = Z$. The main tool is again Theorem 2.

Part (i) is a consequence of the fact that ρ_{Z_{-t},Z_t} is converging to a Fuchsian representation $\rho_{Z,Z}$ which preserves a totally geodesic plane $\mathcal{CH}_{Z,Z} = \mathbb{H}^2$. Since $\mathcal{CH}_{Z_{-t},Z_t} \to \mathcal{CH}_{Z,Z}$, this implies that $\delta^+_t \to 0$. By part (b) of Theorem 2, we have

$$i(\lambda^+, \gamma) \geqslant \cos\left(\delta^+_t\right)^2 |\theta_{\rho_t}(\gamma)|/t$$

and the right hand side converges to $\left|\dot{L}_\gamma\right|$.

Part (ii) is a consequence of the following quantitative relation obtained by combining the inequalities of part (a) and part (b) of Theorem 2:

$$\cosh\left(t \cdot i(\lambda^+, \gamma)\right) - \cosh(\theta_{\rho_t}(\gamma)) \leqslant \cosh(L_{\rho_t}(\gamma)) - \cosh(L_Z(\gamma)).$$

The conclusion follows from basic analysis, essentially the mean value theorem $\cosh(x) - \cosh(y) = \sinh(\xi)(x-y)$ where $\xi \in [x, y]$ and the fact that $(f(-t) + f(t) - 2f(0))/t^2 \to \ddot{f}$ which we apply to $(L_{\rho_t}(\gamma) - L_Z(\gamma))/t^2 = (L_{Z_{-t}}(\gamma) + L_{Z_t}(\gamma) - 2L_Z(\gamma))/2t^2 \to \ddot{L}_\gamma/2$.

Shear-bend parametrization. As an application of our computations on the intrinsic hyperbolic structure and intrinsic bending of a non-convex pleated surface, we also obtain a shear-bend parametrization of the space of Mess 3-manifolds in the spirit of Bonahon's work [6]: Consider the space of Mess representations

$$\mathcal{MR} := \mathcal{T} \times \mathcal{T} \subset \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{B}))/\mathrm{PSL}_2(\mathbb{B}),$$

where $\mathbb{B} := \mathbb{R}[\tau]/(\tau^2-1) = \mathbb{R}\oplus\tau\mathbb{R}$ denotes the ring of *para-complex numbers.* Let $\mathcal{H}(\lambda; \mathbb{B})$ be the finite dimensional $\mathbb{B}$-module of *transverse cocycles* for λ with values in $\mathbb{B}$ as introduced by Bonahon [6]. Notice that there are natural identifications $\mathrm{PSL}_2(\mathbb{B}) = \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ and $\mathcal{H}(\lambda;\mathbb{B}) = \mathcal{H}(\lambda;\mathbb{R}) \oplus \tau\mathcal{H}(\lambda;\mathbb{R})$. We have:

Theorem 5. *Let $\lambda \subset \Sigma$ be a maximal lamination. Then:*

1. *The map*
$$\begin{aligned} \Phi : \mathcal{MR} &\to \mathcal{H}(\lambda;\mathbb{B}) \\ \rho &\to \sigma_\rho := Z_\lambda + \tau\beta_\lambda \end{aligned}$$
that associates to ρ the shear-bend cocycle *of the unique pleated surface $S_\lambda = \hat{S}_\lambda/\rho(\Gamma)$ associated with λ is an analytic para-complex embedding.*
2. *If $\omega_{\mathrm{Th}}(\bullet,\bullet)$ denotes the* Thurston's symplectic form *on $\mathcal{H}(\lambda;\mathbb{B})$, then*
$$\omega^{\mathbb{B}}_{\mathrm{Th}}(\sigma_\rho,\alpha) = L_\rho(\alpha) + \tau\theta_\rho(\alpha)$$
for every measured lamination $\alpha \in \mathcal{ML}_\lambda = \{\alpha \in \mathcal{ML} \mid \mathrm{support}(\alpha) \subset \lambda\}$ and every $\rho \in \mathcal{MR}$.
3. *The image of the embedding is given by*
$$\begin{aligned} \Phi(\mathcal{MR}) &= \{\sigma + \tau\beta \in \mathcal{H}(\lambda;\mathbb{B}) \mid \sigma+\beta, \sigma-\beta \in \mathcal{T} \subset \mathcal{H}(\lambda;\mathbb{R})\} \\ &= \left\{\sigma + \tau\beta \in \mathcal{H}(\lambda;\mathbb{B}) \;\middle|\; |\omega^{\mathbb{B}}_{\mathrm{Th}}(\sigma+\tau\beta,\bullet)|^2_{\mathbb{B}} > 0 \text{ on } \mathcal{ML}_\lambda\right\}. \end{aligned}$$
Here $|x+\tau y|^2_{\mathbb{B}} = x^2 - y^2$ is the para-complex norm.
4. *The pull-back of ω_{Th} to $\mathcal{MR} = \mathcal{T} \times \mathcal{T}$ coincides with*
$$\Phi^*\omega_{\mathrm{Th}} = c \cdot (\omega_{\mathrm{WP}} \oplus -\omega_{\mathrm{WP}})$$
where $\omega_{\mathrm{WP}}(\bullet,\bullet)$ is the Weil-Petersson symplectic form.

Structure of the article. The paper is organized as follows:

- In Section 2 we recall some basic facts in Teichmüller theory and anti de Sitter 3-dimensional geometry.
- In Section 3 we introduce Mess representations and pleated surfaces and recall some of their properties.
- In Section 4 we compute the intrinsic shear-bend cocycles of pleated surfaces and prove Theorems 1 and 5.
- In Section 5 we study the behavior of length functions for Mess representations and prove Theorem 2.
- In Section 6 we discuss the purely anti de Sitter proofs of Theorems 3 and 4.

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2. TEICHMÜLLER AND ANTI DE SITTER SPACE

In this section we recall the amount of basic Teichmüller theory and anti de Sitter 3-dimensional geometry that we will need in the next sections.

2.1. **Teichmüller theory.**

We start with hyperbolic surfaces and (measured) geodesic laminations.

2.1.1. *Hyperbolic surfaces.* We fix once and for all a closed oriented surface Σ of genus $g \geqslant 2$ and denote by $\Gamma := \pi_1(\Sigma)$ its fundamental group.

Definition 2.1 (Hyperbolic Structures)**.** A *marked hyperbolic structure* on Σ is a quotient $\mathbb{H}^2/\rho_X(\Gamma)$ of the hyperbolic plane $\mathbb{H}^2$ by the image of a faithful and discrete representation $\rho_X : \Gamma \to \mathrm{PSL}_2(\mathbb{R})$, the *holonomy* of the structure. Two marked hyperbolic structures X, X' on Σ are equivalent if their holonomies $\rho_X, \rho_{X'}$ are conjugate.

Definition 2.2 (Teichmüller Space)**.** The *Teichmüller space* of Σ, denoted by $\mathcal{T}$, is the space of equivalence classes of marked hyperbolic structures on Σ. It can be realized as a connected component of the space

$$\mathcal{T} \subset \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))/\mathrm{PSL}_2(\mathbb{R})$$

where $\mathrm{PSL}_2(\mathbb{R})$ acts on the space of representations by conjugation.

2.1.2. *Geodesic laminations.* To study the geometry of hyperbolic surfaces it is quite useful to look at the behavior of their geodesic laminations which are 1-dimensional objects generalizing simple closed geodesics.

Definition 2.3 (Space of Geodesics)**.** The space of (unoriented) geodesics on $\mathbb{H}^2$ is naturally identified with the set of pairs of endpoints

$$\mathcal{G} := \{(x, y) \in \mathbb{RP}^1 \times \mathbb{RP}^1 \,|x \neq y\}/(x, y) \sim (y, x)$$

where x, y corresponds to the line $[x, y]$.

Definition 2.4 (Geodesic Lamination)**.** Let $X = \mathbb{H}^2/\rho_X(\Gamma)$ be a hyperbolic surface. A *geodesic lamination* on X is a $\rho_X(\Gamma)$-invariant closed subset $\lambda \subset \mathbb{H}^2$ which can be expressed as a disjoint union of complete geodesics, the *leaves* of the lamination. The complementary regions $\mathbb{H}^2 - \lambda$ are ideal polygons (with possibly infinitely many sides) and are called the *plaques* of λ. The geodesic lamination λ is *maximal* if all its plaques are ideal triangles. A geodesic lamination on X is completely determined by the endpoints on $\mathbb{RP}^1$ of the leaves which form a closed $\rho_X(\Gamma)$-invariant subset of $\mathcal{G}$. We denote by $\mathcal{GL}$ the space of geodesic laminations and by $\mathcal{GL}_m$ the subspace consisting of maximal ones.

For more details, we address the reader to Chapter I.4 of [10].

2.1.3. *Currents and measured laminations.* Both Teichmüller space and measured laminations can be seen inside the space of geodesic currents as introduced by Bonahon (see [5]). This framework is well-suited to study length functions thanks to presence of a natural geometric intersection form as we now explain.

Definition 2.5 (Geodesic Current)**.** Let $X = \mathbb{H}^2/\rho_X(\Gamma)$ be a hyperbolic surface. A *geodesic current* on X is a $\rho_X(\Gamma)$-invariant locally finite Borel measure on $\mathcal{G}$. We denote by $\mathcal{C}$ the space of geodesic currents.

Definition 2.6 (Closed Geodesics)**.** A basic example of geodesic current is the one associated to a (free homotopy class) of a loop $\gamma \in \Gamma - \{1\}$. It is defined as $\delta_\gamma := \sum_{[\alpha]\in\Gamma/\langle\gamma\rangle} \delta_{\ell_\alpha}$ where ℓ_α is the axis of $\rho_X(\alpha)$ and δ_ℓ is the Dirac mass on the point $\ell \in \mathcal{G}$.

Definition 2.7 (Geometric Intersection)**.** On $\mathcal{C}$ there is a natural *intersection form* $i(\bullet, \bullet)$ defined as follows: Let $\alpha, \beta \in \mathcal{C}$ be geodesic currents. Consider the space of intersecting geodesics $\mathcal{I} := \{(\ell, \ell') \in \mathcal{G} \times \mathcal{G} \,|\, \ell \cap \ell' \neq \emptyset\}$. The group $\rho_X(\Gamma)$ acts properly discontinuously and freely on $\mathcal{I}$. By invariance, the measure $\alpha \times \beta$ on $\mathcal{I}$ descends to a Borel measure on $\mathcal{I}/\rho_X(\Gamma)$. Define $i(\alpha, \beta) := \alpha \times \beta(\mathcal{I}/\rho_X(\Gamma))$. An crucial property of the geometric intersection form $i(\alpha, \beta)$ is that it is continuous in α, β.

Definition 2.8 (Measured Lamination)**.** Let $X = \mathbb{H}^2/\rho_X(\Gamma)$ be a hyperbolic surface. A *measured lamination* on X is a geodesic current $\lambda \in \mathcal{C}$ with $i(\lambda, \lambda) = 0$. We denote by $\mathcal{ML}$ the space of measured laminations.

The *support* of a measured lamination support(λ) is a geodesic lamination (see [5]). We denote by $\mathcal{ML}_\lambda := \{\mu \in \mathcal{ML} \mid \mathrm{support}(\mu) \subset \lambda\}$ the space of measured laminations whose support is contained in λ.

2.1.4. *Length functions.* Every hyperbolic surface X has a (marked) length spectrum $\{L_X(\gamma)\}_{\gamma\in\Gamma-\{1\}}$ given by the lengths of its closed geodesics. Conveniently, Bonahon [5] proves that the length function $L_X(\bullet)$ extends continuously to geodesic currents as follows:

Definition 2.9 (Liouville Current)**.** The *Liouville current* $\mathcal{L}$ on $\mathcal{G}$ is the $\mathrm{PSL}_2(\mathbb{R})$-invariant Borel measure on $\mathcal{G}$ defined by

$$\mathcal{L}([a, b] \times [c, d]) := \beta^{\mathbb{R}}(a, b, c, d).$$

on boxes $[a, b] \times [c, d]$ with $[a, b] \cap [c, d] = \emptyset$ (these sets generate the Borel algebra of $\mathcal{G}$). The Liouville current has the property that

$$L_X(\gamma) = i(\mathcal{L}, \delta_\gamma)$$

for every $\gamma \in \Gamma$ (see [5]). Therefore, $i(\mathcal{L}, \bullet)$ extends continuously the length function $L_X(\bullet)$ to the space of geodesic currents.

2.2. **The** $\mathrm{PSL}_2(\mathbb{R})$ **model of** $\mathbb{H}^{2,1}$**.** The second central object that we discuss is the anti de Sitter 3-space $\mathbb{H}^{2,1}$. We will mostly work in its linear and projective models which we now describe. For more details on the material we present here, we refer the reader to [9].

The group $\mathrm{SL}_2(\mathbb{R})$ sits inside the vector space of 2×2 matrices with real entries $M_2(\mathbb{R})$ as the hyperboloid of vectors of norm -1 for the quadratic form $\langle \bullet, \bullet \rangle$ of signature $(2, 2)$ given by

$$4\langle X, Y \rangle := \det(X) + \det(Y) - \det(X + Y) = -\mathrm{tr}(XY^\star).$$

where $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\star := \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Note that for every $X \in \mathrm{SL}_2(\mathbb{R})$, the restriction of the quadratic form to $T_X\mathrm{SL}_2(\mathbb{R}) = X^\perp$ has signature $(2,1)$ and, hence, induces a $(2,1)$-pseudo-Riemannian metric on $\mathrm{SL}_2(\mathbb{R})$ (experts will have recognized the Killing form of $\mathrm{SL}_2(\mathbb{R})$). The group $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ acts on $M_2(\mathbb{R})$ by left and right multiplications as $(A,B)\cdot X := AXB^{-1}$ and the action is isometric with respect to $\langle \bullet, \bullet\rangle$. Passing to the projectivization, $\mathrm{PSL}_2(\mathbb{R}) \subset \mathbb{P}(M_2(\mathbb{R}))$ we obtain the projective model of anti de Sitter 3-space $\mathbb{H}^{2,1}$.

2.2.1. *Boundary at infinity.* In this model, the boundary at infinity $\partial\mathbb{H}^{2,1}$ of $\mathbb{H}^{2,1}$ identifies with the topological boundary of $\mathrm{PSL}_2(\mathbb{R})$ in $\mathbb{P}(M_2(\mathbb{R}))$

$$\partial\mathrm{PSL}_2(\mathbb{R}) = \{[X] \in \mathbb{P}(M_2(\mathbb{R})) \mid \det(X) = 0\}.$$

Observe that $\partial\mathrm{PSL}_2(\mathbb{R})$ consists of rank one matrices and can be naturally $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$-equivariantly identified with $\mathbb{RP}^1 \times \mathbb{RP}^1$ via the map

$$\begin{aligned} \partial\mathrm{PSL}_2(\mathbb{R}) &\to \mathbb{RP}^1 \times \mathbb{RP}^1 \\ [X] &\to ([\mathrm{Im}(X)], [\mathrm{Ker}(X)]). \end{aligned}$$

2.2.2. *Subspaces.* Totally geodesic subspaces in anti de Sitter 3-space $\mathbb{H}^{2,1}$ are of the form $\mathbb{P}(V) \cap \mathrm{PSL}_2(\mathbb{R})$ where $V \subset M_2(\mathbb{R})$ is a linear subspace intersecting $\mathrm{SL}_2(\mathbb{R})$. In particular we have

- *timelike geodesics* isometric to $\mathbb{R}/\pi\mathbb{Z} \Leftrightarrow V$ 2-plane of signature $(0,2)$.
- *spacelike geodesics* isometric to $\mathbb{R} \Leftrightarrow V$ 2-plane of signature $(1,1)$.
- *spacelike planes* isometric to $\mathbb{H}^2 \Leftrightarrow V$ 3-plane of signature $(2,1)$.

