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On the geometry of magnetic flows

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Abstract

To a Riemannian manifold (M, g) endowed with a magnetic form σ and its Lorentz operator Ω we associate an operator M^{Ω}, called the *magnetic curvature operator*. Such an operator encloses the classical Riemannian curvature of the metric g together with terms of perturbation due to the magnetic interaction of σ. From M^{Ω} we derive the *magnetic sectional curvature* Sec^{Ω} and the *magnetic Ricci curvature* Ric^{Ω}. On closed manifolds, with a Bonnet-Myers argument, we show that if Ric^{Ω} is positive on an energy level below the Mañé critical value then, on that energy level, we can recover the Palais-Smale condition and prove the existence of a contractible periodic orbit. In particular, when σ is nowhere vanishing, this implies the existence of contractible periodic orbits on every energy level close to zero. On closed oriented even dimensional manifolds, we discuss about the topological restrictions which appear when one requires Sec^{Ω} to be positive. Finally, we give a magnetic version of the classical Hopf's rigidity theorem for magnetic flows without conjugate points on closed oriented surfaces.

Zusammenfassung

Zu einer riemannschen Mannigfaltigkeit (M, g) versehen mit einer magnetischen Form σ und dem zugehörigen Lorenzoperator Ω assoziieren wir einen Operator M^{Ω}, die magnetische Krümmung. Dieser Operator beinhaltet die riemannsche Krümmung, sowie durch die magnetische Form induzierte Störungsterme. Aus M^{Ω} kann die magnetische Schnittkrümmung Sec^{Ω} sowie die magnetische Ricci-Krümmung Ric^{Ω} abgeleitet werden. Auf einer geschlossenen Mannigfaltigkeit kann mit einem Bonnet-Myers Argument unter der Voraussetzung, dass die magnetische Ricci-Krümmung unter dem Mañé kritischen Wert positiv ist, die Palais-Smale Eigenschaft nachgewiesen werden sowie die Existent eines zusammenziehbaren periodischen Orbits. Insbesondere, wenn σ nicht verschwindend ist, folgt somit die Existenz von zusammenziehbaren periodischen Orbits für Energielevel nahe null. Auf geschlossenen Mannigfaltigkeiten, deren Dimension gerade ist, diskutieren wir die topologischen Einschränkungen, die durch die Positivität der magnetischen Schnittkrümmung induziert werden. Zuletzt werden wir ein Analogon des klassischen Hopf-Starrheits-Theorems für magnetische Flüsse, welche keine konjugierten Punkt haben, auf geschlossenen orientierten Flächen beweisen.

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Contents

1 Introduction

Magnetic systems represent a classical topic which is still of great mathematical interest. They systematically appear in contemporary physics, from collider accelerators and plasma fusion, where the magnetic field confines the particle on a desired region of the space; to Earth's magnetosphere and mass spectrometry, where the magnetic field acts as a mirror or a prism. From a mathematical point of view a magnetic system is the pure toy model for the motion of a charged particle moving on a Riemannian manifold under the action of a magnetic force. In this manuscript we will investigate such dynamics from a variational and geometrical point of view. In particular, we introduce for the first time, the notion of magnetic curvature. This is an operator which encodes the geometrical properties of the Riemannian structure together with the perturbation due to the magnetic interaction, and it generalizes in arbitrary dimension the previously introduced notion of Gaussian magnetic curvature on closed surfaces. We expect that the magnetic curvature carries relevant properties of the magnetic dynamics. For instance, among the main results of this thesis, we show the existence of a periodic orbit when the trace of the magnetic curvature (also called the Ricci magnetic curvature) is positive. A corollary of this result guarantees the existence of periodic orbit for every small energies when the magnetic force is nowhere vanishing. We will also be investigating the relations between this new operator and the topology of the base manifold. In particular, we recover in a magnetic setting some of the classical results known in Riemannian geometry such as the Hopf's rigidity theorem on closed surfaces, the Synge's theorem for oriented even dimensional closed manifolds, and the Bonnet-Myers theorem.

1.1 Magnetic systems

A magnetic system is the data of (M, g, σ), where (M, g) is a closed Riemannian manifold and σ is a closed 2-form on M and in this context it is referred as the *magnetic form*. The condition that σ is closed generalizes the fact that, in Euclidean space, magnetic fields are divergence-free according to Maxwell's equations. A bundle operator $\Omega : TM \to TM$, called Lorentz operator, is associated with σ and the metric $g = \langle\ ,\ \rangle$ in the following sense:

$$\langle v, \Omega(w)\rangle = \sigma(v, w),\ \forall v, w \in T_pM,\ \forall p \in M. \tag{1.1}$$

Observe that Ω is antisymmetric with respect to g. In this setting, denoting by $\frac{D}{dt}$ the covariant derivative of the Levi-Civita connection ∇, the dynamics of a charged particle, moving on M under the influence of the magnetic field σ, is described by the second-order differential equation:

$$\frac{D}{dt}\dot{\gamma} = \Omega(\dot{\gamma}). \tag{1.2}$$

A solution $\gamma : \mathbb{R} \to M$ to (1.2) is called a *magnetic geodesic*. Observe that when there is no magnetic interaction, i.e. $\Omega = 0$, the above equation is reduced to the equation of standard geodesics. By lifting the differential problem (1.2) to TM, we obtain a vector field X_σ also called the *magnetic vector field* of TM. Because the energy $E(p, v) = \frac{1}{2}|v|^2$ is constant along magnetic geodesics, the *magnetic flow* $\varphi_\sigma : \mathbb{R} \times TM \to TM$ associated to X_σ is complete and leaves invariant the level sets $\Sigma_k = \{(p, v),\ |v| = \sqrt{2k}\}$, with $k \in (0, +\infty)$. Observe that the k-level Σ_k coincides with the sphere bundle of radius $\sqrt{2k}$ over M. Hereafter, we denote by φ^k_σ the restriction of φ_σ to Σ_k.

A magnetic system is *exact* if the magnetic form admits a global primitive on the base manifold M, i.e. there exists $\theta \in \Omega^1(M)$ such that $\mathrm{d}\theta = \sigma$. A magnetic system is *weakly exact* if σ is exact on the universal cover $\mathrm{p} : \tilde{M} \to M$, i.e. $\tilde{\sigma} = \mathrm{p}^*\sigma$ is exact on $\tilde{M}$.

Similarly, one can define the magnetic flow by using the Hamiltonian formalism. Indeed, consider the cotangent bundle T^*M endowed with twisted symplectic form given by

$$\omega_\sigma = \omega + \pi^*\sigma;$$

Here ω is the canonical form of T^*M and π its the natural projection onto M. Consider the Hamiltonian $H(p, \lambda) = \frac{1}{2}|\lambda|^2$ (here with abuse of notation we denote by $|\ |$ also the dual norm induced by g). Let $\hat{X}_\sigma$ be the vector field on T^*M given by the symplectic pairing of ω_σ. In particular, $\hat{X}_\sigma$ is the unique vector field on T^*M such that

$$\omega_\sigma(\hat{X}^\sigma, \cdot) = \mathrm{d}H(\cdot).$$

The flow $\hat{\varphi}^\sigma : \mathbb{R} \times T^*M \to T^*M$ associated with $\hat{X}^\sigma$ is conjugate, through the *Legendre transformation*, to the magnetic flow φ_σ defined on TM. Therefore, one can approach the magnetic dynamics on the tangent bundle through φ^σ or on the cotangent bundle through $\hat{\varphi}^\sigma$. For our purposes, we mainly work on TM. We use the Hamiltonian formalism to define the critical value later in the introduction. For more details and properties of twisted cotangent bundles and the duality between φ^σ and $\hat{\varphi}^\sigma$, we refer the reader to [Ben16],[Mer10] or also [AB16].

1.2 The Mañé critical value

Among the values of the energy there is one denoted by $c(g, \sigma) \in [0, +\infty]$ and called Mañé critical value which plays a crucial role in the theory. Indeed the dynamics of φ^k_σ can drastically differ depending on whether $k < c$, $k = c$ or $k > c$. If σ is weakly exact the critical value is defined as

$$c(g, \sigma) = \inf_{\mathrm{d}\tilde{\theta} = \tilde{\sigma}} \sup_{p \in \tilde{M}} \tilde{H}(p, \tilde{\theta}_p), \tag{1.3}$$

where $\tilde{H}$ is the lift of H in $\tilde{M}$ and the infimum is taken over any primitive of $\tilde{\sigma}$. If σ is not weakly exact, we pose $c(g, \sigma) = +\infty$. As pointed out in [Mer10], $c = 0$ if and only if $\sigma = 0$ and c is finite if and only if $\tilde{\sigma}$ admits a bounded primitive, i.e. there exists a primitive $\tilde{\theta}$ of $\tilde{\sigma}$ such that

$$\sup_{\tilde{M}} |\tilde{\theta}_p| < \infty.$$

Observe that in equation (1.2) the growth of the left-hand term and the right-hand term is respectively quadratic and linear in function of the energy. This consideration suggests an insight into how a magnetic system acts on different energy level sets. The intuition is the following: for high energy, namely above c, the Riemannian term is dominant and we weakly perturb the geodesic flow. Instead below c the magnetic interaction is strong and ideally we may expect the evolution of the particle to be lured back in a neighbourhood of the starting point. In this interpretation the critical value plays a role of interlude between these two kind of dynamics. Later in the introduction, we see how c can be characterized also in a variational setting.

1.3 Magnetic Jacobi equation and conjugate points

Along a magnetic geodesic $\gamma : M \to \mathbb{R}$ consider the linearization of equation (1.2) given by the differential problem

$$\Big(\frac{D}{dt}\Big)^2 J + R(J,\dot\gamma)\dot\gamma - (\nabla_J\Omega)(\dot\gamma) - \Omega\Big(\frac{D}{dt}J\Big) = 0, \tag{1.4}$$

where R denotes the Riemann curvature operator. Equation (1.4) is also called the *magnetic Jacobi equation* along γ. An immediate computation shows that if J is solution of (1.4), then $\langle \frac{D}{dt}J, \dot\gamma\rangle$ is constant along γ. A solution J is called *normal solution* if at the starting time t_1, $J(t_1) = 0$ and $\langle \frac{D}{dt}J(t_1), \dot\gamma(t_1)\rangle = 0$. Let $p, q \in M$ be two distinct points and let γ be a magnetic geodesic such that $\gamma(t_1) = p$ and $\gamma(t_2) = q$ for $t_1 \neq t_2$. We say that q is a *conjugate point* with respect to p (through the magnetic geodesic γ) if there exists a vector field J along γ non trivial normal solution of (1.4), such that the orthogonal projection with respect to $\dot\gamma$, $J^\perp(t_2)$ vanishes (i.e. $J(t_2)$ is parallel to $\dot\gamma(t_2)$). A magnetic geodesic is said to be *without conjugate points* if $\gamma(t_1)$ is not conjugate to $\gamma(t_2)$ for any $t_1 \neq t_2$. Finally, the magnetic flow φ^k_σ is said to be *without conjugate points* if every trajectory of φ^k_σ is without conjugate points. Observe again, that in absence of magnetic interaction the equation (1.4) is reduced to the classical Jacobi equation along a geodesic, and the definition of conjugate point coincides with the standard Riemannian one. In Section 3.7, we see how this definition of conjugate point is equivalent to the degeneracy of a suitable exponential map defined for a magnetic flow φ^k_σ. In the case of magnetic systems on closed oriented surface we recover in a magnetic setting, a weaker statement of the Cartan-Hadamard Theorem (see Subsection 3.9 and see [Ada11] and [Ada12] for similar results in this direction). We remind the reader that the classical Cartan-Hadamard theorem (see [dC15]) states that on a complete Riemannian manifold with negative sectional curvatures, the exponential map is a covering map. In particular, it follows that the universal cover of M is $\mathbb{R}^n$.

1.4 Closed magnetic geodesics and main result

A *closed magnetic geodesic* is a periodic magnetic geodesic and one of the most relevant interests in the theory is the study of existence, multiplicity and free homotopy class of closed magnetic geodesic in a given Σ_k (see for instance [Gin04],[Gin96] or also [NT84]). In a variational setting, closed magnetic geodesic with energy k are precisely zeros for a suitable closed 1-form η_k, called the *magnetic action form*, which is defined on $\mathcal{M}$, the space of absolutely continuous loops with L^2-integrable derivative and free period (for more details about the magnetic action form we refer the reader to [Mer10],[AB16] or to Subsection 3.1 of the present manuscript). Such a form always admits a local primitive (as defined in Equation (3.4)) and generalizes the differential of the classical action functional S_k, which is globally defined for example if σ is exact on the base manifold M (see [Abb13]). When the magnetic form is weakly exact, we can extend a local primitive S^σ_k of η_k to the whole connected component of contractible elements of $\mathcal{M}$ which we denote by $\mathcal{M}_0$. Such a primitive it is given for instance by

$$S^\sigma_k : \mathcal{M}_0 \to \mathbb{R}, \quad S^\sigma_k(\gamma) = \int_0^T \Big\{\frac{|\dot\gamma|^2}{2} + k\Big\}\, \mathrm{d}t + \int_{B^2} D^*_\gamma\sigma, \tag{1.5}$$

where $D_\gamma : B^2 \to M$ is an arbitrary capping disk for γ. Since σ is weakly exact if and only if $[\sigma] \in H^2(M,\mathbb{R})$ vanishes over $\pi_2(M)$, definition (1.5) is independent of the choice of D_γ. In this variational setting, we can redefine the Mañé critical value c as

$$c = \inf\{k \geq 0 \mid S_k^\sigma(\gamma) \geq 0,\ \forall \gamma \in \mathcal{M}_0\}. \tag{1.6}$$

The importance of c becomes evident when we look at the existence of closed magnetic geodesics. The next theorem resumes the central results, obtained in the last decades, about the existence of a closed magnetic geodesic in a given k-energy level. We refer the reader to [Osu05],[Tai10] or [Con06] for the case when σ is exact. To [Mer10] for the weakly exact case and to [AB16] for the general case.

Theorem. *Let (M, g, σ) be a magnetic system and $c \in [0, +\infty]$ its Mãné critical value.*

(i) If $\pi_1(M)$ is non trivial and $k > c$, then η_k carries a zero in each non trivial homotopy class.

(ii) If M is simply connected and $k > c$, then η_k carries a contractible zero.

(iii) For almost all $k \in (0, c)$ there exists a contractible zero of η_k.

In fact, when c is finite the magnetic form admits a primitive on each connected component of $\mathcal{M}$. Such a primitive is bounded from below for $k > c$ and satisfies the Palais-Smale condition; thus, the existence of a zero for η_k, which is a minimizer of any primitive, is obtained via classical variational methods through minimization. Below c such compactness conditions are not satisfied anymore and the variational approach becomes extremely delicate. Despite that, for almost all energy k, with a Struwe monotonicity argument [Str90], one can construct converging Palais-Smale sequences using a minimax geometry of η_k on the set of loops with short length (we retrace such a construction in Section 5). In general, it is still an open question if we can extend the existence of closed magnetic geodesic (contractible or not) to the whole interval $(0, c)$. Under stronger assumptions, some progress had been made. For exact magnetic systems on compact oriented surfaces, Taimanov in [Tai92], Contreras, Macarini and Paternain in [CMP04] and later Asselle and Mazzucchelli in [AM19], show the existence of a zero of η_k (minimizer of any global primitive) for all $k \in (0, +\infty)$ in the non simply connected case, and for all $k \in (0, c]$ in the case of the 2-sphere. Another relevant result is in the work of Ginzburg-Gürel [GG08], refined by Usher in [Ush09], where it is showed the existence of contractible closed magnetic geodesics for all energies close to zero when the magnetic form is symplectic. In this work we extend the result of Ginzburg and Usher from symplectic magnetic forms to nowhere vanishing magnetic forms.

Theorem A. *Let (M, g, σ) be a magnetic system. If σ is nowhere vanishing, then there exists a positive real number ρ such that for every $k \in (0, \rho)$, η_k has a contractible zero (i.e. there exists a contractible closed magnetic geodesic with energy k).*

Observe that the assumption on a closed 2-form to be nowhere vanishing presents weak topological obstructions and the set of such systems is very rich. Indeed, in dimension two, being nowhere vanishing is the same as being symplectic, and it is well known that every oriented compact surfaces admits a symplectic form. The situation is drastically different in dimension bigger than 2. There are topological obstructions for a manifold to carry a symplectic form; for instance, it needs to be even-dimensional and all cohomology groups in even degree must be non-zero. On the other hand, as treated in [EM02], every smooth manifolds admits a nowhere vanishing closed 2-form in each cohomology class. Let us point out that Theorem A is an immediate corollary of the central result of this manuscript which shows the existence of contractible closed magnetic geodesic for levels of the energy below c and positively curved in a sense which we describe in the next subsection.

