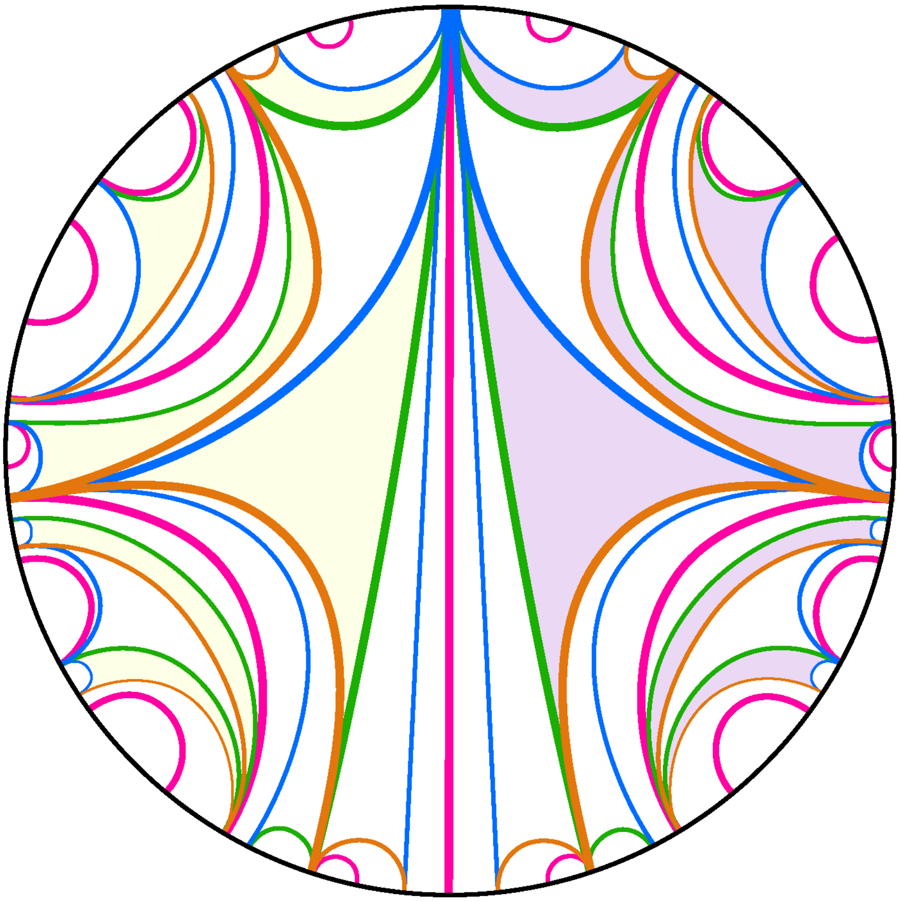
Cataclysm deformations for Anosov representations



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Cataclysm deformations for Anosov representations

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Abstract

In this thesis, we construct special deformations for Anosov representations, so-called *cataclysms*, and investigate their properties. In Teichmüller theory, cataclysms and the closely related shearing coordinates carry information about the structure of Teichmüller space. Therefore, the question arises if cataclysms also exist in higher Teichmüller spaces or more generally for Anosov representations. We construct cataclysms for θ-Anosov representations into a semisimple non-compact connected real Lie group G, where $\theta \subset \Delta$ is a subset of the simple roots that is invariant under the opposition involution.

Important steps in our construction of cataclysm deformations are the definition of the appropriate parameter space as well as the definition of slithering maps in the context of θ-Anosov representations. These maps generalize slithering maps for Hitchin representations which were defined by Bonahon and Dreyer in their parametrization of the Hitchin component. We then construct stretching maps, shearing maps and finally cataclysms. Cataclysms have some natural properties: They are additive and behave well with respect to composing an Anosov representation with a Lie group homomorphism. Moreover, we show how the cataclysm deformation of an Anosov representation affects the associated boundary map.

In Teichmüller space, cataclysm deformations are injective. However, this does not hold true for θ-Anosov representations in general. We give sufficient conditions for injectivity as well as for non-injectivity of the deformation. For certain classes of reducible representations, we explicitly determine the subspace of the parameter space on which the deformation is trivial. These representations include a family of Borel Anosov representations into $\mathrm{SL}(2n+1,\mathbb{R})$, by which we show that cataclysms of Borel Anosov representations are not necessarily injective.

Zusammenfassung

Diese Dissertation behandelt die Konstruktion spezieller Deformationen von Anosov-Darstellungen, sogenannter *Kataklysmen*, und untersucht ihre Eigenschaften. Kataklysmen und die eng verwandten Scherkoordinaten spielen eine wichtige Rolle bei der Untersuchung von Teichmüllerräumen. Daher stellt sich die Frage, ob Kataklysmen auch in höheren Teichmüllerräumen oder allgemeiner für Anosov-Darstellungen existieren. Wir konstruieren Kataklysmen für θ-Anosov-Darstellungen in eine halbeinfache nicht-kompakte zusammenhängende reelle Liegruppe G. Hierbei ist θ eine Teilmenge der einfachen Wurzeln Δ, die invariant unter der Oppositionsinvolution ist.

Ein wichtiger Schritt in unserer Konstruktion von Kataklysmen ist die Definition des passenden Parameterraumes. Desweiteren definieren wir Gleitabbildungen für θ-Anosov-Darstellungen. Diese verallgemeinern entsprechende Abbildungen für Hitchin-Darstellungen in $\mathrm{PSL}(n, \mathbb{R})$, die von Bonahon und Dreyer im Rahmen der Parametrisierung der Hitchin-Komponente definiert wurden. Wir definieren außerdem Scherabbildungen und schließlich Kataklysmen. Im Anschluss zeigen wir einige Eigenschaften von Kataklysmen: Sie sind additiv und verhalten sich natürlich unter der Verknüpfung von Anosov-Darstellungen mit Liegruppenhomomorphismen. Außerdem beschreiben wir, wie ein Kataklysmus die zu einer Anosov-Darstellung gehörige Randabbildung verändert.

Kataklysmen im Teichmüllerraum sind injektiv. Für allgemeine θ-Anosov-Darstellungen gilt dies nicht. Wir geben eine hinreichende Bedingung dafür, dass ein Kataklysmus injektiv ist, sowie hinreichende Bedingungen für das Gegenteil. Für bestimmte Klassen von reduziblen Darstellungen bestimmen wir explizit den Unterraum des Parameterraums, für den der Kataklysmus trivial ist. Diese Darstellungen beinhalten eine Familie von Borel-Anosov-Darstellungen in $\mathrm{SL}(2n + 1, \mathbb{R})$, womit wir zeigen, dass Kataklysmen für Borel-Anosov-Darstellungen im Allgemeinen nicht injektiv sind.

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Contents

1. Introduction

Surfaces are ubiquitous not only in mathematics, but also in other areas, like physics and engineering. The most common examples from everyday life are the boundaries of three-dimensional objects, like the bounding surface of a ball. In mathematics, surfaces are studied from various points of view. Topologically, surfaces are classified by their number of holes. Mathematicians call this the *genus* of a surface. The sphere, i.e. the bounding surface of a ball, has genus 0. The torus, i.e. the bounding surface of a doughnut, has genus 1. A surface of genus 2 has two holes and can be imagined as a blown up eight. A surface of genus 3 resembles the bounding surface of a pretzel, and similarly, one can picture surfaces of higher genus.

Though all surfaces with equal genus are the same from a topological point of view, this is not true anymore if we take a geometric viewpoint. In geometry, we look at metrics on surfaces. This allows us to measure distances and lengths of curves. A surface of fixed genus can admit different metrics. If we restrict to metrics of constant curvature, by the Gauss-Bonnet theorem, all surfaces with genus at least 2 can only admit a metric of negative curvature. If we normalize the curvature to -1, such a metric is called *hyperbolic*. Let us fix a closed connected oriented surface S of genus $g(S) \geq 2$. *Teichmüller space*, denoted by $\mathcal{T}(S)$, is the space of hyperbolic metrics on S up to a suitable equivalence relation. It can be parametrized using *Fenchel-Nielsen coordinates*, which are obtained from decomposing S into pairs of pants. The coordinates for a metric m on S are then given by measuring the lengths of the boundary curves of the pants with respect to the metric m as well as capturing how the pants are combined to give the surface S. The Fenchel-Nielsen coordinates give a homeomorphism between $\mathcal{T}(S)$ and $\mathbb{R}^{6g(S)-6}$.

Teichmüller space has a rich structure. For example, it admits different metrics, one of which is the *Weil-Petersson metric*. With respect to this metric, $\mathcal{T}(S)$ is a Kähler manifold, i.e. a Riemannian manifold that also has a complex structure and a symplectic structure and these structures are compatible [Ahl61]. Another example for an object of interest on Teichmüller space are *length functions*: Given a simple closed curve c on S and a hyperbolic structure $m \in \mathcal{T}(S)$, we can consider the length of c with respect to m. Thurston used length functions to compactify Teichmüller space (see [FLP12, Exposé 8]).

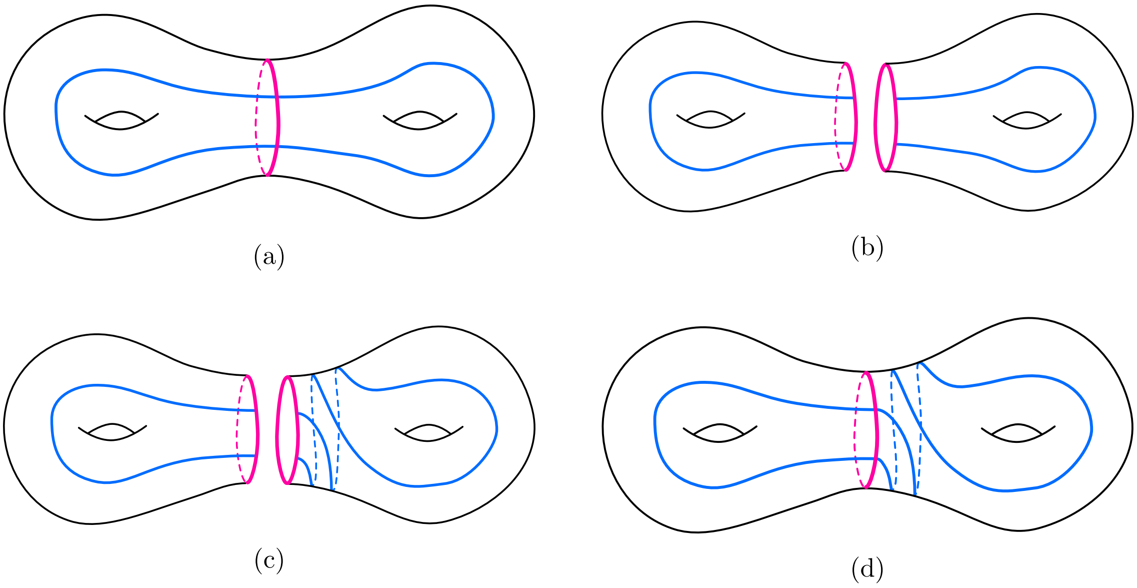


Figure 1.1.: A basic example for an earthquake is a twist along a simple closed geodesic drawn in pink in (a). If we cut the surface along this curve, we obtain two surfaces with boundary (b). We deform the surface on the right by rotating its boundary curve (c) and re-glue the two surfaces back together (d). The blue curve illustrates the deformation. In this picture, the rotation is by 2π.

Apart from Fenchel-Nielsen coordinates, there is another set of coordinates on $\mathcal{T}(S)$ called *shearing coordinates*. They have been first defined by Thurston in [Thu98] and are described in detail by Bonahon in [Bon96]. Shearing coordinates are closely related to deformations of hyperbolic structures. An example for such a deformation is an *earthquake* [Thu86]. In its simplest form, an earthquake is a twist along a simple closed geodesic (see Figure 1.1): Cut the surface along the geodesic, fix one side and rotate the other one to the left by a fixed distance. Since we assume the surface to be oriented, we can talk about *left* in this context. Gluing the two sides together again, we regain the topological surface S, but the metric on it is different. This deformation is called a *left earthquake*. In the same way, we can define a *right earthquake*.

For general earthquakes, the twisting happens not along a simple closed geodesic, but along a *geodesic lamination* λ. A geodesic lamination is a possibly infinite collection of simple disjoint geodesics such that their union is closed. It is called *maximal* if the complement consists of ideal triangles. The amount of twisting for the earthquake is determined by a *transverse measure* for λ, which assigns a non-negative real number to any arc transverse to the lamination λ. This assignment is countably additive and invariant under homotopy relative to the lamination.

If we allow twisting both to the left and to the right, the resulting deformation is called a *cataclysm*. In this case, the amount of twisting is determined by a *transverse cycle*, which, in contrast to a transverse measure, can also attain negative values and is only finitely additive. Positive values correspond to twisting to the left, whereas negative values give a twist to the right. We denote the vector space of transverse cycles for a lamination λ with $\mathcal{H}(\lambda;\mathbb{R})$. Fixing a maximal lamination λ on S, we can assign to any hyperbolic metric m a transverse cycle ε_m and this assignment gives a real analytic homeomorphism from $\mathcal{T}(S)$ to an open convex cone in $\mathcal{H}(\lambda;\mathbb{R})$ [Bon96, Theorem A]. The coordinates on $\mathcal{T}(S)$ obtained in this way are the *shearing coordinates*. Now for any two metrics $m_1, m_2 \in \mathcal{T}(S)$, there is a unique cataclysm along λ that sends m_1 to m_2 and the transverse cycle parametrizing this cataclysm is $\varepsilon_{m_2} - \varepsilon_{m_1}$. This shows the close relation between cataclysms and shearing coordinates.

Deformations of hyperbolic structures and shearing coordinates give us insights in the structure of Teichmüller space: For instance, length functions on $\mathcal{T}(S)$ can be computed in terms of the shearing coordinates [Bon96, Theorem E] and we can study how they behave under deformations. Douady showed that length functions are convex under twists along simple closed curves [FLP12, Exposé 7]. Convexity of length functions along earthquake paths, along Weil-Petersson geodesics and with respect to shearing coordiantes was shown in [Ker83], [Wol87] and [BBFS13], respectively. There is also a relation between deformations of hyperbolic structures and the symplectic structure of Teichmüller space. For example, twist flows along simple closed curves are Hamiltonian flows with respect to its Kähler form [Wol83]. Further, the Weil-Petersson metric can be expressed in terms of the shearing coordinates [SB01] and agrees with Thurston's intersection form on the space of transverse cycles.

Higher Teichmüller spaces and Anosov representations

To a hyperbolic structure on S, one can associate a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$, where $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$ is the group of orientation-preserving isometries of the upper half-plane. This representation is only unique up to conjugation in $\mathrm{PSL}(2,\mathbb{R})$, so an element in $\mathcal{T}(S)$ defines a point in the *character variety* $\chi(S, \mathrm{PSL}(2,\mathbb{R}))$, which is the space of homomorphisms from the fundamental group $\pi_1(S)$ to $\mathrm{PSL}(2,\mathbb{R})$ up to conjugation. In fact, $\mathcal{T}(S)$ forms a connected component of $\chi(S, \mathrm{PSL}(2,\mathbb{R}))$ consisting entirely of discrete and faithful representations.

The viewpoint that Teichmüller space can be understood in terms of homomorphisms into $\mathrm{PSL}(2,\mathbb{R})$ motivates the definition of *higher Teichmüller spaces*, for which we replace the group $\mathrm{PSL}(2,\mathbb{R})$ by a general semisimple Lie group G of higher rank. Higher Teichmüller spaces are connected components of the character variety $\chi(S,G)$ consisting entirely of discrete and faithful representations (see [Wie18] or [Poz20]). One example for this is the Hitchin component. It is the connected component of $\chi(S, \mathrm{PSL}(n, \mathbb{R}))$ that contains representations obtained by composing a discrete and faithful representation into $\mathrm{PSL}(2, \mathbb{R})$ with the unique irreducible representation $\mathrm{PSL}(2, \mathbb{R}) \hookrightarrow \mathrm{PSL}(n, \mathbb{R})$. A *Hitchin representation* is a representation in the Hitchin component. The Hitchin component was first studied by Hitchin using Higgs bundle techniques in [Hit92], and since then, has been investigated extensively. For $n = 3$, it parametrizes real projective structures on S [CG93].

Hitchin representations and all other known higher Teichmüller spaces are examples of a larger class of discrete and faithful representations: *Anosov representations*. They have first been introduced by Labourie in [Lab06] and his definition was extended by Guichard and Wienhard in [GW12]. The original definition is dynamic in nature and involves the geodesic flow on a bundle associated to a representation ρ. In recent years, many equivalent characterizations of Anosov representations have been found and the field is an active area of research (see [GGKW17], [KLP17], [DGK17], [BPS19], [Zhu19], [Tso20], [KP20], [Zhu21], [BCKM21],[CZZ21]). Important examples of Anosov representations are quasi-Fuchsian representations, Hitchin representations, maximal representations [BILW05], θ-positive Anosov representations [GW18] and $(1, 1, 2)$-hyperconvex representations [PSW21].

For a semisimple Lie group G, there are different types of Anosov representations that are related to *parabolic subgroups* P_θ^+ of G, where $\theta \subset \Delta$ is a subset of the simple restricted roots. An Anosov representation is always Anosov *with respect to* a parabolic subgroup P_θ^+ of G. We call such a representation *θ-Anosov*. When we just say that a representation is Anosov, then we mean that it is θ-Anosov with respect to some $\theta \subset \Delta$. In this thesis, we use a definition of Anosov representations through equivariant boundary maps and θ-divergence from [GGKW17].

One approach to understanding higher Teichmüller spaces is to use techniques that have proven helpful in the case of Teichmüller space. An example for this is the symplectic structure on the character variety $\chi(S, G)$ for a reductive Lie group G. Goldman showed that the character variety carries a natural symplectic structure [Gol84] and studied the behavior of the Hamiltonian flows associated to length functions [Gol86]. Since cataclysm deformations are closely related to the structure of Teichmüller space, they might also be a tool to understand higher Teichmüller spaces. First steps in this direction were taken by Dreyer in [Dre13]. He generalizes Thurston's cataclysm deformations to representations into $\mathrm{PSL}(n, \mathbb{R})$ which are Anosov with respect to the minimal parabolic subgroup. One example for such representations are Hitchin representations. Based on work of Fock and Goncharov [FG06], Bonahon and Dreyer define coordinates on the Hitchin component that generalize the shearing coordinates [BD17]. These coordinates depend on a maximal lamination λ and consist of two types of parameters: the triangle invariants, which are associated to the ideal triangles in the complement of the lamination, and the shearing invariants, which are a generalized version of transverse cycles. Cataclysms provide a geometric realization of deformations of Hitchin representations which keep the triangle invariants fixed.

However, the construction of cataclysms by Dreyer is restricted to the group $\mathrm{PSL}(n, \mathbb{R})$ and to representations that are Anosov with respect to the minimal parabolic. Assuming a representation to be Anosov with respect to the minimal parabolic is a strong assumption, which is not satisfied by most Anosov representations (see [CT20]).

In this thesis, we generalize Dreyer's construction and define cataclysm deformations for θ-Anosov representations into a semisimple connected non-compact real Lie group G that are Anosov with respect to a parabolic P_θ^+, where $\theta \subset \Delta$ is a subset invariant under the opposition involution.

The parameter space for cataclysms: transverse twisted cycles

Let $\lambda \subset S$ be a geodesic lamination. It is in general not orientable. This is why we often look at a the *orientation cover* $\widehat{\lambda}$ which is a two-fold cover of λ and can be oriented continuously. Further, we also look at the *universal cover* $\tilde{\lambda} \subset \tilde{S}$, which is the lift of λ to the universal cover $\tilde{S}$ of S. The parameter space for the cataclysm deformation along the lamination λ is the space of $\mathfrak{a}_\theta$*-valued transverse twisted cycles* for the orientation cover $\widehat{\lambda}$ (see Section 3.2). Here, $\mathfrak{a}_\theta$ denotes the Lie algebra of the center of the reductive group $L_\theta = P_\theta^+ \cap P_\theta^-$, where P_θ^+ is the standard parabolic subgroup associated with θ, and P_θ^- is the standard parabolic subgroup transverse to P_θ^+ (see Section 2.1). For the case of $\mathrm{SL}(n, \mathbb{R})$ and $\theta = \Delta$, P_Δ^+ and P_Δ^- consist of all upper and lower triangular matrices in $\mathrm{SL}(n, \mathbb{R})$, respectively. Consequently, L_Δ is the set of diagonal matrices of determinant 1, and $\mathfrak{a}_\Delta$ is the set of diagonal matrices of trace 0.

We only consider transverse cycles that satisfy a certain twist condition (see Definition 3.20):

Definition 1.1 (Definition 3.15 and 3.20)**.** An $\mathfrak{a}_\theta$*-valued transverse twisted cycle* ε assigns to every pair (P, Q) of connected components $P, Q \subset \tilde{S} \setminus \tilde{\lambda}$ an element $\varepsilon(P, Q) \in \mathfrak{a}_\theta$ in a way that is

- $\pi_1(S)$-invariant,
- finitely additive, i.e. $\varepsilon(P, Q) = \varepsilon(P, R) + \varepsilon(R, Q)$ if $R \subset \tilde{S} \setminus \tilde{\lambda}$ lies between P and Q and
- twisted, i.e. $\varepsilon(Q, P) = \iota(\varepsilon(P, Q))$,

where $\iota\colon \mathfrak{a} \to \mathfrak{a}$ is the opposition involution. The space of transverse twisted $\mathfrak{a}_\theta$-valued cycles is denoted by $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$.

Note that in the main text, Definition 3.15 of transverse cycles (not necessarily twisted) is stated slightly differently in terms of arcs transverse to the lamination λ, in order to have a unifying definition of transverse cycles for λ and $\widehat{\lambda}$.

We compute the dimension of the space of transverse twisted cycles, depending on the lamination λ. If λ is maximal, this formula is particularly simple:

Proposition 1.2 (Corollary 3.22)**.** *For a maximal lamination λ on S, the vector space $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ of $\mathfrak{a}_\theta$-valued transverse twisted cycles has dimension $|\theta|(6g(S)-6)+|\theta'|$, where $\theta' \subset \theta$ is a maximal subset satisfying $\theta' \cap \iota(\theta') = \emptyset$ and $g(S)$ is the genus of the surface S.*

Here, $\iota\colon \Delta \to \Delta$ is the opposition involution on the simple roots. Note that we abuse notation and denote by ι both the map on $\mathfrak{a}$ and the induced map on the simple roots. Applying the proposition to $G = \mathrm{SL}(n, \mathbb{R})$ and $\theta = \Delta$, we obtain that the dimension of $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ is $(n-1)(6g(S)-6) + \lfloor \frac{n-1}{2} \rfloor$, recovering a result from Dreyer [Dre13, Lemma 16]. If the cardinality of θ is 2, then the dimension is $12g(S) - 11$.

We determine the dimension of $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ more generally for a geodesic lamination λ that is not necessarily maximal (Proposition 3.21). In general, the dimension is smaller than in the case of a maximal lamination and depends on the number of orientable and non-orientable components of the lamination and on its Euler characteristic.

Slithering, stretching and shearing

Recall that our goal in this thesis is to construct cataclysm deformations for Anosov representations. Fix an Anosov representation $\rho\colon \pi_1(S) \to G$, a lamination λ on S and let $\tilde{S}$ be the universal cover of S. In the course of the construction, we will define three families of elements in G:

- slithering maps Σ_{gh} associated with two geodesics g and h in $\tilde{S}$,
- stretching maps T_g^H associated with one oriented geodesic g in $\tilde{S}$ and an element $H \in \mathfrak{a}_\theta$ and
- shearing maps $\varphi_{PQ}^{\varepsilon}$ associated with two connected components $P, Q \subset \tilde{S} \setminus \tilde{\lambda}$ and an $\mathfrak{a}_\theta$-valued transverse twisted cycle ε.

Abusing notation, we call these group elements *maps*. Note that all of these maps, i.e. Σ_{gh}, T_g^H and $\varphi_{PQ}^{\varepsilon}$, depend on the representation ρ which is not reflected in the notation.

Slithering maps were introduced by Bonahon and Dreyer in [BD17] for Hitchin representations into $\mathrm{PSL}(n,\mathbb{R})$. Using similar techniques, we extend their definition to our more general setting of Anosov representations (see Proposition 4.9). Given a θ-Anosov representation, one can associate to every oriented geodesic g in $\tilde{S}$ a pair (P_g^+, P_g^-) of transverse parabolic subgroups that are conjugate to the pair of standard parabolics (P_θ^+, P_θ^-). The slithering map $\Sigma_{gh} \in G$ maps the pair of parabolics associated to h to the pair of parabolics associated to g. If the two geodesics share an endpoint $g^+ = h^+$, then $P_g^+ = P_h^+$ and Σ_{gh} is the unique unipotent element in P_g^+ that maps P_h^- to P_g^-. In the general case, Σ_{gh} is obtained from concatenating such unipotent elements and taking a limit. We use the slithering map in this thesis as a tool to investigate the stretching maps. It might be useful in other contexts as well. In particular, Bonahon and Dreyer use it in [BD17] to define a parametrization of the Hitchin component.

The second family of elements, the *stretching maps*, form the basic building blocks of the cataclysm deformation. For earthquake maps, basic building blocks are hyperbolic isometries that translate along an axis in $\tilde{S}$. In a similar vein, the basic building blocks for cataclysms are elements $T_g^H \in G$, where g is an oriented geodesic in $\tilde{S}$ and H is an element in $\mathfrak{a}_\theta$ (see Section 4.2). For the case $G = \mathrm{SL}(n,\mathbb{R})$ and $\theta = \Delta$, we can associate to any oriented geodesic g a splitting of $\mathbb{R}^n$ into lines by using the representation ρ. The stretching map T_g^H is the linear transformation that acts on the lines of this splitting as a stretch, where the amount of the stretch is given by the parameter H (see Example 4.15). This motivates the name *stretching map*.

The third family of elements are the *shearing maps*. Those are crucial in the definition of cataclysms: Starting with the representation ρ, we will define a new representation that for every $\gamma \in \pi_1(S)$ is given by left-multiplying $\rho(\gamma)$ with the shearing map $\varphi_{P(\gamma P)}^{\varepsilon}$. The shearing maps are defined as follows: Fix two connected components P and Q in $\tilde{S} \setminus \tilde{\lambda}$ and an $\mathfrak{a}_\theta$-valued transverse twisted cycle $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$. If P and Q are separated by finitely many geodesics $g_1, \ldots, g_m$ in $\tilde{\lambda}$, labeled from P to Q, and if R_i is the connected component of $\tilde{S} \setminus \tilde{\lambda}$ between g_i and g_{i+1} (see Figure 1.2), then $\varphi_{PQ}^{\varepsilon}$ is the element in G obtained as the concatenation of stretching maps

$$\varphi_{PQ}^{\varepsilon} = T_{g_1}^{\varepsilon(P,R_1)} T_{g_2}^{\varepsilon(R_1,R_2)} \cdots T_{g_m}^{\varepsilon(R_{m-1},Q)}. \tag{1.1}$$

In the general case, P and Q are separated by infinitely many geodesics, and the shearing map $\varphi_{PQ}^{\varepsilon}$ is defined using a limit (Proposition 5.3). The shearing maps satisfy some natural properties, namely they behave well under taking the inverse, i.e. ${\varphi_{PQ}^{\varepsilon}}^{-1} = \varphi_{QP}^{\varepsilon}$, they are

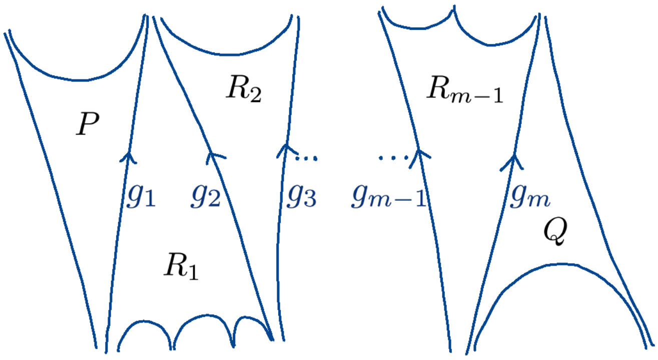


Figure 1.2.: For two connected components P and Q of $\tilde{S} \setminus \tilde{\lambda}$ that are separated by finitely many geodesics $g_1, \dots, g_m$, we label the components between P and Q by $R_1, \dots, R_m$. In this case, the shearing map is a finite concatenation of stretching maps defined in (1.1).

ρ-equivariant, i.e. $\varphi^{\varepsilon}_{(\gamma P)(\gamma Q)} = \rho(\gamma)\varphi^{\varepsilon}_{PQ}\rho(\gamma)^{-1}$ for every $\gamma \in \pi_1(S)$ and they behave well under composition in the sense that for three connected components P, R, Q in $\tilde{S} \setminus \tilde{\lambda}$, it holds that $\varphi^{\varepsilon}_{PQ} = \varphi^{\varepsilon}_{PR}\varphi^{\varepsilon}_{RQ}$.

The cataclysm deformation and its properties

We can now formulate the main result of this thesis:

Theorem 1.3 (Theorem 5.12)**.** *Let ρ be a θ-Anosov representation. Fix a reference component $P_0 \subset \tilde{S} \setminus \tilde{\lambda}$. There exists a neighborhood $\mathcal{V}_\rho$ of 0 in $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ and a continuous map*

$$\begin{aligned} \Lambda_0 \colon \mathcal{V}_\rho &\to \mathrm{Hom}(\pi_1(S), G) \\ \varepsilon &\mapsto \Lambda_0^{\varepsilon}\rho \end{aligned}$$

such that $\Lambda_0^0\rho = \rho$. Up to possibly shrinking the neighborhood $\mathcal{V}_\rho$, the representation $\Lambda_0^{\varepsilon}\rho$ is again θ-Anosov.

The map Λ_0 is called *cataclysm deformation based at ρ along λ*. Note that we mention the lamination λ as well as the Anosov representation ρ we start with, because the construction depends on both.

For a fixed $\mathfrak{a}_\theta$-valued transverse twisted cycle $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$, the deformed representation $\Lambda_0^{\varepsilon}\rho$ is defined by

$$\Lambda_0^{\varepsilon}\rho(\gamma) := \varphi_{P_0(\gamma P_0)}\rho(\gamma) \quad \forall \gamma \in \pi_1(S).$$

From the equivariance and composition properties of the shearing maps, it follows that $\Lambda_0^\varepsilon \rho$ is indeed a homomorphism.

Remark 1.4. For the special case of Δ-Anosov representations into $\mathrm{PSL}(n,\mathbb{R})$, Theorem 1.3 was proven by Dreyer in [Dre13]. We extend their definition of cataclysms to θ-Anosov representations into any semisimple connected non-compact real Lie group G and $\theta \subset \Delta$ with $\iota(\theta) = \theta$, where ι is the opposition involution. The difference to their construction lies in the definition of the parameter space and in the definition of the stretching maps T_g^H. Further, we consider an arbitrary geodesic lamination λ, whereas Dreyer's result is for maximal laminations only. The way in which the shearing maps are constructed as a limit of stretching maps in this thesis is analogous to [Dre13].

Note that $\Lambda_0^\varepsilon \rho$ depends on the choice of the reference triangle P_0, but for different choices, the resulting representations are conjugate. Moreover, if we start with two representations ρ, ρ' that are conjugate in G, the deformed representations $\Lambda_0^\varepsilon \rho$ and $\Lambda_0^\varepsilon \rho'$ are conjugate as well. We thus have the following result:

Theorem 1.5 (Corollary 5.13). *The cataclysm deformation $\Lambda_0 \colon \mathcal{V}_\rho \to \mathrm{Hom}(\pi_1(S), G)$ from Theorem 1.3 descends to a well-defined map $\Lambda \colon \mathcal{V}_\rho \to \chi^{\theta-\mathrm{Anosov}}(S, G)$ on the character variety, where $\chi^{\theta-\mathrm{Anosov}}(S, G)$ denotes the subspace of the character variety consisting entirely of conjugacy classes of θ-Anosov representations.*

The cataclysm deformation Λ_0 on the representation variety has some natural properties: First, it is additive in the sense that $\Lambda_0^{\varepsilon+\eta}\rho = \Lambda_0^\varepsilon (\Lambda_0^\eta \rho)$ for two transverse twisted cycles $\varepsilon, \eta \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ (Corollary 5.17). Further, it behaves well under composition of an Anosov representation with a Lie group homomorphism: Let G and G' be semisimple non-compact connected real Lie groups, let $\rho \colon \pi_1(S) \to G$ be a θ-Anosov representation and let $\kappa \colon G \to G'$ be a Lie group homomorphism. It is a result from Guichard and Wienhard that under an additional assumption on $\theta' \subset \Delta'$ and on the induced map $\kappa_* \colon \mathfrak{a} \to \mathfrak{a}'$, then $\kappa \circ \rho$ is θ'-Anosov (Proposition 5.18). Assuming further that $\kappa_*(\mathfrak{a}_\theta) \subset \mathfrak{a}'_{\theta'}$, we prove that

$$\Lambda^{\kappa_* \varepsilon}(\kappa \circ \rho) = \kappa \left(\Lambda^\varepsilon \rho\right),$$

where $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ and $\kappa_* \varepsilon$ is a transverse cycle induced by κ and ε (see Proposition 5.21).

An Anosov representation has an associated boundary map ζ from $\partial_\infty \tilde{S}$ to the flag space $\mathcal{F}_\theta$ for θ (see Section 2.3). A cataclysm deformation of an Anosov representation ρ can also be understood as a deformation of the associated boundary map ζ (see Theorem 6.1). If $x \in \partial_\infty \tilde{S}$ is an ideal vertex of the connected component $Q \subset \tilde{S} \setminus \tilde{\lambda}$, then the boundary map ζ^ε for the deformed representation $\Lambda_0^\varepsilon\rho$ is at x given by

$$\zeta^\varepsilon(x) = \varphi_{P_0Q} \cdot \zeta(x),$$

where $P_0 \subset \tilde{S} \setminus \tilde{\lambda}$ is the fixed reference component used to define the cataclysm deformation.

In contrast to cataclysm deformations on Teichmüller space, cataclysm deformations for Anosov representations are not injective in general. However, the assignment of the family of shearing maps $\{\varphi^\varepsilon_{PQ}\}_{(P,Q)}$ to a transverse twisted cycle $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ is injective (Proposition 7.1). From this, we can conclude injectivity of the cataclysm deformation under extra assumptions. In particular, we have the following result:

Proposition 1.6 (Corollary 7.7)**.** *If $\rho\colon \pi_1(S) \to \mathrm{SL}(n, \mathbb{R})$ is a Hitchin representation, then the cataclysm deformation based at ρ along λ is injective.*

A class of representations for which the cataclysm deformation is not injective are (n, k)-horocyclic deformations (see Subsection 2.4.3 and Section 7.4). They are obtained from composing a discrete and faithful representation $\rho_0\colon \pi_1(S) \to \mathrm{SL}(2, \mathbb{R})$ with a reducible embedding $\mathrm{SL}(2, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$ and are $\{k, n-k\}$-Anosov.

Proposition 1.7 (Corollary 7.21)**.** *Let $\rho\colon \pi_1(S) \to \mathrm{SL}(n, \mathbb{R})$ be an (n, k)-horocyclic representation with $1 \leq k < n/2$ and let $\theta = \{k, n-k\}$. There is a linear subspace $\mathcal{H}_{\mathrm{trivial}} \subset \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ such that $\Lambda_0^\varepsilon\rho = \rho$ if and only if $\varepsilon \in \mathcal{H}_{\mathrm{trivial}}$.*

For a maximal lamination, the dimension of the subspace $\mathcal{H}_{\mathrm{trivial}}$ is $4g(S) - 5$, where $g(S)$ is the genus of the surface S (Corollary 7.24).

Remark 1.8. The idea of recovering the transverse twisted cycle ε from the family of shearing maps $\{\varphi^\varepsilon_{PQ}\}_{(P,Q)}$ that we use in Section 7.1 is adapted from [Dre13]. They use another viewpoint on Anosov representations, using a flow on a bundle over T^1S associated with ρ. We take a different approach, using the *Busemann cocycle* instead of their flow construction and can thus show that their result also holds in our more general setting. As a corollary, Dreyer states that cataclysm deformations for Borel Anosov representations into $\mathrm{SL}(n, \mathbb{R})$ are injective [Dre13, Corollary]. However, this is not correct. A special case of Proposition 1.7 are representations that are obtained from composing a discrete and faithful representation into $\mathrm{SL}(2, \mathbb{R})$ with a block embedding of $\mathrm{SL}(2, \mathbb{R})$ into $\mathrm{SL}(3, \mathbb{R})$. These representations are Borel Anosov, but not Hitchin. This shows that there exist Borel Anosov representations into $\mathrm{SL}(3, \mathbb{R})$ for which the cataclysm deformation is not injective.

A generalization for representations into $\mathrm{SL}(n,\mathbb{R})$

For a certain class of Anosov representations into $\mathrm{SL}(n,\mathbb{R})$, we can enlarge the parameter space of twisted cycles valued in $\mathfrak{a}_\theta$ to twisted cycles valued in $\mathfrak{a}$. We define a θ-Anosov representation $\rho\colon \pi_1(S) \to \mathrm{SL}(n,\mathbb{R})$ to be λ-*admissible* if we can assign a line splitting to every oriented leaf in $\tilde{\lambda}$ that is ρ-equivariant, subordinate to the splitting obtained from the boundary map and behaves well under reversing the orientation of the leaf (see Definition 8.3). For λ-admissible representations, we can enlarge the parameter space of cataclysm deformations to cycles valued in $\mathfrak{a}$:

Theorem 1.9 (Theorem 8.13)**.** *Let $\rho\colon \pi_1(S) \to \mathrm{SL}(n,\mathbb{R})$ be θ-Anosov and λ-admissible. Then there exist cataclysm deformations of ρ along λ with parameters in the space of $\mathfrak{a}$-valued transverse twisted cycles $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a})$.*

Examples for representations that are λ-admissible can be found in Sections 8.2 and 8.3.

Outline of the thesis

We start in Chapter 2 and 3 with recalling basic definitions - most of this is not original, but a personal selection of concepts and results that will be needed throughout the thesis.

Chapter 2 is concerned with Lie groups and Anosov representations. In Section 2.1, we define parabolic subgroups of Lie groups, the Weyl group and flag manifolds. We illustrate the introduced concepts for the group $\mathrm{SL}(n,\mathbb{R})$. In Section 2.2, we define the Cartan decomposition and the Cartan projection as well as the Iwasawa decomposition and the Busemann cocycle. With these preparations, we can define Anosov representations in Section 2.3 and give examples in Section 2.4.

Chapter 3 introduces geodesic laminations and transverse twisted cycles. In Section 3.1, we define geodesic laminations, the orientation cover of a lamination and look at a classical property of geodesic laminations. In Section 3.2 we define the vector space of transverse twisted cycles and compute its dimension.

In Chapter 4 we define slithering maps and stretching maps. In Section 4.1, we generalize the construction of slithering maps for Hitchin representations to the setting of Anosov representations. The proofs in this section are similar to the ones in [BD17, Section 5.1]. In Section 4.2, we define stretching maps T_g^H. Using the slithering map, we give an estimate on the difference between two stretching maps for geodesics that are close.

Chapter 5 contains the main result of this thesis, Theorem 5.12. In Section 5.1, we define shearing maps as the limit of a composition of stretching maps. Using shearing maps, we can define cataclysm deformations in Section 5.2. In Section 5.3, we show that cataclysm deformations satisfy additivity and behave naturally under composition with Lie group homomorphisms.

In Chapter 6 we show how a cataclysm deformation of an Anosov representation ρ changes the corresponding boundary map ζ. Our proof for this expression of the deformed representation uses different methods from the proof in [Dre13] where they consider for the case $G = \mathrm{PSL}(n, \mathbb{R})$ and $\theta = \Delta$.

In Chapter 7 we ask the question under which conditions a cataclysm deformation is injective. In Section 7.1 we show that the assignment of the family of shearing maps to a transverse twisted cycle is injective. From this, we conclude in Section 7.2 a sufficient condition for injectivity of the cataclysm deformation. In Section 7.3, we show that cataclysms are not injective in general by giving sufficient conditions for the deformation to satisfy $\Lambda_0^{\varepsilon}\rho = \rho$ for a non-trivial transverse twisted cycle ε. We also give examples for representations that satisfy these conditions, among which are horocyclic representations. In Section 7.4 we have a closer look at horocyclic representations and determine the subspace of twisted cycles for which the cataclysm deformation is trivial.

We finish in Chapter 8 by giving a construction for cataclysm deformations of representations into $\mathrm{SL}(n, \mathbb{R})$ which have a bigger parameter space than the deformations from Chapter 5. We define in Subsection 8.1 what it means for a representation to be λ-admissible, and show that λ-admissible representations admit cataclysm deformations bigger parameter space $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a})$. In Subsections 8.2 and 8.3 we give examples for λ-admissible representations into $\mathrm{SL}(4, \mathbb{R})$ and $\mathrm{SL}(2n, \mathbb{R})$, respectively.

In the Appendix A, we recall the definition of a left-invariant and almost right-invariant metric on the group G and collect technical proofs that we need in the thesis, but that would not add significant value when presented in the main part.

2. Lie groups and Anosov representations

In this section, we give a short introduction into parabolic subgroups of Lie groups and Anosov representations. The results and definitions are not original, but a personal selection of concepts that will be relevant throughout the thesis. We define parabolic subgroups of Lie groups in Section 2.1 and explain two different decompositions of Lie groups in Section 2.2. With these preparations, we define Anosov representations in Section 2.3 and give examples for Anosov representations in Section 2.4.

2.1. Parabolic subgroups

We start by recalling some basics about Lie groups. In particular, we define the set of simple roots, parabolic subgroups and flag manifolds. The main references for this section are [GW12, Section 3.2] and [GGKW17, Section 2.2].

Let G be a connected non-compact semisimple real Lie group and let $\mathfrak{g}$ be its Lie algebra. Choose a maximal compact subgroup $K \subset G$. Let $\mathfrak{k} \subset \mathfrak{g}$ be the Lie algebra of K and $\mathfrak{p}$ its orthogonal complement with respect to the Killing form. The killing form is non-degenerate since $\mathfrak{g}$ is semisimple. Let $\mathfrak{a}$ be a maximal abelian subalgebra in $\mathfrak{p}$. The group G acts on its Lie algebra $\mathfrak{g}$ via the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$. Since G is connected, this action is by inner automorphisms of $\mathfrak{g}$ (see [SW73, Cor.7.13]). Taking the derivative of Ad at the identity, we obtain the adjoint representation of the Lie algebra $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. It holds that $\mathrm{ad}_X(Y) = [X, Y]$ for any $X, Y \in \mathfrak{g}$, where $[\cdot,\cdot]$ is the Lie bracket on $\mathfrak{g}$. Let $\mathfrak{a}^*$ be the dual space of $\mathfrak{a}$. For $\alpha \in \mathfrak{a}^*$, we define

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{a} :\ \mathrm{ad}_H(X) = \alpha(H)X\}.$$

If $\mathfrak{g}_\alpha \neq 0$, we say that α is a *restricted root* with associated *root space* $\mathfrak{g}_\alpha$. Let $\Sigma \subset \mathfrak{a}^* \setminus \{0\}$ be the set of restricted roots. The *root space decomposition* of $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_0$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. Let $\Delta \subset \Sigma$ be a simple system, i.e. a subset of Σ such that every element in Σ can be expressed uniquely as a linear combination of elements in Δ where all coefficients are either non-negative or non-positive (see [Kna02, §II.6]). The elements of Δ are called *simple roots*. Denote by Σ^+ the set of *positive roots*, namely the set of all elements in Σ which have non-negative coefficients with respect to the generating set Δ, and by Σ^- the *negative roots*. We have $\Sigma^- = -\Sigma^+$ and $\Sigma = \Sigma^+ \cup \Sigma^-$.

Define the subalgebras

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^- = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha}$$

and consider the corresponding subgroups $N^+ = \exp(\mathfrak{n}^+)$ and $N^- = \exp(\mathfrak{n}^-)$. Further, let $A := \exp(\mathfrak{a})$ and let $Z_K(\mathfrak{a})$ be the centralizer of $\mathfrak{a}$ in K, i.e. the set of all $k \in K$ such that $\mathrm{Ad}_k(H) = H$ for all $H \in \mathfrak{a}$. The *standard minimal parabolic subgroup* is the subgroup $B^+ := Z_K(\mathfrak{a})AN^+$. This is a semidirect product of the abelian subgroup $Z_K(\mathfrak{a})A$ of G and the normal subgroup N^+, where $Z_K(\mathfrak{a})A$ acts on N^+ by conjugation. In the same way, one defines $B^- := Z_K(\mathfrak{a})AN^-$. A subgroup of G that is conjugate to B^+ is called *minimal parabolic subgroup* or *Borel subgroup*. The group N^+ is the *unipotent radical* of B^+, that is, the largest normal subgroup consisting of unipotent elements.

Example 2.1. Consider the special linear group $G = \mathrm{SL}(n, \mathbb{R})$, i.e. the group of $n \times n$ matrices with determinant 1. As maximal compact subgroup we chose $K = \mathrm{SO}(n, \mathbb{R})$, the orthogonal matrices with determinant 1. The Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ is given by all $n \times n$ matrices with trace 0, and $\mathfrak{k} = \mathfrak{so}(n, \mathbb{R})$ are the antisymmetric matrices. The Killing form on $\mathfrak{sl}(n, \mathbb{R})$ is $B(X, Y) = 2n\mathrm{tr}(XY)$ for $X, Y \in \mathfrak{sl}(n, \mathbb{R})$. The orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}$ with respect to B is given by the traceless symmetric matrices. The set of traceless diagonal matrices is a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$. Let $\lambda_i \in \mathfrak{a}^*$ be the evaluation at the i-th entry of an element in $\mathfrak{a}$, i.e. $\lambda_i(\mathrm{diag}(a_1, \ldots, a_n)) = a_i$. A short calculation shows that the root space $\mathfrak{g}_\alpha$ for $\alpha \in \mathfrak{a}^*$ is non-zero if and only if $\alpha = \alpha_{ij} := \lambda_i - \lambda_j$ for $1 \leq i \neq j \leq n$. The corresponding root spaces are $\mathfrak{g}_{\alpha_{ij}} = \mathbb{R} \cdot E_{ij}$, where E_{ij} is the matrix with a 1 in the i-th row and j-th column and 0 everywhere else. Further, $\mathfrak{g}_0 = \mathfrak{a}$ is all traceless diagonal matrices and we have $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{i \neq j} \mathfrak{g}_{\alpha_{ij}}$. We thus find the restricted roots, a system of simple roots and the set of positive roots as

$$\begin{aligned} \Sigma &= \{\alpha_{ij} \mid 1 \leq i \neq j \leq n\}, \\ \Delta &= \{\alpha_{i(i+1)} |\ i = 1, \ldots, n-1\} \text{ and} \\ \Sigma^+ &= \{\alpha_{ij} |\ 1 \leq i < j \leq n\}. \end{aligned}$$

To shorten notation, set $\alpha_i := \alpha_{i(i+1)}$. We sometimes identify the set of simple roots Δ with the set $\{1, \ldots, n-1\}$.

We have

$$\mathfrak{n}^+ = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha = \left\{ \begin{pmatrix} 0 & * & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & * \\ 0 & \ddots & \ddots & 0 \end{pmatrix} \right\} \quad \text{and} \quad N^+ = \exp(\mathfrak{n}^+) = \left\{ \begin{pmatrix} 1 & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right\}.$$

Similarly, $\mathfrak{n}^-$ and N^- are given by the lower triangular matrices with only 0s and only 1s on the diagonal, respectively. Further, $A = \exp(\mathfrak{a})$ is given by all diagonal matrices with positive entries and determinant 1 and the centralizer $Z_K(\mathfrak{a})$ is given by the set of diagonal matrices with all entries ± 1 and determinant 1. The minimal parabolic subgroup is the set of all upper triangular matrices, i.e.

$$B^+ = Z_K(\mathfrak{a})AN^+ = \left\{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \middle| \det = 1 \right\}.$$

Similarly, B^- is the set of all lower triangular matrices.

We now define parabolic subgroups of G A subgroup $P \subset G$ is called *parabolic* if it is conjugate to a subgroup containing B^+. Every parabolic subgroup of G is self-normalizing, i.e. $N_G(P) = P$ (see [Mil17, Corollary 17.49]). Parabolic subgroups can be classified by subset of the set of simple roots Δ as we will now explain.

Given a subset $\theta \subset \Delta$, let

$$\mathfrak{a}_\theta := \bigcap_{\alpha\in\Delta\setminus\theta} \ker(\alpha) \ \subset \mathfrak{a}, \tag{2.1}$$

and let $Z_K(\mathfrak{a}_\theta)$ be its centralizer in K. Define $P_\theta^\pm := Z_K(\mathfrak{a}_\theta)AN^\pm$. Because $\mathfrak{a}_\theta \subset \mathfrak{a}$, we have that $Z_K(\mathfrak{a}) \subset Z_K(\mathfrak{a}_\theta)$ and hence $B^+ \subset P_\theta^+$, so P_θ^+ is a parabolic subgroup. Parabolic subgroups of this form are called *standard parabolic subgroups*. Moreover, these are the only parabolic subgroups containing B^+, and every parabolic subgroup P is conjugate to some P_θ^+ for a unique $\theta \subset \Delta$ (see [BT65, Proposition 5.14]). Thus, conjugacy classes of parabolic subgroups are in one-to-one correspondence with subsets $\theta \subset \Delta$. For example, for $\theta = \Delta$ we have $\mathfrak{a}_\Delta = \mathfrak{a}$, so Δ corresponds to the conjugacy class of the minimal parabolic subgroup $B^+ = Z_K(\mathfrak{a})AN^+$.

Let Σ_θ^+ denote the set of all elements in Σ^+ that do not belong to the span of $\Delta \setminus \theta$. Define the Lie subalgebras

$$\mathfrak{n}_\theta^+ = \bigoplus_{\alpha \in \Sigma_\theta^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}_\theta^- = \bigoplus_{\alpha \in \Sigma_\theta^+} \mathfrak{g}_{-\alpha}$$

and $N_\theta^\pm := \exp(\mathfrak{n}_\theta^\pm)$. The group $N_\theta^\pm$ is the *unipotent radical* of $P_\theta^\pm$, and the standard parabolic subgroups $P_\theta^\pm$ are equal to the semidirect product $L_\theta N_\theta^\pm$, where $L_\theta := P_\theta^+ \cap P_\theta^-$ is the common *Levi subgroup* of P_θ^+ and P_θ^-. Note that $\exp(\mathfrak{a}_\theta) \subset \exp(\mathfrak{a}) \subset L_\theta$ is contained in the center of L_θ.

The quotient $\mathcal{F}_\theta = G/P_\theta^+$ is called *flag manifold*. It is a compact G-homogeneous space and its elements are called *flags*. The group G acts on $\mathcal{F}_\theta$ by left-multiplication, which will be denoted by $(g, F) \mapsto g \cdot F$. In the special case $\theta = \Delta$, we have $\mathfrak{a}_\theta = \mathfrak{a}$, $P_\theta^+ = B^+$ and $\mathcal{F}_\Delta$ is called *complete*. The name flag manifold is motivated by the case of $\mathrm{SL}(n, \mathbb{R})$ as we will see in Example 2.3 below.

Remark 2.2. There is a one-to-one correspondence between $\mathcal{F}_\theta$ and the set of parabolic subgroups conjugate to P_θ^+ as follows: When $F = g \cdot P_\theta^+ \in \mathcal{F}_\theta$ is a flag, then its stabilizer $\mathrm{Stab}_G(gP_\theta^+)$ is $gP_\theta^+ g^{-1}$, i.e. a parabolic subgroup conjugate to P_θ^+. Since P_θ^+ is self-normalizing, this is in fact a one-to-one correspondence. We will often make no distinction between elements in $\mathcal{F}_\theta$ and parabolic subgroups conjugate to P_θ^+ and will switch between the two points of view.

Example 2.3. We consider again the group $\mathrm{SL}(n, \mathbb{R})$ from Example 2.1. Let $\theta = \{i_1, \ldots, i_k\}$ with $1 \leq i_1 < \cdots < i_k \leq n-1$. Also, let $i_0 := 0$ and $i_{k+1} := n$. For $j = 1, \ldots, k+1$, let $m_j := i_j - i_{j-1}$, so the m_j describe the sizes of the gaps between the elements in θ. For $i \in \Delta$, we have that $\ker(\alpha_i)$ is given by all traceless diagonal matrices where the $i$ths and $(i+1)$st entry are equal. Thus $\mathfrak{a}_\theta = \bigcap_{\alpha \in \Delta \setminus \theta} \ker(\alpha)$ is given by block diagonal matrices, where the j-th block is a scalar multiple of the $m_j \times m_j$ identity matrix. The centralizer of $\mathfrak{a}_\theta$ in K is

$$Z_K(\mathfrak{a}_\theta) = \left\{ \begin{pmatrix} A_{m_1} & & 0 \\ & \ddots & \\ 0 & & A_{m_{k+1}} \end{pmatrix} \in \mathrm{SO}(n, \mathbb{R}) \right\},$$

and the standard parabolic subgroup for θ is given by upper block triangular matrices.

More precisely,

$$P_\theta^+ = Z_K(\mathfrak{a}_\theta)AN^+ = \left\{ \begin{pmatrix} A_{m_1} & * & \dots & * \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & * \\ 0 & \dots & 0 & A_{m_{k+1}} \end{pmatrix} \in \mathrm{SL}(n,\mathbb{R}) \right\}.$$

Similarly, P_θ^- consists of lower block triangular matrices, where the blocks are of sizes $m_j \times m_j$. The unipotent radical $N_\theta^\pm$ is the subgroup of $P_\theta^\pm$ where all blocks on the diagonal are identity matrices and the Levi subgroup L_θ is the group of block diagonal matrices where the blocks are of size $m_j \times m_j$.

Elements of the flag manifold $\mathcal{F}_\theta$ are families of nested subspaces of the form

$$F = \left(F^{(i_1)} \subset \dots \subset F^{(i_k)} \subset \mathbb{R}^n\right) \quad \text{with } \dim F^{(i_j)} = i_j,$$

so flags in the sense of linear algebra. This explains the name *flag manifold*. An element in $\mathcal{F}_\theta$ can be seen as element of $\mathrm{Gr}_{i_1}(n) \times \dots \times \mathrm{Gr}_{i_k}(n)$, where $\mathrm{Gr}_l(n)$ is the *Grassmannian*, i.e. the space of all l-dimensional linear subspaces of $\mathbb{R}^n$. Two basic examples of flags are the *standard ascending flag* F^+ and *standard descending flag* F^- that are defined as

$$\begin{aligned} F^+ &= (\langle e_1, \dots, e_{i_1}\rangle \subset \langle e_1, \dots, e_{i_2}\rangle \subset \dots \subset \langle e_1, \dots, e_{i_k}\rangle), \\ F^- &= (\langle e_n, \dots, e_{n-i_1+1}\rangle \subset \langle e_n, e_{n-i_2+1}\rangle \subset \dots \subset \langle e_n, \dots, e_{n-i_k+1}\rangle), \end{aligned}$$

where $e_1, \dots, e_n$ are the standard basis vectors for $\mathbb{R}^n$. The flags F^+ and F^- are stabilized by the standard parabolic subgroup P_θ^+ and P_θ^-, respectively.

As above, let $Z_K(\mathfrak{a})$ be the centralizer of $\mathfrak{a}$ in K. Further, let $N_K(\mathfrak{a})$ be the normalizer of $\mathfrak{a}$ in K. The *Weyl group* of G is defined as

$$W := N_K(\mathfrak{a})/Z_K(\mathfrak{a}).$$

Seen as a subgroup of $\mathrm{GL}(\mathfrak{a})$, it is a finite Coxeter group with system of generators given by the orthogonal reflections in the hyperplanes $\ker(\alpha) \subset \mathfrak{a}$ for $\alpha \in \Delta$. The *Weyl chambers* are the connected components of $\mathfrak{a} \setminus \bigcup_{\alpha\in\Delta} \ker(\alpha)$. A closed Weyl chamber is the closure of a Weyl chamber. The set of positive roots Σ^+ singles out the *closed positive Weyl chamber* of $\mathfrak{a}$ defined by

$$\overline{\mathfrak{a}^+} := \{H \in \mathfrak{a} \mid \alpha(H) \geq 0 \ \forall \alpha \in \Sigma^+\}.$$

The Weyl group W acts simply transitively on the Weyl chambers and thus there exists a unique element $w_0 \in W$ such that $w_0(\overline{\mathfrak{a}^+}) = -\overline{\mathfrak{a}^+}$. It is the unique longest element of W and as such satisfies $w_0 = w_0^{-1}$. Abusing notation, we will denote by w_0 also a representative of the longest element of W in $N_K(\mathfrak{a}) \subset K$.

Definition 2.4. The involution on $\mathfrak{a}$ defined by $\iota := -w_0$ is called *opposition involution.* It induces a dual map on $\mathfrak{a}^*$ which is also denoted by ι and defined by $\iota(\alpha) = \alpha \circ \iota$ for all $\alpha \in \Delta$.

Note that $\iota(\Delta) = \Delta$ and $\iota(\Sigma^+) = \Sigma^+$. For some Lie groups, the opposition involution ι is trivial. One example where it is non-trivial is $\mathrm{SL}(n, \mathbb{R})$.

Example 2.5. For $\mathrm{SL}(n, \mathbb{R})$, $\mathfrak{a}$ is the set of traceless diagonal matrices, i.e. isomorphic to $\mathbb{R}^{n-1}$. The Weyl group is isomorphic to the symmetric group $\mathcal{S}_n$ with generators the orthogonal reflections along the hyperplanes $\ker(\alpha_i)$ for $i = 1, \ldots, n-1$. Here, the hyperplane $\ker(\alpha_i)$ is given by all elements $\mathrm{diag}(a_1, \ldots, a_n) \in \mathfrak{a}$ with $a_i = a_{i+1}$. The closed positive Weyl chamber is

$$\overline{\mathfrak{a}^+} = \left\{ H \in \mathfrak{a} \;\middle|\; \alpha(H) \geq 0 \; \forall \alpha \in \Sigma^+ \right\} = \left\{ \mathrm{diag}(a_1, \ldots, a_n) \;\middle|\; \sum_{i=1}^{n} a_i = 0, a_i \geq a_{i+1} \forall i \right\}.$$

The longest element w_0 of the Weyl group, which sends $\overline{\mathfrak{a}^+}$ to $-\overline{\mathfrak{a}^+}$ acts on $\mathfrak{a}$ by reversing the order of the diagonal elements:

$$w_0 \cdot \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = \begin{pmatrix} a_n & & \\ & \ddots & \\ & & a_1 \end{pmatrix}$$

On Δ, the longest element acts as $w_0(\alpha_i) = -\alpha_{n-i}$, so for the opposition involution $\iota\colon \Delta \to \Delta$, we have $\iota(\alpha_i) = \alpha_{n-i}$.