Two distinct points $x, y \in \mathbb{H}^{2,1}$ are joined by:

- A spacelike geodesic if and only if $|\langle x, y\rangle| > 1$.
- A timelike geodesic if and only if $|\langle x, y\rangle| < 1$.

The geodesic $\gamma(t)$ starting at $x \in \mathbb{H}^{2,1}$ with velocity $v \in T_x\mathbb{H}^{2,1} = x^\perp$ is parametrized by

$$\gamma(t) = \begin{cases} \cosh(t)x + \sinh(t)v & \text{if } \langle v, v\rangle = 1, \\ \cos(t)x + \sin(t)v & \text{if } \langle v, v\rangle = -1. \end{cases}$$

2.2.3. *Acausal sets and pseudo-metrics.* The last concept that we need is the one of acausality:

Definition 2.10 (Acausal Set)**.** A subset $S \subset \mathbb{H}^{2,1} \cup \partial\mathbb{H}^{2,1}$ is *acausal* if for every $x, y \in S$ the geodesic $[x, y]$ is spacelike.

Definition 2.11 (Pseudo Metric)**.** On acausal subsets $S \subset \mathbb{H}^{2,1}$ we have a *pseudo-metric* $d_{\mathbb{H}^{2,1}}(\bullet, \bullet)$ defined as follows

$$\cosh(d_{\mathbb{H}^{2,1}}(x, y)) = |\langle x, y\rangle|.$$

Notice that $d_{\mathbb{H}^{2,1}}$ does not satisfy the triangle inequality in general.

3. Mess representations and pleated surfaces

The goal of the section is to describe Mess representations and the geometry of their pleated surfaces. In particular, at the end of the section, we discuss the structure of the boundary of the convex core associated with a Mess representation.

3.1. **Mess representations.**

First of all we introduce the following class:

Definition 3.1 (Mess Representation)**.** Let $X, Y \in \mathcal{T}$ be hyperbolic structures. The *Mess representation* with parameters X, Y is

$$\rho_{X,Y} := (\rho_X, \rho_Y) : \Gamma \to \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$$

where ρ_X, ρ_Y are the holonomy representations of X, Y.

3.1.1. *Boundary maps.* Every Mess representation $\rho_{X,Y}$ comes with a natural equivariant boundary map

$$\xi : \partial\Gamma \to \partial\mathbb{H}^{2,1}$$

It can be described explicitly as follows: Recall that $\partial\mathrm{PSL}_2(\mathbb{R})$ is naturally identified with $\mathbb{RP}^1 \times \mathbb{RP}^1$. Let $h_X, h_Y : \partial\Gamma \to \mathbb{RP}^1$ be the unique ρ_X, ρ_Y-equivariant homeomorphism. The boundary map $\xi : \partial\Gamma \to \mathbb{RP}^1 \times \mathbb{RP}^1$ is just $\xi = (h_X, h_Y)$.

Its image $\xi(\partial\Gamma) = \Lambda_{X,Y}$ is the graph of the unique $(\rho_X - \rho_Y)$-equivariant homeomorphism $h_{X,Y} : \mathbb{RP}^1 \to \mathbb{RP}^1$.

Checking that $\Lambda_{X,Y}$ has the property that for every $a, b, c \in \mathbb{RP}^1$, the 3-space $\mathrm{Span}\{(a, h_{X,Y}(a)), (b, h_{X,Y}(b)), (c, h_{X,Y}(c))\}$ has signature $(2, 1)$ is not difficult: Let us assume without loss of generalities that $a < b < c$. As $h_{X,Y}$ is an orientation preserving homeomorphism, we have $h_{X,Y}(a) < h_{X,Y}(b) < h_{X,Y}(c)$. Hence, up to the action of $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$, we can assume that $a, b, c = h_{X,Y}(a), h_{X,Y}(b), h_{X,Y}(c) = 0, 1, \infty$. Tracing back the identification with $\partial\mathrm{PSL}_2(\mathbb{R})$ we see that

$$(0,0) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ (1,1) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \ (\infty,\infty) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The conclusion follows by an elementary computation.

3.1.2. *Domain of discontinuity.* From the boundary curve $\Lambda_{X,Y} \subset \partial\mathbb{H}^{2,1}$ one constructs a standard open domain:

$$\Omega_{X,Y} := \{y \in \mathbb{H}^{2,1} \mid [x, y] \text{ spacelike } \forall x \in \Lambda_{X,Y}\}$$

It can also be described as a connected component of

$$\mathbb{H}^{2,1} - \bigcup_{x \in \Lambda_{X,Y}} \{\langle x, \bullet \rangle = 0\}$$

which is a properly convex subset of $\mathbb{P}(M_2(\mathbb{R}))$ whose closure contains $\Lambda_{X,Y}$. In particular, it contains a natural closed $\rho_{X,Y}(\Gamma)$-invariant convex subset, namely the convex hull $\mathcal{CH}_{X,Y}$ of the limit set $\Lambda_{X,Y}$.

As $\Omega_{X,Y}$ does not contain closed timelike geodesics, it has a well defined timelike distance:

Definition 3.2 (Timelike Distance)**.** The *timelike distance* $\delta_{\mathbb{H}^{2,1}}(\bullet,\bullet)$ on $\Omega_{X,Y}$ is defined by

$$\cos(\delta_{\mathbb{H}^{2,1}}(x,y)) := \begin{cases} |\langle x,y\rangle| & \text{if } [x,y] \text{ is timelike} \\ 1 & \text{otherwise.} \end{cases}$$

The group $\rho_{X,Y}(\Gamma)$ acts freely and properly discontinuosly on $\Omega_{X,Y}$ (see [15]). The quotient $M_{X,Y} := \Omega_{X,Y}/\rho_{X,Y}(\Gamma)$ is the *Mess manifold* associated with $X, Y \in \mathcal{T}$.

Let us mention the fact that $M_{X,Y}$ is a so-called *globally hyperbolic maximal Cauchy compact* anti de Sitter 3-manifold (GHMC). In particular, this means that $M_{X,Y}$ contains a closed spacelike surface S homeomorphic to Σ which intersects every inextensible timelike geodesic exactly once. From this property it is not difficult to deduce that $M_{X,Y}$ is diffeomorphic to $\Sigma \times \mathbb{R}$. Mess proves in [15] that, in fact, all GHMC manifolds M where the Cauchy surface is homeomorphic to Σ have the form $M = M_{X,Y}$ for some $X, Y \in \mathcal{T}$.

3.2. **Laminations and pleated surfaces.**

Mess representations are examples of *maximal representations* in $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R}) = \mathrm{PSO}_0(2,2)$ as introduced in [1] (in fact, by a celebrated result of Goldman [12], every maximal representation in $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ is a Mess representation).

As a consequence, we can apply the results of [14] to our setting. In this section, we recall the pleated surface construction from [14] and describe some geometric properties of these objects.

3.2.1. *Pleated sets.* Let $\rho_{X,Y}$ be a Mess representation with boundary map $\xi : \partial\Gamma \to \Lambda_{X,Y}$.

Definition 3.3 (Geometric Realization)**.** Let $\lambda \in \mathcal{GL}$ be a lamination. The *geometric realization* of λ for $\rho_{X,Y}$ is

$$\hat{\lambda} := \bigcup_{(a,b)\in\lambda} [\xi(a), \xi(b)] \subset \mathcal{CH}_{X,Y}$$

where (a,b) is the leaf of λ with endpoints a, b and $[\xi(a), \xi(b)]$ is the spacelike geodesic with endpoints $\xi(a), \xi(b)$.

Definition 3.4 (Pleated Set)**.** Let $\lambda \in \mathcal{GL}_m$ be a maximal lamination. The *pleated set* associated with λ and $\rho_{X,Y}$ is

$$\hat{S}_\lambda := \hat{\lambda} \cup \bigcup_{\Delta(a,b,c)\subset\mathbb{H}^2-\lambda} \Delta(\xi(a), \xi(b), \xi(c)) \subset \mathcal{CH}_{X,Y}$$

where $\Delta(a,b,c)$ is the plaque of λ with vertices a, b, c and $\Delta(\xi(a), \xi(b), \xi(c))$ is the ideal spacelike triangle with endpoints $\xi(a), \xi(b), \xi(c)$.

Proposition 3.5 (Proposition 3.7 of [14])**.** *The pleated set $\widehat{S}_\lambda \subset \mathcal{CH}_{X,Y}$ is a $\rho_{X,Y}(\Gamma)$-invariant topological Lipschitz acausal subsurface.*

Incidentally, combining with classical 3-dimensional topology, Proposition 3.5 has also the following topological corollary:

Corollary 3.6. *Let $\rho_{X,Y}$ be a Mess representation with parameters $X, Y \in \mathcal{T}$. Identify the Mess manifold $M_{X,Y} := \Omega_{X,Y}/\rho_{X,Y}(\Gamma)$ with $\Sigma \times \mathbb{R}$. Let $\alpha \subset \Sigma$ be an essential multicurve. Then, the geodesic realization of α in $M_{X,Y}$ is isotopic to $\alpha \subset \Sigma \times \{0\}$.*

Proof. Let λ_α be a maximal lamination obtained from α by adding finitely many geodesics spiraling around the curves in α. By Proposition 3.5 there exists an embedded π_1-injective (Lipschitz) surface $S_\alpha = \hat{S}_{\lambda_\alpha}/\rho_{X,Y}(\Gamma) \subset M_{X,Y}$ containing the geodesic realization of the curves in α. By Proposition 3.1 and Corollary 3.2 of [20], such surface, being embedded and π_1-injective, is isotopic to $\Sigma \times \{0\}$. □

3.2.2. *Bending locus.* The pleated set $\hat{S}_\lambda$ is not necessarily bent along all the lines in $\hat{\lambda}$.

Definition 3.7 (Bending Locus)**.** Let $\rho_{X,Y}$ be a Mess representation. Consider λ a maximal lamination with geometric realization $\hat{\lambda}$, and denote by $\widehat{S}_\lambda$ the corresponding pleated set. A point $x \in \ell \subset \hat{\lambda}$ is in the *bending locus of* $\widehat{S}_\lambda$ if there is no (necessarily spacelike) geodesic segment k entirely contained in $\widehat{S}_\lambda$ and such that $\operatorname{int}(k) \cap \ell = x$.

We have:

Proposition 3.8 (Proposition 3.11 of [14])**.** *The bending locus is a sublamination of $\hat{\lambda}$, and its complement in $\widehat{S}_\lambda$ is a union of 2-dimensional totally geodesic spacelike regions.*

3.2.3. *1-Lipschitz developing map.* Unfolding pleated sets along the bending locus naturally maps them to $\mathbb{H}^2$. We formalize this heuristic picture as follows:

Definition 3.9 (Developing Map)**.** Let $\rho_{X,Y}$ be a Mess representation. Let $\hat{S}_\lambda \subset \mathcal{CH}_{X,Y}$ be the pleated set associated with the maximal lamination λ. A *1-Lipschitz developing map* is a homeomorphism $f : \hat{S}_\lambda \to \mathbb{H}^2$ with the following properties:

1. It is totally geodesic on every leaf of $\hat{\lambda}$ and every plaque.
2. It is 1-Lipschitz with respect to the intrinsic pseudo-metric on $\hat{S}_\lambda$ and the hyperbolic metric on $\mathbb{H}^2$.

Developing maps have a couple of useful general properties which we now describe. First, they are totally geodesic outside the bending locus.

Lemma 3.10 (Lemma 6.2 of [14])**.** *Let $\rho_{X,Y}$ be a Mess representation, and let $\hat{S}_\lambda$ be the pleated set associated to a maximal lamination λ. Then every 1-Lipschitz developing map $f : \hat{S}_\lambda \to \mathbb{H}^2$ is totally geodesic on the complement of the bending locus of $\hat{S}_\lambda$.*

Secondly, developing maps are contracting with respect to the natural path metric structure on pleated sets.

Definition 3.11 (Regular Path)**.** A *(weakly) regular path* is a map $\gamma : I = [a, b] \to \mathbb{H}^{2,1}$ such that:

- The path is Lipschitz.
- The tangent vector $\dot{\gamma}(t)$ is spacelike (or lightlike) for almost every $t \in I$ (at which $\dot{\gamma}$ is defined).

The length of a weakly regular path is

$$L(\gamma) := \int_I \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} dt.$$

The Lipschitz property implies that the length $L(\gamma)$ is always finite.

Lemma 3.12 (Claim 2 of Lemma 6.4 in [14])**.** *Let $\hat{S} \subset \mathbb{H}^{2,1}$ be an acausal subset. Let $\gamma : I = [a, b] \to \hat{S}$ be a weakly regular path. Then*

$$L(\gamma) = \lim_{\epsilon \to 0} \int_I \frac{d_{\mathbb{H}^{2,1}}(\gamma(t), \gamma(t+\epsilon))}{\epsilon} \mathrm{dt}.$$

Lemma 3.13 (Lemma 6.4 of [14])**.** *Let $\rho_{X,Y}$ be a Mess representation, and let $\hat{S}_\lambda$ be the pleated set associated to a maximal lamination λ. Then every 1-Lipschitz developing map $f : \hat{S}_\lambda \to \mathbb{H}^2$ sends weakly regular paths $\gamma : I \to \hat{S}_\lambda$ to Lipschitz (hence rectifiable) paths $f\gamma : I \to \mathbb{H}^2$ of smaller length $L(\gamma) \geqslant L(f\gamma)$.*

3.2.4. *Pleated surfaces.* The following result makes sure that every pleated set $\hat{S}_\lambda$ admits a natural 1-Lipschitz developing map:

Proposition 3.14 (Proposition 6.6 in [14])**.** *Let $\rho_{X,Y}$ be a Mess representation. For every maximal lamination $\lambda \in \mathcal{GL}_\lambda$ there is:*

- *An intrinsic hyperbolic structure $Z_\lambda \in \mathcal{T}$.*
- *A $(\rho_{X,Y} - \rho_\lambda)$-equivariant 1-Lipschitz developing map $f : \hat{S}_\lambda \to \mathbb{H}^2$ where ρ_λ is the holonomy of Z_λ.*

We can finally define pleated surfaces:

Definition 3.15 (Pleated Surface)**.** Let $\rho_{X,Y}$ be a Mess representation. The *pleated surface* associated with the maximal lamination $\lambda \in \mathcal{GL}$ consists of the following data:

1. The pleated set $\hat{S}_\lambda$.
2. The intrinsic hyperbolic holonomy $\rho_\lambda : \Gamma \to \mathrm{PSL}_2(\mathbb{R})$ of Z_λ.
3. A $(\rho_{X,Y} - \rho_\lambda)$-equivariant 1-Lipschitz developing map $f : \hat{S}_\lambda \to \mathbb{H}^2$.