2 Magnetic curvature and results

2.1 Magnetic curvature

Let SM be the unit sphere bundle and consider E and E^1, the bundles of complementary and unitary complementary direction over SM. In particular, a fiber of E and E^1 at the point $(p,v) \in SM$ is the set $E_{(p,v)} = \{w \in T_pM \mid \langle v,w\rangle = 0\}$ and $E^1_{(p,v)} = \{w \in S_pM \mid \langle v,w\rangle = 0\}$. Let $k \in (0,+\infty)$ and denote by $\Omega^2 = \Omega \circ \Omega$. Consider the bundle operators $R^\Omega_k, A^\Omega : E \to E$ given by

$$R^\Omega_k(v,w) = 2kR(w,v)v - \sqrt{2k}(\nabla_w\Omega)(v), \tag{2.1}$$

$$A^\Omega(v,w) = \frac{3}{4}\langle w,\Omega(v)\rangle\Omega(v) - \frac{1}{4}\Omega^2(w) - \frac{1}{4}\langle\Omega(w),\Omega(v)\rangle v. \tag{2.2}$$

Define the *magnetic curvature operator* $M^\Omega_k : E \to E$ at the energy level k as:

$$M^\Omega_k(v,w) = R^\Omega_k(v,w) + A^\Omega(v,w). \tag{2.3}$$

Definition (2.3) comes naturally by approaching the linearization of equation (1.2) as well as by developing the second variation of a primitive of η_k around one of its zero (see for instance Lemma 13). From M^Ω_k we derive the magnetic curvature functions. Precisely, the *magnetic sectional curvature* $\mathrm{Sec}^\Omega_k : E^1 \to \mathbb{R}$, the *magnetic Ricci curvature* $\mathrm{Ric}^\Omega_k : SM \to \mathbb{R}$ and the *scalar magnetic curvature* $\mathrm{Scal}^\Omega_k : M \to \mathbb{R}$ at the k-level are the functions given by

$$\mathrm{Sec}^\Omega_k(v,w) = \langle M^\Omega_k(v,w),w\rangle, \tag{2.4}$$

$$\mathrm{Ric}^\Omega_k(v) = \mathrm{trace}(M^\Omega_k(v,\cdot)), \tag{2.5}$$

$$\mathrm{Scal}^\Omega_k(p) = \int_{S_pM}\mathrm{Ric}^\Omega_k(v)\mathrm{d}v. \tag{2.6}$$

Observe that if Sec and Ric denote the classical sectional curvature and Ricci curvature, by looking at (2.3), we obtain for definition (2.4) and definition (2.5) the following expression

$$\mathrm{Sec}^\Omega_k(v,w) = 2k\mathrm{Sec}(v,w) - \sqrt{2k}\langle(\nabla_w\Omega)(v),w\rangle + \langle A^\Omega(v,w),w\rangle, \tag{2.7}$$

$$\mathrm{Ric}^\Omega_k(v) = \mathrm{trace}(M^\Omega_k(v,\cdot)) = 2k\mathrm{Ric}(v) - \sqrt{2k}\ \mathrm{trace}\Big((\nabla\Omega)(v)\Big) + \mathrm{trace}\Big(A^\Omega(v,\cdot)\Big). \tag{2.8}$$

Observe that, if (M,g) is closed oriented Riemannian surfaces, we can express the magnetic form σ through smooth function $b \in C^\infty(M)$ which rescale the volume form $\mathrm{d}\mu_g$ induced by the metric g (i.e. $\sigma = b\cdot\mathrm{d}\mu_g$). In this setting the magnetic sectional curvature, as given in (2.4), coincides with the standard (Gaussian) magnetic curvature that can be found in the work of Gabriel and Miguel Paternain [PP97]. In particular, this is a function $\mathcal{K}^b_k \in C^\infty(SM)$ defined as

$$\mathcal{K}^b_k(v) = 2k\mathcal{K} - \sqrt{2k}\mathrm{d}b(i\cdot v) + b^2, \tag{2.9}$$

where $\mathcal{K}$ denotes the Gaussian curvature of (M,g) and i the anticlockwise rotation of an angle $\frac{\pi}{2}$. In [MR23], by assuming $\mathcal{K}^b_k < 0$, James Marshall Reber shows a marked length spectrum rigidity for g and b in the deformative case. In higher dimension, Sec^Ω appeared in the paper of Wojtkowski [Woj00], where it is deduced by studying magnetic Jacobi fields. In particular it is showed that if Sec^Ω_k is negative, then the magnetic flow restricted to Σ_k is of Anosov type (see also [Gou97] and [Gro99] for previous results in this direction). In the context of the second variation of the action, a magnetic Ricci curvature defined as the trace of the operator R^Ω_k was considered by Bahri and Taimanov in [BT98]. To the author's best knowledge, this is the first time the magnetic curvature has been defined in full generality.

2.2 Ricci magnetic curvature and closed magnetic geodesics

One of the main purposes of this manuscript is to put in relation Sec^Ω_k with the second variation of the action (see Lemma 13). In particular, we prove a Bonnet-Myers theorem (see Lemma 18) stating that if γ is a zero of η_k and $\mathrm{Ric}^\Omega_k > 0$, then a bound on the Morse index of γ implies a bound on its period. The magnetic Bonnet-Myers together with index estimates for zeroes of η_k obtained by the Struwe monotonicity argument (Lemma 24 and Lemma 29), allow us to recover a converging Palais-Smale sequence and prove the existence of a contractible closed magnetic geodesic. Before stating the main theorem, we remark that techniques to recover Palais-Smale through index estimates were previously considered for instance by Bahri and Coron in [BC91], where they were approaching the Kazdan-Warner problem; by Coti Zelati and Ekeland in [CZEL90] in a general variational setting and by Benci and Giannoni in [BG89] in the context of billiards.

Theorem B. *Let (M,g,σ) be a magnetic system. If $k \in (0,c)$ is such that $\mathrm{Ric}^\Omega_k > 0$, then η_k carries a contractible zero (i.e. there exists a contractible closed magnetic geodesic with energy k).*

With a different approach, a weaker result was proved by Bahri and Taimanov in [BT98] where, by assuming the trace of R^Ω_k to be positive, they showed the same statement for exact magnetic systems. It is crucial that they did not include A^Ω in their definition of magnetic curvature. In fact, the terms in (2.8) coming from this operator turn out to be always non negative and they are zero at $p \in M$ if and only if $\sigma_p = 0$ (see Lemma 7). This last consideration enlarges the number of situations where we can apply Theorem B and it has a strong impact when we look at magnetic system on low energy levels. Indeed, in Lemma 8, we show that if the magnetic form is symplectic, then $\mathrm{Sec}^\Omega_k > 0$ is positive for k close to zero; analogously if σ is nowhere vanishing, then Ric^Ω_k is positive for values of k close to zero. Intuitively we can look at symplectic magnetic systems and nowhere vanishing magnetic systems with small energies as the magnetic analogue of positively sectionally curved and positively Ricci curved manifolds in Riemannian geometry. Observe that point (ii) of Lemma 8 together with Theorem B immediately implies Theorem A stated in the previous section.

2.3 Magnetic curvature and topology of the base manifold

We also put in evidence some topological restrictions that appear when we require Sec^Ω_k to be positive. For instance, in dimension 2, a symplectic magnetic system can be fully characterized in terms of positive magnetic sectional curvature for low energies.

Theorem C. *Let (M, g, σ) be a magnetic system on a closed oriented surface. If there exists a positive real number k_0 such that $\mathrm{Sec}_k^\Omega > 0$ for every $k \in (0, k_0)$, then either σ is symplectic, or $\sigma = 0$ and g has positive Gaussian curvature (and $M = S^2$).*

The proof of Theorem C is purely topological and the reader can consult it independently from the rest of the paper in Section 3.6. In higher even dimension, we show that Sec_k^Ω can not be positive when k is below c and M is oriented and simply connected. In particular, in Lemma 16 we will give a magnetic version of the classical Synge's theorem [Kli59]. Recall that this says that on even dimensional oriented Riemannian manifold with positive sectional curvature, no closed geodesic is length minimizer. In a magnetic setting one can deduce the following statement.

Theorem D. *Let (M, g, σ) be a magnetic system on an even-dimensional oriented manifold. If $\mathrm{Sec}_k^\Omega > 0$ for some $k > c$ then M is simply connected. Moreover, if M is a compact oriented surface and σ is exact with $\mathrm{Sec}_k^\Omega > 0$, then $M = S^2$ and $k > c$.*

2.4 The magnetic Hopf's rigidity on closed surfaces

One of the classical rigidity theorems in Riemannian geometry stems from E.Hopf [Hop48]: *if a closed Riemannian surface has no conjugate points, then its total Gaussian curvature is non positive and equal to zero if and only if the metric is flat.* The Gauss-Bonnet theorem then implies that every metric without conjugate points on the two dimensional torus T^2 is flat. Hopf's theorem was generalized by Green [Gre58], with the scalar curvature replacing the Gaussian curvature. However, in absence of an higher dimensional argument à la Gauss-Bonnet, the question of whether a metric without conjugate points on a n-torus T^n is flat has remained open for decades. The problem was finally solved (with a different array of techniques) by Burago and Ivanov in [BI94]. In this work we extend the Hopf's rigidity to the case of magnetic system on closed oriented surfaces. By denoting with $\mu_{G(k)}$ the measure induced by μ_g on Σ_k, such a result can be enunciated as follows

Theorem E. *Let (M, g, b) be a magnetic system on a closed oriented surface. If for $k \in (0, +\infty)$, φ_σ^k is without conjugate points, then*

$$\int_{SM} \mathcal{K}_k^b \mathrm{d}\mu_{G(\frac{1}{2})} \leq 0,$$

with the equality if and only if, either $M = \mathrm{T}^2$ and g is the flat metric or (M, g) is a hyperbolic surface, b is constant and k is the Mãné critical value.

A weaker statement was proved by Gouda in [Gou98], in the case of b constant. Observe that, if α is a linear form on M, then with a Fubini argument it holds that for every $k \in (0, +\infty)$,

$$\int_{\Sigma_k} \alpha \mathrm{d}\mu_{G(k)} = \int_M \mathrm{d}\mu_g \int_{\Sigma_k(p)} \alpha(v)\mathrm{d}v = 0.$$

Thus, by looking at the definition (2.9) of $\mathcal{K}_k^b$, as consequence of the Gauss-Bonnet Theorem, it follows that

$$\begin{aligned} \int_{SM} \mathcal{K}_k^b \mathrm{d}\mu_{G(\frac{1}{2})} &= 2\pi \Big[\int_M (2k\mathcal{K} + b^2)\mathrm{d}\mu_g \Big] \\ &= 4\pi^2 \Big[2k(2-2h) + \frac{1}{2\pi} \int_M b^2 \mathrm{d}\mu_g \Big], \end{aligned}$$

where h is the genus of M. Therefore, a first implication of Theorem E is that no magnetic flows on the 2 sphere S^2 is without conjugate points. Moreover, when $h = 1$, we also recover the general result of Bialy in [Bia00] stating that the only magnetic flows without conjugate points on T^n are the ones given when g is flat and $\sigma = 0$. Observe that, the magnetic dynamics on hyperbolic surfaces with a constant multiple of the volume form at the critical value is characterized by an horocycle flow (see for instance [Pat06]) which can now be interpreted as the magnetic analogue of the Riemannian flatness in dimension 2.

In a joint work with James Marshall Reber and Ivo Terek, we generalize Theorem E in arbitrary dimension and we also give a complete classification of the *magnetic flatness*, i.e. when $\mathrm{Sec}_k^\Omega = 0$. The proof and the details are not contained in this manuscript and will appear soon.

Theorem F. *Let (M, g, σ) be a magnetic system and $k \in (0, +\infty)$ such that φ_σ^k is without conjugate points. Then*

$$\int_M \mathrm{Scal}_k^\Omega \mathrm{d}\mu_g \leq 0,$$

with the equality if and only if $\mathrm{Sec}_k^\Omega = 0$.

Theorem G. *Let (M, g, σ) be a magnetic system and $k \in (0, +\infty)$ such that $\mathrm{Sec}_k^\Omega = 0$. Then one of the following holds:*

(i) (M, g) is the flat n-dimensional torus and $\sigma = 0$,

(ii) (M, g) is a Kähler hyperbolic manifold with Kähler form ω, there exists a real non negative constant λ such that the magnetic form $\sigma = \lambda\omega$, and k is the Mãné critical value.

Plan of the thesis

We conclude the introduction by giving an outline of this manuscript. In Section 3 we prepare the variational setting by introducing the space of loop with free period in Subsection 3.1, and by defining the magnetic form in Subsection 3.2. In Subsection 3.3 we present the second variation of the magnetic form together with the definition of Morse index for a vanishing point. In 3.4 we introduce the notion of vanishing sequence for the magnetic form. Concerning the second part, we see in Subsection 3.5 how the requirement of $\mathrm{Sec}_k^\Omega > 0$ and $\mathrm{Ric}_k^\Omega > 0$ on small energies is satisfied respectively by symplectic magnetic form and nowhere vanishing magnetic form. In Subsection 3.6 we give the proof of Theorem C. The magnetic exponential map is defined in Subsection 3.7 and, together with the useful expression of the magnetic Jacobi equation given in Subsection 3.8, leads to the proof of the magnetic Cartan-Hadamard theorem in Subsection 3.9. In Section 4 we relate the magnetic curvature with the second derivative of the magnetic action. In particular, by introducing a magnetic analogue of the classical notion of parallel transport in Subsection 4.1, we prove Theorem D in Subsection 4.2 and a Bonnet-Myers Theorem adapted to the magnetic case in Subsection 4.3. Section 5 is devoted to the proof of Theorem B (and Theorem A). We consider separately the weakly exact case in Subsection 5.1 and the general one in Subsection 5.2. Finally, in section 6 we prove Theorem E.

3 Preliminaries

3.1 Hilbert space of loops with free period and the magnetic action form

Let $\Lambda = H^1(S^1, M)$ be the set of absolutely continuous loops on M parametrized over the unit circle with L^2-integrable tangent vector. The space Λ admits a structure of *Hilbert manifold* modeled as $H^1(S^1, \mathbb{R}^n)$. For more details and properties we refer the reader to [Kli78]. For simplicity, we assume that M is orientable which, up to taking a double cover of M, is no loss of generality for the following constructions. The tangent space at $x \in \Lambda$ is naturally identified with the space of absolutely continuous vector field along x with L^2-integrable covariant derivative and it is naturally isomorphic to $H^1(S^1, \mathbb{R}^n)$. We endow Λ with a metric g_Λ which in each tangent space, by choosing a trivialization $\Psi : S^1 \times \mathbb{R}^n \to x^*TM$ of TM along x, is defined as

$$g_\Lambda(\zeta_1, \zeta_2) = \int_0^1 \Big[\langle \zeta_1, \zeta_2 \rangle_x + \Big\langle \frac{D}{dt}\zeta_1, \frac{D}{dt}\zeta_2 \Big\rangle_x \Big] \,\mathrm{d}s. \tag{3.1}$$

The distance d_Λ induced by g_Λ gives to Λ a structure of *complete Riemannian manifold*. We will denote by e and respectively ℓ the L^2 energy and the length of x which are defined by

$$e(x) = \int_0^1 |\dot{x}|^2 \,\mathrm{d}s, \;\; \ell(x) = \int_0^1 |\dot{x}| \,\mathrm{d}s.$$

The space $C^\infty(S^1, M)$ is dense in Λ, and a local chart centered on a smooth loop x is given by

$$F_\Lambda : H^1(S^1, B_r(0)) \to \Lambda, \;\; F_\Lambda(\zeta)(t) = \exp_{x(t)}(\Psi(t, \zeta(t)));$$

where $B_r(0) \subset \mathbb{R}^n$ is an open ball and exp is the exponential map of (M, g). As pointed out in [Abb13, Remark 2.2], F_Λ is bi-Lipschitz (i.e. F_Λ and F_Λ^{-1} are Lipschitz) with respect to the standard distance d_0 of $H^1(S^1, \mathbb{R}^n)$ and d_Λ. This is a consequence of the fact that the norm $|\ |_0$ of $H^1(S^1, \mathbb{R}^n)$ induced by the Euclidean scalar product of $\mathbb{R}^n$ and the norm $|\ |_\Lambda$ induced by $(F_\Lambda)^* g_\Lambda$ are equivalent on $H^1(S^1, B_r(0))$.

Consider now $\mathcal{M} = \Lambda \times (0, +\infty)$. This set can be interpreted as the set of absolutely continuous loops on M with L^2-integrable tangent vector and free period. Indeed, a point $(x, T) \in \mathcal{M}$ is identified with $\gamma : [0, T] \to M$ through $\gamma(t) = x(\frac{t}{T})$. Vice versa, an absolutely continuous loop with L^2-integrable tangent vector corresponds to the pair (x, T) with $x(s) = \gamma(sT)$. We provide $\mathcal{M}$ with the Riemannian structure obtained from the product of Λ endowed with g_Λ and the Euclidean structure of $(0, +\infty)$. In particular the tangent space splits into

$$T\mathcal{M} = T\Lambda \oplus \mathbb{R}\frac{d}{dT}, \tag{3.2}$$

and in this splitting the metric is given by

$$g_\mathcal{M} = g_\Lambda + dT^2.$$

If $x \in C^\infty(S^1, M)$, a local chart centered at (x, T) is obtained through the product

$$F_\mathcal{M} = F_\Lambda \times \mathrm{Id}_{(0,+\infty)} : H^1(B_r(0), M) \times (0, +\infty) \to \mathcal{M} \quad , \quad F_\mathcal{M}(\zeta, \tau) = (F_\Lambda(\zeta), T) \; . \tag{3.3}$$

With abuse of notation, denote by $|\ |_0$ the standard product norm of $H^1(S^1, \mathbb{R}^n) \times (0, +\infty)$. Observe that $|\ |_0$ and the norm induced by $(F_\mathcal{M})^* g_\mathcal{M}$ are equivalent on $H^1(B_r(0), M) \times (0, +\infty)$.

In analogy with F_Λ, this implies that the local chart defined in (3.3) is bi-Lipschitz (since it is the product of two bi-Lipschitz maps). We point out that the distance induced by $g_\mathcal{M}$ is not complete because dT^2 is not complete in the Euclidean factor. Nevertheless completeness is obtained by restricting g_Λ on sets of the form $\Lambda \times [T_*, +\infty)$ for every arbitrary $T_* > 0$. Finally, since $\mathcal{M}$ is homotopically equivalent to Λ, its connected components are in correspondence with the elements of $[S^1, M]$, namely the set of conjugacy classes of $\pi_1(M)$. If $[\mu]$ is such a class, we denote by $\mathcal{M}_{[\mu]}$ the respective connected component of $\mathcal{M}$. In particular, with $\mathcal{M}_0$ we indicate the connected component of contractible loops with free period.

Hereafter, we often identify γ with the respective (x, T). For simplicity, we also use the "dot" to indicate the Levi-Civita covariant derivative "$\frac{D}{\mathrm{d}t}$". A vector field over x and its respective parametrization over γ is denoted with the same symbol. Observe that the rescaling of the tangent vectors and their time derivatives with respect to the two different parametrization is given by $\dot\gamma(t) = \frac{1}{T}\dot x(\frac{t}{T})$. Analogously, if V is a vector field along x, then $\dot V(t) = \frac{1}{T}\dot V(\frac{t}{T})$.