Remark 2.6. It is useful to have a basis for $\mathfrak{a}$ that is invariant under the opposition involution ι. The elements $\alpha \in \Delta$ form a basis for $\mathfrak{a}^*$. Define by $\{H_\alpha\}_{\alpha \in \Delta}$ the corresponding dual basis for $\mathfrak{a}$ defined by $\alpha(H_\beta) = \delta_{\alpha\beta}$ for all $\alpha, \beta \in \Delta$, where $\delta_{\alpha\beta}$ denotes the Kronecker delta. The basis constructed in this way satisfies $\iota(H_\alpha) = H_{\iota(\alpha)}$ for all $\alpha \in \Delta$, because by definition, for all $\beta \in \Delta$,

$$\beta(\iota(H_\alpha)) = \iota(\beta)(H_\alpha) = \delta_{\iota(\beta)\alpha} = \delta_{\beta\iota(\alpha)} = \beta(H_{\iota(\alpha)}).$$

The first step here is simply the definition of ι on Δ. For the third step, we use that $\iota\colon \Delta \to \Delta$ is an involution, so $\iota(\beta) = \alpha$ if and only if $\beta = \iota(\alpha)$. Further, for every $\beta \neq \alpha$, H_α lies in the kernel of β. It follows that for every $\theta \subset \Delta$, we have $H_\alpha \in \bigcap_{\beta \in \Delta \setminus \theta} \ker(\beta) = \mathfrak{a}_\theta$, so by dimension reasons, $\{H_\alpha\}_{\alpha \in \theta}$ is a basis for $\mathfrak{a}_\theta$. We remark that these elements agree up to a factor with the *coroots*, which are also denoted by H_α in the literature, and are defined using the fundamental weights and the Killing form (see [Hal15, Def. 7.21]).

Example 2.7. If we consider $\mathrm{SL}(n,\mathbb{R})$, then a short computation shows that the basis element for $i \in \Delta$ for $\mathfrak{a}_\theta$ from Remark 2.6 is given by

$$H_i = \frac{1}{n}\begin{pmatrix} (n-i)\mathrm{Id}_i & 0 \\ 0 & (-i)\mathrm{Id}_{n-i} \end{pmatrix}.$$

Remark 2.8. Let $w_0 \in N_K(\mathfrak{a})$ be a representative of the longest element of the Weyl group as in Definition 2.4, and recall that $A = \exp(\mathfrak{a})$. Then, since $w_0 \in N_K(\mathfrak{a})$, we have $w_0 A w_0^{-1} = A$. One can further compute that $w_0 N^+ w_0^{-1} = N^-$ and $w_0 Z_K(\mathfrak{a}_\theta) w_0^{-1} = Z_K(\iota(\mathfrak{a}_\theta)) = Z_K(\mathfrak{a}_{\iota(\theta)})$. Combining this, we obtain

$$\begin{aligned} w_0 P_\theta^+ w_0^{-1} &= \left(w_0 Z_K(\mathfrak{a}_\theta) w_0^{-1}\right)\left(w_0 A w_0^{-1}\right)\left(w_0 N^+ w_0^{-1}\right) \\ &= Z_K(\mathfrak{a}_{\iota(\theta)}) A N^- \\ &= P_{\iota(\theta)}^-, \end{aligned}$$

so $P_{\iota(\theta)}^-$ is conjugate to P_θ^+ by w_0. Since every parabolic subgroup is conjugate to a unique $P_{\theta'}^+$, it follows that P_θ^+ is conjugate to P_θ^- if and only if $\iota(\theta) = \theta$.

Consider the space $\mathcal{F}_\theta \times \mathcal{F}_{\iota(\theta)}$ of pairs of flags. There exists a unique open orbit for the diagonal action of G on $\mathcal{F}_\theta \times \mathcal{F}_{\iota(\theta)}$ (see [GW12, Section 2.1]), and one representative of this orbit is the pair (P_θ^+, P_θ^-). Here, we use that P_θ^- is conjugate to $P_{\iota(\theta)}^+$ by w_0, so P_θ^- is an element of $\mathcal{F}_{\iota(\theta)}$.

Definition 2.9. Two parabolic subgroups P^+ and P^- are *transverse* if the pair (P^+, P^-) lies in the unique open G-orbit of $\mathcal{F}_\theta \times \mathcal{F}_{\iota(\theta)}$ for some $\theta \subset \Delta$. Given $P^+ \in \mathcal{F}_\theta$ and $P^- \in \mathcal{F}_{\iota(\theta)}$, we say that P^+ is *transverse* to P^- (and P^- is transverse to P^-) if the pair (P^+, P^-) is transverse.

Note that the subset θ in this definition is already determined by the parabolic P^+, since every parabolic subgroup is conjugate to P_θ^+ for a unique $\theta \subset \Delta$.

Remark 2.10. Fix the standard parabolic P_θ^+. Then its unipotent radical N_θ^+ acts simply transitively on parabolics transverse to P_θ^+. To see this, let P be a parabolic transverse to P_θ^+. By definition of transversality, G acts transitively on pairs of transverse parabolics, there exists some $g \in G$ such that $(g \cdot P_\theta^+, g \cdot P) = (P_\theta^+, P_\theta^-)$. The element g has to lie in P_θ^+. Recall that $P_\theta^+ = L_\theta N_\theta^+$, where $L_\theta = P_\theta^+ \cap P_\theta^-$. Hence $g = ln$ for unique elements $l \in L_\theta$ and $n \in N_\theta^+$. It follows that $n \cdot P = l^{-1} \cdot P_\theta^- = P_\theta^-$ since l stabilizes P_θ^-. So indeed, N_θ^+ acts simply transitively on parabolics transverse to P_θ^+. Moreover, the element $n \in N_\theta^+$ depends locally Hölder continuously on the flag P transverse to P_θ^+.

In the following, we restrict our attention to subsets $\theta \subset \Delta$ for which P_θ^+ and P_θ^- are conjugate. As see in Remark 2.8, this holds if and only if $\iota(\theta) = \theta$.

Standing Assumption. From now on, we always assume that $\theta \subset \Delta$ is invariant under the opposition involution ι, i.e. $\iota(\theta) = \theta$.

For Lie groups with trivial opposition involution, e.g. for $\mathrm{Sp}(2n, \mathbb{R})$ or $\mathrm{SO}(p, q)$ with $p \neq q$, this assumption is trivially satisfied.

Example 2.11. For $G = \mathrm{SL}(n, \mathbb{R})$ and $\theta = \{i_1, \ldots, i_k\}$, the assumption $\iota(\theta) = \theta$ means that $i_j \in \theta$ if and only if $n - i_j \in \theta$. As in Example 2.3, set $i_0 := 0$ and $i_{k+1} := n$. Two flags $E, F \in \mathcal{F}_\theta = \mathcal{F}_{\iota(\theta)}$ are transverse exactly if

$$E^{(i_j)} \oplus F^{(n-i_j)} = \mathbb{R}^n \quad \forall i_j \in \theta.$$

A pair of transverse flags (E, F) induces a splitting of $\mathbb{R}^n$ in subspaces

$$\mathbb{R}^n = V_1 \oplus \cdots \oplus V_{k+1},$$

where $V_j = E^{(i_j)} \cap F^{(n-i_{j-1})}$. Define m_j as in Example 2.3 as $m_j := i_j - i_{j-1}$. By transversality, V_j is a subspace of dimension m_j. In particular, for $\theta = \Delta$, every pair of transverse flags induces a line splitting of $\mathbb{R}^n$.

2.2. Lie group decompositions, the Cartan projection and the Busemann cocycle

In this section, we define two Lie group decompositions and the Cartan projection that will be important in the definition of Anosov representations. Further, we introduce the Busemann cocycle.

As before, let G be a connected non-compact semisimple real Lie group and $K \subset G$ a maximal compact subgroup. Let $\overline{\mathfrak{a}^+}$ be the closed positive Weyl chamber. The following decomposition of G is the *Cartan decomposition.*

Theorem 2.12 ([Hel01, Ch. IX Theorem 1.1])**.** *Let G be a connected semisimple Lie group. Then*

$$G = K \exp(\overline{\mathfrak{a}^+}) K,$$

i.e. every $g \in G$ ca be written as $g = kak'$ with $k, k' \in K$ and $a \in \exp(\overline{\mathfrak{a}^+})$. Moreover, $a \in \exp(\overline{\mathfrak{a}^+})$ is unique.

Definition 2.13. The Cartan decomposition induces a map $\mu\colon G \to \overline{\mathfrak{a}^+}$, the *Cartan projection*, where for $g \in G$, $\mu(g)$ is defined by $g = k\exp(\mu(g))k'$.

Example 2.14. For $\mathrm{SL}(n,\mathbb{R})$ with maximal compact subgroup $\mathrm{SO}(n)$, $\exp(\overline{\mathfrak{a}^+})$ consists of diagonal matrices with determinant 1 and entries in decreasing order. In this case, the Cartan decomposition is given by the singular value decomposition. Recall that the singular value decomposition of a matrix $M \in \mathrm{SL}(n,\mathbb{R})$ is $M = UDV^T$, where U and V are orthogonal matrices and D is diagonal with eigenvalues the singular values of M, i.e. the square roots of the eigenvalues of MM^T in decreasing order. The matrix D is uniquely determined. With this notation, the Cartan projection $\mu(M)$ is given by $\log(D)$, so its entries are the logarithms of the singular values of M.

There is another decomposition of G that will be useful for us: the *Iwasawa decomposition.*

Theorem 2.15 ([Hel01, Ch. IX Theorem 1.3])**.** *Let G be a connected semisimple Lie group. Then*

$$G = KAN^+,$$

that is, the map $K \times A \times N^+ \to G$, $(k,a,n) \mapsto kan$ is a diffeomorphism.

Since the map is a diffeomorphism, for every $g \in G$ the elements k, a, n such that $g = kan$ are unique.

Example 2.16. For $\mathrm{SL}(n,\mathbb{R})$, the Iwasawa decomposition is given by the QR-decomposition of a matrix in an orthogonal matrix and an upper triangular matrix, i.e. an element in B^+, with positive entries on the diagonal. This upper triangular matrix can be written as a product of a diagonal matrix with positive entries and a unipotent upper triangular matrix.

Using the Iwasawa decomposition, we can define a map $\sigma\colon G \times \mathcal{F}_\Delta \to \mathfrak{a}$ that tells us something about how an element in G acts on a given flag. It is defined as follows: The maximal compact subgroup K acts transitively on $\mathcal{F}_\Delta = G/B^+$. Thus, every element in $\mathcal{F}_\Delta$ can be written as $k \cdot B^+$ for some $k \in K$. By the Iwasawa decomposition, for $g \in G$ and $F = k \cdot B^+ \in \mathcal{F}_\Delta$, there exists a unique element $\sigma(g,F)$ in $\mathfrak{a}$ such that

$$gk \in K\exp(\sigma(g,F))N^+.$$

Definition 2.17. The map $\sigma\colon G \times \mathcal{F}_\Delta \to \mathfrak{a}$ is called the *Busemann cocycle* or *Iwasawa cocycle.*

By definition, the Busemann cocycle satisfies $\sigma(\mathrm{Id}, F) = 0$ for all $F \in \mathcal{F}_\Delta$, and $\sigma(k, B^+) = 0$ for all $k \in K$.

The name *cocycle* refers to the following property:

Lemma 2.18 ([BQ16, Lemma 6.29]). *The Busemann cocycle is continuous and satisfies the* cocycle property*, i.e. for all $g_1, g_2 \in G$ and $F \in \mathcal{F}_\Delta$,*

$$\sigma(g_1 g_2, F) = \sigma(g_1, g_2 \cdot F) + \sigma(g_2, F). \tag{2.2}$$

To gain familiarity with the decomposition and the Busemann cocycle, we include the proof for the cocycle property.

Proof. Let $F \in \mathcal{F}_\Delta$ and let $k \in K$ such that $F = k \cdot B^+ \in \mathcal{F}_\Delta$. Let $g_1, g_2 \in G$, and let $k' \in K$ such that

$$g_2 k \in k' \exp(\sigma(g_2, F)) N^+.$$

Then $g_2 \cdot F = k' \cdot B^+$, so

$$g_1 k' \in K \exp(\sigma(g_1, g_2 \cdot F)) N^+.$$

Combining this, we obtain

$$\begin{aligned} g_1 g_2 k &\in g_1 k' \exp(\exp(\sigma(g_2, F))) N^+ \\ &\subset K \exp(\exp(\sigma(g_1, g_2 \cdot F))) N^+ \exp(\exp(\sigma(g_2, F))) N^+ \\ &= K \exp(\exp(\sigma(g_1, g_2 \cdot F)) + \exp(\sigma(g_2, F))) N^+. \end{aligned}$$

In the last line, we used that $\exp(\mathfrak{a})$ normalizes N^+. It follows that the cocycle property holds. $\square$

Example 2.19. For $G = \mathrm{SL}(n, \mathbb{R})$, $g \in \mathrm{SL}(n, \mathbb{R})$ and $F = \left(F^{(1)} \subset \cdots \subset F^{(n)}\right) \in \mathcal{F}_\Delta$, one can compute that the ith coordinate of $\sigma(g, F)$ of the Busemann cocycle is equal to the logarithm of the norm of the linear transformation induced by g between the 1-dimensional spaces $F^{(i)}/F^{(i-1)}$ and $gF^{(i)}/gF^{(i-1)}$. This can be seen by first looking at the special case that $F = B^+$ is the standard minimal parabolic, and then using the cocycle property.

The Busemann cocycle also exists in the context of partial flag manifolds. Let W_θ be the subgroup of the Weyl group W that fixes $\mathfrak{a}_\theta$ point-wise, i.e.

$$W_\theta := \{w \in W \mid w(H) = H \ \forall H \in \mathfrak{a}_\theta\}.$$

Let $p_\theta \colon \mathfrak{a} \to \mathfrak{a}_\theta$ be the unique projection invariant under W_θ. In terms of the basis $\{H_\alpha\}_{\alpha\in\Delta}$ defined in Remark 2.6, it is given by $p_\theta\left(\sum_{\alpha\in\Delta} x_\alpha H_\alpha\right) = \sum_{\alpha\in\theta}(x_\alpha H_\alpha)$.

Lemma 2.20 ([BQ16, Lemma 8.21])**.** *For every $\theta \subset \Delta$, the map $p_\theta \circ \sigma \colon G \times \mathcal{F}_\Delta \to \mathfrak{a}_\theta$ factors through a map $\sigma_\theta \colon G \times \mathcal{F}_\theta \to \mathfrak{a}_\theta$.*

This map is also called *Busemann cocycle* and satisfies the cocycle property (2.2).

2.3. Anosov representations

We are now ready to define Anosov representations. Anosov representations were first defined by Labourie in [Lab06]. His definition then was generalized by Guichard and Wienhard in [GW12]. Since then, many equivalent characterization of Anosov representations have been found and the field is an active area of research (see [GGKW17], [KLP17], [BPS19], [Tso20], [DGK17], [KP20], [Zhu19], [Zhu21], [BCKM21],[CZZ21]). The definition we are presenting can be found in [GGKW17, Theorem 1.3]. We are only interested in surface groups, but Anosov representations can be defined more generally for any word-hyperbolic group Γ.

Standing Assumption. Throughout this paper, let S be a closed connected oriented surface of genus at least 2. Fix an auxiliary hyperbolic metric m on S. Denote by $\tilde{S}$ the universal cover of S, which then carries a hyperbolic metric as well, and let $\partial_\infty \tilde{S}$ be the boundary at infinity. Fixing base points on S and $\tilde{S}$, the fundamental group $\pi_1(S)$ of S acts on $\tilde{S}$ by isometries.

Before giving the definition of Anosov representations, we need to introduce some more concepts. Let $\theta \subset \Delta$ be a subset of the simple roots and $\mathcal{F}_\theta = G/P_\theta^+$ the flag manifold for the standard parabolic subgroup P_θ^+.

Definition 2.21. Let $\rho \colon \pi_1(S) \to G$ be a representation and $\zeta \colon \partial_\infty \tilde{S} \to \mathcal{F}_\theta$ be a map.

- ζ is called *ρ-equivariant* if $\zeta(\gamma x) = \rho(\gamma) \cdot \zeta(x)$ for all $x \in \partial_\infty \tilde{S}$ and $\gamma \in \pi_1(S)$.
- ζ is called *transverse* if for every pair of points $x \neq y \in \partial_\infty \tilde{S}$, the images $\zeta(x), \zeta(y) \in \mathcal{F}_\theta$ are transverse.
- ζ is called *dynamics-preserving* for ρ if for every non-trivial element $\gamma \in \pi_1(S)$, its unique attracting fixed point in $\partial_\infty \tilde{S}$ is mapped to an attracting fixed point of $\rho(\gamma)$ on $\mathcal{F}_\theta$.

Further, ρ is called *θ-divergent* if for all $\alpha \in \theta$ we have $\alpha(\mu(\rho(\gamma))) \to \infty$ as the word length of $\gamma \in \pi_1(S)$ goes to infinity.

Definition 2.22 ([GGKW17, Theorem 1.3])**.** A representation $\rho\colon \pi_1(S) \to G$ is *θ-Anosov* if it is θ-divergent and there exists a continuous ρ-equivariant map $\zeta\colon \partial_\infty \tilde{S} \to \mathcal{F}_\theta$ that is transverse and dynamics-preserving. The map ζ is called the *boundary map* for ρ.

The set of θ-Anosov representations forms an open subset of the representation variety $\mathrm{Hom}(\pi_1(S), G)$ [GW12, Theorem 5.13]..

Remark 2.23. If there exists a continuous, dynamics-preserving boundary map $\zeta\colon \partial_\infty \tilde{S} \to \mathcal{F}_\theta$ for ρ, then it is unique and ρ-equivariant. This follows from the fact that the set of attracting fixed points of elements $\gamma \in \pi_1(S)$ is dense in $\partial_\infty \tilde{S}$. Further, the boundary map is Hölder continuous [BCLS15, Theorem 6.1].

Remark 2.24. For some settings, the assumption of ρ being θ-divergent is not necessary, since it is already guaranteed by the existence of a continuous transverse ρ-equivariant boundary map. This is the case for example if ρ is Zariski dense or if ρ is an irreducible representation into $\mathrm{SL}(n, \mathbb{R})$ and $\theta = \{1, n-1\}$ [GW12, Theorem 4.11 and Theorem 4.10, respectively].

We now explain how the boundary map ζ can be constructed from the Anosov representation ρ (see [GGKW17, Section 5]): Assume that $\rho\colon \pi_1(S) \to G$ is θ-Anosov. We want to understand $\zeta(x)$ for some $x \in \partial_\infty \tilde{S}$. Define

$$\begin{aligned} \Xi_\theta\colon G &\to G/P_\theta^+ \\ g &\mapsto k_g \cdot P_\theta^+, \end{aligned}$$

where $g = k_g \exp(\mu(g)) k_g'$ is the Cartan decomposition (Theorem 2.12). Note that the element $k_g \in K$ is not unique, but $k_g \cdot P_\theta^+$ is well-defined. Let $x \in \partial_\infty \tilde{S}$ be a point in the boundary of $\tilde{S}$ and $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $\pi_1(S)$ converging to x in the sense that the sequence $(\gamma_n)_{n\in\mathbb{N}}$ and the quasi-geodesic ray defining x have bounded distance. Here, we identify $\partial_\infty \tilde{S}$ with $\partial_\infty \pi_1(S)$. If ρ is θ-Anosov, then the sequence $(\Xi_\theta(\rho(\gamma_m)))_{m\in\mathbb{N}}$ converges and

$$\zeta(x) = \lim_{m\to\infty} \Xi_\theta(\rho(\gamma_m)). \tag{2.3}$$

Example 2.25. Let $G = \mathrm{SL}(n, \mathbb{R})$ and $\theta = \{k, n-k\}$ for some $1 \le k \le \frac{n}{2}$ (see Example 2.3). Then the flag space $\mathcal{F}_\theta$ consists of pairs of nested subspaces $\left(F^{(k)} \subset F^{(n-k)}\right)$ with $\dim F^{(i)} = i$ and can be seen as a subset of $\mathrm{Gr}_k(n) \times \mathrm{Gr}_{n-k}(n)$, where $\mathrm{Gr}_l(n)$ denotes the Grassmannian of l-dimensional subspaces of $\mathbb{R}^n$. We say that $M \in \mathrm{SL}(n, \mathbb{R})$ has a *gap of index j* if $\sigma_j(M) > \sigma_{j+1}(M)$, where $\sigma_j(M)$ and σ_{j+1} are the jth and $(j+1)$st eigenvalues of

M, respectively (see [BPS19, Section 2.2]). If M has a gap of index j, let $U_j(M) \in \mathrm{Gr}_l(d)$ be the subspace of $\mathbb{R}^d$ that contains the j biggest axes of the ellipsoid $\{Mv \mid \|v\| = 1\}$. If $\rho\colon \pi_1(S) \to \mathrm{SL}(n, \mathbb{R})$ is $\{k, n-k\}$-Anosov, then for every $\gamma \in \pi_1(S)$, $\rho(\gamma)$ has gaps of index k and $n-k$. The flag $\Xi_\theta(M)$ is given by

$$\Xi_\theta(M) = (U_k(M) \subset U_{n-k}(m)) .$$

Thus, the k- and $(n-k)$-dimensional parts of the boundary map are given by

$$\zeta^{(k)}(x) = \lim_{m\to\infty} U_k(\rho(\gamma_m)) \quad \text{and} \quad \zeta^{(n-k)}(x) = \lim_{m\to\infty} U_{n-k}(\rho(\gamma_m)),$$

respectively.

Notation 2.26. Let g be an oriented geodesic in the universal cover $\tilde{S}$ and let $\rho\colon \pi_1(S) \to G$ be θ-Anosov with boundary map ζ. Using the boundary map ζ, we can associate to g a pair of opposite parabolics as follows: Denote by g^+ and g^- the positive and negative endpoints of g, respectively. Let $P_g^\pm := \zeta(g^\pm)$ and $L_g := P_g^+ \cap P_g^-$ be the common Levi factor of P_g^+ and P_g^-. Further, let N_g^+ be the unipotent radical of P_g^+. By transversality of ζ, the pair (P_g^+, P_g^-) is transverse.

We want to mention an important class of Anosov representations.

Definition 2.27. A representation $\rho\colon \pi_1(S) \to \mathrm{SL}(n, \mathbb{R})$ that is $\{\alpha_1, \alpha_{n-1}\}$-Anosov is called *projective Anosov.*

Projective Anosov representations play an important role in the study of Anosov representations, because of the following result:

Theorem 2.28 ([GW12, Prop. 4.3 and Remark 4.12])**.** *There exists $d \in \mathbb{N}$ and an irreducible representation $\tau\colon G \to \mathrm{SL}(d, \mathbb{R})$ such that a representation $\rho\colon \pi_1(S) \to G$ is θ-Anosov if and only if $\tau \circ \rho\colon \pi_1(S) \to \mathrm{SL}(d, \mathbb{R})$ is projective Anosov. Furthermore, τ induces maps*

$$\tau^+\colon G/P_\theta^+ \to \mathbb{R}P^{d-1} \text{ and } \tau^-\colon G/P_\theta^+ \to \mathrm{Gr}_{d-1}(d).$$

If $\zeta\colon \partial_\infty \tilde{S} \to G/P_\theta^+$ is the boundary map for ρ, then the 1- and $(d-1)$-dimensional parts of the boundary map for $\tau \circ \rho$ are given by $\tau^+ \circ \zeta$ and $\tau^- \circ \zeta$, respectively.

Note that τ is not unique. If for $M \in \mathrm{SL}(d, \mathbb{R})$, we denote by $U_1(M)$ the eigenspace of MM^T with respect to its biggest eigenvalue as in Example 2.25, then we have

$$\tau^+ (\Xi_\theta(g)) = U_1(\tau(g)).$$

Example 2.29. Let $G = \mathrm{SL}(n, \mathbb{R})$, $\theta = \{k, n-k\}$ for some $1 \leq k \leq \frac{n}{2}$. Then the *exterior power representation* $\bigwedge_n^k \colon \mathrm{SL}(n, \mathbb{R}) \to \mathrm{SL}\left(\bigwedge^k \mathbb{R}^n\right)$ is an irreducible representation as in Theorem 2.28. It is defined by

$$\bigwedge\nolimits_n^k(A)(v_1 \wedge \cdots \wedge v_k) := Av_1 \wedge \cdots \wedge Av_k$$

for every $A \in \mathrm{SL}(n, \mathbb{R})$ and every decomposable element $v_1 \wedge \cdots \wedge v_k \in \bigwedge^k \mathbb{R}^n$. In this case, $d = \dim\left(\bigwedge^k \mathbb{R}^n\right) = \binom{n}{k}$. The induced map $\iota^+ \colon G/{P_\theta}^+ \to \mathbb{R}P^{d-1}$ is called the *Plücker embedding* and is given by

$$\iota^+(V) = \langle v_1 \wedge \cdots \wedge v_k \rangle$$

if $V = (V^{(k)} \subset V^{(n-k)}) \in G/{P_\theta}^+ \subset \mathrm{Gr}_k(n) \times \mathrm{Gr}_{n-k}(n)$ and $V^{(k)} = \langle v_1, \ldots, v_k \rangle$. This is independent of the choice of $v_1, \ldots, v_k$, because if $w_1, \ldots, w_n$ is another generating system for V, then $w_i = Av_i$ for all i for some $A \in \mathrm{GL}(V^{(k)})$. Hence $\langle w_1 \wedge \cdots \wedge w_k \rangle = \langle \det(A) v_1 \wedge \cdots \wedge v_k \rangle$.f

If a representation ρ is θ-Anosov, then also every representation ρ' that is conjugate to ρ by an element in G is θ-Anosov. Thus, it makes sense to talk about θ-Anosov representations also on the character variety.

Definition 2.30. The space

$$\chi(S, G) := \mathrm{Hom}(\pi_1(S), G)//G$$

of representations of the fundamental group of S into G up to G-conjugation is called *G-character variety.* For a subset $\theta \subset \Delta$ of the simple roots, denote by $\chi^{\theta\text{-Anosov}}(S, G)$ the subset consisting of the equivalence classes of θ-Anosov representations.

Note that here, $\chi(S, G)$ is not the usual quotient, which is in general not Hausdorff. Instead, it is the largest Hausdorff quotient, which is equivalent to restricting to the subset of reductive representation (see [CLM18, Theorem 2.2]).

2.4. Examples for Anosov representations

In this section we give examples of Anosov representations for different families of Lie groups G.

2.4.1. Teichmüller space

This section on Teichmüller space is built on [FM12, Section 10.3], which gives a nice treatment on the viewpoint on Teichmüller space as discrete and faithful representations

A *marked hyperbolic structure* on S is a pair (X, f_X), where X is a hyperbolic surface and $f_X : S \to X$ is a homeomorphism, called *marking*. We say that two marked hyperbolic structures (X, f_X) and (Y, f_Y) are *equivalent* if there exists an isometry $h : X \to Y$ such that f_Y and $h \circ f_X$ are homotopic. *Teichmüller space* $\mathcal{T}(S)$ is the space of marked hyperbolic structures on S up to this equivalence relation.

To a marked hyperbolic structure, we can assign a representation ρ from $\pi_1(S)$ to $\mathrm{PSL}(2,\mathbb{R})$ as follows: There exists an isometry from the universal cover $\tilde{X}$ of X to the hyperbolic plane $\mathbb{H}^2$. The fundamental group $\pi_1(X)$ identifies with the group of deck transformations for the universal cover $\tilde{X}$, so it acts on $\tilde{X}$ properly discontinuously and by isometries. Hence, $\pi_1(X)$ identifies with a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}(2,\mathbb{R})$. Combining this with the isomorphism $(f_X)_* : \pi_1(S) \to \pi_1(X)$ induced from the marking $f_X : S \to X$ we obtain a discrete and faithful representation $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$. This construction depends on various choices, e.g. on the choice of representative (X, f_X) of the equivalence class $[(X, f_X)]$ and on the identification of $\tilde{X}$ with $\mathbb{H}^2$, but the representations we obtain for different choices are the same up to conjugation. We obtain a continuous and injective map

$$\mathcal{T}(S) \to \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))//\mathrm{PSL}(2,\mathbb{R}),$$

whose image is a connected component of the $\mathrm{PSL}(2,\mathbb{R})$-character variety. Goldman showed in his thesis [Gol80] that there are two components of the $\mathrm{PSL}(2,\mathbb{R})$-character variety that can be identified with $\mathcal{T}(S)$. They correspond to the two possible orientations of the surface S. A representation ρ that represents an element in $\mathcal{T}(S)$ is Anosov and the boundary map is constructed as follows: Consider the orbit map $\tau_\rho : \pi_1(S) \to \mathbb{H}^2$ for ρ which is defined by $\tau_\rho(\gamma) = \rho(\gamma) \cdot x_0$, where $x_0 \in \mathbb{H}^2$ is a fixed base point. It induces a ρ-equivariant homeomorphism on the boundary $\zeta : \partial\pi_1(S) \to \partial\mathbb{H}^2 \cong \mathbb{R}P^1$. This boundary map is transverse and dynamics-preserving.

2.4.2. Hitchin representations

One way of constructing θ-Anosov representations into a Lie group G is to take a discrete and faithful representation $\rho_0 : \pi_1(S) \to \mathrm{SL}(2,\mathbb{R})$, a so-called *Fuchsian representation*, and post-compose it with a suitable representation $\mathrm{SL}(2,\mathbb{R}) \hookrightarrow G$. *Hitchin representations* arise in this way. They are one of the first examples of Anosov representations and have been studied extensively. We first introduce Hitchin representations for $G = \mathrm{SL}(n, \mathbb{R})$.

Consider the n-dimensional irreducible representation $j_n \colon \mathrm{SL}(2, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$. It can be realized by considering the action of $\mathrm{SL}(2, \mathbb{R})$ on the n-dimensional vector space of homogeneous polynomials of degree $n-1$ in 2 variables x and y. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$, then $j_n(A)$ acts on this space by sending x to $ax + by$ and y to $cx + dy$. A basis for the space of homogeneous polynomials of degree $n-1$ is $x^{n-1}, x^{n-2}y, \ldots, xy^{n-2}, y^{n-1}$. Scaling this basis appropriately, we can arrange that $j_n(\mathrm{SO}(2)) \subset \mathrm{SO}(n)$, where we consider $\mathrm{SO}(n)$ with respect to the standard inner product on $\mathbb{R}^n$. For example, for $n = 3$ we choose the basis $\frac{1}{\sqrt{2}}x^2, xy, \frac{1}{\sqrt{2}}x^2$ and $j_3 \colon \mathrm{SL}(2, \mathbb{R}) \to \mathrm{SL}(3, \mathbb{R})$ is given by

$$j_3\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a^2 & \sqrt{2}ac & c^2 \\ \sqrt{2}ab & ad + bc & \sqrt{2}cd \\ b^2 & \sqrt{2}bd & d^2 \end{pmatrix}.$$

A *principal Fuchsian representation* in $\mathrm{SL}(n, \mathbb{R})$ is a representation of the form $\rho = j_n \circ \rho_0$, where $\rho_0 \colon \pi_1(S) \to \mathrm{SL}(2, \mathbb{R})$ is a Fuchsian representation. A *Hitchin representation* is a representation that lies in the same connected component as a principal Fuchsian representation. The *Hitchin component* $\mathrm{Hit}(\pi_1(S), \mathrm{SL}(n, \mathbb{R}))$ is the component of the character variety $\mathcal{X}(S, \mathrm{SL}(n, \mathbb{R}))$ consisting of conjugacy classes of Hitchin representations.

The construction of the Hitchin component can be generalized to any split real semi-simple Lie group G using an embedding $j \colon \mathrm{SL}(2, \mathbb{R}) \hookrightarrow G$. For the classical groups $\mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{SO}(n, n+1)$, the embedding j is given by j_{2n} and j_{2n+1}, respectively, which has image contained in these subgroups. Just as in the $\mathrm{SL}(n, \mathbb{R})$-case, the *Hitchin component* is the connected component of $\mathcal{X}(S, G)$ containing a principal Fuchsian representation of the form $\rho = j \circ \rho_0$.

The Hitchin component in $\chi(S, \mathrm{PSL}(n, \mathbb{R}))$ was first studied through Higgs bundles by Hitchin in [Hit92], who called it *Teichmüller component*. He showed that it consists entirely of irreducible representations and is diffeomorphic to $\mathbb{R}^{-\chi(S)(n^2-1)}$, where $\chi(S$ is the Euler characteristic of the surface S. Labourie showed that any Hitchin representation ρ is discrete and faithful and that for any $\gamma \in \pi_1(S)$, $\rho(\gamma)$ is diagonalizable with distinct eigenvalues of the same sign [Lab06]. Bonahon and Dreyer construct coordinates for the Hitchin component in $\mathrm{PSL}(n, \mathbb{R})$ that generalize shearing coordinates on Teichmüller space [BD17].

To motivate that Hitchin representations are Δ-Anosov, we look at the special case of principal Fuchsian representations. For general Hitchin representations, this was proved by Labourie [Lab06]. The following treatment is adapted from lecture notes by Canary [Can20, Chapter 32]. Let $\rho = j_n \circ \rho_0$ be a principal Fuchsian representation. The irreducible representation j_n induces a continuous map $j_n^+ \colon \mathbb{R}P^1 \to \mathrm{Flag}(\mathbb{R}^n)$ as follows: Every element in $\mathbb{R}P^1$ can be written uniquely as $L \cdot \langle e_1 \rangle$, where $e_1 \in \mathbb{R}^2$ is the first standard basis vector and L is in $\mathrm{SO}(2)$. Define

$$j_n^+(L \cdot \langle e_1 \rangle) = j_n(L) \cdot F^+,$$

where F^+ is the standard ascending flag in $\mathbb{R}^n$ as in Example 2.3. One can check that this defines a continuous j_n-equivariant injective map which is dynamics-preserving, i.e. if $\langle x \rangle \in \mathbb{R}P^1$ is the attracting eigenline for $A \in \mathrm{SL}(2,\mathbb{R})$, then $j_n^+(\langle x \rangle)$ is the attracting flag for $j_n(A)$. It is a generalized version of the *Veronese embedding* of $\mathbb{R}P^1$ in $\mathbb{R}P^{n-1}$. Define

$$\zeta \colon \partial_\infty \tilde{S} \to \mathrm{Flag}(\mathbb{R}^n), \quad \zeta := j_n^+ \circ \zeta_0,$$

where $\zeta_0 \colon \partial_\infty \tilde{S} \to \mathbb{R}P^1$ is the flag curve for ρ_0. Then ζ is a continuous, ρ-equivariant, dynamics-preserving and transverse boundary map.

The fact that ρ is θ-divergent can be seen as follows: If the singular value decomposition for $A \in \mathrm{SL}(2,\mathbb{R})$ is $A = LDK$, then the singular value decomposition for $j_n(A)$ is $j_n(L)j_n(D)j_n(K)$. By examining how j_n acts on diagonal matrices, we see that the ith singular value σ_i of $j_n(A)$ is given by $\sigma_i(j_n(A)) = \sigma_1(A)^{n+1-2i}$. Recall from Example 2.14 that for $\mathrm{SL}(n,\mathbb{R})$, the Cartan projection $\mu \colon \mathrm{SL}(n,\mathbb{R}) \to \overline{\mathfrak{a}^+_{\mathrm{SL}(n,\mathbb{R})}}$ is given by the logarithms of the singular values. It follows that for all $\alpha_i \in \Delta$

$$\alpha_i\left(\mu(\rho(\gamma))\right) = \log \sigma_i(\rho(\gamma)) - \log \sigma_{i+1}(\rho(\gamma)) = \log \frac{\sigma_1(\rho_0(\gamma))^{n+1-2i}}{\sigma_1(\rho_0(\gamma))^{n+1-2(i+1)}} = \log \sigma_1(\rho_0(\gamma))^2,$$

which, since ρ_0 is Anosov, goes to infinity as the word length of γ goes to infinity. In total, ρ is θ-divergent and has a continuous, ρ-equivariant, transverse and dynamics-preserving boundary map, so is Δ-Anosov.

2.4.3. Horocyclic representations and other reducible representations

For constructing principal Fuchsian representations in Subsection 2.4.2, we used an irreducible embedding $j \colon \mathrm{SL}(2,\mathbb{R}) \hookrightarrow G$. We can also consider embeddings $\mathrm{SL}(2,\mathbb{R}) \hookrightarrow G$ that are reducible. Define

$$\begin{aligned} \iota\colon \mathrm{SL}(2,\mathbb{R}) &\to \mathrm{SL}(3,\mathbb{R}),\\ \begin{pmatrix} a & b\\ c & d\end{pmatrix} &\mapsto \begin{pmatrix} a & 0 & b\\ 0 & 1 & 0\\ c & 0 & d\end{pmatrix}. \end{aligned} \tag{2.4}$$

The image of ι in $\mathrm{SL}(3,\mathbb{R})$ fixes the line $\langle e_2\rangle$ and the hyperplane $\langle e_1, e_3\rangle$. Let $\rho_0\colon \pi_1(S) \to \mathrm{SL}(2,\mathbb{R})$ be discrete and faithful. Then a representation of the form

$$\rho := \iota \circ \rho_0 \colon \pi_1(S) \to \mathrm{SL}(3,\mathbb{R})$$

is called *horocyclic representation.* In other words, a horocyclic representation is the direct sum of a Fuchsian representation with a trivial representation. The name *horocyclic* is motivated by the fact that the image of ρ is contained in a parabolic subgroup of $\mathrm{SL}(3,\mathbb{R})$, namely the one that stabilizes the line $\langle e_2\rangle$ and the complementary hyperplane $\langle e_1, e_3\rangle$. Horocyclic representations and deformations thereof were defined by Barbot [Bar10] and are sometimes also called *Barbot representations.* Barbot also proved that horocyclic representations are Δ-Anosov.

To see this, we construct the boundary map $\zeta\colon \partial_\infty \tilde{S} \to \mathrm{Flag}(\mathbb{R}^3)$ for a horocyclic representation ρ. Define a continuous, ι-equivariant and dynamics-preserving embedding by

$$\begin{aligned} &\iota^+\colon \mathbb{R}P^1 \to \mathrm{Flag}(\mathbb{R}^3),\\ &\left\langle \begin{pmatrix} a\\ b\end{pmatrix}\right\rangle \mapsto \left(\left\langle \begin{pmatrix} a\\ 0\\ b\end{pmatrix}\right\rangle \subset \left\langle \begin{pmatrix} a\\ 0\\ b\end{pmatrix}, \begin{pmatrix} 0\\ 1\\ 0\end{pmatrix}\right\rangle \right). \end{aligned}$$

Let $\rho = \iota \circ \rho_0$ be a horocyclic representation and $\zeta_0\colon \partial_\infty \tilde{S} \to \mathbb{R}P^1$ the boundary map for ρ_0. Then

$$\zeta\colon \partial_\infty \tilde{S} \to \mathrm{Flag}(\mathbb{R}^3), \quad \zeta(x) := \iota^+ \circ \zeta_0(x)$$

is a continuous, ρ-equivariant, transverse and dynamics-preserving boundary map for ρ. This can be seen analogous to the case of Hitchin representations in Subsection 2.4.2. To show that ρ is Δ-Anosov, the only thing that is left to show is Δ-divergence, which follows from a short computation.

This construction of horocyclic representations not only works in $\mathrm{SL}(3,\mathbb{R})$. We can generalize it to obtain reducible $\{k, n-k\}$-Anosov representations in $\mathrm{SL}(n,\mathbb{R})$ for any $k \leq \frac{n}{2}$ as follows: Identify $\mathbb{R}^n$ with $\left(\mathbb{R}^2\right)^k \times \mathbb{R}^{n-2k}$ and embed $\mathrm{SL}(2,\mathbb{R})$ as $\mathrm{SL}(2,\mathbb{R})^k \times \{\mathrm{Id}_{n-2k}\}$.

Reordering the basis of $\mathbb{R}^n$, this gives an embedding

$$\begin{aligned} \iota_{n,k} \colon \mathrm{SL}(2,\mathbb{R}) &\to \mathrm{SL}(n,\mathbb{R}), \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \begin{pmatrix} a\mathrm{Id}_k & 0 & b\mathrm{Id}_k \\ 0 & \mathrm{Id}_{n-2} & 0 \\ c\mathrm{Id}_k & 0 & d\mathrm{Id}_k \end{pmatrix}, \end{aligned} \tag{2.5}$$

where Id_m denotes the $m \times m$-identity matrix, and the 0s denote zero matrices of the appropriate sizes. For $n = 3$ and $k = 1$, we recover $\iota_{3,1} = \iota$ from above. An (n, k)-horocyclic representation is a representation of the form $\rho = \iota_{n,k} \circ \rho_0$ where $\rho_0 \colon \pi_1(S) \to \mathrm{SL}(2,\mathbb{R})$ is discrete and injective. Sometimes we will just say a representation is *horocyclic*, meaning that it is (n, k)-horocyclic for some n and k. Again, the motivation for the name comes from the fact that such a representation stabilizes an element in $\mathcal{F}_{\{k,n-k\}}$. In the special case $n = 3$ and $k = 1$ considered above, we have $\{k, n-k\} = \{1, 2\} = \Delta_{\mathrm{SL}(3,\mathbb{R})}$, so $\mathcal{F}_{\{1,2\}}$ is the complete flag variety. In the general case, however, $\mathbb{F}_{\{k,n-k\}}$ is a partial flag variety.

A (n, k)-horocyclic representation is $\{k, n-k\}$-Anosov. This can be seen analogous to the special case $n = 3$ and $k = 1$ above, using the embedding

$$\begin{aligned} (\iota_{n,k})^+ : \mathbb{R}P^1 &\to \mathrm{SL}(n,\mathbb{R})/P^+_{\{k,n-k\}}, \\ M \cdot \langle e_1 \rangle &\mapsto \iota_{n,k}(M) \cdot (\langle e_1, \dots, e_k \rangle \subset \langle e_1, \dots, e_{n-k} \rangle) \end{aligned}$$

for $M \in \mathrm{SO}(2)$. The boundary map for $\rho = \iota_{n,k} \circ \rho_0$ is then given by $(\iota_{n,k})^+ \circ \zeta_0$.

In a similar way, we can construct reducible representations into $\mathrm{SL}(2n + 1,\mathbb{R})$ that are $\Delta_{\mathrm{SL}(2n+1,\mathbb{R})}$-Anosov. For that, consider the representation

$$\begin{aligned} \iota_{2n\to 2n+1} \colon \mathrm{SL}(2n,\mathbb{R}) &\to \mathrm{SL}(2n + 1,\mathbb{R}), \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} &\mapsto \begin{pmatrix} A & 0 & B \\ 0^T & 1 & 0^T \\ C & 0 & D \end{pmatrix}. \end{aligned} \tag{2.6}$$

Here, 0 and 0^T denote the 0 vector in $\mathbb{R}^n$ and $(\mathbb{R}^n)^T$, respectively. Now let $\rho_0 \colon \pi_1(S) \to \mathrm{Sp}(2n,\mathbb{R}) \subset \mathrm{SL}(2n,\mathbb{R})$ be a $\Delta_{\mathrm{Sp}(2n,\mathbb{R})}$-Anosov representation. In particular, ρ_0 is $\Delta_{\mathrm{SL}(2n,\mathbb{R})}$-Anosov and for every $\gamma \in \pi_1(S)$, the eigenvalues of $\rho(\gamma)$ are pairwise distinct and different from 1. Then $\rho := \iota_{2n\to 2n+1} \circ \rho_0$ is a reducible representation into $\mathrm{SL}(2n + 1,\mathbb{R})$. The map $\iota_{2n\to 2n+1}$ induces a continuous, $\iota_{2n\to 2n+1}$-equivariant embedding $\iota^+_{2n\to 2n+1} \colon \mathrm{Flag}(\mathbb{R}^{2n}) \to \mathrm{Flag}(\mathbb{R}^{2n+1})$. Since $\mathrm{Sp}(2n,\mathbb{R})/B^+$ can be embedded in $\mathrm{Flag}(\mathbb{R}^{2n})$, we obtain a continuous, transverse, ρ-equivariant map $\zeta := \iota^+_{2n\to 2n+1} \circ \zeta_0 \colon \partial_\infty \tilde{S} \to \mathrm{Flag}(\mathbb{R}^{2n+1})$. By similar considerations as above, ρ is $\Delta_{\mathrm{SL}(2n+1,\mathbb{R})}$-divergent. Thus, ρ is a $\Delta_{\mathrm{SL}(2n+1,\mathbb{R})}$-Anosov representation into $\mathrm{SL}(2n+1,\mathbb{R})$. It is reducible, so does not lie in the Hitchin component. Note that in this construction we need to start with a $\Delta_{\mathrm{Sp}(2n,\mathbb{R})}$-Anosov representation ρ_0 into $\mathrm{Sp}(2n,\mathbb{R})$ and cannot replace it by a $\Delta_{\mathrm{SL}(2n,\mathbb{R})}$-Anosov representation into $\mathrm{SL}(2n,\mathbb{R})$. The reason is that for the latter, the elements $\rho_0(\gamma)$ for $\gamma \in \pi_1(S)$ can have 1 as an eigenvalue. In that case, the composed representation $\iota_{2n\to 2n+1} \circ \rho_0$ is not $\Delta_{\mathrm{SL}(2n+1,\mathbb{R})}$-divergent.

2.4.4. Maximal representations

In addition to Hitchin representations, maximal representations were among the first known examples of Anosov representations. Their definition originates in the observation that the two Teichmüller components for representations into $\mathrm{PSL}(2,\mathbb{R})$ are distinguished from the other components by the Euler number $E(\rho)$ [Gol88]. A representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ lies in one of the Teichmüller components if and only if the modulus of its Euler number is maximal. Let now G be a Lie group of Hermitian type, i.e. connected semisimple with finite center, without compact factors and such that the associated symmetric space $\mathcal{X}$ has a G-invariant complex structure. For a representation $\rho\colon \pi_1(S) \to G$, one can define an invariant $T(\rho)$, called *Toledo invariant*, which agrees with the Euler number for $G = \mathrm{PSL}(2,\mathbb{R})$. The map $\rho \mapsto T(\rho)$ on $\mathrm{Hom}(\pi_1(S), G)$ is continuous and takes values in a discrete bounded set. Consequently, it is constant on connected components [BIW10]. A representation $\rho \in \mathrm{Hom}(\pi_1(S), G)$ is *maximal* if $|T(\rho)|$ is maximal. Maximal representations are P-Anosov for a certain maximal parabolic $P \subset G$ (see [BILW05, Theorem 6.1] for $G = \mathrm{Sp}(2n,\mathbb{R})$ and [GW12, Theorem 6.6 and Remark 6.7] for a classical group G of Hermitian type).

An important example for a Lie group of Hermitian type is $\mathrm{Sp}(2n,\mathbb{R})$. In this case, a maximal representation is Anosov with respect to the maximal parabolic that stabilizes a Lagrangian, i.e. a maximal isotropic subspace of $\mathbb{R}^{2n}$. The group $\mathrm{Sp}(2n,\mathbb{R})$ is also split, hence we can also define Hitchin representations into $\mathrm{Sp}(2n,\mathbb{R})$. Hitchin representations into $\mathrm{Sp}(2n,\mathbb{R})$ are maximal, the converse is not true see [GW12, p.581].

2.4.5. Positive representations

Recently, Guichard and Wienhard discovered another class of representations that are θ-Anosov for some $\theta \subset \Delta$ and that contains both Hitchin representations and maximal representations [GW18]. The starting point for their definition is the observation that the boundary maps for both Hitchin representations and maximal representations are *positive* in the sense that they map positive triples in $\partial_\infty \tilde{S}$ to positive triples in G/P_θ^+. In order to say what it means for a triple in G/P_θ^+ to be positive, Guichard and Wienhard define what a θ-positive structure on a simple Lie group G is. They show that there are four families of Lie groups admitting a θ-positive structure. Two of these families are split real Lie groups and Lie groups of Hermitian types, for which one can define Hitchin representations and maximal representations, respectively. The definition of positivity from Guichard and Wienhard generalizes the notion of positivity in split real Lie groups introduced by Lusztig [Lus94].

Definition 2.31. A representation $\rho\colon \pi_1(S) \to G$ is *θ-positive* if there exists a continuous ρ-equivariant map $\zeta\colon \partial_\infty \tilde{S} \to G/P_\theta^+$ which sends positive triples to positive triples.

The fact that Hitchin representations and maximal representations are positive for some θ is proven in [FG06, Theorem 1.15] and [BIW10, Theorem 8], respectively. In unpublished work, Guichard, Labourie and Wienhard [GLW16] show that θ-positive representations are θ-Anosov and that the subset of θ-positive representations is open in $\mathrm{Hom}(\pi_1(S), G)$. Further, they conjecture that it is also closed, so θ-positive representations form a union of connected components of $\mathrm{Hom}(\pi_1(S), G)$.

3. Laminations and transverse twisted cycles

In this chapter we introduce two concepts that are important for the definition of cataclysm deformations. The first one is a so-called lamination, which is a fixed topological object on the surface S. Recall that S is a closed connected oriented surface of genus at least 2. The second one is called twisted cycle and serves as a parameter for the cataclysm deformation. It determines how much the cataclysm changes a representation ρ.

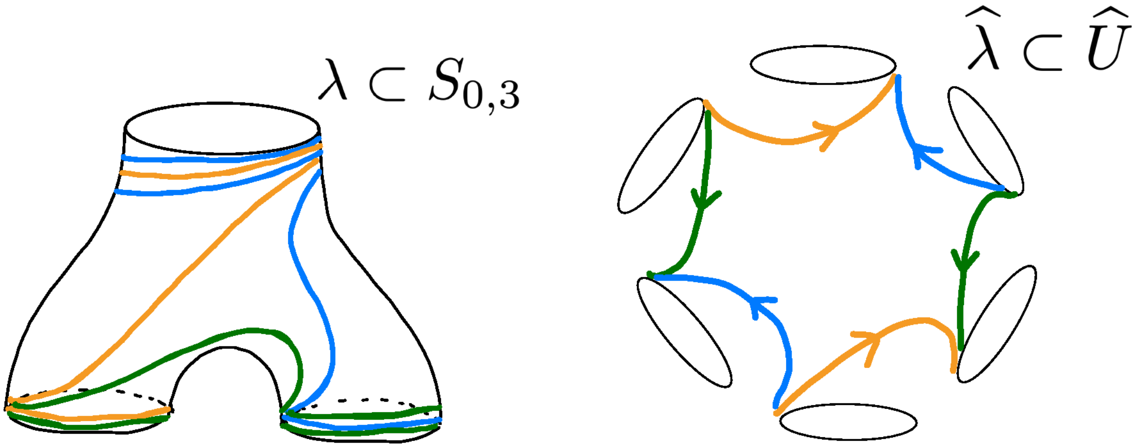
3.1. Geodesic laminations

Geodesic laminations have been introduced by Thurston and are powerful tool in low-dimensional topology and geometry. In this section we define the concepts that are important for the construction of cataclysms and give examples and properties. An overview of geodesic laminations can be found in [Bon01].

Definition 3.1. A *geodesic lamination* λ on S is a collection of simple complete disjoint geodesics such that their union is a closed subset of S. The geodesics contained in the lamination λ are called *leaves*. A geodesic lamination λ is *maximal* if every connected component of $S \setminus \lambda$ is isometric to an ideal triangle. We denote by $\tilde{\lambda}$ the lift of λ to the universal cover $\tilde{S}$.

Although Definition 3.1 makes use of the auxiliary hyperbolic metric m on S, geodesic laminations can be defined independently on the choice of a metric (see [Bon97, Section 3]).

Example 3.2. A basic example for a geodesic lamination is just one closed simple geodesic on S. More generally, every collection of simple closed disjoint geodesics is a geodesic lamination. A pair of pants decomposition, for example, is a geodesic lamination consisting of $3 \cdot g(S) - 3$ closed leaves, where $g(S)$ is the genus of the surface S.



(a) A maximal lamination λ on $S_{0,3}$

(b) The orientation cover $\widehat{\lambda}$ of λ

Figure 3.1.: On a pair of pants $S_{0,3}$, a maximal lamination λ consists of three bi-infinite geodesics spiraling pairwise to the boundary curves. The orientation cover $\widehat{\lambda}$ consists of six bi-infinite geodesics, two lifts for every geodesic in λ. Here, the ambient surface $\widehat{U}$ is not drawn for the sake of clarity.

Example 3.3. On a pair of pants $S_{0,3}$ we can consider three bi-infinite disjoint geodesics that pairwise spiral towards the boundary components of the pants (see Figure 3.1a). The complement of these geodesics consists of two ideal triangles. If we have a pair of pants decomposition of S, we can construct a maximal lamination by subdividing every pants using three bi-infinite geodesics (see Figure 3.2a).

Remark 3.4. For a maximal lamination λ, the complement $S \setminus \lambda$ always consists of $4 \cdot g(S) - 4$ ideal triangles, where $g(S)$ denotes the genus of the surface S. This follows from the fact that every ideal triangle has area π and that by the Gauss-Bonnet theorem, a closed surface S has area $2\pi\,(2 \cdot g(S) - 2)$. Consequently, if λ has only finitely many leaves, the number of bi-infinite leaves, i.e. leaves that are in the boundary of an ideal triangle, is $6 \cdot g(S) - 6$. The number s of closed leaves lies between 1 and $3 \cdot g(S) - 3$. Generically, a lamination is more complicated and consists of infinitely many leaves. For examples for generic laminations, see [Bon01, p.4 and p.9].

We can equip the geodesics of a lamination λ with an orientation – but if the lamination is maximal, this cannot be done in a continuous way. For this reason, we look at the *orientation cover* $\widehat{\lambda}$ of λ.

Definition 3.5. The *orientation cover* $\widehat{\lambda}$ of a lamination λ is a 2-cover of λ whose geodesics are oriented in a continuous fashion. More precisely, $\widehat{\lambda}$ consists of all pairs (x, o) where $x \in \lambda$ and o is a continuous orientation of the leaves of λ near x.

For example, if λ is orientable, then $\widehat{\lambda}$ is consists of two disjoint copies of λ with opposite orientations. In order to consider arcs transverse to the lamination $\widehat{\lambda}$, we need an ambient surface $\widehat{U}$ in which $\widehat{\lambda}$ is embedded. Let $U \subset S$ be an open neighborhood of λ that avoids at least one point in the interior of each ideal triangle in $S \setminus \lambda$. The orientation cover $\widehat{\lambda} \to \lambda$ extends to a cover $\widehat{U} \to U$. On $\widehat{U}$, we can consider the orientation reversing involution $\mathfrak{R}\colon \widehat{U} \to \widehat{U}$ that interchanges the two sheets of the cover.

Example 3.6. If λ consists of m disjoint simple closed geodesics $c_1, \dots, c_m$, then it is orientable. Its orientation cover consists of $2m$ disjoint simple closed geodesics $c_1^+, \dots, c_m^+$, $c_1^-, \dots, c_m^-$, where $c_i^\pm$ are the two lifts of c_i with opposite orientations.

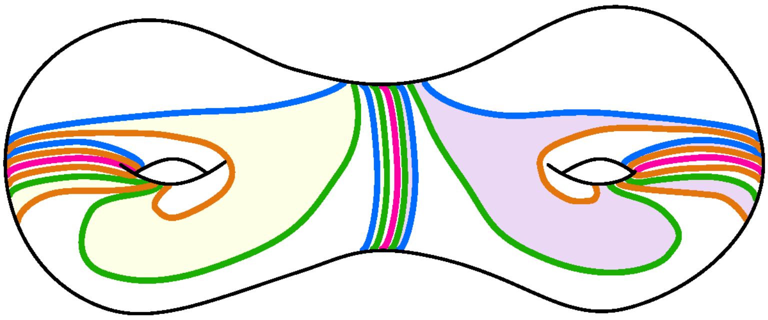
Example 3.7. Let P be a pair of pants. Consider a maximal lamination λ on P, consisting of three bi-infinite geodesics that pairwise spiral towards the boundary curves (see Figure 3.1a). It is not possible to orient the bi-infinite geodesics in a continuous way. The orientation cover $\widehat{\lambda}$ consists of six bi-infinite geodesics, two lifts for every geodesics g_i in λ. They spiral pairwise to lifts of the boundary curves, and can be oriented continuously as sketched in Figure 3.1a.

Example 3.8. An example for a finite maximal lamination λ on a surface S of genus 2 is sketched in Figure 3.2a. It is obtained from a pair of pants decomposition of S, where every pants is divided into two ideal triangles by three infinite geodesics that spiral towards the closed leaves. The complements $S \setminus \lambda$ consists of four ideal triangles. Figure 3.2b shows a sketch of the orientation cover $\widehat{\lambda}$ with a choice of orientation of the leaves. The orientation cover $\widehat{\lambda}$ is drawn without the ambient surface $\widehat{U}$ to keep the picture clean. Figure 3.2c shows a sketch of the lift $\tilde{\lambda}$ of λ to the universal cover $\tilde{S}$.

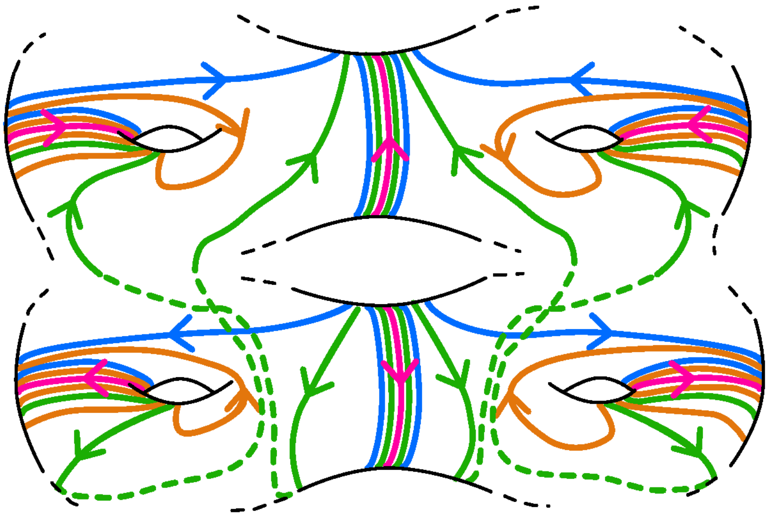
We will be interested in arcs transverse to a lamination λ, that are well-behaved in the following sense.

Definition 3.9. A arc k is *tightly transverse* to λ if it is simple, compact, transverse to λ, has non-empty intersection with λ, and every connected component $d \subset k \setminus \lambda$ either contains an endpoint of k, or separates one of the ideal vertices of the ideal triangle $P \subset S \setminus \lambda$ containing d from the other two vertices. We use the same notation for arcs $\tilde{k}$ transverse to the universal cover $\tilde{\lambda}$ or to the orientation cover $\widehat{\lambda}$ of λ.