Let us conclude this discussion by observing that pleated surfaces for a fixed Mess representation $\rho_{X,Y}$ have some useful compactness properties:

Lemma 3.16. *Let $\rho_{X,Y}$ be the Mess representation with parameters $X, Y \in \mathcal{T}$. Then the space of intrinsic hyperbolic structures on the pleated sets*

$$\{Z_\lambda\}_{\lambda \in \mathcal{GL}_m}$$

is pre-compact in $\mathcal{T}$.

Proof. Recall that $\rho_{X,Y}(\Gamma)$ acts cocompactly on $\mathcal{CH}_{X,Y}$. Let $F \subset \mathcal{CH}_{X,Y}$ be a compact fundamental domain. For every maximal lamination $\lambda \in \mathcal{GL}_m$ with associated pleated set $\hat{S}_\lambda \subset \mathcal{CH}_{X,Y}$ choose a basepoint $x_\lambda \in \hat{S}_\lambda \cap F$. Let $f_\lambda : \hat{S}_\lambda \to \mathbb{H}^2$ be a $(\rho_{X,Y} - \rho_\lambda)$-equivariant 1-Lipschitz developing map normalized so that $f_\lambda(x_\lambda) = o \in \mathbb{H}^2$, a fixed basepoint. The equivariance and the 1-Lipschitz property tell us that

$$d_{\mathbb{H}^2}(o, \rho_\lambda(\gamma)o) \leqslant d_{\mathbb{H}^{2,1}}(x_\lambda, \rho_{X,Y}(\gamma)x_\lambda)$$

for every $\gamma \in \Gamma$. Notice that the right hand side is bounded from above by a uniform constant K_γ independent of λ as $x_\lambda \in F$ is contained in a compact set and

$$\cosh(d_{\mathbb{H}^{2,1}}(x_\lambda, \rho_{X,Y}(\gamma)x_\lambda)) = |\langle x_\lambda, \rho_{X,Y}(\gamma)x_\lambda\rangle|.$$

Therefore the set of representations $\{\rho_\lambda\}_{\lambda \in \mathcal{GL}_m} \subset \mathcal{T} \subset \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ is pre-compact. ☐

3.2.5. *Convex core.* An example of pleated surface is given by the two connected components of the boundary of the convex core $\partial\mathcal{CH}_{X,Y} = \partial^+\mathcal{CH}_{X,Y} \cup \partial^-\mathcal{CH}_{X,Y}$. Each of them has the structure of a pleated set with bending loci λ^+ and λ^- and intrinsic hyperbolic structures $Z_{\lambda^+}, Z_{\lambda^-} \in \mathcal{T}$. As we mentioned in the introduction, measuring the total turning angles along paths $\alpha : I \to \partial^\pm\mathcal{CH}_{X,Y}$ equips the geodesic laminations $\lambda^\pm$ with a transverse measure and, hence identifies a pair of points $\lambda^\pm \in \mathcal{ML}$. Mess proves that we have the following relations

$$\begin{tikzcd}
 & Z_{\lambda^+} \arrow[dl, "E^r_{\lambda^+}"'] \arrow[dr, "E^l_{\lambda^+}"] & \\
X & & Y \\
 & Z_{\lambda^-} \arrow[ul, "E^r_{\lambda^-}"] \arrow[ur, "E^l_{\lambda^-}"'] &
\end{tikzcd}$$

where $E^l_{\lambda^+}, E^l_{\lambda^-}, E^r_{\lambda^+}, E^r_{\lambda^-}$ are the left and right earthquakes induced by the measured laminations λ^+, λ^-. Heuristically speaking, an earthquake is the generalization to laminations of a *twist deformation* along a simple closed geodesic. Given a closed geodesic γ on a hyperbolic surface X and a real parameter $\theta > 0$ we do the following operation: We lift γ to a $\rho_X(\Gamma)$-invariant family of pairwise disjoint geodesics $\lambda \subset \mathbb{H}^2$. We cut $\mathbb{H}^2$ along λ. We reglue all the ideal polygons $P \subset \mathbb{H}^2 - \lambda$ by composing all the initial identifications $\ell \subset \partial P \to \ell' \subset \partial P'$ (left-to-right) with the isometry of ℓ' given by $t \to t + \theta$ (the identification $\ell' = \mathbb{R}$ is determined by the orientation). The result is still isometric to $\mathbb{H}^2$ but the action of Γ on it is the holonomy of a different hyperbolic structure, which, depending on the choices of orientations, is $E^l_{\theta\gamma}(X)$ or $E^r_{\theta\gamma}(X)$.

We will describe more carefully the various elements that enter this picture in the next section where we will prove a generalization of the result of Mess.

3.2.6. *Initial and terminal singularities.* We end this section by describing the *initial* and *terminal singularities* of $\Omega_{X,Y}$ which are subsets of $\partial\Omega_{X,Y}$ dual to the boundary components of the convex core. Duality is understood in the sense of the duality induced by the quadratic form $\langle \bullet, \bullet \rangle_{(2,2)}$ on $\mathbb{P}(M_2(\mathbb{R}))$. Explicitly, we have

$$\mathbb{P}(L) \leftrightarrow \mathbb{P}(L^{\perp})$$

where $L^{\perp} \subset M_2(\mathbb{R})$ is the linear subspace orthogonal to L with respect to the quadratic form.

Define the following:

Definition 3.17 (Initial and Terminal Singularities)**.** The sets $\mathcal{S}^{\pm}$ of dual points of supporting planes of $\partial^{\pm}\mathcal{CH}_{X,Y}$ are the *initial* and *terminal singularities.*

Let us start with the following observation:

Lemma 3.18. *Let* $H = P(V) \cap \mathbb{H}^{2,1}$ *be a supporting plane of* $\partial^{\pm}\mathcal{CH}_{X,Y}$*. Then:*

- *H is spacelike and defines a dual point $P(V^{\perp}) \in \mathbb{H}^{2,1}$. Let $w \in V^{\perp}$ be a unit timelike vector pointing outside $\mathcal{CH}_{X,Y}$.*
- *For every $x \in H \cap \mathcal{CH}_{X,Y}$, the timelike geodesic $\gamma(t) = \cos(t)x - \sin(t)w$ with $t \in [0, \pi/2)$ is contained in $\Omega_{X,Y}$ while $w = \gamma(\pi/2) \in \partial\Omega_{X,Y}$.*

Any two distinct supporting planes H_1, H_2 of $\partial^{+}\mathcal{CH}_{X,Y}$ intersect in a spacelike geodesic $H_1 \cap H_2$. If w_1, w_2 are the dual points of H_1, H_2, then $[w_1, w_2]$ is spacelike.

Proof. The first point: Recall that $\partial\mathbb{H}^{2,1} = \mathbb{RP}^1 \times \mathbb{RP}^1$ and that $\Lambda_{X,Y}$ is the graph of an orientation preserving homeomorphism $h_{X,Y} : \mathbb{RP}^1 \to \mathbb{RP}^1$. If H is a supporting hyperplane for $\mathcal{CH}_{X,Y}$ then ∂H does not intersect $\Lambda_{X,Y}$ transversely. The fact that H must be spacelike follows from the following observations: The boundary of a lightlike plane has the form $\{t\} \times \mathbb{RP}^1$ or $\mathbb{RP}^1 \times \{t\}$. The boundary of a timelike plane is the graph of an orientation reversing linear transformation $\mathbb{RP}^1 \to \mathbb{RP}^1$. In both cases the boundary intersects $\Lambda_{X,Y}$ transversely.

The second point: Recall that $\Omega_{X,Y}$ is the set of points that can be connected to every point in $\Lambda_{X,Y}$ by a spacelike geodesic. A point $x \in \mathbb{H}^{2,1}$ and a point $p \in \partial\mathbb{H}^{2,1}$ are connected by a spacelike geodesic if and only if $\langle x, p \rangle \neq 0$. Let us show that $\gamma(t) \in \Omega_{X,Y}$ for every $t \in [0, \pi/2)$. In order to do so, lift $\Lambda_{X,Y}$ continuously to $M_2(\mathbb{R})$. As $x \in \Omega_{X,Y}$, we have $\langle x, p \rangle \neq 0$ for every $p \in \Lambda_{X,Y}$ and, by continuity, we can assume that it is negative for every $p \in \Lambda_{X,Y}$. As H is a supporting hyperplane and w is timelike, orthogonal to H, and pointing outside $\mathcal{CH}_{X,Y}$, we have $\langle p, w \rangle \geqslant 0$ for every $p \in \Lambda_{X,Y}$. Therefore $\langle \gamma(t), p \rangle = \cos(t)\langle x, p \rangle - \sin(t)\langle w, p \rangle < 0$ for every

$p \in \Lambda_{X,Y}$ and $t < \pi/2$. In order to conclude, it is enough to observe that $w = \gamma(\pi/2) \notin \Omega_{X,Y}$ as $\langle w, p\rangle = 0$ for every $p \in \partial H \cap \Lambda_{X,Y} \neq \emptyset$.

For the last part notice that $H_1 \cap H_2$ is either empty or a spacelike geodesic. Suppose that $H_1 \cap H_2 = \emptyset$. Then $\mathbb{H}^{2,1} - (H_1 \cup H_2)$ consists of two connected components one of them containing $\mathcal{CH}_{X,Y}$. As H_1, H_2 lies on opposite sides of $\mathcal{CH}_{X,Y}$ in such component, they cannot be supporting hyperplanes for the same boundary component of $\partial\mathcal{CH}_{X,Y}$. This is a contradiction. □

Notice that, by Lemma 3.18, the initial and terminal singularities $\mathcal{S}^{\pm}$ are $\rho_{X,Y}(\Gamma)$-invariant, acausal, and contained in $\partial\Omega_{X,Y}$. Benedetti and Guadagnini [3] prove that they have the structure of a $\mathbb{R}$-tree and relate them to the bending laminations $\lambda^{\pm}$.

Definition 3.19 ($\mathbb{R}$-tree)**.** A $\mathbb{R}$-*tree* is a geodesic metric space $(\mathcal{S}, d_{\mathcal{S}}(\bullet, \bullet))$ such that between two points $x, y \in \mathcal{S}$ there is a unique (up to reparametrization) injective path $\alpha : [0,1] \to \mathcal{S}$ with $\alpha(0) = x, \alpha(1) = y$.

Benedetti and Guadagnini [3] show the following:

Proposition 3.20. *Let $\rho_{X,Y}$ be a Mess representation. Let $\mathcal{S}^{\pm} \subset \partial\Omega_{X,Y}$ be the initial and terminal singularities. Then:*

- *$\mathcal{S}^{\pm}$ is $\rho_{X,Y}(\Gamma)$-invariant, acausal, and path connected by regular paths. In particular, it has an intrinsic path metric*
$$d_{\mathcal{S}^{\pm}}(x, y) = L(\alpha)$$
where $\alpha : [0,1] \to \mathcal{S}^{\pm}$ is a regular path joining x to y.
- *For every pair of points $w, w' \in \mathcal{S}^{\pm}$ there is a unique continuous injective path connecting them.*
- *For every $\gamma \in \Gamma - \{1\}$, the minimal displacement*
$$\min_{x \in \mathcal{S}^{\pm}} \{d_{\mathcal{S}^{\pm}}(x, \rho_{X,Y}(\gamma)x)\}$$
coincides with $i(\gamma, \lambda^{\pm})$ and is realized by some point $x \in \mathcal{S}^{\pm}$.

Here $\lambda^{\pm} \in \mathcal{ML}$ is the bending lamination of $\partial^{\pm}\mathcal{CH}_{X,Y}$ and $i(\bullet, \bullet)$ is the geometric intersection form.

For a proof we refer to [2].

4. A generalization of a result of Mess

The goal of the section is to define the shear-bend cocycles of pleated surfaces and prove Theorem 1.

We begin by recalling the Thurston-Bonahon shear parametrization of Teichmüller space (as discussed by Bonahon in [6]) which we will generalize to the space of Mess representations in Theorem 5 at the end of the section.

4.1. Shear coordinates.

We refer to Bonahon [6] for more details on the material presented in this section.

4.1.1. *Transverse cocycles.* Shear-bend cocycles are a special case of transverse cocycles for λ.

Definition 4.1 (Transverse Cocycle)**.** Let $\mathbb{A}$ be a commutative ring. Let $\lambda \subset \mathbb{H}^2$ be a maximal lamination. An $\mathbb{A}$-*transverse cocycle* for λ is a function $\sigma(\bullet, \bullet)$ of pairs of plaques satisfying the following properties:

- Invariance: $\sigma(\gamma P, \gamma Q) = \sigma(P, Q)$ for every $\gamma \in \Gamma$ and plaques P, Q.
- Symmetry: $\sigma(P, Q) = \sigma(Q, P)$ for every plaques P, Q.
- Additivity: $\sigma(P, R) = \sigma(P, Q) + \sigma(Q, R)$ for every plaques P, Q, R such that R separates P from Q.

The space of $\mathbb{A}$-transverse cocycles is denoted by $\mathcal{H}(\lambda; \mathbb{A})$. It has a natural structure of $\mathbb{A}$-module isomorphic to $\mathbb{A}^{-3\chi(\Sigma)}$ (see Bonahon [6]).

4.1.2. *Measured laminations.* Every measured lamination $\mu \in \mathcal{ML}_\lambda$ determines a natural transverse cocycle which, with a little abuse of notation, we will still denote by $\mu \in \mathcal{H}(\lambda; \mathbb{R})$. It is defined as follows: Let P, P' be plaques of λ. Let $\ell \subset P, \ell' \subset \ell'$ be the (oriented) edges that separate P, P'. Then

$$\mu(P, P') := \mu([\ell, \ell']),$$

the measure, determined by μ, of the box $[\ell, \ell'] \subset \mathcal{G}$ consisting of those geodesics separating ℓ and ℓ'.

4.1.3. *Hyperbolic structures.* Every hyperbolic structure X on Σ also determines a transverse cocycle $\sigma_\lambda^X \in \mathcal{H}(\lambda; \mathbb{R})$, the so-called *shear cocycle* of X. It is defined as follows: Let P, P' be plaques of λ. Let $\ell \subset P, \ell' \subset \ell'$ be the (oriented) edges that separate P, P'. Denote by $x \in \ell, x' \in \ell'$ the orthogonal projections of the opposite vertices in P, P'.

Consider the partial foliation $\lambda_{PP'}$ of the region $[\ell, \ell']$ bounded by ℓ, ℓ' given by all the leaves that separate P from P' and note that $[\ell, \ell'] - \lambda_{PP'}$ is a union of wedges, that is regions bounded by a pair of leaves of $\lambda_{PP'}$ that are asymptotic in one or the other direction. Each of the wedges can be foliated by adding all the geodesics separating the boundary leaves and to their common endpoint at infinity. Thus, we get a natural geodesic foliation of $[\ell, \ell']$. The line field on $[\ell, \ell']$ which is orthogonal to this foliation is integrable and following its leaves provides a natural isometric identification $\pi : \ell \to \ell'$. Define

$$\sigma_\lambda^X(P, P') := d_{\ell'}(\pi(x), x')$$

where $d_{\ell'}$ is the signed distance along ℓ'.