3.2 Magnetic action form

Fix $k \in (0, +\infty)$ and consider the k-kinetic action $A_k : \mathcal{M} \to R$ defined as:

$$\begin{aligned} A_k(x,T) &= T\int_0^1 \Big\{\frac{|\dot x|^2}{2T^2} + k\Big\}\mathrm{d}s \\ &= \int_0^T \Big\{\frac{|\dot\gamma|^2}{2} + k\Big\}\mathrm{d}t. \end{aligned}$$

Let $\Theta \in \Omega^1(\Lambda)$ be such that

$$\Theta_x(V) = \int_0^1 \langle \Omega(V), \dot x\rangle \mathrm{d}s.$$

Denoting with $\pi_\Lambda : \mathcal{M} \to \Lambda$ the projection into the first factor of $\mathcal{M}$, the *k-magnetic action form* $\eta_k \in \Omega^1(\mathcal{M})$ is defined by

$$\eta_k(\gamma) = d_\gamma A_k + \pi_\Lambda^* \Theta_x.$$

If x is of class C^2 and $(V, \tau) \in T_x\Lambda \oplus \mathbb{R}\frac{\partial}{\partial T}$, then η_k acts as follows:

$$\eta_k(\gamma)(V, 0) = -\int_0^T \langle \nabla_{\dot\gamma}\dot\gamma - \Omega(\dot\gamma), V\rangle \mathrm{d}t,\ \ \eta_k(\gamma)(0,\tau) = \frac{\tau}{T}\int_0^T \Big\{k - \frac{|\dot\gamma|^2}{2}\Big\}\mathrm{d}t.$$

In agreement with [AB16, Lemma 2.2] and [Abb13, Lemma 3.1], η_k is a smooth a section of $T\mathcal{M}$ and $\eta_k(\gamma) = 0$ if and only if γ is a closed magnetic geodesic contained in Σ_k. We will denote by $\mathcal{Z}(\eta_k)$ the zero set of η_k. An important property of the magnetic action form concerns its behavior when integrated over a loop of $\mathcal{M}$. Indeed if $u : [0, 1] \to \mathcal{M}$ is a closed path, the integral $\int_0^1 u^*\eta_k$ depends only on the homotopy class of u. In this sense η_k is said to be *closed* [AB16, Corollary 2.4].

Let $\mathcal{U} \subseteq \mathcal{M}$ be an open set diffeomorphic to a ball and $(x_0, T_0) \in \mathcal{U}$. A local primitive $S_k^\sigma : \mathcal{U} \to \mathbb{R}$ of η_k, centered in (x_0, T_0), is defined by

$$S_k^\sigma(x,T) = A_k(x,T) + \int_{S^1\times[0,1]} c_{x_0,x}^*\ \sigma. \tag{3.4}$$

Here, $c_{x_0,x} : S^1 \times [0,1] \to M$ is a cylinder which connects x_0 and x such that $c_{x_0,x}(\cdot, s) \in \pi_\Lambda(\mathcal{U})$ for every s. Indeed, the closedness of η_k makes the above definition independent from the choice of $c_{x_0,x}$ and $dS_k^\sigma = \eta_k$ on $\mathcal{U}$. It is obvious that, if $\gamma \in \mathcal{Z}(\eta_k)$, then γ is a critical point for every primitive of η_k.

3.3 Hessian of η_k and the index of a vanishing point

Let γ be a zero of η_k and S_k^σ be a local primitive of η_k centered at γ. If $\zeta \in T_\gamma\mathcal{M}$, we naturally identify the tangent space $T_\zeta(T_\gamma\mathcal{M})$ with $T_\gamma\mathcal{M}$ itself. Under this identification we can look at the second (Fréchet) derivative $\mathrm{d}^2_\gamma S_k^\sigma$ of S_k^σ at the point γ, as a bilinear form on $T_\gamma\mathcal{M}$. Because the connection on M is the Levi-Civita connection, such a bilinear form is symmetric. The Hessian $\mathrm{Hess}_\gamma(S_k^\sigma)$ of S_k^σ at γ is the quadratic form associated with $\mathrm{d}^2_\gamma S_k^\sigma$. We refer the reader to [Cha93] for more details. The fact that two local primitives of η_k differ by a constant makes natural the following definition.

Definition 1. Let $\gamma \in \mathcal{Z}(\eta_k)$. The Hessian $\mathrm{Q}_\gamma(\eta_k)$ of η_k at γ is the quadratic form given by

$$\mathrm{Q}_\gamma(\eta_k) = \mathrm{Hess}_\gamma(S_k^\sigma),$$

where S_k^σ is any primitive of η_k defined on a neighbourhood of γ.

Lemma 2. *Let $\gamma \in \mathcal{Z}(\eta_k)$ and $\zeta = (V, \tau) \in T_\gamma\mathcal{M}$. Then*

$$\mathrm{Q}_\gamma(\eta_k)(V,\tau) = \int_0^T \left\{ \langle \dot V - \Omega(V), \dot V \rangle - \langle R(V,\dot\gamma)\dot\gamma - \nabla_V \Omega(\dot\gamma), V\rangle - \frac{\langle \dot V, \dot\gamma\rangle^2}{|\dot\gamma|^2} \right\} \mathrm{d}t + \int_0^T \left(\frac{\langle \dot V, \dot\gamma\rangle}{|\dot\gamma|} - \frac{\tau}{T}|\dot\gamma| \right)^2 \mathrm{d}t.$$

Proof. The computation follows by adopting [Gou97, Section 2] to a primitive of η_k as given by (3.4). ☐

If γ is a vanishing point, the signature of $\mathrm{Q}_\gamma(\eta_k)$ is well-defined because it is independent of the local coordinate chart. Therefore, there exist two vector subspaces $H_\gamma, E_\gamma \subset T_\gamma\mathcal{M}$ such that the tangent space splits into

$$T_\gamma\mathcal{M} = H_\gamma \oplus E_\gamma,$$

and E_γ is the maximal subspace, in terms of dimension, for which the restriction $\mathrm{Q}_\gamma(\eta_k)|_{E_\gamma}$ is negative definite.

Definition 3. Let $\gamma \in \mathcal{Z}(\eta_k)$. The *Morse index* of γ, which is denoted by $\mathrm{index}(\gamma)$, is the non negative integer given by

$$\mathrm{index}(\gamma) = \dim E_\gamma.$$

As argued in [AS09, Proposition 3.1], the self adjoint operator associated with $\mathrm{d}^2_\gamma S_k^\sigma$ is a compact deformation of a Fredholm operator, which implies that the index of a vanishing point of η_k is always finite.

We end the section by constructing a system of local coordinates around a zeroes of η_k which isolates the direction of $T_\gamma\mathcal{M}$ where $\mathrm{Q}_\gamma(\eta_k)$ is negative definite. Such construction is used in the proof of Lemma 24 and Lemma 29 in Section 5.

Let γ such that $\mathrm{index}(\gamma) = d$ and consider $(\mathcal{U}_\gamma, F_\mathcal{M})$ local coordinates centered at γ as given in Equation (3.3). By the assumption on the index and by the equivalence of the norms $|\ |_0$ and $|\ |_\mathcal{M}$ there exist two vector spaces $H_\gamma, E_\gamma \subset H^1(S^1, \mathbb{R}^n) \times (0, +\infty)$ and a positive constant D_γ such that

$$H^1(S^1, \mathbb{R}^n) \times (0, +\infty) = H_\gamma \oplus E_\gamma,$$

the space E_γ is of dimension d and

$$\mathrm{Q}_0(F^*_\mathcal{M}\eta_k)(\zeta, \tau) \leq -4D_\gamma |(\zeta, \tau)|_0^2 \ , \quad \forall (\zeta, \tau) \in E_\gamma. \tag{3.5}$$

Hereafter, we denote by B_r^H and B_r^E respectively the open ball of radius r centered at the origin of H_γ and E_γ.

Lemma 4. *There exists a local chart $(\Phi_\gamma^{-1}(\mathcal{V}_\gamma), \Phi_\gamma)$ centered at γ such that $\mathcal{V}_\gamma = B_r^H \times B_r^E$. Moreover, this system of local coordinates enjoys the following properties:*

(i) Φ_γ is bi-Lipschitz (with respect to d_0 and $d_\mathcal{M}$) and $\Phi_\gamma(\mathcal{Z}(\eta_k)) \subseteq B_r^H \times \{0\}$.

(ii) If S_k^σ is an arbitrarily primitive of η_k on $\Phi_\gamma^{-1}(\mathcal{V}_\gamma)$, then for every $(y_h, y_e) \in B_r^H \times B_r^E$ and for every $0 < \lambda < r - |y_e|_0$ it holds

$$(S_k^\sigma \circ \Phi_\gamma^{-1})\Big(y_h, y_e + \lambda \frac{y_e}{|y_e|_0}\Big) \leq (S_k^\sigma \circ \Phi_\gamma^{-1})\Big(y_h, y_e\Big) - D_\gamma \lambda^2.$$

Proof. Let $(\mathcal{U}_\gamma, F_\mathcal{M})$, H_γ and E_γ as above and S_k^σ a local primitive defined on $\mathcal{U}_\gamma$. By (3.5) and by the smoothness of S_k^σ there exists $\mathcal{V}_\gamma \subset U_\gamma$ open neighborhood of γ such that

$$\mathrm{Q}_\theta(F^*_\mathcal{M}\eta_k)(\zeta, \tau) \leq -2D_\gamma |(\zeta, \tau)|_0^2 \ , \ \forall \theta \in F_\mathcal{M}^{-1}(V_\gamma) \ , \ \forall (\zeta, \tau) \in E_\gamma. \tag{3.6}$$

Without loss of generality, for a positive real R, we can assume $\mathcal{V}_\gamma$ diffeomorphic through $F_\mathcal{M}$ to $B_R^H \times B_R^E$ with coordinates (y_h, y_e) centered in γ. Consider now the function $G_\gamma : B_R^H \times B_R^E \to E_\gamma$ defined as

$$G_\gamma(y_h, y_e) = \partial_{y_e}(S_k^\sigma \circ \Phi_\gamma^{-1})(y_h, y_e).$$

As consequence of the Implicit Function Theorem, there exists a function $g_\gamma : B_R^H \to B_R^E$ such that, over $B_R^H \times B_R^E$, the equality $G_\gamma(y_h, y_e) = 0$ holds if and only if $y_e = g_\gamma(y_h)$. In particular, all the critical points of S_k^σ in this local coordinate chart belong to the graph of g_γ which we denote by $\mathcal{G}(g_\gamma)$. Consider the map $T(y_h, y_e) = (y_h, y_e - g_\gamma(y_h))$ which pointwise translates $\mathcal{G}(g_\gamma)$ to $B_R^H \times \{0\}$ in the direction of y_e. Let $r \in (0, R)$ such that $B_r^H \times B_r^E$ is the image through T of an open neighbourhood of $\mathcal{G}(g_\gamma)$ and set $\Phi_\gamma = F_\mathcal{M} \circ T$ defined on $\mathcal{V}_\gamma = \Phi_\gamma^{-1}(B_r^H \times B_r^E)$. Observe that Φ_γ is bi-Lipschitz because composition of bi-Lipschitz maps. Moreover, if $\theta \in \mathcal{Z}(\eta_k)$, then $\Phi_\gamma(\theta) \in B_R^H \times \{0\}$ since F_M sends zeroes of η_k to $\mathcal{G}(g_\gamma)$ and T sends the graph of g_γ to $B_r^H \times \{0\}$.

Finally let $\lambda < r - |y_e|_0$ and for simplicity write $q(t) = \Big(y_h, y_e + t\lambda \frac{y_e}{|y_e|_0}\Big)$. Up to shrinking r

we have that

$$
\begin{aligned}
(S_k^\sigma \circ \Phi_\gamma^{-1})(q(1)) - (S_k^\sigma \circ \Phi_\gamma^{-1})(q(0)) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(S_k^\sigma \circ \Phi_\gamma^{-1})(q(t))\mathrm{d}t \\
&= \int_0^1 \mathrm{d}_{q(t)}(S_k^\sigma \circ \Phi_\gamma^{-1})(\dot{q}(t))\mathrm{d}t \\
&= \int_0^1 \int_0^t \mathrm{d}(S_k^\sigma \circ \Phi_\gamma^{-1})_{(y_h, y_e)}(\dot{q}(0))\mathrm{d}s\mathrm{d}t \\
&\qquad + \int_0^1 \int_0^t \left[\frac{\mathrm{d}}{\mathrm{d}s} d(S_k^\sigma \circ \Phi_\gamma^{-1})_{q(t)}(\dot{q}(0))\right] \mathrm{d}s\mathrm{d}t \\
&\leq \int_0^1 \int_0^t \mathrm{Hess}_{q(t)}(S_\sigma^k \circ \Phi_\gamma^{-1})(\dot{q}(0))\mathrm{d}s\mathrm{d}t \\
&\leq -D_\gamma \lambda^2,
\end{aligned}
$$

where in the above inequalities we first use the fact that the first derivative of $(S_k^\sigma \circ \Phi_\gamma^{-1})$ with respect to y_e is non positive for points close to $B_r^H \times \{0\}$ and then the inequality pointed out in (3.6) □

3.4 Vanishing sequences

Our goal is to find zeroes of η_k via minimax methods. The next definition generalizes to the magnetic action form the notion of Palais-Smale sequence.

Definition 5. A *vanishing sequence* for η_k is a sequence $(x_n, T_n) = (\gamma_n)$ of $\mathcal{M}$ such that

$$|\eta_k(\gamma_n)|_{\mathcal{M}} \to 0.$$

Because η_k is continuous, $\mathcal{Z}(\eta_k)$ coincides with the limit point set of vanishing sequence. Thus understanding whether a vanishing sequence admits or not a converging subsequence becomes of crucial importance. Generally if the sequence of periods T_n diverges to infinity or tends to zero there is no hope to find a limit point for γ_n. Nevertheless the next theorem shows that this is exactly the case to avoid in order to have compactness.

Theorem 6. *Let γ_n be a vanishing sequence on a given connected component of $\mathcal{M}$. If $T_n \in [T_*, T^*]$ for positive constants $T_* < T^*$ then γ_n admits a limit point.*

3.5 Symplectic and nowhere vanishing magnetic forms

Let us evidence the expression in a basis for the magnetic Ricci curvaure. Let v be a unit tangent vector and complete v to an orthonormal basis $\{v, e_2, ..., e_n\}$. Then

$$
\begin{aligned}
\mathrm{Ric}_k^\Omega(v) &= 2k\mathrm{Ric}(V) - \sqrt{2k}\ \mathrm{trace}(\nabla\Omega(V)) + \mathrm{trace}(A_\Omega^V) \\
&= \sum_{i=2}^n \left\{2k\mathrm{Sec}(v, e_i) - \sqrt{2k}\langle(\nabla_{e_i}\Omega)(v), e_i\rangle + \langle A^\Omega(v, e_i), e_i\rangle\right\} \\
&= \sum_{i=2}^n \mathrm{Sec}_k^\Omega(v, e_i).
\end{aligned}
$$

Thus, as in the Riemannian case, Ric_k^Ω is the sum of the magnetic sectional curvature computed by fixing any orthonormal basis of the complement of v. A question which emerges naturally is whereas a magnetic system shows up a k-level of the energy positively curved in terms of Sec_k^Ω or Ric_k^Ω. For our purpose we are interested in magnetic systems positively curved and in a range of energy close to zero. This requirement is satisfied by two standard classes of magnetic systems: symplectic magnetic systems respect to the magnetic sectional curvature and nowhere vanishing magnetic systems respect to the magnetic Ricci curvature. We need the following relevant lemma.

Lemma 7. *Let A^Ω the operator defined in (2.2). Then the following statements hold*

(i) $\langle A^\Omega(v,w),w\rangle \geq 0$ for every $(v,w)\in E^1$ and it is equal to zero if and only if $\Omega(w)=0$,

(ii) $\mathrm{trace}\Big(A^\Omega(v,\cdot)\Big)\geq 0$ for every $v\in SM$ and it is equal to zero if and only if $\Omega_p=0$.

Proof. By definition of A^Ω, if $(v,w)\in E^1$, then

$$\langle A^\Omega(v,w),w\rangle = \frac{3}{4}\langle w,\Omega(v)\rangle^2+\frac{1}{4}|\Omega(w)|^2, \tag{3.7}$$

The above formula immediately implies point (i). Moreover, if we extend v to an orthonormal basis $\{v,e_2,...,e_n\}$ then by (3.7) we get

$$\begin{aligned}\mathrm{trace}\Big(A^\Omega(v,\cdot)\Big) &= \sum_{i=2}^n\langle A^\Omega(v,e_i),e_i\rangle\\ &= \sum_{i=2}^n\Big\{\frac{3}{4}\langle\langle e_i,\Omega(v)\rangle\Omega(v),e_i\rangle+\frac{1}{4}\langle\Omega(e_i),\Omega(e_i)\rangle\Big\}\\ &= \sum_{i=2}^n\frac{3}{4}\langle e_i,\Omega(v)\rangle^2+\frac{1}{4}\Big(\sum_{i,j=2}^n\langle\Omega(e_i),e_j\rangle^2+\sum_{i=2}^n\langle\Omega(e_i),v\rangle^2\Big)\\ &= \sum_{i=2}^n\langle e_i,\Omega(v)\rangle^2+\frac{1}{4}\sum_{i,j=2}^n\langle\Omega(e_i),e_j\rangle^2,\end{aligned} \tag{3.8}$$

which is always non negative and equal to zero if and only if $\Omega_p=0$. Thus, also point (ii) holds. □

Lemma 8. *Let (M,g,σ) be a magnetic system. It holds that*

(i) if σ is symplectic then there exists a $k_0>0$ such that $\mathrm{Sec}_k^\Omega>0$ for every $k\in(0,k_0)$;

(ii) if σ is nowhere vanishing then there exists a $k_0>0$ such that $\mathrm{Ric}_k^\Omega>0$ for every $k\in(0,k_0)$.

Proof. First observe that, if σ is symplectic then by the compatibility condition (1.1) it follows that $\Omega(w)\neq 0$ for every $(p,w)\in TM$. In particular, by Equation (3.7), this implies that $\langle A^\Omega(v,w),w\rangle$ is strictly positive for every $(v,w)\in E^1$. Analogously, if σ is nowhere vanishing, then at least one term in (3.8) is different from zero which implies that $\mathrm{trace}\Big(A^\Omega(v,\cdot)\Big)$ is strictly positive for every $v\in SM$. Since E^1 and SM are compact and since the operator A^Ω is independent of k, we deduce that, if σ is symplectic then for small values of k

$$\Big\langle R_k^\Omega(v,w)+A^\Omega(v,w),w\Big\rangle>0\ ,\ \forall(v,w)\in E^1.$$

If σ is nowhere vanishing, then for small values of k

$$\mathrm{trace}\Big(R_k^\Omega(v,\cdot)+A^\Omega(v,\cdot)\Big)>0\ ,\ \forall v\in SM.$$

The statement follows. □

3.6 Symplectic magnetic forms in dimension 2: proof of Theorem C

In this subsection (M,g) is a closed oriented Riemannian surface and we denote by $\mathcal{K}: M \to \mathbb{R}$ its Gaussian curvature. If σ is a closed two form on M, then there exists a smooth function $b: M \to \mathbb{R}$ such that

$$\sigma = b \cdot \mathrm{d}\mu_g. \tag{3.9}$$

For simplicity we refer to a magnetic form σ and in case to its Lorentz operator Ω through the function b given by (3.9). Observe that, due to dimensional reasons, Ric_k^b coincides with Sec_k^b and its expression in terms of b is pointed out by the next lemma.