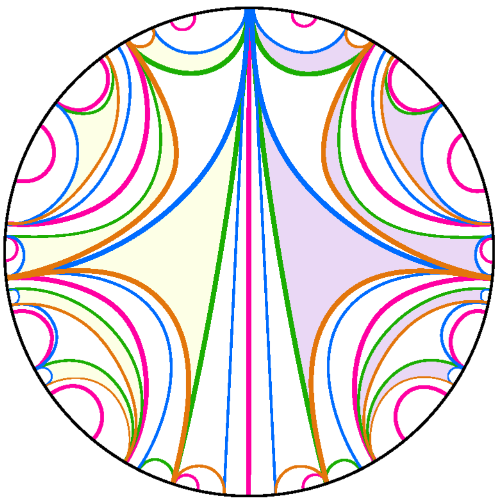
Notation 3.10. Throughout the paper, we use the following notation: By λ, we denote a lamination on S, by $\tilde{\lambda}$ its lift to the universal cover $\tilde{S}$ and by $\widehat{\lambda}$ the orientation cover. We denote by k an oriented arc tightly transverse to λ, by $\tilde{k}$ an oriented an arc tightly transverse to $\tilde{\lambda}$ and by $\widehat{k}$ an oriented arc tightly transverse to $\widehat{\lambda}$. Since the universal cover $\tilde{S}$ is oriented, the leaves of $\tilde{\lambda}$ intersecting $\tilde{k}$ have a well-defined transverse orientation determined by $\tilde{k}$, i.e. if g is a leaf of $\tilde{\lambda}$ intersecting $\tilde{k}$, we orient g such that the intersection $\tilde{k} \pitchfork g$ is positive. Denote by P and Q the connected components of $\tilde{k}$ containing the negative and



(a) The lamination λ



(b) The orientation cover $\widehat{\lambda}$



(c) The universal cover $\tilde{\lambda}$ in $\tilde{S}$

Figure 3.2.: A finite maximal lamination on S that is subordinate to a pair of pants decomposition and sketches of its universal cover $\tilde{\lambda}$ and orientation cover $\widehat{\lambda}$. The regions in yellow and purple highlight connected components in the complement of the lamination. For the orientation cover, details are omitted for the sake of clarity.

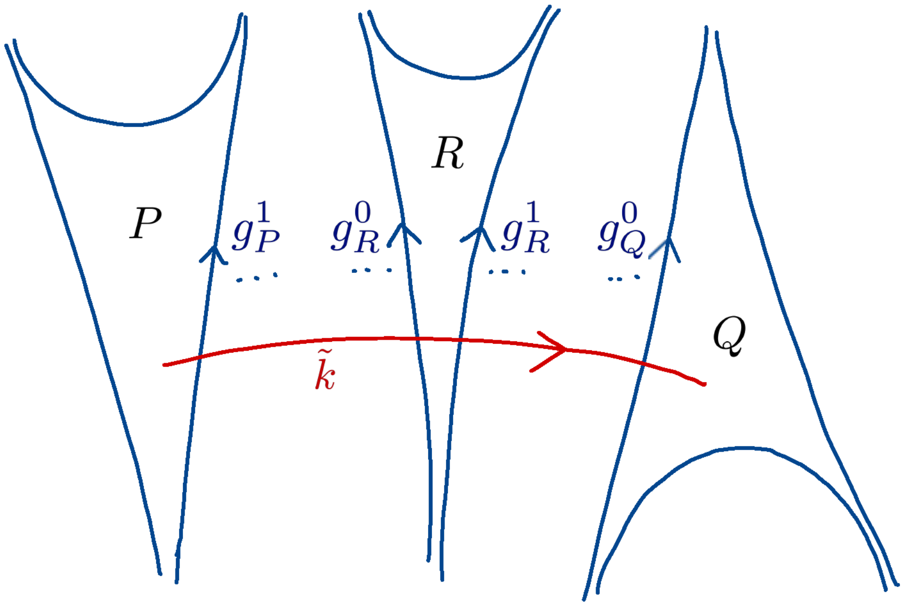


Figure 3.3.: For an arc $\tilde{k}$ tightly transverse to $\tilde{\lambda}$ and an ideal triangle $R \subset \tilde{S} \setminus \tilde{\lambda}$ such that $\tilde{k} \cap R \neq \emptyset$, denote by g_R^0, and g_R^1 the oriented geodesics in $\tilde{\lambda}$ passing through the negative and positive endpoint of $\tilde{k} \cap R$, respectively.

positive endpoint of $\tilde{k}$, respectively. Let $\mathcal{C}_{PQ}$ be the set of connected components of $\tilde{S} \setminus \tilde{\lambda}$ separating P and Q. Likewise, for two geodesics g and h in $\tilde{\lambda}$, let $\mathcal{C}_{gh}$ be the set of all connected components in $\tilde{S} \setminus \tilde{\lambda}$ lying between g and h. If $R \subset \tilde{S} \setminus \tilde{\lambda}$ is in $\mathcal{C}_{PQ}$, denote by g_R^0 and g_R^1 the oriented geodesics passing through the negative and positive endpoint of $\tilde{k} \cap R$, respectively (Figure 3.3). The fact that $\tilde{k}$ is tightly transverse guarantees that the geodesics g_R^0 and g_R^1 are disjoint.

Standing Assumption. Throughout the thesis, we will assume that all arcs k, $\tilde{k}$, $\widehat{k}$ are tightly transverse.

The following is classical property of geodesic laminations. It will be important later when proving existence of certain maps that are defined by a limit.

Lemma 3.11 ([BD17, Lemma 5.3])**.** *Let g and h be two geodesics in $\tilde{\lambda}$. There is a function $r\colon \mathcal{C}_{gh} \to \mathbb{N}$, called* divergence radius*, and constants $C_1, C_2, A_1, A_2 > 0$ such that the following conditions hold:*

1. $C_1 e^{-A_1 r(R)} \leq \ell(\tilde{k} \cap R) \leq C_2 e^{-A_2 r(R)}$ *for every $R \in \mathcal{C}_{gh}$;*
2. *for every $N \in \mathbb{N}$, the number of triangles $R \in \mathcal{C}_{gh}$ with $r(R) = N$ is uniformly bounded, independent of N.*

Here, ℓ denotes the length function on $\tilde{S}$ induced by the fixed metric on S.

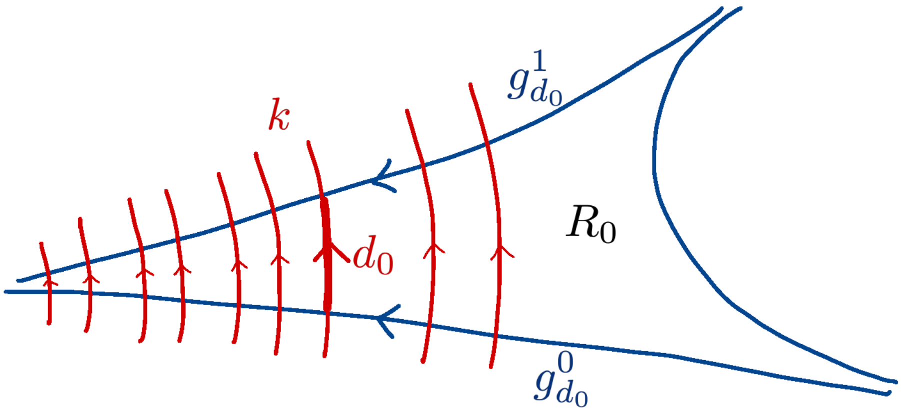


Figure 3.4.: This picture shows the situation on the surface S. The compact arc k crosses the connected component R_0 of $S \setminus \lambda$ several times. The arc $d_0 \subset k \setminus \lambda$ divides the component R_0 into two regions. The divergence radius $r(d_0)$ is the minimal number of components of $d \subset k \setminus \lambda$ that have their endpoints on the same geodesics $g^0_{d_0}$ and $g^1_{d_0}$ as d_0. In this picture, we have $r(d_0) = 2$.

For $R \in \mathcal{C}_{gh}$, the divergence radius $r(R)$ is defined the projection of the arc $\tilde{k}$ to the surface S. So first, let k be an arc in S transverse to λ and $d_0 \subset k \setminus \lambda$ a connected component. We define the divergence radius $r(d_0)$.. Let $g^0_{d_0}, g^1_{d_0}$ be the leaves of λ passing through the endpoints of d_0 (see Figure 3.4). Let $R_0 \subset S \setminus \lambda$ be the connected component containing d_0. The arc d_0 divides R_0 into two regions. Consider all components $d \subset k \setminus \lambda$ that have their negative endpoint on $g^0_{d_0}$ and their positive endpoints on $g^1_{d_0}$. Since k is compact, at least one of the two regions of R_0 defined by d_0 contains only finitely many such components $d \subset k \setminus \lambda$. Define $r(d_0) \in \mathbb{N}$ as the minimal number of components of $k \setminus \lambda$ contained in one of the regions that have their endpoints on $g^0_{d_0}$ and $g^1_{d_0}$. Now for $R \in \mathcal{C}_{gh}$, define $r(R) := r(d)$, where $d \subset k \setminus \lambda$ is the projection of $\tilde{k} \cap R$ to S.

Example 3.12. If $\tilde{k}$ meets exactly one lift R of R_0, i.e. if k crosses the projection R_0 of R to S only once, then $r(R) = 0$. If $\tilde{k}$ meets R and R' that are lifts of the same component R_0, and if $\tilde{k}$ does not meet any other lift of R_0 between R and R', then $|r(R) - r(R')| \leq 1$.

For the rest of this section, **assume that the lamination λ is maximal and finite**. In particular, all components in $\tilde{S} \setminus \tilde{\lambda}$ are ideal triangles and for every $R \subset \mathcal{C}_{gh}$ there exist unique components R^- and R^+ of $\tilde{S} \setminus \tilde{\lambda}$ that are adjacent to R and lie before and after R along $\tilde{k}$, respectively.

Definition 3.13. A component $R \in \mathcal{C}_{gh}$ is an *inner component* if the components R^- and R^+ that are adjacent to R and lie along $\tilde{k}$ before and after R, respectively, do not contain an endpoint of $\tilde{k}$ (see Figure 3.5). In other words, R^+, R^- are elements of $\mathcal{C}_{gh}$. An inner component $R \subset \mathcal{C}_{gh}$ is *pinched* if the geodesics $g^0_{R^-}, g^0_R, g^1_R$ and $g^1_{R^+}$ all have an endpoint in common.

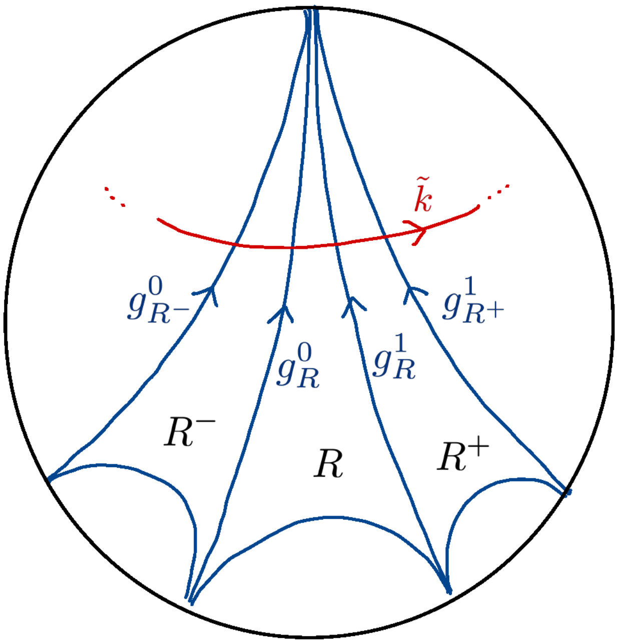


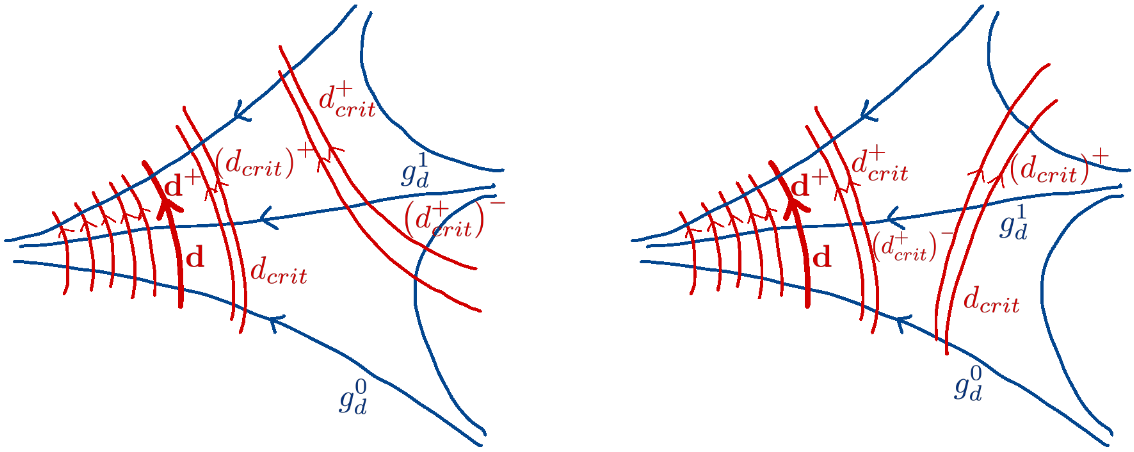
Figure 3.5.: An element $R \in \mathcal{C}_{gh}$ is an inner component if its neighboring components R^- and R^+ of $\tilde{S} \setminus \tilde{\lambda}$ along $\tilde{k}$ do not contain an endpoint of $\tilde{k}$. The component R in this picture is in addition pinched, i.e. the geodesics $g^0_{R^-}, g^0_R, g^1_R$ and $g^1_{R^+}$ all have a common endpoint.

Note that all but two components in $\mathcal{C}_{gh}$ are inner components. Further, because we consider a finite lamination, all but finitely many components in $\mathcal{C}_{gh}$ are pinched.

If $R \in \mathcal{C}_{gh}$ is a pinched inner component, we can relate the divergence radii $r(R)$ and $r(R^+)$ by a constant depending on $\tilde{k}$.

Lemma 3.14. *If $R \in \mathcal{C}_{gh}$ is an inner component and R^+, R^- are its neighboring components as in Definition 3.13, then $r(R)$ and $r(R^+)$ differ by a constant depending on the arc $\tilde{k}$. The same holds for R^-.*

Proof. By definition of r, is suffices to look at the situation on the surface S, and to show that $r(d)$ and $r(d^+)$ are bounded by a constant depending on k, where k is the projection of $\tilde{k}$ to S and d and d^+ are the projections of $\tilde{k} \cap R$ and $\tilde{k} \cap R^+$ to S, respectively. Let R_0 and R_0^+ be the components of $S \setminus \lambda$ that are the projections of R and R^+ to S, respectively. Let $d_{\text{crit}} \subset k \cap R_0$ be the component of $k \setminus \tilde{\lambda}$ with endpoints on g^0_d and g^1_d and such that $r(d_{\text{crit}}) = 0$, i.e. one of the two regions of $R_0 \setminus d_{\text{crit}}$ does not contain any other component of $k \setminus \lambda$ with endpoints on the same geodesics as d_{crit} (see Figure 3.6). Let $(d_{\text{crit}})^+$ be the connected component of $k \setminus \lambda$ in R_0^+ right after d_{crit}. It has its negative endpoint on g^1_d and its positive endpoint either on $g^1_{d^+}$ or on the other geodesic bounding R_0^+. In the



(a) The case where $r(d^+) \geq r(d)$. (b) The case where $r(d^+) \leq r(d)$.

Figure 3.6.: The critical component $d_{crit} \subset k \setminus \lambda$ has its endpoints on the same geodesics g_d^0 and g_d^1 as d. The component $(d_{crit})^+$ either has its positive endpoint on the same geodesics as d^+, in which case $r(d^+) \geq r(d)$, or on another geodesic, in which case $r(d^+) \leq r(d)$. In these pictures, $|r(d) - r(d^+)| = 2$ in both cases.

first case, $r(d^+) \geq r(d)$ as in Figure 3.6a, in the second case, $r(d^+) \leq r(d)$ as shown in Figure 3.6b. If $r(d^+) \geq r(d)$, then $r(d^+) = r(d) + r((d_{\text{crit}})^+)$. Similarly, if $r(d^+) \leq r(d)$, then $r(d^+) = r(d) - r((d^+_{\text{crit}})^-)$, where d^+_{crit} for R_0^+ is the critical component for R_0^+ and $(d^+_{\text{crit}})^- \subset k \setminus \lambda$ denotes the component that lies in R_0 and lies before d^+_{crit} when traveling along k. Thus, $|r(R^+) - r(R)| \leq \max\{r((d_{\text{crit}})^+), r((d^+_{\text{crit}})^-)\}$. The values $r((d_{\text{crit}})^+)$ and $r((d^+_{\text{crit}})^-)$ only depend on k, R_0 and on the geodesics, g_d^0, g_d^1, but not on d. Since $S \setminus \lambda$ has only finitely many components, it follows that $\max\{r((d_{\text{crit}})^+), r((d^+_{\text{crit}})^-)\}$ is bounded by a constant depending on k. In total, this shows that the divergence radii $r(d)$ and $r(d^+)$ differ at most by a constant depending on k, so also $r(R)$ and $r(R^+)$ differ at most by a constant depending on $\tilde{k}$. The proof for R^- works analogous.

$\square$

3.2. Transverse twisted cycles

In the following, we use the notation introduced in Section 2.1. We fix a geodesic lamination λ on S, not necessarily maximal. The following definition is taken from [Bon96, Section 1].

Definition 3.15. An *$\mathfrak{a}_\theta$-valued transverse cycle for* λ is a map associating to each unoriented arc k transverse to λ an element $\varepsilon(k) \in \mathfrak{a}_\theta$, which satisfies the following properties:

1. ε is finitely additive, i.e. $\varepsilon(k) = \varepsilon(k_1) + \varepsilon(k_2)$ if we split k in two subarcs k_1, k_2 with disjoint interiors.
2. ε is λ-invariant, i.e. $\varepsilon(k) = \varepsilon(k')$ whenever the arcs k and k' are homotopic via a homotopy respecting the lamination λ.

We denote the vector space of $\mathfrak{a}_\theta$-valued transverse cycles for λ by $\mathcal{H}(\lambda; \mathfrak{a}_\theta)$.

In the same way, one can define $\mathbb{R}$-valued transverse cycles and transverse cycles for the orientation cover $\widehat{\lambda}$.

Remark 3.16. We will mostly be interested in transverse cycles for the orientation cover $\widehat{\lambda}$, i.e. elements in $\mathcal{H}(\widehat{\lambda}; \mathfrak{a}_\theta)$. This is because cycles for $\widehat{\lambda}$ allow us to assign values to *oriented* arcs transverse to the lamination λ, rather than unoriented arcs as for cycles for the lamination λ. The correspondence between oriented arcs transverse to λ and arcs transverse to $\widehat{\lambda}$ works as follows: If k is an oriented arc transverse to λ, then there exists a unique lift $\widehat{k}$ of k to $\widehat{U}$ such that the intersection $\widehat{k} \pitchfork \widehat{\lambda}$ is positive (see Figure 3.7). This gives a one-to-one correspondence between oriented arcs transverse to λ and unoriented arcs transverse to $\widehat{\lambda}$. If $\varepsilon \in \mathcal{H}(\lambda; \mathfrak{a}_\theta)$ is a transverse cycle and k an oriented arc, denote by $\overline{k}$ the arc with opposite orientation. Then, since ε does not take into account the orientation of k, $\varepsilon(k) = \varepsilon(\overline{k})$. If, in contrast, we consider the corresponding arcs in the orientation cover, then $\widehat{\overline{k}} = \mathfrak{R}(\widehat{k})$, where $\mathfrak{R}\colon \widehat{U} \to \widehat{U}$ is the orientation reversing involution. So for $\widehat{\varepsilon} \in \mathcal{H}(\widehat{\lambda}; \mathfrak{a}_\theta)$, we have $\widehat{\varepsilon}(\widehat{\overline{k}}) = \widehat{\varepsilon}(\mathfrak{R}(\widehat{k})) \neq \widehat{\varepsilon}(\widehat{k})$ in general.

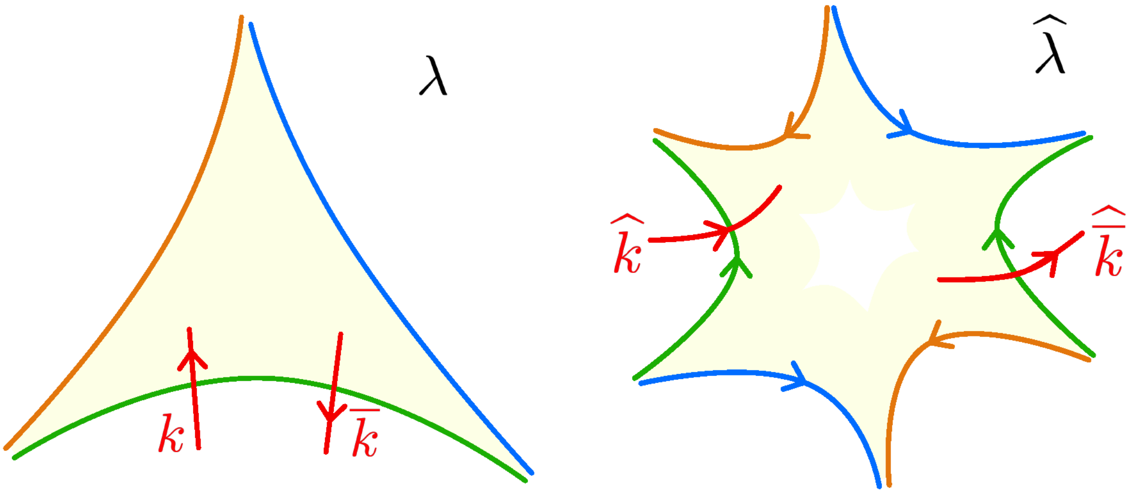
Remark 3.17. If $\{H_\alpha\}_{\alpha \in \theta}$ is the basis for $\mathfrak{a}_\theta$ introduced in Remark 2.6, we can express $\varepsilon \in \mathcal{H}(\widehat{\lambda}; \mathfrak{a}_\theta)$ as

$$\varepsilon = \sum_{\alpha \in \theta} \varepsilon_\alpha H_\alpha, \tag{3.1}$$

where for all $\alpha \in \theta$, $\varepsilon_\alpha \in \mathcal{H}(\widehat{\lambda}; \mathbb{R})$ is an $\mathbb{R}$-valued transverse cycle.

Remark 3.18. There are different viewpoints on transverse cycles which are useful to work with in different contexts. One of them is by using *train tracks*: Fixing a sufficiently nice train track τ that carries the lamination λ, there is a one-to-one correspondence between transverse cycles for λ and weights on the train track τ that satisfy the so-called switch relations. For details on the different viewpoints on transverse cycles, see [Bon97].

In the following, if not stated otherwise, a *transverse cycle* will always be an $\mathfrak{a}_\theta$-valued transverse cycle. A lamination λ is *connected* if it cannot be written as a disjoint union of two sublaminations, i.e. subsets that are themselves laminations. Using train tracks, one can show:



(a) Oriented arcs transverse to λ.

(b) Lifts of oriented arcs transverse to $\widehat{\lambda}$.

Figure 3.7.: For an oriented arc k transverse to the lamination λ that intersects a leaf, drawn in green here, there exists a unique lift $\widehat{\lambda}$ transverse to $\widehat{\lambda}$ such that the intersection $\widehat{k} \pitchfork \widehat{\lambda}$ is positive. If $\overline{k}$ is the same arc with opposite orientation, then its lift $\widehat{\overline{k}}$ intersects the other lift of the leaf.

Proposition 3.19 ([Bon96, Proposition 1] and [Bon97, Theorem 15])**.** *The vector space $\mathcal{H}(\lambda; \mathfrak{a}_\theta)$ is isomorphic to $\mathfrak{a}_\theta^{-\chi(\lambda)+n_o(\lambda)}$, where $n_o(\lambda)$ denotes the number of connected components of λ that are orientable.*

Here, $\chi(\lambda)$ denotes the Euler characteristic of a geodesic lamination, which is the same as the Euler characteristic of a train track for λ (see [Bon97, Section 5]). For a maximal lamination λ, it satisfies $\chi(\lambda) = 3\chi(S)$, and for the orientation cover $\widehat{\lambda}$ of a maximal lamination, we have $\chi(\widehat{\lambda}) = 6\chi(S)$. In particular, if λ is maximal, the space $\mathcal{H}(\lambda; \mathfrak{a}_\theta)$ is isomorphic to $\mathfrak{a}_\theta^{6g(S)-6}$ and $\mathcal{H}(\widehat{\lambda}; \mathfrak{a}_\theta)$ is isomorphic to $\mathfrak{a}_\theta^{12g(S)-11}$.

The orientation reversing involution $\mathfrak{R}\colon \widehat{U} \to \widehat{U}$ on the orientation cover induces a pullback endomorphism $\mathfrak{R}^*\colon \mathcal{H}(\widehat{\lambda}; \mathfrak{a}_\theta) \to \mathcal{H}(\widehat{\lambda}; \mathfrak{a}_\theta)$ by $\mathfrak{R}^*\varepsilon(\widehat{k}) := \varepsilon(\mathfrak{R}(\widehat{k}))$ for $\varepsilon \in \mathcal{H}(\widehat{\lambda}; \mathfrak{a}_\theta)$. We require the transverse cycles to satisfy the following twist condition.

Definition 3.20. The space of *$\mathfrak{a}_\theta$-valued transverse twisted cycles* for the orientation cover $\widehat{\lambda}$ is

$$\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta) := \{ \ \varepsilon \in \mathcal{H}(\widehat{\lambda}; \mathfrak{a}_\theta) \mid \mathfrak{R}^*\varepsilon = \iota(\varepsilon) \ \}, \tag{3.2}$$

where $\iota\colon \mathfrak{a}_\theta \to \mathfrak{a}_\theta$ is the opposition involution (see Definition 2.4).

Here, we have $\iota(\mathfrak{a}_\theta) = \mathfrak{a}_\theta$ since $\iota(\theta) = \theta$ by assumption. The motivation for this special twist condition will become apparent in Section 5.1, when we use transverse cycles as parameters for the shearing map (see Remark 5.5).

We now calculate the dimension of the space of twisted cycles. Let $\theta' \subset \theta$ be a maximal subset satisfying $\theta' \cap \iota(\theta') = \emptyset$. For example, if ι is trivial, then θ' is empty. If $G = \mathrm{SL}(n, \mathbb{R})$ and $\theta = \Delta$, then θ' can be chosen to be $\{\alpha_1, \ldots, \alpha_{\lfloor \frac{n}{2} \rfloor}\}$. The subset θ' is not unique, but its cardinality $|\theta'|$ is independent on a choice.

Proposition 3.21. *Let G be a connected non-compact semisimple real Lie group, $\theta \subset \Delta$ and let $\theta' \subset \theta$ be a maximal subset satisfying $\iota(\theta') \cap \theta' = \emptyset$. Then*

$$\dim \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta) = |\theta| \left(-\chi(\lambda) + n_o(\lambda)\right) + |\theta'| \left(n(\lambda) - n_o(\lambda)\right),$$

where $\chi(\lambda)$ is the Euler characteristic of λ, $n(\lambda)$ is the number of connected components of λ and $n_o(\lambda)$ is the number of components that are orientable.

Proof. Using (3.1), we can write $\varepsilon \in \mathcal{H}(\widehat{\lambda}; \mathfrak{a}_\theta)$ as $\varepsilon = \sum \varepsilon_\alpha H_\alpha$ with $\varepsilon_\alpha \in \mathcal{H}(\widehat{\lambda}; \mathbb{R})$. Since the pullback endomorphism $\mathfrak{R}^* \colon \mathcal{H}(\widehat{\lambda}; \mathbb{R}) \to \mathcal{H}(\widehat{\lambda}; \mathbb{R})$ is an involution, its eigenvalues are ± 1 and we can write the vector space $\mathcal{H}(\widehat{\lambda}; \mathbb{R})$ as direct sum $\mathcal{H}(\widehat{\lambda}; \mathbb{R})^+ \oplus \mathcal{H}(\widehat{\lambda}; \mathbb{R})^-$, where $\mathcal{H}(\widehat{\lambda}; \mathbb{R})^\pm$ is the (± 1)-eigenspace with respect to $\mathfrak{R}^*$. We write every $\varepsilon_\alpha \in \mathcal{H}(\widehat{\lambda}; \mathbb{R})$ as $\varepsilon_\alpha = \varepsilon_\alpha^+ + \varepsilon_\alpha^-$, where ε_α^+ and ε_α^- lie in $\mathcal{H}(\widehat{\lambda}; \mathbb{R})^+$ and $\mathcal{H}(\widehat{\lambda}; \mathbb{R})^-$, respectively. By definition of $\varepsilon_\alpha^\pm$, we have

$$\mathfrak{R}^* \varepsilon = \sum_{\alpha \in \theta} (\mathfrak{R}^* \varepsilon_\alpha^+ + \mathfrak{R}^* \varepsilon_\alpha^-) H_\alpha = \sum_{\alpha \in \theta} (\varepsilon_\alpha^+ - \varepsilon_\alpha^-) H_\alpha.$$

Let $\theta' \subset \theta$ be as in the Proposition and let $\mathrm{Fix}(\iota) \subset \Delta$ denote the set of fixed points of ι. Then we have $\theta = (\mathrm{Fix}(\iota) \cap \theta) \dot\cup \theta' \dot\cup \iota(\theta')$. If we apply the opposition involution ι to ε, we obtain

$$\begin{aligned}
\iota(\varepsilon) &= \sum_{\alpha \in \theta} (\varepsilon_\alpha^+ + \varepsilon_\alpha^-) \iota(H_\alpha) = \sum_{\alpha \in \theta} (\varepsilon_\alpha^+ + \varepsilon_\alpha^-) H_{\iota(\alpha)} = \sum_{\alpha \in \theta} (\varepsilon_{\iota(\alpha)}^+ + \varepsilon_{\iota(\alpha)}^-) H_\alpha \\
&= \sum_{\alpha \in \mathrm{Fix}(\iota) \cap \theta} (\varepsilon_\alpha^+ + \varepsilon_\alpha^-) H_\alpha + \sum_{\alpha \in \theta'} (\varepsilon_{\iota(\alpha)}^+ + \varepsilon_{\iota(\alpha)}^-) H_\alpha + \sum_{\alpha \in \theta'} (\varepsilon_\alpha^+ + \varepsilon_\alpha^-) H_{\iota(\alpha)}.
\end{aligned}$$

By the two equations above, the twist condition $\mathfrak{R}^* \varepsilon = \iota(\varepsilon)$ becomes

$$\begin{aligned}
\varepsilon_\alpha^- &= -\varepsilon_\alpha^- \quad \forall \alpha \in \mathrm{Fix}(\iota) \cap \theta, \\
\varepsilon_\alpha^+ &= \varepsilon_{\iota(\alpha)}^+ \quad \forall \alpha \in \theta' \text{ and} \\
\varepsilon_\alpha^- &= -\varepsilon_{\iota(\alpha)}^- \quad \forall \alpha \in \theta'.
\end{aligned}$$

Thus, it follows that $\varepsilon_\alpha^- = 0$ for all $\alpha \in \mathrm{Fix}(\iota) \cap \theta$ and hence

$$\dim \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta) = \big(|\mathrm{Fix}(\iota) \cap \theta| + |\theta'|\big) \cdot \dim \mathcal{H}(\widehat{\lambda}; \mathbb{R})^+ + |\theta'| \cdot \dim \mathcal{H}(\widehat{\lambda}; \mathbb{R})^-.$$

Using the one-to-one correspondence between arcs transverse to $\widehat{\lambda}$ and oriented arcs transverse to λ, we can identify the subspace $\mathcal{H}(\widehat{\lambda}; \mathbb{R})^+$ with the space $\mathcal{H}(\lambda; \mathbb{R})$ of transverse cycles for the lamination λ. We know from Proposition 3.19 that $\mathcal{H}(\widehat{\lambda}; \mathbb{R})$ has dimension $-\chi(\widehat{\lambda}) + n_o(\widehat{\lambda}) = -2\chi(\lambda) + n(\lambda) + n_o(\lambda)$ and $\mathcal{H}(\widehat{\lambda}; \mathbb{R})^+$ has dimension $-\chi(\lambda) + n_o(\lambda)$. Thus,

$$\begin{aligned}
\dim \mathcal{H}(\widehat{\lambda}; \mathbb{R})^- &= \dim \mathcal{H}(\widehat{\lambda}; \mathbb{R}) - \dim \mathcal{H}(\widehat{\lambda}; \mathbb{R})^+ \\
&= (-2\chi(\lambda) + n(\lambda) + n_o(\lambda)) - (-\chi(\lambda) + n_o(\lambda)) \\
&= -\chi(\lambda) + n(\lambda).
\end{aligned}$$

Combining this with the fact that $|\mathrm{Fix}(\iota) \cap \theta| + |\theta'| = |\theta| - |\theta'|$, we obtain

$$\begin{aligned}
\dim \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta) &= \big(|\theta| - |\theta'|\big) \cdot (-\chi(\lambda) + n_o(\lambda)) + |\theta'| \cdot (-\chi(\lambda) + n(\lambda)) \\
&= |\theta| \left(-\chi(\lambda) + n_o(\lambda)\right) + |\theta'| \left(n(\lambda) - n_o(\lambda)\right),
\end{aligned}$$

which finishes the proof. □

Corollary 3.22. *If the lamination λ is maximal, we have*

$$\dim \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta) = |\theta|(6 \cdot g(S) - 6) + |\theta'|.$$

Proof. This follows from Proposition 3.21 using the fact that a maximal lamination is connected and not orientable, so $n(\lambda) = 1$ and $n_o(\lambda) = 0$, and that $\chi(\lambda) = 3\chi(S) = 6 - 6 \cdot g(S)$. □

For the case $G = \mathrm{SL}(n, \mathbb{R})$ and $\theta = \Delta$, i.e. when $P_\theta = P_\Delta$ is the minimal parabolic, Corollary 3.22 is exactly Lemma 16 in [Dre13].

We conclude this section with an estimate for transverse cycles. To formulate it, let $\tilde{k}$ be an oriented arc transverse to $\tilde{\lambda}$, k its projection to S and let $\widehat{k}$ be the unique lift of k to $\widehat{U}$ as in Remark 3.16. Let $\|\cdot\|_{\mathfrak{a}_\theta}$ be the maximum norm with respect to the basis $\{H_\alpha\}_{\alpha \in \theta}$ from Remark 2.6, i.e. if $X = \sum_{\alpha \in \theta} x_\alpha H_\alpha$ with $x_\alpha \in \mathbb{R}$, then $\|X\|_{\mathfrak{a}_\theta} = \max_{\alpha \in \theta} |x_\alpha|$. Further, on $\mathcal{H}(\widehat{\lambda}; \mathfrak{a}_\theta)$ consider the norm $\|\varepsilon\|_{\mathcal{H}(\widehat{\lambda}; \mathfrak{a}_\theta)} = \max_{\alpha \in \theta} \|\varepsilon_\alpha\|_{\mathcal{H}(\widehat{\lambda}; \mathbb{R})}$ for a transverse cycle $\varepsilon = \sum_{\alpha \in \theta} \varepsilon_\alpha H_\alpha$ for some fixed norm on $\mathcal{H}(\widehat{\lambda}; \mathbb{R})$.

Lemma 3.23. *There exists some constant $C > 0$, depending on $\tilde{k}$, such that for every transverse cycle $\varepsilon \in \mathcal{H}(\widehat{\lambda}; \mathfrak{a}_\theta)$, for every $R \in \mathcal{C}_{PQ}$,*

$$\left\| \varepsilon(\widehat{k}_R) \right\|_{\mathfrak{a}_\theta} \leq C \left\| \varepsilon \right\|_{\mathcal{H}(\widehat{\lambda};\mathfrak{a}_\theta)} (r(R) + 1),$$

where $r(R)$ is the divergence radius, $\tilde{k}_R$ is a subarc of $\tilde{k}$ joining the negative endpoint of $\tilde{k}$ to an arbitrary point contained in R and $\widehat{k}_R$ is the corresponding arc in $\widehat{U}$.

Proof. In [Bon96, Lemma 6], the statement is proven for $\mathbb{R}$-valued transverse cycles. For the more general case of $\mathfrak{a}_\theta$-valued transverse cycles, let $\varepsilon = \sum_{\alpha\in\theta} \varepsilon_\alpha H_\alpha$. Then, with [Bon96, Lemma 6],

$$\begin{aligned}
\left\| \varepsilon(\widehat{k}_d) \right\|_{\mathfrak{a}_\theta} &= \max_{\alpha\in\theta} \left| \varepsilon_\alpha(\widehat{k}_d) \right| \\
&\leq \max_{\alpha\in\theta} \Big(C \left\| \varepsilon_\alpha \right\|_{\mathcal{H}(\widehat{\lambda};\mathbb{R})} (r(R) + 1) \Big) \\
&= C \left\| \varepsilon \right\|_{\mathcal{H}(\widehat{\lambda};\mathfrak{a}_\theta)} (r(R) + 1).
\end{aligned}$$

This finishes the proof. $\square$

4. Slithering and stretching

In this chapter we define two more important concepts that will become relevant for the definition of cataclysm deformations. The first is the *slithering map*, which is an element in G associated to two geodesics g, h that relates the pairs of flags associated to the geodesics. The second one are *stretching maps*. They are associated with one oriented geodesic g, fix the pair of flags associated with g and have an additional parameter $H \in \mathfrak{a}_\theta$. Throughout this chapter, let $\theta \subset \Delta$ and let $\rho\colon \pi_1(S) \to G$ be θ-Anosov with boundary map ζ.

4.1. The slithering map

In this section, if not stated otherwise, the lamination λ **is assumed to be maximal.** Recall from Notation 2.26 that to every oriented geodesic g in $\tilde{S}$ we can associate a pair $(P_g^+, P_g^-) = (\zeta(g^+), \zeta(g^-))$ of transverse flags. For two geodesics g and h in $\tilde{\lambda}$, we want to construct an element in G that identifies the pairs of transverse flags (P_h^+, P_h^-) and (P_g^+, P_g^-). This element will be called *slithering map*, referring to [BD17, Section 5.1], where this map is constructed for the special case of Hitchin representations into $\mathrm{SL}(n, \mathbb{R})$. In [BD17, Section 5.1], they already mention that their construction possibly extends to θ-Anosov representations. Indeed, the proofs presented here are analogous to theirs, transferred to our more general setting of Lie groups. We give the details for completeness and for later reference.

In this thesis, we need the slithering maps only for an estimate in Proposition 4.18. However, we treat them in this chapter in detail, since they might be useful in other contexts as well. In particular, slithering maps play a crucial role in the construction of the Bonahon-Dreyer coordinates for Hitchin representations [BD17] and our generalization of the slithering maps to semisimple Lie groups might be useful to define such coordinates in a more general context.

We first construct the slithering map in the special case that g and h have a common endpoint.

Lemma 4.1. *Let g and h be two oriented geodesics in $\tilde{\lambda}$ that share their positive endpoint, i.e. $g^+ = h^+$. Then there exists a unique element Σ_{gh} in the unipotent radical $N_g^+ \subset P_g^+$ that sends P_h^- to P_g^-, where g^- and h^- are the other endpoints of g and h, respectively. Further, Σ_{gh} depends locally Hölder continuously on the flags P_g^- and P_h^-.*

Proof. The existence and uniqueness are a direct consequence of the transversality of the boundary map and of the fact that the unipotent radical $N_g^+ \subset P_g^+$ acts simply transitively on flags transverse to P_g^+ by Remark 2.10. Also the local Hölder continuity follows from the local Hölder continuity in Remark 2.10. □

An analogous result holds if the geodesics share their negative endpoint. The map Σ_{gh} only depends on the geodesics g and h, not on an orientation. The orientation of g and h is only used to distinguish their endpoints.

If g, h are arbitrary geodesics in $\tilde{\lambda}$, and $R \in \mathcal{P}_{gh}$, set $\Sigma_R := \Sigma_{g_R^0 g_R^1}$, where g_R^0, g_R^1 are the oriented geodesics bounding R as in Notation 3.10 (see Figure 3.3). Since λ is maximal by assumption, g_R^0 and g_R^1 share an endpoint, so we can define $\Sigma_{g_R^0 g_R^1}$. Let $\mathcal{C} = \{R_1, \ldots, R_m\}$ be a finite subset of $\mathcal{C}_{gh}$, where the indexing is from g to h, i.e. such that each R_i separates g from R_{i+1}. Consider the finite composition

$$\Sigma_{\mathcal{C}} = \Sigma_{R_1}\Sigma_{R_2}\cdots\Sigma_{R_m}.$$

We show in Lemma 4.5 below that $\Sigma_{\mathcal{C}}$ converges when $\mathcal{C}$ tends to the whole set $\mathcal{C}_{gh}$. To do so, we first need an estimate for the basic slithering maps from Lemma 4.1.

Let $\tilde{k}$ be an arc transverse to $\tilde{\lambda}$ that intersects both g and h. Let d_∞ be a distance on $\partial_\infty\tilde{S}$. Define a distance d on the space all oriented geodesics in the universal cover $\tilde{S}$ as the sum of the distances in $\partial_\infty\tilde{S}$ between the positive and negative endpoints. For two oriented geodesics g and h, their distance is given by

$$\mathrm{d}(g,h) := \mathrm{d}_\infty(g^+,h^+) + \mathrm{d}_\infty(g^-,h^-), \tag{4.1}$$

where d_∞ is a distance on $\partial_\infty\tilde{S}$.

Remark 4.2. Let $\mathcal{G}_{\tilde{k}}$ be the set of geodesics in $\tilde{\lambda}$ that intersect $\tilde{k}$. On $\mathcal{G}_{\tilde{k}}$, there exists a distance $\mathrm{d}_{\tilde{k}}$ defined by $\mathrm{d}_{\tilde{k}}(g,h) = \ell(\tilde{k}_{gh})$, where $\ell(\tilde{k}_{gh})$ is the the subarc of $\tilde{k}$ from g to h. Since $\mathcal{G}_{\tilde{k}}$ is compact, the distance d from above and $\mathrm{d}_{\tilde{k}}$ are equivalent, so there exist constants $B_1, B_2 > 0$ depending on $\tilde{k}$, such that for all geodesics $g, h \in \mathcal{G}_{\tilde{k}}$, we have

$$B_1\ell(\tilde{k}_{gh}) \leq \mathrm{d}(g,h) \leq B_2\ell(\tilde{k}_{gh}),$$

Let d_G be the left-invariant and almost-right invariant metric on G introduced in Appendix A.1 and $r\colon \mathcal{C}_{gh} \to \mathbb{N}$ the divergence radius from Lemma 3.11.

Lemma 4.3. *There exist constants $C, A > 0$ depending on $\tilde{k}$ and ρ such that*

$$\mathrm{d}_G(\Sigma_R, Id) \leq Ce^{-Ar(R)}$$

for every component $R \in \mathcal{C}_{gh}$.

Proof. Fix $R \in \mathcal{C}_{gh}$ and orient g_R^0 and g_R^1 such that they share the positive endpoint $(g_R^0)^+ = (g_R^1)^+ \in \partial_\infty \tilde{S}$. The element Σ_R depends locally Hölder continuously on the $P^-_{g_R^0}$ and $P^-_{g_R^1}$ and is the identity if the two flags agree. As R varies in $\mathcal{C}_{gh}$, these flags stay in a compact subset of the space of pairs of transverse flags, because the arc $\tilde{k}$ is compact. Note that this compact subset depends on $\tilde{k}$ and ρ. Thus, there exist constants $C_1, A_1 > 0$ depending on $\tilde{k}$ and ρ such that

$$\mathrm{d}_G(\Sigma_R, \mathrm{Id}) \leq C_1 \mathrm{d}_{\mathcal{F}_\theta}\left(P^-_{g_R^0}, P^-_{g_R^1}\right)^{A_1}, \tag{4.2}$$

where $\mathrm{d}_{\mathcal{F}_\theta}$ is a metric on the space of flags $\mathcal{F}_\theta$. By Hölder continuity of the flag curve ζ, there exist constants $C_2, A_2 > 0$ depending on ρ such that

$$\mathrm{d}_{\mathcal{F}_\theta}\left(P^-_{g_R^0}, P^-_{g_R^1}\right) \leq C_2 \ \mathrm{d}_{\partial_\infty \tilde{S}}\left((g_R^0)^-, (g_R^1)^-\right)^{A_2} = C_2 \ \mathrm{d}\left(g_R^0, g_R^1\right)^{A_2}, \tag{4.3}$$

where we use that the distance between g_R^0 and g_R^1 is given as the sum of the distances of the endpoints as in (4.1). Combining (4.2) and (4.3) with Remark 4.2 and Lemma 3.11, we have

$$\mathrm{d}_G(\Sigma_R, \mathrm{Id}) \leq C_3 \ell(\tilde{k} \cap R)^{A_3} \leq Ce^{-Ar(R)}$$

for constants $C, A > 0$ depending on ρ and $\tilde{k}$. This finishes the proof. □

Before we can show convergence of the $\Sigma_{\mathcal{C}}$ as $\mathcal{C}$ tends to $\mathcal{C}_{gh}$ (Lemma 4.5), we need one more lemma.

Lemma 4.4. *As $\mathcal{C} = \{R_1, \dots, R_m\}$ ranges over all finite subsets of $\mathcal{C}_{gh}$, the distance $\mathrm{d}_G(\Sigma_{\mathcal{C}}, Id)$ remains uniformly bounded.*

Proof. Using the triangle inequality and left-invariance of the metric d_G, we have

$$\begin{aligned}\mathrm{d}_G\left(\Sigma_{\mathcal{C}}, Id\right) &\leq \mathrm{d}_G(\Sigma_{R_1}\dots\Sigma_{R_m}, \Sigma_{R_1}\dots\Sigma_{R_{m-1}}) + \mathrm{d}_G(\Sigma_{R_1}\dots\Sigma_{R_{m-1}}, \mathrm{Id}) \\ &\leq \mathrm{d}_G(\Sigma_{R_m}, \mathrm{Id}) + \mathrm{d}_G(\Sigma_{R_1}\dots\Sigma_{R_{m-1}}, \Sigma_{R_1}\dots\Sigma_{R_{m-2}}) + \mathrm{d}_G(\Sigma_{R_1}\dots\Sigma_{R_{m-2}}, \mathrm{Id}) \\ &\leq \sum_{i=1}^{m} \mathrm{d}_G\left(\Sigma_{R_i}, Id\right) \\ &\leq \sum_{i=1}^{m} Ce^{-Ar(R_i)},\end{aligned}$$

The constants C and A are obtained from Lemma 4.3 and depend on $\tilde{k}$ and ρ. Since the number of all triangles with a fixed divergence radius is uniformly bounded by some $D \in \mathbb{N}$ (Lemma 3.11), we have

$$\sum_{i=1}^{m} Ce^{-Ar(R_i)} \leq CD\sum_{r=0}^{\infty} e^{-Ar} < \infty,$$

This bound does not depend on $\mathcal{C}$, which proves the Lemma. □

Now we have the necessary preparation to define the slithering map in general.

Lemma 4.5. *As the finite subset $\mathcal{C}$ tends to $\mathcal{C}_{gh}$, the limit*

$$\Sigma_{gh} := \lim_{\mathcal{C}\to\mathcal{C}_{gh}} \Sigma_{\mathcal{C}}$$

exists.

Proof. Choose a sequence $(\mathcal{C}_m)_{m\in\mathbb{N}}$ of finite subsets of $\mathcal{C}_{gh}$ tending to $\mathcal{C}_{gh}$ such that $\mathcal{C}_m$ has cardinality m and $\mathcal{C}_m \subset \mathcal{C}_{m+1}$ for all $m \in \mathbb{N}$. Let $\mathcal{C}_m = \{R_1, \dots, R_m\}$ and let $R \in \mathcal{C}_{gh}$ such that $\mathcal{C}_{m+1} = \mathcal{C}_m \cup \{R\}$. Let R separate R_i from R_{i+1}. Set $\mathcal{C} := \{R_1, \dots, R_i\}$ and $\mathcal{C}' = \{R_{i+1}, \dots, R_m\}$. Then

$$\Sigma_{\mathcal{C}_{m+1}} = \Sigma_{\mathcal{C}}\Sigma_{\mathcal{C}'} \quad \text{and} \quad \Sigma_{\mathcal{C}_{m+1}} = \Sigma_{\mathcal{C}}\Sigma_R\Sigma_{\mathcal{C}'}.$$

Let $\|\cdot\|_{\mathrm{op}(\mathfrak{g})}$ be the operator induced by a norm $\|\cdot\|_{\mathfrak{g}}$ as in Lemma A.2. Using left-invariance and almost right-invariance of the the metric d_G, we have

$$\begin{aligned}
\mathrm{d}_G\left(\Sigma_{\mathcal{C}_{m+1}}, \Sigma_{\mathcal{C}_m}\right) &= \mathrm{d}_G\left(\Sigma_{\mathcal{C}}\Sigma_{\mathcal{C}'}, \Sigma_{\mathcal{C}}\Sigma_R\Sigma_{\mathcal{C}'}\right) \\
&= \mathrm{d}_G\left(\Sigma_{\mathcal{C}'}, \Sigma_R\Sigma_{\mathcal{C}'}\right) \\
&\leq \left\| \mathrm{Ad}_{\Sigma_{\mathcal{C}'}^{-1}} \right\|_{\mathrm{op}(\mathfrak{g})} \mathrm{d}_G(\Sigma_R, Id) \\
&\leq Ce^{-Ar(R)}
\end{aligned} \tag{4.4}$$

where in the last inequality, the constants C and A come from the constants in Lemma 4.3 and from the fact that $\left\| \mathrm{Ad}_{\Sigma_{\mathcal{C}'}^{-1}} \right\|_{\mathrm{op}(\mathfrak{g})}$ is uniformly bounded by Lemma 4.4. By Lemma 3.11, the number of triangles $R \in \mathcal{C}_{gh}$ with divergence radius $r(R) \leq r$ is uniformly bounded. Together with (4.4), this shows that $\mathrm{d}_G\left(\Sigma_{\mathcal{C}_{m+1}}, \Sigma_{\mathcal{C}_m}\right)$ goes to zero as m goes to infinity, hence $\left(\Sigma_{\mathcal{C}_m}\right)_{m\in\mathbb{N}}$ is a Cauchy sequence and converges. $\square$

Note that Σ_{gh} is well-defined: When g and h have a common endpoint $g^+ = h^+ \in \partial_\infty \tilde{S}$ and other endpoints g^- and h^-, respectively, we can either define Σ_{gh} as the unique element in the unipotent radical $N_g^+ < P_g^+$ sending P_h^- to P_g^- or as the limit $\lim_{\mathcal{C}\to\mathcal{C}_{gh}} \Sigma_{\mathcal{C}}$. Since all geodesics separating g from h have g^+ as an endpoint and N_g^+ is closed, this limit lies in N_g^+ and sends P_h^- to P_g^-. By uniqueness, it agrees with Σ_{gh}.

The slithering map $\Sigma_{gh} \in G$ satisfies some composition properties.

Lemma 4.6. *For any two leaves g and h of $\tilde{\lambda}$, we have $\Sigma_{gg} = \mathrm{Id}$ and $\Sigma_{hg} = \Sigma_{gh}^{-1}$. In addition, if g, h, h' are three leaves such that one of them separates the other two, we have $\Sigma_{gh'} = \Sigma_{gh}\Sigma_{hh'}$.*

Proof. The fact that $\Sigma_{gg} = Id$ is trivial. The other two properties follow from the construction of Σ_{gh} by showing them first for finite compositions $\Sigma_{\mathcal{C}}$ and then taking the limit. For completeness, we write them down here. For showing the behavior under taking inverses, note that, as sets $\mathcal{C}_{gh} = \mathcal{C}_{hg}$. If $\mathcal{C} = \{R_1, \ldots, R_m\} \subset \mathcal{C}_{gh}$ is a finite subset, let $\mathcal{C}' := \{R_1', \ldots, R_m'\} \subset \mathcal{C}_{hg}$ be defined by $R_i' := R_{m-i+1}$. As sets, $\mathcal{C}$ and $\mathcal{C}'$ agree, but the labelings are reversed. For the basic slithering maps, we have $\Sigma_{R_i'} = \Sigma_{R_{m-i+1}}^{-1}$ by construction, so

$$\Sigma_{\mathcal{C}'} = \Sigma_{R_1'} \cdots \Sigma_{R_m'} = \Sigma_{R_m}^{-1} \cdots \Sigma_{R_1}^{-1} = \left(\Sigma_{R_1} \cdots \Sigma_{R_m}\right)^{-1} = \Sigma_{\mathcal{C}}^{-1}.$$

If $\mathcal{C}$ goes to $\mathcal{C}_{gh}$, we have that $\mathcal{C}'$ goes to $\mathcal{C}_{hg}$ and by taking the limit it follows that $\Sigma_{hg} = \Sigma_{gh}^{-1}$.

For the composition property, first assume that h separates g and h'. Then we have $\mathcal{C}_{gh'} = \mathcal{C}_{gh} \cup \mathcal{C}_{hh'}$. Let $\mathcal{C} \subset \mathcal{C}_{gh}$ be a finite subset and $\mathcal{C}_1 := \mathcal{C} \cap \mathcal{C}_{gh'}$, $\mathcal{C}_2 := \mathcal{C} \cap \mathcal{C}_{hh'}$.

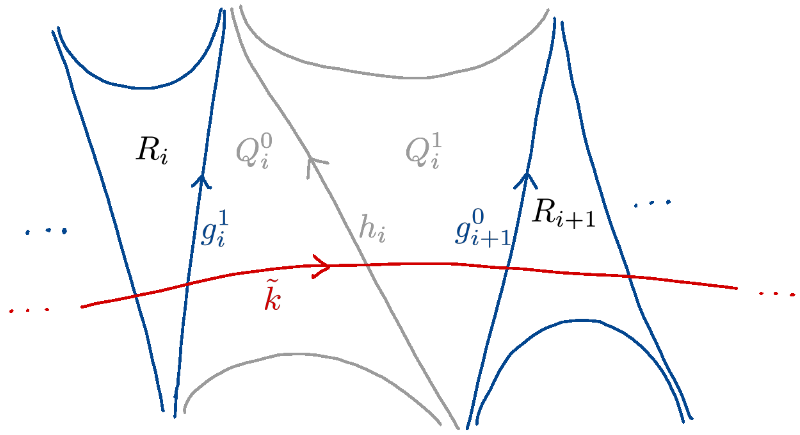


Figure 4.1.: In between the two triangles R_i and R_{i+1}, insert an auxiliary geodesic h_i and two triangles Q_i^0 and Q_i^1 to approximate the part of the lamination $\tilde{\lambda}$ between R_i and R_{i+1} by a finite lamination.

Then $\Sigma_{\mathcal{C}} = \Sigma_{\mathcal{C}_1}\Sigma_{\mathcal{C}_2}$ and if $\mathcal{C}$ tends to $\mathcal{C}_{gh'}$, we have that $\mathcal{C}_1$ tends to $\mathcal{C}_{gh}$ and $\mathcal{C}_2$ tends to $\mathcal{C}_{hh'}$. By taking the limit, it follows that $\Sigma_{gh'} = \Sigma_{gh}\Sigma_{hh'}$. If h' separates g and h, we have $\Sigma_{gh} = \Sigma_{gh'}\Sigma_{h'h}$, which by the behavior under taking inverses is equivalent to $\Sigma_{gh'} = \Sigma_{gh}\Sigma_{hh'}$. The case that g separates h and h' can be concluded in the same way. $\square$

We say that two geodesics $g, h \subset \tilde{\lambda}$ are *oriented in parallel* if exactly one of the orientations of g and h agrees with the boundary orientation of the connected component of $\tilde{S} \setminus (g \cup h)$ that separates g from h.

Lemma 4.7. *Let g, h be two geodesics in $\tilde{\lambda}$ that are oriented in parallel and denote by $g^{\pm}$ and $h^{\pm}$ their positive and negative endpoints in $\partial_\infty \tilde{S}$. Then the slithering map Σ_{gh} sends $P_h^{\pm}$ to $P_g^{\pm}$.*

Proof. The idea of the proof is to approximate the part of $\tilde{\lambda}$ that separates g from h by a finite lamination. This finite approximation of Σ_{gh} sends $P_h^{\pm}$ to $P_g^{\pm}$. Passing to the limit will prove the lemma.

Let $\mathcal{C} := \{R_1, \ldots, R_m\} \subset \mathcal{C}_{gh}$ be a finite subset. For $i = 1, \ldots, m$, let $g_i^0 := g_{R_i}^0$ and $g_i^1 := g_{R_i}^1$. Orient the geodesics $g_i^{0/1}$ parallel to g and h. Without loss of generality, the positive endpoints are to the left as seen from g. Let h_i be the geodesic that has as endpoints $(g_{i+1}^0)^-$ and $(g_i^1)^+$ (see Figure 4.1). Note that the geodesics h_i are not part of the lamination $\tilde{\lambda}$. Let Q_i^0 be the unique ideal triangle that is bounded by g_i^1 and h_i, and Q_i^1 the unique ideal triangle bounded by h_i and g_{i+1}^0. When going along an arc from g to h, the triangles $R_i, Q_i^0, Q_i^1, R_{i+1}$ are met in this order and are adjacent to the respective next one. It may happen that one or both of Q_i^0 and Q_i^1 are degenerate, i.e. a geodesic, if h_i agrees with g_i^1 or g_{i+1}^0.

As in Lemma 4.1, define $\Sigma_{Q_i^0}$ as the unique element in the unipotent radical of $P^+_{g_i^1}$ that sends $P^-_{g_i^1}$ to $P^-_{h_i} = P^-_{g_{i+1}^0}$ and $\Sigma_{Q_i^1}$ as the unique element in the unipotent radical of $P^-_{h_i} = P^-_{g_{i+1}^0}$ that sends $P^+_{h_i} = P^+_{g_i^1}$ to $P^+_{g_{i+1}^0}$. Set

$$\widehat{\Sigma}_{\mathcal{C}} := \Sigma_{Q_0^0}\Sigma_{Q_0^1}\Sigma_{R_1}\Sigma_{Q_1^0}\Sigma_{Q_1^1}\Sigma_{R_2}\cdots\Sigma_{R_m}\Sigma_{Q_m^0}\Sigma_{Q_m^1}.$$

By construction, since the triangles are pairwise adjacent when going from g to h, $\widehat{\Sigma}_{\mathcal{C}}$ maps $P_h^{\pm}$ to $P_g^{\pm}$. We want to show that $\widehat{\Sigma}_{\mathcal{C}}$ and $\Sigma_{\mathcal{C}}$ have the same limit as $\mathcal{C}$ goes to $\mathcal{C}_{gh}$.