A straightforward computation in $\mathbb{H}^2$ shows the following:

Lemma 4.2. *Let $\beta^{\mathbb{R}}$ be the cross ratio on $\mathbb{RP}^1$. We have*

- *If P, P' are adjacent triangles and $\ell = \ell'$, then*

$$\sigma_\lambda^X(P, P') = \beta^{\mathbb{R}}(\ell^+, \ell^-, u, u')$$

where $u \in P, u' \in P'$ are the vertices opposite to $\ell = \ell'$.

- *If P, P' are asymptotic to a leaf $\ell \subset \lambda$, then*
$$\sigma_\lambda^X(P, P') = \beta^{\mathbb{R}}(\ell^+, \ell^-, u, u')$$
where $u \in e, u' \in e'$ are the vertices not on ℓ on the edges $e \subset P, e' \subset P'$ which separate the plaques P, P'.

Bonahon proves the following:

Theorem 4.3 (Theorems A and B of [6])**.** *Let λ be a maximal lamination. For every $X \in \mathcal{T}$ the function $\sigma_\lambda^X(\bullet, \bullet)$ is a transverse cocycle. The map*

$$\begin{aligned} \Phi : \mathcal{T} &\to \mathcal{H}(\lambda; \mathbb{R}) \\ X &\to \sigma_\lambda^X \end{aligned}$$

is a real analytic diffeomorphism. The image $\Phi(\mathcal{T})$ is the open convex cone

$$\Phi(\mathcal{T}) = \{\sigma \in \mathcal{H}(\lambda, \mathbb{R}) \mid \omega_{\mathrm{Th}}(\sigma, \bullet) > 0 \text{ on } \mathcal{ML}_\lambda\}$$

where $\omega_{\mathrm{Th}}(\bullet, \bullet)$ is the Thurston's symplectic form *on $\mathcal{H}(\lambda; \mathbb{R})$.*

The resulting set of coordinates for Teichmüller space are called *shear coordinates* relative to λ.

The Thurston's symplectic form $\omega_{\mathrm{Th}}(\bullet, \bullet)$ is a natural symplectic form on the vector space $\mathcal{H}(\lambda; \mathbb{R})$. For our purposes we don't need a precise definition of this object (we refer to Bonahon [6] for details), as we will only use the following property:

Theorem 4.4 (Theorem E of [6])**.** *Let λ be a maximal lamination. Let $\omega_{\mathrm{Th}}(\bullet, \bullet)$ is the Thurston's symplectic form on $\mathcal{H}(\lambda; \mathbb{R})$. Then, for every $\mu \in \mathcal{ML}_\lambda$ and $X \in \mathcal{T}$ we have*

$$\omega_{\mathrm{Th}}(\sigma_\lambda^X, \mu) = L_X(\mu).$$

4.1.4. *Continuity of cocycles.* In order to talk about continuity properties of cocycles we need to compare $\mathcal{H}(\lambda'; \mathbb{R})$ with $\mathcal{H}(\lambda; \mathbb{R})$ for λ' close to λ. This can be done using the *weights system* $\mathcal{W}(\tau; \mathbb{R})$ of a *train track* τ carrying λ. For us it is not important the definition of these objects, but rather the following facts (see the proof of Lemma 13 in Bonahon [7] or Proposition 5.10 and Corollary 5.11 in [14]):

- τ determines an open set $U_\tau \subset \mathcal{GL}_m$ containing λ.
- $\mathcal{W}(\tau; \mathbb{R})$ is a real vector space and there is a canonical linear isomorphism $\mathcal{H}(\lambda'; \mathbb{R}) \to \mathcal{W}(\tau; \mathbb{R})$ for every $\lambda' \in U_\tau$.
- For every $\lambda_1, \lambda_2 \in U_\tau$ the following diagram commutes

$$\begin{tikzcd} \mathcal{T} \arrow[r] \arrow[d] & \mathcal{H}(\lambda_2; \mathbb{R}) \arrow[d] \\ \mathcal{H}(\lambda_1; \mathbb{R}) \arrow[r] & \mathcal{W}(\tau; \mathbb{R}). \end{tikzcd}$$

- For every $X \in \mathcal{T}$ the map $\lambda \in U_\tau \to \sigma_\lambda^X \in \mathcal{W}(\tau; \mathbb{R})$ is continuous.

4.2. **Para-complex numbers.**

In order to define the shear-bend cocycle of pleated surfaces it is convenient to exploit the natural para-complex cross-ratio on the boundary of $\mathbb{H}^{2,1}$ (see Section 2 of Danciger [11]).

Definition 4.5 (Para-complex Numbers)**.** The ring of para-complex numbers is $\mathbb{B} := \mathbb{R}[\tau]/(\tau^2 - 1)$. Similarly to the case of complex numbers, every element $z = x + \tau y$ has:

- A conjugate $\bar{z} := x - \tau y$.
- A pseudo-norm $|z|^2 := z\bar{z} = x^2 - y^2 \in \mathbb{R}$.

However $\mathbb{B}$ has also non trivial zero-divisors: An element $z \in \mathbb{B}$ is invertible if and only if $|z|^2 \neq 0$, in which case $z^{-1} = \bar{z}/|z|^2$. We denote by $\mathbb{B}^*$ the set of invertible elements of $\mathbb{B}$.

It is convenient to decompose $\mathbb{B}$ as $\mathbb{R} \times \mathbb{R}$: Consider

$$e_l := \frac{1+\tau}{2}, e_r := \frac{1-\tau}{2}.$$

The elements e_l, e_r are idempotent $e_j^2 = e_j$, orthogonal $e_l e_r = 0$, and conjugate $\bar{e}_l = e_r$. This implies that the map $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R} \to \lambda e_l + \mu e_r \in \mathbb{B}$ is a ring isomorphism. In these coordinates, the conjugate of an element is $\overline{\lambda e_l + \mu e_r} = \mu e_l + \lambda e_r$ and its norm is $|\lambda e_l + \mu e_r| = \lambda\mu$.

4.2.1. *Exponential and logarithm.* The para-complex exponential function $\exp : \mathbb{B} \to \mathbb{B}$ is given by $\exp(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}$. In terms of the classical exponential we have $e^{x+\tau y} = e^x(\cosh(y) + \tau \sinh(y))$. The para-complex exponential map is injective, but not surjective. Its image coincides with

$$\mathbb{B}^+ := \{x + \tau y \in \mathbb{B} \mid x > 0 \text{ and } |x + \tau y|^2 > 0\}.$$

The inverse of the exponential is the para-complex logarithm $\log : \mathbb{B}^+ \to \mathbb{B}$.

In coordinates $\mathbb{B} = \mathbb{R} \times \mathbb{R}$, we have: $\mathbb{B}^+ = \{(\lambda, \mu) \in \mathbb{R} \times \mathbb{R} \mid \lambda, \mu > 0\}$. The exponential is $\exp(\lambda e_l + \mu e_r) = \exp(\lambda) e_l + \exp(\mu) e_r$. The logarithm is $\log(\lambda e_l + \mu e_r) = \log(\lambda) e_l + \log(\mu) e_r$.

4.2.2. *Projective para-complex line.* The boundary $\partial\mathbb{H}^{2,1} = \mathbb{RP}^1 \times \mathbb{RP}^1$ can be identified with the para-complex projective line $\mathbb{BP}^1 = (\mathbb{B}^2 - \{0\})/\mathbb{B}^*$ via

$$([u], [v]) \in \mathbb{RP}^1 \times \mathbb{RP}^1 \to \left[\frac{1+\tau}{2} u + \frac{1-\tau}{2} v\right] \in \mathbb{BP}^1$$

and $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ can be thought of as the para-complex projective linear transformations $\mathrm{PSL}_2(\mathbb{B}) = \mathrm{SL}_2(\mathbb{B})/\mathbb{B}^*$ via the isomorphism

$$([A], [B]) \in \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R}) \to \left[\frac{1+\tau}{2} A + \frac{1-\tau}{2} B\right] \in \mathrm{PSL}_2(\mathbb{B}).$$

The para-complex projective line $\mathbb{BP}^1$ is equipped with a natural *para-complex cross-ratio*:

Definition 4.6 (Cross Ratio)**.** The *para-complex cross-ratio* is defined by

$$\beta^{\mathbb{B}}(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3} \in \mathbb{B}.$$

The following is an elementary computation:

Lemma 4.7. *For every* $a, b, c, d \in \mathbb{BP}^1 = \mathbb{RP}^1 \times \mathbb{RP}^1$ *we have*

$$\beta^{\mathbb{B}}(a, b, c, d) = \frac{1+\tau}{2}\beta^{\mathbb{R}}(a_l, b_l, c_l, d_l) + \frac{1-\tau}{2}\beta^{\mathbb{R}}(a_r, b_r, c_r, d_r).$$

4.3. **Shear-bend cocycle.** We now recall the natural shear-bend cocycle and its geometric interpretation as given in Sections 4 and 5 of [14].

Let $\rho_{X,Y}$ be a Mess representation with limit curve $\Lambda_{X,Y}$.

4.3.1. *Elementary shear.* Let us start with an elementary shear-bend.

Lemma 4.8. *Let* $\Delta = (u, \ell^-, \ell^+), \Delta' = (u', \ell^+, \ell^-) \subset \mathbb{H}^{2,1}$ *be ideal triangles sharing a common edge* $\ell = [\ell^-, \ell^+]$ *and with vertices on* $\Lambda_{X,Y}$ *ordered as* $u < \ell^- < u' < \ell^+$. *Then* $\beta^{\mathbb{B}}(\ell^+, \ell^-, u_l, u_r) \in \mathbb{B}^+$.

Proof. Recall that $\Lambda_{X,Y}$ is the graph of the unique $(\rho_X - \rho_Y)$-equivariant homeomorphism $h_{X,Y} : \mathbb{RP}^1 \to \mathbb{RP}^1$. For a point $p \in \mathbb{RP}^1 \times \mathbb{RP}^1$ denote by p_l, p_r the left and right components. Then we have $u_j < \ell_j^+ < u_j' < \ell_j^-$ on $\mathbb{RP}^1$ for $j = l, r$. The conclusion follows from Lemma 4.7. □

We define:

$$\sigma^{\mathbb{B}}(\Delta, \Delta') := \log \beta^{\mathbb{B}}(\ell^+, \ell^-, u_l, u_r) \in \mathbb{B}.$$

4.3.2. *Maximal laminations with countably many leaves.* We then consider the case of maximal laminations with countably many leaves.

These laminations always have the following structure: There is a canonical collection of simple sublaminations

$$\lambda' = \lambda_1 \sqcup \cdots \sqcup \lambda_n \subset \lambda$$

where each λ_j consists of the orbit of the axis of an element $\gamma_j \in \Gamma - \{1\}$ representing a simple closed curve. The complement $\lambda - \lambda'$ is made of isolated geodesics asymptotic to leaves of λ'.

Let $\lambda \subset \mathbb{H}^2$ be a maximal lamination with countably many leaves. Let $P, Q \subset \mathbb{H}^2 - \lambda$ be a pair of plaques. Denote by $\mathcal{P}_{PQ}$ the set of plaques separating P from Q. Let $\ell_1, \cdots, \ell_m$ be the leaves of λ' separating P from Q. For each of those leaves ℓ_j select plaques R_j^+, R_j^- asymptotic to it from the left and from the right. The elementary shear between them is

$$\sigma^{\mathbb{B}}(R_j^-, R_j^+) := \sigma^{\mathbb{B}}(\Delta(u_j^-, \ell_j^-, \ell_j^+), \Delta(\ell_j^+, \ell_j^-, u_j^+)).$$

where u_j^-, u_j^+ are the vertices of R_j^-, R_j^- that lie on the edges separating the two plaques and are not endpoints of ℓ_j. Note that between R_{j-1}^+ and R_j^- there are only finitely many consecutive adjacent plaques

$$R_{j-1}^+ = T_{j,0}, \cdots, T_{j,k_j} = R_j^-.$$

Define

$$\sigma_\rho(P,Q) := \sum_{j=1}^{m}\left(\sigma^{\mathbb{B}}(R_j^-,R_j^+) + \sum_{i=0}^{k_j-1}\sigma^{\mathbb{B}}(T_{j,i},T_{i+1})\right).$$

As observed in [14], a simple computation shows that a different choice of plaques R_j^+, R_j^- asymptotic to the lifts of the leaves $\ell_j \in \lambda'$ separating P from Q gives the same value for $\sigma_\rho(P,Q)$. The fact that $\sigma_\rho(P,Q)$ is well-defined immediately implies that it is also satisfies the properties of a transverse cocycle. Therefore:

Definition 4.9 (Intrinsic Shear-Bend I)**.** Let $\rho_{X,Y}$ be a Mess representation. Let λ be a maximal lamination with countably many leaves. The cocycle $\sigma_\rho(\bullet,\bullet) \in \mathcal{H}(\lambda;\mathbb{B})$ is the *intrinsic shear-bend cocycle* of the pleated set $\hat{S}_\lambda$.

Furthermore we have:

Proposition 4.10 (Proposition 6.7 in [14])**.** *Let $\rho_{X,Y}$ be a Mess representation. Let λ be a maximal lamination with countably many leaves. Then $(\sigma_\rho+\bar{\sigma}_\rho)/2 \in \mathcal{H}(\lambda;\mathbb{R})$ is the shear cocycle of the intrinsic hyperbolic structure $Z_\lambda \in \mathcal{T}$ of the pleated set $\hat{S}_\lambda$.*

4.3.3. *General maximal laminations.* Lastly, we describe the natural finite approximation process that defines the shear-bend cocycle in general extending the previous case: Let $\lambda \subset \mathbb{H}^2$ be an arbitrary maximal lamination. As before, let $P,Q \subset \mathbb{H}^2 - \lambda$ be a pair of plaques and let $\mathcal{P}_{PQ}$ be the set of plaques separating P from Q. Let

$$\mathcal{P} = \{P_1,\cdots,P_m\} \subset \mathcal{P}_{PQ}$$

be a finite subset of plaques ordered from P to Q. Any two consecutive P_j, P_{j+1} cobound a (possibly empty) region U_j. We decompose its boundary as $\partial U_j = \ell_j \cup \ell_{j+1}$ with $\ell_j \subset \partial P_j$ and $\ell_{j+1} \subset \partial P_{j+1}$. We add to the finite collection $\mathcal{P}$ of plaques the triangles

$$\Delta(\ell_j^+,\ell_j^-,\ell_{j+1}^+), \Delta(\ell_j^-,\ell_{j+1}^-,\ell_{j+1}^+)$$

obtaining a chain of triangles $P = T_1, T_2, \cdots, T_{3m-2}, T_{3m-1} = Q$.

We then define

$$\sigma_\rho(P,Q) := \sum_{j=1}^{3m-2}\sigma^{\mathbb{B}}(T_j,T_{j+1}).$$

We then carefully choose an exhaustion $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ of $\mathcal{P}_{PQ}$ by an finite subsets and we set

$$\sigma_\rho(P,Q) := \lim_{n\to\infty}\sigma^{\mathbb{B}}_{\mathcal{P}_n}(P,Q).$$

The existence of the limit as well as the independence of the choices made to define it and the fact that the limit object is a $\mathbb{B}$-transverse cocycle are proved in [14]:

Theorem 4.11 (Theorem B of [14]). *Let $\rho_{X,Y}$ be a Mess representation. For every maximal geodesic lamination $\lambda \in \mathcal{GL}$, the finite approximation process converges and defines a $\mathbb{B}$-transverse cocycle $\sigma_\rho \in \mathcal{H}(\lambda;\mathbb{B})$.*

Definition 4.12 (Intrinsic Shear-Bend II). Let $\rho_{X,Y}$ be a Mess representation. Let λ be a maximal lamination. The cocycle $\sigma_\rho \in \mathcal{H}(\lambda;\mathbb{B})$ provided by Theorem 4.11 is the *intrinsic shear-bend cocycle* of the pleated set $\hat{S}_\lambda$.