Lemma 9. *Let (M,g,b) be a magnetic system on a closed oriented surface. Then*

$$\mathrm{Sec}_k^b(v) = 2k \cdot \mathcal{K} - \sqrt{2k} \cdot \mathrm{d}b(i \cdot v) + b^2,$$

where by i we denote the anticlockwise rotation of an angle $\frac{\pi}{2}$.

Proof. In any orthonormal basis we have that

$$\Omega = b \cdot i \text{ and } \nabla_v \Omega = \mathrm{d}b(v) \cdot i. \tag{3.10}$$

Therefore, by definition (2.7), by using (3.10) and an orthonormal basis $\{v, e_2 = i \cdot v\}$, we obtain

$$\begin{aligned}\mathrm{Sec}_k^b(v) &= 2k\mathrm{Sec}(v,e_2) - \sqrt{2k}\langle(\nabla_{e_2}\Omega)(v), e_2\rangle + \langle A^{\Omega}(v,e_2), e_2\rangle \\ &= 2k \cdot \mathcal{K} - \sqrt{2k}\langle \mathrm{d}b(e_2) i \cdot v, e_2\rangle + \frac{3}{4}b^2 + \frac{1}{4}b^2 \\ &= 2k \cdot \mathcal{K} - \sqrt{2k} \cdot \mathrm{d}b(i \cdot v) + b^2.\end{aligned}$$

□

In this setting, Theorem C is reformulated as follows

Theorem C. *Let (M,g,b) be a magnetic system on a closed oriented surface. If there exists a positive real constant k_0 such that $\mathrm{Sec}_k^b > 0$ for every $k \in (0,k_0)$, then either b is nowhere zero or b is constantly zero and $\mathcal{K} > 0$ (and $M = S^2$).*

Proof. Suppose that for small values of k we have that $\mathrm{Sec}_k^b > 0$. Denote by S the subset of M defined as

$$S = \{p \in M \mid b(p) = 0\}.$$

This subset is closed by definition. If we show that S is also open, then the statement follows. Let $p \in S$ and for an arbitrary small radius r, let $B_r(p)$ be an open ball of M centered at p. Let $q \in B_r(p)$ be such that

$$|b(q)| = \max_{z \in B_r(p)} |b(z)| \ .$$

Assume that $|b(q)| \neq 0$ and denote by $|\mathrm{d}b|_\infty$ the uniform norm of $\mathrm{d}b$. In particular, by assumption, for small values of k, it yields

$$2k\mathcal{K} + b^2 > \sqrt{2k}|\mathrm{d}b|_\infty,$$

which implies that

$$|b(q) - b(p)| \leq \int_0^{d(x,y)} |\mathrm{d}b|_\infty \mathrm{d}t \leq \frac{r}{\sqrt{2k}}(2k \max_{z \in B_r(p)} \mathcal{K}(z) + |b(q)|^2) = r\sqrt{2k} \max_{z \in B_r(p)} \mathcal{K}(z) + \frac{r}{\sqrt{2k}}|b(q)|^2.$$

Since $b(p)=0$, up to shrinking r, we can choose k such that $\sqrt{2k}=|b(q)|$. Thus, from the above inequality, we can deduce that

$$|b(q)| \leq (r \max_{z\in B_r(p)} \mathcal{K}(z) + r)|b(q)|.$$

Up to shrinking again r and k, we also have that

$$r \max_{z\in B_r(p)} \mathcal{K}(z) + r < 1,$$

which implies that $|b(q)|=0$. Therefore, for every $p\in S$ we can find an open ball $B_r(p)$ centered such that $b|_{B_r(p)}=0$ which concludes the proof. ☐

3.7 Magnetic exponential map and conjugate points

Let $k\in(0,+\infty)$ and $p\in M$. We define the map $\exp_k^\sigma(p): T_pM \to M$ as

$$\exp_k^\sigma(p)(v) = \gamma_{\sqrt{2k}\hat{v}}(|v|) \text{ for } v\neq 0, \text{ and } \exp_k^\sigma(p)(0)=0; \tag{3.11}$$

where $\hat{v}=\frac{v}{|v|}$ and $\gamma_{\sqrt{2k}\hat{v}}$ is the unique solution of (1.2) such that $\gamma_{\sqrt{2k}\hat{v}}(0)=p$ and $\dot{\gamma}_{\sqrt{2k}\hat{v}}(0)=\sqrt{2k}\hat{v}$. As pointed out in [DPSU07], $\mathrm{d}(\exp_k^\sigma(p))_0$ is the identity of T_pM and, in particular, $\exp_k^\sigma(p)$ is a C^1 local diffeomorphism at $v=0$. Moreover, $\exp_k^\sigma(p)$ is smooth on $T_pM\setminus\{0\}$ and it is C^2 on the whole T_pM if and only if $\sigma=0$. The next lemma shows how conjugate points with respect to p correspond to the points where $\mathrm{d}(\exp_k^\sigma(p))$ is degenerate.

Lemma 10. *Let $q=\exp_k^\sigma(p)(v)$ for some $v\neq 0$. Then $\mathrm{d}(\exp_k^\sigma(p))_v$ is degenerate if and only if q is conjugate to p.*

Proof. We first point out that the differential of the magnetic exponential map is expressed by:

$$\mathrm{d}(\exp_k^\sigma(p))_v(w) = J(|v|) + \frac{1}{|v|}\langle w,v\rangle \dot{\gamma}_{\sqrt{2k}\hat{v}}(|v|), \tag{3.12}$$

where J is the unique solution of (1.4) along $\gamma_{\sqrt{2k}\hat{v}}$ with initial conditions

$$J(0)=0 \text{ and } \frac{D}{dt}J(0) = \frac{w}{|v|} - \frac{1}{|v|}\langle w,\hat{v}\rangle\hat{v}.$$

Indeed, let $\alpha:(-\varepsilon,\varepsilon)\to T_pM\setminus\{0\}$ be such that $\alpha(0)=v$ and $\alpha'(0)=0$ (here we write $\alpha'(s)=\frac{\mathrm{d}}{\mathrm{d}s}\alpha(s)$). Denote by $\widehat{\alpha(s)}=\frac{\alpha(s)}{|\alpha(s)|}$ and consider the two-parameter variation $f:(-\varepsilon,\varepsilon)\times[0,+\infty)\to M$ defined as

$$f(s,t) = \exp_k^\sigma(p)(t\widehat{\alpha(s)}).$$

Because for a fixed $s\in(-\varepsilon,\varepsilon)$, $f(s,\cdot)$ is a magnetic geodesic, the variational vector field $\frac{\mathrm{d}f}{\mathrm{d}s}(0,t)=J(t)$ is a magnetic Jacobi field along $\gamma_{\sqrt{2k}\hat{v}}$. Observe that

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}s} &= \mathrm{d}(\exp_k^\sigma(p))_{t\widehat{\alpha(s)}}\Big(t\frac{\alpha'(s)^\perp}{|\alpha(s)|}\Big)\\ &= \frac{t}{|\alpha(s)|}\mathrm{d}(\exp_k^\sigma(p))_{t\widehat{\alpha(s)}}(\alpha'(s)^\perp),\end{aligned}$$

where $\alpha^{'}(s)^{\perp} = \alpha^{'}(s) - \langle \alpha^{'}(s), \widehat{\alpha(s)}\rangle \widehat{\alpha(s)}$. The above equation implies that

$$J(t) = \frac{t}{|v|}\mathrm{d}(\exp_k^{\sigma}(p))_{t\hat{v}}(w^{\perp}). \tag{3.13}$$

Thus, we deduce that $J(0) = 0$. By deriving along $\gamma_{\sqrt{2k}\hat{v}}$ expression (3.13), we obtain

$$\frac{D}{dt}J(t) = \frac{1}{v}\mathrm{d}(\exp_k^{\sigma}(p))_{t\hat{v}}(w^{\perp}) + \frac{t}{|v|}\frac{D}{dt}\Big(\mathrm{d}(\exp_k^{\sigma}(p))_{t\hat{v}}(w^{\perp})\Big),$$

for which we also deduce that $\frac{D}{dt}J(0) = \frac{w^{\perp}}{|v|}$. Let us point out that, by (3.13), for $t = |v|$ we get

$$J(|v|) = \mathrm{d}(\exp_k^{\sigma}(p))_{v}(w^{\perp}). \tag{3.14}$$

Moreover,

$$\mathrm{d}(\exp_k^{\sigma}(p))_{v}(v) = \frac{\mathrm{d}}{\mathrm{d}t}\Big[\exp_k^{\sigma}(p)\Big((1+t)v\Big)\Big]_{t=0} = |v|\dot{\gamma}_{\sqrt{2k}\hat{v}}(|v|). \tag{3.15}$$

Thanks to (3.14) and (3.15) we conclude that

$$\begin{aligned}\mathrm{d}(\exp_k^{\sigma}(p))_{v}(w) &= \mathrm{d}(\exp_k^{\sigma}(p))_{v}\Big(w^{\perp} + \frac{1}{|v|^2}\langle w, v\rangle v\Big)\\ &= \mathrm{d}(\exp_k^{\sigma}(p))_{v}(w^{\perp}) + \mathrm{d}(\exp_k^{\sigma}(p))_{v}\Big(\frac{1}{|v|^2}\langle w, v\rangle v\Big)\\ &= J(|v|) + \frac{1}{|v|}\langle w, v\rangle\dot{\gamma}_{\sqrt{2k}\hat{v}}(|v|).\end{aligned}$$

Keeping in mind the expression (3.12), we can immediately conclude that if $\mathrm{d}(\exp_k^{\sigma}(p))_{v}(w) = 0$ for some $w \neq 0$, then $q = \exp_k^{\sigma}(p)(v)$ is a conjugate point with respect to p. Conversely, let J be a non trivial magnetic Jacobi field such that $J(|v|) = \lambda\dot{\gamma}_{\sqrt{2k}\hat{v}}(|v|)$ for a real $\lambda \in \mathbb{R}$ and observe again that

$$\sqrt{2k}\hat{v} = \mathrm{d}(\exp_k^{\sigma}(p))_{0}(v) = \dot{\gamma}_{\sqrt{2k}\hat{v}}(0),$$

so that, the normal condition on J implies that $\langle \frac{D}{dt}J(0), v\rangle = 0$. By computing

$$\mathrm{d}(\exp_k^{\sigma}(p))_{v}\Big(|v|\frac{D}{dt}J(0) - \lambda v\Big) = |v|J(|v|) - \lambda|v|\lambda\dot{\gamma}_{\sqrt{2k}\hat{v}}(|v|) = 0.$$

we deduce that $\mathrm{d}(\exp_k^{\sigma}(p))_{v}$ is degenerate which concludes the proof. □

3.8 Magnetic Jacobi fileds in dimension 2

In this subsection we consider a magnetic system (M, g, b) on a closed oriented surface. Let γ be a magnetic geodesic with energy k. Consider the frame

$$\{\dot{\gamma}_k = \frac{\dot{\gamma}}{\sqrt{2k}}, e_2\}, \tag{3.16}$$

where e_2 satisfies $\dot{e}_2 = \Omega(e_2)$. We write $J = J_1\dot{\gamma}_k + J_2 e_2$ a vector field along γ, where J_1 and J_2 are two smooth functions over $\mathbb{R}$. The next lemma gives a useful expression of the magnetic Jacobi equation evaluated on the frame (3.16).

Lemma 11. *With respect to the frame (3.16), the magnetic Jacobi equation (1.4) with normal condition takes the form*

$$\begin{cases} \ddot{J}_2 + J_2\mathcal{K}^b_k(\dot\gamma_k) = 0 \\ J_1(t) = \int_0^t J_2 b \ \mathrm{d}s \end{cases} \tag{3.17}$$

Proof. First we compute

$$\frac{D}{dt}\Big(J_1\dot\gamma_k + J_2 e_2\Big) = \dot J_1\dot\gamma_k + J_1\Omega(\dot\gamma_k) + \dot J_2 e_2 + J_2\Omega(e_2), \tag{3.18}$$

and

$$\begin{aligned}\frac{D}{dt}^2\Big(J_1\dot\gamma_k + J_2 e_2\Big) = \ddot J_1\dot\gamma_k + 2\dot J_1\Omega(\dot\gamma_k) + J_1\Big(\frac{D}{dt}\Omega\Big)(\dot\gamma_k) + J_1\Omega^2(\dot\gamma_k) + \ddot J_2 e_2 \\ +2\dot J_2\Omega(e_2) + J_2\Big(\frac{D}{dt}\Omega\Big)(e_2) + J_2\Omega^2(e_2)\end{aligned} \tag{3.19}$$

Then we also have

$$R(J_1\dot\gamma_k + J_2 e_2, \dot\gamma)\dot\gamma = J_2 2k\mathcal{K}(\dot\gamma_k)e_2, \tag{3.20}$$

$$(\nabla_J\Omega)(\dot\gamma) = J_1\mathrm{d}b(\dot\gamma)e_2 + J_2\mathrm{d}b(\sqrt{2k}e_2)e_2, \tag{3.21}$$

and, by using (3.18)

$$\Omega(\dot J) = \dot J_1\Omega(\dot\gamma_k) + J_1\Omega^2(\dot\gamma_k) + \dot J_2\Omega(e_2) + J_2\Omega^2(e_2). \tag{3.22}$$

By substituting the equalities (3.19),(3.20),(3.21) and (3.22) in (1.4), we obtain along the component of $\dot\gamma_k$

$$\ddot J_1 - (\dot J_2 b + J_2\mathrm{d}b(\dot\gamma)) = 0, \tag{3.23}$$

which, by using the normal condition, implies

$$\dot J_1 = J_2 b. \tag{3.24}$$

Along the orthogonal component we have

$$\ddot J_2 + \dot J_1 b + 2kJ_2\mathcal{K} - \sqrt{2k}J_2\mathrm{d}b(\dot\gamma_k^\perp) = \ddot J_2 + \mathcal{K}^b_k(\dot\gamma_k) = 0,$$

where in the last equality we substituted $\dot J_1$ as in (3.24). By integrating (3.24) the proof follows. ☐

This lemma plays a central role in the next Subsection as well as in the proof of the Hopf's rigidity.

3.9 A Cartan-Hadamard Theorem for magnetic flows on closed surfaces

In this Subsection, with the setting acquired in Subsections 3.7 and 3.8, we show a weaker version of the magnetic analogue of the classical Cartan-Hadamard theorem on closed oriented surfaces.

Theorem 12. *Let (M, g, b) be a magnetic system on closed oriented surface, and let $k \in (0, +\infty)$ be such that $\mathcal{K}^b_k < 0$. Then, for every $p \in M$, $\exp^\sigma_k(p)$ is a surjective local diffeomorphism.*

Proof. First observe that the condition $\mathcal{K}^b_k < 0$ implies by [Woj00] that φ^k_σ is of Anosov type. By [Con06], we deduce that $k > c$ so that for every point $p \in M$, $\exp^\sigma_k(p)$ is surjective. Now let $\gamma(t) = \exp^\sigma_k(p)(t\hat{v})$ be a magnetic geodesic starting from p for some $\hat{v} \in T_pM$ of unit norm. In accord with the splitting given in subsection 3.8, let $J = (J_1, J_2)$ be a normal magnetic Jacobi field along γ and suppose that $J_2(0) = 0$. By Lemma 11 we deduce that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\dot{J}_2 J_2 &= \ddot{J}_2 J_2 + (J_2)^2\\ &= -\mathcal{K}^b_k (J_2)^2 + (J_2)^2 \geq 0.\end{aligned}$$

Thus, $\dot{J}_2 J_2$ is increasing. Finally observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}(J_2)^2\Big) = \dot{J}_2 J_2,$$

and because $J_2(0) = 0$, $J_2(t)$ vanishes for another time $t \neq 0$ if and only if J_2 is constantly zero. We conclude that every magnetic geodesic starting from p has no conjugate points. By Lemma 10, the map $\exp^\sigma_k(p)$ is a local diffeomorphism for every $v \in T_pM$. □

We remark that to have a complete statement of the Cartan-Hadamard Theorem for magnetic systems, we would still need to prove that the magnetic exponential map is a covering map.

4 Magnetic curvature and Hessian of the magnetic form

Let $\gamma \in \mathcal{Z}(\eta_k)$ and consider the splitting $\dot{\gamma} \oplus \{\dot{\gamma}\}^\perp$ of $\gamma^* TM$, where $\{\dot{\gamma}\}^\perp$ is the orthogonal complement of $\dot{\gamma}$. We decompose a vector field along γ into $V = V_1 + V_2$, where V_1 denotes its component along $\dot{\gamma}$ and V_2 its orthogonal component. To avoid any kind of confusion we adopt the following notations: $\frac{D}{\mathrm{d}t}V_1 = \dot{V}_1$ and $(\dot{V})_1 = \langle \dot{V}, \dot{\gamma}\rangle \frac{\dot{\gamma}}{|\dot{\gamma}|^2}$. Analogously, $\frac{D}{\mathrm{d}t}V_2 = \dot{V}_2$ while $(\dot{V})_2$ indicates the orthogonal projection of $\dot{V}$ on $\{\dot{\gamma}\}^\perp$. The next crucial lemma shows how such a splitting let us write the Hessian of η_k in terms of the sectional magnetic curvature.