By the triangle inequality, left-invariance and almost-right invariance of d_G, we have

$$\begin{aligned}
\mathrm{d}_G\left(\widehat{\Sigma}_{\mathcal{C}}, \Sigma_{\mathcal{C}}\right) &= \mathrm{d}_G\left(\Sigma_{Q_0^0}\Sigma_{Q_0^1}\prod_{i=1}^m\left(\Sigma_{R_i}\Sigma_{Q_i^0}\Sigma_{Q_i^1}\right), \prod_{i=1}^m \Sigma_{R_i}\right)\\
&\leq \mathrm{d}_G\left(\Sigma_{Q_0^0}\Sigma_{Q_0^1}\prod_{i=1}^m\left(\Sigma_{R_i}\Sigma_{Q_i^0}\Sigma_{Q_i^1}\right), \Sigma_{Q_0^0}\Sigma_{Q_0^1}\prod_{i=1}^m \Sigma_{R_i}\right)\\
&\qquad + \mathrm{d}_G\left(\Sigma_{Q_0^0}\Sigma_{Q_0^1}\prod_{i=1}^m \Sigma_{R_i}, \prod_{i=1}^m \Sigma_{R_i}\right)\\
&\leq \mathrm{d}_G\left(\Sigma_{Q_1^0}\Sigma_{Q_1^1}\prod_{i=2}^m\left(\Sigma_{R_i}\Sigma_{Q_i^0}\Sigma_{Q_i^1}\right), \prod_{i=2}^m \Sigma_{R_i}\right)\\
&\qquad + \left\|\mathrm{Ad}_{\left(\prod_{i=1}^m \Sigma_{R_i}\right)^{-1}}\right\|_{\mathrm{op}(\mathfrak{g})} \mathrm{d}_G\left(\Sigma_{Q_0^0}\Sigma_{Q_0^1}, \mathrm{Id}\right)\\
&\leq \quad \cdots\\
&\leq \sum_{i=0}^m \left\|\mathrm{Ad}_{(\Sigma_{R_{i+1}}\cdots\Sigma_{R_m})^{-1}}\right\|_{\mathrm{op}(\mathfrak{g})} \mathrm{d}_G\left(\Sigma_{Q_i^0}\Sigma_{Q_i^1}, Id\right).
\end{aligned}$$

Since $\mathrm{d}_G(\Sigma_{\mathcal{C}}, Id)$ is uniformly bounded for every finite subset $\mathcal{C}$ of $\mathcal{C}_{gh}$ (Lemma 4.4), also $\left\|\mathrm{Ad}_{(\Sigma_{R_{i+1}}\cdots\Sigma_{R_m})^{-1}}\right\|_{\mathrm{op}(\mathfrak{g})}$ is uniformly bounded and we have that

$$\mathrm{d}_G\left(\widehat{\Sigma}_{\mathcal{C}}, \Sigma_{\mathcal{C}}\right) \leq C\sum_{i=0}^m \mathrm{d}_G\left(\Sigma_{Q_i^0}\Sigma_{Q_i^1}, Id\right).$$

Again by the triangle inequality and left-invariance of d_G,

$$\begin{aligned}
\mathrm{d}_G\left(\Sigma_{Q_i^0}\Sigma_{Q_i^1}, Id\right) &\leq \mathrm{d}_G\left(\Sigma_{Q_i^0}\Sigma_{Q_i^1}, \Sigma_{Q_i^0}\right) + \mathrm{d}_G\left(\Sigma_{Q_i^0}, Id\right)\\
&= \mathrm{d}_G\left(\Sigma_{Q_i^1}, Id\right) + \mathrm{d}_G\left(\Sigma_{Q_i^0}, Id\right)
\end{aligned}$$

As in the proof of Lemma 4.3, $\mathrm{d}_G\left(\Sigma_{Q_i^{0/1}}, Id\right) \leq C\ \ell(\tilde{k} \cap Q_i^{0/1})^A$ for constants $C, A > 0$. The constant A comes from the Hölder continuity of the flag curve ζ. Without loss of generality we can assume that $A \leq 1$. Thus, if Q_i^* is either Q_i^0 or Q_i^1, we have

$$\begin{aligned}
\ell(\tilde{k} \cap Q_i^{0/1})^A &\leq \ell(\tilde{k} \cap (Q_i^0 \cup Q_i^1)^A \\
&\leq \ell\left(\tilde{k} \cap \bigcup_{R \in \mathcal{C}_{g_i^1 g_{i+1}^0}} R\right)^A \\
&\leq \sum_{R \in \mathcal{C}_{g_i^1 g_{i+1}^0}} \ell(\tilde{k} \cap R)^A,
\end{aligned}$$

where $\mathcal{C}_{g_i^1 g_{i+1}^0}$ is the set of all ideal triangles in $\tilde{S} \setminus \tilde{\lambda}$ between R_i and R_{i+1}. Combining the above estimates and Lemma 3.11, we obtain

$$\mathrm{d}_G\left(\widehat{\Sigma}_{\mathcal{C}}, \Sigma_{\mathcal{C}}\right) \leq C \sum_{R \in \mathcal{C}_{gh} \setminus \mathcal{C}} e^{-Ar(R)}.$$

The right hand side converges to zero as $\mathcal{C}$ converges to $\mathcal{C}_{gh}$, so in total, $\widehat{\Sigma}_{\mathcal{C}}$ and $\Sigma_{\mathcal{C}}$ have the same limit. This implies that Σ_{gh} maps $P_h^{\pm}$ to $P_g^{\pm}$. $\square$

We now turn to the regularity of the slithering map.

Lemma 4.8. *The slithering map Σ_{gh} depends locally separately Hölder continuously on the leaves g, h in $\tilde{\lambda}$ in the following sense: Let $\tilde{k}$ be a compact arc transverse to $\tilde{\lambda}$, and let $\mathcal{G}_{\tilde{k}}$ denote the set of all geodesics in $\tilde{\lambda}$ that intersect $\tilde{k}$. Then for all $g \in \mathcal{G}_{\tilde{k}}$, the assignment $h \mapsto \Sigma_{gh}$ is Hölder continuous, i.e. there there exist constants $C, A > 0$ depending on $\tilde{k}$ and ρ such that*

$$\mathrm{d}_G\left(\Sigma_{gh'}, \Sigma_{gh}\right) \leq C\mathrm{d}(h, h')^A$$

for every $h, h' \in \mathcal{G}_{\tilde{k}}$. In particular, $\mathrm{d}_G(\Sigma_{gh}, Id) \leq C\mathrm{d}(g, h)^A$.

Proof. Let $\tilde{k}$ be an arc transverse to $\tilde{\lambda}$. Fix a geodesic g in $\mathcal{G}_{\tilde{k}}$ and let $h, h' \in \mathcal{G}_{\tilde{k}}$ be two other geodesics. Since $\tilde{k}$ is tightly transverse, one of the three geodesics separates the other two. By Lemma 4.6, $\Sigma_{gh'} = \Sigma_{gh}\Sigma_{hh'}$, and using left-invariance of d_G,

$$\mathrm{d}_G\left(\Sigma_{gh'}, \Sigma_{gh}\right) = \mathrm{d}_G\left(\Sigma_{hh'}, Id\right) \tag{4.5}$$

Now, using Lemma 4.3, there exist constants $C, A > 0$ such that

$$\begin{aligned}
\mathrm{d}_G\left(\Sigma_{hh'}, Id\right) &= \lim_{\mathcal{C}\to\mathcal{C}_{hh'}} \mathrm{d}_G\left(\Sigma_{\mathcal{C}}, Id\right) \\
&\leq \lim_{\mathcal{C}\to\mathcal{C}_{hh'}} \sum_{i=1}^{m} \mathrm{d}_G\left(\Sigma_{R_i}, Id\right) \\
&\leq C \sum_{R\in\mathcal{C}_{hh'}} e^{-Ar(R)} \\
&\leq CD \sum_{r=N}^{\infty} e^{-Ar},
\end{aligned}$$

where $N := \min_{R\in\mathcal{C}_{hh'}} r(R)$ and $D \in \mathbb{N}$ is the bound on the number of components with fixed divergence radius from Lemma 3.11. The last expression is the remainder term of a geometric series and bounded by a constant times e^{-AN}. Thus, we have that

$$\mathrm{d}_G\left(\Sigma_{hh'}, Id\right) \leq C_1 e^{-A\min_{R\in N} r(R)} = C_1 \max_{R\in\mathcal{C}_{hh'}} e^{-Ar(R)}.$$

Together with Lemma 3.11 and Remark 4.2, there exist constants $C_i, A_i > 0$ such that

$$\max_{R\in\mathcal{C}_{hh'}} e^{-Ar(R)} \leq C_2 \max_{R\in\mathcal{C}_{hh'}} \ell(\tilde{k}\cap R)^A \leq C_2\ \ell(\tilde{k}^A_{hh'}) \leq C_3\mathrm{d}(h,h')^A.$$

Therefore, $\mathrm{d}_G\left(\Sigma_{hh'}, Id\right) \leq C\mathrm{d}\left(h,h'\right)^A$. Together with (4.5), it follows that there exist constants $C, A > 0$ such that

$$\mathrm{d}_G\left(\Sigma_{gh'}, \Sigma_{gh}\right) \leq C_1C_3\mathrm{d}(h,h')^A,$$

which proves that $h \mapsto \Sigma_{gh}$ is locally Hölder continuous. The constants only depend on $\tilde{k}$ and ρ, but not on g. $\square$

We summarize the results from this section in a Proposition.

Proposition 4.9. *There exists a unique family $\{\Sigma_{gh}\}_{(g,h)}$ of elements in G, indexed by all pairs of leaves g, h in $\tilde{\lambda}$, that satisfies the following conditions:*

(1) $\Sigma_{gg} = \mathrm{Id}$, $\Sigma_{hg} = \left(\Sigma_{gh}\right)^{-1}$, and $\Sigma_{gh'} = \Sigma_{gh}\Sigma_{hh'}$ when one of the three geodesics g, h, h' separates the others;

(2) Σ_{gh} depends locally separately Hölder continuously on g and h in the sense of Lemma 4.8;

(3) if g and h are oriented and have a common positive endpoint $g^+ = h^+ \in \partial_\infty\tilde{S}$ and if g^- and h^- are the negative endpoints of g and h, respectively, then Σ_{gh} is the unique

element in the unipotent radical of P_g^+ that sends P_h^- to P_g^-.

From (1)-(3) *it follows that if g and h are oriented in parallel, Σ_{gh} sends the pair (P_h^+, P_h^-) to the pair (P_g^+, P_g^-).*

Proof. The only thing left to show is uniqueness of the family $\{\Sigma_{gh}\}_{(g,h)}$. Let $\{\Sigma'_{gh}\}_{(g,h)}$ be a family of elements in G satisfying the conditions (1)-(3) in Proposition 4.9. We claim that for all geodesics g, h in $\tilde{\lambda}$, Σ'_{gh} is equal to Σ_{gh}. Let $\mathcal{C} = \{R_1, \ldots, R_m\} \subset \mathcal{C}_{gh}$ be a finite subset. By the composition condition (1) we have

$$\Sigma'_{gh} = \Sigma'_{gg_1^0}\Sigma'_{g_1^0 g_1^1}\Sigma'_{g_1^1 g_2^0}\Sigma'_{g_2^0 g_2^1}\cdots\Sigma'_{g_m^0 g_m^1}\Sigma'_{g_m^1 h}.$$

By condition (3), we have $\Sigma'_{g_i^0 g_i^1} = \Sigma_{g_1^0 g_1^1} = \Sigma_{R_i}$ for all i, so

$$\Sigma'_{gh} = \Sigma'_{gg_1^0}\Sigma_{R_1}\Sigma'_{g_1^1 g_2^0}\Sigma_{R_2}\cdots\Sigma_{R_m}\Sigma'_{g_m^1 h}.$$

By the Hölder continuity (2), there exists constants $C, A > 0$ such that

$$\mathrm{d}_G\left(\Sigma'_{g_i^1 g_{i+1}^0}, Id\right) \leq C\mathrm{d}\left(g_i^1, g_{i+1}^0\right)^A.$$

From Remark 4.2, we know that

$$\mathrm{d}\left(g_i^1, g_{i+1}^0\right) \leq C_1 \ell\left(\tilde{k}_{g_i^1 g_{i+1}^0}\right) = C_1 \sum_{R \in \mathcal{C}_{g_i^1 g_{i+1}^0}} \ell\left(\tilde{k} \cap R\right)$$

Using the same techniques as in the proof of Lemma 4.7, we see that

$$\mathrm{d}_G\left(\Sigma'_{gh}, \Sigma_{\mathcal{C}}\right) \leq C \sum_{R \in \mathcal{C}_{gh} \setminus \mathcal{C}} e^{-Ar(R)},$$

where the right hand side goes to zero as $\mathcal{C}$ tends to $\mathcal{C}_{gh}$. This shows that $\Sigma'_{gh} = \lim_{\mathcal{C} \to \mathcal{C}_{gh}} \Sigma_{\mathcal{C}} = \Sigma_{gh}$. □

Remark 4.10. The requirement of being locally separately Hölder continuous is crucial for the uniqueness in Proposition 4.9. One can construct families of maps satisfying all requirements of the proposition except for Hölder continuity. For the special case $G = \mathrm{PSL}(n, \mathbb{R})$ and $\theta = \Delta$, this is explained in [BD17, Remark 5.10].

The construction of the slithering map depends on the fact that λ is maximal. However, from the proofs above, we can deduce a weaker statement also for non-maximal laminations. In this case, we cannot construct Σ_{gh} for all pairs of geodesics, but we can still construct it in the case that the part of $\tilde{\lambda}$ between g and h consists only of "wedges". Let λ be a lamination, **not necessarily maximal**.

Definition 4.11. Let g and h be two geodesics in $\tilde{\lambda}$. Then $R \in \mathcal{C}_{gh}$ is a *wedge component* if the geodesics g_R^0 and g_R^1 share an endpoint. Else, it is a *non-wedge component*. We say that g and h are *separated by wedges* if $\mathcal{C}_{gh}$ consists entirely of wedge components.

In particular, if λ is maximal, every $R \in \mathcal{C}_{gh}$ is a wedge component and every pair of geodesics in $\tilde{\lambda}$ is separated by wedges. In general, the number of non-wedge components is finite, because the arc k has finite length.

Corollary 4.12. *Let g and h be two geodesics in $\tilde{\lambda}$ oriented in parallel and separated by wedges. Then there exists $\Sigma_{gh} \in G$ sending $P_h^{\pm}$ to $P_g^{\pm}$ and constants $C, A > 0$ such that*

$$\mathrm{d}_G\left(\Sigma_{gh}, Id\right) \leq C\mathrm{d}(g,h)^A.$$

Proof. The proofs of Lemma 4.1 to 4.8 do not use that λ is maximal, but only the weaker assumption that g and h are separated by wedges. Thus, they are valid also in this setting. □

4.2. The stretching map

Let $\rho\colon \pi_1(S) \to G$ be a θ-Anosov representation and $\zeta\colon \partial_\infty \tilde{S} \to \mathcal{F}_\theta$ the boundary map. The goal of this section is to associate to an oriented geodesic g in the universal cover $\tilde{S}$ and $H \in \mathfrak{a}_\theta$ an element T_g^H in G, the so-called stretching map. These maps are the basic building blocks for the cataclysm deformation.

Definition 4.13. Let g be an oriented geodesic in $\tilde{S}$ and $H \in \mathfrak{a}_\theta$. Let (P_g^+, P_g^-) be the pair of transverse parabolics associated with g and let $m_g \in G$ such that $m_g \cdot P_\theta^{\pm} = P_g^{\pm}$. We define the *$H$-stretching along g* as

$$T_g^H := m_g \exp(H) m_g^{-1}.$$

By construction, T_g^H lies in $L_g = P_g^+ \cap P_g^-$. The choice of m_g from above is only unique up to an element in L_θ. However, the definition of T_g^H is independent of the choice of m_g, which follows from the fact that $\exp(\mathfrak{a}_\theta)$ is contained in the centralizer of L_θ.

Example 4.14. For $G = \mathrm{SL}(2,\mathbb{R})$, we have $\theta = \Delta = \{\alpha_1\}$ (see Example 2.3). In this case, an element $H \in \mathfrak{a}_\theta$ is a diagonal matrix with entries a and $-a$ for some $a \in \mathbb{R}$, and $\exp(H)$

is a diagonal matrix with entries e^a and e^{-a}. For the oriented geodesic g, the element m_g maps the oriented geodesic from 0 to ∞ to g, and thus

$$T_g^H = m_g \begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix} m_g^{-1}$$

is the hyperbolic element that acts as translation along the oriented geodesic g with translation length $2a$.

Example 4.15. The case $G = \mathrm{SL}(n, \mathbb{R})$ and $\theta = \Delta$, i.e. $P_\Delta^+ = B^+$ is the minimal parabolic, is treated in [Dre13]. We have that $\mathfrak{a}_\Delta = \mathfrak{a} \cong \{(a_1, \ldots, a_n) \in \mathbb{R}^n | \sum_{i=1}^n a_i = 0\}$ and a pair of transverse flags gives rise to a line splitting of $\mathbb{R}^n$ (see Example 2.11). If we look at the pair of standard minimal parabolics (B^+, B^-), the ith line of the splitting is given by the ith standard vector of $\mathbb{R}^n$. For an oriented geodesic g, the element $m_g \in \mathrm{SL}(n, \mathbb{R})$ maps the standard splitting to the splitting given by the pair (P_g^+, P_g^-). With respect to a basis adapted to this splitting, the stretching map T_g^H for $H = (\lambda_1, \ldots, \lambda_n) \in \mathfrak{a}_\Delta$ is diagonal with entries $e^{a_1}, \ldots, e^{a_n}$. Geometrically speaking, T_g^H is the linear map that acts on the ith line of the splitting as a stretch by the factor e^{a_i}. This motivates the term *stretching map*.

For an oriented geodesic g in $\tilde{S}$, denote by $\overline{g}$ the geodesic in $\tilde{S}$ with the same image, but opposite orientation, and let $\iota\colon \mathfrak{a} \to \mathfrak{a}$ be the opposition involution.

Lemma 4.16. *The stretching map T_g^H for an oriented geodesic g and elements $H, H_1, H_2 \in \mathfrak{a}_\theta$ has the following properties:*

1. $T_g^{H_1} T_g^{H_2} = T_g^{H_1+H_2}$,
2. $\left(T_g^H\right)^{-1} = T_g^{-H}$,
3. $T_{\overline{g}}^H = T_g^{-\iota(H)}$ *and*
4. *ρ-equivariance, i.e.* $T_{\gamma g}^H = \rho(\gamma) T_g^H \rho(\gamma)^{-1}$ *for all* $\gamma \in \pi_1(S)$.

Proof. For the first property we have

$$\begin{aligned} T_g^{H_1} T_g^{H_2} &= \left(m_g \exp(H_1) m_g^{-1}\right)\left(m_g \exp(H_2) m_g^{-1}\right) \\ &= m_g \exp(H_1)\exp(H_2) m_g^{-1} \\ &= m_g \exp(H_1 + H_2) m_g^{-1}. \end{aligned}$$

The behavior under inverse holds since

$$T_g^{-H} = m_g \exp(-H) m_g^{-1} = \left(m_g exp(H) m_g^{-1}\right)^{-1} = \left(T_g^H\right)^{-1}.$$

For the third property, the behavior under reversing the orientation of g, let $w_0 \in G$ be an element satisfying $w_0(\overline{\mathfrak{a}^+}) = -\overline{\mathfrak{a}^+}$ as in Definition 2.4. We have for $H \in \mathfrak{a}_\theta$

$$w_0 \exp(H) w_0^{-1} = \exp(\mathrm{Ad}_{w_0}(H)) = \exp(-\iota(H))$$

by definition of the opposition involution ι. Further, the element w_0 satisfies $w_0 P_\theta^+ = P_\theta^-$, so if $m_g \in G$ maps $P_\theta^\pm$ to $P_g^\pm$, then $m_g w_0 \in G$ maps $P_\theta^\pm$ to $P_g^\mp = P_{\bar{g}}^\pm$. It follows that

$$T_{\bar{g}}^H = (m_g w_0)\exp(H)(m_g w_0)^{-1} = m_g \exp(-\iota(H)) m_g^{-1} = T_g^{-\iota(H)}.$$

Note that this is independent og the choice of the representative w_0, because two representatives differ by an element in $Z_K(\mathfrak{a})$. For the ρ-equivariance, note that the flag curve ζ is ρ-equivariant, i.e. $(P_{\gamma g}^+, P_{\gamma g}^-) = (\rho(\gamma)\cdot P_g^+, \rho(\gamma)\cdot P_g^-)$. Thus, we can choose $m_{\gamma g} = \rho(\gamma) m_g$, which implies the ρ-equivariance of the stretching map. □

Remark 4.17. If ρ and ρ' are two representations that are conjugate by an element $M \in G$, then also the stretching maps T_g^H and ${T'}_g^H$ are conjugate by M. This follows from the fact that the corresponding boundary maps ζ and ζ' satisfy $\zeta' = M \cdot \zeta$, using the same argument as for the ρ-equivariance in Lemma 4.16. Since on the character variety, conjugate representations define the same point, we will sometimes switch from ρ to a representation conjugate to ρ if convenient.

The following Proposition gives an estimate on the distance of the stretching maps T_g^H and T_h^H depending on the distance of the oriented geodesics g and h for the case that g and h are separated by wedges.

Proposition 4.18. *There exist constants $C, A > 0$, depending on $\tilde{k}$ and ρ, such that for all geodesics g, h in $\tilde{\lambda}$ that intersect $\tilde{k}$, separated by wedges and oriented positively with respect to the orientation of $\tilde{k}$, for every $H \in \mathfrak{a}_\theta$,*

$$\mathrm{d}_G(T_g^H, T_h^H) \leq C\left((e^{\|H\|_{\mathfrak{a}}} + 1)\mathrm{d}(g,h)^a\right),$$

where $\|\cdot\|_{\mathfrak{a}}$ is the norm on $\mathfrak{a}$ introduced in Lemma A.3.

Proof. Without loss of generality we can assume that the pair of transverse parabolics (P_g^+, P_g^-) associated with g agrees with the pair (P_θ^+, P_θ^-) of standard transverse parabolics. Since G acts transitively on pairs of transverse parabolics, there always exists a representation conjugate to ρ with this property. Then the slithering map Σ_{hg} sends $P_\theta^\pm$ to $P_h^\pm$. Thus, in the definition of the stretching maps, we can choose $m_h := \Sigma_{hg}$. Using left-invariance and almost right-invariance of d_G (Lemma A.2), we have

$$\begin{aligned}\mathrm{d}_G(T_g^H, T_h^H) &= \mathrm{d}_G(\exp(H), \Sigma_{hg}\exp(H)\Sigma_{gh}) \\ &\leq \left\|\mathrm{Ad}_{\exp(-H)}\right\|_{\mathrm{op}(\mathfrak{g})} \mathrm{d}_G(\mathrm{Id}, \Sigma_{hg}) + \mathrm{d}_G(\mathrm{Id}, \Sigma_{gh}). \end{aligned} \tag{4.6}$$

By left-invariance of the metric and Hölder continuity of the slithering map (Proposition 4.9 (2)), there exist constants $C, A > 0$ depending on $\tilde{k}$ and ρ such that

$$\mathrm{d}_G(\mathrm{Id}, \Sigma_{gh}) = \mathrm{d}_G(\mathrm{Id}, \Sigma_{hg}) \leq C\mathrm{d}(g,h)^A. \tag{4.7}$$

Combining (4.6) and (4.7) with the estimate for $\left\|\mathrm{Ad}_{\exp(-H)}\right\|_{\mathrm{op}(\mathfrak{g})}$ provided by Lemma A.3, there exist constants $C, A > 0$ depending on $\tilde{k}$ and ρ such that

$$\mathrm{d}_G(T_g^H, T_h^H) \leq C\left(e^{\|H\|_{\mathfrak{a}}} + 1\right)\mathrm{d}(g,h)^A.$$

□

We want to remark here that the proof of Proposition 4.18 is the only point in the construction of cataclysms where we need the slithering map. The Hölder continuity of the slithering map allows us to prove the desired estimate, whereas the definition of the stretching map itself does not use the slithering map.

The following corollary covers the special case of Proposition 4.18 when the two geodesics bound the same connected component in $\tilde{S} \setminus \tilde{\lambda}$.

Corollary 4.19. *Let $\tilde{k}$ be an oriented arc transverse to $\tilde{\lambda}$, let $R \subset \tilde{S} \setminus \tilde{\lambda}$ be a wedge component such that $\tilde{k} \cap R \neq \emptyset$ and let $r(R)$ be the divergence radius (see Lemma 3.11). There exist constants $C, A > 0$ depending on $\tilde{k}$ and ρ such that*

$$\mathrm{d}_G\left(T_{g_R^0}^H T_{g_R^1}^{-H}, \mathrm{Id}\right) \leq C\left(e^{\|H\|_{\mathfrak{a}}} + 1\right)e^{-Ar(R)}.$$

Proof. This is a direct consequence of Proposition 4.18, using left-invariance of the metric, Remark 4.2 and Lemma 3.11. □

5. Cataclysms

In the first part of this chapter we compose the stretching maps constructed in Section 4.2 to obtain *shearing maps*. Those are central in the second part of the chapter where we introduce *cataclysms*. In the third part we prove additivity of cataclysms and natural behavior under composing an Anosov representation with a Lie group homomorphism. Cataclysms were first introduced for Teichmüller space by Thurston [Thu98] and under the name of *shear maps* studied by Bonahon [Bon96]. Dreyer extended the construction to Δ-Anosov representations into $\mathrm{PSL}(n,\mathbb{R})$ [Dre13].

We generalize his results to θ-Anosov representations into any semisimple connected non-compact Lie group G for $\theta \subset \Delta$ with $\iota(\theta) = \theta$. The construction of the shearing maps as a composition of stretching maps in Section 5.1 and the definition of cataclysms in Section 5.2 are parallel to the construction in [Dre13]. The main difference lies in the definition of the stretching maps. Further, in contrast to [Dre13] we do not assume the lamination λ to be maximal.

5.1. The shearing map between two connected components

Fix a θ-Anosov representation $\rho\colon \pi_1(S) \to G$. Let $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ be an $\mathfrak{a}_\theta$-valued transverse twisted cycle (see Definition 3.20) and $R \in \mathcal{C}_{PQ}$. Set

$$\varepsilon(P,R) := \varepsilon(\widehat{k}_{PR}) \in \mathfrak{a}_\theta, \tag{5.1}$$

where $\tilde{k}_{PR}$ is the oriented subarc of $\tilde{k}$ from P to R, k_{PR} its projection to S and $\widehat{k}_{PR}$ is the distinguished lift transverse to $\widehat{\lambda}$ as in Remark 3.16. Let $\mathcal{C} = \{R_1, \ldots, R_m\} \subset \mathcal{C}_{PQ}$ be a finite set of connected components labeled from P to Q and set $g_i^{0/1} := g_{R_i}^{0/1}$. Set

$$\psi_{\mathcal{C}}^{\varepsilon} := \left(T_{g_1^0}^{\varepsilon(P,R_1)} T_{g_1^1}^{-\varepsilon(P,R_1)}\right) \cdots \left(T_{g_m^0}^{\varepsilon(P,R_m)} T_{g_m^1}^{-\varepsilon(P,R_m)}\right), \tag{5.2}$$

and let

$$\varphi_{\mathcal{C}}^{\varepsilon} := \psi_{\mathcal{C}}^{\varepsilon} \cdot T_{g_Q^0}^{\varepsilon(P,Q)} \tag{5.3}$$

We will show that $\varphi^{\varepsilon}_{\mathcal{C}}$ converges when the finite set $\mathcal{C}$ goes to the possibly infinite set $\mathcal{C}_{PQ}$.

Remark 5.1. This construction of the shearing maps is very similar to the construction of the slithering maps Σ_{gh} in Section 4.1: For a finite subset $\mathcal{C} = \{R_1, \ldots, R_m\} \subset \mathcal{C}_{PQ}$, for every component $R_i \in \mathcal{C}$, we have an element $\left(T^{\varepsilon(P,R_1)}_{g_1^0} T^{-\varepsilon(P,R_1)}_{g_1^1}\right)$, and composing those gives us $\psi^{\varepsilon}_{\mathcal{C}}$. As in the construction of the slithering maps, we then let $\mathcal{C}$ go to $\mathcal{C}_{PQ}$ and show that the limit exists (Proposition 5.3). Also the ideas of the proofs are similar to the ones for the slithering map. However, the two families of maps are different and should not be confused. In particular, the slithering maps only depend on the representation ρ, whereas the shearing maps highly depend on the twisted cycle $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$.

Before showing convergence, we prove a technical lemma. Similar to Lemma 4.4, we show that $\psi_{\mathcal{C}}$ is uniformly bounded for all finite subsets $\mathcal{C} \subset \mathcal{C}_{PQ}$.

Lemma 5.2. *For $\|\varepsilon\|_{\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a}_\theta)}$ small enough, for every finite subset $\mathcal{C} \subset \mathcal{C}_{PQ}$, the distance $\mathrm{d}_G(\psi^{\varepsilon}_{\mathcal{C}}, Id)$ is uniformly bounded. The bound depends on $\tilde{k}$ and ρ.*

Proof. Recall that the connected components of $\tilde{S} \setminus \tilde{\lambda}$ that lie in $\mathcal{C}_{PQ}$ are either wedge components or non-wedge components (Definition 4.11) and that the number of non-wedge components in $\mathcal{C}_{PQ}$ is finite. Let $Q_1, \ldots, Q_l \in \mathcal{C}_{PQ}$ be the finitely many non-wedge components. It might happen that all components are wedge components, for instance if the lamination λ is maximal. Every finite subset $\mathcal{C} \subset \mathcal{C}_{PQ}$ can contain wedge components and non-wedge components. Let $\mathcal{C} = \{R_1, \ldots, R_{m_1}, Q_{i_1}, \ldots, Q_{i_{m_2}}, \}$, where the R_j are wedge components and the Q_{i_j} are non-wedge components. In particular, $\{i_1, \ldots, i_{m_2}\} \subset \{1, \ldots, l\}$. By the triangle inequality and left-invariance of d_G, we have

$$\begin{aligned}
\mathrm{d}_G(\psi^{\varepsilon}_{\mathcal{C}}, Id) &\leq \sum_{j=1}^{m_1} \mathrm{d}_G\left(T^{\varepsilon(P,R_j)}_{g_j^0} T^{-\varepsilon(P,R_j)}_{g_j^1}, Id\right) + \sum_{j=1}^{m_2} \mathrm{d}_G\left(T^{\varepsilon(P,Q_{i_j})}_{g^0_{Q_{i_j}}} T^{-\varepsilon(P,Q_{i_j})}_{g^1_{Q_{i_j}}}, Id\right) \\
&\leq \sum_{j=1}^{m_1} \mathrm{d}_G\left(T^{\varepsilon(P,R_j)}_{g_j^0} T^{-\varepsilon(P,R_j)}_{g_j^1}, Id\right) + \sum_{j=1}^{l} \mathrm{d}_G\left(T^{\varepsilon(P,Q_j)}_{g^0_{Q_j}} T^{-\varepsilon(P,Q_j)}_{g^1_{Q_j}}, Id\right).
\end{aligned}$$

The second sum is finite and depends only on $\tilde{k}$, but not on $\mathcal{C}$. For the first sum, we can use Corollary 4.19 and obtain

$$\sum_{j=1}^{m_1} \mathrm{d}_G\left(T^{\varepsilon(P,R_j)}_{g_j^0} T^{-\varepsilon(P,R_j)}_{g_j^1}, Id\right) \leq C_1 \sum_{j=1}^{m_1} \left(e^{\|\varepsilon(P,R_j)\|_{\mathfrak{a}}} + 1\right) e^{-Ar(R_j)}.$$

Since $\mathfrak{a}_\theta$ is a finite-dimensional vector space, the norms $\|\cdot\|_{\mathfrak{a}}$ restricted to $\mathfrak{a}_\theta$ and the norm $\|\cdot\|_{\mathfrak{a}_\theta}$ from Lemma 3.23 are equivalent. Hence, Lemma 3.23 gives $\|\varepsilon(P,R_j)\|_{\mathfrak{a}} \leq C\,\|\varepsilon\|\,(r(R_j)+1)$ for a norm $\|\cdot\|$ on $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$. Using the fact that the number of connected components with fixed divergence radius is uniformly bounded by some $D \in \mathbb{N}$

(Lemma 3.11), it follows that there exists $C_2 > 0$ such that

$$\sum_{j=1}^{m_1} \left(e^{\|\varepsilon(P,R_j)\|_{\mathfrak{a}}} + 1\right) e^{-Ar(R_j)} \leq C_2 \sum_{j=1}^{m_1} \left(e^{C\|\varepsilon\|(r(R_j)+1)}\right) e^{-Ar(R_j)} + C_2 \sum_{j=1}^{m_1} e^{-Ar(R_j)}$$
$$\leq C_2 D \sum_{r=0}^{\infty} \left(e^{C\|\varepsilon\|(r+1)}\right) e^{-Ar} + C_2 D \sum_{r=0}^{\infty} e^{-Ar}.$$

The second sum converges as $A > 0$. For $\|\varepsilon\| < A/C$, also the first sum converges, independent of m_1. Thus, $\mathrm{d}_G(\psi^{\varepsilon}_{\mathcal{C}}, Id)$ is uniformly bounded, the bound depending on $\tilde{k}$ and ρ. □

We remark that in the above proof, the distinction between wedge components and non-wedge components is only necessary if the lamination λ is non-maximal.

We can now prove convergence of the maps $\varphi_{\mathcal{C}}$ as the finite sets $\mathcal{C}$ converge to $\mathcal{C}_{PQ}$.

Proposition 5.3. *There exists a constant $B > 0$ depending on $\tilde{k}$ and ρ such that for $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ with $\|\varepsilon\|_{\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)} < B$, the limit*

$$\varphi^{\varepsilon}_{PQ} := \lim_{\mathcal{C} \to \mathcal{C}_{PQ}} \varphi^{\varepsilon}_{\mathcal{C}}$$

exists.

Proof. If $\mathcal{C}_{PQ}$ is finite, we have nothing to prove. So assume that $\mathcal{C}_{PQ}$ is infinite. For a finite subset $\mathcal{C} \subset \mathcal{C}_{PQ}$ as above, consider $\psi^{\varepsilon}_{\mathcal{C}}$ as in (5.2). To show that $\psi^{\varepsilon}_{\mathcal{C}}$ converges for $\mathcal{C} \to \mathcal{C}_{PQ}$, analogous to the proof of Lemma 4.5, choose a sequence $(\mathcal{C}_m)_{m \in \mathbb{N}}$ of subsets of $\mathcal{C}_{PQ}$ such that $\mathcal{C}_m$ has cardinality m and such that $\mathcal{C}_m \subset \mathcal{C}_{m+1}$ for all $m \in \mathbb{N}$. Fix $m \in \mathbb{N}$, let $\mathcal{C}_m = \{R_1, \ldots, R_m\}$ and let $\mathcal{C}_{m+1} = \mathcal{C}_m \cup \{R\}$ for a connected component $R \subset \tilde{S} \setminus \tilde{\lambda}$. Let R separate R_i from R_{i+1} and let $\mathcal{C} := \{R_1, \ldots R_i\}$ and $\mathcal{C}' := \{R_{i+1}, \ldots, R_m\}$. Then

$$\psi^{\varepsilon}_{\mathcal{C}_{m+1}} = \psi^{\varepsilon}_{\mathcal{C}} \left(T^{\varepsilon(P,R)}_{g^0_R} T^{-\varepsilon(P,R)}_{g^1_R}\right) \psi^{\varepsilon}_{\mathcal{C}'}.$$

As in (4.4), we have that

$$\mathrm{d}_G(\psi^{\varepsilon}_{\mathcal{C}_{m+1}}, \psi^{\varepsilon}_{\mathcal{C}_m}) \leq C e^{-Ar(R)},$$

where we use the fact that $\psi^{\varepsilon}_{\mathcal{C}}$ is uniformly bounded (Lemma 5.2) and Corollary 4.19. We can apply Corollary 4.19, because for m big enough, $R \in \mathcal{C}_{m+1} \backslash \mathcal{C}_m$ is a wedge component. It follows that $\left(\psi^{\varepsilon}_{\mathcal{C}_m}\right)_{m \in \mathbb{N}}$ is a Cauchy sequence and thus converges. Thus, also $\varphi^{\varepsilon}_{\mathcal{C}_m}$ converges as m goes to infinity. The constant B here is the same as in Lemma 5.2. □

Definition 5.4. For $P, Q \in \tilde{S} \backslash \tilde{\lambda}$ and $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ small enough, the element $\varphi^\varepsilon_{PQ} \in G$ is called *shearing map* from P to Q.

If no confusion is possible, we omit the superscript and simply write φ_{PQ} for φ^ε_{PQ}.

Remark 5.5. We can now motivate the twist condition in Definition 3.20: It guarantees that the shearing maps behave well under taking the inverse. We illustrate this for the case that P and Q are adjacent. The general case is treated below in Proposition 5.6. If $P, Q \subset \tilde{S} \setminus \tilde{\lambda}$ are two connected components, then, by definition of $\widehat{k}_{PQ}$, we have $\widehat{k}_{QP} = \mathfrak{R}(\widehat{k}_{PQ})$, where $\mathfrak{R}\colon \widehat{U} \to \widehat{U}$ is the orientation-reversing involution. It follows that for $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$,

$$\iota(\varepsilon(P, Q)) = \mathfrak{R}^*\varepsilon(\widehat{k}_{PQ}) = \varepsilon(\mathfrak{R}(\widehat{k}_{PQ})) = \varepsilon(\widehat{k}_{QP}) = \varepsilon(Q, P).$$

Let P and Q be adjacent components, separated by a geodesic g, oriented to the left as seen from P. Denote by $\overline{g}$ the same geodesics, but with opposite orientation. Using the properties of the stretching map (Lemma 4.16)

$$(\varphi_{PQ})^{-1} = \left(T_g^{\varepsilon(P,Q)}\right)^{-1} = T_g^{-\varepsilon(P,Q)} = T_{\overline{g}}^{\iota(\varepsilon(P,Q))} = T_{\overline{g}}^{\varepsilon(Q,P)} = \varphi_{QP}. \tag{5.4}$$

Thus, the twist condition $\mathfrak{R}^*\varepsilon = \iota(\varepsilon)$ guarantees that $\varphi_{PQ}^{-1} = \varphi_{QP}$.

The shearing maps φ_{PQ} have some natural properties.

Proposition 5.6. *For connected components P, Q, R in $\tilde{S} \setminus \tilde{\lambda}$, and $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ small enough, the shearing maps satisfy*

$$\varphi_{QP} = \varphi_{PQ}^{-1}, \tag{5.5}$$

$$\varphi_{PR}\, \varphi_{RQ} = \varphi_{PQ}. \tag{5.6}$$

Further, the shearing maps are ρ-equivariant, i.e. for all $\gamma \in \pi_1(S)$

$$\varphi_{(\gamma P)(\gamma Q)} = \rho(\gamma)\varphi_{PQ}\rho(\gamma)^{-1}. \tag{5.7}$$

Proof. We first show ρ-equivariance (5.7). Observe that $\mathcal{C}_{(\gamma P)(\gamma Q)} = \gamma\mathcal{C}_{PQ}$, and that the twisted cycle $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ satisfies $\varepsilon(\gamma R, \gamma R') = \varepsilon(R, R')$ for all $R, R' \subset \tilde{S} \setminus \tilde{\lambda}$. Let $\mathcal{P}' = \{R'_1, \ldots, R'_m\} \subset \mathcal{C}_{(\gamma P)(\gamma Q)}$ be a finite subset and $R_i \in \mathcal{C}_{PQ}$ such that $R'_i = \gamma R_i$. Then, using the ρ-equivariance of the stretching maps (Lemma 4.16), we have

$$\begin{aligned}
\varphi_{\mathcal{C}'} &:= \prod_{i=1}^{m} \left(T_{g^0_{R'_i}}^{\varepsilon(\gamma P, R'_i)} T_{g^1_{R'_i}}^{-\varepsilon(\gamma P, R'_i)} \right) T_{g^0_{\gamma Q}}^{\varepsilon(\gamma P, \gamma Q)} \\
&= \prod_{i=1}^{m} \left(T_{\gamma g^0_{R_i}}^{\varepsilon(P, R_i)} T_{\gamma g^1_{R_i}}^{-\varepsilon(P, R_i)} \right) T_{\gamma g^0_{Q}}^{\varepsilon(P, Q)} \\
&= \prod_{i=1}^{m} \left(\rho(\gamma) T_{g^0_{R_i}}^{\varepsilon(P, R_i)} T_{g^1_{R_i}}^{-\varepsilon(P, R'_i)} \rho(\gamma)^{-1} \right) \rho(\gamma) T_{g^0_{Q}}^{\varepsilon(P, Q)} \rho(\gamma)^{-1} \\
&= \rho(\gamma) \left(\prod_{i=1}^{m} \left(T_{g^0_{R_i}}^{\varepsilon(P, R_i)} T_{g^1_{R_i}}^{-\varepsilon(P, R'_i)} \right) T_{g^0_{Q}}^{\varepsilon(P, Q)} \right) \rho(\gamma)^{-1} \\
&= \rho(\gamma) \varphi_{\mathcal{C}} \rho(\gamma)^{-1}.
\end{aligned}$$

If we let $\mathcal{C}'$ tend to $\mathcal{C}_{PQ}$, it follows that $\varphi_{\gamma P \gamma Q} = \rho(\gamma)\varphi_{PQ}\rho(\gamma)^{-1}$.

For the behavior under taking the inverse, we first assume that $\mathcal{C}_{PQ} = \{R_1, \dots, R_m\}$ is finite. Set $R_0 := P$ and $R_{m+1} := Q$. Since all the connected components are adjacent, we have $g^1_{R_i} = g^0_{R_{i+1}}$. Using the additivity of the transverse cycle, we have $-\varepsilon(P, R_i) + \varepsilon(P, R_{i+1}) = \varepsilon(R_i, R_{i+1})$. Further, the stretching maps satisfy $T_g^{H_1} T^{H_2} = T_g^{H_1+H_2}$ for $H_1, H_2 \in \mathfrak{a}_\theta$ by Lemma 4.16. This simplifies the expression of the shearing map to

$$\varphi_{PQ} = T_{g^0_{R_1}}^{\varepsilon(P, R_1)} T_{g^0_{R_2}}^{\varepsilon(R_1, R_2)} \cdots T_{g^0_{R_{m+1}}}^{\varepsilon(R_m, Q)}. \tag{5.8}$$

Let $\mathcal{C}_{QP} = \{R'_1, \dots, R'_m\}$, where $R'_i = R_{m-i+1}$. As sets, $\mathcal{C}_{PQ}$ and $\mathcal{C}_{QP}$ agree, but the elements are labeled in opposite order. Then $g^0_{R'_i} = \overline{g}^1_{R_{m-i+1}} = \overline{g}^0_{R_{m-i+2}}$, where the last equality holds because the triangles R_{m-i+1} and R_{m-i+2} are adjacent. Using the equality from (5.4) for all the stretching maps in the composition, we have

$$\begin{aligned}
\varphi_{PQ}^{-1} &= \left(T_{g^0_{R_1}}^{\varepsilon(P, R_1)} \cdots T_{g^0_{R_{m+1}}}^{\varepsilon(R_m, Q)} \right)^{-1} \\
&= T_{\overline{g}^0_{R_{m+1}}}^{\varepsilon(Q, R_m)} \cdots T_{\overline{g}_{R^0_1}}^{\varepsilon(R_1, P)} \\
&= T_{g^0_{R'_1}}^{\varepsilon(Q, R'_1)} \cdots T_{g^0_{R'_{m+1}}}^{\varepsilon(R'_m, P)} \\
&= \varphi_{QP}.
\end{aligned} \tag{5.9}$$

In the general case, $\mathcal{C}_{PQ}$ is infinite. In this case, to show the behavior under taking inverses, we use the same idea as in the proof of Lemma 4.7. We express φ_{PQ} as limit of different maps $\widehat{\varphi}_{\mathcal{C}}$, that are constructed using collections of triangles that are adjacent to the respective next one. For $\mathcal{C} = \{R_1, \dots, R_m\} \subset \mathcal{C}_{PQ}$, define auxiliary geodesics h_i and triangles Q^0_i, Q^1_i as in the proof of Lemma 4.7 (see Figure 4.1). In this way, we approximate the part of the geodesic lamination between two components R_i and R_{i+1} by triangles. Set

$$\begin{aligned}\widehat{\psi}_{\mathcal{C}} := &\left(T_{g^0_{Q^0_0}}^{\varepsilon(P,Q^0_0)} T_{g^1_{Q^0_0}}^{-\varepsilon(P,Q^0_0)}\right)\left(T_{g^0_{Q^1_0}}^{\varepsilon(P,Q^1_0)} T_{g^1_{Q^1_0}}^{-\varepsilon(P,Q^1_0)}\right)\\ &\prod_{i=1}^{m}\left(\left(T_{g^0_i}^{\varepsilon(P,R_i)} T_{g^1_i}^{-\varepsilon(P,R_i)}\right)\left(T_{g^0_{Q^0_i}}^{\varepsilon(P,Q^0_i)} T_{g^1_{Q^0_i}}^{-\varepsilon(P,Q^0_i)}\right)\left(T_{g^0_{Q^1_i}}^{\varepsilon(P,Q^1_i)} T_{g^1_{Q^1_i}}^{-\varepsilon(P,Q^1_i)}\right)\right).\end{aligned}$$

and $\widehat{\varphi}_{\mathcal{C}} := \widehat{\psi}_{\mathcal{C}} T_{g^0_Q}^{\varepsilon(P,Q)}$. We remark here that the value of $\varepsilon(P, Q_i^{0/1})$ is not well-defined, because the auxiliary triangles $Q_i^{0/1}$ are not in the complement of the lamination. To define $\varepsilon(P, Q_i^{0/1})$ we have to choose an arc from P to $Q_i^{0/1}$ and $\varepsilon(P, Q_i^{0/1})$ is determined by the endpoint of this arc. However, when we take the limit $\mathcal{C} \to \mathcal{C}_{PQ}$, this choice becomes irrelevant.

With the same techniques as in the proof of Lemma 4.7, using that $\mathrm{d}_G(\psi_{\mathcal{C}}, \mathrm{Id})$ is uniformly bounded by Lemma 5.2, we can show that

$$\mathrm{d}_G\left(\widehat{\psi}_{\mathcal{C}}, \psi_{\mathcal{C}}\right) \leq C \sum_{R \in \mathcal{C}_{PQ} \setminus \mathcal{C}} e^{-Ar(R)},$$

so $\widehat{\psi}_{\mathcal{C}}$ and $\psi_{\mathcal{C}}$ have the same limit as $\mathcal{C}$ tends to $\mathcal{C}_{PQ}$. It follows that also $\widehat{\varphi}_{\mathcal{C}}$ converges to φ_{PQ}.

For a finite subset $\mathcal{C} \subset \mathcal{C}_{PQ}$, we can show the behavior under taking inverse for $\widehat{\varphi}_{\mathcal{C}}$ just as in (5.9), because all the triangles appearing in the composition are adjacent to their respective neighbors. Since $\widehat{\varphi}_{\mathcal{C}}$ converges to φ_{PQ}, also φ_{PQ} satisfies (5.5) by continuity.

For the composition property (5.6), assume first that $P, R, Q \subset \tilde{S} \setminus \tilde{\lambda}$ are such that R separates P and Q and that $\mathcal{C}_{PQ}$, $\mathcal{C}_{QR}$ and $\mathcal{C}_{PR}$ are finite. We have that $\mathcal{C}_{PQ} = \mathcal{C}_{PR} \cup \{R\} \cup \mathcal{C}_{RQ}$ and the composition property directly follows from writing the shearing maps as in (5.8). An according property for the maps $\widehat{\varphi}_{\mathcal{C}}$ can be shown in the same way, and the composition property (5.6) for the shearing maps φ_{PQ} follows by taking the limit. If P separates R from Q, then we have by the above argument $\varphi_{RQ} = \varphi_{RP}\varphi_{PQ}$, which implies $\varphi_{PQ} = \varphi_{PR}\varphi_{RQ}$, since $\varphi_{RP}^{-1} = \varphi_{PR}$. A similar argument works if Q separates P from R. In general, none of P, Q and R separates the other two, but there exists a component P' that separates P, R and Q pairwise (see Figure 5.1). Then we have

$$\varphi_{PQ} = \varphi_{PP'}\varphi_{P'Q} = \varphi_{PR}\varphi_{RP'}\varphi_{P'Q} = \varphi_{PR}\varphi_{RQ}.$$

This shows that the composition property (5.6) holds for any three connected components $P, R, Q \subset \tilde{S} \setminus \tilde{\lambda}$. $\square$

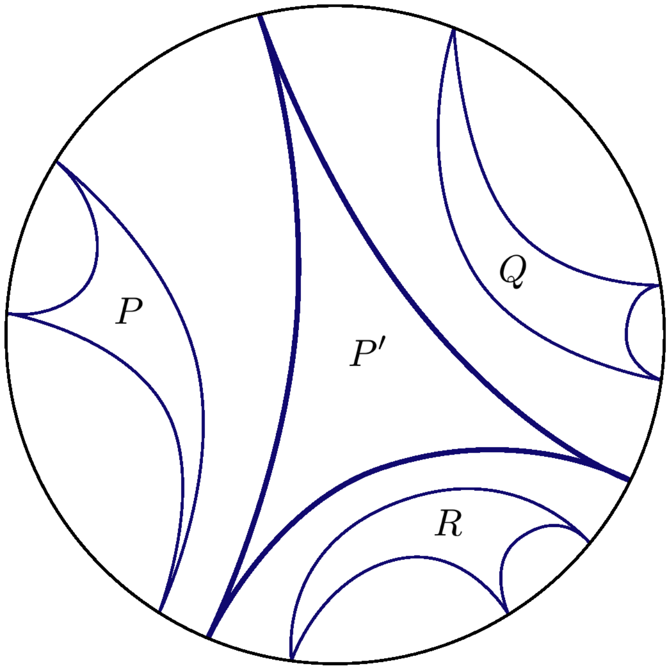


Figure 5.1.: If three components $P, Q, R \subset \tilde{S} \setminus \tilde{\lambda}$ are such that none of the three separates the others, then there exists a fourth component $P' \subset \tilde{S} \setminus \tilde{\lambda}$ that separates P, Q and R pairwise.

Remark 5.7. Let $\psi_{PQ} := \varphi_{PQ} \cdot T_{g_Q^0}^{-\varepsilon(P,Q)}$. By (5.6), for any connected component $R \in \mathcal{C}_{PQ}$, we can write the shearing map φ_{PQ} as $\varphi_{PQ} = \varphi_{PR}\psi_{RQ}T_{g_Q^0}^{\varepsilon(R,Q)}$.

The following estimate will be useful in Chapter 7, when we look at injectivity properties of cataclysms.

Lemma 5.8. *Let $r \in \mathbb{N}$ and let $P, Q \subset \tilde{S} \setminus \tilde{\lambda}$ be such that all $R \in \mathcal{C}_{PQ}$ have divergence radius bigger than r, i.e. satisfy $r(R) > r$. Then, for $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ small enough, there exist constants $C', A' > 0$ such that $\mathrm{d}_G(\psi_{PQ}, Id) \leq Ce^{-A'r}$.*

Proof. As in the proof of Lemma 5.2, we have

$$\begin{aligned}\mathrm{d}_G(\psi_{PQ}, \mathrm{Id}) &\leq C_1 \sum_{R \in \mathcal{C}_{PQ}} \left(e^{\|\varepsilon(P,R)\|_{\mathfrak{a}}} + 1\right) e^{-Ar(R)} \\ &\leq C_1 \sum_{R \in \mathcal{C}_{PQ}} \left(e^{C\|\varepsilon\|(r(R)+1)} + 1\right) e^{-Ar(R)}.\end{aligned}$$

Using that all components in $\mathcal{C}_{PQ}$ have divergence radius at least $r+1$ and that the number of components with fixed divergence radius is bounded by some $D \in \mathbb{N}$, this gives us

$$
\begin{aligned}
\mathrm{d}_G(\psi_{PQ}, \mathrm{Id}) &\leq C_1 D \sum_{l=r+1}^{\infty} \left(e^{C\|\varepsilon\|(l+1)} + 1\right) e^{-Al} \\
&\leq C_1 D \sum_{l=r+1}^{\infty} \left(e^{C\|\varepsilon\|(l+1)}\right) e^{-Al} + C_1 \sum_{l=r+1}^{\infty} e^{-Al}.
\end{aligned}
$$

Both sums are the remainder term of the geometric series and for $\|\varepsilon\| < A/C$, they are bounded by a constant times $e^{-A'r}$, where $A' := -(C\,\|\varepsilon\| - A)$. This finishes the proof. $\square$

Up to now, the bound on the cycle $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ that guarantees the convergence depends on the arc $\tilde{k}$ transverse to $\tilde{\lambda}$. We now show that there exists a constant depending on the representation ρ only.

Proposition 5.9. *There exists a constant $B > 0$ depending on the representation ρ only such that for all connected components $P, Q, R \subset \tilde{S} \setminus \tilde{\lambda}$, for all $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ with $\|\varepsilon\|_{\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)} < B$, the limit $\varphi_{PQ} = \lim_{\mathcal{C} \to \mathcal{C}_{PQ}} \varphi_{PQ}$ exists and satisfies* (5.5)*,* (5.6) *and* (5.7)*.*

Proof. The only thing left to show is that the constant B from Proposition 5.3 can be made independent of the arc $\tilde{k}$. Choose a collection $k_1, \dots, k_m$ of arcs on the surface S transverse to λ such that every connected component in $S \setminus \lambda$ and every leaf of λ is met by at least one arc k_j. For every j, let $B_j > 0$ be the constant as in Proposition 5.3 for a lift $\tilde{k}_j$ of k_j. This is independent of the choice of lift. Let $B := \min\{B_1, \dots, B_m\}$. Let $P, Q \subset \tilde{S} \setminus \tilde{\lambda}$ be connected components. Then there exists a finite sequence of components $P = R_0, R_1, \dots, R_N, R_{N+1} = Q$ such that R_j separates R_{j-1} from R_{j+1} and R_j, R_{j+1} are met by the same lift $\widetilde{k}_{i_j}$ of a transverse arc k_{i_j}. For $\|\varepsilon\|_{\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)} < B$, the maps $\varphi_{R_j R_{j+1}}$ exist and thus also $\varphi_{PQ} = \varphi_{PR_1} \cdots \varphi_{R_N Q}$. $\square$

From Proposition 5.9, it follows that there exists a neighborhood $\mathcal{V}_\rho \subset \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ around 0, depending on ρ, and a map

$$
\begin{aligned}
\mathcal{V}_\rho &\to G^{\{(P,Q) \;|\; P,Q \subset \tilde{S} \setminus \tilde{\lambda}\}} \\
\varepsilon &\mapsto \{\varphi^{\varepsilon}_{PQ}\}_{(P,Q)}
\end{aligned}
$$

that assigns to ε the associated family of shearing maps. By construction, the family of shearing maps $\{\varphi^{\varepsilon}_{PQ}\}_{(P,Q)}$ depends continuously on the shearing parameter ε.

Remark 5.10. The family of shearing maps $\{\varphi_{PQ}\}_{(P,Q)}$ depends on the representation ρ. If two representations $\rho, \rho' \colon \pi_1(S) \to G$ are conjugate, $\rho'(\gamma) = M\rho(\gamma)M^{-1}$ for some $M \in G$, then for every oriented geodesic g in $\tilde{S}$ and $H \in \mathfrak{a}_\theta$, the associated stretching maps T_g^H and ${T'}_g^H$ are conjugated by M. Consequently, also the shearing maps φ_{PQ} and φ'_{PQ} are conjugated by M. Thus, the conjugacy class of the shearing map φ_{PQ} only depends on the conjugacy class of ρ.

5.2. Cataclysm deformations

We have now developed the concepts that we need to define cataclysm deformations for θ-Anosov representations.

Let $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a}_\theta)$ be sufficiently small such that the shearing map φ_{PQ} exists for all connected components $P, Q \subset \tilde{S} \setminus \tilde{\lambda}$. Fix a reference component $P_0 \subset \tilde{S} \setminus \tilde{\lambda}$. For $\gamma \in \pi_1(S)$ set

$$\Lambda_0^\varepsilon \rho(\gamma) := \varphi_{P_0(\gamma P_0)} \rho(\gamma) \ .$$

Lemma 5.11. *The map $\Lambda_0^\varepsilon \rho \colon \pi_1(S) \to G$ is a group homomorphism. Further, if $P_1 \subset \tilde{S} \setminus \tilde{\lambda}$ is another reference component, then $\Lambda_1^\varepsilon \rho$ is conjugate to $\Lambda_0^\varepsilon \rho$ by $\varphi_{P_1 P_0}$.*

Proof. Using the ρ-equivariance and the composition property of the shearing map (Proposition 5.6) we have for $\gamma_1, \gamma_2 \in \pi_1(S)$

$$\begin{aligned}
\Lambda_0^\varepsilon \rho(\gamma_1\gamma_2) &= \varphi_{P_0(\gamma_1\gamma_2 P_0)} \rho(\gamma_1\gamma_2) \\
&= \varphi_{P_0(\gamma_1 P_0)} \varphi_{(\gamma_1 P_0)(\gamma_1\gamma_2 P_0)} \rho(\gamma_1)\rho(\gamma_2) \\
&= \varphi_{P_0(\gamma_1 P_0)} \left(\rho(\gamma_1) \varphi_{P_0(\gamma_2 P_0)} \rho(\gamma_1)^{-1}\right) \rho(\gamma_1)\rho(\gamma_2) \\
&= \left(\varphi_{P_0(\gamma_1 P_0)} \rho(\gamma_1)\right) \left(\varphi_{P_0(\gamma_2 P_0)} \rho(\gamma_2)\right) \\
&= \Lambda_0^\varepsilon \rho(\gamma_1) \Lambda_0^\varepsilon \rho(\gamma_2).
\end{aligned}$$

If P_1 is another reference triangle, then for $\gamma \in \pi_1(S)$,

$$\begin{aligned}
\Lambda_1^\varepsilon \rho(\gamma) &= \varphi_{P_1 \gamma P_1} \rho(\gamma) \\
&= \varphi_{P_1 P_0} \varphi_{P_0(\gamma P_0)} \varphi_{(\gamma P_0)(\gamma P_1)} \rho(\gamma) \\
&= \varphi_{P_1 P_0} \varphi_{P_0(\gamma P_0)} \rho(\gamma) \varphi_{P_0 P_1} \rho(\gamma)^{-1} \rho(\gamma) \\
&= \varphi_{P_1 P_0} \Lambda_0^\varepsilon \rho(\gamma) \left(\varphi_{P_1 P_0}\right)^{-1},
\end{aligned}$$

so the two cataclysms with respect to the reference triangles P_0 and P_1 are conjugate by $\varphi_{P_1 P_0}$. □

We have now constructed a map $\Lambda_0^\varepsilon \colon \mathcal{V}_\rho \to \mathrm{Hom}(\pi_1(S), G)$, defined on a small neighborhood of 0 in $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ that depends on the reference triangle P_0.