The following is a summary of the results in Sections 4 and 5 of [14].

Proposition 4.13. *We have the following properties:*

(i) If λ has countably many leaves the definitions I and II coincide.

(ii) $(\sigma_\rho + \bar{\sigma}_\rho)/2$ is the shear cocycle of the intrinsic hyperbolic structure $Z_\lambda \in \mathcal{T}$.

(iii) The map $\lambda \in \mathcal{GL}_m \to \sigma_\rho \in \mathcal{W}(\tau;\mathbb{B})$ is continuous with respect to the Hausdorff topology on $\mathcal{GL}_m$. Here $\mathcal{W}(\tau;\mathbb{R})$ is the weight space of a train track τ carrying λ.

4.4. **Gauss map.** In order to prove Theorem 1 we study the *Gauss map* of the pleated set $\hat{S}_\lambda$ which we now describe. To this purpose let us begin with some general observations.

The group $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ acts transitively on oriented timelike geodesic. The stabilizer of the oriented timelike geodesic $\gamma(t) = \cos(t)I + \sin(t)J \in \mathrm{PSO}(2)$ where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is $\mathrm{PSO}(2) \times \mathrm{PSO}(2)$.

Therefore, the space of oriented timelike geodesics is naturally $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$-equivariantly identified with $\mathrm{PSL}_2(\mathbb{R})/\mathrm{PSO}(2) \times \mathrm{PSL}_2(\mathbb{R})/\mathrm{PSO}(2)$.

We identify $\mathbb{RP}^1$ with $\mathbb{P}\{A \in M_2(\mathbb{R}) \mid \mathrm{rk}(A) = 1\}/\mathrm{PSO}(2)$ and $\mathbb{H}^2$ with $\mathrm{PSL}_2(\mathbb{R})/\mathrm{PSO}(2)$.

Lemma 4.14. *Let $H \subset \mathbb{H}^{2,1}$ be a spacelike plane. Consider the map $g = (g_l, g_r) : H \to \mathbb{H}^2 \times \mathbb{H}^2$ where $g(x)$ is the future pointing timelike geodesic orthogonal to H at x. Then g_j is isometric and extends continuously to the map $g_j : \partial H \subset \mathbb{RP}^1 \times \mathbb{RP}^1 \to \mathbb{RP}^1$ sending $g_j(a_l, a_r) = a_j$ for $j = l, r$.*

Proof. By equivariance it is enough to check the claim for a specific hyperplane $H \subset \mathbb{H}^{2,1} = \mathrm{PSL}_2(\mathbb{R})$. We choose H to be the dual plane of I, that is $H = \mathbb{P}\{M \in \mathrm{SL}_2(\mathbb{R}) \mid \mathrm{tr}(M) = 0\}$. As above, let $\gamma = \mathrm{PSO}(2)$.

Notice that $J = H \cap \gamma$ and, hence, $g(J) = \gamma = ([I],[I])$. As the diagonal group of $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ preserves H and acts transitively on it, by equivariance we have $g_j(AJ) = [A]$, that is, the components g_j are the restrictions of the standard projection $\pi : \mathrm{PSL}_2 \to \mathrm{PSL}_2/\mathrm{PSO}(2)$ to H. Also observe that, as γ is orthogonal to H at J, the differential $\mathrm{d}\pi_J$ is isometric. Thus, by equivariance, $\mathrm{d}\pi$ is isometric everywhere.

The boundary of H is $\partial H = \mathbb{P}\{M \in M_2(\mathbb{R}) \mid \mathrm{tr}(M) = 0, \mathrm{rk}(M) = 1\}$. Notice that, by Hamilton-Cayley, every $M \in M_2(\mathbb{R})$ satisfies $M^2 - \mathrm{tr}(M)M + \det(M) = 0$. Therefore, if $M \in \partial H$, then $M^2 = 0 \Longleftrightarrow \mathrm{Im}(M) = \mathrm{Ker}(M)$. The map $g_j(AJ) = [A]$ extends continuously to a map $\partial H \to \mathbb{RP}^1$ sending $g_j(\mathrm{Im}(M), \mathrm{Ker}(M)) = [\mathrm{Im}(M)] = [\mathrm{Ker}(M)]$. □

Let $\rho_{X,Y}$ be a Mess representation with limit curve $\Lambda_{X,Y} \subset \mathbb{RP}^1 \times \mathbb{RP}^1$.

Lemma 4.15. *Consider two ideal spacelike adjacent triangles* $\Delta = \Delta(a, b, c)$ *and* $\Delta' = (c, b, a')$ *sharing a common edge* $[b, c]$ *and with vertices ordered as* $a < b < a' < c$ *along* $\Lambda_{X,Y}$*. Let* $g = (g_l, g_r) : \mathrm{int}(\Delta) \cup \mathrm{int}(\Delta') \to \mathbb{H}^2 \times \mathbb{H}^2$ *be the map sending* x *to the future pointing timelike normal* $g(x) \in \mathbb{H}^2 \times \mathbb{H}^2$*. Then*

$$\sigma(\Delta, \Delta') = \frac{\sigma_{\mathbb{H}^2}(g_l(\Delta), g_l(\Delta')) + \sigma_{\mathbb{H}^2}(g_r(\Delta), g_r(\Delta'))}{2}$$
$$\beta(\Delta, \Delta') = \frac{\sigma_{\mathbb{H}^2}(g_l(\Delta), g_l(\Delta')) - \sigma_{\mathbb{H}^2}(g_r(\Delta), g_r(\Delta'))}{2}$$

where $\sigma^2_{\mathbb{H}}(\Delta_1, \Delta_2)$ *denotes the hyperbolic shear of the adjacent ideal triangles* $\Delta_1, \Delta_2 \subset \mathbb{H}^2$.

Proof. Identify $\mathbb{BP}^1$ with $\mathbb{RP}^1 \times \mathbb{RP}^1$. By Lemma 4.14, their left and right projections of Δ, Δ' are the ideal triangles $g_j(\Delta) = \Delta(a_j, b_j, c_j), g_j(\Delta') = \Delta(c_j, b_j, a'_j)$ where $j = l, r$ respectively. Notice that we have $a_j < b_j < a'_j < c_j$ on $\mathbb{RP}^1$ because the set $\Delta \cup \Delta'$ is acausal. In particular, $\sigma_{\mathbb{H}^2}(g_j(\Delta), g_j(\Delta')) = \beta^{\mathbb{R}}(b_j, c_j, a_j, a'_j)$ by Lemma 4.2. Recall that $\sigma(\Delta, \Delta'), \beta(\Delta, \Delta')$ are the real and para-complex parts of $\sigma^{\mathbb{B}}(\Delta, \Delta') = \sigma^{\mathbb{B}}(a, b, c, d)$ and that, by definition, $\sigma^{\mathbb{B}}(b, c, a, a') = \beta^{\mathbb{B}}(b, c, a, a')$. The conclusion follows from Lemma 4.7. □

We are ready to prove Theorem 1.

4.5. The proof of Theorem 1.

Let $\rho_{X,Y}$ be a Mess representation.

Consider the pleated set $\hat{S}_\lambda$ associated with the maximal lamination λ. Every point $x \in \hat{S}_\lambda - \hat{\lambda}$ lies in a plaque and, therefore, has a well-defined future pointing timelike unit normal direction $g(x)$. The map $g = (g_l, g_r) : \hat{S}_\lambda - \hat{\lambda} \to \mathbb{H}^2 \times \mathbb{H}^2$ is the *Gauss map* of the pleated set $\hat{S}_\lambda$. By Lemma 4.14, it is $\rho_{X,Y}$-equivariant and totally geodesic on each plaque.

Proof of Theorem 1. We split the proof into two cases.

Maximal laminations with countably many leaves. Let P, Q be distinct plaques. By definition and by Lemma 4.2, it is enough to consider the case where P, Q are either adjacent or asymptotic to the same leaf. The claim then follows from the computations of Lemmas 4.14 and 4.15.

General maximal laminations. The general case follows density of finite leaved maximal laminations in $\mathcal{GL}_m$ and continuity properties of the cocycles as given in Theorem 4.13. □

4.6. Shear-bend parametrization.

The proof of Theorem 5 is a combination of Theorem 1 and some properties of the classical shear coordinates $\Phi : \mathcal{T} \to \mathcal{H}(\lambda; \mathbb{R})$.

Proof of Theorem 5. We have

$$\mathcal{H}(\lambda; \mathbb{B}) = \frac{1 + \tau}{2} \mathcal{H}(\lambda; \mathbb{R}) \oplus \frac{1 - \tau}{2} \mathcal{H}(\lambda; \mathbb{R})$$

as $\mathbb{B}$-modules.

Part (1). Recall that $\sigma_\lambda^{\mathbb{B}} = \sigma + \tau\beta$ and that, by Theorem 1, we have $\sigma = (\sigma_\lambda^X + \sigma_\lambda^Y)/2$ and $\beta = (\sigma_\lambda^X - \sigma_\lambda^Y)/2$. Therefore, in terms of the above splitting, the shear-bend map decomposes as

$$\Psi : \rho_{X,Y} \to \sigma_\lambda^{\mathbb{B}} = \frac{1+\tau}{2}\sigma_\lambda^X \oplus \frac{1-\tau}{2}\sigma_\lambda^Y.$$

The single components $\Phi(X), \Phi(Y) = \sigma_\lambda^X, \sigma_\lambda^Y$ are analytic by Theorem 4.3. Injectivity also follows from the injectivity in the same theorem since:

$$\sigma_\rho^{\mathbb{B}} = \sigma_{\rho'}^{\mathbb{B}} \Longleftrightarrow \sigma_\lambda^X = \sigma_\lambda^{X'} \text{ and } \sigma_\lambda^Y = \sigma_\lambda^{Y'}.$$

It remains to be checked that the map respects the para-complex structures of $\mathcal{T} \times \mathcal{T}$ and $\mathcal{H}(\lambda; \mathbb{B})$. The para-complex structure $\mathbb{J}$ acts on $T_X\mathcal{T} \oplus T_Y\mathcal{T}$ simply as $\mathbb{J}(u,v) = (u,-v)$ and acts on $\mathcal{H}(\lambda;\mathbb{B})$ as the multiplication by τ. Denote by $\Phi : \mathcal{T} \to \mathcal{H}(\lambda;\mathbb{R})$ the classical shear coordinates, we have:

$$\begin{aligned} \mathrm{d}\Psi\mathbb{J}(u,v) = \mathrm{d}\Psi(u,-v) &= \frac{1+\tau}{2}\mathrm{d}\Phi(u) \oplus \frac{1-\tau}{2}(-\mathrm{d}\Phi(v)) \\ &= \tau\left(\frac{1+\tau}{2}\mathrm{d}\Phi(u) \oplus \frac{1-\tau}{2}\mathrm{d}\Phi(v)\right) = \tau\mathrm{d}\Psi(u,v). \end{aligned}$$

Part (2). The Thurston's symplectic form on $\mathcal{H}(\lambda;\mathbb{B})$ splits as

$$\omega_{\mathrm{Th}}^{\mathbb{B}} = \frac{1+\tau}{2}\omega_{\mathrm{Th}}^{\mathbb{R}} \oplus \frac{1-\tau}{2}\omega_{\mathrm{Th}}^{\mathbb{R}},$$

with respect to the above decomposition. Thus, by Theorem 4.4, we have

$$\begin{aligned} \omega_{\mathrm{Th}}^{\mathbb{B}}(\sigma_\rho^{\mathbb{B}}, \mu) &= \frac{1+\tau}{2}\omega_{\mathrm{Th}}^{\mathbb{R}}(\sigma_\lambda^X, \mu) + \frac{1-\tau}{2}\omega_{\mathrm{Th}}^{\mathbb{R}}(\sigma_\lambda^Y, \mu) \\ &= \frac{1+\tau}{2}L_X(\mu) + \frac{1-\tau}{2}L_Y(\mu) \\ &= \frac{L_X(\mu) + L_Y(\mu)}{2} + \tau\frac{L_X(\mu) - L_Y(\mu)}{2} \end{aligned}$$

for every $\mu \in \mathcal{ML}_\lambda$. We will see in the next section that $L_\rho = (L_X + L_Y)/2$ and $\theta_\rho = (L_X - L_Y)/2$.

Part (3). By part (1), the image of Ψ is

$$\{\sigma + \tau\beta \in \mathcal{H}(\lambda;\mathbb{B}) \mid \sigma + \beta, \sigma - \beta \in \mathcal{T} \subset \mathcal{H}(\lambda;\mathbb{R})\}.$$

By Theorem 4.3, we have

$$\mathcal{T} = \{\sigma \in \mathcal{H}(\lambda, \mathbb{R}) \mid \omega_{\mathrm{Th}}(\sigma, \bullet) > 0 \text{ on } \mathcal{ML}_\lambda\},$$

Thus

$$\begin{aligned} &\sigma + \tau\beta \in \Psi(\mathcal{T} \times \mathcal{T}) \Leftrightarrow \omega_{\mathrm{Th}}(\sigma \pm \beta, \mu) > 0 \\ &\Leftrightarrow \omega_{\mathrm{Th}}(\sigma,\mu)^2 - \omega_{\mathrm{Th}}(\beta,\mu)^2 = \left|\omega_{\mathrm{Th}}^{\mathbb{B}}(\sigma + \tau\beta, \mu)\right|_{\mathbb{B}}^2 > 0 \end{aligned}$$

for every $\mu \in \mathcal{ML}_\lambda$.

Part (4). By work of Bonahon and Sözen [8], we have that $\Phi^*\omega_{\mathrm{Th}} = c\cdot\omega_{\mathrm{WP}}$. The conclusion comes from the fact that Ψ splits as $\frac{1+\tau}{2}\Phi\oplus\frac{1-\tau}{2}\Phi$ and $\omega^{\mathbb{B}}_{\mathrm{Th}}$ splits as $\frac{1+\tau}{2}\omega^{\mathbb{R}}_{\mathrm{Th}}\oplus\frac{1-\tau}{2}\omega^{\mathbb{R}}_{\mathrm{Th}}$. □

5. LENGTH FUNCTIONS IN ANTI DE SITTER 3-MANIFOLDS

In this section we study the anti de Sitter length functions associated with Mess representations and prove Theorem 2.

5.1. Moving endpoints orthogonally.

Let us start with some estimates in $\mathbb{H}^{2,1}$ on how the length of a spacelike segment changes if we move its endpoints orthogonally in timelike directions. The following is an elementary computation:

Lemma 5.1. *Let $[x,y]$ be a spacelike segment. Let $v\in T_x\mathbb{H}^{2,1}, w\in T_y\mathbb{H}^{2,1}$ be unit timelike vectors orthogonal to $[x,y]$. Consider $p=\cos(t)x+\sin(t)v$ and $q=\cos(t)y+\sin(t)w$. Then*

1. *$[v,w]$ lies on the dual geodesic of $[x,y]$. Hence, it is spacelike.*
2. *We have*

$$-\langle p,q\rangle=\cos(t)^2\cosh(d_{\mathbb{H}^{2,1}}(x,y))+\sin(t)^2\cosh(d_{\mathbb{H}^{2,1}}(v,w)).$$

As $-\langle p,q\rangle>1$, $[p,q]$ is spacelike and $\cosh(d_{\mathbb{H}^{2,1}}(p,q))=-\langle p,q\rangle$.