Lemma 13. *Let $V = V_1 + V_2$ be a variation along γ and $\tau \in \mathbb{R}$. Then*

$$\begin{aligned}\mathrm{Q}_\gamma(\eta_k)(V,\tau) = \int_0^T \Big\{\Big|(\dot{V})_2 - \frac{1}{2}(\Omega(V_1) + \Omega(V))_2\Big|^2 + \Big(\frac{\langle \dot{V}, \dot{\gamma}\rangle}{|\dot{\gamma}|} - \frac{\tau}{T}|\dot{\gamma}|\Big)^2\Big\}\mathrm{d}t\\ - \int_0^T |V_2|^2 \mathrm{Sec}^\Omega_k\Big(\frac{\dot{\gamma}}{|\dot{\gamma}|}, \frac{V_2}{|V_2|}\Big)\mathrm{d}t\ .\end{aligned} \tag{4.1}$$

Proof. Preliminarily, we need the following identities:

$$(\nabla_{V_1}\Omega)(\dot{\gamma}) = (\nabla_{\dot{\gamma}}\Omega)(V_1), \tag{4.2}$$

$$\dot{V}_1 = (\dot{V})_1 - \Omega(V)_1 + \Omega(V_1), \tag{4.3}$$

$$\frac{d}{\mathrm{d}t}\langle \Omega(V_1), V\rangle = \langle (\nabla_{\dot{\gamma}}\Omega)(V_1), V\rangle + \langle \Omega(\dot{V}_1), V\rangle + \langle \Omega(V_1), \dot{V}\rangle. \tag{4.4}$$

In the expression of $\mathrm{Q}_\gamma(\eta_k)$ in Lemma 2, we decompose V into its component V_1 and V_2 and we use the identity (4.2) to obtain:

$$
\begin{aligned}
\mathrm{Q}_\gamma(\eta_k)(V,\tau) = \int_0^T P(V)\mathrm{d}t - \int_0^T \Big\{\langle R(V_2,\dot\gamma)\dot\gamma - (\nabla_{V_2}\Omega)(\dot\gamma), V_2\rangle\Big\}\mathrm{d}t \\
+ \int_0^T \Big(\frac{\langle \dot V,\dot\gamma\rangle}{|\dot\gamma|} - \frac{\tau}{T}|\dot\gamma|\Big)^2 \mathrm{d}t,
\end{aligned}
\tag{4.5}
$$

where $P(V) = |(\dot V)_2|^2 + \langle(\nabla_{\dot\gamma}\Omega)(V_1), V\rangle - \langle \dot V, \Omega(V)\rangle$. With the help of identities (4.3) and (4.4) and a Stokes argument we obtain:

$$
\begin{aligned}
\int_0^T P(V)\mathrm{d}t &= \int_0^T \Big\{|(\dot V)_2|^2 + \frac{d}{dt}\langle\Omega(V_1),V\rangle - \langle\Omega(\dot V_1),V\rangle - \langle\Omega(V_1),\dot V\rangle - \langle\dot V,\Omega(V)\rangle\Big\}\mathrm{d}t \\
&= \int_0^T \Big\{|(\dot V)_2|^2 + \langle\dot V_1,\Omega(V)\rangle - \langle\Omega(V_1),(\dot V)_2\rangle - \langle(\dot V)_2,\Omega(V)\rangle - \langle(\dot V)_1,\Omega(V)\rangle\Big\}\mathrm{d}t \\
&= \int_0^T \Big\{|(\dot V)_2|^2 - \langle\Omega(V_1)+\Omega(V)_2,(\dot V)_2\rangle + \langle\dot V_1 - (\dot V)_1,\Omega(V)\rangle\Big\}\mathrm{d}t \\
&= \int_0^T \Big\{|(\dot V)_2 - \frac{1}{2}(\Omega(V_1)+\Omega(V))_2|^2 - H(V)\Big\}\mathrm{d}t\ .
\end{aligned}
$$

where we write $H(V) = \frac{1}{4}|\Omega(V_1)+\Omega(V)_2|^2 + \langle\Omega(V)_1,\Omega(V)\rangle - \langle\Omega(V_1),\Omega(V)_2\rangle$. Observe that

$$
\begin{aligned}
H(V) &= \frac{1}{4}|2\Omega(V_1)+\Omega(V_2)_2|^2 + |\Omega(V_2)_1|^2 + \langle\Omega(V_1),\Omega(V_1)\rangle - \langle\Omega(V_1),\Omega(V_2)_2\rangle \\
&= \frac{1}{4}|\Omega(V_2)_2|^2 + |\Omega(V_2)_1|^2 \\
&= \frac{1}{4}|\Omega(V_2)|^2 + \frac{3}{4}|\Omega(V_2)_1|^2 \\
&= |V_2|^2\Big\langle A^\Omega\big(\frac{\dot\gamma}{|\dot\gamma|},\frac{V_2}{|V_2|}\big),\frac{V_2}{|V_2|}\Big\rangle,
\end{aligned}
$$

where in the last equality we use Definition (2.2) of A^Ω. Therefore, we finally obtain that

$$
\int_0^T P(V)\mathrm{d}t = \int_0^T \Big\{|(\dot V)_2 + \frac{1}{2}(\Omega(V_1)+\Omega(V))_2|^2 - |V_2|^2\Big\langle A^\Omega\big(\frac{\dot\gamma}{|\dot\gamma|},\frac{V_2}{|V_2|}\big),\frac{V_2}{|V_2|}\Big\rangle\Big\}\mathrm{d}t.
$$

By substituting $P(V)$ in (4.5), the statement follows. □

The next lemma shows how we can always construct variations along γ such that their evaluation in $\mathrm{Q}_\gamma(\eta_k)$ has no terms which depend on the tangential component.

Lemma 14. *Let V be a variation along γ such that $\langle V,\dot\gamma\rangle = 0$. Then there exists a periodic function $g : [0,T] \to \mathbb{R}$ and a real constant τ depending linearly on V such that, if we write $W = V + g\dot\gamma$, then*

$$
\mathrm{Q}_\gamma(\eta_k)(W,\tau) = \int_0^T \Big|(\dot V)_2 - \frac{1}{2}\Omega(V)_2\Big|^2\mathrm{d}t - \int_0^T |V|^2\mathrm{Sec}^\Omega_k\Big(\frac{\dot\gamma}{|\dot\gamma|},\frac{V}{|V|}\Big)\mathrm{d}t. \tag{4.6}
$$

Proof. Consider g and τ defined as follows

$$
g(t) = -\int_0^t \Big\{\frac{\langle\dot V,\dot\gamma\rangle}{|\dot\gamma|^2} - \frac{\tau}{T}\Big\}\mathrm{d}t,\ \tau = \int_0^T \frac{\langle\dot V,\dot\gamma\rangle}{|\dot\gamma|^2}\mathrm{d}t.
$$

In particular, the couple (g, τ) satisfies the differential problem along γ given by

$$\begin{cases} \dot{g} + \frac{\langle \dot{V}, \dot{\gamma}\rangle}{|\dot{\gamma}|^2} - \frac{\tau}{T} = 0 \\ g(0) = g(T) = 0 \end{cases} \tag{4.7}$$

Thus, if $W = V + g\dot{\gamma}$, then

$$\Big(\frac{\langle \dot{W}, \dot{\gamma}\rangle}{|\dot{\gamma}|} - \frac{\tau}{T}|\dot{\gamma}|\Big)^2 = |\dot{\gamma}|^2\Big(\frac{\langle \dot{V}, \dot{\gamma}\rangle}{|\dot{\gamma}|^2} + \dot{g} - \frac{\tau}{T}\Big)^2 = 0, \tag{4.8}$$

and

$$(\dot{W})_2 - \frac{1}{2}(\Omega(W_1) + \Omega(W))_2 = (\dot{V})_2 + g\ddot{\gamma} - g\Omega(\dot{\gamma}) - \frac{1}{2}\Omega(V)_2 = (\dot{V})_2 - \frac{1}{2}\Omega(V)_2. \tag{4.9}$$

By using (4.8) and (4.9) in the expression (4.1) of Lemma 13 the statement follows.

□

4.1 Magnetic parallel transport

Along γ, with the same splitting as above, we define the operator $\tilde{\Omega} : \gamma^*TM \to \gamma^*TM$ as

$$\tilde{\Omega}(V) = \Omega(V_1) + \Omega(V)_1 + \frac{1}{2}\Omega(V_2)_2. \tag{4.10}$$

Consider now the differential problem given by

$$\dot{V} = \tilde{\Omega}(V). \tag{4.11}$$

This is an ordinary linear system of first order differential equations which allows us to define a linear isomorphism $P_\gamma : T_{\gamma(0)}M \to T_{\gamma(0)}M$ as follows

$$P_\gamma(v) = V(T),$$

where $V(t)$ is the unique solution of (4.11) with initial vector v.

Lemma 15. *The operator $\tilde{\Omega}$ is antisymmetric with respect to the metric g. This in particular implies that the linear isomorphism P_γ is orthogonal.*

Proof. Let V and W be variations along γ. Then, by definition (4.10) of $\tilde{\Omega}$

$$\begin{aligned} \langle \tilde{\Omega}(V), W\rangle &= \langle \Omega(V_1) + \Omega(V)_1 + \frac{1}{2}\Omega(V_2)_2, W\rangle \\ &= -\langle V_1, \Omega(W)\rangle - \langle V, \Omega(W_1)\rangle - \frac{1}{2}\langle V_2, \Omega(W_2)\rangle \\ &= -\langle V, \Omega(W)_1\rangle - \langle V, \Omega(W_1)\rangle - \frac{1}{2}\langle V, \Omega(W_2)_2\rangle \\ &= -\langle V, \tilde{\Omega}(W)\rangle. \end{aligned}$$

This implies that if V and W are solution of (4.11), then

$$\frac{d}{\mathrm{d}t}\langle V, W\rangle = \langle \tilde{\Omega}(V), W\rangle + \langle V, \tilde{\Omega}(W)\rangle = 0.$$

By definition of P_γ, the statement follows. □

Thus P_γ is an orthogonal operator and in this sense the above construction generalizes the notion of Riemannian parallel transport to the magnetic case.

4.2 Proof of Theorem D

We can now proceed to the proof of Theorem D which is an immediate consequence of the next statement.

Lemma 16. *Let (M, g, σ) be a magnetic system over an oriented even dimensional manifold and γ a zero of η_k. If* $\mathrm{Sec}_k^\Omega > 0$*, then* $\mathrm{index}(\gamma) \geq 1$.

Proof. Assume $\mathrm{Sec}_k^\Omega > 0$. Consider $G \subset \gamma^*TM$ the orthogonal subbundle respect to $\dot\gamma$ together with its projection map $\mathrm{p}_G : \gamma^*TM \to G$. Because $\tilde\Omega(\dot\gamma) = \Omega(\dot\gamma)$, $\dot\gamma$ is a periodic solution of (4.11) and the map P_γ leaves invariant $G(0)$. Since P_γ is orthogonal and since M is oriented, we deduce that $\tilde P_\gamma = P_\gamma \circ \mathrm{p}_G$ is a special orthogonal isomorphism of $G(0)$. By assumption $G(0)$ is odd dimensional so that 1 is eigenvalue of $\tilde P_\gamma$ i.e. there exists $v \in G$ such that $\tilde P_\gamma(v) = v$. By definition of $\tilde P_\gamma$, this is equal to say that there exists V periodic solution of (4.11), orthogonal to $\dot\gamma$. Let V such a solution and observe that $\tilde\Omega(V)_2 = \frac{1}{2}\Omega(V)_2$. Consider $W = V + g\dot\gamma$ and τ as in Lemma 14. It follows that

$$\begin{aligned}
\mathrm{Q}_\gamma(\eta_k)(W,\tau) &= \int_0^T \Big|(\dot V)_2 - \frac{1}{2}\Omega(V)_2\Big|^2 \mathrm{d}t - \int_0^T |V|^2 \mathrm{Sec}_k^\Omega\Big(\frac{\dot\gamma}{|\dot\gamma|}, \frac{V}{|V|}\Big)\mathrm{d}t \\
&= -\int_0^T |V|^2 \mathrm{Sec}_k^\Omega\Big(\frac{\dot\gamma}{|\dot\gamma|}, \frac{V}{|V|}\Big)\mathrm{d}t < 0.
\end{aligned}$$

Therefore the index of γ is at least one. □

Proof of Theorem D. As argued in [Abb13], η_k carries a minimizer in each non trivial free homotopy class of loops of M for $k > c$. If M is oriented and even dimensional and $\mathrm{Sec}_k^\Omega > 0$, then by Lemma 16, $\pi_1(M)$ has to be necessarily trivial. In the case when M is closed oriented surfaces, as showed in [CMP04], if the magnetic form is exact then η_k carries a minimizer for every $k \in (0, c]$. Thus with the same argument one can deduce that if Sec_k^Ω is positive on a level k then $M = S^2$ and $k > c$. □

4.3 A magnetic Bonnet-Myers theorem

The next lemma gives an explicit formula of $\mathrm{Q}_\gamma(\eta_k)$ when evaluated on vector fields obtained by rescaling solutions of equation (4.11) with a T-periodic function.

Lemma 17. *Let V be a solution of (4.11) of unit norm and orthogonal to $\dot\gamma$. Consider $V^f = fV$, where $f : [0,T] \to \mathbb{R}$ with $f(0) = f(T) = 0$. There exists a variation W^f and a real number τ such that*

$$\mathrm{Q}_\gamma(\eta_k)(W^f,\tau) = \int_0^T \Big\{\dot f^2 - f^2 \mathrm{Sec}_k^\Omega\Big(\frac{\dot\gamma}{|\dot\gamma|}, V\Big)\Big\}\mathrm{d}t. \tag{4.12}$$

Proof. Let V^f as in the statement. Then by Lemma 14, there exist g and τ such that, writing $W^f = V^f + g\dot\gamma$, it follows that

$$\begin{aligned}
\mathrm{Q}_\gamma(\eta_k)(W^f,\tau) &= \int_0^T \Big|(\dot V^f)_2 - \frac{1}{2}\Omega(V^f)_2\Big|^2 - |V^f|^2\ \mathrm{Sec}_k^\Omega\Big(\frac{\dot\gamma}{|\dot\gamma|}, V\Big)\mathrm{d}t \\
&= \int_0^T \Big|\dot f V + f(\dot V)_2 - f\frac{1}{2}\Omega(V)_2\Big|^2 - |fV|^2\ \mathrm{Sec}_k^\Omega\Big(\frac{\dot\gamma}{|\dot\gamma|}, V\Big)\mathrm{d}t \\
&= \int_0^T \Big\{\dot f^2 - f^2 \mathrm{Sec}_k^\Omega\Big(\frac{\dot\gamma}{|\dot\gamma|}, V\Big)\Big\}\mathrm{d}t.
\end{aligned}$$

□

With the previous lemma in one hand, we conclude the section by stating the magnetic version of the classical Bonnet-Myers theorem.

Lemma 18. *Let $\gamma = (x, T)$ be a zero of η_k with* $\operatorname{index}(\gamma) = m$. *If* $\mathrm{Ric}^{\Omega}_k \geq \frac{1}{r^2} > 0$ *for a positive constant r, then*

$$T \leq r\pi(m+1).$$

Proof. Let $\{V_1, ..., V_{n-1}\}$ be a family of linearly independent solutions for the differential problem (4.11), with unit norm and orthogonal to $\dot\gamma$. For $j = 0, 1, ..., m$ and $i = 1, ..., n-1$ look at $V_i^{f_j} = f_j V_i$, where

$$f_j(t) = \begin{cases} \sin\Big(\dfrac{(m+1)\pi t}{T}\Big) & , \ t \in \Big[\dfrac{jT}{m+1}, \dfrac{(j+1)T}{m+1}\Big] \\ 0 & \text{otherwise} \end{cases}$$

Assume $\mathrm{Ric}^{\Omega}_k \geq \frac{1}{r^2} > 0$ and suppose by contradiction $T > \pi r(m+1)$. Let $W_i^{f_j} = V_i^{f_j} + g_{ij}\dot\gamma$ and τ_{ij} given by Lemma 17, fix j and observe that the sum

$$\begin{aligned}
\sum_{i=1}^{n-1} \mathrm{Q}_\gamma(\eta_k)(W_i^{f_j}, \tau_{ij}) &= \sum_{i=1}^{n-1} \int_{\frac{jT}{m+1}}^{\frac{(j+1)T}{m+1}} \frac{(m+1)^2\pi^2}{T^2} \cos^2\Big(\frac{(m+1)\pi t}{T}\Big)\mathrm{d}t \\
&\quad - \int_{\frac{jT}{m+1}}^{\frac{(j+1)T}{m+1}} \sin^2\Big(\frac{(m+1)\pi t}{T}\Big)\mathrm{Sec}^{\Omega}_k\Big(\frac{\dot\gamma}{|\dot\gamma|}, V_i\Big)\mathrm{d}t \\
&= (n-1)\Big[\frac{(m+1)^2\pi^2}{2T(m+1)} - \int_{\frac{jT}{m+1}}^{\frac{(j+1)T}{m+1}} \Big\{\sin^2\Big(\frac{(m+1)\pi t}{T}\Big)\mathrm{Ric}^{\Omega}_k\Big(\frac{\dot\gamma}{|\dot\gamma|}\Big)\Big\}\mathrm{d}t\Big] \\
&\leq (n-1)\Big[\frac{(m+1)^2\pi^2}{2T(m+1)} - \frac{T}{2r^2(m+1)}\Big] \\
&= (n-1)\Big[\frac{(m+1)^2\pi^2 r^2 - T^2}{2T(m+1)r^2}\Big] < 0.
\end{aligned}$$

Therefore for every j there exists $i_j \in \{0, 1, ..., n-1\}$ such that $\mathrm{Q}_\gamma(\eta_k)(W_{i_j}^{f_j}, \tau_{i_j j}) < 0$. Moreover, by construction, for every $s \neq l$ the supports of f_{is} and f_{il} are disjoint which implies that $W_{i_j}^{f_j}$ are linearly independent. For real coefficients $\lambda_1, ..., \lambda_{m+1}$ define

$$V = \sum_{j=1}^{m+1} \lambda_j V_{i_j}^{f_j}, \ g = \sum_{j=1}^{m+1} \lambda_j g_{i_j j} \text{ and } \tau = \sum_{j=1}^{m+1} \lambda_j \tau_{i_j j}.$$

Observe that, by linearity of Equation (4.7) and Equation (4.9), g and τ are exactly the ones associated to V and $W = V + g\dot\gamma$ in Lemma 14 in order to obtain that

$$\begin{aligned}
\mathrm{Q}_\gamma(\eta_k)\Big(\sum_{j=1}^{m+1}(W^{f_j}, \tau_{i_j j})\Big) &= \int_0^T \Big|(\dot V)_2 - \frac{1}{2}\Omega(V)_2\Big|^2 \mathrm{d}t - \int_0^T |V|^2 \mathrm{Sec}^{\Omega}_k\Big(\frac{\dot\gamma}{|\dot\gamma|}, \frac{V}{|V|}\Big)\mathrm{d}t \\
&= \sum_{j=1}^{m+1} \lambda_j^2 \int_{\frac{jT}{m+1}}^{\frac{(j+1)T}{m+1}} \Big\{\dot f_j^2 - f_j^2 \ \mathrm{Sec}^{\Omega}_k\Big(\frac{\dot\gamma}{|\dot\gamma|}, V_{i_j}\Big)\Big\}\mathrm{d}t \qquad (4.13) \\
&= \lambda_j^2 \sum_{j=1}^{m+1} \mathrm{Q}_\gamma(\eta_k)(W^{f_j}, \tau_{i_j j}) \leq 0, \qquad (4.14)
\end{aligned}$$

where in (4.13) we used again that the support of f_j are disjoint. Observe that, the equality in (4.14) holds if and only if $\lambda_j = 0$ for every j. Summing up the family $\{(W^j_{i_j}, \tau_{i_j j})\}$ is a family of linear independent vectors of $T_\gamma\mathcal{M}$ which generate a vector subspace Z, of dimension $m+1$, such that $\mathrm{Q}_\gamma(\eta_k)|_Z$ is negative definite. We conclude that the index of γ is equal or greater than $m+1$ in contradiction with the assumption. $\square$

5 Proof of Theorem B (and Theorem A)

5.1 Weakly exact case

Minimax geometry below $c(g,\sigma)$

In this paragraph we assume σ to be weakly exact. Let S^σ_k be the primitive of η_k defined on $\mathcal{M}_0$ as given in (1.5). Below the critical value S^σ_k enjoys a minimax geometry on the set of loops with short length which we discuss now. Denote by $M^+ = M \times (0,+\infty)$ the set of constant loops with free period and consider

$$\Gamma^k_0 = \{\phi : [0,1] \to \mathcal{M}_0,\ \phi(0) \in M^+ \text{ and } S^k_\sigma(\phi(1)) < 0\}.$$

By definition (1.3) of c, if $k < c$, then Γ^k_0 is non empty and the minimax value function is defined as follows

$$\mathbf{s} : (0,c) \to \mathbb{R},\ \mathbf{s}(k) = \inf_{\Gamma^k_0} \max_{[0,1]} S^\sigma_k(\phi(t)). \tag{5.1}$$

The minimax geometry of S^σ_k on Γ^k_0 is highlighted by the next lemma.