Theorem 5.12. *Let ρ be a θ-Anosov representation. Fix a reference component $P_0 \subset \tilde{S} \backslash \tilde{\lambda}$. There exists a neighborhood $\mathcal{V}_\rho$ of 0 in $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ and a continuous map*

$$\begin{aligned} \Lambda_0 \colon \mathcal{V}_\rho &\to \mathrm{Hom}(\pi_1(S), G) \\ \varepsilon &\mapsto \Lambda_0^\varepsilon \rho \end{aligned}$$

such that $\Lambda_0^0 \rho = \rho$. Further, there exists a neighborhood $\mathcal{U}_\rho \subset \mathcal{V}_\rho$ such that for all $\varepsilon \in \mathcal{U}_\rho$, $\Lambda_0^\varepsilon \rho$ is θ-Anosov.

Proof. The only thing left to show for the first statement is continuity, which follows from the fact that the shearing maps φ_{PQ}^ε depend continuously on the shearing parameter ε. The second statement follows from the fact that the set of θ-Anosov representations is open in $\mathrm{Hom}(\pi_1(S), G)$. $\square$

If we change the representation ρ by conjugation, or if we change the reference triangle, we obtain conjugate representations. Thus, the map Λ_0 descends to the character variety.

Corollary 5.13. *Let ρ be a θ-Anosov representation and denote by $[\rho]$ the corresponding element in the character variety $\chi^{\theta-\mathrm{Anosov}}(S, G)$. There exists a neighborhood $\mathcal{U}_\rho$ of 0 in $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ and a continuous map*

$$\begin{aligned} \Lambda \colon \mathcal{U}_\rho &\to \chi^{\theta\text{-}Anosov}(S, G) \\ \varepsilon &\mapsto [\Lambda_0^\varepsilon \rho] \end{aligned}$$

such that $\Lambda^0[\rho] = [\rho]$.

Proof. Since conjugate representations give conjugate shearing maps, there exists a well-defined map Λ on the representation variety. This map does not depend on the choice of reference triangle $P_0 \subset \tilde{S} \setminus \tilde{\lambda}$. The neighborhood $\mathcal{U}_\rho$ is the same as in Theorem 5.12. $\square$

Abusing notation, we call both maps Λ_0 and Λ *cataclysm deformation based at ρ along λ* with respect to the *shearing parameter* $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$. When talking about cataclysms on the level of homomorphisms with respect to a fixed reference component P_0, we write Λ_0. Without the subscript 0, we refer to the deformation on the character variety.

Example 5.14. Let $G = \mathrm{PSL}(2,\mathbb{R})$. In this case, $\mathfrak{a} = \{\mathrm{diag}(a,-a) | a \in \mathbb{R}\} \cong \mathbb{R}$ and the simple root system consist of only one element, $\Delta = \{\alpha_1\}$ with $\alpha\,(\mathrm{diag}(a,-a)) = 2a$. The opposition involution ι is trivial, so the space $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a}_\theta)$ of twisted transverse cycles consist exactly of those transverse cycles for $\widehat{\lambda}$ that are invariant under the orientation reversing involution $\mathfrak{R}$, which can be identified with $\mathcal{H}(\lambda;\mathbb{R})$ as in the proof of Proposition 3.21. For $H = \mathrm{diag}(a,-a) \in \mathfrak{a}$ and an oriented geodesic g, the stretching map T_g^H is the hyperbolic isometry with oriented axis g and translation length $\alpha(H) = 2a$. In this case cataclysms are *shearing deformations* for hyperbolic metrics, that were first described by Thurston [Thu98] and investigated in detail by Bonahon [Bon96]. They can be used to define *shearing coordinates* for Teichmüller space. In particular, shearing deformations are injective, i.e. deforming with respect to different shearing parameters gives different hyperbolic metrics. Further, if we restrict to cycles with values in $\mathfrak{a}^+$, then a transverse cycle is a transverse measure, all the stretching maps T_g^H move to the left with respect to the orientation of g and the resulting deformations are Thurston's *earthquake maps* [Thu86].

5.3. Properties of cataclysm deformations

We now show two properties of cataclysms: additivity with respect to the transverse twisted cycles and natural behavior with respect to composing an Anosov representation with a Lie group homomorphism.

5.3.1. Additivity

In this subsection we show that the cataclysm deformation with respect to a fixed reference component P_0 behaves well under addition of cycles, i.e. $\Lambda_0^{\varepsilon+\eta}\rho = \Lambda_0^{\varepsilon}\,(\Lambda_0^{\eta}\rho)$ for $\eta, \varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a}_\theta)$. We have to keep in mind here that a cataclysm deformation is *based at* ρ for some representation ρ, i.e. the whole construction depends on the representation we start with. In this case, we look at the $(\varepsilon+\eta)$-cataclysm deformation based at ρ and at the ε-cataclysm deformation based at $\Lambda_0^{\eta}\rho$.

To shorten notation, let $P = P_0$ be the fixed reference component. Let $\eta \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a}_\theta)$ be small and $\rho' := \Lambda_0^{\eta}\rho$. We will denote all elements constructed with respect to ρ' with a prime, in particular the stretching maps ${T'}_g^H$ and shearing maps ${\varphi'}_{PQ}^{\varepsilon}$. The first step to show additivity is to relate the stretching maps T_g^H and ${T'}_g^{H'}$ for ρ and ρ'. Let g be an oriented leaf bounding a component Q. Let $m_g \in G$ be such that

$$m_g \cdot P_\theta^{\pm} = P_g^{\pm} = \zeta(g^{\pm}).$$

In Theorem 6.1 below we prove that if x is a vertex of the component Q, then that $\zeta'(x) = \varphi^{\eta}_{PQ} \cdot \zeta(x)$. Define $m'_g := \varphi^{\eta}_{PQ} m_g$. Then it follows that

$$m'_g \cdot P^{\pm}_{\theta} = \left(\varphi^{\eta}_{PQ} m_g\right) \cdot P^{\pm}_{\theta} = \varphi^{\eta}_{PQ} \cdot P^{\pm}_{g} = \varphi^{\eta}_{PQ} \cdot \zeta(g^{\pm}) = \zeta'(g^{\pm}) = {P'}^{\pm}_{g}.$$

Hence, for $H \in \mathfrak{a}_\theta$, we have

$$ {T'}^{H}_{g} = m'_g \exp(H)(m'_g)^{-1} = \varphi^{\eta}_{PQ} \left(m_g \exp(H) m_g^{-1}\right) \left(\varphi^{\eta}_{PQ}\right)^{-1} = \varphi^{\eta}_{PQ} T^{H}_{g} \left(\varphi^{\eta}_{PQ}\right)^{-1}, \quad (5.10)$$

so ${T'}^{H}_{g}$ is conjugated to T^{H}_{g} by φ^{η}_{PQ}.

Proposition 5.15. *Let P be the fixed reference component for the cataclysm deformation Λ_0. Then for $\varepsilon, \eta \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ small enough, for every component $Q \subset \tilde{S} \setminus \tilde{\lambda}$ it holds that*

$$ {\varphi'}^{\varepsilon}_{PQ} = \varphi^{\varepsilon+\eta}_{PQ} \left(\varphi^{\eta}_{PQ}\right)^{-1}.$$

Proof. The details of the proof can be found in the Appendix A.2. The ideas are very similar to the proofs for the shearing maps in Section 5.1 and use (5.10) . In particular, we approximate $\mathcal{C}_{PQ}$ by a sequence $(\mathcal{C}_r)_{r \in \mathbb{N}}$ of finite subsets, where $\mathcal{C}_r$ consists of all components of $\tilde{S} \setminus \tilde{\lambda}$ of divergence radius at most r. Then we compute ${\varphi'}^{\varepsilon}_{\mathcal{C}_r}$ and show that it converges to $\varphi^{\varepsilon+\eta}_{PQ} \left(\varphi^{\eta}_{PQ}\right)^{-1}$ as r goes to infinity. ☐

Remark 5.16. Proposition 5.15 relies on the fact that P is the fixed reference component for the cataclysm deformation Λ_0. If we look at components $R, Q \subset \tilde{S} \setminus \tilde{\lambda}$ where R is not equal to the reference component P, we have, using the composition property of the shearing maps,

$$\begin{aligned} {\varphi'}^{\varepsilon}_{RQ} &= {\varphi'}^{\varepsilon}_{RP} {\varphi'}^{\varepsilon}_{PQ} \\ &= \left({\varphi'}^{\varepsilon}_{PR}\right)^{-1} {\varphi'}^{\varepsilon}_{PQ} \\ &= \left(\varphi^{\varepsilon+\eta}_{PR} \left(\varphi^{\eta}_{PR}\right)^{-1}\right)^{-1} \varphi^{\varepsilon+\eta}_{PQ} \left(\varphi^{\eta}_{PQ}\right)^{-1} \\ &= \varphi^{\eta}_{PR} \varphi^{\varepsilon+\eta}_{RP} \varphi^{\varepsilon+\eta}_{PQ} \left(\varphi^{\eta}_{PQ}\right)^{-1} \\ &= \varphi^{\eta}_{PR} \left(\varphi^{\varepsilon+\eta}_{RQ} \left(\varphi^{\eta}_{RQ}\right)^{-1}\right) \left(\varphi^{\eta}_{PR}\right)^{-1}. \end{aligned}$$

So in this case, ${\varphi'}^{\varepsilon}_{RQ}$ is conjugate to $\left(\varphi^{\varepsilon+\eta}_{RQ} \left(\varphi^{\eta}_{RQ}\right)^{-1}\right)$ by φ^{η}_{PR}.

Corollary 5.17. *Let $\rho\colon \pi_1(S) \to G$ be θ-Anosov and let $\varepsilon, \eta \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ be small* *enough. Then*

$$\Lambda_0^{\varepsilon+\eta}\rho = \Lambda_0^{\varepsilon}\left(\Lambda_0^{\eta}\rho\right).$$

Proof. Set $\rho' := \Lambda_0^{\eta}\rho$ and use the same notation as before. Then for every $\gamma \in \pi_1(S)$ we have using Proposition 5.15,

$$\Lambda_0^{\varepsilon}\left(\Lambda_0^{\eta}\rho\right)(\gamma) = {\varphi'}^{\varepsilon}_{P(\gamma P)}\left(\Lambda_0^{\eta}\rho(\gamma)\right) = {\varphi'}^{\varepsilon}_{P(\gamma P)}\left(\varphi^{\eta}_{P(\gamma P)}\rho(\gamma)\right) = \varphi^{\varepsilon+\eta}_{P(\gamma P)}\rho(\gamma) = \Lambda_0^{\varepsilon+\eta}\rho(\gamma),$$

which proves the claim. □

5.3.2. Behavior under composition with Lie group homomorphisms

We now examine how cataclysms behave with respect to composing an Anosov representation $\rho\colon \pi_1(S) \to G$ with a Lie group homomorphism $\kappa\colon G \to G'$ for some other Lie group G'. For that, we first need to know how the property of being Anosov behaves under composition with Lie groups homomorphisms. This question is adressed by Guichard and Wienhard in [GW12]. In particular, composing an Anosov representation with a Lie group homomorphism does not always give an Anosov representation. An example where this is not the case can be found in [GW12, Section 4.3]. We first recall the results from Guichard and Wienhard before giving the main result of this subsection.

Let $\kappa\colon G \to G'$ be a homomorphism between semisimple Lie groups and $\kappa_*\colon \mathfrak{g} \to \mathfrak{g}'$ the associated Lie algebra homomorphism. We denote with primes the objects associated with G', e.g. Δ' is the set of simple roots for G'. We can assume that the maximal compact subgroup $K' < G'$ and the maximal abelian subalgebra $\mathfrak{a}'$ are compatible with κ in the sense that $\kappa(K) \subset K'$ and $\kappa_*(\mathfrak{a}) \subset \mathfrak{a}'$ (see [GW12, Section 4.1] and references therein). As in the previous sections, let $\mathfrak{a}_\theta = \bigcup_{\alpha\in\Delta\setminus\theta}\ker(\alpha)$. Let $W'_{\theta'}$ be the subgroup of the Weyl group W' for G' that fixes $\mathfrak{a}'_{\theta'}$ pointwise. There there is the following result:

Proposition 5.18 ([GW12, Proposition 4.4])**.** *Let $\kappa\colon G \to G'$ be a Lie group homomorphism and assume that $\kappa(K) \subset K'$ and $\kappa(\mathfrak{a}) \subset \mathfrak{a}'$. Let $\theta \subset \Delta$ and suppose that there exist $w' \in W'$ and $\theta' \subset \Delta'$ such that*

$$\kappa_*\left(\overline{\mathfrak{a}^+} \setminus \bigcup_{\alpha\in\theta}\ker(\alpha)\right) \subset w' \cdot W'_{\theta'} \cdot \left(\overline{\mathfrak{a}'^+} \setminus \bigcup_{\alpha\in\theta'}\ker(\alpha')\right). \tag{5.11}$$

Then for any θ-Anosov representation $\rho\colon \pi_1(S) \to G$, the representation $\kappa\circ\rho$ is θ'-Anosov. Furthermore $\kappa(P_\theta^\pm) \subset w'{P'}^{\pm}_{\theta'}{w'}^{-1}$, and there is an induced map $\kappa^+\colon \mathcal{F}_\theta \to \mathcal{F}'_{\theta'}$. If $\zeta\colon \partial_\infty\tilde{S} \to \mathcal{F}_\theta$ is the boundary map associated to ρ, then the boundary map for $\kappa\circ\rho$ is $\kappa^+\circ\zeta$.

The assumption (5.11) guarantees that if $H \in \overline{\mathfrak{a}^+}$ does not lie in a wall of the Weyl chamber corresponding to an element $\alpha \in \theta$, then also its image under κ_* stays away from the walls corresponding to the elements $\alpha' \in \theta'$.

Remark 5.19. In [GW12], they use a different notational convention than we do – what they call θ-Anosov is $(\Delta \setminus \theta)$-Anosov in our notation. Moreover, in their paper, there is a typing error in the statement and proof of Proposition 5.18: In the assumption (5.11), θ and $\Delta \setminus \theta$ are reversed. The version we state here is adapted to our notational convention of θ-Anosov and has the corrected assumption (5.11).

For the special case that G has rank 1, e.g. for $G = \mathrm{SL}(2,\mathbb{R})$, then $|\Delta| = 1$ and being Anosov always means being Δ-Anosov. In this case, there is a simple description for $\theta' \subset \Delta'$ that guarantees that $\kappa \circ \rho$ is θ'-Anosov:

Proposition 5.20 ([GW12, Proposition 4.7])**.** *Let G be a Lie group of real rank 1. Let $\rho\colon \pi_1(S) \to G$ be an Anosov representation and $\kappa\colon G \to G'$ a homomorphism of Lie groups. Assume that the Weyl chambers of G and G' are arranged so that $\kappa_*(\overline{\mathfrak{a}^+}) \subset \overline{\mathfrak{a}'^+}$. Then $\kappa \circ \rho$ is θ'-Anosov, where $\theta' = \{\alpha' \in \theta' \mid \kappa^*\alpha' \neq 0\}$, where $\kappa^*\colon \mathfrak{a}'^* \to \mathfrak{a}^*$ is the map induced by κ.*

Recall that $\mathfrak{a}^*$ is the dual vector space to $\mathfrak{a}$ and the induced homomorphism $\kappa^*\colon \mathfrak{a}'^* \to \mathfrak{a}^*$ is the dual of the homomorphism $\kappa_*\colon \mathfrak{a} \to \mathfrak{a}'$. It is defined by

$$(\kappa^*\alpha')(H) = \alpha'(\kappa_*(H))$$

for every $\alpha' \in \mathfrak{a}'^*$ and every $H \in \mathfrak{a}$.

If κ_* satisfies $\kappa_*(\mathfrak{a}_\theta) \subset \mathfrak{a}'_{\theta'}$, then it induces a linear map $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta) \hookrightarrow \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}'_{\theta'})$ that we also denote by κ_*.

The cataclysm deformation Λ_0 with respect to a fixed reference component in $\tilde{S} \setminus \tilde{\lambda}$ is natural in the following sense:

Proposition 5.21. *Let $\kappa\colon G \to G'$ be a Lie group homomorphism and assume that $\kappa(K) \subset K'$ and $\kappa(\mathfrak{a}) \subset \mathfrak{a}'$. Let $\theta \subset \Delta$ and $\theta' \subset \Delta'$ such that* (5.11) *is satisfied. Further, assume that $\kappa_*(\mathfrak{a}_\theta) \subset \mathfrak{a}'_{\theta'}$. Let $\rho\colon \pi_1(S) \to G$ be θ-Anosov and let $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ be sufficiently small such that $\Lambda_0^\varepsilon\rho$ exists. Then $\kappa \circ \rho$ is θ'-Anosov and*

$$\Lambda_0^{\kappa_*\varepsilon}(\kappa \circ \rho) = \kappa\left(\Lambda_0^\varepsilon\rho\right). \tag{5.12}$$

We want to remark that the notation Λ_0 for the cataclysm deformation does not encode the group G containing the image of the representation ρ. This is given implicitly by the cycle. In particular, in Proposition 5.21, $\Lambda_0^{\kappa_*\varepsilon}$ is the deformation of a representation with values in G', and Λ_0^{ε} is the deformation of a representation with values in G.

Remark 5.22. In Proposition 5.21 we need to make the assumption $\kappa_*(\mathfrak{a}_\theta) \subset \mathfrak{a}'_{\theta'}$. If the Lie group G is of rank 1 and if θ' is as in Proposition 5.20, then this assumption automatically satisfied. Indeed, we have $\theta = \Delta$, so $\mathfrak{a}_\theta = \mathfrak{a}$ and for all $\alpha' \in \Delta' \setminus \theta'$ we have $\kappa^*\alpha' = 0$ by definition of θ'. Hence,

$$\alpha'(\kappa_*(H)) = (\kappa^*\alpha')(H) = 0 \quad \forall \alpha' \in \Delta' \setminus \theta',\ H \in \mathfrak{a},$$

so $\kappa_*(\mathfrak{a}) \subset \mathfrak{a}'_{\theta'}$.

Proof of Proposition 5.21. The fact that $\kappa \circ \rho$ is θ'-Anosov is guaranteed by Proposition 5.18. Up to changing the Weyl chamber of G', we can assume that $w' = \mathrm{Id}$. Hence, by Proposition 5.18 we know that $\kappa(P_\theta^{\pm}) = {P'}_{\theta'}^{\pm}$. Let g be an oriented geodesic in $\tilde{S}$. If $m_g \in G$ such that $m_g \cdot P_\theta^{\pm} = \zeta(g^{\pm})$, then by equivariance of κ^+ and the definition of the boundary map for $\kappa \circ \rho$, we have

$$\kappa(m_g) \cdot {P'}_{\theta'}^{\pm} = \kappa(m_g) \cdot \kappa^+(P_\theta^{\pm}) = \kappa^+(m_g \cdot P_\theta^{\pm}) = \kappa^+(P_g^{\pm}) = \kappa^+ \circ \zeta(g^{\pm}).$$

Further, for $H \in \mathfrak{a}_\theta$ it holds that $\exp(\kappa_* H) = \kappa(\exp(H))$. Thus, the stretching maps $T_g^H \in G$ and $T_g^{\kappa_* H} \in G'$ satisfy

$$T_g^{\kappa_* H} = \kappa(m_g)\exp(\kappa_* H)\kappa(m_g)^{-1} = \kappa(T_g^H).$$

It follows that for $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ the shearing maps satisfy $\varphi_{PQ}^{\kappa_*\varepsilon} = \kappa(\varphi_{PQ}^{\varepsilon})$. Consequently, we have for every $\gamma \in \pi_1(S)$,

$$\begin{aligned}
\Lambda_0^{\kappa_*\varepsilon}(\kappa \circ \rho(\gamma)) &= \varphi_{P\gamma P}^{\kappa_*\varepsilon}(\kappa \circ \rho(\gamma)) \\
&= \kappa(\varphi_{P\gamma P}^{\varepsilon})\kappa(\rho(\gamma)) \\
&= \kappa(\varphi_{P\gamma P}^{\varepsilon}\rho(\gamma)) \\
&= \kappa\left(\Lambda_0^{\varepsilon}\rho(\gamma)\right).
\end{aligned}$$

This finishes the proof. □

We finish this subsection with examples for Lie group homomorphisms κ for which the prerequisites in Proposition 5.21 are satisfied, as well as with a counterexample.

Example 5.23. An example where we can apply Proposition 5.21 are (n, k)-horocyclic representations as introduced in Subsection 2.4.3. They stabilize a k-dimensional subspace of $\mathbb{R}^n$ and are obtained from composing a discrete and faithful representation $\rho_0 \colon \pi_1(S) \to$

$\mathrm{SL}(2,\mathbb{R})$ with the reducible representation $\iota_{n,k}\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(k,\mathbb{R})$ defined in Equation (2.5). In this case, $G = \mathrm{SL}(2,\mathbb{R})$, $\theta = \Delta_{\mathrm{SL}(2,\mathbb{R})}$ and $G' = \mathrm{SL}(n,\mathbb{R})$. The induced map $(\iota_{n,k})_* : \mathfrak{a} \to \mathfrak{a}'$ is

$$(\iota_{n,k})_* \left(\begin{pmatrix} a & \\ & -a \end{pmatrix} \right) = \begin{pmatrix} a\mathrm{Id}_k & & \\ & 0_{n-2k} & \\ & & -a\mathrm{Id}_k \end{pmatrix}.$$

Here, 0_{n-2k} is the $(n-2k)\times(n-2k)$ matrix with all entries zero. The abelian subalgebra $\mathfrak{a}$ consists of all diagonal matrices $\mathrm{diag}(a, -a \in \mathfrak{a})$ with $a \in \mathbb{R}$. The positive Weyl chamber $\mathfrak{a}^+$ is given by all elements $\mathrm{diag}(a,-a)$ with $a > 0$. We have $\overline{\mathfrak{a}^+} \setminus \bigcup_{\alpha\in\theta} \ker(\alpha) = \mathfrak{a}^+$. Set $\theta' = \{k, n-k\}$. Then for $H = \mathrm{diag}(a,-a) \in \mathfrak{a}^+$ it holds that

$$\begin{aligned} \alpha_j((\iota_{n,k})_*(H)) &= 0 \quad \forall j \neq k, n-k, \\ \alpha_k((\iota_{n,k})_*(H)) &= a - 0 = a > 0 \quad \text{and} \\ \alpha_{n-k}((\iota_{n,k})_*(H)) &= 0 - (-a) = a > 0. \end{aligned}$$

Hence, $(\iota_{n,k})_*(\mathfrak{a}^+) \subset \overline{\mathfrak{a}'^+} \setminus \cup_{\alpha'\in\theta'} \ker(\alpha')$. Further, $(\iota_{n,k})_*(\mathfrak{a}_\theta) \subset \mathfrak{a}'_{\theta'}$, so the assumptions from Proposition 5.21 are satisfied and we have $\Lambda_0^{(\iota_{n,k})_*\varepsilon}(\iota_{n,k}\circ\rho) = \iota_{n,k}(\Lambda_0^{\varepsilon}\rho)$. In this case, as $\mathrm{SL}(2,\mathbb{R})$ is of rank 1, we could have drawn the same conclusion by applying Proposition 5.20 and Remark 5.22. We consider cataclysm deformations of horocyclic representation in more detail in Section 7.4.

Example 5.24. We now give an example where G is not of rank 1. Let $G = \mathrm{SL}(4,\mathbb{R})$, $\theta = \{\alpha_2\}$, $G' = \mathrm{SL}(6,\mathbb{R})$ and let $\kappa := \bigwedge_4^2\colon \mathrm{SL}(4,\mathbb{R}) \to \mathrm{SL}(6,\mathbb{R})$ be the exterior power representation from Example 2.29. The induced map on $\mathfrak{a}$ is given by

$$\left(\bigwedge\nolimits_4^2\right)_* \left(\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4 \end{pmatrix} \right) = \begin{pmatrix} a_1+a_2 & & & & & \\ & a_1+a_3 & & & & \\ & & a_1+a_4 & & & \\ & & & a_2+a_3 & & \\ & & & & a_2+a_4 & \\ & & & & & a_3+a_4 \end{pmatrix}.$$

Let $\theta' := \{\alpha_1, \alpha_5\} \subset \Delta'$ and $H = \mathrm{diag}(a_1,a_2,a_3,a_4) \in \overline{\mathfrak{a}^+} \setminus \ker(\alpha_2)$. We have that $\overline{\mathfrak{a}^+} \setminus \ker(\alpha_2)$ consists of all diagonal matrices $\mathrm{diag}(a_1,a_2,a_3,a_4)$ with trace zero and $a_1 \geq a_2 > a_3 \geq a_4$. Then

$$a_1 + a_3 = \max\{a_1+a_3, a_1+a_4, a_2+a_3, a_2+a_4\} \quad \text{and}$$
$$a_2 + a_4 = \min\max\{a_1+a_3, a_1+a_4, a_2+a_3, a_2+a_4\}.$$

Note that $\left(\bigwedge_4^2\right)_* (H) \notin \overline{\mathfrak{a}'^+}$ in general, since we do not know if $\alpha_3(H) = (a_1+a_4)-(a_2+a_3)$ is non-negative. By applying an element in $W'_{\theta'}$ if necessary we can arrange that the entries are in descending order. Further, we have

$$\begin{aligned}\alpha_1\left(\left(\bigwedge_4^2\right)_* (H)\right) &= (a_1+a_2)-(a_1+a_3) = a_2 - a_3 > 0 \text{ and}\\ \alpha_5\left(\left(\bigwedge_4^2\right)_* (H)\right) &= (a_2+a_4)-(a_3+a_4) = a_2 - a_3 > 0.\end{aligned}$$

Thus, $\left(\bigwedge_4^2\right)_* (H) \notin \ker(\alpha_1) \cup \ker(\alpha_5)$. This shows that $\left(\bigwedge_5^2\right)_* \left(\overline{\mathfrak{a}^+} \setminus \ker(\alpha_2)\right) \subset W'_{\theta'} \cdot \left(\overline{\mathfrak{a}^+} \setminus \bigcup_{\alpha' \in \theta'} \ker(\alpha')\right)$, so the prerequisite (5.11) from Proposition 5.18 is satisfied. Moreover, we have $\mathfrak{a}_\theta = \{\operatorname{diag}(a, a, -a, -a) \mid a \in \mathbb{R}\}$ and for every $a \in \mathbb{R}$

$$\left(\bigwedge_4^2\right)_* \left(\begin{pmatrix} a &&& \\ & a && \\ && -a & \\ &&& -a \end{pmatrix}\right) = \begin{pmatrix} 2a &&&&& \\ & 0 &&&& \\ && 0 &&& \\ &&& 0 && \\ &&&& 0 & \\ &&&&& -2a \end{pmatrix} \in \mathfrak{a}'_{\theta'},$$

so $\left(\bigwedge_4^2\right)_* (\mathfrak{a}_\theta) \subset \mathfrak{a}'_{\theta'}$. Thus, all prerequisites from Proposition 5.21 are satisfied and it holds that $\Lambda_0^{\kappa_*\varepsilon}(\kappa \circ \rho) = \kappa\left(\Lambda_0^{\varepsilon}\rho\right)$ for $\kappa = \bigwedge_4^2$.

Example 5.25. We now give an example where Proposition 5.18 holds, but the additional assumption $\kappa_*(\mathfrak{a}_\theta) \subset \mathfrak{a}'_{\theta'}$ in Proposition 5.21 is not satisfied. Let $G = \mathrm{SL}(5, \mathbb{R})$, $\theta = \{\alpha_2, \alpha_3\}$ and let $\kappa := \bigwedge_5^2 \colon \mathrm{SL}(5, \mathbb{R}) \to \mathrm{SL}(10, \mathbb{R})$ be the exterior power representation from Example 2.29. Analogous to Example 5.24 we have

$$\left(\bigwedge_5^2\right)_* \left(\begin{pmatrix} a_1 &&&& \\ & a_2 &&& \\ && a_3 && \\ &&& a_4 & \\ &&&& a_5 \end{pmatrix}\right) = \begin{pmatrix} a_1 + a_2 &&&& \\ & \ddots &&& \\ && (a_i + a_j)_{i<j} && \\ &&& \ddots & \\ &&&& a_4 + a_5 \end{pmatrix}.$$

Let $\theta' := \{\alpha_1, \alpha_9\} \subset \Delta'$. As in Example 5.24, we see that $\alpha_1\left(\left(\bigwedge_4^2\right)_* (H)\right) > 0$ and $\alpha_9\left(\left(\bigwedge_4^2\right)_* (H)\right) > 0$ for all $H \in \overline{\mathfrak{a}^+} \setminus (\ker(\alpha_2) \cup \ker(\alpha_3))$ and up to applying an element in $W'_{\theta'}$, $\left(\bigwedge_4^2\right)_* (H)$ lies in $\overline{\mathfrak{a}'^+}$. Thus, (5.11) is satisfied, so for every $\{\alpha_2, \alpha_3\}$-Anosov representation $\rho\colon \pi_1(S) \to \mathrm{SL}(5, \mathbb{R})$ the representation $\bigwedge_5^2 \circ \rho\colon \pi_1(S) \to \mathrm{SL}(5, \mathbb{R})$ is $\{\alpha_1, \alpha_9\}$-Anosov. However, the additional assumption $\left(\bigwedge_5^2\right)_* (\mathfrak{a}_\theta) \subset \mathfrak{a}'_{\theta'}$ from Proposition 5.21 is not satisfied, since

$$\left(\bigwedge\nolimits_5^2\right)_* \left(\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & -1 & \\ & & & & -1 \end{pmatrix}\right) = \operatorname{diag}(2,1,0,0,1,0,0,-1,-1,-2) \notin \mathfrak{a}'_{\theta'}.$$

Note that there does not exist $w' \in W'$ such that $\left(\bigwedge_5^2\right)_*(\mathfrak{a}_\theta) \subset w' \cdot \mathfrak{a}'_{\theta'}$. This shows that the additional assumption $\kappa_*(\mathfrak{a}_\theta) \subset \mathfrak{a}'_{\theta'}$ in Proposition 5.21 is not implied by the assumption (5.11) from Proposition 5.18.

In this example, we see that for the bigger subset $\theta'' := \{\alpha_1, \alpha_3, \alpha_7, \alpha_9\}$, there exists $w' \in W'$ such that $\left(\bigwedge_5^2\right)_*(\mathfrak{a}_\theta) \subset w' \cdot \mathfrak{a}'_{\theta''}$. If a representation $\rho\colon \pi_1(S) \to \mathrm{SL}(4,\mathbb{R})$ is such that $\bigwedge_5^2 \circ \rho$ is not only θ'-Anosov, but satisfies the stronger requirement of being θ''-Anosov, then if we choose the positive Weyl chamber accordingly and if additionally $\bigwedge_5^2(P_\theta^+) \subset P'^+_{\theta''}$, the conclusion (5.12) from Proposition 5.21 holds.

6. Cataclysms and boundary maps

A cataclysm deformation of a θ-Anosov representation ρ can also be understood in terms of a deformation of the associated ρ-equivariant flag curve $\zeta\colon \partial_\infty\tilde{S} \to \mathcal{F}_\theta$. Let $\rho^\varepsilon := \Lambda_0^\varepsilon\rho$ for $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ be the θ-Anosov representation obtained from ρ by a cataclysm deformation with respect to a fixed reference component $P_0 \subset \tilde{S}\setminus\tilde{\lambda}$. Denote by $\zeta^\varepsilon\colon \partial_\infty\tilde{S} \to \mathcal{F}_\theta$ the boundary map associated with ρ^ε. Our goal is to express ζ^ε in terms of the flag curve ζ and the family of shearing maps $\{\varphi^\varepsilon_{PQ}\}_{(P,Q)}$ that were used to define ρ^ε. The expression for ζ^ε in Theorem 6.1 and most of the results in Section 6.1 are analogous to results obtained by Dreyer in [Dre13], while for the proof of Theorem 6.1 in Section 6.2 we use a different approach.

Let $\mathcal{V}_\lambda$ be the subset of $\partial_\infty\tilde{\lambda}$ consisting of the vertices of connected components in $\tilde{S}\setminus\tilde{\lambda}$. In general, $\mathcal{V}_\lambda$ is not equal to $\partial_\infty\tilde{\lambda}$, since the endpoints of non-isolated leaves are not necessarily contained in $\mathcal{V}_\lambda$. The main result of this section is the following:

Theorem 6.1. *Let $\rho\colon \pi_1(S) \to G$ be θ-Anosov, let $\varepsilon \in \mathcal{U}_\rho$ and let $\rho^\varepsilon := \Lambda_0^\varepsilon\rho$ be the ε-cataclysm deformation of ρ along λ with respect to a reference component $P_0 \subset \tilde{S}\setminus\tilde{\lambda}$. Here, $\mathcal{U}_\rho \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ is as in Theorem 5.12 such that ρ^ε is θ-Anosov. Let ζ and ζ^ε be the boundary maps associated with ρ and ρ^ε, respectively. Then for every $x \in \mathcal{V}_\lambda$, the boundary map ζ^ε is given by*

$$\zeta^\varepsilon(x) = \varphi^\varepsilon_{P_0Q_x}\cdot\zeta(x), \tag{6.1}$$

where Q_x is a component of $\tilde{S}\setminus\tilde{\lambda}$ having x as a vertex.

To prove Theorem 6.1, we first show that the right hand side of (6.1) defines a ρ^ε-equivariant boundary map on $\mathcal{V}_\lambda$. We omit the superscript ε in the notation of the shearing maps. Let $\zeta^\lambda\colon \mathcal{V}_\lambda \to \mathcal{F}_\theta$ be defined by the right hand side of (6.1), i.e.

$$\zeta^\lambda(x) := \varphi_{P_0Q_x}\cdot\zeta(x). \tag{6.2}$$

Note that the map ζ^λ depends on the parameter $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$, since the shearing maps φ_{P_0Q} depend on ε. Further, the notation ζ^λ is motivated by the fact that ζ^λ is not defined on all of $\partial_\infty \tilde{S}$, but only on $\mathcal{V}_\lambda$. We will later see that ζ^λ can be extended to the boundary $\partial_\infty \tilde{\lambda}$ of the lifted lamination $\tilde{\lambda}$.

With this definition of ζ^λ in (6.2), the statement of Theorem 6.1 translates to $\zeta^\varepsilon|_{\mathcal{V}_\lambda} = \zeta^\lambda$. Before proving this, we have a closer look at the map ζ^λ.

6.1. The boundary map ζ^λ

First, we prove that ζ^λ is well-defined and ρ^ε-equivariant. To shorten notation, let $P := P_0$ be the fixed reference component of $\tilde{S} \setminus \tilde{\lambda}$ used to define the deformed representation ρ^ε.

Lemma 6.2. *The map $\zeta^\lambda \colon \partial_\infty \tilde{\lambda} \to \mathcal{F}_\theta$ defined in* (6.2) *is well-defined and ρ^ε-equivariant.*

Proof. To show well-definedness, we have to show that $\zeta^\lambda(x)$ does not depend on the choice of component $Q_x \subset \tilde{S} \setminus \tilde{\lambda}$ having x as a vertex. If $x \in \mathcal{V}_\lambda$ and Q_x, R_x are two connected components of $\tilde{S} \setminus \tilde{\lambda}$ having x as a vertex, then all components between Q_x and R_x also have x as a vertex. By definition, $\varphi_{Q_x R_x}$ is a (possibly infinite) concatenation of stretching maps that all stabilize $\zeta(x)$. Hence, also $\varphi_{Q_x R_x}$ stabilizes $\zeta(x)$. Using the composition property of the shearing maps from Proposition 5.6, we have

$$\varphi_{PR_x} \cdot \zeta(x) = (\varphi_{PQ_x}\ \varphi_{Q_x R_x}) \cdot \zeta(x) = \varphi_{PQ_x} \cdot \zeta(x),$$

so ζ^λ is well-defined.

To show ρ^ε-equivariance, let $x \in \mathcal{V}_\lambda$ be a vertex of Q_x and $\gamma \in \pi_1(S)$. Using the ρ-equivariance of ζ and of the shearing maps (Proposition 5.6), we have

$$\begin{aligned}
\zeta^\lambda(\gamma x) &= \varphi_{PQ_{\gamma x}} \cdot \zeta(\gamma x) \\
&= (\varphi_{P(\gamma P)}\ \varphi_{(\gamma P)(\gamma Q_x)}\ \rho(\gamma)) \cdot \zeta(x) \\
&= \big(\varphi_{P(\gamma P)}(\rho(\gamma)\ \varphi_{PQ_x}\ \rho(\gamma)^{-1})\rho(\gamma)\big) \cdot \zeta(x) \\
&= \big((\varphi_{P(\gamma P)}\ \rho(\gamma))\, \varphi_{PQ_x}\big) \cdot \zeta(x) \\
&= \rho^\varepsilon(\gamma) \cdot \zeta^\lambda(x).
\end{aligned}$$

This finishes the proof. □

Note that we cannot say anything about continuity of ζ^λ here. We give an alternative definition of ζ^λ that uses oriented leaves of the lamination $\tilde{\lambda}$ instead of connected components of $\tilde{S} \setminus \tilde{\lambda}$. This will allow us to show a continuity result in Lemma 6.5. Further, with this alternative definition, we can define ζ^λ on all of $\partial_\infty \tilde{\lambda}$.

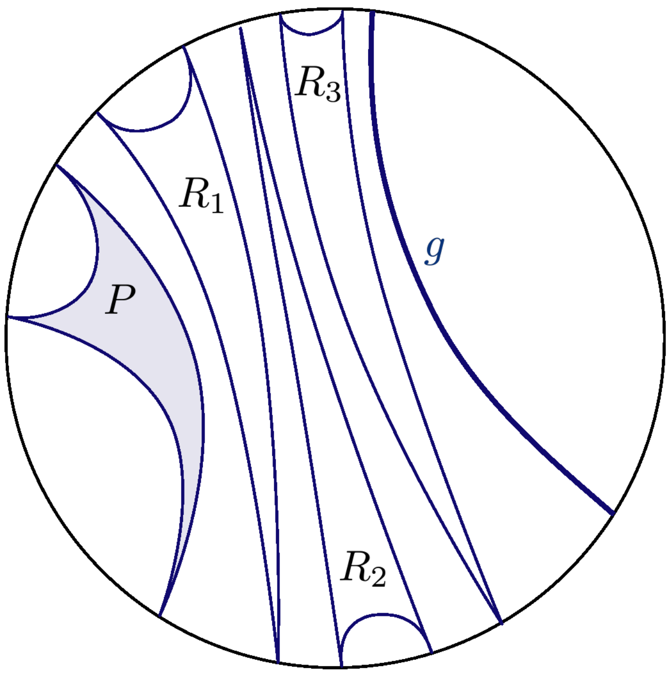


Figure 6.1.: If $P \subset \tilde{S} \backslash \tilde{\lambda}$ is a connected component and g a leaf in $\tilde{\lambda}$, consider a finite subset $\{R_1, \ldots, R_m\}$ of connected components separating P and g, labeled from P to g. In this picture, we have $m = 3$.

As a preparation, we define maps ψ_{Pg} associated to the reference component P and a leaf g of $\tilde{\lambda}$ as follows: Let g be a leaf of $\tilde{\lambda}$. Define $\mathcal{C}_{Pg}$ as the set of all connected components of $\tilde{S} \setminus \tilde{\lambda}$ between P and g. Choose a finite subset $\mathcal{C} = \{R_1, \ldots, R_m\} \subset \mathcal{C}_{Pg}$, ordered from P to g, i.e. such that R_{i+1} separates R_i from g for each i (see Figure 6.1). We define analogously to (5.2)

$$\psi_{\mathcal{C}} := \prod_{i=1}^{m} \left(T_{g_i^0}^{\varepsilon(P,R_i)} \cdot T_{g_i^1}^{-\varepsilon(P,R_i)} \right).$$

Let $\psi_{Pg} := \lim_{\mathcal{C} \to \mathcal{C}_{Pg}} \psi_{\mathcal{C}}$. This limit exists by the same arguments as in Proposition 5.3, since if we chose an arc k that starts in P, intersects g and ends in some ideal triangle Q, then $\mathcal{C}_{Pg} \subset \mathcal{C}_{PQ}$ and by Proposition 5.3, the limit φ_{PQ} exists. The map ψ_{Pg} is obtained from almost the same sequence with some of the factors replaced by the identity, so it converges as well.

The map ψ_{Pg} is ρ-equivariant. To see this, let $\mathcal{C} = \{R_1, \ldots, R_m\} \subset \mathcal{C}_{Pg}$ be a finite subset and for $\gamma \in \pi_1(S)$, let $\gamma\mathcal{C} = \{\gamma R_1, \ldots, \gamma R_m\} \subset \mathcal{C}_{(\gamma P)(\gamma g)}$. By ρ-equivariance of the stretching maps (Lemma 4.16), we have $\psi_{\gamma\mathcal{C}} = \rho(\gamma)\psi_{\mathcal{C}}\rho(\gamma)^{-1}$. By taking the limit, we observe that also ψ_{Pg} is ρ-equivariant, i.e. $\psi_{(\gamma P)(\gamma g)} = \rho(\gamma)\psi_{Pg}\rho(\gamma)^{-1}$. Further, if $Q \subset \tilde{S} \backslash \tilde{\lambda}$ is a connected component of $\tilde{S} \setminus \tilde{\lambda}$, then $\psi_{Pg} = \varphi_{PQ}\psi_{Qg}$. This can be seen by the same techniques as in the proof of the composition property of the shearing maps in Proposition 5.6.

The maps ψ_{Pg} depend continuously on the leaf g in the following sense:

Lemma 6.3. *Let g be an oriented leaf in $\tilde{\lambda}$ and let $(g_n)_{n\in\mathbb{N}}$ be a sequence of oriented leaves of $\tilde{\lambda}$ converging to g, i.e. $\lim_{n\to\infty} g_n^+ = g^+$ and $\lim_{n\to\infty} g_n^- = g^-$, where g_n^+ and g_n^- denote the positive and negative endpoints of g_n, respectively. Then $\lim_{n\to\infty} \psi_{Pg_n} = \psi_{Pg}$.*

Proof. Up to passing to a subsequence, we can assume that the geodesics g_n all lie on the same side of g. Since the sequence $(g_n)_{n\in\mathbb{N}}$ converges to g, there exists $N \in \mathbb{N}$ such that for all $n > N$ either g_n separates P and g, or g separates P and g_n. In the first case, P and g_n lie on the same side of g, in the second case, they lie on different sides. Assume first that g_n separates P and g for all $n > N$. Then $\mathcal{C}_{Pg_n} \subset \mathcal{C}_{Pg}$ and $\lim_{n\to\infty} \mathcal{C}_{Pg_n} = \mathcal{C}_{Pg}$ in the sense that for all $R \in \mathcal{C}_{Pg}$, there exists some $M \in \mathbb{N}$ such that $R \in \mathcal{C}_{Pg_n}$ for all $n > M$. By definition of ψ_{Pg}, we have

$$\lim_{n\to\infty} \psi_{Pg_n} = \lim_{n\to\infty} \lim_{\mathcal{C}\to\mathcal{C}_{Pg_n}} \psi_{\mathcal{C}} = \lim_{\mathcal{C}\to\mathcal{C}_{Pg}} \psi_{\mathcal{C}} = \psi_{Pg}.$$

If P and g_n lie on different sides of g, let $Q \subset \tilde{S} \setminus \tilde{\lambda}$ be a component that lies on the same side of g as the geodesics g_n, and let g_n separate Q from g for all $n \in \mathbb{N}$. Then by the composition property of ψ_{Pg} and by what we have shown above

$$\lim_{n\to\infty} \psi_{Pg_n} = \lim_{n\to\infty} \varphi_{PQ}\psi_{Qg_n} = \varphi_{PQ}\psi_{Qg} = \psi_{Pg}$$

which finishes the proof. □

We can use the maps ψ_{Pg} to give an alternative definition of ζ^λ: For $x \in \partial_\infty\tilde{\lambda}$, let g be a geodesic having x as endpoint. Define

$$\zeta^\lambda(x) := \psi_{Pg} \cdot \zeta(x). \tag{6.3}$$

On $\mathcal{V}_\lambda$, this definition agrees with the one from Lemma 6.2:

Lemma 6.4. *The map $\zeta^\lambda \colon \partial_\infty\tilde{\lambda} \to \mathcal{F}_\theta$ is ρ^ε-equivariant and agrees with the map ζ^λ from Lemma 6.2 when restricted to $\mathcal{V}_\lambda$.*

Proof. The ρ^ε-equivariance follows from the ρ-equivariance of ζ and of the maps ψ_{Pg}. To show that ζ^λ restricts to the map from Lemma 6.2, we have to show that for a point $x \in \mathcal{V}_\lambda$ that is a vertex of a connected component Q_x and an endpoint of a leaf g bounding Q_x, it holds that $\psi_{Pg} \cdot \zeta(x) = \varphi_{PQ_x} \cdot \zeta(x)$. With Notation 3.10, we have $g = g^0_{Q_x}$ or $g = g^1_{Q_x}$. In both cases, for every $H \in \mathfrak{a}_\theta$, T^H_g fixes $\zeta(x)$. If $g = g^0_Q$, we have

$$\psi_{Pg} \cdot \zeta(x) = \psi_{PQ_x} \cdot \zeta(x) = \left(\psi_{PQ_x} T_g^{\varepsilon(P,Q)}\right) \cdot \zeta(x) = \varphi_{PQ_x} \cdot \zeta(x).$$

Similarly, if $g = g^1_Q$, then

$$\psi_{Pg} \cdot \zeta(x) = \left(\psi_{PQ_x} T_{g^0_Q}^{\varepsilon(P,Q_x)} T_g^{-\varepsilon(P,Q_x)}\right) \cdot \zeta(x) = \left(\varphi_{PQ_x} T_g^{-\varepsilon(P,Q_x)}\right) \cdot \zeta(x) = \varphi_{PQ_x} \cdot \zeta(x).$$

Thus, on $\mathcal{V}_\lambda$, ζ^λ agrees with the map from Lemma 6.2. ☐

Lemma 6.3 and the alternative definition of ζ^λ in (6.3) allow us to show a continuity property of ζ^λ.

Lemma 6.5. *Let g be an oriented non-isolated leaf of $\tilde{\lambda}$ with positive endpoint g^+ and let $(g_n)_{n\in\mathbb{N}}$ be a sequence of oriented leaves of $\tilde{\lambda}$ converging to g. For $n \in \mathbb{N}$, let g_n^+ be the positive endpoint of g_n. Then $\lim_{n\to\infty} \zeta^\lambda(g_n^+) = \zeta^\lambda(g^+)$.*

Proof. By continuity of ζ and by Lemma 6.3 we have

$$\lim_{n\to\infty} \zeta^\lambda(g_n^+) = \lim_{n\to\infty} \psi_{Pg_n} \cdot \zeta(g_n^+) = \psi_{Pg} \cdot \zeta(g^+) = \zeta^\lambda(g^+).$$

☐

Note that we do not show continuity of ζ^λ for an arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ in $\partial_\infty\tilde{\lambda}$ converging to $x \in \partial_\infty\tilde{\lambda}$, only for the special situation described in the Lemma. Once we have proven Theorem 6.1, the continuity of ζ^ε implies continuity of ζ^λ - but this will only follow a posteriori and we cannot use it at this point.

We finish this section with the dynamical behavior of the map ζ^λ. Recall that the shearing maps φ^ε_{PQ} depend on the shearing parameter ε, so also ζ^λ depends on ε.

Lemma 6.6. *For $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ small enough, the map $\zeta^\lambda \colon \mathcal{V}_\lambda \to \mathcal{F}_\theta$ is dynamics-preserving with respect to ρ^ε, i.e. if $x \in \mathcal{V}_\lambda$ is an attracting fixed point of $\gamma \in \pi_1(S)$, then $\zeta^\lambda(x) \in \mathcal{F}_\theta$ is an attracting fixed point of $\rho^\varepsilon(\gamma)$.*

Proof. Let $x \in \mathcal{V}_\lambda$ and $\gamma \in \pi_1(S)$ such that x is an attracting fixed point of γ. By ρ^ε-equivariance of ζ^λ, $\zeta^\lambda(x)$ is a fixed point of $\rho^\varepsilon(\gamma)$. For $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ small enough, since $\rho^\varepsilon(\gamma)$ depends continuously on ε, this fixed point is again attracting. ☐

Remark 6.7. If the lamination λ is finite, Theorem 6.1 is a direct consequence of Lemma 6.6. In this case, all bi-infinite leaves of λ spiral towards a closed leaf, so all points in $\mathcal{V}_\lambda$ are endpoints of lifts of closed geodesics. In particular, they are attracting fixed points for some element $\gamma \in \pi_1(S)$. By Lemma 6.6, since ζ^ε and ζ^λ are dynamics-preserving for ρ^ε and since attracting fixed points are unique, we have $\zeta^\varepsilon(x) = \zeta^\lambda(x)$ for every vertex x of $\mathcal{V}_\lambda$.

6.2. Proof of Theorem 6.1

We now prove Theorem 6.1, i.e. show that ζ^λ and ζ^ε agree on $\mathcal{V}_\lambda$. The key observation is that the two maps agree on vertices of the fixed reference component P.

Proposition 6.8. *Let $\rho, \rho^\varepsilon, \zeta$ and ζ^ε be as in Theorem 6.1. If $x \in \mathcal{V}_\lambda$ is a vertex of the reference component $P \subset \tilde{S} \backslash \tilde{\lambda}$ used to define the cataclysm deformation, then $\zeta^\varepsilon(x) = \zeta(x)$.*

In other words, a cataclysm deformation does not change the flags associated to vertices of the reference component P.

Using Proposition 6.8, we can prove Theorem 6.1.

Proof of Theorem 6.1. Let $x \in \mathcal{V}_\lambda$. We want to show that $\zeta^\varepsilon(x) = \zeta^\lambda(x)$, where ζ^λ is as in (6.2). If x is a vertex of the fixed reference component P that is used to define ρ^ε, then, by Proposition 6.8, we have

$$\zeta^\varepsilon(x) = \zeta(x) = \varphi_{PP} \cdot \zeta(x) = \zeta^\lambda(x).$$

If x is not a vertex of P, let $Q_x \subset \tilde{S} \setminus \tilde{\lambda}$ be a component having x as a vertex. Then we look at the cataclysm deformation with respect to the reference component Q_x. Let $\rho_x^\varepsilon = \Lambda_x^\varepsilon \rho$ be the ε-cataclysm deformation of ρ with respect to the triangle Q_x and let ζ_x^ε be the corresponding boundary map. By Lemma 5.11, ρ_x^ε and ρ^ε are conjugated, $\rho^\varepsilon(\gamma) = \varphi_{PQ_x} \rho_x^\varepsilon(\gamma) \left(\varphi_{PQ_x}\right)^{-1}$ for all $\gamma \in \pi_1(S)$. Consequently, the boundary maps satisfy $\zeta^\varepsilon = \varphi_{PQ_x} \cdot \zeta_x^\varepsilon$. By Proposition 6.8, we have $\zeta_x^\varepsilon(x) = \zeta(x)$, since x is a vertex of the reference component Q_x. Thus, we have

$$\zeta^\varepsilon(x) = \varphi_{PQ_x} \cdot \zeta_x^\varepsilon(x) = \varphi_{PQ_x} \cdot \zeta(x) = \zeta^\lambda(x).$$

It follows that ζ^λ and ζ^ε agree on all of $\mathcal{V}_\lambda$, which finishes the proof of Theorem 6.1. $\square$

It remains to prove Proposition 6.8. In the proof, we will work in the setting of projective Anosov representations into $\mathrm{SL}(d,\mathbb{R})$ and use a result from Bochi, Potrie and Sambarino [BPS19]. By Theorem 2.28 there exists an irreducible representation $\tau\colon G \to \mathrm{SL}(d,\mathbb{R})$ for some $d \in \mathbb{N}$ such that ρ is θ-Anosov if and only if $\tau \circ \rho$ is projective Anosov. Let $\tau^+\colon G/P_\theta \to \mathbb{RP}^{d-1}$ and $\tau^-\colon G/P_\theta \to \mathrm{Gr}_{d-1}(d)$ be the τ-equivariant embeddings induced by τ. Since τ^+ is injective, to prove Proposition 6.8 it is sufficient to show that $\tau^+ \circ \zeta(x) = \tau^+ \circ \zeta^\varepsilon(x)$. We will now have a closer look at the situation in $\mathrm{SL}(d,\mathbb{R})$.

Recall from Example 2.25 that $A \in \mathrm{SL}(d,\mathbb{R})$ has a *gap of index k* if $\sigma_k(A) > \sigma_{k+1}(A)$, where $\sigma_j(A)$ is the $j$the singular value of A, i.e. the square root of the jth eigenvalue of AA^T, where the eigenvalues are in descending order. If A has a gap of index k, let $U_k(A) \in \mathrm{Gr}_k(d)$ be the subspace of $\mathbb{R}^d$ that contains the k biggest axes of the ellipsoid $\{Mv \mid \|v\| = 1\}$.

We define the angle between two subspaces $E, F \subset \mathbb{R}^d$, not necessarily of the same dimension, as

$$\measuredangle(E,F) := \min_{\substack{v\in E\setminus\{0\}\\ w\in F\setminus\{0\}}} \measuredangle(v,w).$$

In particular, $\measuredangle(E,F) = 0$ if and only if $E \cap F \neq \{0\}$.

The following estimate will be the key element in the proof of Proposition 6.8.

Lemma 6.9 ([BPS19, Lemma A.6])**.** *Let $A \in \mathrm{SL}(d,\mathbb{R})$ have a gap of index k. Then A^{-1} has a gap of index $d-k$ and for all $F \in \mathrm{Gr}_k(d)$ transverse to $U_{d-k}(A^{-1})$,*

$$\mathrm{d}_{\mathrm{Gr}_k(d)}(A \cdot F, U_k(A)) \leq \frac{\sigma_{k+1}(A)}{\sigma_k(A)} \frac{1}{\sin \measuredangle(F, U_{d-k}(A^{-1}))},$$

where $\mathrm{d}_{\mathrm{Gr}_k(d)}$ is a suitable distance on $\mathrm{Gr}_k(d)$ as in [BPS19, Equation A.2].

With theses preliminary remarks, we can now prove Proposition 6.8.

Proof of Proposition 6.8. First note that $\zeta^\lambda(x) = \varphi_{PP} \cdot \zeta(x) = \zeta(x)$ for every $x \in \mathcal{V}_\lambda$ that is a vertex of P. Assume that every boundary leaf of P is isolated. Then all boundary leaves of P spiral towards a closed leaf, so x is an endpoint of a lift of a closed geodesics. In particular, x is a fixed point of an element in $\pi_1(S)$ and we have by Remark 6.7 $\zeta^\varepsilon(x) = \zeta^\lambda(x) = \zeta(x)$ for every vertex x of P.

In the general case, x is not a fixed point of an element in $\pi_1(S)$. Let g be the leaf bounding P with endpoint x. Denote by $g^\pm$ the endpoints of g and let $x = g^+$. Since g is not isolated, the orbit of g is dense in $\tilde{\lambda}$ and there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in $\pi_1(S)$ such that $(\gamma_n \cdot g)_{n\in\mathbb{N}}$

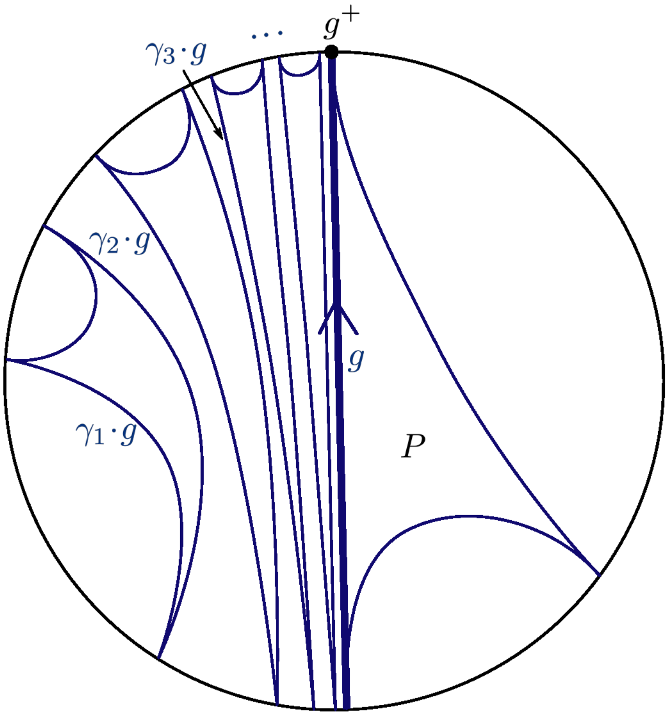


Figure 6.2.: If the leaf g bounding P is non-isolated and its endpoint g^+ is not a fixed point of an element in $\pi_1(S)$, then there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ such that the sequence of geodesics $(\gamma_n \cdot g)_{n\in\mathbb{N}}$ converges to g.

converges to g with $\lim_{n\to\infty} \gamma_n \cdot g^+ = g^+$ and $\lim_{n\to\infty} \gamma_n \cdot g^- = g^-$ (see Figure 6.2). By Lemma A.5, the sequence $(\gamma_n)_{n\in\mathbb{N}}$ in $\pi_1(S)$ converges to $g^+ \in \partial_\infty \tilde{S}$.

Let $\tau\colon G \to \mathrm{SL}(d,\mathbb{R})$ be the irreducible representation from Theorem 2.28 and $\tau^\pm$ be the induced maps on $\mathcal{F}_\theta$ as explained above. We want to apply Lemma 6.9 to the following situation:

- $A = \tau \circ \rho^\varepsilon(\gamma_n)$ and
- $F = \tau^+ \circ \zeta(g^+)$.