In order to manipulate better some inequalities, later on we will use several times the following estimates on hyperbolic trigonometric functions:

Lemma 5.2. *We have:*

1. *For every $\epsilon,\delta>0$ there exists $\kappa>0$ such that*

$$\cos(\delta)^2\cosh(x)+\sin(\delta)^2\cosh(x-\epsilon)\leqslant\cosh(x-\kappa)$$

for every $x\geqslant\epsilon$.

2. *For every $\epsilon, L_0>0$ there exists $a_0<1$ such that*

$$a\cosh(L)\geqslant\cosh(L-\epsilon)$$

for every $L\geqslant L_0$ and $a\in(a_0,1)$.

3. *For every $a>1$ we have $\cosh(ax)\geqslant a\cosh(x)$.*

Proof. A straightforward computation shows that for every fixed $b>0$ the function $\cosh(x-b)/\cosh(x)$ defined on the interval $[\epsilon,\infty)$ is decreasing, so that we have $e^{-b}\leqslant\cosh(x-b)/\cosh(x)\leqslant\cosh(\epsilon-b)/\cosh(\epsilon)$.

Inequality (2). As $\cosh(x-\epsilon)/\cosh(x)$ is decreasing on $[\epsilon,\infty)$, it is bounded from above by $\cosh(L_0-\epsilon)/\cosh(L_0)<1$. It is enough to choose a_0 in the interval $[\cosh(L_0-\epsilon)/\cosh(L_0),1]$.

Inequality (1). We first rewrite it as

$$\cos(\delta)^2+\sin(\delta)^2\frac{\cosh(x-\epsilon)}{\cosh(x)}\leqslant\frac{\cosh(x-\kappa)}{\cosh(x)}.$$

As $\cosh(x-a)/\cosh(x)$ is decreasing, we have

$$\cos(\delta)^2+\sin(\delta)^2\frac{\cosh(x-\epsilon)}{\cosh(x)}\leqslant\cos(\delta)^2+\sin(\delta)^2\frac{1}{\cosh(\epsilon)}<1,$$

So, it is enough to choose $\kappa>0$ so that $\cos(\delta)^2+\sin(\delta)^2/\cosh(\epsilon)<e^{-\kappa}$.

Inequality (3). The function $\cosh(ax)-a\cosh(x)$ has derivative $a(\cosh(ax)-\cosh(x))$ which is positive when $a>1$. ☐

5.2. **Length and pleated surfaces.** We now introduce loxodromic transformations of $\mathbb{H}^{2,1}$ and the length functions associated to Mess representations.

Definition 5.3 (Loxodromic)**.** An isometry $\gamma=(A,B)\in\mathrm{PSL}_2(\mathbb{R})\times\mathrm{PSL}_2(\mathbb{R})$ is *loxodromic* if A,B are both loxodromic transformations of $\mathrm{PSL}_2(\mathbb{R})$. A loxodromic transformation γ preserves two disjoint (dual) lines

$$\ell=[(x_A^+,x_B^+),(x_A^-,x_B^-)],\ell^*=[(x_A^+,x_B^-),(x_A^-,x_B^+)]\subset\mathbb{H}^{2,1},$$

where $x_A^\pm,x_B^\pm$ are the attracting and repelling fixed points of A,B on $\mathbb{RP}^1$, and acts on them by translations by

$$L(\gamma)=\frac{L(A)+L(B)}{2}\quad\text{and}\quad\theta(\gamma)=\frac{|L(A)-L(B)|}{2}$$

respectively where $L(A),L(B)$ are the translation lengths of A,B. The quantities $L(\gamma)$ and $\theta(\gamma)$ are the *translation length* and *torsion* of γ.

Notice that if $\rho_{X,Y}$ is a Mess representation, then for every $\gamma\in\Gamma-\{1\}$ the transformation $\rho_{X,Y}(\gamma)=(\rho_X(\gamma),\rho_Y(\gamma))$ is loxodromic because ρ_X,ρ_Y are holonomies of hyperbolic structures. Furthermore, as $\Lambda_{X,Y}\subset\mathbb{RP}^1\times\mathbb{RP}^1$ is the graph of the unique $(\rho_X-\rho_Y)$-equivariant homeomorphism $h_{X,Y}:\mathbb{RP}^1\times\mathbb{RP}^1$, we see that the axis ℓ_γ of $\rho_{X,Y}(\gamma)$, having the endpoints on $\Lambda_{X,Y}$, is contained in $\mathcal{CH}_{X,Y}$.

We are now ready to prove the first part of Theorem 2.

Proposition 5.4. *Let $\rho_{X,Y}$ a Mess representation. Let $\gamma\in\Gamma-\{1\}$ be a non-trivial element, denote by $\ell\subset\mathcal{CH}_{X,Y}$ the axis of $\rho_{X,Y}(\gamma)$. Let $\lambda\subset\Sigma$ be a maximal lamination, let $Z_\lambda\in\mathcal{T}$ be the intrinsic hyperbolic structure on $\hat{S}_\lambda/\rho_{X,Y}(\Gamma)$ where $\hat{S}_\lambda\subset\mathcal{CH}_{X,Y}$ is the pleated set associated with λ. Let δ be the maximal timelike distance of ℓ from $\hat{S}_\lambda$. Then:*

$$\cosh(L_Z(\gamma))\leqslant\cos(\delta)^2\cosh(L_\rho(\gamma))+\sin(\delta)^2\cosh(\theta_\rho(\gamma)).$$

Proof. Let $x\in\ell,y\in\hat{S}_\lambda$ be points that realize the maximal timelike distance δ. Notice that the timelike segment $[x,y]$ is orthogonal to ℓ at x. Denote by $v\in T_x\mathbb{H}^{2,1}$ the unit timelike vector tangent to $[x,y]$. We can write $y=\cos(\delta)x+\sin(\delta)v$.

We now apply Lemma 5.1 to the spacelike segment $[x, \rho_{X,Y}(\gamma)x] \subset \ell$ and the timelike unit tangent vectors $v, \rho_{X,Y}(\gamma)v$. We have:

$$\begin{aligned}&\cosh\left(d_{\mathbb{H}^{2,1}}(y, \rho_{X,Y}(\gamma)y)\right)\\ &= \cos(\delta)^2 \cosh\left(d_{\mathbb{H}^{2,1}}(x, \rho_{X,Y}(\gamma)x)\right) + \sin(\delta)^2 \cosh\left(d_{\mathbb{H}^{2,1}}(v, \rho_{X,Y}(\gamma)v)\right).\end{aligned}$$

Notice that $d_{\mathbb{H}^{2,1}}(x, \rho_{X,Y}(\gamma)x) = L_\rho(\gamma)$ and $d_{\mathbb{H}^{2,1}}(v, \rho_{X,Y}(\gamma)v) = \theta_\rho(\gamma)$.

The conclusion then follows from Proposition 3.14 which says that the intrinsic hyperbolic distance between $y, \rho_{X,Y}(\gamma)y$ on $\hat{S}_\lambda$ is smaller than $d_{\mathbb{H}^{2,1}}(y, \rho_{X,Y}(\gamma)y)$ and the fact that $L_Z(\gamma)$ coincides with the minimal displacement of $\rho_{X,Y}(\gamma)$ with respect to the hyperbolic metric on $\hat{S}_\lambda$. □

5.3. Intersection and pleated surfaces.

We then prove the second part of Theorem 2.

Proposition 5.5. *Let $\rho_{X,Y}$ be a Mess representation. Let $\gamma \in \Gamma - \{1\}$ be a non-trivial element, denote by $\ell \subset \mathcal{CH}_{X,Y}$ the axis of $\rho_{X,Y}(\gamma)$. Let $\delta^\pm, \Delta^\pm$ be the maximal timelike distances of ℓ from $\lambda^\pm$ and $\partial^\pm \mathcal{CH}_{X,Y}$. Then:*

(1) We have

$$\cosh\big(i(\lambda^\pm, \gamma)\big) \leqslant \sin\big(\delta^\pm\big)^2 \cosh(L_\rho(\gamma)) + \cos\big(\delta^\pm\big)^2 \cosh(\theta_\rho(\gamma)).$$

(2) We have

$$i(\lambda^\pm, \gamma) \geqslant \cos\big(\Delta^\pm\big)^2 \theta_\rho(\gamma).$$

Proof of part (1) of Proposition 5.5. Let $[x, x^\pm]$ be a timelike segment, with $x \in \ell, x^\pm \in \ell^\pm \subset \lambda^\pm$ that realizes the maximal timelike distance $\delta^\pm$. Notice that $[x, x^\pm]$ is orthogonal to both $\ell, \ell^\pm$. Let $v \in T_x\mathbb{H}^{2,1}, v^\pm \in \mathcal{T}_{x^\pm}\mathbb{H}^{2,1}$ be the unit speed timelike vectors tangent to the geodesic $[x, x^\pm]$ at the endpoints.

Claim 1. We have

$$\cosh\big(d_{\mathbb{H}^{2,1}}(v^\pm, \rho_{X,Y}(\gamma)v^\pm)\big) = \sin\big(\delta^\pm\big)^2 \cosh(L_\rho(\gamma)) + \cos\big(\delta^\pm\big)^2 \theta_\rho(\gamma).$$

Proof of the claim. Note that

$$v^\pm = \cos\big(\pi/2 - \delta^\pm\big)x + \sin\big(\pi/2 - \delta^\pm\big)v$$

and that v and $\rho_{X,Y}(\gamma)v$ are both orthogonal to the segment $[x, \rho_{X,Y}(\gamma)x] \subset \ell$. The claim follows from Lemma 5.1. □

Claim 2. Let $v, v', v'' \in \mathbb{H}^{2,1}$ be dual to the supporting planes H, H', H'' of $\partial^+\mathcal{CH}_{X,Y}$.

(i) If v, v', v'' are aligned along $\mathcal{S}^+$ then $H \cap H' \cap H'' = \varnothing$.

(ii) If $v < v' < v''$ along $\mathcal{S}^+$, then the reverse triangle inequality holds

$$d_{\mathbb{H}^{2,1}}(v, v'') \geqslant d_{\mathbb{H}^{2,1}}(v, v') + d_{\mathbb{H}^{2,1}}(v', v'').$$

Proof of the claim. The first part: Consider the faces $F, F', F'' = H, H', H'' \cap \partial^+\mathcal{CH}_{X,Y}$. As $\mathcal{S}^+$ is an $\mathbb{R}$-tree, there are two possibilities: Either one of the faces separates the other two on $\partial^+\mathcal{CH}_{X,Y}$ or there is a unique face $G \subset \partial^+\mathcal{CH}_{X,Y}$ that separates every pair of them. The first case corresponds to the configuration where the dual points v, v', v'' are aligned. The second case corresponds to the configuration where v, v', v'' are the vertices of a tripod in $\mathcal{S}^+$ with center w, the dual point of G. Let us consider the first case. In addition, let us assume that $v < v' < v''$ without loss of generalities. Then F' separates $H \cap H'$ from $H' \cap H''$ in H'. Hence the triple intersection $H \cap H' \cap H''$ is empty.

The second part of the claim follows from Lemma 6.3.5 of [2]. $\square$

Claim 3. Let $v, w \in \mathcal{S}^+$ be distinct points. Then

$$d_{\mathcal{S}^+}[v, w] \leqslant d_{\mathbb{H}^{2,1}}(v, w).$$

Proof of the claim. Let $\alpha : I = [0,1] \to \mathcal{S}^+$ be an injective weakly regular path joining v and w. By Lemma 3.12, we have

$$L = \int_I |\dot{\alpha}(t)| \mathrm{dt} = \lim_{\epsilon \to 0} \int_I \frac{d_{\mathbb{H}^{2,1}}(\alpha(t), \alpha(t+\epsilon))}{\epsilon} \mathrm{dt}.$$

If $\epsilon < \epsilon_0$ then

$$\left| \int_I \frac{d_{\mathbb{H}^{2,1}}(\alpha(t), \alpha(t+\epsilon))}{\epsilon} \mathrm{dt} - L \right| < \delta$$

Choose $\epsilon = 1/2^k$. For convenience, we take dyadic approximations of the integral with Riemann sums:

$$\int_I \frac{d_{\mathbb{H}^{2,1}}(\alpha(t), \alpha(t+1/2^k))}{1/2^k} \mathrm{dt} = \lim_{n \to \infty} \sum_{p=0}^{2^n - 2^{n-k}} \frac{d_{\mathbb{H}^{2,1}}(\alpha(p/2^n), \alpha(p/2^n + 1/2^k))}{1/2^k} \cdot \frac{1}{2^n}.$$

We reorganize the sum as

$$\begin{aligned}
&\frac{2^k}{2^n} \sum_{p=0}^{2^{n-k}} d_{\mathbb{H}^{2,1}} \left(\alpha\left(\frac{p}{2^n}\right), \alpha\left(\frac{p}{2^n} + \frac{1}{2^k}\right) \right) \\
&= \frac{2^k}{2^n} \sum_{j=1}^{2^{n-k}} \sum_{q=1}^{2^k - 1} d_{\mathbb{H}^{2,1}} \left(\alpha\left(\frac{j}{2^n} + \frac{q}{2^k}\right), \alpha\left(\frac{j}{2^n} + \frac{q+1}{2^k}\right) \right) \\
&\leqslant \frac{2^k}{2^n} \sum_{j=1}^{2^{n-k}} d_{\mathbb{H}^{2,1}} \left(\alpha(0), \alpha(L)\right) = d_{\mathbb{H}^{2,1}} \left(\alpha(0), \alpha(L)\right) \qquad \text{by Claim 2.}
\end{aligned}$$

$\square$

We have:

$$\begin{aligned}
\cosh(i(\lambda, \gamma)) &\leqslant \cosh(d_{\mathcal{S}^+}(v, \rho_{X,Y}(\gamma)v)) \\
&\leqslant \cosh(d_{\mathbb{H}^{2,1}}(v, \rho_{X,Y}(\gamma)v)) && \text{by Claim 3} \\
&= \sin\left(\delta^{\pm}\right)^2 \cosh(L_\rho(\gamma)) + \cos\left(\delta^{\pm}\right)^2 \cosh(\theta_\rho(\gamma)) && \text{by Claim 1.}
\end{aligned}$$

$\square$

Proof of part (2) of Proposition 5.5. We start with the following observation: Let $H, \ell \subset \mathbb{H}^{2,1}$ by a spacelike plane and a spacelike geodesic. If H and ℓ have disjoint closures, then there exists a unique timelike segment $[x, y]$ with $x \in \ell, y \in H$ which is orthogonal to both.

We apply the above to the following configuration: Consider ℓ, the axis of $\rho_{X,Y}(\gamma)$. Being contained in $\mathcal{CH}_{X,Y}$, ℓ has disjoint closure from every support plane H of $\partial^+\mathcal{CH}_{X,Y}$ and, hence, it is connected to H by a timelike segment $[x, y]$ with $x \in \ell, y \in H$ which is orthogonal to both. Denote by v, w respectively the unit timelike velocity of the segment $[x, y]$ at x, y respectively.