Lemma 19. *Let I be an open interval with compact closure fully contained in $(0,c)$. Then there exists a positive $\varepsilon = \varepsilon(I)$ such that*

$$\mathbf{s}(k) \geq \varepsilon,\ \forall k \in I. \tag{5.2}$$

Proof. For the proof we refer the reader to [AB16, Section 5]. $\square$

Consider the negative gradient vector field $-\nabla S^\sigma_k$ on $\mathcal{M}_0$ induced by the Riemannian metric $g_\mathcal{M}$. Since $-\nabla S^\sigma_k$ is smooth, it admits a local flow for positive time which in general is not complete. Indeed, source of non-completeness mainly originates from the non-completeness of $(0,+\infty)$ and the fact that $|\nabla S^\sigma_k|_\mathcal{M}$ is not bounded on $\mathcal{M}_0$. One can avoid these situations by considering the pseudo gradient given by

$$\mathrm{X}_k = (h \circ S^\sigma_k)\frac{\nabla S^\sigma_k}{\sqrt{1 + |\nabla S^\sigma_k|^2_\mathcal{M}}}. \tag{5.3}$$

Here $h : \mathbb{R} \to [0,1]$ is a cut-off function such that $h^{-1}\{0\} = (-\infty, \frac{s(k)}{4}]$ and $h^{-1}\{1\} = [\frac{s(k)}{2}, +\infty)$. As argued in [Abb13, Section 8], because X_k has bounded norm and vanishes on the set $\{S^\sigma_k < \frac{s(k)}{4}\}$, it admits a positively complete flow $\Psi_k : [0,+\infty) \times \mathcal{M}_0 \to \mathcal{M}_0$. The next lemma evidences two important properties of Ψ_k.

Lemma 20. *Let $u = (x,T) : (0,+\infty) \to \mathcal{M}_0$ be an integral curve of Ψ_k. Then for every $s \in (0,+\infty)$ the following statements hold:*

(i) $S^\sigma_k(u(s)) \leq S^\sigma_k(u(0))$;

(ii) $|T(s) - T(0)|^2 \leq s\Big(S_k^\sigma(u(0)) - S_k^\sigma(u(s))\Big)$.

Proof. For the proof we refer the reader to [Mer10, Section 5]. □

Lemma 21. *The set Γ_0^k is Ψ_k-invariant.*

Proof. Since for every $(p,T) \in M^+$, $\nabla S_k^\sigma(p,T) = k\frac{\partial}{\partial T}$ then $\Psi_k(t, M^+) \subseteq M^+$ for every positive time t. This fact together with point (i) of Lemma 20 implies that Γ_0^k is Ψ_k-invariant. □

Existence almost everywhere and index estimates

Hereafter we fix an interval I with compact closure fully contained in $(0,c)$. Because S_k^σ is monotone increasing with respect to the parameter k, one can easily deduce that also $\mathbf{s}$ is monotone increasing. Thus, there exists $J \subset I$ of full Lebesgue measure such that $\mathbf{s}$ is differentiable on J. Let us point out that the almost every where differentiability of $\mathbf{s}$ is a crucial assumption in the Struwe monotonicity argument. Indeed, when $k \in J$, we can recover compactness condition for vanishing sequences related to the minimax geometry pointed out in Section 5.1. For more details about this construction, we address the reader to [Abb13, Section 8] and [Mer10, Section 5]. Here we will need a more precise version of this statement (see [AMP15] and [AMMP17]), in order to prepare the ground to prove, in Lemma 24, that vanishing points of η_k coming from this minimax geometry have Morse index bounded by 1. We proceed by showing that at the values of the energy where $\mathbf{s}$ is differentiable we can bound the period of points in the image of elements of Γ_0^k which almost realize the minimax value.

Lemma 22. *Let $k \in J$. Then there exists a constant A such that for every small $\varepsilon > 0$, by writing $k_\varepsilon = k + \varepsilon$, if $\phi \in \Gamma_0^k$ and $\max S_{k_\varepsilon}^\sigma(\phi) \leq \mathbf{s}(k_\varepsilon) + \varepsilon$, then for every $t \in [0,1]$ satisfying also $S_k^\sigma(\phi(t)) > \mathbf{s}(k) - \varepsilon$, the inequality $T_{\phi(t)} \leq A + 2$ holds.*

Proof. Because $\mathbf{s}$ is differentiable at k, there exists a positive constant $A = A(k)$ such that for every positive ε

$$|\mathbf{s}(k_\varepsilon) - \mathbf{s}(k)| \leq A \cdot |k_\varepsilon - k|. \tag{5.4}$$

Thus, if $\phi \in \Gamma_0^k$ and $t \in [0,1]$ are as in the statement, then

$$T_{\phi(t)} = \frac{S_\sigma^{k_\varepsilon}(\phi(t)) - S_k^\sigma(\phi(t))}{k_\varepsilon - k} \leq \frac{\mathbf{s}(k_\varepsilon) + \varepsilon - \mathbf{s}(k) + \varepsilon}{\varepsilon} \leq A + 2.$$

□

Let $T^* > \sqrt{2\mathbf{s}(k)} + 3(A+1)$ for which the role is clarified later. Consider the set

$$\mathcal{B}_k = \{S_k^\sigma \geq \frac{\mathbf{s}(k)}{2}\} \cap \{T < T^*\}.$$

By Lemma 19, $\mathbf{s}(k)$ is positive and, as argued in [Mer10, Proposition 5.8], S_k^σ restricted to $\mathcal{B}_k$ satisfies Palais-Smale. Therefore the set $\mathcal{C}_k = \mathrm{Crit}(S_k^\sigma) \cap \mathcal{B}_k = \mathcal{Z}(\eta_k) \cap \mathcal{B}_k$ is compact. The next statement shows that $\mathcal{C}_k$ is non empty and that we can always find an element of Γ_0^k which passes arbitrarily close to $\mathcal{C}_k$.

Lemma 23. *Let $k \in J$. For every V open neighborhood of $\mathcal{C}_k$ and for every small $\varepsilon > 0$ there exists an element $\varphi_\varepsilon \in \Gamma_0^k$ such that*

$$\varphi_\varepsilon([0,1]) \subset \{S_k^\sigma < \mathbf{s}(k)\} \cup \Big(\{\mathbf{s}(k) \leq S_k^\sigma < \mathbf{s}(k) + \varepsilon\} \cap V\Big).$$

In particular, $\mathcal{C}_k$ is non empty.

Proof. Let k be a point of differentiability for $\mathbf{s}$ and let V be an open neighborhood of $\mathcal{C}_k$. Since $\mathcal{C}_k$ consists of fixed points for Ψ_k and S_k^σ satisfies Palais-Smale on $\mathcal{B}_k$, there exists an open set V' and a positive $\delta = \delta(V')$ such that V' still contains $\mathcal{C}_k$, $\Psi_k([0,1] \times V') \subset V \cap \mathcal{B}_k$ and

$$|\nabla S_k^\sigma|_{\mathcal{M}} \geq \delta > 0 \quad \text{on} \quad \forall (x,T) \in \mathcal{B}_k \setminus V'. \tag{5.5}$$

Fix a positive ε such that $\varepsilon < \delta^2$. By definition of $\mathbf{s}$, for every $\tilde{\varepsilon} \in (0,\varepsilon)$ there exists $\phi_{\tilde{\varepsilon}} \in \Gamma_0^{k_{\tilde{\varepsilon}}}$ such that

$$\max_{t \in [0,1]} S_{k_{\tilde{\varepsilon}}}^\sigma(\phi_{\tilde{\varepsilon}}(t)) \leq \mathbf{s}(k_{\tilde{\varepsilon}}) + \tilde{\varepsilon}.$$

Moreover, up to shrinking $\tilde{\varepsilon}$, we can assume that $\tilde{\varepsilon}(A+1) < \varepsilon$ and that $\phi_{\tilde{\varepsilon}} \in \Gamma_0^k$. By using (5.4) in Lemma 22, we deduce that

$$S_k^\sigma(\phi_{\tilde{\varepsilon}}(t)) < S_{k_{\tilde{\varepsilon}}}^\sigma(\phi_{\tilde{\varepsilon}}(t)) \leq \mathbf{s}(k_{\tilde{\varepsilon}}) + \tilde{\varepsilon} - \mathbf{s}(k) + \mathbf{s}(k) \leq \mathbf{s}(k) + (A+1)\tilde{\varepsilon} < \mathbf{s}(k) + \varepsilon.$$

We claim that the element $\varphi_\varepsilon \in \Gamma_0^k$ defined by

$$\varphi_\varepsilon = \Psi_k(1, \phi_{\tilde{\varepsilon}}(t)), \tag{5.6}$$

is the desired one. Indeed, first observe that because S_k^σ decreases along the flow lines of Ψ_k, if $t \in [0,1]$ is such that $S_k^\sigma(\phi_{\tilde{\varepsilon}}(t)) < s(k)$ then $S_k^\sigma(\varphi_\varepsilon(t)) < s(k)$. By the choice of T^*, by Lemma 22 and by Lemma 20, if $S_k^\sigma(\phi_{\tilde{\varepsilon}}(t)) \in (\mathbf{s}(k), \mathbf{s}(k) + \varepsilon)$, then either $S_k^\sigma(\varphi_\varepsilon(t)) < s(k)$ or $\Psi_k(s, \phi_{\tilde{\varepsilon}}(t)) \in \mathcal{B}_k \cap (\mathbf{s}(k), \mathbf{s}(k) + \varepsilon)$ for every $s \in [0,1]$. Let us focus on the second case. If there exists a time $s_0 \in [0,1]$ such that $\Psi_k(s_0, \phi_{\tilde{\varepsilon}}(t)) \in V'$ then, by the assumptions on V', we deduce that $\Psi_k(1, \phi_{\tilde{\varepsilon}}(t)) \in V'$. Suppose by contradiction that for every t such that $\Psi_k(s, \phi_{\tilde{\varepsilon}}(t)) \in \mathcal{B}_k \cap (\mathbf{s}(k), \mathbf{s}(k) + \varepsilon)$ and for every $s \in [0,1]$, $\Psi_k(s, \phi_{\tilde{\varepsilon}}(t))$ does not enter V'. Then, by using (5.5), we deduce that

$$\begin{aligned}
S_k^\sigma(\varphi_\varepsilon(t)) &= S_k^\sigma(\phi_{\tilde{\varepsilon}}(t)) + \int_0^1 \frac{d}{ds} S_k^\sigma(\Psi_k(s, \phi_{\tilde{\varepsilon}}(t)))\mathrm{d}s \\
&\leq \mathbf{s}(k) + \varepsilon - \int_0^1 |\nabla S_k^\sigma|_{\mathcal{M}}^2 \mathrm{d}s \\
&\leq \mathbf{s}(k) + \varepsilon - \delta^2 \\
&< \mathbf{s}(k).
\end{aligned}$$

In this way we find an element of Γ_0^k such that $\varphi_\varepsilon([0,1]) \subset \{S_k^\sigma < \mathbf{s}(k)\}$ which contradicts the definition of $\mathbf{s}(k)$. We can conclude that φ_ε, as defined in (5.6), is such that either $S_k^\sigma(\varphi_\varepsilon(t)) < \mathbf{s}(k)$ or $S_k^\sigma(\varphi_\varepsilon(t)) \in (\mathbf{s}(k), \mathbf{s}(k) + \varepsilon)$ and $\varphi_\varepsilon(t) \in V' \subset V$. The claim is proved. □

Lemma 24. *Let $k \in J$. There exists $\gamma \in \mathcal{Z}(\eta_k)$ such that* $\mathrm{index}(\gamma) \leq 1$.

Proof. By Lemma 23 the set $\mathcal{C}_k$ is not empty. Let $\gamma \in \mathcal{C}_k$ and consider $(\mathcal{V}_\gamma, \Phi_\gamma)$ and D_γ as in Lemma 4. Let $\rho^t_\gamma : B^H_{r_\gamma} \times (B^E_{r_\gamma} \setminus \{0\}) \to B^H_{r_\gamma} \times B^E_{r_\gamma}$ be the deformation given by

$$\rho^t_\gamma(y_h, y_e) = \Big(y_h, y_e + \min\{t, r_\gamma - |y_e|_0\}\frac{y_e}{|y_e|_0}\Big).$$

Observe that ρ^t_γ pushes points of $B^H_{r_\gamma} \times (B^E_{r_\gamma} \setminus \{0\})$ towards the boundary, in the direction of E. In particular, if $t < r_\gamma - |y_e|_0$, then by point (ii) of Lemma 4, it holds

$$(S^\sigma_k \circ \Phi^{-1}_\gamma)(\rho^t_\gamma(y_h, y_e)) \leq (S^\sigma_k \circ \Phi^{-1}_\gamma)(y_h, y_e) - D_\gamma t^2. \tag{5.7}$$

Denote by $\mathcal{W}_\gamma = \Phi^{-1}_\gamma(B^H_{r_\gamma/4} \times B^E_{r_\gamma/4})$ and by $\mathcal{U}_\gamma = \Phi^{-1}_\gamma(B^H_{r_\gamma/2} \times B^E_{r_\gamma/2})$. Let χ_γ be a bump function on $\mathcal{M}_0$ supported in $\mathcal{V}_\gamma$ such that $\chi_\gamma(\mathcal{U}_\gamma) = 1$. By contradiction suppose that every $\gamma \in \mathcal{C}_k$ has index greater than 1. Since $\mathcal{C}_k$ is compact, there exist $\gamma_1, ..., \gamma_n \in \mathcal{C}_k$ and local charts $(\mathcal{V}_{\gamma_i}, \Phi_{\gamma_i})$ such that $\mathcal{C}_k \subset \bigcup_{i=1}^n \mathcal{W}_{\gamma_i}$. Without loss of generality, we can assume that $M^+ \cap \Phi^{-1}_{\gamma_i}(\mathcal{V}_{\gamma_i}) = \emptyset$ for every i. Let us point out that the change of coordinates is bi-Lipschitz because composition of bi-Lipschitz map. We will denote by C_{ij} the Lipschitz constant of the change of coordinates $\Phi_{\gamma_j} \circ \Phi^{-1}_{\gamma_i}$ and by $|\cdot|_{0,i}$ the norm on $\Phi_{\gamma_i}(\mathcal{V}_{\gamma_i})$ induced by the chart Φ_{γ_i}.
Let $\delta = \min_i D_{\gamma_i}$, $R = \min_i r_{\gamma_i}$, $C = \max_{i \neq j} C_{ij}$ and fix

$$\varepsilon \in \Big(0, \min\Big\{\frac{R}{4nC}, \frac{R}{4n}\Big\}\Big).$$

For $j = 1, ..., n$ consider the set

$$\mathcal{W}_{\gamma_i, j} = \Phi^{-1}_{\gamma_i}\Big(\bigcup_{z \in \Phi_{\gamma_i}(\mathcal{W}_{\gamma_i})} B(z, j\varepsilon C)\Big),$$

where $B(z, j\varepsilon C)$ is the ball in $\Phi_{\gamma_i}(\mathcal{W}_{\gamma_i})$ (with respect to the norm $|\cdot|_{0,i}$) centered at z with radius $j\varepsilon C$. The following inclusions hold

$$\mathcal{W}_{\gamma_i} \subset \mathcal{W}_{\gamma_i,1} \subset \mathcal{W}_{\gamma_i,2} \subset ... \subset \mathcal{W}_{\gamma_i,j} \subset \mathcal{W}_{\gamma_i,j+1} \subset ... \subset \mathcal{W}_{\gamma_i,n} \subset \mathcal{U}_{\gamma_i} \subset \mathcal{V}_{\gamma_i}.$$