Since $\tau \circ \rho^\varepsilon$ is projective Anosov, $\tau \circ \rho^\varepsilon(\gamma_n)$ has a gap of index 1 and the 1-dimensional part of the boundary map is given by $\tau^+ \circ \zeta^\varepsilon$. To apply Lemma 6.9, we need to ensure that $\tau^+ \circ \zeta(g^+)$ and $U_{d-1}(\tau \circ \rho^\varepsilon(\gamma_n^{-1}))$ are transverse. Assume for now that this holds true and that $\sin\measuredangle\left(\tau^+ \circ \zeta(g^+), U_{d-1}(\tau \circ \rho^\varepsilon(\gamma_n^{-1}))\right) \geq \delta$ for some $\delta > 0$. Then by Lemma 6.9, we have

$$\begin{aligned}
\mathrm{d}\left((\tau\circ\rho^\varepsilon(\gamma_n))\cdot\left(\tau^+\circ\zeta(g^+)\right), U_1(\tau\circ\rho^\varepsilon(\gamma_n))\right) &\leq \frac{\sigma_2(\tau\circ\rho^\varepsilon(\gamma_n))}{\sigma_1(\tau\circ\rho^\varepsilon(\gamma_n))}\frac{1}{\sin\measuredangle\left(\tau^+\circ\zeta(g^+), U_{d-1}(\tau\circ\rho^\varepsilon(\gamma_n^{-1}))\right)} \\
&\leq \frac{\sigma_2(\tau\circ\rho^\varepsilon(\gamma_n))}{\sigma_1(\tau\circ\rho^\varepsilon(\gamma_n))}\frac{1}{\delta}.
\end{aligned}$$

Since $\tau \circ \rho^{\varepsilon}$ is projective Anosov, the singular value gap $\frac{\sigma_2(\tau\circ\rho^{\varepsilon}(\gamma_n))}{\sigma_1(\tau\circ\rho^{\varepsilon}(\gamma_n))}$ tends to zero as n goes to infinity. Thus, using Lemma 6.5, we have

$$\lim_{n\to\infty} (\tau \circ \rho^{\varepsilon}(\gamma_n)) \cdot \left(\tau^{+} \circ \zeta(g^{+})\right) = \lim_{n\to\infty} U_1(\tau \circ \rho^{\varepsilon}(\gamma_n)). \tag{6.4}$$

Further, with the continuity of ζ^{λ} from Lemma 6.5 it holds that

$$\begin{aligned}
\zeta(g^{+}) = \zeta^{\lambda}(g^{+}) &= \lim_{n\to\infty} \zeta^{\lambda}(\gamma_n \cdot g^{+}) \\
&= \lim_{n\to\infty} \varphi_{P(\gamma_n P)} \cdot \zeta(\gamma_n \cdot g^{+}) \\
&= \lim_{n\to\infty} \left(\varphi_{P(\gamma_n P)} \rho(\gamma_n)\right) \cdot \zeta(g^{+}) \\
&= \lim_{n\to\infty} \rho^{\varepsilon}(\gamma_n) \cdot \zeta(g^{+}).
\end{aligned} \tag{6.5}$$

Using equation (6.5) and τ-equivariance of τ^{+}, we obtain

$$\tau^{+} \circ \zeta(g^{+}) = \lim_{n\to\infty} \tau^{+}\left(\rho^{\varepsilon}(\gamma_n) \cdot \zeta(g^{+})\right) = \lim_{n\to\infty} \left((\tau \circ \rho^{\varepsilon})(\gamma_n)\right) \cdot \tau^{+}\left(\zeta(g^{+})\right). \tag{6.6}$$

Combining (6.6) with (6.4) and the fact that for any $h \in G$, we have $\tau^{+}\left(\Xi_{\theta}(h)\right) = U_1(\tau(h))$ gives us

$$\begin{aligned}
\tau^{+} \circ \zeta(g^{+}) &= \lim_{n\to\infty} (\tau \circ \rho^{\varepsilon}(\gamma_n)) \cdot \left(\tau^{+}\left(\zeta(g^{+})\right)\right) \\
&= \lim_{n\to\infty} U_1(\tau \circ \rho^{\varepsilon}(\gamma_n)) \\
&= \lim_{n\to\infty} \tau^{+}\left(\Xi_{\theta}(\rho^{\varepsilon}(\gamma_n)\right) \\
&= \tau^{+} \circ \zeta^{\varepsilon}(g^{+}).
\end{aligned}$$

For the last step, we use that the boundary map ζ^{ε} satisfies $\zeta^{\varepsilon}(g^{+}) = \lim_{n\to\infty} \Xi_{\theta}(\rho^{\varepsilon}(\gamma_n)$ (see (2.3) in Section 2.3). Since τ^{+} is injective, this shows that $\zeta(g^{+}) = \zeta^{\varepsilon}(g^{+})$ as claimed.

It remains to prove that the prerequisites of Lemma 6.9 are satisfied, i.e. that the angle $\measuredangle\left(\tau^{+} \circ \zeta(g^{+}), U_{d-1}(\tau \circ \rho^{\varepsilon}(\gamma_n^{-1}))\right)$ is uniformly bounded away from zero for n big enough. By Lemma A.5, the sequence $(\gamma_n^{-1})_{n\in\mathbb{N}}$ converges to $g^{-} \in \partial_{\infty}\tilde{S}$. The $(d-1)$-dimensional part of the boundary map for $\tau \circ \rho^{\varepsilon}$ is given by $\tau^{-} \circ \zeta^{\varepsilon}$ and we have with Example 2.25

$$\tau^{-} \circ \zeta^{\varepsilon}(g^{-}) = \lim_{n\to\infty} U_{d-1}(\tau \circ \rho^{\varepsilon}(\gamma_n^{-1})).$$

By transversality of the boundary maps $\tau^{+} \circ \zeta$ and $\tau^{-} \circ \zeta$ and since $g^{+} \neq g^{-}$, we know that the angle between $\tau^{+} \circ \zeta(g^{+})$ and $\tau^{-} \circ \zeta(g^{-})$ is positive. Since ρ^{ε} is a small deformation of ρ, we have that $\tau^{-} \circ \zeta^{\varepsilon}(g^{-})$ is close to $\tau^{-} \circ \zeta(g^{-})$. It follows that for $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_{\theta})$ small enough also the angle between $\tau^{+} \circ \zeta(g^{+})$ and $\tau^{-} \circ \zeta^{\varepsilon}(g^{-})$ is positive. Using the fact that $\tau^- \circ \zeta^\varepsilon(g^-) = \lim_{n\to\infty} U_{d-1}(\tau \circ \rho^\varepsilon(\gamma_n^{-1}))$, we conclude that for n large enough, the angle $\measuredangle\left(\tau^+ \circ \zeta(g^+), U_{d-1}(\tau \circ \rho^\varepsilon(\gamma_n^{-1}))\right)$ is bounded away from zero. This completes the proof. $\square$

7. Injectivity properties of cataclysms

For a fixed lamination λ, the cataclysm deformation Λ cannot be surjective onto a small neighborhood of ρ. This can be seen by looking at the dimensions: Consider for instance Hitchin representations (Subsection 2.4.2) for which we have $\theta = \Delta$. The Hitchin component in $\mathrm{PSL}(n, \mathbb{R})$ is diffeomorphic to $\mathbb{R}^{2(g-1)(n^2-1)}$, whereas the dimension of the parameter space $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}, \mathfrak{a}_\Delta)$ grows only linearly in n by Corollary 3.22. Another way to look at it is in Bonahon-Dreyer coordinates [BD17], where cataclysms only change the shearing cycle, but not the triangle invariants.

We can also ask if cataclysms are injective. We have to distinguish between looking at cataclysms on the representation variety, i.e. Λ_0 with respect to a fixed reference component, or on the character variety, i.e. Λ, where we do not need to specify a reference component. If Λ is injective, then so is Λ_0. In the following, we only consider the cataclysm deformation Λ_0 on the representation variety.

We will see that Λ_0 is in general not injective, not even for Δ-Anosov representations. Some parts of the construction of cataclysms are injective': The assignment of the family of shearing maps to a twisted cycle is injective as we will see in Section 7.1. The results in Section 7.1 are adapted from [Dre13, Section 5.1], see Remark 1.8. In Section 7.2, we give a sufficient condition on the representation ρ that guarantees that the cataclysm deformation is injective. In Section 7.3, we give a sufficient condition for Λ_0 not to be injective. Examples where Λ_0 is not injective include horocyclic representations and reducible Δ-Anosov representations into $\mathrm{SL}(n, \mathbb{R})$. These are investigated in Section 7.4.

7.1. Different twisted cycles give different families of shearing maps

In the construction of cataclysms, the family of shearing maps $\{\varphi^{\varepsilon}_{PQ}\}_{(P,Q)}$ defined in Section 5.1 plays an important role. The goal of this section is to recover the parameter $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ from the family of shearing maps. In other words, we show the following:

Proposition 7.1. *If two transverse twisted cycles* $\varepsilon, \eta \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ *have the same family of shearing maps, i.e.* $\varphi^\varepsilon_{PQ} = \varphi^\eta_{PQ}$ *for all* $P, Q \subset \tilde{S} \setminus \tilde{\lambda}$*, then* $\varepsilon = \eta$*. In other words, the map*

$$\mathcal{V}_\rho \to G^{\{(P,Q) \mid P,Q \subset \tilde{S} \setminus \tilde{\lambda}\}}$$
$$\varepsilon \mapsto \{\varphi^\varepsilon_{PQ}\}_{(P,Q)}$$

is injective. Here, $\mathcal{V}_\rho \subset \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ *is the neighborhood of* 0 *consisting of all twisted cycles such that the family of shearing maps* $\{\varphi^\varepsilon_{PQ}\}_{(P,Q)}$ *is defined (see Proposition 5.9).*

Remark 7.2. The idea of the proof is to recover the shearing parameter from a sum depending on the shearing maps (see Equation (7.1)). This construction is adapted from [Dre13, Section 5.1], where it is done for the case of Δ-Anosov representations into $\mathrm{PSL}(n, \mathbb{R})$. The proof in [Dre13] uses the dynamical viewpoint on Anosov representations through bundles. We take a different approach and use the Busemann cocycle, which allows us to generalize their construction to any semisimple Lie group G and $\theta \subset \Delta$. Further, in [Dre13] they conclude from their result that the cataclysm deformation is injective for Δ-Anosov representations into $\mathrm{PSL}(n, \mathbb{R})$. However, this statement is wrong as we will see in Examples 7.12 and 7.14.

Throughout the section, we fix $\varepsilon \in \mathcal{V}_\rho$, where $\mathcal{V}_\rho$ is the subset of $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ for which the family of shearing maps $\{\varphi^\varepsilon_{PQ}\}_{(P,Q)}$ exists (see Proposition 7.5).

Let $\sigma_\theta \colon G \times \mathcal{F}_\theta \to \mathfrak{a}_\theta$ be the Busemann cocycle (see Lemma 2.20). We first make the following observation.

Lemma 7.3. *Let* g *be an oriented geodesic in* $\tilde{S}$ *and let* $H \in \mathfrak{a}_\theta$*. Then* $\sigma_\theta\left(T^H_g, P^+_g\right) = H$*.*

Proof. First, note that $\sigma_\theta\left(\exp(H), P^+_\theta\right) = H$, since $\exp(H) \in \exp(\mathfrak{a})$. By definition, $T^H_g = m_g \exp(H) m_g^{-1}$, where $m_g \cdot P^\pm_\theta = P^\pm_g$. With the cocycle property of σ_θ, we have

$$\begin{aligned}
\sigma_\theta\left(T^H_g, P^+_g\right) &= \sigma_\theta\left(m_g \exp(H), m_g^{-1} \cdot P^+_g\right) + \sigma_\theta\left(m_g^{-1}, P^+_g\right) \\
&= \sigma_\theta\left(m_g \exp(H), P^+_\theta\right) + \sigma_\theta\left(m_g^{-1}, m_g \cdot P^+_\theta\right) \\
&= \sigma_\theta\left(m_g, \exp(H) \cdot P^+_\theta\right) + \sigma_\theta\left(\exp(H), P^+_\theta\right) + \sigma_\theta\left(m_g^{-1}, m_g \cdot P^+_\theta\right) \\
&= H,
\end{aligned}$$

where for the last equality, we used the cocycle property together with $\exp(H) \cdot P^+_\theta = P^+_\theta$ and $\sigma_\theta\left(m_g^{-1} m_g, P^+_\theta\right) = 0$. $\square$

In particular, if $P, Q \subset \tilde{S} \setminus \tilde{\lambda}$ are two adjacent components separated by the oriented geodesic g, then it holds that $\varphi^\varepsilon_{PQ} = T^{\varepsilon(P,Q)}_g$, so $\sigma_\theta\left(\varphi^\varepsilon_{PQ}, P^+_g\right) = \sigma_\theta\left(T^{\varepsilon(P,Q)}_g, P^+_g\right) = \varepsilon(P, Q)$. In this way, we can recover the value of the twisted cycle ε from the shearing map $\varphi^{\varepsilon}_{PQ}$. This does not only work for adjacent components, as we will now see.

Let P and Q be arbitrary components in $\tilde{S} \setminus \tilde{\lambda}$ and let $\tilde{k}$ be an oriented arc transverse to $\tilde{\lambda}$ joining P to Q. We now define an element $\delta(\tilde{k})$ in $\mathfrak{a}_\theta$ associated with $\tilde{k}$. In Proposition 7.5, we will see that $\delta(\tilde{k}) = \varepsilon(P, Q)$, so δ recovers the shearing parameter $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$. In the following, we omit the superscript ε in the notation of the shearing maps. All shearing maps are with respect to the fixed twisted cycle $\varepsilon \in \mathcal{V}_\rho \subset \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$. Further, for a component $R \subset \tilde{S} \setminus \tilde{\lambda}$, let $\varphi_R := \varphi_{PR}$. We define

$$\delta(\tilde{k}) := \sum_{R \in \mathcal{C}_{PQ}} \left(\sigma_\theta \left(\varphi_R, P^+_{g^0_R} \right) - \sigma_\theta \left(\varphi_R, P^+_{g^1_R} \right) \right) - \sigma_\theta \left(\mathrm{Id}, P^+_{g^1_P} \right) + \sigma_\theta \left(\varphi_Q, P^+_{g^0_Q} \right) \tag{7.1}$$

Note that δ depends on the family of shearing maps $\{\varphi^{\varepsilon}_{PQ}\}_{(P,Q)}$ and only indirectly on ε itself. If two different cycles $\varepsilon, \eta \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ have the same families of shearing maps, i.e. $\varphi^{\varepsilon}_{PQ} = \varphi^{\eta}_{PQ}$ for all $P, Q \subset \tilde{S} \setminus \tilde{\lambda}$, then they give the same value $\delta(\tilde{k})$ for every transverse arc $\tilde{k}$.

First, we prove that δ is well-defined, i.e. that the sum in (7.1) is convergent.

Lemma 7.4. *For every transverse twisted cycle $\varepsilon \in \mathcal{V}_\rho \subset \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ as in Proposition 5.9 and for every oriented arc $\tilde{k}$ transverse to the orientation cover $\tilde{\lambda}$, the sum defining $\delta(\tilde{k})$ as in* (7.1) *is absolutely convergent.*

Proof. Fix a norm $\|\cdot\|$ on $\mathfrak{a}$. The Busemann cocycle is analytic, so in particular, it is locally Lipschitz. Since the arc $\tilde{k}$ is compact, all the shearing maps φ_R lie within a compact subset of G, and the flags $P^+_{g_d^{0/1}}$ lie within a compact subset of $\mathcal{F}_\theta$. Thus, using Hölder continuity of the boundary map, Remark 4.2 and Lemma 3.11, we have constant $C_i, A_i > 0$ depending on the arc $\tilde{k}$ and the representation ρ such that

$$\begin{aligned} \left\| \sigma_\theta \left(\varphi_R, P^+_{g^0_R} \right) - \sigma_\theta \left(\varphi_R, P^+_{g^1_R} \right) \right\| &\leq C_1 \, \mathrm{d}_{\mathcal{F}_\theta} \left(P^+_{g^0_R}, P^+_{g^1_R} \right) \\ &\leq C_2 \, \mathrm{d} \left(g^0_R, g^1_R \right)^{A_1} \\ &\leq C_3 \, \ell \left(\tilde{k} \cap R \right)^{A_1} \\ &\leq C_4 e^{-A_2 r(R)}. \end{aligned}$$

It follows that

$$\sum_{R\in\mathcal{C}_{PQ}} \left\|\sigma_\theta\left(\varphi_R, P^+_{g^0_R}\right) - \sigma_\theta\left(\varphi_R, P^+_{g^1_R}\right)\right\| \leq C_4 \sum_{R\in\mathcal{C}_{PQ}} e^{-A_2 r(R)}$$
$$\leq C_5 \sum_{r=0}^{\infty} e^{-A_2 r} < \infty$$

where for the last equality, we use the fact that the number of all components with a fixed divergence radius is uniformly bounded (Lemma 3.11). In total, the sum defining δ is absolutely convergent. □

We are now ready to show that δ recovers the shearing parameter ε. For the special case when the components P and Q are adjacent, this was shown above as a consequence of Lemma 7.3.

Proposition 7.5. *For every transverse twisted cycle $\varepsilon \in \mathcal{V}_\rho \subset \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ as in Proposition 5.9 and for every oriented arc $\tilde{k}$ transverse to the orientation cover $\tilde{\lambda}$, we have $\delta(\tilde{k}) = \varepsilon(P, Q)$, where P and Q are the connected components of $\tilde{k}$ containing the endpoints of $\tilde{k}$.*

Proof. Let $\mathcal{C} = \{R_1, \dots, R_m\}$ be a finite subset of $\mathcal{C}_{PQ}$. Set $R_0 := P$ and $R_{m+1} := Q$. To shorten notation, let $\varphi_i := \varphi_{R_i}$ and $g_i^{0/1} := g_{R_i}^{0/1}$. Reordering the sum defining δ, we have

$$\delta(\tilde{k}) = \lim_{m\to\infty} \sum_{i=1}^{m} \left(\sigma_\theta\left(\varphi_i, P^+_{g^0_i}\right) - \sigma_\theta\left(\varphi_i, P^+_{g^1_i}\right)\right) - \sigma_\theta\left(\varphi_0, P^+_{g^1_0}\right) + \sigma_\theta\left(\varphi_{m+1}, P^+_{g^0_{m+1}}\right)$$
$$= \lim_{m\to\infty} \sum_{i=0}^{m} \left(\sigma_\theta\left(\varphi_{i+1}, P^+_{g^0_{i+1}}\right) - \sigma_\theta\left(\varphi_i, P^+_{g^1_i}\right)\right).$$

Remember that by Remark 5.7, $\varphi_{i+1} = \varphi_i \psi_i T^{\varepsilon(R_i, R_{i+1})}_{g^0_{i+1}}$, where $\psi_i := \psi_{R_i R_{i+1}}$. Using the cocycle property (2.2), we have

$$\sigma_\theta\left(\varphi_{i+1}, P^+_{g^0_{i+1}}\right) = \sigma_\theta\left(\varphi_i\psi_i, T^{\varepsilon(R_i,R_{i+1})}_{g^0_{i+1}} P^+_{g^0_{i+1}}\right) + \sigma_\theta\left(T^{\varepsilon(R_i,R_{i+1})}_{g^0_{i+1}}, P^+_{g^0_{i+1}}\right)$$
$$= \sigma_\theta\left(\varphi_i\psi_i, P^+_{g^0_{i+1}}\right) + \varepsilon(R_i, R_{i+1}),$$

where in the last step we use Lemma 7.3 and the fact that $T^{\varepsilon(R_i,R_{i+1})}_{g^0_{i+1}}$ stabilizes $P^+_{g^0_{i+1}}$.

Thus, by additivity of ε and the cocycle property,

$$\begin{aligned}\delta(\tilde{k}) &= \lim_{m\to\infty}\sum_{i=0}^{m}\left(\sigma_\theta\left(\varphi_i\psi_i, P^+_{g^0_{i+1}}\right) + \varepsilon(R_i, R_{i+1}) - \sigma_\theta\left(\varphi_i, P^+_{g^1_i}\right)\right)\\ &= \varepsilon(P,Q) + \lim_{m\to\infty}\sum_{i=0}^{m}\left(\sigma_\theta\left(\varphi_i, \psi_i P^+_{g^0_{i+1}}\right) - \sigma_\theta\left(\varphi_i, P^+_{g^1_i}\right) + \sigma_\theta\left(\psi_i, P^+_{g^0_{i+1}}\right)\right)\end{aligned}$$

It remains to show that the limit on the right side equals zero. By local Lipschitz continuity of the Busemann cocycle, there exists a constant $C_1 > 0$ depending on $\tilde{k}$ and ρ such that

$$\begin{aligned}\left\|\sigma_\theta\left(\varphi_i, \psi_i P^+_{g^0_{i+1}}\right) - \sigma_\theta\left(\varphi_i, P^+_{g^1_i}\right)\right\| \quad &\leq C_1 \,\mathrm{d}_{\mathcal{F}_\theta}\left(\psi_i P^+_{g^0_{i+1}}, P^+_{g^1_i}\right)\\ &\leq C_1\left(\mathrm{d}_{\mathcal{F}_\theta}\left(\psi_i P^+_{g^0_{i+1}}, P^+_{g^0_{i+1}}\right) + \mathrm{d}_{\mathcal{F}_\theta}\left(P^+_{g^0_{i+1}}, P^+_{g^1_i}\right)\right).\end{aligned}$$

Let $N_{\mathcal{C}} := \min_{R\in\mathcal{C}_{PQ}\setminus\mathcal{C}} r(R)$ be the minimal divergence radius of all components of $\mathcal{C}_{PQ}$ that are not contained in $\mathcal{C}$. $N_{\mathcal{C}}$ goes to infinity as $\mathcal{C}$ tends to $\mathcal{C}_{PQ}$, since for fixed n, there are only finitely many components in $\mathcal{C}_{PQ}$ with $r(R) = n$. By Hölder continuity of the boundary map and by Remark 4.2, there are constants $A_i, C_i > 0$ depending on $\tilde{k}$ and ρ such that

$$\begin{aligned}\mathrm{d}_{\mathcal{F}_\theta}\left(P^+_{g^0_{i+1}}, P^+_{g^1_i}\right) &\leq C_2\mathrm{d}\left(g^0_{i+1}, g^1_i\right)^{A_1}\\ &\leq C_3\ell\left(\tilde{k}\cap\bigcup_{R\in\mathcal{C}_{g^0_{i+1}g^1_i}} R\right)^{A_1}\\ &= C_3\left(\sum_{R\in\mathcal{C}_{g^0_{i+1}g^1_i}}\ell\left(\tilde{k}\cap R\right)\right)^{A_1}\\ &\leq C_4\left(\sum_{R\in\mathcal{C}_{g^0_{i+1}g^1_i}} e^{-A_2 r(R)}\right)^{A_1}\\ &\leq C_5 e^{-A_3 N_{\mathcal{C}}},\end{aligned}$$

where for the last step, we used that the series can be estimated by the remainder term of a geometric series and is bounded by a constant times $e^{-A_3 N_{\mathcal{C}}}$ as in the proof of Lemma 4.8.

Further, since the action of G on $\mathcal{F}_\theta$ is smooth, it is in particular locally Lipschitz, we have by Lemma 5.8

$$\mathrm{d}_{\mathcal{F}_\theta}\left(\psi_i P^+_{g^0_{i+1}}, P^+_{g^0_{i+1}}\right) \leq C_6 \mathrm{d}_G(\psi_i, \mathrm{Id}) \leq C_7 e^{-A_4 N_{\mathcal{C}}}.$$

Combining these estimates gives us

$$\left\|\sigma_\theta\left(\varphi_i, \psi_i P^+_{g^0_{i+1}}\right) - \sigma_\theta\left(\varphi_i, P^+_{g^1_i}\right)\right\| \leq C_8 e^{-A_5 N_{\mathcal{C}}}. \tag{7.2}$$

In addition, again by Lemma 5.8, we have

$$\begin{aligned}\left\|\sigma_\theta\left(\psi_i, P^+_{g^0_{i+1}}\right)\right\| &= \left\|\sigma_\theta\left(\psi_i, P^+_{g^0_{i+1}}\right) - \sigma_\theta\left(\mathrm{Id}, P^+_{g^0_{i+1}}\right)\right\| \\ &\leq C_9 \mathrm{d}_G(\psi_i, \mathrm{Id}) \\ &\leq C_{10} e^{-A_6 N_{\mathcal{C}}}.\end{aligned} \tag{7.3}$$

Combining the estimates from (7.2) and (7.3), we have

$$\sum_{r=0}^{m}\left\|\sigma_\theta\left(\varphi_i\psi_i, P^+_{g^0_{i+1}}\right) - \sigma_\theta\left(\varphi_i, P^+_{g^1_i}\right)\right\| \leq \sum_{i=0}^{m} C_{11} e^{-A_7 N_{\mathcal{C}}} \leq C(N_{\mathcal{C}}+1)e^{-AN_{\mathcal{C}}},$$

where we use the fact that $m = |\mathcal{C}|$ is bounded by a constant times $N_{\mathcal{C}}+1$ (Lemma 3.11). If $\mathcal{C}$ goes to $\mathcal{C}_{PQ}$, $N_{\mathcal{C}}$ goes to infinity, so the right hand side converges to zero. This finishes the proof. □

As a direct consequence of Proposition 7.5, we obtain that different cycles in $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ give different families of shearing maps, i.e. Proposition 7.1. In other words, the assignment of the family of shearing maps $\{\varphi^\varepsilon_{PQ}\}_{(P,Q)}$ to a transverse cycle ε is injective.

Proof of Proposition 7.1. This follows from Proposition 7.5, using that δ only depends on the family of shearing maps $\{\varphi^\varepsilon_{PQ}\}_{(P,Q)}$ and not on ε itself. □

Now one would like to conclude from that that the cataclysm map on the level of homomorphisms, $\Lambda_0\colon \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta) \to \mathrm{Hom}(\pi_1(S), G)$, is injective for a fixed reference component P. Assume that there exist $\varepsilon \neq \eta \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ with $\Lambda^\varepsilon_0\rho = \Lambda^\eta_0\rho$. This implies $\varphi^\varepsilon_{P(\gamma P)} = \varphi^\eta_{P(\gamma P)}$ for all $\gamma \in \pi_1(S)$, but in general not that $\varphi^\varepsilon_{PQ} = \varphi^\eta_{PQ}$ for all components $Q \subset \tilde{S} \setminus \tilde{\lambda}$ as we will see in Section 7.3. In particular, we cannot apply Proposition 7.1 since for that, we need $\varphi^\varepsilon_{PQ} = \varphi^\eta_{PQ}$ for all pairs of components $P, Q \subset \tilde{S} \setminus \tilde{\lambda}$.

7.2. A sufficient condition for injectivity of the cataclysm deformation

We will see in Sections 7.3 and 7.4 below that the cataclysm deformation is not injective in general. Under an additional assumption injectivity follows from the injectivity of the assignment of the family of shearing maps to a twisted cycle, i.e. Proposition 7.5. For a connected component $Q \subset \tilde{S} \setminus \tilde{\lambda}$, let $\partial Q \subset \partial_\infty \tilde{S}$ be the set of ideal vertices of Q in the boundary of $\tilde{S}$.

Proposition 7.6. *Let $\rho\colon \pi_1(S) \to G$ be θ-Anosov such that for every connected component $Q \subset \tilde{S} \setminus \tilde{\lambda}$, the stabilizer of the set of all flags $\zeta(x)$ with $x \in \partial Q$, $\mathrm{Stab}_G\{\zeta(x) | x \in \partial Q\}$, is trivial. Then for a fixed reference triangle P_0, the cataclysm deformation based at ρ, $\Lambda_0\colon \mathcal{U}_\rho \to \mathrm{Hom}(\pi_1(S), G)$ is injective.*

Here, $\mathcal{U}_\rho$ is the neighborhood of 0 in $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ for which the deformed representation $\Lambda_0^\varepsilon \rho$ is again an Anosov representation. We restrict to these representations, since it ensures that $\Lambda_0^\varepsilon \rho$ has a continuous, transverse, dynamics-preserving boundary map that is central to the proof.

Proof. Let $\varepsilon, \eta \in \mathcal{U}_\rho$ be such that $\Lambda_0^\varepsilon \rho = \Lambda_0^\eta \rho$. We want to show that $\varepsilon = \eta$. Let ζ^ε and ζ^η denote the boundary maps corresponding to $\Lambda_0^\varepsilon \rho$ and $\Lambda_0^\eta \rho$, respectively. By assumption, $\zeta^\varepsilon = \zeta^\eta$. Let $Q \subset \tilde{S} \setminus \tilde{\lambda}$ be a connected component and let $x \in \partial Q$ be a vertex of Q. Then, by Theorem 6.1, we can express ζ^ε and ζ^η in terms of ζ and the families of shearing maps $\{\varphi^\varepsilon_{PQ}\}_{(P,Q)}$ and $\{\varphi^\eta_{PQ}\}_{(P,Q)}$, respectively. This gives us

$$\varphi^\varepsilon_{P_0Q} \cdot \zeta(x) = \zeta^\varepsilon(x) = \zeta\eta(x) = \varphi^\eta_{P_0Q} \cdot \zeta(x),$$

so $\left(\varphi^\varepsilon_{P_0Q}\right)^{-1} \varphi^\eta_{P_0Q} \in \mathrm{Stab}(\zeta(x))$. This holds for all $x \in \partial Q$, hence

$$\left(\varphi^\varepsilon_{P_0Q}\right)^{-1} \varphi^\eta_{P_0Q} \in \mathrm{Stab}\{\zeta(x) \mid x \in \partial Q\} = \{\mathrm{Id}\}$$

by assumption. Thus, $\varphi^\varepsilon_{P_0Q} = \varphi^\eta_{P_0Q}$ for all $Q \subset \tilde{S} \setminus \tilde{\lambda}$. If $P \subset \tilde{S} \setminus \tilde{\lambda}$ is another component different from P_0 and Q, then

$$\varphi^\varepsilon_{PQ} = \varphi^\varepsilon_{PP_0}\varphi^\varepsilon_{P_0Q} = \left(\varphi^\varepsilon_{P_0P}\right)^{-1} \varphi^\varepsilon_{PQ} = \left(\varphi^\eta_{P_0P}\right)^{-1} \varphi^\eta_{PQ} = \varphi^\eta_{PP_0}\varphi^\eta_{P_0Q} = \varphi^\eta_{PQ},$$

so the families of shearing maps associated with ε and η agree. With Proposition 7.1, we can conclude that $\varepsilon = \eta$. □

For every $Q \subset \tilde{S} \setminus \tilde{\lambda}$, the boundary ∂Q consists of at least three points. In particular, Proposition 7.6 holds under the stronger assumption that for any triple of distinct points $x, y, z \in \partial_\infty \tilde{S}$, the stabilizer of the triple $(\zeta(x), \zeta(y), \zeta(z))$ is trivial.

Corollary 7.7. *If ρ is a Hitchin representation into $\mathrm{PSL}(n, \mathbb{R})$ or into $\mathrm{SL}(n, \mathbb{R})$, then the cataclysm deformation based at ρ is injective.*

Proof. For Hitchin representations into $\mathrm{PSL}(n, \mathbb{R})$, this directly follows from Proposition 7.6, since the stabilizer of every triple of flags $(\zeta(x), \zeta(y), \zeta(z))$ for $x, y, z \in \partial_\infty \tilde{S}$ is trivial and ∂Q consists of at least three points for every $Q \subset \tilde{S} \setminus \tilde{\lambda}$. For Hitchin representations into $\mathrm{SL}(n, \mathbb{R})$ with n odd, the same holds. If n is even, then the stabilizer of a triple $(\zeta(x), \zeta(y), \zeta(z))$ is $\{\pm \mathrm{Id}\}$. In this case, we can look at the projection $\pi\colon \mathrm{Hom}(\pi_1(S), \mathrm{SL}(n, \mathbb{R})) \to \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(n, \mathbb{R}))$ and see that it commutes with the cataclysm deformation in the sense that $\pi\left(\Lambda_0^\varepsilon \rho\right) = \Lambda_0^\varepsilon(\pi(\rho))$. If $\eta, \varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ are such that $\Lambda_0^\varepsilon \rho = \Lambda_0^\eta \rho$, then

$$\Lambda_0^\varepsilon(\pi(\rho)) = \pi\left(\Lambda_0^\varepsilon \rho\right) = \pi\left(\Lambda_0^\eta \rho\right) = \Lambda_0^\eta(\pi(\rho)).$$

From injectivity of the cataclysm deformation for Hitchin representations into $\mathrm{PSL}(n, \mathbb{R})$, it follows that $\varepsilon = \eta$, so the cataclysm deformation is injective. □

Corollary 7.7 shows in particular that the injectivity of cataclysm deformations claimed by Dreyer in [Dre13, Corollary 35] is correct if we restrict to Hitchin representations.

7.3. A sufficient condition for non-injectivity of the cataclysm deformation

The goal of this section is to show that the cataclysm deformation in general not injective. We consider the cataclysm $\Lambda_0\colon \mathcal{U}_\rho \to \mathrm{Hom}(\pi_1(S), G)$ on the level of homomorphisms. We construct a maximal lamination λ with reference triangle P_0 and give sufficient conditions on G, θ and the representation ρ that ensure that there exists a transverse twisted cycle $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ that is non-zero, but such that $\Lambda_0^\varepsilon \rho = \rho$. By definition, we have $\Lambda_0^\varepsilon \rho = \rho$ if and only if $\varphi^\varepsilon_{P \gamma P} = \mathrm{Id}$ for all $\gamma \in \pi_1(S)$.

Let S be a surface of genus 2. Consider the finite maximal lamination λ from Example 3.8 (see Figure 3.2). As fixed reference triangle, we choose a lift of the yellow triangle and denote it by P (see Figure 7.1). Further, denote by g, h and g' the oriented geodesics bounding a triangle adjacent to P as in Figure 7.1. Fix another component $\tilde{P}$ as in Figure 7.1. The component $\tilde{P}$ is separated from P by exactly one lift of a closed leaf.

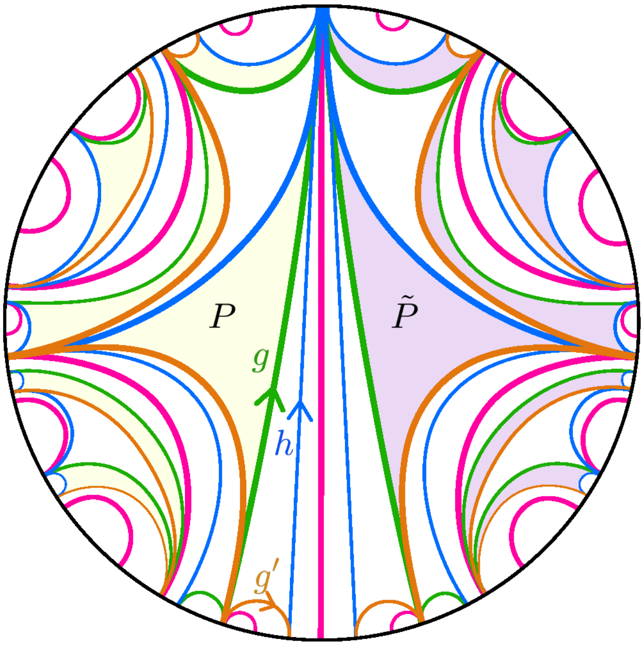


Figure 7.1.: The lift of the lamination λ from Example 3.8, with distinguished components P and $\tilde{P}$ as well as oriented geodesics g, h, g'.

Proposition 7.8. *Consider the lamination λ described in Example 3.8 and with the notation from Figure 7.1. Let $P_\theta^* \in \mathcal{F}_\theta$ be a flag transverse to both P_θ^+ and P_θ^- and assume that there exists a representative $w_0 \in N_K(\mathfrak{a}_\theta)$ of the longest element of the Weyl group such that $w_0 P_\theta^* = P_\theta^*$. Let $m \in N_\theta^+$ be the unique element such that $m \cdot P_\theta^- = P_\theta^*$. Let $H \in \mathfrak{a}_\theta \setminus \{0\}$ be such that $m \in Z_G(\exp(H))$ and $\iota(H) = -H$. Let $\rho\colon \pi_1(S) \to G$ be a representation such that the triple of flags (P_g^+, P_g^-, P_h^-) is conjugated in G to $(P_\theta^+, P_\theta^-, P_\theta^*)$. Then there exists $0 \neq \varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ such that $\varphi^\varepsilon_{P(\gamma P)} = \mathrm{Id}$ for all $\gamma \in \pi_1(S)$.*

We will see examples for which the assumptions of Proposition 7.8 are satified below (Examples 7.12 – 7.16).

Proposition 7.8 has the following consequence.

Corollary 7.9. *The cataclysm deformations $\Lambda_0\colon \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta) \to \mathrm{Hom}(\pi_1(S), G)$ from Theorem 5.12 and $\Lambda\colon \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta) \to \chi^{\theta\text{-}Anosov}(S, G)$ from Corollary 5.13 are in general not injective.*

Proof. Let the representation $\rho\colon \pi_1(S) \to G$, the lamination λ and the transverse twisted cycle $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ be as in Proposition 7.8 and let P be the fixed reference triangle. Then $\varepsilon \neq 0$, but for every $\gamma \in \pi_1(S)$, we have $\Lambda_0^\varepsilon \rho(\gamma) = \varphi^\varepsilon_{P(\gamma P)} \rho(\gamma) = \rho(\gamma)$, so Λ_0 is not injective. On the character variety, we have $\Lambda^\varepsilon[\rho] = [\Lambda_0^\varepsilon \rho] = [\rho]$, so also Λ is not injective. $\square$

As a first step in the proof of Proposition 7.8 we have a closer look at the stretching maps in our specific situation.

Lemma 7.10. *Let the representation $\rho\colon \pi_1(S) \to G$, the lamination λ and $H \in \mathfrak{a}_\theta$ be as in Proposition 7.8. Let g, h and g' be the oriented geodesics as in Figure 7.1. Then the stretching maps satisfy $T_g^H = T_h^H = T_{g'}^H$.*

Proof. Without loss of generality we can assume $(P_g^+, P_g^-, P_h^-) = (P_\theta^+, P_\theta^-, P_\theta^*)$. In particular, $(P_g^+, P_g^-) = (P_\theta^+, P_\theta^-)$, $(P_h^+, P_h^-) = (P_\theta^+, P_\theta^*)$ and $(P_{g'}^+, P_{g'}^-) = (P_h^-, P_g^-) = (P_\theta^*, P_\theta^-)$. In this case we have $T_g^H = \exp(H)$. Let $m \in N_\theta^+$ as in Proposition 7.8 be the unique element such that $m \cdot P_\theta^- = P_\theta^*$. By the assumption in Proposition 7.8 on m and H, we have $m \in Z_G(\exp(H))$. It follows that

$$T_h^H = m\exp(H)m^{-1} = \exp(H) = T_g^H.$$

Since by assumption $w_0 \in \mathrm{Stab}(P_\theta^*)$, we have that $w_0 m w_0^{-1}(P_\theta^+, P_\theta^-) = (P_\theta^*, P_\theta^-)$. Recall further that $w_0 \exp(H) w_0^{-1} = \exp(-\iota(H))$ as seen in the proof of Lemma 4.16. This gives

$$\begin{aligned}
T_{g'}^H &= \left(w_0 m w_0^{-1}\right)\exp(H)\left(w_0 m w_0^{-1}\right)^{-1}\\
&= w_0 m \exp(-\iota(H)) m^{-1} w_0^{-1}\\
&= w_0 \exp(H) w_0^{-1}\\
&= \exp(-\iota(H))\\
&= \exp(H)\\
&= T_g^H,
\end{aligned}$$

where we used $\iota(H) = -H$ and $m \in Z_G(\exp(H))$. This concludes the proof. □

Remark 7.11. We now define a transverse twisted cycle $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$. We will use this cycle in the proof of Proposition 7.8 and show that it satisfies $\varphi^\varepsilon_{P(\gamma P)} = \mathrm{Id}$ for all $\gamma \in \pi_1(S)$. Let $H \in \mathfrak{a}_\theta$ satisfy the prerequisites from Proposition 7.8, i.e. $\iota(H) = -H$ and $m \in Z_G(\exp(H))$. We define the cycle on the level of arcs on S transverse to λ. Abusing notation, we denote the leaves of λ on S with the same letters as their lifts to the universal cover as in Figure 7.1. Let k_g, k_h, $k_{g'}$, $k_{\tilde{h}}$, k_1 and k_2 be the oriented arcs shown in Figure 7.2. For example, k_g is an oriented arc crossing a projection of the leaf g, drawn in green, and has its negative endpoint in the yellow triangle, its positive endpoint in the white

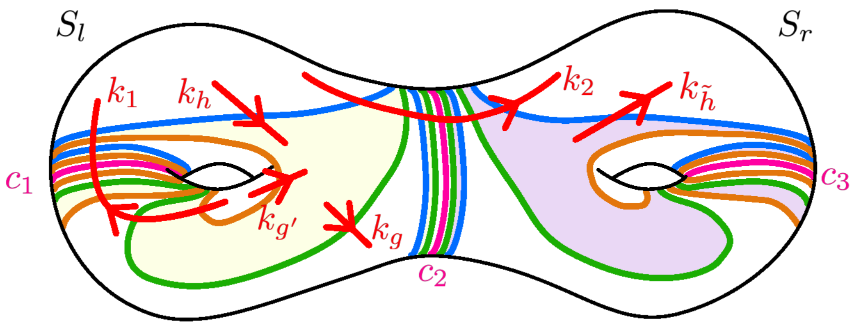


Figure 7.2.: The lamination λ contains three closed leaves c_1, c_2 and c_3, where c_2 is separating. We fix oriented arcs k_g, k_h, $k_{g'}$, $k_{\tilde{h}}$, k_1 and k_2 that we use to define a twisted cycle in Remark 7.11. Further, the closed leaf c_2 divides S into two subsurfaces S_l and S_r.

triangle. Then set

$$\begin{aligned}\varepsilon(k_g) &= H,\\ \varepsilon(k_h) &= \varepsilon(k_{g'}) = -H,\\ \varepsilon(k_1) &= 0 \text{ and}\\ \varepsilon(k_2) &= -H.\end{aligned}$$

The closed leaf c_2 divides S into two connected components S_l and S_r (see Figure 7.2). If k is an oriented arc lying entirely in the connected component $S_r \subset S \setminus c_2$, then define $\varepsilon(k) = 0$. For example, $\varepsilon(k_{\tilde{h}}) = 0$. If $\overline{k}$ is any of the above arcs k with opposite orientation, then $\varepsilon(\overline{k}) = \iota(\varepsilon(k))$. If k is an arbitrary oriented arc transverse to λ, then $\varepsilon(k)$ is given from the above definition using homotopy invariance and additivity of ε. For example, if k is an arc crossing c_2 with negative endpoint in the yellow triangle and positive endpoint in the purple triangle, then

$$\varepsilon(k) = \varepsilon(k_2) - \varepsilon(k_h) - \varepsilon(\overline{k_{\tilde{h}}}) = -H - (-H) + 0 = 0. \tag{7.4}$$

Similarly, if k is an arc crossing c_1 from bottom to top, with negative and positive endpoint in the yellow triangle, then

$$\varepsilon(k) = \varepsilon(k_1) - \varepsilon(k_{g'}) - \varepsilon(\overline{k_h}) = 0 - (-H) - \iota(-H) = 0. \tag{7.5}$$

By the one-to-one correspondence between oriented arcs transverse to λ and arcs transverse to the orientation cover (Remark 3.16), we obtain an $\mathfrak{a}_\theta$-valued transverse cycle for the orientation cover $\widehat{\lambda}$. By construction, since we defined $\varepsilon(\overline{k}) = \iota(\varepsilon(k))$, the twist condition is satisfied, so ε is an element in $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$.

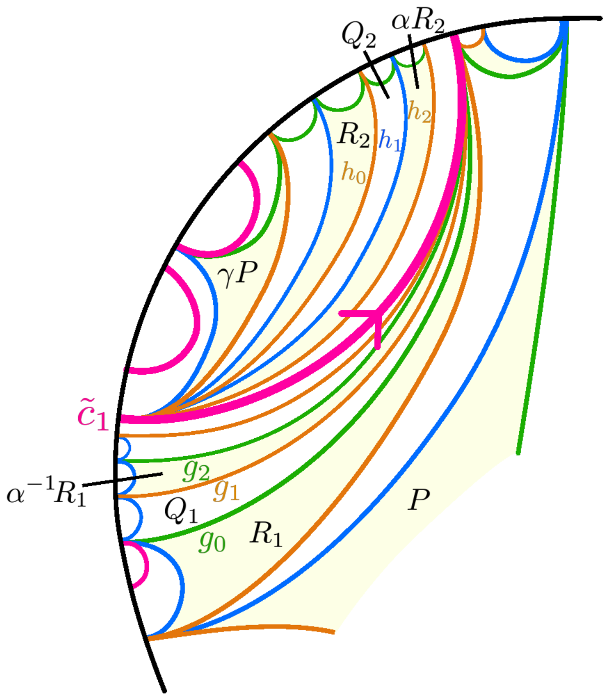


Figure 7.3.: If P and γP are separated by exactly one lift $\tilde{c}_1$ of the closed leaf c_1, define auxiliary triangles R_1 and R_2 separating P and γP from $\tilde{c}_1$, respectively. In the case pictured here, if α denotes the element in $\pi(S)$ that acts as translation along $\tilde{c}_1$ as indicated by the arrow, then the the triangles $\alpha^{-i}R_1$ and $\alpha^i R_2$ converge to $\tilde{c}_1$.

We are now ready to prove Proposition 7.8, i.e. show that there exists $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ such that $\varphi^{\varepsilon}_{P(\gamma P)} = \mathrm{Id}$ for all $\gamma \in \pi_1(S)$.

Proof of Proposition 7.8. Since there is no ambiguity, we omit the superscript ε in the notation of the shearing maps. First, we observe that by Lemma 7.10

$$T_g^H T_h^{-H} = \mathrm{Id}, \quad T_g^H T_{g'}^{-H} = \mathrm{Id} \quad \text{and} \quad T_{\overline{h}}^{\iota(H)} T_{g'}^{-H} = T_h^H T_{g'}^{-H} = \mathrm{Id}. \tag{7.6}$$

Let $\gamma \in \pi_1(S)$ and assume first that $\mathcal{C}_{P(\gamma P)}$ consists of finitely many triangles, so P and γP are not separated by a non-isolated leaf. In this case, an arc $\tilde{k}$ from P to γP can be subdivided into finitely many arcs $\tilde{k}_1, \ldots, \tilde{k}_l$ that have both their negative and positive endpoint in a component of $\tilde{S} \setminus \tilde{\lambda}$ that lie in the same $\pi_1(S)$-orbit as P. In Figure 7.1, these are the components shaded in yellow. Further, we can assume that every $\tilde{k}_i$ crosses exactly two leaves of $\tilde{\lambda}$. These leaves then have to be g and h or g and g' or $\overline{h}$ and g'. Hence, $\varphi_{P(\gamma P)}$ is a finite concatenation of conjugates of maps as in (7.6), so the identity.

Next, assume that P and γP are separated by exactly one non-isolated leaf $\tilde{c}_1$ which is a lift of the closed leaf c_1 (see Figure 7.3). Denote by α the element of $\pi_1(S)$ that acts as translation along c_1 in the direction indicated in Figure 7.3. Let $R_1 \subset \tilde{S} \setminus \tilde{\lambda}$ be such that R_1 separates P from $\tilde{c}_1$ and that it has an endpoint of $\tilde{c}_1$ as one of its vertices (see Figure 7.3). Further, we require that R_1 is in the same $\pi_1(S)$-orbit as P, so there exists some $\gamma_1 \in \pi_1(S)$ such that $R_1 = \gamma_1 P$. The component R_1 is not unique. In particular, if P itself has an endpoint of $\tilde{c}_1$ as vertex, we can set $R_1 = P$. Any other choice of R_1 works as well and the argument below is independent of the choice. Similarly, let $R_2 \subset \tilde{S} \setminus \tilde{\lambda}$ be such that it separates $\tilde{c}_1$ from γP, that it has an endpoint of $\tilde{c}_1$ as one of its vertices and such that $R_2 = \gamma_2 P$ for some $\gamma_2 \in \pi_1(S)$. Then $\varphi_{P(\gamma P)} = \varphi_{PR_1}\varphi_{R_1R_2}\varphi_{R_2\gamma P}$. There are only finitely many ideal triangles between P and $R_1 = \gamma_1 P$ and the same holds for $R_2 = \gamma_2 P$. As above, φ_{PR_1} and $\varphi_{R_2(\gamma P)}$ are finite concatenations of conjugates of the maps as in (7.6), so $\varphi_{P(\gamma P)} = \varphi_{R_1R_2}$. Denote by Q_1 and Q_2 the ideal triangles in $\tilde{S} \setminus \tilde{\lambda}$ adjacent to R_1 and R_2, respectively, and separating them from $\tilde{c}_1$. The ideal triangles $\alpha^{-m}R_1$ and $\alpha^m R_2$ accumulate to $\tilde{c}_1$. The set $\mathcal{C}_{R_1R_2}$ is infinite. We can approximate it by a sequence $\mathcal{C}_m$ defined as

$$\mathcal{C}_m = \{Q_1, \alpha^{-1}R_1, \alpha^{-1}Q_1, \ldots, \alpha^{-m}R_1, \alpha^m R_2, \alpha^{m-1}Q_2, \cdots, \alpha R_2, Q_2\}.$$

So $\mathcal{C}_m$ contains all triangles separating R_1 from $\tilde{c}_1$ up to $\alpha^{-m}R_1$ and all triangles separating R_2 from $\tilde{c}_1$ up to $\alpha^m R_2$. Denote by g_{2i} the oriented geodesic separating $\alpha^{-i}R_1$ from $\alpha^i Q_1$, and by g_{2i+1} the oriented geodesic separating $\alpha^{-i}Q_1$ from $\alpha^{-(i+1)}R_1$ (see Figure 7.3). Analogously, define the geodesics h_{2i} and h_{2i+1} for R_2 and Q_2. Then we have

$$\begin{aligned}
\varphi_{\mathcal{C}_m} &= \left(T_{g_0}^{-\varepsilon(R_1,Q_1)}T_{g_1}^{-\varepsilon(R_1,Q_1)}\right)\left(T_{g_1}^{\varepsilon(R_1,\alpha^{-1}R_1)}T_{g_2}^{-\varepsilon(R_1,\alpha^{-1}R_1)}\right)\cdots \\
&\quad \left(T_{g_m}^{\varepsilon(R_1,\alpha^{-m}R_1)}T_{g_{m+1}}^{-\varepsilon(R_1,\alpha^{-m}R_1)}\right)\left(T_{h_m}^{\varepsilon(R_1,\alpha^m R_2)}T_{h_{m-1}}^{-\varepsilon(R_1,\alpha^m R_2)}\right)\cdots \\
&\quad \left(T_{h_1}^{\varepsilon(R_1,Q_2)}T_{h_0}^{-\varepsilon(R_1,Q_2)}\right)T_{h_0}^{\varepsilon(R_1,R_2)} \\
&= \varphi_{R_1(\alpha^m R_1)}T_{g_{m+1}}^{-\varepsilon(R_1,\alpha^m R_1)}\left(T_{h_m}^{\varepsilon(R_1,\alpha^m R_2)}T_{h_{m-1}}^{-\varepsilon(R_1,\alpha^m R_2)}\right)\cdots \\
&\quad \left(T_{h_1}^{\varepsilon(R_1,Q_2)}T_{h_0}^{-\varepsilon(R_1,Q_2)}\right)T_{h_0}^{\varepsilon(R_1,R_2)} \\
&= T_{g_{m+1}}^{-\varepsilon(R_1,\alpha^m R_1)}\left(T_{h_m}^{\varepsilon(R_1,\alpha^m R_2)}T_{h_{m-1}}^{-\varepsilon(R_1,\alpha^m R_2)}\right)\cdots\left(T_{h_1}^{\varepsilon(R_1,Q_2)}T_{h_0}^{-\varepsilon(R_1,Q_2)}\right)T_{h_0}^{\varepsilon(R_1,R_2)},
\end{aligned}$$

where we use that $\varphi_{R_1\alpha^m R_1}$ is the identity. Note that we have $\varepsilon(R_1, \alpha^m R_1) = 0$ and $\varepsilon(R_1, \alpha^m R_2) = 0$ by definition of ε as seen in (7.5). This implies that for all $j = 1, \ldots, m$ we have

$$\begin{aligned}
&\varepsilon(R_1, \alpha^j R_2) = \varepsilon(R_1, \alpha^m R_2) + \varepsilon(\alpha^m R_2, \alpha^j R_2) = \varepsilon(\alpha^m R_2, \alpha^j R_2) \quad \text{and} \\
&\varepsilon(R_1, \alpha^j Q_2) = \varepsilon(R_1, \alpha^m R_2) + \varepsilon(\alpha^m R_2, \alpha^j Q_2) = \varepsilon(\alpha^m R_2, \alpha^j Q_2).
\end{aligned}$$

As a consequence, the composition defining $\varphi_{\mathcal{C}_m}$ simplifies to

$$\begin{aligned}\varphi_{\mathcal{C}_m} &= T_{g_{m+1}}^{-\varepsilon(R_1,\alpha^m R_1)} T_{h_m}^{\varepsilon(R_1,\alpha^m R_2)} T_{h_{m-1}}^{-\varepsilon(R_1,\alpha^m R_2)} \varphi_{(\alpha^m R_2)R_2} \\ &= \varphi_{(\alpha^m R_2)R_2} \\ &= \mathrm{Id}\end{aligned}$$

This holds for all $m \in \mathbb{N}$, so also $\lim_{m\to\infty} \varphi_{\mathcal{C}_m} = \mathrm{Id}$. In total, $\varphi_{P(\gamma P)} = \mathrm{Id}$ if there is exactly one non-isolated leaf between P and γP.

With the same techniques, we can show that $\varphi_{P\tilde{P}} = \mathrm{Id}$, where $\tilde{P} \subset \tilde{S} \setminus \tilde{\lambda}$ is the connected component shaded in purple in Figure 7.1. Here, we use that $\varepsilon(k) = 0$ if k is an arc contained in S_r and that $\varepsilon(P, \tilde{P}) = 0$ by (7.4). Further, $\varphi_{\tilde{P}(\gamma\tilde{P})} = \mathrm{Id}$ if $\tilde{P}$ and $\gamma\tilde{P}$ are separated by exactly one lift of the non-isolated leaf c_3. This follows again from the fact that ε is zero on all arcs that lie entirely in S_r. If P and γP are separated by several non-isolated leaves, we can write $\varphi_{P(\gamma P)}$ as a finite concatenation of shearing maps that are conjugate to map that is of one of the following forms:

- $\varphi_{P(\gamma' P)}$ where P and $\gamma' P$ are separated by finitely many leaves or by exactly one lift of the non-isolated leaf c_1,

- $\varphi_{P\tilde{P}}$ or

- $\varphi_{\tilde{P}(\gamma'\tilde{P})}$ where $\tilde{P}$ and $\gamma'\tilde{P}$ are separated by finitely many leaves or by exactly one lift of the non-isolated leaf c_3.

We have considered these cases above and have seen that the shearing maps are the identity. Hence, also $\varphi_{P(\gamma P)}$ is the identity, which finishes the proof. □

We now give explicit examples where Proposition 7.8 applies. These examples show that there are representations $\rho\colon \pi_1(S) \to G$ with different properties for which the cataclysm deformation is not injective.

Example 7.12. Let $G = \mathrm{SL}(3, \mathbb{R})$. In this case, $|\Delta| = 1$, so every Anosov representation is Δ-Anosov. In particular, being projective Anosov is the same as being Δ-Anosov. Denote by e_1, e_2, e_3 the standard basis vectors for $\mathbb{R}^3$. Then $P_\Delta^+ = (\langle e_1\rangle \subset \langle e_1, e_2\rangle)$, $P_\Delta^- = (\langle e_3\rangle \subset \langle e_2, e_3\rangle)$ and define $P_\Delta^* := (\langle e_1 + e_3\rangle \subset \langle e_1 + e_3, e_2\rangle)$. The flag P_Δ^* is transverse to both P_Δ^+ and P_Δ^- and satisfies $w_0 \cdot P_\Delta^* = P_\Delta^*$, where $w_0 = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}$ represents the longest element in the Weyl group. The unique element $m \in N_\Delta^+$ satisfying $m \cdot P_\Delta^- = P_\Delta^*$ is

$$m = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Set

$$H = \begin{pmatrix} a & 0 & 0 \\ 0 & -2a & 0 \\ 0 & 0 & a \end{pmatrix} \quad \text{for } a \in \mathbb{R}.$$

Then $\iota(H) = -H$ and $m \in Z_G(\exp(H))$. Let $\rho\colon \pi_1(S) \to \mathrm{SL}(3,\mathbb{R})$ be a horocyclic representation (see Subsection 2.4.3). Choosing $a \in \mathbb{R}$ small, ε as in Remark 7.11 defines a cycle such that the cataclysm deformation along λ based at ρ exists. For all $x \in \partial_\infty \tilde{S}$, the flag curve ζ for ρ is of the form

$$\zeta(x) = \left(\left\langle \begin{pmatrix} a \\ 0 \\ b \end{pmatrix} \right\rangle \subset \left\langle \begin{pmatrix} a \\ 0 \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle \right) \tag{7.7}$$

as seen in Subsection 2.4.3. Since $\mathrm{SL}(2,\mathbb{R})$ acts transitively on triples of points in $\mathbb{RP}^1$, $\mathrm{SL}(3,\mathbb{R})$ acts transitively on triples of flags of the form (7.7). In particular, the triple of flags (P_g^+, P_g^-, P_h^-) is conjugated to the triple $(P_\Delta^+, P_\Delta^-, P_\Delta^*)$. Thus, the prerequisites of Proposition 7.8 are satisfied and by Corollary 7.9, the cataclysm deformation is not injective in this case. We will have a closer look at deformations of horocyclic representations in Section 7.4 and determine explicitly for which they are trivial.

Example 7.13. We now give an example for projective Anosov representations in $\mathrm{SL}(n,\mathbb{R})$ for $n \geq 3$ for which Proposition 7.8 applies: We can directly generalize the counterexample 7.12 to $(1, n-1)$-horocyclic representations into $\mathrm{SL}(n,\mathbb{R})$. In this case, the parabolic subgroups satisfy $P_\theta^+ = (\langle e_1 \rangle \subset \langle e_1, \dots, e_{n-1} \rangle)$ and $P_\theta^- = (\langle e_n \rangle \subset \langle e_2, \dots, e_n \rangle)$. We define $P_\theta^* := (\langle e_1 + e_n \rangle \subset \langle e_1 + e_n, e_2, \dots, e_{n-1} \rangle)$ and

$$H = \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{-2a}{n-2}\mathrm{Id}_{n-2} & 0 \\ 0 & 0 & a \end{pmatrix} \quad \text{for } a \in \mathbb{R}.$$

The representation $\rho\colon \pi_1(S) \to \mathrm{SL}(n,\mathbb{R})$ can be chosen as $(1, n-1)$-horocyclic representation as defined in Subsection 2.4.3. Analogous to Example 7.12, the triple of flags (P_g^+, P_g^-, P_h^-) is conjugated to the triple $(P_\theta^+, P_\theta^-, P_\theta^*)$, and we can apply Proposition 7.8.