Fix $\epsilon > 0$. We prove that

$$i(\gamma, \lambda^\pm) + \epsilon \geqslant \cos\big(\Delta^\pm\big)^2 \theta(\gamma).$$

Let $\alpha : I \to \mathcal{S}^\pm$ be a geodesic of length $\ell(\alpha) = i(\gamma, \lambda^\pm)$ with $\alpha(1) = \rho_{X,Y}(\gamma)\alpha(0)$. Recall that, by Lemma 3.12, we have

$$i(\gamma, \lambda^\pm) = \ell(\alpha) = \lim_{\epsilon \to 0} \int_I \frac{d_{\mathbb{H}^{2,1}}(\alpha(t), \alpha(t+\epsilon))}{\epsilon} \mathrm{dt}.$$

We deduce that:

Claim 1. There exists a dyadic subdivision with

$$\frac{2^k}{2^n} \sum_{j=1}^{2^{n-k}} \sum_{q=1}^{2^k} d_{\mathbb{H}^{2,1}}(\alpha(j/2^n + q/2^k), \alpha(j/2^n + (q+1)/2^k)) \leqslant i(\lambda, \gamma) + \epsilon.$$

We simplify the notation by introducing $w_{j,q} := \alpha(j/2^n + q/2^k)$ and $d_{j,q} := d_{\mathbb{H}^{2,1}}(w_{j,q}, w_{j,q+1})$. By Claim 1, we have:

$$\frac{2^k}{2^n} \sum_{j=1}^{2^{n-k}} \sum_{q=1}^{2^k} d_{\mathbb{H}^{2,1}}(w_{j,q}, w_{j,q+1}) = \frac{2^k}{2^n} \sum_{j=1}^{2^{n-k}} \sum_{q=1}^{2^k L} d_{j,q} \leqslant i(\lambda, \gamma) + \epsilon.$$

Each $w_{j,q}$ is dual to a support plane $H_{j,q}$ of $\partial^+\mathcal{CH}$. We associate to it the segment $[x_{j,q}, y_{j,q}]$ of length $\delta_{j,q} > 0$ and the velocities $v_{j,q}, w_{j,q}$ as provided by the initial observations. Consider the spacelike distances

- $\theta_{j,q} := d_{\mathbb{H}^{2,1}}(v_{j,q}, v_{j,q+1})$.
- $D_{j,q} := d_{\mathbb{H}^{2,1}}(x_{j,q}, x_{j,q+1})$.

Claim 2. We have

- $\theta_\rho(\gamma) \leqslant \theta_{j,1} + \cdots + \theta_{j,2^{n-k}}$.
- $L_\rho(\gamma) \leqslant D_{j,1} + \cdots + D_{j,2^{n-k}}$.

Proof of the claim. Recall that $x_{j,q}$ and $v_{j,q}$ are respectively aligned along the axis ℓ of γ and its dual line ℓ^*. Thus, we have

$$\theta_\rho(\gamma) = d_{\mathbb{H}^{2,1}}(v_{j,1}, \rho(\gamma)v_{j,1} = v_{j,2^{n-k}}) \leqslant \sum_{q \leqslant 2^{n-k}} d_{\mathbb{H}^{2,1}}(v_{j,q}, v_{j,q+1})$$

and

$$L_\rho(\gamma) = d_{\mathbb{H}^{2,1}}(x_{j,1}, \rho(\gamma)x_{j,1} = x_{j,2^{n-k}}) \leqslant \sum_{q \leqslant 2^{n-k}} d_{\mathbb{H}^{2,1}}(x_{j,q}, x_{j,q+1})$$

□

By Lemma 5.1, we have

$$\cosh(d_{j,q}) = \sin(\delta_{j,q})\sin(\delta_{j,q+1})\cosh(D_{j,q}) + \cos(\delta_{j,q})\cos(\delta_{j,q+1})\cosh(\theta_{j,q}).$$

Notice that $\sin(\delta_{j,q})\sin(\delta_{j,q+1}) + \cos(\delta_{j,q})\cos(\delta_{j,q+1}) = \cos(\delta_{j,q} - \delta_{j,q+1})$. By part (3) of Lemma 5.2, we have

$$\cosh\left(\frac{d_{j,q}}{\cos(\delta_{j,q} - \delta_{j,q+1})}\right) \geqslant \frac{\cosh(d_{j,q})}{\cos(\delta_{j,q} - \delta_{j,q+1})}.$$

The right hand side is equal to

$$\frac{\sin(\delta_{j,q})\sin(\delta_{j,q+1})\cosh(D_{j,q}) + \cos(\delta_{j,q})\cos(\delta_{j,q+1})\cosh(\theta_{j,q})}{\cos(\delta_{j,q} - \delta_{j,q+1})},$$

therefore, by convexity of $\cosh(\bullet)$, it is greater than

$$\geqslant \cosh\left(\frac{\sin(\delta_{j,q})\sin(\delta_{j,q+1})D_{j,q} + \cos(\delta_{j,q})\cos(\delta_{j,q+1})\theta_{j,q}}{\cos(\delta_{j,q} - \delta_{j,q+1})}\right).$$

As $\cosh(\bullet)$ is increasing, we conclude

$$d_{j,q} \geqslant \sin(\delta_{j,q})\sin(\delta_{j,q+1})D_{j,q} + \cos(\delta_{j,q})\cos(\delta_{j,q+1})\theta_{j,q}.$$

Denoting by $\Delta := \max\{\delta_{j,q}\}$ and $\delta := \min\{\delta_{j,q}\}$ and using the monotonicity of $\cos(\bullet)$ and $\sin(\bullet)$, we can continue the previous inequality to

$$\geqslant \cos(\Delta)^2\theta_{j,q} + \sin(\delta)^2 D_{j,q}.$$

Summing all the contributions, by Claim 2, we get

$$\sum_{q=1}^{2^k L} d_{j,q} \geqslant \cos(\Delta)^2\theta_\rho(\gamma) + \sin(\delta)^2 L_\rho(\gamma).$$

In conclusion,

$$\begin{aligned} i(\lambda, \gamma) + \epsilon &\geqslant \frac{2^k}{2^n}\sum_{j=1}^{2^{n-k}}\sum_{q=1}^{2^k} d_{j,q} \\ &\geqslant \frac{2^k}{2^n}\sum_{j=1}^{2^{n-k}} \cos(\Delta)^2\theta_\rho(\gamma) + \sin(\delta)^2 L_\rho(\gamma) \\ &= \cos(\Delta)^2\theta_\rho(\gamma) + \sin(\delta)^2 L_\rho(\gamma). \end{aligned}$$

As $\epsilon > 0$ was arbitrary, this finishes the proof. □

6. Length functions in Teichmüller space

In this section we carry out an anti de Sitter analysis of length function on Teichmüller space on both global and infinitesimal scales and prove Theorems 3 and 4.

6.1. Orthogonal projection to a line.

We begin with some explicit computations on the orthogonal projection $\pi : \mathbb{H}^{2,1} \to \ell$ to a spacelike geodesic.

Lemma 6.1. *Let y, ℓ be a point and a spacelike line in $\mathbb{H}^{2,1}$ such that the rays $[y, \ell^{\pm}]$ are spacelike. Then*

$$m_{y,\ell} = \min_{x \in \ell}\{-\langle y, x\rangle\}) = \sqrt{\frac{2\langle y, \ell^+\rangle\langle y, \ell^-\rangle}{-\langle \ell^+, \ell^-\rangle}}$$

and it is realized at the unique point

$$x = \frac{1}{\sqrt{-2\langle \ell^+, \ell^-\rangle}}\left(\sqrt{\frac{\langle y, \ell^-\rangle}{\langle y, \ell^+\rangle}}\ell^+ + \sqrt{\frac{\langle y, \ell^+\rangle}{\langle y, \ell^-\rangle}}\ell^-\right) \in \ell$$

such that $[y, x]$ is orthogonal to ℓ.

Proof. Write $\ell(t) = (e^t\ell^+ + e^{-t}\ell^-)/\sqrt{-2\langle \ell^+, \ell^-\rangle}$ and consider the function $f(t) = -\langle \ell(t), y\rangle$. As $[y, \ell^+], [y, \ell^-]$ are spacelike, we have $f(t) \to \infty$ as $|t| \to \infty$. Hence, $f(t)$ has a minimum which is a critical point. The unique critical point of the function is at $e^{2t} = \langle y, \ell^-\rangle/\langle y, \ell^+\rangle$. The conclusion follows by elementary computations. □

6.2. Convexity of length functions.

We now describe the purely anti de Sitter proof of (strict) convexity of length functions on Teichmüller space $\mathcal{T}$ in shear coordinates for an arbitrary maximal lamination $\lambda \subset \Sigma$.

We prove separately the two parts of Theorem 3.

Proposition 6.2. *Let $\lambda \subset \Sigma$ be a maximal lamination. Let $\gamma \in \Gamma - \{1\}$ be a non-trivial loop. The length function $L_\gamma : \mathcal{T} \subset \mathcal{H}(\lambda; \mathbb{R}) \to (0, \infty)$ is convex. Moreover, convexity is strict if γ intersects essentially every leaf of λ.*

Proof. Recall that a function $L : U \subset \mathbb{R}^n \to \mathbb{R}$ defined on an open convex subset $U \subset \mathbb{R}^n$ is (strictly) convex if and only if for every $x, y \in U$ we have a (strict) inequality

$$L\left(\frac{x+y}{2}\right) \leqslant \frac{L(x) + L(y)}{2}.$$

Consider $X, Y \in \mathcal{T}$. Let $\rho_{X,Y}$ be the corresponding Mess representation. Let $\hat{S}_\lambda \subset \mathcal{CH}_{X,Y}$ be the pleated set associated with λ and let $\rho_\lambda : \Gamma \to \mathrm{PSL}_2(\mathbb{R})$ be the holonomy of the intrinsic hyperbolic structure $Z_\lambda \in \mathcal{T}$ on $\hat{S}_\lambda/\rho_{X,Y}(\Gamma)$. By Theorem 1 we have $Z_\lambda = (X+Y)/2$ in $\mathcal{H}(\lambda; \mathbb{R})$. By Theorem 2, we have

$$\cosh(L_{Z_\lambda}(\gamma)) \leqslant \cos(\delta)^2 \cosh(L_\rho(\gamma)) + \sin(\delta)^2 \cosh(\theta_\rho(\gamma))$$

where δ is the maximal timelike distance of the axis ℓ of $\rho_{X,Y}(\gamma)$ from the pleated set $\hat{S}_\lambda$. Notice that $\delta > 0$ unless $\ell \subset \hat{S}_\lambda$ in which case ℓ does not intersect the bending locus. Also observe that unless ρ is Fuchsian, which happens precisely when $X = Y$, the bending locus cannot be empty. Therefore, if X, Y are distinct and the support of $\gamma \in \mathcal{C}$ intersects essentially every leaf of λ we have $\delta > 0$ and

$$\cos(\delta)^2 \cosh(L_\rho(\gamma)) + \sin(\delta)^2 \cosh(\theta_\rho(\gamma)) < \cosh(L_\rho(\gamma))$$

as $L_\rho(\gamma) < \theta_\rho(\gamma)$. Since $\cosh(\bullet)$ is strictly increasing on $(0, \infty)$, we conclude $L_Z(\gamma) < L_\rho(\gamma)$. □

Proposition 6.3. *Let $\lambda \subset \Sigma$ be a maximal lamination. Let $\gamma \in \mathcal{ML}$ be a measured lamination. The length function $L_\gamma : \mathcal{T} \subset \mathcal{H}(\lambda; \mathbb{R}) \to (0, \infty)$ is convex. Furthermore, convexity is strict if the support of γ intersects transversely each leaf of λ.*

Proof. We immediately deduce convexity by Proposition 6.2 and density of weighted simple curves in $\mathcal{ML}$ and $\mathcal{C}^\infty$-convergence of length functions $L_{\gamma_n} \to L_\gamma$ if $\gamma_n \to \gamma$ in $\mathcal{ML}$.

We now discuss strict convexity.

Consider $X, Y \in \mathcal{T}$ and the Mess representation $\rho := \rho_{X,Y}$. Let $\hat{S}_\lambda \subset \mathcal{CH}_{X,Y}$ be the pleated set associated with λ. Let $\gamma \in \mathcal{ML}$ be a measured lamination whose support contains a leaf ℓ that intersects the bending locus of $\hat{S}_\lambda$ (which is non-empty unless the representation is Fuchsian).

Since ℓ intersects the bending locus, its geometric realization $\hat{\ell}$ is not contained on $\hat{S}_\lambda$. Let $x \in \hat{\ell}$ and $y \in \hat{S}_\lambda$ be points that realize the maximal timelike distance $\delta = \max\{\delta_{\mathbb{H}^{2,1}}(z, t) \mid z \in \hat{S}_\lambda, t \in \hat{\ell}\} > 0$.

Let $K := I \times J$ denote the neighborhood of ℓ in the space of geodesics $\mathcal{G}$ consisting of those lines with one endpoint in I and another endpoint in J.

Recall that, by Lemma 6.1, we have

$$m_{z,\ell} := \min_{t \in \ell}\{-\langle z, t\rangle\}) = \sqrt{\frac{2\langle z, \ell^+\rangle\langle z, \ell^-\rangle}{-\langle \ell^+, \ell^-\rangle}}$$

and that the minimum is realized at a point $\pi(z) \in \ell$, the orthogonal projection of z to ℓ, described explicitly by

$$\pi(z) = \frac{1}{\sqrt{-2\langle \ell^+, \ell^-\rangle}} \left(\sqrt{\frac{\langle z, \ell^-\rangle}{\langle z, \ell^+\rangle}} \ell^+ + \sqrt{\frac{\langle z, \ell^+\rangle}{\langle z, \ell^-\rangle}} \ell^- \right).$$

As y is connected to ℓ by a timelike segment of length δ orthogonal to ℓ, we have $m_{y,\ell} = \cos(\delta)$. As $x \in \ell$ we have $m_{x,\ell} = 1$. By continuity of the above expressions, we have:

Claim 1. There exist a neighborhood $K = I \times J$ of $\ell \in \mathcal{G}$ and a neighborhood U of x in $\mathbb{H}^{2,1}$ with the following properties:

(i) $m_{y,\ell'} \in (\cos(2\delta), \cos(\delta/2))$ for every $\ell' \in K$. In particular, y is connected to every $\ell' \in K$ by a timelike segment of length at least $\delta/2$ and, hence, $\delta_{\mathbb{H}^{2,1}}(y, \ell) \geqslant \delta/2$.
(ii) Every $\ell' \in K$ intersects U.
(iii) If $\ell_1, \ell_2 \in K$, $\ell_1 \cup \ell_2$ is acausal, and $z_j \in \ell_j \cap U$, then $d_{\mathbb{H}^{2,1}}(z_1, z_2) < \epsilon$.
(iv) $m_{z,\ell'} \in (\cos(\epsilon), \cosh(\epsilon))$ for every $\ell' \in K$ and $z \in U$.
(v) For every $x, y \in U$ and $\ell' \in K$, we have $d_{\mathbb{H}^{2,1}}(\pi(x), \pi(y)) < \epsilon$ where π is the orthogonal projection to ℓ'.

Let K and U be the neighborhoods provided by the claim.