By Lemma 23, there exists an element $\phi \in \Gamma^k_0$ such that

$$\phi([0,1]) \subset \{S^\sigma_k < \mathbf{s}(k)\} \cup \Big(\{\mathbf{s}(k) \leq S^\sigma_k < \mathbf{s}(k) + \varepsilon^2\delta\} \cap \bigcup_{i=1}^n \mathcal{W}_{\gamma_i}\Big). \tag{5.8}$$

Up to refining the cover, we can assume that the endpoints $\phi(0), \phi(1) \notin \bigcup_{i=1}^n \mathcal{V}_{\gamma_i}$. By point (i) of Lemma 4, $\Phi_{\gamma_i}(\mathcal{C}_k \cap \Phi^{-1}_{\gamma_i}(\mathcal{V}_{\gamma_i})) \subseteq B^H_{r_{\gamma_i}} \times \{0\}$ for every i, and by contradiction we are assuming that $B^H_{r_{\gamma_i}} \times \{0\}$ has codimension bigger than one. Since the domain of ϕ has dimension one, with the help of the transversality theorem [GP74, Section 5], we can find an element ϕ_0 arbitrarily C^0 close to ϕ such that its image $\phi_0([0,1])$ does not intersect $B^H_{r_\gamma} \times \{0\}$ and it still contained in the right-hand term of the inclusion (5.8).
Now define

$$\phi_1(t) = \begin{cases} \Phi^{-1}_{\gamma_1}\Big(\rho^{\varepsilon\chi_{\gamma_1}(\phi_0(t))}_{\gamma_1}(\Phi_{\gamma_1}(\phi_0(t)))\Big) & \text{if } \phi_0(t) \in \mathcal{V}_{\gamma_1}, \\ \phi_0(t) & \text{otherwise.} \end{cases} \tag{5.9}$$

Observe that $\phi_1 \in \Gamma_0^k$ because it is obtained by deforming continuously the segment of ϕ_0 contained in V_{γ_1} by keeping the endpoints fixed. Moreover, if $\phi_0(t) \in \{\mathbf{s}(k) \le S_k^\sigma < \mathbf{s}(k) + \varepsilon^2\delta\} \cap \mathcal{W}_{\gamma_1}$, then the choice of ε and (5.7) implies that

$$\begin{aligned} S_k^\sigma(\phi_1(t)) &\le S_k^\sigma(\phi_0(t)) - D_{\gamma_1}\varepsilon^2 \\ &\le \mathbf{s}(k) + \delta\varepsilon^2 - D_{\gamma_1}\varepsilon^2 \\ &< \mathbf{s}(k). \end{aligned}$$

Finally, because $\rho_{\gamma_1}^{\varepsilon\chi_{\gamma_1}}$ pushes points of $\Phi_{\gamma_1}(\mathcal{W}_{\gamma_1})$ at distance at most ε, if $\phi_0(t) \in \mathcal{W}_{\gamma_1} \cap \mathcal{W}_{\gamma_i}$ then

$$\Big|\Phi_{\gamma_i}(\phi_1(t)) - \Phi_{\gamma_i}(\phi_0(t))\Big|_{0,i} \le C_{1i}\Big|\Phi_{\gamma_1}(\phi_1(t)) - \Phi_{\gamma_1}(\phi_0(t))\Big|_{0,1} \le C\varepsilon, \tag{5.10}$$

so that $\phi_1(t) \in \mathcal{W}_{\gamma_i,1}$.
Summing up the element ϕ_1 enjoys that

$$\phi_1([0,1]) \subset \{S_k^\sigma < \mathbf{s}(k)\} \cup \Big(\{\mathbf{s}(k) \le S_k^\sigma < \mathbf{s}(k) + \varepsilon^2\delta\} \cap \bigcup_{i=2}^{n} \mathcal{W}_{\gamma_i,1}\Big).$$

We repeat the construction as follows. Define $\phi_2 \in \Gamma_0^k$ as

$$\phi_2(t) = \begin{cases} \Phi_{\gamma_2}^{-1}\Big(\rho_{\gamma_2}^{\varepsilon\chi_{\gamma_2}(\phi_1(t))}(\Phi_{\gamma_2}(\phi_1(t)))\Big) & \text{if } \phi_1(t) \in \mathcal{V}_{\gamma_1}, \\ \phi_1(t) & \text{otherwise.} \end{cases}$$

By using again the inequality (5.7), if $\phi_1(t) \in \mathcal{W}_{\gamma_2,1}$ then

$$\begin{aligned} S_k^\sigma(\phi_2(t)) &\le S_k^\sigma(\phi_1(t)) - D_{\gamma_2}\varepsilon^2 \\ &\le \mathbf{s}(k) + \delta\varepsilon^2 - D_{\gamma_2}\varepsilon^2 \\ &< \mathbf{s}(k), \end{aligned}$$

Moreover, if $\phi_1(t) \in \mathcal{W}_{\gamma_1,1} \cap \mathcal{W}_{\gamma_i,1}$ for some $i \in \{2, ..., n\}$, then

$$\Big|\Phi_{\gamma_i}(\phi_2(t)) - \Phi_{\gamma_i}(\phi_1(t))\Big|_{0,i} \le C_{2i}\Big|\Phi_{\gamma_2}(\phi_2(t)) - \Phi_{\gamma_2}(\phi_1(t))\Big|_{0,2} \le C\varepsilon,$$

which implies that $\phi_2(t) \in \mathcal{W}_{\gamma_i,2}$. Thus the element ϕ_2 enjoys that

$$\phi_2([0,1]) \subset \{S_k^\sigma < \mathbf{s}(k)\} \cup \Big(\{\mathbf{s}(k) \le S_k^\sigma < \mathbf{s}(k) + \varepsilon^2\delta\} \cap \bigcup_{i=3}^{n} \mathcal{W}_{\gamma_i,2}\Big).$$

Iterating the process, at the step m we find ϕ_m such that

$$\phi_m([0,1]) \subset \{S_k^\sigma < \mathbf{s}(k)\} \cup \Big(\{\mathbf{s}(k) \le S_k^\sigma < \mathbf{s}(k) + \varepsilon^2\delta\} \cap \bigcup_{i=m+1}^{n} \mathcal{W}_{\gamma_i,m}\Big),$$

at the step n, we find an element ϕ_n such that

$$\phi_n([0,1]) \subset \{S_k^\sigma < \mathbf{s}(k)\},$$

in contradiction with the definition of $\mathbf{s}(k)$. □

5.2 Non-weakly exact case

Variation of η_k along paths

If σ is not weakly exact, then $c(g,\sigma)=+\infty$ and η_k does not admit a primitive globally defined on $\mathcal{M}_0$. Despite that, if $u:[0,1]\to\mathcal{M}_0$ is a continuous path the action variation $\Delta\eta_k(u):[0,1]\to\mathbb{R}$ of η_k along u is always well defined and it is given by

$$\Delta\eta_k(u)(t)=\int_0^t u^*\eta_k.$$

Moreover, as argued in [AB16, Section 2.3], if $\mathcal{U}\subseteq\mathcal{M}_0$ is an open subset and $u([0,1])\subset\mathcal{U}$, then for every local primitive $S_k^\sigma:\mathcal{U}\to\mathbb{R}$ we have

$$\Delta\eta_k(u)(t)=S_k^\sigma(u(t))-S_k^\sigma(u(0)),\ \forall t\in[0,1]. \tag{5.11}$$

Let $u:[0,1]\times[0,R]\to\mathcal{M}_0$ be a homotopy of u with the starting point fixed and write $u_s=u(\cdot,s)$ and by $u^t=u(t,\cdot)$. The fact that η_k is closed implies the following formula

$$\Delta\eta_k(u_s)(t)=\Delta\eta_k(u_0)(t)+\Delta\eta_k(u^t)(R). \tag{5.12}$$

Minimax geometry (non-weakly exact case)

Henceforth, we take into account M^+ and $\mathcal{V}_\delta$ as in the paragraph 5.1. Observe that, if δ is small enough, then we can restrict definition (1.5) of S_k^σ to set of short loops $\mathcal{V}_\delta\subset\mathcal{M}_0$. Moreover, as argued in [AB16, Lemma 3.2], such a primitive also enjoys that

$$\inf_{\mathcal{V}_\delta} S_k^\sigma=0.$$

By identifying the north and the south pole of the unit sphere S^2 with the end points of the interval $[0,1]$, we have a 1-1 correspondence between

$$\Big\{f:S^2\to M\Big\}\xrightarrow{\ \mathrm{F}\ }\Big\{\phi:\Big([0,1],\{0,1\}\Big)\to\Big(\mathcal{M}_0,M^+\Big)\Big\}.$$

It is a well known fact that F descends to the homotopy quotient (see for instance [Kli78]). Because σ is not weakly exact, $\pi_2(M)\neq\{0\}$ so that given a non trivial element $\mathfrak{u}\in\pi_2(M)$ one can consider the following set:

$$\Gamma_{\mathfrak{u}}=\Big\{\phi:\Big([0,1],\{0,1\}\Big)\to\Big(\mathcal{M}_0,M^+\Big),\ F^{-1}(\phi)\in\mathfrak{u}\Big\}.$$

For $\phi\in\Gamma_{\mathfrak{u}}$, we can define a primitive $S_k^\sigma(\phi):[0,1]\to\mathbb{R}$ of η_k along ϕ by

$$S_k^\sigma(\phi)(t)=\Delta\eta_k(\phi)(t)+T_{\phi(0)}k; \tag{5.13}$$

in this setting, the minimax value function is given by

$$\mathbf{s}^{\mathfrak{u}}:(0,+\infty)\to(0,+\infty),\ \mathbf{s}^{\mathfrak{u}}(k)=\inf_{\phi\in\Gamma^{\mathfrak{u}}}\max_{t\in[0,1]}\Delta S_k^\sigma(\phi)(t).$$

In analogy with the weakly exact case, the following lemma holds.

Lemma 25. *Let I be an open interval with compact closure fully contained in $(0,+\infty)$. There exists a positive ε such that*

$$\mathbf{s}^{\mathrm{u}}(k) \geq \varepsilon, \ \forall t \in I. \tag{5.14}$$

Moreover $\mathbf{s}^{\mathrm{u}}$ is monotone increasing on I.

Proof. The proof is contained in [AB16, Section 4]. □

Let $N_k = \{(x,T) \in \mathcal{V}_\delta \mid S_k^\sigma(x,T) < \frac{\varepsilon}{4}\}$ and h a cut-off function such that $h^{-1}(1) = [\frac{\varepsilon}{2}, +\infty)$ and $h^{-1}(0) = (-\infty, \frac{\varepsilon}{4}]$. Define $\tilde{h} : \mathcal{M}_0 \to \mathbb{R}$ as follows

$$\tilde{h}(x,T) = \begin{cases} 1 \ \text{ if } (x,T) \in \mathcal{M}_0 \setminus N_k \\ h \circ S_k^\sigma \ \text{ if } (x,T) \in N_k \end{cases}.$$

Let X_k be the vector field on $\mathcal{M}_0$ given by

$$X_k = \tilde{h} \cdot \frac{\eta_k^\#}{\sqrt{1 + |\eta_k^\#|_{\mathcal{M}}^2}} \ ,$$

where $\eta_k^\#$ is the dual vector field of the 1-form η_k obtained through the Riemannian metric $g_{\mathcal{M}}$. In accordance with the weakly exact case, the pseudo gradient X_k enjoys the following properties

Lemma 26. *The flow Ψ_k of X_k is positively complete and Γ_{u} is a Ψ_k-invariant set. Moreover, if $u = (x,T) : (0,+\infty) \to \mathcal{M}_0$ is a flow line of Ψ_k, then for every $s \in (0,+\infty)$ it yields that*

(i) $\Delta\eta_k(u)(s) \leq 0$,

(ii) $|T(s) - T(0)|^2 \leq -s\Delta\eta_k(u)(s)$.

Proof. The details of the proof are in [AB16, Proposition 2.8 and Lemma 2.9]. □

Existence almost everywhere and index estimates

As pointed out in Lemma 25, $\mathbf{s}^{\mathrm{u}}$ is monotone increasing. Therefore there exists $J \subset I$ of full Lebesgue measure such that $\mathbf{s}^{\mathrm{u}}$ is differentiable on J. In the following, we readapt Lemma 22, Lemma 23 and Lemma 24 to the minimax geometry of η_k on Γ^{u}.

Lemma 27. *Let $k \in J$. Then there exists a constant $A > 0$ such that $\forall \varepsilon > 0$, by writing $k_\varepsilon = k + \varepsilon$, if $\phi \in \Gamma_{\mathrm{u}}$ is such that $S_k^\sigma(\phi)(t) \leq \mathbf{s}^{\mathrm{u}}(k_\varepsilon) + \varepsilon$ for all $t \in [0,1]$, then when $\phi(t)$ satisfies also $S_k^\sigma(\phi)(t) > \mathbf{s}^{\mathrm{u}}(k) - \varepsilon$, the inequality $T_{\phi(t)} \leq A$ holds.*

Proof. The fact that k is a point of differentiability for $\mathbf{s}^{\mathrm{u}}$ implies the existence of a positive constant $A = A(k)$ such that

$$|\mathbf{s}^{\mathrm{u}}(k_\varepsilon) - \mathbf{s}^{\mathrm{u}}(k)| \leq A \cdot |k_\varepsilon - k| \ , \ \forall k_\varepsilon \in I.$$

Fix $\varepsilon > 0$ and $\phi \in \Gamma_{\mathrm{u}}$. If $\phi(t)$ is taken as in the statement, then

$$T_{\phi(t)} = \frac{S_{k_\varepsilon}^\sigma(\phi)(t) - S_k^\sigma(\phi)(t)}{k_\varepsilon - k} \leq \frac{\mathbf{s}^{\mathrm{u}}(k_\varepsilon) + \varepsilon - \mathbf{s}^{\mathrm{u}}(k) + \varepsilon}{\varepsilon} \leq A + 2.$$

□

Let $T^* = \sqrt{2\mathbf{s}^{\mathrm{u}}(k)} + 3(A+1)$ and define the set

$$\mathcal{B}_k = (\mathcal{M}_0 \setminus N_k) \cap \{T \leq T^*\}.$$

Since every vanishing sequence of η_k contained in $\mathcal{B}_k$ is compact the set $\mathcal{C}_k = \mathcal{B}_k \cap \mathcal{Z}(\eta_k)$ is compact.

Lemma 28. *Let $k \in J$. Then for every V open neighborhood of $\mathcal{C}_k$ and for every $\varepsilon > 0$ there exists an element $\varphi_\varepsilon \in \Gamma_{\mathrm{u}}$ such that, $\forall t \in [0,1]$ either $S_k^\sigma(\varphi_\varepsilon)(t) < s^{\mathrm{u}}(k)$ or $S_k^\sigma(\varphi_\varepsilon)(t) \in [s^{\mathrm{u}}(k), \mathbf{s}^{\mathrm{u}} + \varepsilon)$ and $\varphi_\varepsilon(t) \in V$. In particular, $\mathcal{C}_k$ is non empty.*

Proof. Because $\mathcal{C}_k$ consists of zeros for η_k and because vanishing sequences of $\mathcal{B}_k$ converge, there exists $V' \subset V$ and a positive $\delta = \delta(V')$ such that V' still contains $\mathcal{C}_k$, $\Psi_k([0,1] \times V') \subset V \cap \mathcal{B}_k$ and

$$|\eta_k^{\#}|_{\mathcal{M}_0} \geq \delta, \quad \forall (x,T) \in \mathcal{B}_k \setminus V'. \tag{5.15}$$

Fix $\varepsilon > 0$ such that $\varepsilon < \delta^2$. By definition of $\mathbf{s}^{\mathrm{u}}$, for every $\tilde{\varepsilon} \in (0,\varepsilon)$ there exists $\phi_{\tilde{\varepsilon}} \in \Gamma_{\mathrm{u}}$ such that

$$\max_{t\in[0,1]} S_k^\sigma(\phi_{\tilde{\varepsilon}})(t) < \mathbf{s}^{\mathrm{u}}(k_{\tilde{\varepsilon}}) + \tilde{\varepsilon}.$$

For $s \in [0,1]$ define

$$\varphi_\varepsilon^s(t) = \Psi_k(s, \varphi_{\tilde{\varepsilon}}(t)), \tag{5.16}$$

and observe that, by (5.12)

$$S_k^\sigma(\varphi_\varepsilon^s)(t) = S_k^\sigma(\phi_{\tilde{\varepsilon}})(t) + \Delta\eta_k\Big(\Psi_k(\cdot, \varphi_{\tilde{\varepsilon}}(t))\Big)(s). \tag{5.17}$$

As in the weakly exact case, we claim that the element $\varphi_\varepsilon = \varphi_\varepsilon^1 \in \Gamma_{\mathrm{u}}$ is the desired one. Indeed, if $t \in [0,1]$ is such that $S_k^\sigma(\phi_{\tilde{\varepsilon}})(t) < \mathbf{s}^{\mathrm{u}}(k)$, because $\Delta\eta_k$ is non positive along flow lines of Ψ_k (point (i) of Lemma 26), by (5.17) we deduce that $S_k^\sigma(\varphi_\varepsilon)(t) < \mathbf{s}^{\mathrm{u}}(k)$. On the other hand, if $S_k^\sigma(\phi_{\tilde{\varepsilon}})(t) \in [\mathbf{s}^{\mathrm{u}}(k), \mathbf{s}^{\mathrm{u}}(k)+\varepsilon)$, by the choice of T^*, by Lemma 27 and Lemma 26, it follows that, either $S_k^\sigma(\varphi_\varepsilon)(t) < \mathbf{s}^{\mathrm{u}}(k)$ or $S_k^\sigma(\varphi_\varepsilon)(t) \in [\mathbf{s}^{\mathrm{u}}(k), \mathbf{s}^{\mathrm{u}}(k)+\varepsilon)$ and $\varphi_\varepsilon^s(t) \in \mathcal{B}_k$ for every $s \in [0,1]$. Let us focus on the second case. By the assumptions on V', if for some time $s_0 \in [0,1]$ we have that $\varphi_\varepsilon^{s_0}(t) \in V'$, then $\varphi_\varepsilon(t)$ also belongs to V'. Suppose by contradiction that $\varphi_\varepsilon^s(t)$ does not enter V' for every $s \in [0,1]$. Write $\varphi_\varepsilon(s,t) = \varphi_\varepsilon^s(t)$ and observe that by (5.15), it yields

$$\Delta\eta_k(\varphi_\varepsilon(\cdot,t))(1) = \int_0^1 \Psi_k(\cdot, \phi_0(t))^*\eta_k \leq -\int_0^1 |\nabla\eta_k|^2_{\mathcal{M}_0} \leq -\delta^2. \tag{5.18}$$

From (5.17) and (5.18), we conclude that

$$S_k^\sigma(\phi)(t) \leq S_k^\sigma(\phi_0)(t) - \delta^2 \leq \mathbf{s}^{\mathrm{u}}(k) + \varepsilon - \delta^2 < \mathbf{s}^{\mathrm{u}}(k).$$

Summing up the element $\varphi_\varepsilon \in \Gamma_{\mathrm{u}}$ is such that for every $t \in [0,1]$, $S_k^\sigma(\varphi_\varepsilon)(t) < \mathbf{s}^{\mathrm{u}}(k)$ in contradiction with the definition of $\mathbf{s}^{\mathrm{u}}(k)$. Therefore, the element φ_ε is such that either $S_k^\sigma(\varphi_\varepsilon)(t) < s^{\mathrm{u}}(k)$ or $S_k^\sigma(\varphi_\varepsilon)(t) \in [s^{\mathrm{u}}(k), \mathbf{s}^{\mathrm{u}} + \varepsilon)$ and $\varphi_\varepsilon(t) \in V' \subset V$. The claim holds. □

Lemma 29. *Let $k \in J$. There exists $\gamma \in \mathcal{Z}(\eta_k)$ such that* $\mathrm{index}(\gamma) \leq 1$.