Example 7.14. We now show that Proposition 7.8 not only applies to projective Anosov representations, but that there exist also Δ-Anosov representations into $\mathrm{SL}(2n+1,\mathbb{R})$ for which the cataclysm deformation is not injective. Let $j_4\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(4,\mathbb{R})$ be the unique irreducible representation introduced in Subsection 2.4.2. For a suitable choice of symplectic form, its image lies in $\mathrm{Sp}(4,\mathbb{R})$. Consider a principal Fuchsian representation $j_4 \circ \rho_0\colon \pi_1(S) \to \mathrm{Sp}(4,\mathbb{R})$. It is $\Delta_{\mathrm{Sp}(4,\mathbb{R})}$-Anosov. Post-composing it with the reducible representation $\iota_{4\to 5}\colon \mathrm{Sp}(4,\mathbb{R}) \to \mathrm{SL}(5,\mathbb{R})$ from Subsection 2.4.3, we obtain a reducible $\Delta_{\mathrm{SL}(5,\mathbb{R})}$-Anosov representation into $\mathrm{SL}(5,\mathbb{R})$ that we call ρ. The embedding $\iota_{4\to 5} \circ j_4\colon \mathrm{SL}(4,\mathbb{R}) \to \mathrm{SL}(5,\mathbb{R})$ induces a map $\iota^+_{4\to 5}\colon \mathbb{RP}^1 \to \mathrm{Flag}(\mathbb{R}^5)$ between the corresponding flag spaces. Let $P^\pm_\Delta$ be the standard parabolic subgroups. We have that $P^+_\Delta = \iota^+_{4\to 5} \circ j^+_4(\langle e_1\rangle)$, $P^-_\Delta = \iota^+_{4\to 5} \circ j^+_4(\langle e_2\rangle)$ and define $P^*_\Delta := \iota^+_{4\to 5} \circ j^+_4(\langle e_1 + e_2\rangle)$. By construction, for the $\Delta_{\mathrm{SL}(5,\mathbb{R})}$-Anosov representation $\rho = \iota_{4\to 5} \circ j_4 \circ \rho_0\colon \pi_1(S) \to \mathrm{SL}(5,\mathbb{R})$ as above the triple of flags (P^+_g, P^-_g, P^-_h) is conjugate to the triple $(P^+_\Delta, P^-_\Delta, P^*_\Delta)$. Further, one can check that $w_0 \cdot P^*_\Delta = P^*_\Delta$, where w_0 is the representative of the longest element of the Weyl group that has only entries on the anti-diagonal, all equal to 1 except the central one which is -1. Set

$$H = \begin{pmatrix} a & & & & \\ & a & & & \\ & & -4a & & \\ & & & a & \\ & & & & a \end{pmatrix} \quad \text{for } a \in \mathbb{R}.$$

Then $\iota(H) = -H$, and $m \in Z_G(\exp(H))$, where $m \in N^+_\Delta$ is the unique element such that $m \cdot P^-_\Delta = P^*_\Delta$. Hence, all prerequisites of Proposition 7.8 are satisfied, so this provides an example of a reducible $\Delta_{\mathrm{SL}(5,\mathbb{R})}$-Anosov representation into $\mathrm{SL}(5,\mathbb{R})$ for which the cataclysm deformation based at ρ is not injective.

Remark 7.15. Example 7.14 shows that there exist Δ-Anosov representations into $\mathrm{SL}(2n+1,\mathbb{R})$ for which the cataclysm deformation is not injective. In particular, the injectivity of the cataclysm deformation stated in [Dre13, Corollary 35] is not correct.

Example 7.16. The examples considered so far are all reducible. We now show that there also exist irreducible representations that satisfy the prerequisites of Proposition 7.8. We consider *hybrid representations*. These were constructed for representations into $\mathrm{Sp}(4,\mathbb{R})$ in [GW10, §3.3.1]. We use the same technique for $\mathrm{SL}(3,\mathbb{R})$ to obtain an irreducible representation whose restriction to a subsurface is reducible. Let S be a closed connected oriented surface of genus at least 2 and let c be a simple closed separating curve (see Figure 7.4). Let S_l and S_r be the connected components of $S \setminus c$. Then the fundamental group of S is the amalgamated product $\pi_1(S) = \pi_1(S_l) *_c \pi_1(S_r)$. Let $\rho_0\colon \pi_1(S) \to \mathrm{SL}(2,\mathbb{R})$ be discrete and faithful. We can assume that $\rho_0(c)$ is diagonal with eigenvalues e^a and e^{-a}. For $t \in [0,1]$

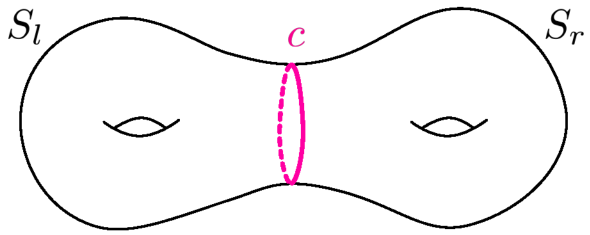


Figure 7.4.: The simple closed curve c divides the surface S into two subsurfaces S_l and S_r. The fundamental group $\pi_1(S)$ is the amalgamated product of the fundamental groups $\pi_1(S_l)$ and $\pi_1(S_r)$.

let $\rho_t\colon \pi_1(S) \to \mathrm{SL}(2,\mathbb{R})$ be a continuous path of representations starting at ρ_0 such that $\rho_1(c)$ is diagonal with entries e^{2a} and e^{-2a}. Such a path can for example be constructed using Fenchel-Nielsen coordinates. Set $\rho_l := \iota_3 \circ \rho_1$, where $\iota_3\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(3,\mathbb{R})$ is the reducible representation introduced in Subsection 2.4.3. Further, let $\rho_r := j_3 \circ \rho_0$, where $j_3\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(3,\mathbb{R})$ is the irreducible representation introduced in Subsection 2.4.2. Then

$$\rho_l(c) = \begin{pmatrix} e^{2a} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2a} \end{pmatrix} = \rho_r(c),$$

so we can define $\rho := \rho_l|_{\pi_1(S_l)} *_c \rho_r|_{\pi_1(S_r)}$. In Proposition 7.8, only $\rho|_{\pi_1(S_l)}$ is relevant, because the cycle ε is trivial on S_r. On S_l, the representation $\rho|_{\pi_1(S_l)} = \rho_l|_{\pi_1(S_l)}$ is reducible and we are in the same situation as in Example 7.12. Thus, we can apply Proposition 7.8. Since $\rho|_{\pi_1(S_r)}$ is irreducible also ρ is irreducible. Hence we have constructed an example of an irreducible representation for which the cataclysm deformation with respect to ε is trivial.

7.4. Cataclysm deformations for reducible representations

In this section, we have a closer look at cataclysm deformations for the two classes of reducible representations introduced in Subsection 2.4.3: for (n,k)-horocyclic representations and for reducible Δ-Anosov representations. In particular, we determine the subspace of $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ along which the cataclysm deformation is trivial. First, we concentrate on (n,k)-horocyclic representations in Subsection 7.4.1. Then we turn to reducible Δ-Anosov representations in Subsection 7.4.2. We prove a technical needed in both cases in Subsection 7.4.3.

7.4.1. Cataclysms deformations for (n,k)-horocyclic representations

Recall from Subsection 2.4.3 that an (n,k)-horocyclic representation is obtained by composing a Fuchsian representation $\rho_0\colon \pi_1(S) \to \mathrm{SL}(2,\mathbb{R})$ with an embedding $\iota_{n,k}\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(k,\mathbb{R})$ whose image stabilizes a flag in $\mathcal{F}_{\{k,n-k\}}$ for some $k \leq \frac{n}{2}$. In this subsection, we assume that $k < \frac{n}{2}$ to exclude the case where $k = n-k$. Let $\theta := \{k, n-k\}$. Then $\mathfrak{a}_\theta$ is two-dimensional.

We will see that if ρ is a horocyclic representation, then for the cataclysm deformation based at ρ there are two directions in the parameter space $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a}_\theta)$ with different behavior of the deformations: In one direction, the deformed representation remains horocyclic, so has image contained in $\iota_{n,k}(\mathrm{SL}(2,\mathbb{R}))$ in $\mathrm{SL}(n,\mathbb{R})$. In the other direction, it does not remain horocyclic and the deformed representation is obtained from ρ by right-multiplication with elements in the centralizer of $\iota_{n,k}(\mathrm{SL}(2,\mathbb{R}))$. For $n=3$ and $k=1$, these deformations agree with *linear u-deformations* defined by Barbot [Bar10, Section 4.1]. Deformations of this form also appear as *bulging deformations* of convex projective structures in [Gol13] and [WZ18].

In Example 7.12, we have already seen that the cataclysm deformation Λ_0^ε for horocyclic representations into $\mathrm{SL}(3,\mathbb{R})$ is not injective in general. In that example, we fixed a special lamination λ. Now, we consider an arbitrary lamination λ and explicitly describe the space of cycles $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a}_\theta)$ for which the deformation Λ_0^ε is trivial, i.e. satisfies $\Lambda_0^\varepsilon\rho = \rho$.

Let $P \subset \tilde{S} \setminus \tilde{\lambda}$ be a fixed reference component and let Λ_0 as in Theorem 5.12 be the cataclysm deformation with respect to P. Let $\rho_0\colon \pi_1(S) \to \mathrm{SL}(2,\mathbb{R})$ be discrete and faithful, and let $\iota_{n,k}\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$ be the reducible representation introduced in (2.5). Let $\mathfrak{a}_2$ be the maximal abelian subalgebra for $\mathrm{SL}(2,\mathbb{R})$. The inclusion $\iota_{n,k}$ induces an inclusion $(\iota_{n,k})_*\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{sl}(n,\mathbb{R})$. On the abelian subalgebra $\mathfrak{a}_2$, it is given by

$$(\iota_{n,k})_*\colon \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \mapsto \begin{pmatrix} a\mathrm{Id}_k & & \\ & 0_{n-2k} & \\ & & -a\mathrm{Id}_k \end{pmatrix}.$$

Thus, $(\iota_{n,k})_*$ satisfies $(\iota_{n,k})_*(\mathfrak{a}_2) \subset \mathfrak{a}_\theta$. As seen in Example 5.23 we can apply Proposition 5.21 and obtain

$$\Lambda_0^{(\iota_{n,k})_*\varepsilon}(\iota_{n,k} \circ \rho_0) = \iota_{n,k}\left(\Lambda_0^\varepsilon\rho_0\right) \tag{7.8}$$

for every $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}\left(\widehat{\lambda};\mathfrak{a}_2\right)$, where $(\iota_{n,k})_*$ is the embedding $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a}_2) \hookrightarrow \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a}_\theta)$ induced by $(\iota_{n,k})_*$.

Define the 1-dimensional subspace

$$\mathfrak{a}' := \left\{ \begin{pmatrix} a\mathrm{Id}_k & & \\ & -2a\mathrm{Id}_{n-2k} & \\ & & a\mathrm{Id}_k \end{pmatrix} \middle| a \in \mathbb{R} \right\} \subset \mathfrak{a}.$$

We can split $\mathfrak{a}_\theta$ into the two 1-dimensional subspaces

$$\begin{aligned} \mathfrak{a}_\theta &= \left\{ \begin{pmatrix} a\mathrm{Id}_k & & \\ & 0 & \\ & & -a\mathrm{Id}_k \end{pmatrix} \middle| a \in \mathbb{R} \right\} \oplus \left\{ \begin{pmatrix} a\mathrm{Id}_k & & \\ & -2a\mathrm{Id}_{n-2k} & \\ & & a\mathrm{Id}_k \end{pmatrix} \middle| a \in \mathbb{R} \right\} \\ &= (\iota_{n,k})_* (\mathfrak{a}_2) \oplus \mathfrak{a}'. \end{aligned} \tag{7.9}$$

We know by (7.8) how cycles with values in $(\iota_{n,k})_* (\mathfrak{a}_2)$ deform a representation. We now examine what happens for cycles with values in $\mathfrak{a}'$. Let $H \in \mathfrak{a}'$, let g be an oriented geodesic in $\tilde{S}$ and let $m_g \in \mathrm{SL}(2,\mathbb{R})$ be such that $m_g \cdot P_\Delta^\pm = \zeta_0(g^\pm)$, where $P_\Delta^\pm$ are the Borel subgroups in $\mathrm{SL}(2,\mathbb{R})$ and ζ_0 is the boundary map for the Fuchsian representation ρ_0. Note that $\exp(H) \in Z_{\mathrm{SL}(n,\mathbb{R})}(\iota_{n,k}(\mathrm{SL}(2,\mathbb{R})))$. Hence,

$$T_g^H = \iota_{n,k}(m_g)\exp(H)\iota_{n,k}(m_g)^{-1} = \exp(H).$$

In particular, all stretching maps T_g^H with $H \in \mathfrak{a}'$ are diagonal and independent of the oriented geodesic g. It follows that for all components $P, Q \subset \tilde{S} \setminus \tilde{\lambda}$, the shearing maps satisfy

$$\varphi_{PQ}^\varepsilon = \prod_{R \in \mathcal{P}_{PQ}} \left(T_{g_R^0}^{\varepsilon(P,R)} T_{g_R^1}^{-\varepsilon(P,R)} \right) T_{g_Q^0}^{\varepsilon(P,Q)} = T_{g_Q^0}^{\varepsilon(P,Q)} = \exp(\varepsilon(P,Q)). \tag{7.10}$$

By definition of Λ_0, for $\gamma \in \pi_1(S)$, we have $\Lambda_0^\varepsilon \rho(\gamma) = \varphi_{P\gamma P}^\varepsilon \rho(\gamma)$, so by (7.10), we have

$$\Lambda_0^\varepsilon \rho = \rho \quad \text{if and only if } \varepsilon(P, \gamma P) = 0 \ \forall \gamma \in \pi_1(S). \tag{7.11}$$

We now have a closer look at $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}')$ and determine its dimension. Since $\mathfrak{a}'$ is 1-dimensional, we can write every $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}')$ as $\varepsilon = \varepsilon_1 H$, where $\varepsilon_1 \in \mathcal{H}(\widehat{\lambda}; \mathbb{R})$ and $H = \mathrm{diag}(1, -2, 1)$. Recall from Section 3.1 the orientation reversing involution $\mathfrak{R} \colon \widehat{U} \to \widehat{U}$. By the twist condition, we have $\mathfrak{R}^* \varepsilon_1 = -\varepsilon_1$, so ε_1 lies in the (-1)-eigenspace of $\mathcal{H}(\widehat{\lambda}; \mathbb{R})$ with respect to $\mathfrak{R}^*$. Thus, we can identify $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}')$ with $\mathcal{H}(\widehat{\lambda}; \mathbb{R})^-$, where $\mathcal{H}(\widehat{\lambda}; \mathbb{R})^-$ denotes the (-1)-eigenspace of $\mathcal{H}(\widehat{\lambda}; \mathbb{R})$ with respect to $\mathfrak{R}$. As seen in the proof of Proposition 3.21, $\mathcal{H}(\widehat{\lambda}; \mathbb{R})^-$ has dimension $-\chi(\lambda) + n(\lambda)$, where $n(\lambda)$ is the number of connected components of the lamination λ and $\chi(\lambda)$ is its Euler characteristic. In particular, if λ is maximal, $\dim\left(\mathcal{H}(\widehat{\lambda};\mathbb{R})^-\right) = 6g(S)-5$, where $g(S)$ denotes the genus of the surface S. In the following, we always identify $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a}')$ with $\mathcal{H}(\widehat{\lambda};\mathbb{R})^-$.

By (7.11), to understand for which cycles the deformation is trivial, we have to understand which cycles satisfy $\varepsilon(P,\gamma P)=0$ for all $\gamma\in\pi_1(S)$. As a first step, we construct a vector space homomorphism from $\mathcal{H}(\widehat{\lambda};\mathbb{R})^-$ into the group homomorphisms from $\pi_1(S)$ to $\mathbb{R}$.

Lemma 7.17. *Let $P\subset\tilde{S}\setminus\tilde{\lambda}$ be a fixed reference component. The map*

$$\begin{aligned} f\colon \mathcal{H}(\widehat{\lambda};\mathbb{R})^- &\to \mathrm{Hom}(\pi_1(S),\mathbb{R}),\\ \varepsilon &\mapsto (u_\varepsilon\colon \gamma\mapsto \varepsilon(P,\gamma P)) \end{aligned}$$

is a well-defined vector space homomorphism, where we consider $\mathbb{R}$ as a group with addition. Moreover, f does not depend on the component $P\subset\tilde{S}\setminus\tilde{\lambda}$.

The proof of Lemma 7.17 is given in the Subsection 7.4.3 below, where we will also show that f is surjective if the lamination λ is maximal.

Remark 7.18. In [Bar10, Section 4.1], Barbot considers for $u\in\mathrm{Hom}(\pi_1(S),\mathbb{R})$ deformations of a $(3,1)$-horocyclic representation ρ, called *u-deformations*, that are defined as

$$\rho_u(\gamma) = \begin{pmatrix} e^{\frac{u(\gamma)}{3}} & & \\ & e^{-\frac{2u(\gamma)}{3}} & \\ & & e^{\frac{u(\gamma)}{3}} \end{pmatrix}\rho(\gamma),$$

for every $\gamma\in\pi_1(S)$. For $\varepsilon\in\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a}')$, we see using (7.10) that $\Lambda_0^\varepsilon\rho=\rho_{\frac{f(\varepsilon)}{3}}$ with f as in Lemma 7.17. So in this case, a cataclysm with cycle ε is a linear u-deformation. Barbot moreover gives a precise condition for which u the deformation ρ_u is Anosov [Bar10, Theorem 4.2]. Further, he not only defines linear u-deformations, but also deformations of $(3,1)$-horocyclic representations that are different from cataclysm deformations. The resulting representations that he calls *radial representations* are Anosov under an extra assumption. Whereas all cataclysm deformations of horocyclic representations stabilize the line $\langle e_2\rangle$ as well as the complementary hyperplane $\langle e_1,e_3\rangle$, Barbot's radial representations only stabilize $\langle e_2\rangle$, but not the hyperplane $\langle e_1,e_3\rangle$. Thus, for reducible representations, there exist interesting deformations that are not cataclysms.

Given the homomorphism f, we can now determine for which cycles with values in $\mathfrak{a}'$ the cataclysm deformation is trivial:

Lemma 7.19. *For $\varepsilon\in\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a}')$ and an (n,k)-horocyclic representation $\rho=\iota_{n,k}\circ\rho_0$, the following are equivalent:*

(1) $\varepsilon \in \ker(f)$*,*

(2) $\Lambda_0^\varepsilon \rho = \rho$*,*

(3) $\Lambda_0^\varepsilon \rho$ *is* (n,k)*-horocyclic.*

Proof. The equivalence between (1) and (2) is given in (7.11). The fact that (2) implies (3) is trivial, since ρ is horocyclic. To show that (3) implies (1), assume that $\Lambda_0^\varepsilon \rho$ is horocyclic and let $\gamma \in \pi_1(S)$. Let $u_\varepsilon = f(\varepsilon)$ as in Lemma 7.17. For $\gamma \in \pi_1(S)$, let $\rho_0(\gamma) = \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix}$. Then

$$\begin{aligned}\Lambda_0^\varepsilon \rho(\gamma) &= \varphi^\varepsilon_{P\gamma P} \rho(\gamma) \\ &= \begin{pmatrix} e^{u_\varepsilon(\gamma)}\mathrm{Id}_k & & \\ & e^{-2u_\varepsilon(\gamma)}\mathrm{Id}_{n-2k} & \\ & & e^{u_\varepsilon(\gamma)}\mathrm{Id}_k \end{pmatrix} \begin{pmatrix} a(\gamma)\mathrm{Id}_k & 0 & b(\gamma)\mathrm{Id}_k \\ 0 & \mathrm{Id}_{n-2k} & 0 \\ c(\gamma)\mathrm{Id}_k & 0 & d(\gamma)\mathrm{Id}_k \end{pmatrix}.\end{aligned}$$

Since this is (n,k)-horocyclic by assumption, the middle block has to equal Id_{n-k}, which implies that $u_\varepsilon(\gamma) = 0$. This holds for all $\gamma \in \pi_1(S)$, so $\varepsilon \in \ker(f)$. □

We can now conclude for which cycles $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a})$ the cataclysm deformation is trivial.

Proposition 7.20. *Let* $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a})$*. Then* $\Lambda_0^\varepsilon \rho = \rho$ *if and only if* ε *takes values in* $\mathfrak{a}'$ *and lies in the kernel of* f*.*

Proof. Assume first that ε takes values in $\mathfrak{a}'$ and that $\varepsilon \in \ker(f)$. Then $\Lambda_0^\varepsilon \rho = \rho$ by Lemma 7.19. For the other direction, let $\varepsilon = \varepsilon' + (\iota_{n,k})_*(\varepsilon_0)$, with $\varepsilon_0 \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_2)$ and $\varepsilon' \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}') \cong \mathcal{H}(\widehat{\lambda}; \mathbb{R})^-$. Assume that $\Lambda_0^\varepsilon \rho = \rho$. By additivity of the cataclysm deformation (Corollary 5.17),

$$\rho = \Lambda_0^\varepsilon \rho = \Lambda_0^{\varepsilon'} \left(\Lambda_0^{(\iota_{n,k})_*(\varepsilon_0)} \rho \right) = \Lambda_0^{\varepsilon'} \left(\iota_{n,k} \left(\Lambda_0^{\varepsilon_0} \rho_0 \right) \right),$$

where we use Proposition 5.21. By Lemma 7.19, since both ρ and $\iota_{n,k}(\Lambda_0^{\varepsilon_0}\rho_0)$ are horocyclic, we have $\varepsilon' \in \ker(f)$ and $\rho = \Lambda_0^{(\iota_{n,k})_*(\varepsilon_0)} \rho = \iota_{n,k}(\Lambda_0^{\varepsilon_0}\rho_0)$. Since cataclysm deformations in $\mathrm{SL}(2,\mathbb{R})$ are injective by Corollary 7.7, we have $\varepsilon_0 = 0$, so $\varepsilon = \varepsilon' \in \ker(f)$. □

Proposition 7.20 has as a consequence the following corollary.

Corollary 7.21. *There is a subspace $\mathcal{H}_{\text{trivial}} \subset \mathcal{H}(\widehat{\lambda};\mathfrak{a})$ such that $\Lambda_0^\varepsilon \rho = \rho$ if and only if $\varepsilon \in \mathcal{H}_{\text{trivial}}$. The subspace $\mathcal{H}_{\text{trivial}}$ depends only on the maximal lamination λ, not on the reference triangle P or the representation ρ. Moreover, the dimension of $\mathcal{H}_{\text{trivial}}$ can be estimated by*

$$-\chi(\lambda) + n(\lambda) \geq \dim \mathcal{H}_{\text{trivial}} \geq -\chi(\lambda) + n(\lambda) - 2g, \tag{7.12}$$

where $\chi(\lambda)$ is the Euler characteristic of λ and $n(\lambda)$ the number of connected components.

Proof. Set $\mathcal{H}_{\text{trivial}} := \ker(f)$. From Proposition 7.20, we know that $\Lambda_0^\varepsilon \rho = \rho$ if and only of $\varepsilon \in \mathcal{H}_{\text{trivial}}$. From Lemma 7.17, we know that $\mathcal{H}_{\text{trivial}}$ is independent of the reference triangle P. The only thing to check is the dimension of $\mathcal{H}_{\text{trivial}} = \ker(f)$. We already know that the dimension of $\dim \mathcal{H}(\widehat{\lambda};\mathbb{R})^-$ is $-\chi(\lambda) + n(\lambda)$. Further, $\dim \operatorname{Hom}(\pi_1(S),\mathbb{R}) = 2g$, where $g := g(S)$ is the genus of S, since $\pi_1(S)$ has $2g$ generators $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ and the relation $\prod_i [\alpha_i, \beta_i] = \mathrm{Id}$ becomes trivial in $\mathbb{R}$. Thus

$$-\chi(\lambda) + n(\lambda) \geq \dim \ker(f) = \dim \mathcal{H}(\widehat{\lambda};\mathbb{R})^- - \dim \operatorname{Im}(f) \geq -\chi(\lambda) + n(\lambda) - 2g,$$

which finishes the proof. □

To summarize, there are different possibilities for deforming an (n,k)-horocyclic representation by a cataclysms: If we deform with a cycle valued in $(\iota_{n,k})_*(\mathfrak{a}_2)$, we obtain a different (n,k)-horocyclic representation. If we deform with a cycle in $\mathcal{H}_{\text{trivial}}$, we do not change the representation at all. Lastly, if we deform with a cycle valued in $\mathfrak{a}'$ that does not lie in $\mathcal{H}_{\text{trivial}}$, we move out of the copy of $\mathrm{SL}(2,\mathbb{R})$ and the deformed representation ρ' is obtained from ρ by right-multiplication with elements in the centralizer of $\iota_{n,k}(\mathrm{SL}(2,\mathbb{R}))$. For $(n,k) = (3,1)$, these deformations are linear u-deformations in the sense of Barbot [Bar10, Section 4.1]. If we use any combination of cycles valued in $(\iota_{n,k})_*(\mathfrak{a}_2)$ and in $\mathfrak{a}'$, then by additivity of the deformation (Corollary 5.17), it is irrelevant in which direction we deform first – the resulting representation will be the same. Note that the crucial points in the considerations in this subsection are that twisted cycles with values in $\mathfrak{a}'$ can be identified with $\mathcal{H}(\widehat{\lambda};\mathbb{R})^-$ and that the 2-dimensional space $\mathfrak{a}_\theta$ can be split up as in (7.9). In the following subsection, we consider reducible representations where the dimension of $\mathfrak{a}_\theta$ is bigger than 2.

7.4.2. Cataclysm deformations for reducible Borel Anosov representations

Now let us consider reducible Δ-Anosov representations into $\mathrm{SL}(2n+1,\mathbb{R})$ that are obtained from composing a $\Delta_{\mathrm{Sp}(2n,\mathbb{R})}$-Anosov representation $\rho_0\colon \pi_1(S) \to \mathrm{Sp}(2n,\mathbb{R})$ with a reducible embedding $\iota_{2n\to 2n+1}\colon \mathrm{Sp}(2n,\mathbb{R}) \to \mathrm{SL}(2n+1,\mathbb{R})$ (see (2.6) in Subsection 2.4.3). In this case, $\Delta = \Delta_{\mathrm{SL}(2n+1,\mathbb{R})}$ is $2n$-dimensional. The maximal abelian subalgebra $\mathfrak{a}_\Delta$ can be decomposed as

$$\mathfrak{a}_\Delta = (\iota_{2n\to 2n+1})_* \left(\mathfrak{a}_{\mathrm{Sp}(2n,\mathbb{R})}\right) \oplus \mathfrak{a}' \oplus \mathfrak{a}'', \tag{7.13}$$

where

$$\mathfrak{a}' := \left\{ \left. \begin{pmatrix} a\mathrm{Id}_n & & \\ & -2n\cdot a & \\ & & a\mathrm{Id}_n \end{pmatrix} \right| a \in \mathbb{R} \right\}$$

and $\mathfrak{a}'' \subset \mathfrak{a}_\Delta$ is an $(n-1)$-dimensional subalgebra such that the sum in (7.13) is direct. As in the case of (n,k)-horocyclic representations, $\exp(\mathfrak{a}')$ lies in the centralizer of $\iota_{2n\to 2n+1}(\mathrm{Sp}(2n,\mathbb{R}))$, so for $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a}')$ the shearing maps satisfy $\varphi^\varepsilon_{PQ} = \varepsilon(P,Q)$ for all $P,Q \subset \tilde{S}\setminus\tilde{\lambda}$. It follows that for cycles valued in $\mathfrak{a}'$, $\Lambda^\varepsilon_0\rho = \rho$ if and only if $\varepsilon(P,\gamma P) = 0$ for all $\gamma \in \pi_1(S)$, i.e. (7.11) holds also in this case. Further, transverse twisted cycles with values in $\mathfrak{a}'$ are in one-to-one correspondence with elements in $\mathcal{H}(\widehat{\lambda};\mathbb{R})^-$. With the same arguments as for the case of (n,k)-horocyclic representations, we have the following result:

Proposition 7.22. *Let $\rho\colon \pi_1(S) \to \mathrm{SL}(2n+1,\mathbb{R})$ be Δ-Anosov of the form $\iota_{2n\to 2n+1}\circ\rho_0$, where $\rho_0\colon \pi_1(S) \to \mathrm{Sp}(2n,\mathbb{R})$ is $\Delta_{\mathrm{Sp}(2n,\mathbb{R})}$-Anosov. Then there exists a subspace $\mathcal{H}_{\mathrm{trivial}} \subset \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a}_\Delta)$ such that $\Lambda^\varepsilon_0\rho = \rho$ for all $\varepsilon \in \mathcal{H}_{\mathrm{trivial}}$. The dimension of $\mathcal{H}_{\mathrm{trivial}}$ is at least $-\chi(\lambda) + n(\lambda) - 2g$, where $\chi(\lambda)$ is the Euler characteristic of λ and $n(\lambda)$ the number of connected components.*

Proof. Set $\mathcal{H}_{\mathrm{trivial}} := \ker(f)$ with $f\colon \mathcal{H}(\widehat{\lambda},\mathbb{R})^- \to \mathrm{Hom}(\pi_1(S),\mathbb{R})$ as in Lemma 7.17. Then $\Lambda^\varepsilon_0\rho = \rho$ for all $\varepsilon \in \mathcal{H}_{\mathrm{trivial}}$, and the lower bound on the dimension follows as in the proof of Corollary 7.21. □

Note that here, in contrast to Corollary 7.21 we cannot prove the statement as *if and only if*, because we cannot describe in detail how cataclysm deformations with values in $\mathfrak{a}''$ behave. What we can say, however, is that cataclysm deformations of Δ-Anosov representations into $\mathrm{SL}(2n+1,\mathbb{R})$ are not injective in general.

7.4.3. The homomorphism $f\colon \mathcal{H}(\widehat{\lambda};\mathbb{R})^- \to \mathrm{Hom}(\pi_1(S),\mathbb{R})$

In this subsection, we proof Lemma 7.17 and show that for a maximal lamination, the homomorphism f is surjective. Recall that $f\colon \mathcal{H}(\widehat{\lambda};\mathbb{R})^- \to \mathrm{Hom}(\pi_1(S),\mathbb{R})$ is defined by $f(\varepsilon) = u_\varepsilon$, where for every $\gamma \in \pi_1(S)$, $u_\varepsilon(\gamma) = \varepsilon(P, \gamma P)$ for a fixed component P of $\tilde{S} \setminus \tilde{\lambda}$.

Proof of Lemma 7.17. We have to check that u_ε indeed is a group homomorphism. Let $\gamma_1, \gamma_2 \in \pi_1(S)$, and assume first that $\gamma_1 P$ separates P from $\gamma_1\gamma_2 P$. Then by additivity of ε and $\pi_1(S)$-invariance, we have

$$\begin{aligned} u_\varepsilon(\gamma_1\gamma_2) &= \varepsilon(P, \gamma_1\gamma_2 P) \\ &= \varepsilon(P, \gamma_1 P) + \varepsilon(\gamma_1 P, \gamma_1\gamma_2 P) \\ &= \varepsilon(P, \gamma_1 P) + \varepsilon(P, \gamma_2 P) \\ &= u_\varepsilon(\gamma_1) + u_\varepsilon(\gamma_2). \end{aligned}$$

If P separates $\gamma_1 P$ from $\gamma_1\gamma_2 P$, then we use that $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathbb{R})^-$, so $\varepsilon(Q,P) = -\varepsilon(P,Q)$ for all $Q \subset \tilde{S} \setminus \tilde{\lambda}$. This gives us

$$\begin{aligned} u_\varepsilon(\gamma_1\gamma_2) &= \varepsilon(P, \gamma_1\gamma_2 P) \\ &= -\varepsilon(\gamma_1 P, P) + \varepsilon(\gamma_1 P, \gamma_1\gamma_2 P) \\ &= \varepsilon(P, \gamma_1 P) + \varepsilon(P, \gamma_2 P) \\ &= u_\varepsilon(\gamma_1) + u_\varepsilon(\gamma_2). \end{aligned}$$

The case where $\gamma_1\gamma_2 P$ separates P from $\gamma_1 P$ works analogous.

If none of P, $\gamma_1 P$, $\gamma_1\gamma_2 P$ separates the other two, there exists some $Q \subset \tilde{S} \setminus \tilde{\lambda}$ that pairwise separates P, $\gamma_1 P$ and $\gamma_1\gamma_2 P$ (as in Figure 5.1). Using Q together with additivity of ε, $\pi_1(S)$-invariance and the fact that ε lies in $\mathcal{H}(\widehat{\lambda};\mathbb{R})^-$, we can conclude that $u_\varepsilon(\gamma_1\gamma_2) = u_\varepsilon(\gamma_1) + u_\varepsilon(\gamma_2)$ for all $\gamma_1, \gamma_2 \in \pi_1(S)$. So indeed, u_ε is a group homomorphism and $f\colon \mathcal{H}(\widehat{\lambda};\mathbb{R}) \to \mathrm{Hom}(\pi_1(S),\mathbb{R})$ is a well-defined vector space homomorphism. To show independence of P, let $Q \subset \tilde{S} \setminus \tilde{\lambda}$ be another connected component. Then, by the same arguments as above,

$$\begin{aligned} \varepsilon(Q, \gamma Q) &= \varepsilon(Q, P) + \varepsilon(P, \gamma P) + \varepsilon(\gamma P, \gamma Q) \\ &= -\varepsilon(P, Q) + \varepsilon(P, \gamma P) + \varepsilon(P, Q) \\ &= \varepsilon(P, \gamma P). \end{aligned}$$

Thus, f is independent of the choice of the component $P \subset \tilde{S} \setminus \tilde{\lambda}$. $\square$

We have estimated the dimension of the kernel of f in (7.12). For a maximal lamination λ, we can compute the dimension of $\ker f$ exactly in terms of the genus $g(S)$ of the surface.

Standing Assumption. For the rest of this subsection, we assume that the lamination λ is maximal.

If λ is maximal, then (7.12) becomes

$$6g-5 \geq \dim\ker(f) \geq 4g-5$$

We will show that f is onto, i.e. $\ker(f)$ is as small as possible. Since λ is maximal, $S\setminus\lambda$ consists of $4g-4$ connected components that are ideal triangles. Let $\mathcal{C}'_\lambda := \{P = Q_0, Q_1, \dots, Q_{4g-5}\}$ be a set of representatives of connected components of $\tilde{S}\setminus\tilde{\lambda}$ such that each component of $S\setminus\lambda$ has a lift contained in $\mathcal{C}'_\lambda$. Let P be the fixed reference component, and let $\mathcal{C}_\lambda := \mathcal{C}'_\lambda \setminus \{P\}$. Further, let α_i, β_i for $i = 1, \dots, 4g(S)$ be generators of $\pi_1(S)$.

Lemma 7.23. *The map*

$$\begin{aligned} V\colon \mathcal{H}(\widehat{\lambda};\mathbb{R})^- &\to \mathbb{R}^{2g}\times\mathbb{R}^{\mathcal{C}_\lambda}, \\ \varepsilon &\mapsto \Big(\left(\varepsilon(P,\alpha_i P), \varepsilon(P,\beta_i P)\right)_{i=1,\dots,g}, \left(\varepsilon(P,Q)\right)_{Q\in\mathcal{C}_\lambda}\Big) \end{aligned}$$

is an isomorphism of vector spaces.

Proof. It is straightforward to check that V is a vector space homomorphism. We want to show that $V(\varepsilon)$ uniquely determines ε, i.e. that V is injective. Let $\widehat{k}$ be an arc transverse to $\widehat{\lambda}$, let k be the corresponding oriented arc transverse to λ (as in Remark 5.5) and let $\tilde{k}$ be a lift of k to the universal cover. Let R_1 and $R_2 \in \tilde{S}\setminus\tilde{\lambda}$ be the components containing the negative and positive endpoint of $\tilde{k}$. Then $\varepsilon(\widehat{k}) = \varepsilon(R_1, R_2)$. In particular, if we know $\varepsilon(R_1,R_2)$ for all $R_1, R_2 \subset \tilde{S}\setminus\tilde{\lambda}$, then we know $\varepsilon(\widehat{k})$ for every arc $\widehat{k}$ transverse to $\widehat{\lambda}$. We have seen in the proof of Lemma 7.17 that for any three components $R_1, R_2, R_3 \subset \tilde{S}\setminus\tilde{\lambda}$, we have $\varepsilon(R_1,R_3) = \varepsilon(R_1,R_2) + \varepsilon(R_2,R_3)$. Note that here it is important that $\varepsilon \in \mathcal{H}(\widehat{\lambda};\mathbb{R})^-$. Let $R_1, R_2 \subset \tilde{S}\setminus\tilde{\lambda}$ be arbitrary. Then there exists $\gamma_1,\gamma_2 \in \pi_1(S)$ and $i_1, i_2 \in \{0,\dots,4g-5\}$ such that $R_1 = \gamma_1 Q_{i_1}$ and $R_2 = \gamma_2 Q_{i_2}$, where we set $P = Q_0$. We have

$$\begin{aligned} \varepsilon(R_1,R_2) &= \varepsilon(\gamma_1 Q_{i_1}, P) + \varepsilon(P, \gamma_2 Q_{i_2}) \\ &= -\varepsilon(P,\gamma_1 Q_{i_1}) + \varepsilon(P,\gamma_2 Q_{i_2}) \\ &= -\varepsilon(P,\gamma_1 P) - \varepsilon(\gamma_1 P, \gamma_1 Q_{i_1}) + \varepsilon(P,\gamma_2 P) + \varepsilon(\gamma_2 P, \gamma_2 Q_{i_2}) \\ &= -\varepsilon(P,\gamma_1 P) - \varepsilon(P, Q_{i_1}) + \varepsilon(P,\gamma_2 P) + \varepsilon(P, Q_{i_2}). \end{aligned}$$

The right hand side, and thus also $\varepsilon(R_1, R_2)$, is uniquely determined by $V(\varepsilon)$. In particular, if $V(\varepsilon) = 0$, then $\varepsilon(R_1, R_2) = 0$ for all $R_1, R_2 \subset \tilde{S} \setminus \tilde{\lambda}$, so $\varepsilon = 0$. This shows that V is injective. As $\dim \mathcal{H}(\widehat{\lambda}; \mathbb{R})^- = 6g-5 = \dim \left(\mathbb{R}^{2g} \times \mathbb{R}^{\mathcal{C}_\lambda}\right)$, it follows that V is also surjective, so an isomorphism. $\square$

Corollary 7.24. *If the lamination λ is maximal, then the homomorphism from f is surjective, and $\mathcal{H}_{trivial}$ from Corollary 7.21 has dimension $4g-5$.*

Proof. With $f \colon \mathcal{H}(\widehat{\lambda}; \mathbb{R})^- \to \mathrm{Hom}(\pi_1(S), \mathbb{R})$ as above, we have $\mathcal{H}_{trivial} = \ker(f)$. Since V as in Lemma 7.23 is an isomorphism, $\dim \ker(f) = \dim \ker(f \circ V^{-1})$. Identifying $\mathrm{Hom}(\pi_1(S), \mathbb{R})$ with $\mathbb{R}^{2g}$, we see that

$$V^{-1} \circ f \colon \mathbb{R}^{2g} \times \mathbb{R}^{\mathcal{C}_\lambda} \to \mathbb{R}^{2g} \cong \mathrm{Hom}(\pi_1(S), \mathbb{R}),$$
$$(x_1, \ldots, x_{2g}, y_1, \ldots, y_{4g-5}) \mapsto (x_1, \ldots, x_{2g}),$$

and $\ker(V^{-1} \circ f) = \{0\}^{2g} \times \mathbb{R}^{\mathcal{C}_\lambda}$. Hence, $\ker(f)$ has dimension $|\mathcal{C}_\lambda| = 4g-5$, so $\mathrm{Im}(f)$ has dimension $2g$. This shows that f is surjective, and that $\mathcal{H}_{\mathrm{trivial}} = \ker(f)$ has dimension $4g-5$. $\square$

8. Generalized cataclysms for representations into $\mathrm{SL}(n,\mathbb{R})$

The goal of this chapter is to enlarge the parameter space for the cataclysm deformations of Anosov representations into $\mathrm{SL}(n,\mathbb{R})$. The stretching maps T_g^H that are the basic building blocks for cataclysm deformations have the important property that they lie in the stabilizer of the pair of flags (P_g^+, P_g^-) associated with the oriented geodesic g. This stabilizer is conjugate to the Levi subgroup $L_\theta = P_\theta^+ \cap P_\theta^-$. The size of L_θ depends on the size of the set of simple roots $\theta \subset \Delta$ and becomes bigger if θ becomes smaller. For example, for $G = \mathrm{SL}(4,\mathbb{R})$ and $\theta = \{1,3\} \subset \Delta$, we have

$$L_\theta = \left\{ \begin{pmatrix} x & & \\ & A & \\ & & y \end{pmatrix} \middle| \, x, y \in \mathbb{R} \setminus \{0\}, A \in \mathrm{GL}(2,\mathbb{R}), \det(A) = x^{-1}y^{-1} \right\}.$$

This space is bigger than L_Δ, which consists of all diagonal matrices in $\mathrm{SL}(4,\mathbb{R})$.

In contrast, the parameter space for cataclysms, i.e. the space of transverse twisted $\mathfrak{a}_\theta$-valued cycles $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$, becomes smaller if θ becomes smaller by Proposition 3.21. The reason for that is we can only use parameters H that lie in $\mathfrak{a}_\theta$, which is the centralizer of L_θ in $\mathfrak{a}$. This is necessary for the stretching map T_g^H to be well-defined. Ideally, we would like to define deformations where the parameter space is not restricted to this small set.

In this section we make a first step in this direction and enlarge the parameter space $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ of cataclysms deformations for θ-Anosov representations into $\mathrm{SL}(n,\mathbb{R})$ to $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a})$. To do so, we require the representation ρ to satisfy an additional requirement, namely λ-*admissibility* (see Definition 8.3). In the first part of this section, we define what it means for a representation to be λ-admissible and show that λ-admissibility allows us to define cataclysm representations with parameters in $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a})$.

Standing Assumption. Throughout this section, λ is a maximal lamination with finitely many leaves.

In this case, the non-isolated leaves of $\tilde{\lambda}$ are exactly lifts of closed leaves of λ, and every bi-infinite leaf is isolated.

8.1. Admissible representations

Let $\rho\colon \pi_1(S) \to \mathrm{SL}(n,\mathbb{R})$ be θ-Anosov with $\theta = \{i_1, \dots, i_k\} \subset \Delta$ (see Example 2.3) and let $\zeta\colon \partial_\infty \tilde{S} \to \mathcal{F}_\theta$ be its boundary map. For an oriented geodesic g in $\tilde{S}$, the pair of transverse flags $(P_g^+, P_g^-) = (\zeta(g^+), \zeta(g^-))$ induces a splitting

$$\mathbb{R}^n = V_1(g) \oplus \cdots \oplus V_{k+1}(g) \tag{8.1}$$

as in Example 2.11, where the subspaces $V_j(g)$ are not necessarily 1-dimensional. For $H \in \mathfrak{a}_\theta$, the stretching map T_g^H from Definition 4.13 acts on every subspace $V_j(g)$ as a multiple of the identity. We now define stretching maps that act on $V_j(g)$ by stretching in different directions. To do so, we need to find preferred directions within the subspaces $V_j(g)$, i.e. a line splitting of $\mathbb{R}^n$ that is subordinate to the splitting from (8.1). We distinguish between isolated and non-isolated leaves of $\tilde{\lambda}$.

We start with considering the non-isolated leaves. Since the lamination λ is finite, those are exactly the lifts of closed leaves of λ on S. Up to the action of $\pi_1(S)$, there exist only finitely many non-isolated leaves in $\tilde{\lambda}$. Using the fixed hyperbolic metric on S, every closed leaf γ_c in λ defines an element of $\pi_1(S)$ and acts on $\tilde{S}$ as translation along a geodesic g_c. This geodesic g_c is a non-isolated leaf of $\tilde{\lambda}$. Assume that the representation ρ has the property that $\rho(\gamma_c)$ is diagonalizable with distinct eigenvalues for all of the finitely many closed leaves γ_c of the lamination λ, i.e.

$$\rho(\gamma_c) \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \quad \lambda_1 > \cdots > \lambda_n.$$

We naturally obtain a line splitting

$$\mathbb{R}^n = \ell_1(g_c) \oplus \cdots \oplus \ell_n(g_c) \tag{8.2}$$

associated to g_c, where $\ell_i(g_c)$ is the eigenspace of $\rho(\gamma_c)$ with eigenvalue λ_i. If we consider the same geodesic $\overline{g_c}$ with reversed orientation, then we obtain the line splitting $\bigoplus_i \ell_i(\overline{g_c})$ with $\ell_i(\overline{g_c}) = \ell_{n-i+1}(g_c)$. If g_c' is another lift of γ_c, $g_c' = \eta \cdot g_c$ for some $\eta \in \pi_1(S)$, set $\ell_i(g_c') := \rho(\eta)\ell_i(g_c)$. The splitting obtained in this way is ρ-equivariant by construction and subordinate to the splitting $\bigoplus_j V_j(g_c)$ defined by ρ in the sense that $\ell_j(g_c) \subset V_l(g_c)$ for $i_{l-1} < j \le i_l$. This follows from the fact that the boundary map ζ for ρ is dynamics-preserving. Thus, under the assumption that $\rho(\gamma_c)$ is diagonalizable with distinct eigenvalues for all closed leaves γ_c in λ, we obtain for every non-isolated leaf in $\tilde{\lambda}$ a line splitting of $\mathbb{R}^n$ that is subordinate to the splitting from (8.1).

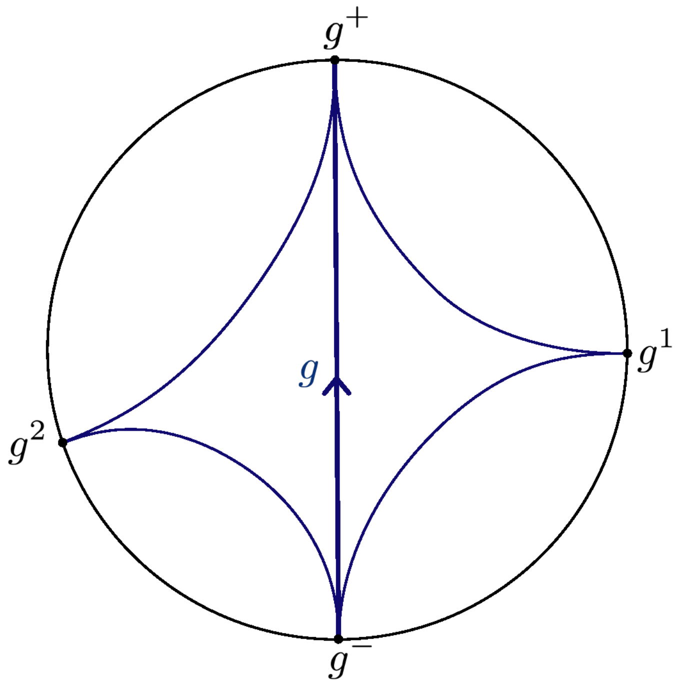


Figure 8.1.: To an oriented geodesic g that is an isolated leaf of the lamination $\tilde{\lambda}$, we associate a quadruple (g^+, g^-, g^1, g^2) of points in $\partial_\infty \tilde{S}$.

If g is an isolated leaf of $\tilde{\lambda}$, we use additional information on the lamination to construct a line splitting. To an oriented isolated geodesic g in $\tilde{S}$, we associate a quadruple of flags as follows.

Notation 8.1. Every isolated leaf g of $\tilde{\lambda}$ is part of the boundary of two ideal triangles in $\tilde{S} \setminus \tilde{\lambda}$. One of these triangles has vertices g^-, g^+, g^1, and the other one has vertices. g^-, g^+, g^2 (see Figure 8.1). We fix an orientation on $\partial_\infty \tilde{S}$ and choose the notation such that the points g^-, g^1, g^+, g^2 appear in this order along the boundary. We assign to g the quadruple (g^+, g^-, g^1, g^2). Using the flag curve ζ, we obtain a quadruple of pairwise transverse flags $\mathcal{Q}_\rho(g) := (P_g^+, P_g^-, P_g^1, P_g^2) := \big(\zeta(g^+), \zeta(g^-), \zeta(g^1), \zeta(g^2)\big)$.

Definition 8.2. A quadruple (E, F, G_1, G_2) of pairwise transverse elements in $\mathcal{F}_\theta$ is *splitting admissible* if there exists a line splitting

$$\mathbb{R}^n = \bigoplus_{i=1}^{n} \ell_i(E, F, G_1, G_2)$$

such that

- the line splitting is subordinate to the splitting $\mathbb{R}^n = V_1 \oplus \cdots \oplus V_{k+1}$ induced from the pair of transverse flags (E, F) as in Example 2.11 and

- the lines satisfy $\ell_i(F,E,G_2,G_1) = \ell_{n-i+1}(E,F,G_1,G_2)$.

Further, we require the assignment $(E,F,G_1,G_2) \mapsto \ell_i(E,F,G_1,G_2)$ from all splitting admissible quadruples to $\mathbb{R}P^{n-1}$ to be smooth and equivariant, i.e. for $M \in \mathrm{SL}(n,\mathbb{R})$, $\ell_i(M\cdot E, M\cdot F, M\cdot G_1, M\cdot G_2) = M\cdot \ell_i(E,F,G_1,G_2)$.

We can now define what it means for a representation ρ to be λ-admissible.

Definition 8.3. A θ-Anosov representation $\rho\colon \pi_1(S)\to \mathrm{SL}(n,\mathbb{R})$ is λ-*admissible* if

1. for every closed leaf γ_c of λ, $\rho(\gamma_c)$ is diagonalizable with distinct eigenvalues, and
2. for every isolated leaf g of $\tilde{\lambda}$, the associated quadruple $\mathcal{Q}_\rho(g)$ of pairwise transverse flags as in Notation 8.1 is splitting admissible.

Note that Condition 1 is an open condition in all θ-Anosov representations, since being diagonalizable with distinct eigenvalues is an open condition in $\mathrm{SL}(n,\mathbb{R})$.

Example 8.4. Every Δ-Anosov representation into $\mathrm{SL}(n,\mathbb{R})$ is trivially λ-admissible: We know that for Δ-Anosov representations, $\rho(\gamma)$ is diagonalizable with distinct eigenvalues for all $\gamma\in\pi_1(S)$. Further, the splitting $\mathbb{R}^n = \bigoplus_{i=1}^n V_i(g)$ is a line splitting, since in this case, the spaces $V_i(g)$ are 1-dimensional. However, we are not interested in Δ-Anosov representations here, where $\mathfrak{a}_\Delta = \mathfrak{a}$ and the parameter space of the cataclysm deformation is already equal to $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a})$.

Given a λ-admissible representation ρ, we can assign to every oriented leaf g of $\tilde{\lambda}$ a line splitting of $\mathbb{R}^n$.

Lemma 8.5. *Let $\rho\colon \pi_1(S)\to\mathrm{SL}(n,\mathbb{R})$ be θ-Anosov and λ-admissible. Then we can assign to every oriented leaf g of the lamination $\tilde{\lambda}$ a line splitting*

$$\mathbb{R}^n = \bigoplus_{i=0}^{n} \ell_i(g) \tag{8.3}$$

that is ρ-equivariant and satisfies $\ell_i(\overline{g}) = \ell_{n-i+1}(g)$, where $\overline{g}$ denotes the geodesic g with opposite orientation.

Proof. For the case that g is non-isolated, we can construct a line splitting as in (8.2). For the case that g is isolated, we obtain a line splitting from the splitting admissible quadruple $\mathcal{Q}_\rho(g)$ by setting $\ell_i(g) := \ell_i(\mathcal{Q}_\rho(g))$. The ρ-equivariance of the lines $\ell_i(g)$ follows from the ρ-equivariance of the flag curve ζ and the equivariance of the lines ℓ_i. Further, the quadruple $\mathcal{Q}_\rho(\overline{g})$ is obtained from $\mathcal{Q}_\rho(g)$ by switching the first two and the last two flags. The behavior of the line splitting under reversing orientation then follows from the behavior of the lines ℓ_i under permutations of the flags. $\square$

Remark 8.6. If a representation ρ is Δ-Anosov, then it is also θ-Anosov for any $\theta \subset \Delta$ by definition. Thus, we can consider a Δ-Anosov representation ρ as θ-Anosov representation. In terms of boundary maps, this means that we forget about some parts of the flag. Assume that ρ , considered as θ-Anosov representation, is λ-admissible. For every oriented geodesic g in $\tilde{\lambda}$, we now have two line splittings: On one hand, the line splitting $\mathbb{R}^n = \bigoplus_{i=1}^n V_i(g)$ coming from the boundary map in the complete flags, when viewing ρ as Δ-Anosov and on the other hand the line splitting $\mathbb{R}^n = \bigoplus_{i=1}^n \ell_i(g)$ from Lemma 8.5, when viewing ρ as θ-Anosov. Those line splittings do not agree in general. This results form the fact that the line splitting $\mathbb{R}^n = \bigoplus_{i=1}^n V_i(g)$ is independent of the vertices g^1 and g^2, whereas those are crucial for the construction in Lemma 8.5. See also Example 8.18 for an explicit example in $\mathrm{SL}(4, \mathbb{R})$.

Remark 8.7. The existence of a line splitting associated to every oriented geodesic g in $\tilde{\lambda}$ as in (8.3) now raises the question if we can use this splitting to define a boundary map from the boundary of the lamination $\partial_\infty \tilde{\lambda}$ into the variety of complete flags $\mathrm{Flag}(\mathbb{R}^n)$. This is not possible. The boundary map into the partial flag variety $\zeta\colon \partial_\infty \tilde{S} \to \mathcal{F}_\theta$ can be reconstructed from the splitting in (8.1) as

$$\zeta^{(i_j)}(g^+) = \bigoplus_{l=1}^{i_l} V_l(g), \tag{8.4}$$

where g^+ is the positive endpoint of g. However, we cannot construct a boundary map into the variety of complete flags in the same way. The problem is that for a boundary point $x \in \partial_\infty \tilde{\lambda}$, there are infinitely many oriented geodesics g having x as positive endpoints and a construction as in (8.4) depends on the choice of g. See also Example 8.20 for an explicit example in $\mathrm{SL}(4, \mathbb{R})$.

One main ingredient in the definition of cataclysms are the stretching maps from Section 4.2 with parameter $H \in \mathfrak{a}_\theta \subset \mathfrak{a}$. Using the line splitting for g constructed in Lemma (8.5), we can enlarge the parameter space and define stretching maps with parameter $H \in \mathfrak{a}$.

Definition 8.8. Let $H \in \mathfrak{a}$ and let g be an oriented geodesic in $\tilde{\lambda}$. Let $m_g \in \mathrm{SL}(n, \mathbb{R})$ be an element that maps the line splitting given by the standard basis vectors $e_1, \dots, e_n$ to the line splitting given by g as in Lemma 8.5. Then $\widehat{T}_g^H := m_g \exp(H) m_g^{-1}$ is the *full-stretching map* along g by H.

We use the notation $\widehat{T}$ to distinguish the full-stretching maps from the stretching maps from Section 4.2.

Remark 8.9. In a basis adapted to the line splitting for g, the full-stretching map $\widehat{T}_g^H$ is diagonal with entries given by $H \in \mathfrak{a}$. By construction, the properties from Lemma 4.16

also hold for full-stretching maps, i.e. fro an oriented geodesic g in $\tilde{S}$ and $H, H_1, H_2 \in \mathfrak{a}$ we have

1. $\widehat{T}_g^{H_1}\widehat{T}_g^{H_2} = \widehat{T}_g^{H_1+H_2}$
2. $\left(\widehat{T}_g^{H}\right)^{-1} = \widehat{T}_g^{-H}$,
3. $\widehat{T}_{\bar{g}}^{H} = \widehat{T}_g^{-\iota(H)}$ and
4. ρ-equivariance, i.e. $\widehat{T}_{\gamma g}^{H} = \rho(\gamma)\widehat{T}_g^{H}\rho(\gamma)^{-1}$ for all $\gamma \in \pi_1(S)$.

For the proof of convergence of the shearing maps (Proposition 5.3) we needed the fact that if two geodesics are close, also the corresponding stretching maps are close (Proposition 4.18). We want to establish the same result for full-stretching maps. Let g, h be oriented geodesics in $\tilde{\lambda}$ and let $\mathcal{C}_{gh}$ be the set of all connected components of $\tilde{S} \setminus \tilde{\lambda}$ between g and h (see Notation 3.10). Let $\tilde{k}$ be an oriented arc tightly transverse to $\tilde{\lambda}$ that crosses first g, then h. The proof of Proposition 4.18 uses the slithering map. In the situation at hand, we cannot use slithering maps as we did for Proposition 4.18, because the element Σ_{gh} maps the spaces $V_j(h)$ to $V_j(g)$, but not necessarily the lines $\ell_i(h)$ from Lemma 8.5 to $\ell_i(g)$. Our first step is to show that the lines $\ell_i(g)$ from Lemma 8.5 are close for two oriented geodesics g_R^0 and g_R^1 that bound a pinched inner component. Recall from Definition 3.13 that a component $R \in \mathcal{C}_{gh}$ is *pinched* if there are other components R^- and R^+ in $\mathcal{C}_{gh}$ lying along $\tilde{k}$ directly before and after R, respectively, and such that the oriented geodesics $g_{R^-}^0$, $g_{R^-}^1$, g_R^0, g_R^1, $g_{R^+}^0$ and $g_{R^+}^1$ all share an endpoint (see Figure 3.5).

Lemma 8.10. *There exist constants $C, A > 0$, depending on $\tilde{k}$ and ρ, such that for every pinched inner component $R \in \mathcal{C}_{gh}$, for all i,*

$$\mathrm{d}_{\mathbb{R}P^{n-1}}\left(\ell_i(g_R^0), \ell_i(g_R^1)\right) \leq Ce^{-Ar(R)}, \tag{8.5}$$

where $r(R)$ is the divergence radius (Lemma 3.11).

Proof. Let $R \in \mathcal{C}_{gh}$ be a pinched inner component. Without loss of generality assume that g_R^0, g_R^1 are oriented towards their common endpoint which we denote by $\left(g_R^0\right)^+ = \left(g_R^1\right)^+$. Likewise, let$\left(g_R^0\right)^-$ and $\left(g_R^1\right)^-$ be the negative endpoints of the oriented geodesics g_R^0 and g_R^1, respectively. Let d_∞ be a distance function on $\partial_\infty\tilde{S}$ and, abusing notation, also on $\left(\partial_\infty\tilde{S}\right)^4$, which is defined as sum of the element-wise distances in $\partial_\infty\tilde{S}$. For the distance function on the space of oriented geodesics, we use the sum of the distances of the endpoints, i.e. $\mathrm{d}(g,h) := \mathrm{d}_\infty(g^+,h^+) + \mathrm{d}_\infty(g^-,h^-)$, where $g^\pm$ are the endpoints of g, and the same for h. Fix a distance function $\mathrm{d}_{\mathbb{R}P^{n-1}}$ on $\mathbb{R}P^{n-1}$.