As ℓ lies in the support of γ, we have $m := \gamma(K) > 0$.

We approximate γ in $\mathcal{ML}$ with a sequence of weighted simple closed curves $a_n\gamma_n$. By convergence of $a_n\gamma_n$ to γ, we have $a_n m_n := a_n\gamma_n(K) \to m$. Notice that m_n is the number of distinct leaves of the geometric realization $\hat{\gamma}_n$ contained in K. Let ℓ_n be one of those leaves.

Claim 2. Fix $\epsilon > 0$. We can find elements

$$\alpha_1, \cdots, \alpha_{m_n} \in \Gamma$$

and corresponding points

$$z_1 < \cdots < z_{m_n}$$

on $\hat{\ell}_n$ with the following properties:

(i) $\alpha_{m_n} \cdots \alpha_1 = \gamma_n$.
(ii) $d_{\mathbb{H}^{2,1}}(z_j, \rho(\alpha_j \cdots \alpha_1)x) < \epsilon$ for every $j \leqslant m_n$.
(iii) The axis of α_j lies in $\alpha_{j-1} \cdots \alpha_1(K)$.

Proof of the claim. Consider the m_n translates of ℓ_n contained in K

$$\{\ell_n = \ell_n^1, \cdots, \ell_n^{m_n}\}.$$

For each of them there exists a point $x_n^j \in \ell_n^j \cap U$ and an element β_n^j such that $z_n^j := \rho(\beta_n^j)x_n^j$ lies in $[x, \rho(\gamma_n)x] \subset \ell_n$. We assume that the numbering agrees with the linear order along $[x, \rho(\gamma_n)x]$, that is

$$x = z_0 < z_1 < \cdots < z_{m_n} = \rho(\gamma_n)x.$$

Set $\alpha_j := \beta_j\beta_{j-1}^{-1}$ with $\beta_0 = 1$.

Property (i) follows by construction.

Property (ii) follows from Claim 1.

Property (iii) follows from stability of quasi-geodesics on $\hat{S}_n$, the pleated set associated with the lamination λ_n consisting of the closed geodesic γ_n suitably completed to a maximal lamination of Σ by adding finitely many leaves spiraling around γ_n.

Let us explain how: Consider the concatenation of the translates

$$l = \bigcup_{k \in \mathbb{Z}} \rho(\alpha_j)^k \left([x_{j-1}, z_j]_{\hat{S}_n} \cup [z_j, x_j]_{\hat{S}_n} \right).$$

By basic hyperbolic geometry, l is a uniform quasi geodesic on $\hat{S}_n$ with respect to the intrinsic hyperbolic metric, with quasi-geodesic constants that are $O(\epsilon)$-close to 1. Hence, the invariant axis of $\rho(\alpha_j)$ on $\hat{S}_n$ lies in the $O(\epsilon)$-neighborhood of l with respect to the hyperbolic metric. In particular such endpoints are close to the endpoints of ℓ_n on the Gromov boundary $\partial_\infty \hat{S}_n$.

Let $\phi_n : \partial_\infty \hat{S}_n \to \partial\Gamma$ be the unique equivariant homeomorphism. By Lemma 3.16, the hyperbolic structures $\hat{S}_n/\rho(\Gamma)$ lie in a compact subspace of Teichmüller space $\mathcal{T}$. Thus, as the boundary maps ϕ_n depend continuously on S_n, they are uniformly continuous. This implies that if ϵ is small enough, the endpoints of α_j are contained in K. □

Define $x_j := \rho(\alpha_j \cdots \alpha_1)x$ and $y_j = \rho(\alpha_j \cdots \alpha_1)y$.

Let $\delta_j := \delta_{\mathbb{H}^{2,1}}(y_{j-1}, \ell_{\alpha_j})$ be the timelike distance of $y_j \in \hat{S}_\lambda$ from ℓ_{α_j}, the axis of $\rho(\alpha_j)$. By Property (ii) of Claim 2, we have $(\alpha_{j-1} \cdots \alpha_1)^{-1}\ell_{\alpha_j} \in K$, hence, by Claim 1, we deduce that

$$\delta_j = \delta_{\mathbb{H}^{2,1}}(y_{j-1}, \ell_{\alpha_j}) = \delta_{\mathbb{H}^{2,1}}(y, \rho(\alpha_{j-1} \cdots \alpha_1)^{-1}\ell_{\alpha_j}) > \delta/2.$$

Claim 3. We have

$$\begin{aligned}\cosh(d_{\mathbb{H}^{2,1}}(y_{j-1}, y_j)) = \cos(\delta_j)^2 \cosh(L_\rho(\alpha_j)) + \sin(\delta_j)^2 \cosh(\theta_\rho(\alpha_j))\\ \leqslant \cosh(L_\rho(\alpha_j) - \kappa).\end{aligned}$$

Proof of the claim. Let $\pi_j(p) \in \ell_{\alpha_j}$ be the unique point such that $[p, \pi_j(p)]$ is orthogonal to ℓ_{α_j}. Observe that $\pi_j(\rho(\alpha_j)p) = \rho(\alpha_j)\pi_j(p)$. The conclusion follows from Lemma 5.1 applied to the spacelike segment $[\pi_j(y_j), \rho(\alpha_j)\pi_j(y_j)]$ of length $L_\rho(\alpha_j)$ and the orthogonal timelike segments $[y_j, \pi_j(y_j)], [y_{j+1} = \rho(\alpha_j)y_j, \rho(\alpha_j)\pi_j(y_j)]$ combined with part (1) of Lemma 5.2. □

Claim 4. We have

$$L_\rho(\alpha_j) - d_{\mathbb{H}^{2,1}}(z_{j-1}, z_j) \leqslant O(\epsilon).$$

Proof of the claim. Let π_j be the orthogonal projection to ℓ_{α_j}. By Claim 1, since $\ell_{\alpha_j} \in \alpha_{j-1} \cdots \alpha_1(K)$ and $x_{j-1}, z_{j-1} \in \rho(\alpha_{j-1} \cdots \alpha_1)(U)$, we have that

$$d_{\mathbb{H}^{2,1}}(\pi_j(x_{j-1}), \pi_j(z_{j-1})) \leqslant \epsilon.$$

Therefore $d_{\mathbb{H}^{2,1}}(\pi_j(x_{j-1}), \pi_j(x_j)) - d_{\mathbb{H}^{2,1}}(z_{j-1}, z_j) \leqslant 2\epsilon$.

By Claim 1, we also have,

$$D_j = \min_{t \in \ell_{\alpha_j}} \{-\langle x_{j-1}, t\rangle\} = \sqrt{\frac{2\langle x_{j-1}, \ell^+_{\alpha_j}\rangle\langle x_{j-1}, \ell^-_{\alpha_j}\rangle}{-\langle \ell^+_{\alpha_j}, \ell^-_{\alpha_j}\rangle}} = -\langle x_{j-1}, \pi_j(x_{j-1})\rangle.$$

is contained in the interval $(\cos(\epsilon), \cosh(\epsilon))$.

We prove that $d_{\mathbb{H}^{2,1}}(\pi_j(x_{j-1}), \pi_j(x_j)) \geqslant \ell(\alpha_j) - O(\epsilon)$.

If $D_j > 1$, then the segment $[x_{j-1}, \pi_j(x_{j-1})]$ is spacelike. Write $D_j = \cosh(d_j)$ and $x_{j-1} = \cosh(d_j)\pi_j(x_{j-1}) + \sinh(d_j)v_j$ with v_j orthogonal to ℓ_{α_j} at $\pi_j(x_{j-1})$. Recall that $x_{j+1} = \rho(\alpha_j)x_{j-1}$ and

$$\begin{aligned}&\cosh(d_{\mathbb{H}^{2,1}}(x_{j-1}, x_j)) = -\langle x_{j-1}, x_j\rangle\\ &= \cosh(d_j)^2 \cosh(L_\rho(\alpha_j)) - \sinh(d_j)^2 \cosh(\theta_\rho(\alpha_j)).\end{aligned}$$

Hence, as $L_\rho(\alpha_j) > \theta_\rho(\alpha_j)$, we get $\cosh(d_{\mathbb{H}^{2,1}}(x_{j-1}, x_j)) > \cosh(L_\rho(\alpha_j))$.

If $D_j < 1$, then the segment $x_j = \pi_j(p)$ is timelike. Write $D_j = \cos(d_j)$ and $x_{j-1} = \cos(d_j)\pi_j(x_{j-1}) + \sin(d_j)v_j$ with v_j orthogonal to ℓ_{α_j} at $\pi_j(x_{j-1})$. Recall that $x_j = \rho(\alpha_j)x_{j-1}$ and

$$\begin{aligned} &\cosh(d_{\mathbb{H}^{2,1}}(x_{j-1}, x_j)) = -\langle x_{j-1}, x_j\rangle \\ &= \cos(d_j)^2\cosh(L_\rho(\alpha_j)) + \sin(d_j)^2\cosh(\theta_\rho(\alpha_j)). \end{aligned}$$

Thus $\cosh(d_{\mathbb{H}^{2,1}}(x_{j-1}, x_j)) > \cos(d_j)^2\cosh(L_\rho(\alpha_j))$. The conclusion follows from part (2) of Lemma 5.2. □

Conclusion:

$$\begin{aligned} L_Z(\gamma_n) &\leqslant d_{\hat{S}}(y, \gamma_n y) \\ &\leqslant \sum_j d_{\hat{S}}(y_{j-1}, y_j) && \text{triangle inequality} \\ &\leqslant \sum_j d_{\mathbb{H}^{2,1}}(y_{j-1}, y_j) && \text{by Theorem 1.1} \\ &\leqslant \sum_j L_\rho(\alpha_j) - \kappa && \text{by Claim 3} \\ &\leqslant \sum_j d_{\mathbb{H}^{2,1}}(z_{j-1}, z_j) + O(\epsilon) - \kappa && \text{by Claim 4} \\ &= L_\rho(\gamma_n) + m_n(O(\epsilon) - \kappa) && . \end{aligned}$$

Multiply by a_n and take a limit to finish the proof (assuming that we chose ϵ so that $O(\epsilon) - \kappa < 0$).

□

6.3. **First and second variations along earthquakes.** In the case of earthquakes, we make quantitative estimates and compute the first and second variations of length functions as given in Theorem 4.

As before, we prove separately the two parts of the theorem.

Proposition 6.4. *Let $\lambda \in \mathcal{ML}$ be a measured lamination. Let $E_\lambda : [a, b] \to \mathcal{T}$ be an earthquake path driven by λ. Let $\gamma \in \Gamma - \{1\}$ be a non-trivial loop. Set $L_\gamma(t) := L_\gamma(E_\lambda(t))$. Then:*

$$\left|\dot{L}_\gamma\right| \leqslant i(\gamma, \lambda).$$

Proof. Let $Z_t := E_\lambda(t)$, consider the Mess representation $\rho_t := \rho_{Z_{-t}, Z_t}$ with parameters $Z_{-t}, Z_t \in \mathcal{T}$. Notice that, by Theorem 1, we have $\lambda_t^+ = t\lambda$.

For convenience, introduce $\theta_t := \theta_{\rho_t}(\gamma)$.

By Proposition 5.5, we have

$$i(\gamma, t\lambda) \geqslant \cos\left(\Delta_t^+\right)^2\theta_t(\gamma).$$

As $t \to 0$, we have $\Delta_t^+ \to 0$ since ρ_t is converging to the Fuchsian representation ρ_0. Moreover, we also have $\theta_t(\gamma)/t = |L_\gamma(t) - L_\gamma(-t)|/2t \to |\dot{L}_\gamma|$.

Dividing by t and passing to the limit, we obtain

$$i(\gamma, \lambda) \geqslant |\dot{L}_\gamma|.$$

□

Proposition 6.5. *Let $\lambda \in \mathcal{ML}$ be a measured lamination. Let $E_\lambda : [a, b] \to \mathcal{T}$ be an earthquake path driven by λ. Let $\gamma \in \Gamma - \{1\}$ be a non-trivial loop. Set $L_\gamma(t) := L_\gamma(E_\lambda(t))$. Then:*

$$\ddot{L}_\gamma \geqslant \frac{1}{\sinh(L_\gamma)} \left|\dot{L}_\gamma\right| \left(i(\gamma, \lambda) - \left|\dot{L}_\gamma\right|\right).$$

Proof. Let $Z_t := E_\lambda(t)$, consider the Mess representation $\rho_t := \rho_{Z_{-t}, Z_t}$ with parameters $Z_{-t}, Z_t \in \mathcal{T}$. Notice that, by Theorem 1, we have $\lambda_t^+ = t\lambda$ and $Z_{\lambda_t^+} = Z$ is constant.

For convenience, we introduce $L_t := L_{\rho_t}(\gamma)$ and $\theta_t := \theta_{\rho_t}(\gamma)$.

By Propositions 5.4 and 5.5, we have

$$\cosh(L_Z(\gamma)) \leqslant \cos\big(\delta_t^\pm\big)^2 \cosh(L_t) + \sin\big(\delta_t^\pm\big)^2 \cosh(\theta_t)$$

and

$$\cosh\big(i(\lambda_t^\pm, \gamma)\big) \leqslant \sin\big(\delta_t^\pm\big)^2 \cosh(L_t) + \cos\big(\delta_t^\pm\big)^2 \cosh(\theta_t).$$

Summing the inequalities, we get

$$\cosh\big(t \cdot i(\lambda^+, \gamma)\big) - \cosh(\theta_t) \leqslant \cosh(L_t) - \cosh(L_Z(\gamma)).$$

By the mean value theorem, we can write

$$\cosh\big(t \cdot i(\lambda^+, \gamma)\big) - \cosh(\theta_t) = \sinh(\xi_t) \left(t \cdot i(\lambda^+, \gamma) - |\theta_t|\right)$$

where $\xi_t \in [|\theta_t|, t \cdot i(\lambda^+, \gamma)]$, and

$$\cosh(L_t) - \cosh(L_Z(\gamma)) = \sinh(\zeta_t)\left(L_t - L_Z(\gamma)\right)$$

where $\zeta_t \in [\ell_Z(\gamma), L_t]$.

We now divide both right and left hand side by t^2 as follows

$$\frac{\sinh(\xi_t)}{t}\left(i(\lambda^+, \gamma) - \frac{|\theta_t|}{t}\right) \leqslant \sinh(\zeta_t) \frac{L_t - \ell_Z(\gamma)}{t^2}$$

and we observe that as $t \to 0$ the terms converge to: In the left hand side,

- $|\theta_t|/t = |L_\gamma(t) - L_\gamma(-t)|/2t \to \dot{L}_\gamma$.
- $\sinh(\xi_t)/t \geqslant \sinh(|\theta_t|)/t$ as $\xi_t \geqslant |\theta_t|$.
- $\sinh(|\theta_t|)/t \to \cosh(\theta_0)\dot{\theta}_0 = \dot{L}_\gamma$.

In the right hand side,

- $\sinh(\zeta_t) \to \sinh(L_\gamma(Z))$ as $L_t = (L_\gamma(t) + L_\gamma(-t))/2 \to L_Z(\gamma)$.
- $(L_t - L_Z(\gamma))/t^2 = (L_\gamma(t) + L_\gamma(-t) - 2L_Z(\gamma))/2t^2 \to \ddot{L}_\gamma/2$.

The conclusion follows.

□

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