Proof. We proceed by contradiction by following the same scheme and by adopting the same setting and the same notation as in the proof of Lemma 24. By Lemma 28 and by transversality theorem [GP74, Section 5], we can find an element $\phi_0 \in \Gamma_{\mathfrak{u}}$ such that its image does not intersect $B^H_{r_{\gamma_i}} \times \{0\}$ for every i and it satisfies that for every $t \in [0,1]$, either $S^\sigma_k(\phi_0)(t) < \mathbf{s}^{\mathfrak{u}}(k)$ or $S^\sigma_k(\phi_0)(t) \in [\mathbf{s}^{\mathfrak{u}}(k), \mathbf{s}^{\mathfrak{u}}(k) + \varepsilon^2\delta)$ and $\phi_0(t) \in \bigcup_{i=1}^n \mathcal{W}_{\gamma_i}$. Define ϕ_1 by (5.9) and look at $\phi_0(t) \in \mathcal{W}_{\gamma_1}$. Let S^σ_k be a primitive of η_k defined on $\mathcal{W}_{\gamma_1}$ and $\alpha(\phi_0(t)) : [0,\varepsilon] \to \mathcal{W}_{\gamma_1}$ given by

$$\alpha(\phi_0(t))(s) = \Phi^{-1}_{\gamma_1}\Big(\rho^{s\varepsilon\chi_{\gamma_1}(\phi_0(t))}_{\gamma_1}(\Phi_{\gamma_1}(\phi_0(t)))\Big).$$

By (5.11) and by point (ii) of Lemma 4

$$\begin{aligned}\Delta\eta_k(\alpha(\phi_0(t))(1) &= S^\sigma_k(\alpha(\phi_0(t)(1)) - S^\sigma_k(\alpha(\phi_0(t)(0))\\ &\le \Big(S^\sigma_k \circ \Phi^{-1}_{\gamma_1}\Big)\Big(\rho^{\varepsilon\chi_{\gamma_1}(\phi_0(t)))}_{\gamma_1}(\Phi_{\gamma_1}(\phi_0(t))\Big) - \Big(S^\sigma_k \circ \Phi^{-1}_{\gamma_1}\Big)\Big(\Phi_{\gamma_1}(\phi_0(t))\Big)\\ &\le -\varepsilon^2 D_{\gamma_1},\end{aligned}$$

so that

$$\begin{aligned}S^\sigma_k(\phi_1)(t) &= S^\sigma_k(\phi_0)(t) + \Delta\eta_k(\alpha(\phi_0(t))(1)\\ &\le \mathbf{s}^{\mathfrak{u}}(k) + \varepsilon^2\delta - \varepsilon^2 D_{\gamma_1}\\ &< \mathbf{s}^{\mathfrak{u}}(k).\end{aligned}$$

Moreover, if $\phi_0(t) \in \mathcal{W}_{\gamma_1} \cap \mathcal{W}_{\gamma_i}$ for some $i \in \{2, ..., n\}$, due to (5.10), we can deduce that $\phi_1(t) \in \mathcal{W}_{\gamma_i,1}$. Thus the element ϕ_1 enjoys that for every $t \in [0,1]$ either $S^\sigma_k(\phi_1)(t) < \mathbf{s}^{\mathfrak{u}}(k)$ or $S^\sigma_k(\phi_1)(t) \in [\mathbf{s}^{\mathfrak{u}}(k), \mathbf{s}^{\mathfrak{u}}(k) + \varepsilon^2\delta)$ and $\phi_1(t) \in \bigcup_{i=2}^n \mathcal{W}_{\gamma_i,1}$. Iterating the procedure, at the step n we find an element $\phi_n \in \Gamma_{\mathfrak{u}}$ such that for every $t \in [0,1]$, $S^\sigma_k(\phi_n)(t) < \mathbf{s}^{\mathfrak{u}}(k)$ which contradicts the definition of $\mathbf{s}^{\mathfrak{u}}$. □

Proof of Theorem B (and Theorem A)

Let $k < c$ such that $\mathrm{Ric}^\Omega_k > 0$. By continuity of Ric^Ω_k, there exists an open interval $I_k \subset (0,c)$ centered at k and a positive constant r such that $\mathrm{Ric}^\Omega_s \ge \frac{1}{r^2}$, for all $s \in I_k$. With the help of Lemma 24 or Lemma 29, let $\{k_n\}$ be a sequence of energies such that $k_n \to k$ and let $\{\gamma_n = (x_n, T_n)\}$ be a sequence of $\mathcal{M}_0$ such that γ_n is a contractible zero of η_{k_n} with index smaller or equal than 1. By applying Lemma 18 to $\{\gamma_n\}$ we deduce that $T_n \le 2\pi r$, for every n. Moreover,

$$|\eta_k(\gamma_n)|_{\mathcal{M}} = |(\eta_k - \eta_{k_n})(\gamma_n)|_{\mathcal{M}} + |\eta_{k_n}(\gamma_n)|_{\mathcal{M}} = |k_n - k|T_n \le |k_n - k|2\pi r.$$

Thus $\{\gamma_n\}$ is a vanishing sequence for η_k and T_n is bounded from above. Because for every energy k the magnetic flow φ^k_σ does not have rest points, by [HZ12, Proposition 1, Section 4.1], up to subsequence γ_n converges in $\mathcal{M}$ to a point γ which is a contractible zero of η_k. Finally, if σ is nowhere vanishing, by point (ii) of Lemma 8, the magnetic Ricci curvature is positive for values close to zero and Theorem A follows.

6 The Hopf's rigidity: proof of Theorem E

We conclude this manuscript by giving the proof of the magnetic version of the Hopf's rigidity theorem. First we introduce or reintroduce some notation. In this final subsection (M, g, b) is a magnetic system on a closed oriented surface. We denote by X^σ the magnetic vector field on TM which is always tangent to the k-energy level

$$\Sigma_k = \{(p,v) \in TM,\ |v| = \sqrt{2k}\}.$$

Observe that the set Σ_k is the sphere bundle of radius $\sqrt{2k}$ over M. We write $P : \Sigma_k \to SM$ the map $P_k(v) = \frac{v}{|v|}$. Again, $\varphi_\sigma^k : \mathbb{R} \times \Sigma_k \to \Sigma_k$ is the restriction of the magnetic flow φ^σ to Σ_k. By μ_{Gk} we denote the Riemannian measure of Σ_k which disintegrates as the product of the Riemannian measure μ_g induced by g on M and the Lebesgue measure of the fiber circles of Σ_k. In particular, if $f : M \to \mathbb{R}$ is an integrable function for μ_g, then

$$\int_{\Sigma_k} f \mathrm{d}\mu_{Gk} = \sqrt{2k}2\pi \int_M f\mu_g.$$

It is a classical fact that the magnetic flow φ_σ^k leaves the measure μ_{Gk} invariant (see for instance [Pat06]); so that if $u : \Sigma_k \to \mathbb{R}$ is a function differentiable with respect to X^σ such that $X^\sigma u$ is integrable, then

$$\int_{\Sigma_k} \Big(X^\sigma u\Big) \mathrm{d}\mu_{Gk} = 0. \tag{6.1}$$

For the proof of Theorem E we need the following lemmas.

Lemma 30. *Let $k \in (0,+\infty)$ such that φ_σ^k is without conjugate points. Then the (magnetic) Riccati equation on Σ_k*

$$X^\sigma u + u^2 + (\mathcal{K}_k^b \circ P) = 0, \tag{6.2}$$

admits a measurable solution $u : \Sigma_k \to \mathbb{R}$

Proof. Let $p \in M$ and consider along a magnetic geodesic $\gamma_v(t) = \varphi_\sigma^k(t,p,v)$ the orthogonal component with respect to $\dot\gamma_v$ of the magnetic Jacobi equation, which by Lemma 11, is given by

$$\ddot q + \mathcal{K}_k^b(P(\dot\gamma_{(p,v)}(t))q = 0. \tag{6.3}$$

The absence of conjugate points for φ_σ^k is equivalent to the fact that any non trivial solution of (6.3) vanishes at most at one point; more generally, for all times $t_1 < t_2$ and for all reals c_1, c_2, there exists a unique solution a of (6.3) such that $a(t_1) = c_1$ and $a(t_2) = c_2$. Let $s \neq 0$ and denote by a_s^v the solution of (6.3) such that $a_s^v(0) = 1$ and $a_s^v(s) = 0$. Observe that the absence of conjugate points implies that for every $s \neq \tilde s$ we have that $\dot a_s^v(0) \neq \dot a_{\tilde s}^v(0)$ as well as $a_s^v(t) \neq a_{\tilde s}^v(t)$. From these considerations it follows that

$$\dot a_{s_2}^v(0) < \dot a_{s_3}^v(0) < \dot a_{s_1}^v(0), \quad \forall s_1 < 0 < s_2 < s_3. \tag{6.4}$$

Indeed, if $\dot a_{s_2}^v(0) > \dot a_{s_3}^v(0)$ for $s_3 > s_2 > 0$, because $a_{s_3}^v$ vanishes at the point s_3, we would find a time $\tau \in (0, s_2]$ such that $a_{s_3}^v(\tau) = a_{s_2}^v(\tau)$. Analogously, if $\dot a_{s_3}^v(0) > \dot a_{s_1}^v(0)$ for some $s_3 > 0 > s_1$, we would find a time $\tau \in [s_1, 0)$ such that $a_{s_1}^v(\tau) = a_{s_3}^v(\tau)$. In particular, the inequality (6.4) together with the absence of conjugate points implies also

$$a_{s_2}^v(t) < a_{s_3}^v(t) < a_{s_1}^v(t),\ \forall s_1 < 0 < s_2 < s_3, \quad \forall t \in (0,+\infty), \tag{6.5}$$

and equivalently

$$-a_{s_2}^v(t) < -a_{s_3}^v(t) < -a_{s_1}^v(t),\ \forall s_1 < 0 < s_2 < s_3, \quad \forall t \in (-\infty, 0). \tag{6.6}$$

By (6.4), $\dot{a}_s^v$ is monotone increasing and bounded from above which implies that the limit

$$u(v) := \lim_{s\to+\infty} \dot{a}_s^v(0), \tag{6.7}$$

exists and it is finite. The function u on Σ_k given by (6.7) is measurable, being the pointwise limit of the function $\dot{a}_s^v(0)$ depending smoothly on $(p,v) \in \Sigma_k$. Let a^v be the solution of (6.3) such that $a^v(0) = 1$ and $\dot{a}^v(0) = u(v)$. By (6.4), we can deduce that

$$a^v(t) = \lim_{s\to+\infty} a_s^v(t). \tag{6.8}$$

Moreover, with the help of (6.5) and (6.6), we also deduce that a^v is everywhere strictly positive on $\mathbb{R}$. Therefore, we define the function $z^v : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ as

$$z^v(t) = \frac{\dot{a}^v(t)}{a^v(t)}. \tag{6.9}$$

By differentiating (6.9) along γ_v, using the fact that a^v is a solution of (6.3), we have that

$$\begin{aligned}\dot{z}^v &= \frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{\dot{a}^v(t)}{a^v(t)}\Big] \\ &= \frac{\ddot{a}^v(t)}{a^v(t)} - \Big(\frac{\dot{a}^v(t)}{a^v(t)}\Big)^2 \\ &= -\mathcal{K}_k^b(P(\dot{\gamma}_v)) - (z^v)^2. \end{aligned} \tag{6.10}$$

Thus, z^v satisfies the Riccati equation (6.2) along γ_v. Now denote by $v(t) = \varphi_\sigma^k(t,p,v)$ and observe that, by the uniqueness of the solution of (6.3) with prescribed boundary condition, we have that

$$a_s^{v(t)}(\tau) = \frac{a_{s+t}^v(\tau+t)}{a_{s+t}^v(t)},$$

which implies

$$a^{v(t)}(\tau) = \frac{a^v(\tau+t)}{a^v(t)}. \tag{6.11}$$

By differentiation of (6.11) at $\tau = 0$, we obtain

$$\dot{a}^{v(t)}(0) = \frac{\dot{a}^v(t)}{a^v(t)}. \tag{6.12}$$

By using (6.12), we can deduce that

$$z^{v(t)}(0) = \frac{\dot{a}^{v(t)}(0)}{a^{v(t)}(0)} = \dot{a}^{v(t)}(0) = \frac{\dot{a}^v(t)}{a^v(t)} = z^v(t). \tag{6.13}$$

Finally, notice that $u(v) = z^v(0)$ and, with the help of (6.10) and (6.13), we can conclude that

$$X^\sigma u(v) = \Big(\frac{D}{dt} z^{v(t)}\Big)(0) = -\Big(z^v(0)\Big)^2 - \mathcal{K}_k^b(P(\dot{\gamma}_v(0))) = -u(v)^2 - \mathcal{K}_k^b(P(v)).$$

Thus u is a measurable solution of (6.2). □

Lemma 31. *Let u be the measurable solution given by Lemma 30, then u satisfies that*

$$\sup|u| \leq \sqrt{-\max\{0, \min\{\mathcal{K}_k^b \circ P\}}$$

Proof. We prove the lemma by using a variation of the classical Sturm's argument (see for instance [dC15]). Let $r > 0$ such that

$$\min \mathcal{K}_k^b \geq -r^2 \tag{6.14}$$

Consider the differential problem along γ_v given by

$$\ddot{x} - r^2 x = 0. \tag{6.15}$$

For $s \neq 0$, the function x given by

$$x_s(t) = \frac{e^{r(s-t)} + e^{r(t-s)}}{e^{rs} - e^{-rs}},$$

is the unique solution of (6.15) with boundary condition $x_s(0) = 1$ and $x_s(s) = 0$. Observe that

$$\dot{x}_{s(0)} = -r\frac{e^{rs} + e^{-rs}}{e^{rs} - e^{-rs}},$$

so that $\lim_{s\to+\infty} \dot{x}_{s(0)} = -r$ and $\lim_{s\to-\infty} \dot{x}_{s(0)} = +r$. Let a^v be as defined by (6.8) and consider the function

$$h_s = x_s\dot{a}^v - \dot{x}_s a^v.$$

In particular, it satisfies $h_s(s) = -\dot{x}_s(s)\dot{a}^v(s)$ and because a^v is strictly positive we also get that

$$\dot{h}_s(t) = x_s(t)\ddot{a}^v(t) - \ddot{x}_s a^v(s) = -(\mathcal{K}_k^b(P(\dot{\gamma}_v)) + r^2)x_s(t)a^v(t) < 0, \ \forall |t| < |s|. \tag{6.16}$$

If $s > 0$, then $\dot{x}_s(s) < 0$ and $h_s(s) > 0$ which, by (6.16), implies that $h_s(0) > 0$. Therefore, we get the lower bound

$$u(v) = \dot{a}^v(0) > \dot{x}_s(0) \to -r \quad \text{for } s \to -\infty.$$

In the same manner, if $s < 0$, then $\dot{x}_s(s) > 0$ and $h_s(s) < 0$ which, by (6.16), implies that $h_s(0) < 0$ and

$$u(v) = \dot{a}^v(0) < \dot{x}_s(0) \to r \quad \text{for } s \to +\infty.$$

□

Proof of Theorem E

By Lemma 30 and Lemma 31, there exists a measurable and bounded function $u : \Sigma_k \to \mathbb{R}$ solution of the Riccati equation (6.2). Then u^2 and $X^\sigma u$ are integrable functions on Σ_k and we get

$$\int_{\Sigma_k} \left[X^\sigma u + u^2 + (\mathcal{K}_k^b \circ P)\right] \mathrm{d}\mu_{Gk} = 0. \tag{6.17}$$

By (6.1) and by the fact that $\mathrm{d}\mu_{Gk} = \sqrt{2k}\mathrm{d}\mu_{G\frac{1}{2}}$, we get that

$$\sqrt{2k}\int_{SM} \mathcal{K}_k^b \mathrm{d}\mu_{G\frac{1}{2}} = \int_{\Sigma_k} (\mathcal{K}_k^b \circ P)\mathrm{d}\mu_{Gk} = -\int_{\Sigma_k} u^2 \mathrm{d}\mu_{Gk} \leq 0. \tag{6.18}$$

The equality in (6.18) holds if and only if $u = 0$ which implies that $\mathcal{K}_k^b = 0$. By looking at the definition (2.9) of $\mathcal{K}_k^b$, if follows immediately that if $\mathcal{K}_k^b = 0$, then $\mathrm{d}b = 0$, i.e. the function b is constant. By the Gauss-Bonnet Theorem, either (M, g) is the flat torus and $b = 0$ or (M, g) is a closed surface of genus $h > 1$ with $\mathcal{K} = -\frac{b^2}{2k}$. By [Pat06], if (M, g, b) is a magnetic system on a closed oriented surface of genus bigger than one such that $\mathcal{K}$ and b are constants, then the Mañé critical value $c(g, b) = \frac{b^2}{2\mathcal{K}}$ and $\varphi^{\sigma}_{c(g,b)}$ is a horocycle flow.

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