For every oriented isolated leaf g, the lines $\ell_i(g)$ depend smoothly on the quadruple $\mathcal{Q}_\rho(g)$ by the definition of being splitting admissible. Further, since the boundary map ζ is Hölder continuous, the quadruple $\mathcal{Q}_\rho(g)$ depends Hölder continuously on the four points (g^+, g^-, g^1, g^2) from Notation 8.1. Thus, the assignment from the quadruple of points (g^+, g^-, g^1, g^2) to the lines $\ell_i(g)$ is locally Hölder continuous, i.e. there exist constants $C, A > 0$ depending on $\tilde{k}$ and ρ such that

$$\begin{aligned}
&\mathrm{d}_{\mathbb{R}P^{n-1}}\left(\ell_i(g_R^0), \ell_i(g_R^1)\right) \\
&\quad \leq C\mathrm{d}_\infty\left(\left(\left(g_R^0\right)^+, \left(g_R^0\right)^-, \left(g_R^0\right)^1, \left(g_R^0\right)^2\right), \left(\left(g_R^1\right)^+, \left(g_R^1\right)^-, \left(g_R^1\right)^1, \left(g_R^1\right)^2\right)\right)^A \\
&\quad = C\left(\mathrm{d}_\infty\left(\left(g_R^0\right)^+, \left(g_R^1\right)^+\right) + \mathrm{d}_\infty\left(\left(g_R^0\right)^-, \left(g_R^1\right)^-\right)\right. \\
&\qquad\qquad \left. + \mathrm{d}_\infty\left(\left(g_R^0\right)^1, \left(g_R^1\right)^1\right) + \mathrm{d}_\infty\left(\left(g_R^0\right)^2, \left(g_R^1\right)^2\right)\right)^A, \\
&\quad \leq C\left(\mathrm{d}\left(g_R^0, g_R^1\right)\right) + \mathrm{d}\left(g_{R^+}^0, g_{R^+}^1\right) + \mathrm{d}\left(g_{R^-}^0, g_{R^-}^1\right))^A.
\end{aligned}$$

Here, R^- and R^+ are the connected component of $\tilde{S} \setminus \tilde{\lambda}$ adjacent to R and lying before and after R, respectively, in the direction of $\tilde{k}$. In the last step, we used the definition of the distance of two oriented geodesics as sum of the distances between the endpoints. Thus, using Remark 4.2 and Lemma 3.11, there exist constants $C', C'', A' > 0$ such that

$$\begin{aligned}
\mathrm{d}_{\mathbb{R}P^{n-1}}\left(\ell_i(g_R^0), \ell_i(g_R^1)\right) &\leq C'\left(\mathrm{length}(\tilde{k} \cap R) + \mathrm{length}(\tilde{k} \cap R^+) + \mathrm{length}(\tilde{k} \cap R^-)\right)^A \\
&\leq C''\left(e^{-A'r(R)} + e^{-A'r(R^+)} + e^{-A'r(R^-)}\right)^A.
\end{aligned}$$

Since the divergence radii $r(R), r(R^+)$ and $r(R^-)$ differ by a constant depending on $\tilde{k}$ by Lemma 3.14, the claim follows. ☐

Lemma 8.11. *Let $R \in \mathcal{C}_{gh}$ be a pinched inner component. Then there exists and element $m_{g_R^0 g_R^1} \in \mathrm{SL}(n, \mathbb{R})$ sending the line splitting for g_R^1 to the line splitting for g_R^0, and constants $C, A > 0$ depending on $\tilde{k}$ and ρ such that*

$$\mathrm{d}_{\mathrm{SL}(n,\mathbb{R})}(\mathrm{Id}, m_{g_R^0 g_R^1}) \leq Ce^{-Ar(R)}.$$

Note that the element $m_{g_R^0 g_R^1}$ plays the role of the slithering map in the proof of Proposition 4.18. The estimates follows from Lemma 8.10 by carefully choosing $m_{g_R^0 g_R^1}$ and suitable norms. The details can be found in Section A.4.

We can now prove that, if two geodesics bound the same component and share an endpoint, then the corresponding full-stretching maps are close. This is an analogue of Corollary 4.19.

Lemma 8.12. *Let $\tilde{k}$ be an oriented arc transverse to $\tilde{\lambda}$ and let $R \subset \tilde{S} \setminus \tilde{\lambda}$ be a pinched inner component. Then there exist constants $C, A > 0$ depending on $\tilde{k}$ and ρ such that*

$$\mathrm{d}_{\mathrm{SL}(n,\mathbb{R})}\left(\widehat{T}^{H}_{g_R^0}\widehat{T}^{-H}_{g_R^1}, \mathrm{Id}\right) \leq C\left(e^{\|H\|_{\mathfrak{a}}}+1\right)e^{-Ar(R)}.$$

Proof. As in the proof of Prop 4.18, up to conjugation ρ by an element in $\mathrm{SL}(n,\mathbb{R})$, we can assume that the splitting associated to g_R^0 is the standard splitting given by the basis vectors $e_1, \ldots, e_n$. Let $m = m_{g_R^0 g_R^1} \in \mathrm{SL}(n,\mathbb{R})$ be as in Lemma 8.11. Then

$$\begin{aligned}
\mathrm{d}_{\mathrm{SL}(n,\mathbb{R})}\left(\widehat{T}^{H}_{g_R^0}\ \widehat{T}^{-H}_{g_R^1}, \mathrm{Id}\right) &= \mathrm{d}_{\mathrm{SL}(n,\mathbb{R})}\left(\widehat{T}^{-H}_{g_R^0}, \widehat{T}^{-H}_{g_R^1}\right)\\
&= \mathrm{d}_{\mathrm{SL}(n,\mathbb{R})}\left(\exp(-H), m\exp(-H)m^{-1}\right)\\
&\leq \mathrm{d}_{\mathrm{SL}(n,\mathbb{R})}\left(\exp(-H), m\exp(-H)\right)\\
&\qquad + \mathrm{d}_{\mathrm{SL}(n,\mathbb{R})}\left(m\exp(-H), m\exp(-H)m^{-1}\right)\\
&\leq \left\|\mathrm{Ad}_{\exp(H)}\right\|_{\mathrm{op}(\mathfrak{g})}\mathrm{d}_{\mathrm{SL}(n,\mathbb{R})}\left(\mathrm{Id}, m\right) + \mathrm{d}_{\mathrm{SL}(n,\mathbb{R})}\left(\mathrm{Id}, m^{-1}\right)\\
&\leq C\left(e^{\|H\|_{\mathfrak{a}}}+1\right)e^{-Ar(R)},
\end{aligned}$$

where for the second last inequality, we use left-invariance and almost right-invariance of the metric $\mathrm{d}_{\mathrm{SL}(n,\mathbb{R})}$ (Lemma A.2), and for the last inequality we use Lemma A.3 and Lemma 8.11. $\square$

We can now define shearing maps as in Section 5.1 as a limit of a composition of full-stretching maps. Their existence is guaranteed by Lemma 8.12. Note that it is sufficient that Lemma 8.12 holds for pinched inner components only, because all but finitely many components are pinched. If ρ is λ-admissible, and $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\Delta)$ sufficiently small, we can define $\widehat{\Lambda}_0^{\varepsilon}\rho$, the *generalized ε-cataclysm deformation of the Anosov representation ρ along the maximal geodesic lamination* λ with coefficients in $\varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\Delta)$ with respect to a fixed reference triangle P_0.

In total, we have the following result.

Theorem 8.13. *Let $\rho\colon \pi_1(S) \to \mathrm{SL}(n,\mathbb{R})$ be θ-Anosov and λ-admissible. Fix a reference triangle $P_0 \subset \tilde{S} \setminus \tilde{\lambda}$. Then exists a neighborhood $\mathcal{V}_\rho$ of 0 in $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a})$ and a continuous map*

$$\begin{aligned}
\widehat{\Lambda}_0\colon \mathcal{V}_\rho &\to \mathrm{Hom}(\pi_1(S), \mathrm{SL}(n,\mathbb{R}))\\
\varepsilon &\mapsto \widehat{\Lambda}_0^{\varepsilon}\rho
\end{aligned}$$

such that $\widehat{\Lambda}_0^{0}\rho = \rho$.

Note that the difference to Theorem 5.12 lies in the parameter space: The parameter space in Theorem 5.12 is a subset of $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a}_\theta)$, which has dimension $|\theta|(6\cdot\mathrm{genus}(S)-6)+|\theta'|$ by Corollary 3.22, where $\theta'\subset\theta$ is a maximal subset satisfying $\theta'\cap\iota(\theta)=\emptyset$. Now, for λ-admissible representations, we can shear with parameters in $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a})$, which has dimension $(n-1)(6\cdot\mathrm{genus}(S)-6)+\lfloor\frac{n}{2}\rfloor$. In particular, the dimension of $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a}_\theta)$ can be independent on n, but $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a})$ grows linearly in n.

Remark 8.14.

(1) One can ask if it is possible to construct a parameter space that is bigger than $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a})$ by using more general deformations of the subspaces $V_j(g)$, not just diagonal stretches. However, our construction does not give a basis for the spaces $V_j(g)$, but only a line splitting. Thus, we cannot use it to specify transformations on $V_j(g)$ that are different from diagonal stretches.

(2) The cataclysms defined in Section 5 can also be seen as a special case of the deformations defined in this section: Since $\mathfrak{a}_\theta\subset\mathfrak{a}$, we can identify $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a}_\theta)$ with a subset of $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a})$. If we only look at $H\in\mathfrak{a}_\theta$, then the stretching maps T_g^H from Section 4.2 agree with the full-stretching maps $\widehat{T}_g^H$, so for $\varepsilon\in\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a}_\theta)$, the resulting cataclysm deformation is the same.

In the following two sections, we give examples for λ-admissible representations.

8.2. Admissibility for projective Anosov representations in $\mathrm{SL}(4,\mathbb{R})$

In this section we give an example for λ-admissible projective Anosov representations into $\mathrm{SL}(4,\mathbb{R})$, so $\theta=\{1,3\}$. The flag curve is of the form $\zeta(x)=\big(\zeta^{(1)}(x)\subset\zeta^{(3)}(x)\big)$ for $x\in\partial_\infty\tilde{S}$, with $\zeta^{(1)}(x)\in\mathbb{R}P^3$ and $\zeta^{(3)}(x)\in\mathrm{Gr}_3(4)$. Our construction of the line splitting is based on the observation that generically, a quadruple of flags in $\mathcal{F}_{\{1,3\}}$ satisfies additional transversality properties.

Lemma 8.15. *Let (E,F,G_1,G_2) be a quadruple of pairwise transverse flags in $\mathcal{F}_{\{1,3\}}$ such that*

$$E^{(3)}\cap F^{(3)}\cap G_1^{(3)}\cap G_2^{(3)}=\{0\}. \tag{8.6}$$

Then (E,F,G_1,G_2) is splitting admissible.

Proof. Define $\ell_1(E,F,G_1,G_2) := E^{(1)}$ and $\ell_4(E,F,G_1,G_2) := F^{(1)}$. Note that, since the flags are pairwise transverse, the intersection $E^{(3)} \cap F^{(3)}$ has dimension two and likewise for the other flags. By looking at the dimension of the subspaces, we have

$$\begin{aligned}
&\dim\left(E^{(3)} \cap F^{(3)} \cap G_1^{(3)}\right)\\
&\quad = \dim\left(E^{(3)} \cap F^{(3)} \cap G_1^{(3)} \cap G_2^{(3)}\right) + \dim\left((E^{(3)} \cap F^{(3)} \cap G_1^{(3)}) + G_2^{(3)}\right) - \dim(G_2^{(3)})\\
&\quad = 0 + 4 - 3 = 1.
\end{aligned}$$

Set $\ell_2(E,F,G_1,G_2) := \left(E^{(3)} \cap F^{(3)} \cap G_1^{(3)}\right)$. Analogously, we can compute that

$$\dim\ \left(E^{(3)} \cap F^{(3)} \cap G_2^{(3)}\right) = 1 \quad \text{and} \quad \left(E^{(3)} \cap F^{(3)} \cap G_1^{(3)}\right) \neq \left(E^{(3)} \cap F^{(3)} \cap G_2^{(3)}\right).$$

Set $\ell_3(E,F,G_1,G_2) := \left(E^{(3)} \cap F^{(3)} \cap G_2^{(3)}\right)$. The lines $\ell_i(E,F,G_1,G_2)$ give a line splitting of $\mathbb{R}^4$. Let the spaces V_j be the subspaces of the splitting given by E and F as in Example (2.11). We have

$$\begin{aligned}
V_1 &= E^{(1)} = \ell_1(E,F,G_1,G_2),\\
V_2 &= E^{(3)} \cap F^{(3)} = \ell_2(E,F,G_1,G_2) \oplus \ell_3(E,F,G_1,G_2) \quad \text{and}\\
V_3 &= F^{(1)} = \ell_4(E,F,G_1,G_2),
\end{aligned}$$

so the line splitting is subordinate to the splitting given by the V_j. Further, the behavior under permutations and the equivariance follow directly from the definition. As the intersection of subspaces in $\mathbb{R}^4$ is smooth away from non-generic situations, the lines depend smoothly on the quadruple. ☐

From Lemma 8.15, we can conclude the following:

Corollary 8.16. *Let $\rho\colon \pi_1(S) \to \mathrm{SL}(4,\mathbb{R})$ be projective Anosov such that $\rho(\gamma_c)$ is diagonalizable with distinct eigenvalues for every closed leaf γ_c of λ and that for every isolated leaf g of $\tilde{S}$, the quadruple $\mathcal{Q}_\rho(g)$ satisfies the intersection condition* (8.6)*. Then ρ is λ-admissible and Theorem 8.13 applies, i.e. there exists a cataclysm deformation $\widehat{\Lambda}_0$ with parameter space $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda};\mathfrak{a})$.*

Lemma 8.17. *The subset of λ-admissible representation in all projective Anosov representations into $\mathrm{SL}(4,\mathbb{R})$ is open.*

Proof. We already know that the first condition is open. Further, the condition (8.6) is an open condition in the space of quadruples of flags that are pairwise transverse. Since the boundary map ζ is continuous and depends continuously on the representation ρ it follows that, for a fixed oriented geodesic g the set of all projective Anosov representations such that $\mathcal{Q}_\rho(g)$ is splitting admissible is open. Since the lamination λ is finite and being splitting admissible is invariant under $\mathrm{SL}(4,\mathbb{R})$, the set of projective Anosov representations such that $\mathcal{Q}_\rho(g)$ is splitting admissible for every isolated leaf g is open as a finite intersection of open sets. In total, the set of λ-admissible projective Anosov representations is open. $\square$

Example 8.18. We can now give an explicit example for Remark 8.6, i.e. show that if we consider a Δ-Anosov representation in $\mathrm{SL}(4,\mathbb{R})$ as projective Anosov, then the two line splittings we obtain do not agree. Consider the irreducible representation $j_4\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(4,\mathbb{R})$ introduced in Subsection 2.4.2 and let $j_4^+\colon \mathbb{R}P^1 \to \mathrm{Flag}(\mathbb{R}^4)$ be the induced j_4-equivariant map. Let $E := j_4^+(\langle e_1\rangle)$, $F := j_4^+(\langle e_2\rangle)$, $G_1 = j_4^+(\langle e_1+e_2\rangle)$ and $G_2 := j_4^+(\langle e_1-e_2\rangle)$. Then E is standard ascending flag , F is the standard descending flag F and $G_1^{(i)}$ is spanned by $v_1,\ldots,v_i$, $G_2^{(i)}$ is spanned by $w_1,\ldots,w_i$, where

$$v_1 = \begin{pmatrix} 1 \\ -\sqrt{3} \\ \sqrt{3} \\ -1 \end{pmatrix},\quad v_2 = \begin{pmatrix} \sqrt{3} \\ -1 \\ -1 \\ \sqrt{3} \end{pmatrix},\quad v_3 = \begin{pmatrix} \sqrt{3} \\ 1 \\ -1 \\ -\sqrt{3} \end{pmatrix},$$

$$w_1 = \begin{pmatrix} -1 \\ -\sqrt{3} \\ -\sqrt{3} \\ -1 \end{pmatrix},\quad w_2 = \begin{pmatrix} \sqrt{3} \\ 1 \\ -1 \\ -\sqrt{3} \end{pmatrix},\quad w_3 = \begin{pmatrix} -\sqrt{3} \\ 1 \\ 1 \\ -\sqrt{3} \end{pmatrix}.$$

Since these flags come from the elements in $\mathbb{R}P^1$, the triangle invariants for the triples (E,F,G_1) and (F,E,G_2) are all positive, so using Bonahon-Dreyer coordinates [BD17], there exists a Hitchin representation $\rho\colon \pi_1(S) \to \mathrm{SL}(4,\mathbb{R})$ such that the quadruple of flags (E,F,G_1,G_2) agrees with the quadruple $\mathcal{Q}\rho(g)$ for some oriented geodesic g in $\tilde{S}$. On one hand, considering ρ as Δ-Anosov, we obtain a line splitting with lines $V_i(g) = \langle e_i\rangle$. On the other hand, one computes that the line $\ell_2(g)$ is given as

$$\ell_2(g) = E^{(3)} \cap F^{(3)} \cap G_1^{(3)} = \left\langle \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\rangle \neq \langle e_2\rangle.$$

A similar computation shows $\ell_3(g) \neq \langle e_3\rangle$, so the line splitting from Lemma 8.5 does not agree with the splitting given by the full flags E and F, which is the standard line splitting.

Remark 8.19. For two adjacent geodesics, the lines $\ell_i(g)$ are related: Let g and h be two oriented geodesics in $\tilde{\lambda}$ that bound the same triangle and assume that g and h converge to the same endpoint (see Figure 8.2). Further assume that the points g^+, g^-, h^- appear in

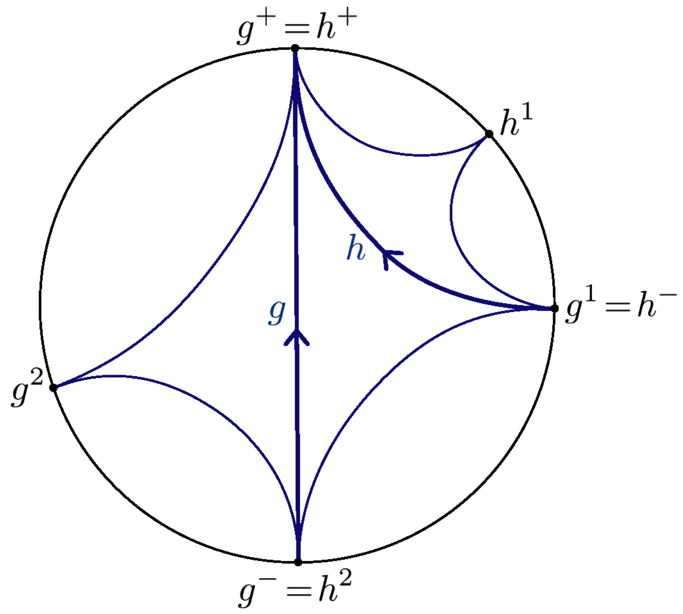


Figure 8.2.: If two oriented geodesics g and h share their positive endpoint $g^+ = h^+$, then some of the points associated to g and h agree. In the situation pictured here, we have $g^- = h^2$ and $g^1 = h^-$.

this order along the circle. We have $h^+ = g^+$, $h^- = g^1$ and $h^2 = g^-$. Thus

$$\begin{aligned} \ell_3(h) &= \zeta^{(3)}(h^+) \cap \zeta^{(3)}(h^-) \cap \zeta^{(3)}(h^2) \\ &= \zeta^{(3)}(g^+) \cap \zeta^{(3)}(g^1) \cap \zeta^{(3)}(g^-) \\ &= \ell_2(g). \end{aligned}$$

The case where g and h diverge works analogously.

Example 8.20. In this example, we illustrate Remark 8.7, i.e. show that the line splitting $\mathbb{R}^4 = \bigoplus_{i=1}^4 \ell_i(g)$ obtained from Lemma 8.5 does not induce a well-defined boundary map into the complete flag variety $\mathrm{Flag}(\mathbb{R}^4)$. Let $x \in \partial_\infty \tilde{\lambda}$ and let g, h be two geodesics sharing the positive endpoint $g^+ = h^+$ as in Figure 8.2. Let ρ be a projective Anosov representation with flag curve $\zeta \colon \partial_\infty \tilde{S} \to \mathcal{F}_{\{1,3\}}$. Considering the oriented geodesic h with positive endpoint h^+, the construction from (8.4) yields for the 2-dimensional part of the flag

$$\ell_1(h) \oplus \ell_2(h) = \zeta^{(1)}(h^+) \oplus \ell_2(h). \tag{8.7}$$

Using instead the oriented geodesic g with endpoint $g^+ = h^+$, we have by Remark 8.19

$$\ell_1(g) \oplus \ell_2(g) = \zeta^{(1)}(h^+) \oplus \ell_3(h). \tag{8.8}$$

Since $\ell_2(h)$ and $\ell_3(h)$ together span $V_2(h)$, the planes (8.7) and (8.8) do not coincide. In particular, the line splitting from Lemma 8.5 does not induce a well-defined boundary map into the full flags.

One can now ask if we can find projective Anosov representations into $\mathrm{SL}(n,\mathbb{R})$ for $n > 4$ that are λ-admissible. In this case, the subspace $V_2(g)$ of the splitting from (8.1) has dimension $n-2$. To obtain a line splitting of $V_2(g)$ it is not sufficient to consider the quadruple of flags $\mathcal{Q}_\rho(g)$. Instead, we need an n-tuple of pairwise transverse flags associated to an oriented isolated geodesic. Further, these flags need to satisfy additional genericity conditions in order to obtain a line splitting. It might be be possible to extend the definition of λ-admissibility in this way to projective Anosov representations into $\mathrm{SL}(n,\mathbb{R})$, but this requires additional choices and does not seem natural to us.

8.3. Admissibility for $\{\alpha_n\}$-Anosov representations in $\mathrm{SL}(2n,\mathbb{R})$

In this section we give an example for λ-admissible representations into $\mathrm{SL}(2n,\mathbb{R})$ that are $\{\alpha_n\}$-Anosov representations, so the boundary map ζ maps into $\mathrm{Gr}_n(2n)$.

As a preparation, we have a look of the action of $\mathrm{SL}(2n,\mathbb{R})$ on quadruples of pairwise transverse n-planes. Let $e_1,\dots,e_{2n}$ be the standard unit vectors in $\mathbb{R}^{2n}$, set $f_i := e_{n+i}$ and let $P_1 := \langle e_1,\dots,e_n\rangle$ and $P_2 := \langle f_1,\dots,f_n\rangle$ be fixed transverse n-planes that we use for reference.

Lemma 8.21. *The group* $\mathrm{SL}(2n,\mathbb{R})$ *acts transitively on pairs of transverse elements in* $\mathrm{Gr}_n(2n)$ *and*

$$\mathrm{Stab}_{\mathrm{SL}(2n,\mathbb{R})}(P_1,P_2) = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \;\middle|\; A, D \in \mathrm{GL}(n,\mathbb{R}), \det(A)\det(D) = 1 \right\}.$$

Proof. This is a direct consequence of the fact that $\mathrm{SL}(2n,\mathbb{R})$ acts transitively on projective bases of $\mathbb{R}^{2n}$. □

Next, we want to see how $\mathrm{SL}(2n,\mathbb{R})$ acts on triples of pairwise transverse n-planes.

Remark 8.22. Let $\mathrm{Gr}_n(2n)^{P_1,P_2}$ be the set of all elements in $\mathrm{Gr}_n(2n)$ that are transverse to both P_1 and P_2, i.e.

$$\mathrm{Gr}_n(2n)^{P_1,P_2} := \left\{ P \in \mathrm{Gr}_n(2n) \;\middle|\; P \oplus P_1 = \mathbb{R}^{2n} = P \oplus P_2 \right\}.$$

We can identify $\mathrm{Gr}_n(2n)^{P_1,P_2}$ with $\mathrm{GL}(n,\mathbb{R})$ as follows: If $P \in \mathrm{Gr}_n(2n)$ is transverse to P_2, there exists a unique matrix $X_P \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ such that the columns of the $2n\times n$-matrix

$$\begin{pmatrix} \mathrm{Id}_n \\ X_P \end{pmatrix} \tag{8.9}$$

form a basis for P. If P is in addition transverse to P_1, then $\det(X_P) \neq 0$, so $X_P \in \mathrm{GL}(n,\mathbb{R})$. Vice versa, if $X \in \mathrm{GL}(n,\mathbb{R})$, then the columns of the matrix (8.9) span an n-dimensional subspace transverse to P_1 and P_2. Note that this construction agrees with classical coordinate charts for the Grassmannians, considered for the special case of n-planes transverse to P_1 and P_2.

We sometimes abuse notation and identify an element $P \in \mathrm{Gr}_n(2n)^{P_1,P_2}$ with the corresponding matrix $X_P \in \mathrm{GL}(n,\mathbb{R})$.

We define two more reference planes in $\mathrm{Gr}_n(2n)^{P_1,P_2}$: $P_3^+ := \langle e_1+f_1,\ldots,e_n+f_n\rangle$ and $P_3^- := JP_3^+ = \langle e_1-f_1, e_2+f_2,\ldots,e_n+f_n\rangle$, where

$$J := \begin{pmatrix} -1 & 0 \\ 0 & \mathrm{Id}_{n-1} \end{pmatrix} \in \mathrm{GL}_n(\mathbb{R}). \tag{8.10}$$

Lemma 8.23. *The action of* $\mathrm{Stab}_{\mathrm{SL}(2n,\mathbb{R})}(P_1,P_2)$ *on* $\mathrm{Gr}_n(2n)^{P_1,P_2}$ *has two orbits represented by* P_3^+ *and* P_3^- *and*

$$\begin{aligned}
\mathrm{Stab}_{\mathrm{SL}(2n,\mathbb{R})}(P_1,P_2,P_3^+) &= \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \;\middle|\; \det(A)^2 = 1 \right\},\\
\mathrm{Stab}_{\mathrm{SL}(2n,\mathbb{R})}(P_1,P_2,P_3^-) &= \left\{ \begin{pmatrix} A & 0 \\ 0 & JAJ \end{pmatrix} \;\middle|\; \det(A)^2 = 1 \right\}.
\end{aligned}$$

Proof. First, note that no element in $\mathrm{Stab}_{\mathrm{SL}(2n,\mathbb{R})}(P_1,P_2)$ can map P_3^+ to P_3^-, so the two elements lie in different orbits. For $P \in \mathrm{Gr}_n(2n)^{P_1,P_2}$ we either have $\det(X_P) > 0$ or $\det(X_P) < 0$. Let $P \in \mathrm{Gr}_n(2n)^{P_1,P_2}$ with $\det(X_P) > 0$. The matrix

$$A := (\det X_P)^{\frac{1}{2n}} \begin{pmatrix} \mathrm{Id}_n & 0 \\ 0 & X_P^{-1} \end{pmatrix}$$

lies in $\mathrm{Stab}_{\mathrm{SL}(2n,\mathbb{R})}(P_1,P_2)$ and sends P to P_3^+. This shows that $\mathrm{SL}(2n,\mathbb{R})$ acts transitively on elements in $\mathrm{Gr}_n(2n)^{P_1,P_2}$ with $\det(X_P) > 0$. Similarly, for $P \in \mathrm{Gr}_n(2n)^{P_1,P_2}$ with

$\det(X_P) < 0$, define

$$A := (-\det(X_P))^{\frac{1}{2n}} \begin{pmatrix} \mathrm{Id}_n & 0 \\ 0 & JX_P^{-1} \end{pmatrix}.$$

Then A has determinant 1, so is in $\mathrm{Stab}_{\mathrm{SL}(2n,\mathbb{R})}(P_1, P_2)$ and sends P to P_3^-. Thus, there are two $\mathrm{Stab}_{\mathrm{SL}(2n,\mathbb{R})}(P_1, P_2)$-orbits in $\mathrm{Gr}_n(2n)^{P_1,P_2}$ distinguished by the sign of $\det(X_P)$ with representatives P_3^+ and P_3^-. The $\mathrm{SL}(2n,\mathbb{R})$-stabilizers follow from a short computation. $\square$

As a consequence of Lemma 8.21 and Lemma 8.23, we obtain the following:

Corollary 8.24. *The* $\mathrm{SL}(2n,\mathbb{R})$*-action on triples of pairwise transverse planes has two orbits represented by the triples* (P_1, P_2, P_3^+) *and* (P_1, P_2, P_3^-).

Definition 8.25. A triple of pairwise transverse planes in $\mathbb{R}^4$ is of *positive type*, if it lies in the $\mathrm{SL}(2n,\mathbb{R})$-orbit represented by (P_1, P_2, P_3^+). It is of *negative type* if it lies in the $\mathrm{SL}(2n,\mathbb{R})$-orbit represented by (P_1, P_2, P_3^-).

Note that the type of a triple depends continuously on the triple.

Remark 8.26. Let $\rho\colon \pi_1(S) \to \mathrm{SL}(2n,\mathbb{R})$ be an $\{\alpha_n\}$-Anosov representation with boundary map $\zeta\colon \partial_\infty\tilde{S} \to \mathrm{Gr}_n(2n)$ and let $(x, y, z) \in \partial_\infty\tilde{S}$ be a triple of pairwise distinct points in positive order. Then $(\zeta(x), \zeta(y), \zeta(z))$ is a triple of pairwise transverse n-planes, and its type only depends on the connected component of the representation variety containing ρ. This follows from the fact that the triple depends continuously on ρ and the type of a triple depends continuously on the triple, so the type of $(\zeta(x), \zeta(y), \zeta(z))$ is constant on connected components of the representation variety. Further, it is independent of the triple (x, y, z), since the space of triples of pairwise distinct points in positive order is connected.

Example 8.27. We want to understand what types of triples can appear for a maximal representation $\rho\colon \pi_1(S) \to \mathrm{Sp}(2n,\mathbb{R}) \subset \mathrm{SL}(2n,\mathbb{R})$ with boundary map $\zeta\colon \partial_\infty\tilde{S} \to \mathrm{Gr}_n(2n)$ (see Subsection 2.4.4). In this case, ζ maps into the *Lagrangians* $\mathcal{L}(\mathbb{R}^{2n}) \subset \mathrm{Gr}_n(2n)$, i.e. n-dimensional subspaces on which the symplectic form ω vanishes. To a triple of Lagrangians, one can associate the *Maslov index* which takes values in $\{-n, -n+2, \ldots, n-2, n\}$ (see [BILW05, Section 8.1]). The group $\mathrm{Sp}(2n,\mathbb{R})$ acts transitively on triples of Lagrangians with the same Maslov index. If $x, y, z \in \partial_\infty\tilde{S}$ are distinct points in the boundary, then the Maslov index of $(\zeta(x), \zeta(y), \zeta(x))$ is n if the triple (x, y, z) is positively ordered and $-n$ if it is negatively ordered by [BILW05, Lemma 8.3]. Consider the triple $(\langle e_1\rangle, \langle e_2\rangle, \langle e_1 - e_2\rangle)$ in $\mathbb{R}P^1$. It is positively ordered.
Since Hitchin representations into $\mathrm{Sp}(2n,\mathbb{R})$ are maximal it follows that the triple of Lagrangians $\left(j_{2n}^+(\langle e_1\rangle), j_{2n}^+(\langle e_2\rangle), j_{2n}^+(\langle e_1 - e_2\rangle)\right)$ has Maslov index n. Here, $j_{2n}\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Sp}(2n,\mathbb{R})$ denotes the irreducible representation introduced in Subsection 2.4.2 and j_{2n}^+ the

induced map on flags.

Note that we only care about the n-dimensional parts of the flags and forget the others, and that $j_{2n}^{+}(\langle e_1 \rangle) = P_1$ and $j_{2n}^{+}(\langle e_2 \rangle) = P_2$. Analogously, the triple $(P_1, P_2, j_{2n}^{+}(\langle e_1 + e_2 \rangle))$ has Maslov index $-n$. Now let (x, y, z) be a positively oriented triple in $\partial_\infty \tilde{S}$. Then, since ζ is maximal, the triple $(\zeta(x), \zeta(y), \zeta(z))$ has Maslov index n. As $\mathrm{Sp}(2n, \mathbb{R})$ acts transitively on triples with the same Maslov index, we know that $(\zeta(x), \zeta(y), \zeta(z))$ is of the same type as $(P_1, P_2, j_{2n}^{+}(\langle e_1 - e_2 \rangle))$. Analogously, if (x, y, z) is negatively oriented, then $(\zeta(x), \zeta(y), \zeta(z))$ is of the same type as $(P_1, P_2, j_{2n}^{+}(\langle e_1 + e_2 \rangle))$.

For $\mathrm{Sp}(4, \mathbb{R})$, one computes that both triples $(P_1, P_2, j_{4}^{+}(\langle e_1 \pm e_2 \rangle))$ are of positive type. In particular, if $\rho\colon \pi_1(S) \to \mathrm{Sp}(4, \mathbb{R})$ is maximal, then for any triple (x, y, z) in $\partial_\infty \tilde{S}$, the triple $(\zeta(x), \zeta(y), \zeta(z))$ is of positive type.

Let $\mathrm{Gr}_n(2n)^{P_1, P_2, P_3^{\pm}}$ be the set of all elements in $\mathrm{Gr}_n(2n)$ that are transverse to P_1, P_2 and $P_3^{\pm}$, i.e.

$$\mathrm{Gr}_n(2n)^{P_1, P_2, P_3^{\pm}} := \left\{ P \in \mathrm{Gr}_n(2n) \;\middle|\; P \oplus P_i = \mathbb{R}^{2n} \text{ for } i = 1, 2,\ P \oplus P_3^{\pm} = \mathbb{R}^{2n} \right\}.$$

Remark 8.28. We have seen in Remark 8.22 that every $P \in \mathrm{Gr}_n(2n)^{P_1, P_2}$ can be identified with a matrix $X_P \in \mathrm{GL}(n, \mathbb{R})$. Transversality to P_3^{+} and P_3^{-} gives additional conditions on the determinant, namely $\mathrm{Gr}_n(2n)^{P_1, P_2, P_3^{+}}$ can be identified with the set of all matrices $X \in \mathrm{GL}_n(\mathbb{R})$ satisfying $\det(X - \mathrm{Id}_n) \neq 0$ and $\mathrm{Gr}_n(2n)^{P_1, P_2, P_3^{-}}$ can be identified with the set of all matrices $X \in \mathrm{GL}_n(\mathbb{R})$ satisfying $\det(X - J) \neq 0$. This can be seen from the fact that the $(2n \times n)$-matrix from (8.9) and the columns of $(\mathrm{Id}_n, \mathrm{Id}_n)^T$ for the case of P_3^{+} and $(\mathrm{Id}_n, J)^T$ for the case of P_3^{-} have to be linearly independent.

We now examine how $\mathrm{Stab}_{\mathrm{SL}(2n, \mathbb{R})}(P_1, P_2, P_3^{+})$ acts on $\mathrm{Gr}_n(2n)^{P_1, P_2, P_3^{+}}$.

Lemma 8.29. *Let* $M = \mathrm{diag}(A, A) \in \mathrm{Stab}_{\mathrm{SL}(2n, \mathbb{R})}(P_1, P_2, P_3^{+})$ *with* $A \in \mathrm{GL}(n, \mathbb{R})$, $P \in \mathrm{Gr}_n(2n)^{P_1, P_2, P_3^{+}}$ *and let* $X_P \in \mathrm{GL}(4, \mathbb{R})$ *as in Remark 8.22. Then the matrix representing* $M \cdot P$ *is* AX_PA^{-1}. *Vice versa, for any* $A \in \mathrm{GL}(n, \mathbb{R})$, AX_PA^{-1} *represents an element in* $\mathrm{Gr}_n(2n)^{P_1, P_2, P_3^{+}}$ *that lies in the same* $\mathrm{SL}(2n, \mathbb{R})$*-orbit as* P.

In other words, the $\mathrm{Stab}_{\mathrm{SL}(2n, \mathbb{R})}(P_1, P_2, P_3^{+})$-orbits of $\mathrm{Gr}_n(2n)^{P_1, P_2, P_3^{+}}$ can be identified with conjugacy classes of matrices in $\mathrm{GL}(n, \mathbb{R})$.

Proof. Use the notation as in the lemma. The n-plane $M \cdot P$ is spanned by the columns of the $(2n \times n)$-matrix

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} \mathrm{Id}_n \\ X_P \end{pmatrix} = \begin{pmatrix} A \\ AX_P \end{pmatrix}.$$

By right-multiplying both blocks by A^{-1}, which is a change of basis for P, we see that P is spanned by the columns of the matrix $\begin{pmatrix} \mathrm{Id}_n \\ AX_PA^{-1} \end{pmatrix}$, which proves the first claim. Vice versa, if we consider the matrix AX_PA^{-1} for some $A \in \mathrm{GL}(n,\mathbb{R})$, then the matrix $\mathrm{diag}(A,A)$ is, after appropriate scaling, in $\mathrm{Stab}_{\mathrm{SL}(2n,\mathbb{R})}(P_1,P_2,P_3^+)$ and maps P to the n-plane defined by AX_PA^{-1}. □

An analogous result holds for the action of $\mathrm{Stab}_{\mathrm{SL}(2n,\mathbb{R})}(P_1,P_2,P_3^-)$ on $\mathrm{Gr}_n(2n)^{P_1,P_2,P_3^-}$.

Lemma 8.30. *Let $M = \mathrm{diag}(A,JA) \in \mathrm{Stab}_{\mathrm{SL}(2n,\mathbb{R})}(P_1,P_2,P_3^-)$ with $A \in \mathrm{GL}(n,\mathbb{R})$, $P \in \mathrm{Gr}_n(2n)^{P_1,P_2,P_3^+}$. Let $X_P \in \mathrm{GL}(n,\mathbb{R})$ such that JX_P represents the n-plane P as in Remark 8.22. Then the matrix representing $M \cdot P$ is JAX_PA^{-1}. Vice versa, for any $A \in \mathrm{GL}(n,\mathbb{R})$, JAX_PA^{-1} represents an element in $\mathrm{Gr}_n(2n)^{P_1,P_2,P_3^-}$ that lies in the same $\mathrm{Stab}_{\mathrm{SL}(2n,\mathbb{R})}(P_1,P_2,P_3^-)$-orbit as P.*

Proof. The proof is analogous to the proof of Lemma 8.29. □

Remark 8.31. If $P,P' \in \mathrm{Gr}_n(2n)^{P_1,P_2,P_3^+}$ are in the same $\mathrm{Stab}_{\mathrm{SL}(2n,\mathbb{R})}(P_1,P_2,P_3^+)$-orbit, then by Lemma 8.29, $X_{P'} = AX_PA^{-1}$ for some $A \in \mathrm{GL}(n,\mathbb{R})$. If ℓ_i is an eigenspace for X_P, then $A\ell_i$ is an eigenspace for $X_{P'}$. Similarly, if $P,P' \in \mathrm{Gr}_n(2n)^{P_1,P_2,P_3^-}$ lie in the same $\mathrm{Stab}_{\mathrm{SL}(2n,\mathbb{R})}(P_1,P_2,P_3^-)$-orbit, then $X_{P'} = J\left(AJX_PA^{-1}\right)$ for some $A \in \mathrm{GL}(n,\mathbb{R})$.

With this preliminary work on the action of $\mathrm{SL}(2n,\mathbb{R})$ on the space of quadruples of pairwise transverse n-planes, we can now give a sufficient condition for a quadruple (E,F,G_1,G_2) to be splitting admissible.

Lemma 8.32. *Let (E,F,G_1,G_2) be a quadruple of pairwise transverse elements in $\mathrm{Gr}_n(2n)$ such that either*

- *(E,F,G_1) is of positive type and (E,F,G_1,G_2) is in the $\mathrm{SL}(2n,\mathbb{R})$-orbit of a standard quadruple of the form (P_1,P_2,P_3^+,P) where $X_P \in \mathrm{GL}(n,\mathbb{R})$ is diagonalizable with distinct real eigenvalues, or*

- *(E,F,G_1) is of negative type and (E,F,G_1,G_2) is in the $\mathrm{SL}(2n,\mathbb{R})$-orbit of a standard quadruple (P_1,P_2,P_3^-,P), where $JX_P \in \mathrm{GL}(n,\mathbb{R})$ is diagonalizable with distinct real eigenvalues.*

Then (E,F,G_1,G_2) is splitting admissible.

Before we proof the Lemma, we make a remark.

Remark 8.33. The condition in Lemma 8.32 is well-defined, i.e. does not depend on the standard form $(P_1, P_2, P_3^{\pm}, P)$: If M and $M' \in \mathrm{SL}(2n,\mathbb{R})$ both bring (E, F, G_1, G_2) to standard form $(P_1, P_2, P_3^{\pm}, P)$ and $(P_1, P_2, P_3^{\pm}, P')$, respectively, then $(M')^{-1}M$ lies in the stabilizer of the triple $(P_1, P_2, P_3^{\pm})$ and $X_{P'} = AX_PA^{-1}$ for some $A \in \mathrm{GL}(n,\mathbb{R})$ by Lemma 8.29 if we consider quadruples of positive type. In particular, $X_{P'}$ is diagonalizable with distinct eigenvalues if and only if X_P is diagonalizable with distinct eigenvalues. The argument for quadruples of negative type is similar. Further, this shows that the condition in Lemma 8.32 is invariant under the $\mathrm{SL}(2n,\mathbb{R})$-action.

Proof of Lemma 8.32. First, we look at quadruples of the form (P_1, P_2, P_3^+, P), i.e. standard quadruples where the first triple is of positive type. We would like to construct a line splitting of $\mathbb{R}^n$ that is subordinate to the splitting given by the first two flags, which in this case simply is $P_1 \oplus P_2$. We first find n distinct lines in P_1. By assumption, X_P is diagonalizable with distinct eigenvalues. For $i = 1, \ldots, n$, let $\ell_i(P_1, P_2, P_3^+, P) \in \mathbb{R}P^{n-1}$ be the eigenline of X_P corresponding to the ith eigenvalue, where the eigenvalues are ordered by descending absolute value. They give a line splitting $\mathbb{R}^n = \bigoplus_{i=1}^n \ell_i(P_1, P_2, P_3^+, P)$. We can identify $\mathbb{R}^n$ with P_1 using the inclusion

$$\mathrm{incl}_{P_1} : \mathbb{R}^n \to \mathbb{R}^{2n}, \quad x \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix} \in P_1 \subset \mathbb{R}^{2n}. \tag{8.11}$$

In this way, we obtain a line splitting

$$P_1 = \bigoplus_{i=1}^n \mathrm{incl}_{P_1}(\ell_i(P_1, P_2, P_3^+, P)).$$

In the following, we will omit the identification through incl_{P_1} in the notation and interpret ℓ_i directly as a line in $P_1 \subset \mathbb{R}^{2n}$.

Analogously, if (P_1, P_2, P_3^-, P) is a standard quadruple where the first triple is of negative type, then the eigenspaces for JX_P define a line splitting of $\mathbb{R}^n$ which under the identification of $\mathbb{R}^n$ with P_1 as in (8.11) gives a line splitting of P_1. Since the eigenspaces of a matrix depend smoothly on the matrix entries if the matrix remains diagonalizable with distinct eigenvalues, the lines $\ell_i(P_1, P_2, P_3^+, P)$ depend smoothly on the quadruple.

Now look at a general quadruple (E, F, G_1, G_2) satisfying the condition from Lemma 8.32. Let $M \in \mathrm{SL}(2n,\mathbb{R})$ be such that $M \cdot (E, F, G_1, G_2) = (P_1, P_2, P_3^{\pm}, P)$, i.e. M brings the quadruple in standard position. Such an M exists because $\mathrm{SL}(2n,\mathbb{R})$ acts transitively on triples of positive and negative type by Corollary 8.24. By what we have shown above, we have a line splitting $P_1 = \bigoplus_{i=1}^n \ell_i(P_1, P_2, P_3^{\pm}, P)$. Applying M^{-1}, we obtain a line splitting of E by

$$E = \bigoplus_{i=1}^{n} M^{-1}\ell_i(P_1, P_2, P_3^{\pm}, P).$$

We set $\ell_i(E, F, G_1, G_2) := M^{-1}\ell_i(P_1, P_2, P_3^{\pm}, P)$ for $i = 1, \dots, n$. This splitting does not depend on the choice of M by Remark 8.31. So indeed, the splitting of E is independent of the choice of M. To see the regularity, notice that if we vary the quadruple, the matrix M varies smoothly with it. In total, the lines $\ell_i(E, F, G_1, G_2)$ depend smoothly on the quadruple. If $j = n + i > n$, set

$$\ell_j(E, F, G_1, G_2) := \ell_{n-i+1}(F, E, G_2, G_1).$$

Then $F = \bigoplus_{j=n+1}^{2n} \ell_j(E, F, G_1, G_2)$, and the lines $\ell_j(E, F, G_1, G_2)$ depend smoothly on the quadruple as seen above. In total, this gives a line splitting of $\mathbb{R}^{2n}$ that is subordinate to the splitting $\mathbb{R}^{2n} = V_1 \oplus V_2$, since $V_1 = E$ and $V_2 = F$. The behavior under permutation as well as equivariance are satisfied by construction. In total, the quadruple (E, F, G_1, G_2) is splitting admissible. $\square$

From Lemma 8.32, we can deduce a sufficient condition for an $\{\alpha_n\}$-Anosov representation to be λ-admissible.

Corollary 8.34. *Let $\rho\colon \pi_1(S) \to \mathrm{SL}(2n,\mathbb{R})$ be an $\{\alpha_n\}$-Anosov representation such that $\rho(\gamma_c)$ is diagonalizable with distinct eigenvalues for every closed leaf γ_c of λ and that for every isolated leaf g of $\tilde{S}$, the quadruple $\mathcal{Q}_\rho(g)$ satisfies the condition in Lemma 8.32. Then ρ is λ-admissible and Theorem 8.13 holds, i.e. there exists a cataclysm deformation $\widehat{\Lambda}_0$ with parameter space $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a})$.*

By similar arguments as in Lemma 8.17, the set of representations satisfying the prerequisites of Corollary 8.34 is open in all $\{\alpha_n\}$-Anosov representations.

Corollary 8.34 increases the parameter space for λ-admissible representations to the space $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a})$, which is significantly bigger than $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_{\{\alpha_n\}})$. In particular, the dimension of $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_{\{\alpha_n\}})$ is independent on n, whereas the dimension of $\mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a})$ grows linearly in n by Corollary 3.22.

A. Appendix

In the appendix, we collect technical proofs and results that we need in the thesis, but that would not add significant value when presented in the main part.

A.1. An almost-right invariant distance on G

In this section we introduce a distance on the Lie group G that we use for computations. This distance is induced by a Riemannian metric, that itself is induced by an inner product on the Lie group $\mathfrak{g}$.

Remember that a distance $\mathrm{d}\colon G \times G \to \mathbb{R}$ on G is *left-invariant* respectively *right-invariant* if for all $a_1, a_2, b \in G$,

$$\mathrm{d}(ba_1, ba_2) = \mathrm{d}(a_1, a_2) \quad \text{respectively} \quad \mathrm{d}(a_1 b, a_2 b) = \mathrm{d}(a_1, a_2).$$

A distance is *bi-invariant* if it is both left- and right-invariant. It is *conjugation-invariant* if $\mathrm{d}(ba_1b^{-1}, ba_2b^{-1}) = \mathrm{d}(a_1, a_2)$ holds for all $a_1, a_2, b \in G$. Bi-invariance implies conjugation-invariance. In general, a bi-invariant distance on G does not exist.

This is why we introduce a weaker requirement.

Definition A.1. A distance $\mathrm{d}_G\colon G \times G \to \mathbb{R}$ on G is *almost right-invariant* if there exists a continuous function $f\colon G \to \mathbb{R}_{>0}$ such that for all $a_1, a_2, b \in G$, we have

$$\mathrm{d}_G(a_1 b, a_2 b) \leq f(b) \mathrm{d}_G(a_1, a_2).$$

Similarly, one can define what it means for a distance d_G to be *almost conjugation-invariant.*

Remember that an inner product on $\mathfrak{g}$ induces a complete left-invariant Riemannian metric on G. Fix such an inner product on $\mathfrak{g}$, let $\|\cdot\|_{\mathfrak{g}}$ be the norm induced by this inner product and let $\|\cdot\|_{\mathrm{op}(\mathfrak{g})}$ be the operator norm induced by $\|\cdot\|_{\mathfrak{g}}$. Remember that $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ is the adjoint representation, i.e. for $b \in G$, Ad_b is the derivative of the conjugation $a \mapsto bab^{-1}$ at the identity.

Lemma A.2 ([SS01, Lemma A]). *Let G be a connected Lie group. The distance d_G induced by the Riemannian metric on G is left-invariant, almost conjugation-invariant and almost right-invariant. More precisely, for every $a_1, a_2, b \in G$ we have*

$$\mathrm{d}_G(ba_1b^{-1}, ba_2b^{-1}) \leq \|\mathrm{Ad}_b\|_{\mathrm{op}(\mathfrak{g})} \mathrm{d}_G(a_1, a_2) \quad \text{and}$$
$$\mathrm{d}_G(a_1b, a_2b) \leq \|\mathrm{Ad}_{b^{-1}}\|_{\mathrm{op}(\mathfrak{g})} \mathrm{d}_G(a_1, a_2).$$

Proof. The left-invariance is a direct consequence of the left-invariance of the Riemannian metric. The almost right-invariance follows from left-invariance and almost conjugation-invariance. Hence, we show that the d_G is almost conjugation invariant, i.e. that

$$\mathrm{d}_G(ba_1b^{-1}, ba_2b^{-1}) \leq \|\mathrm{Ad}_b\|_{\mathrm{op}(\mathfrak{g})} \mathrm{d}_G(a_1, a_2).$$

Let $a_1, a_2 \in G$ and $c\colon [0,1] \to G$ be a differentiable curve from a_1 to a_2. Then bcb^{-1} is a differentiable curve from ba_1b^{-1} to ba_2b^{-1}, and for every $t \in [0,1]$, $(\dot{bc(t)b^{-1}}) = \mathrm{Ad}_b\dot{c}(t)$. Thus

$$\begin{aligned}
\ell(bcb^{-1}) &= \int_0^1 \left\|(\dot{bc(t)b^{-1}})\right\|_{\mathfrak{g}} \mathrm{d}t \\
&= \int_0^1 \|\mathrm{Ad}_b\dot{c}(t)\|_{\mathfrak{g}} \mathrm{d}t \\
&\leq \int_0^1 \|\mathrm{Ad}_b\|_{\mathrm{op}(\mathfrak{g})} \|\dot{c}(t)\|_{\mathfrak{g}} \mathrm{d}t \\
&= \|\mathrm{Ad}_b\|_{\mathrm{op}(\mathfrak{g})} \ell(c).
\end{aligned}$$

Since the distance between two points is given as the infimum of the length of piecewise differentiable curves from a_1 to a_2, it follows that d_G is almost conjugation-invariant. $\square$

We now want to estimate the factor $\|\mathrm{Ad}_{b^{-1}}\|_{\mathrm{op}(\mathfrak{g})}$ for a specific choice of $b \in G$. To do so, we define a norm on $\mathfrak{a}$ as follows: For $H \in \mathfrak{a}$,

$$\|H\|_{\mathfrak{a}} := \max_{\alpha \in \Delta} |\alpha(H)|. \tag{A.1}$$

It is easy to check that this is indeed a norm. Note that this norm does in general not agree with the restriction of the norm $\|\cdot\|_{\mathfrak{g}}$ to $\mathfrak{a}$.

Lemma A.3. *There exists a constant $C > 0$ such that for all $H \in \mathfrak{a}$, we have*

$$\left\|\mathrm{Ad}_{\exp(H)}\right\|_{\mathrm{op}(\mathfrak{g})} \leq Ce^{\|H\|_{\mathfrak{a}}}.$$

Proof. For computational purposes, we consider a norm on $\mathrm{GL}(\mathfrak{g})$ different from $\|\cdot\|_{\mathrm{op}(\mathfrak{g})}$. Recall the decomposition of $\mathfrak{g}$ into root spaces

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha$$

and choose a basis of $\mathfrak{g}$ that is adapted to this decomposition. With respect to this basis, we can identify $\mathfrak{gl}(\mathfrak{g})$ with quadratic matrices. Consider the infinity norm $\|\cdot\|_\infty$ on $\mathfrak{gl}(\mathfrak{g})$ that is given by the maximal absolute row sum of the matrix. It restricts to a norm on $\mathrm{GL}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$. Let $H \in \mathfrak{a}$, and $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ be the adjoint representation on the level of Lie algebras. With respect to the chosen basis, $\mathrm{ad}_H \in \mathfrak{gl}(\mathfrak{g})$ is a diagonal matrix with entries 0 and $\alpha_i(H)$, where $\Sigma = \{\alpha_1, \dots, \alpha_l\}$, with multiplicity if the root spaces are more than 1-dimensional. Thus, $\mathrm{Ad}_{\exp(H)} = \exp(\mathrm{ad}_H)$ is diagonal with entries 1 and $e^{\alpha_i(H)}$, $i = 1, \dots, l$, possibly with multiplicity. On a finite dimensional vector space, all norms are equivalent, so there exists a constant $C > 0$ such that $\|A\|_{\mathrm{op}(\mathfrak{g})} \leq C\,\|A\|_\infty$ for all $A \in \mathfrak{gl}(\mathfrak{g})$. In particular,

$$\left\|\mathrm{Ad}_{\exp(H)}\right\|_{\mathrm{op}(\mathfrak{g})} \leq C\left\|\mathrm{Ad}_{\exp(H)}\right\|_\infty = C\left\|\exp(\mathrm{ad}_H)\right\|_\infty \leq C\max_{\alpha\in\Sigma} e^{|\alpha(H)|} = Ce^{\|H\|_{\mathfrak{a}}}.$$

This proves the Lemma. □

A.2. Proof of additivity of the shearing maps

Here, we give the proof of additivity of the shearing maps (Proposition 5.15). We recall the statement for the reader's convenience.

Proposition A.4 (Proposition 5.15)**.** *Let P be the fixed reference component for the cataclysm deformation Λ_0. Then for $\eta, \varepsilon \in \mathcal{H}^{\mathrm{Twist}}(\widehat{\lambda}; \mathfrak{a}_\theta)$ small enough, for every component $Q \subset \tilde{S} \setminus \tilde{\lambda}$ it holds that*

$${\varphi'}^{\varepsilon}_{PQ} = \varphi^{\varepsilon+\eta}_{PQ}\left(\varphi^{\eta}_{PQ}\right)^{-1}.$$

The ideas in the proof are similar to the ones for the slithering and shearing maps.

Proof. Recall that the shearing map φ^ε_{PQ} is defined as $\lim_{\mathcal{P}\to\mathcal{C}_{PQ}} \varphi^\varepsilon_{\mathcal{C}}$, where $\mathcal{C} \subset \mathcal{C}_{PQ}$ is a finite subset and $\varphi^\varepsilon_{\mathcal{C}}$ is as in (5.3). We consider a specific sequence $(\mathcal{C}_r)_{r\in\mathbb{N}}$ going to $\mathcal{C}_{PQ}$. Namely, for $r \in \mathbb{N}$, let $\mathcal{C}_r := \{R \in \mathcal{C}_{PQ} \mid r(R) \leq r\}$ be the set of all components between P and Q that have divergence radius at most r. By Lemma 3.11, $\mathcal{C}_r$ is finite and there exists a constant D such that $n_r := |\mathcal{C}_r| \leq D \cdot r$. We use the same notation as in Section 5.1. In particular, $\psi^{\varepsilon}_{PQ} = \varphi^{\varepsilon}_{PQ} T^{-\varepsilon(P,Q)}_{g^0_Q}$. To compute ${\varphi'}^{\varepsilon}_{\mathcal{C}_r}$, we first look at a one-element subset $\{R\} \subset \mathcal{C}_{PQ}$. Using the relation between the stretching maps T^H_g and $T^{H'}_g$ from (5.10), the fact that for $H_1, H_2 \in \mathfrak{a}_\theta$ we have $T^{H_1}_g T^{H_2}_g = T^{H_1+H_2}_g$ and additivity of the cycle, we compute

$$
\begin{aligned}
{\varphi'}^{\varepsilon}_{\{R\}} &= \left({T'}^{\varepsilon(P,R)}_{g^0_R} {T'}^{-\varepsilon(P,R)}_{g^1_R}\right) {T'}^{\varepsilon(P,Q)}_{g^0_Q} \\
&= \varphi^{\eta}_{PR} \left(T^{\varepsilon(P,R)}_{g^0_R} T^{-\varepsilon(P,R)}_{g^1_R}\right) \left(\varphi^{\eta}_{PR}\right)^{-1} \varphi^{\eta}_{PQ} T^{\varepsilon(P,Q)}_{g^0_Q} \left(\varphi^{\eta}_{PQ}\right)^{-1} \\
&= \left(\psi^{\eta}_{PR} T^{\eta(P,R)}_{g^0_R}\right) \left(T^{\varepsilon(P,R)}_{g^0_R} T^{-\varepsilon(P,R)}_{g^1_R}\right) \varphi^{\eta}_{RQ} T^{\varepsilon(P,Q)}_{g^0_Q} \left(\varphi^{\eta}_{PQ}\right)^{-1} \\
&= \psi^{\eta}_{PR} \left(T^{\varepsilon+\eta(P,R)}_{g^0_R} T^{-(\varepsilon+\eta)(P,R)}_{g^1_R}\right) \left(T^{\eta(P,R)}_{g^1_R} \psi^{\eta}_{RQ} T^{\eta(R,Q)}_{g^0_Q}\right) T^{\varepsilon(P,Q)}_{g^0_Q} \left(\varphi^{\eta}_{PQ}\right)^{-1} \\
&= \psi^{\eta}_{PR} \left(T^{\varepsilon+\eta(P,R)}_{g^0_R} T^{-(\varepsilon+\eta)(P,R)}_{g^1_R}\right) \left(T^{\eta(P,R)}_{g^1_R} \psi^{\eta}_{RQ} T^{-\eta(P,R)}_{g^0_Q}\right) T^{\varepsilon+\eta(P,Q)}_{g^0_Q} \left(\varphi^{\eta}_{PQ}\right)^{-1}.
\end{aligned}
$$

In the last equality, we used that $\eta(R,Q) = \eta(P,Q) - \eta(P,R)$. Using the same techniques on the finite subset $\mathcal{C}_r = \{R_1, \ldots, R_{n_r}\} \subset \mathcal{C}_{PQ}$ we find

$$
\begin{aligned}
{\varphi'}^{\varepsilon}_{\mathcal{C}_r} &= \prod_{i=1}^{n_r} \left({T'}^{\varepsilon(P,R_i)}_{g^0_i} {T'}^{-\varepsilon(P,R_i)}_{g^1_i}\right) \quad {T'}^{\varepsilon(P,Q)}_{g^0_Q} \\
&= \psi^{\eta}_{PR_1} \prod_{i=1}^{n_r} \left(\left(T^{\varepsilon+\eta(P,R_i)}_{g^0_i} T^{-(\varepsilon+\eta)(P,R_i)}_{g^1_i}\right) \left(T^{\eta(P,R_i)}_{g^1_i} \psi^{\eta}_{R_iR_{i+1}} T^{-\eta(P,R_i)}_{g^0_{i+1}}\right)\right) \\
&\qquad T^{\varepsilon+\eta(P,Q)}_{g^0_Q} \left(\varphi^{\eta}_{PQ}\right)^{-1}.
\end{aligned}
$$

Set

$$
\widehat{\psi}^{\eta,\varepsilon}_{\mathcal{C}_r} := {\varphi'}^{\varepsilon}_{\mathcal{C}_r} \left(T^{\varepsilon+\eta(P,Q)}_{g^0_Q} \left(\varphi^{\eta}_{PQ}\right)^{-1}\right)^{-1}.
$$

We claim that $\lim_{r\to\infty} \widehat{\psi}^{\eta,\varepsilon}_{\mathcal{C}_r} = \psi^{\varepsilon+\eta}_{PQ}$. Recall that

$$
\psi^{\varepsilon+\eta}_{PQ} = \lim_{r\to\infty} \psi^{\varepsilon+\eta}_{\mathcal{C}_r} = \lim_{r\to\infty} \prod_{i=1}^{n_r} \left(T^{\varepsilon+\eta(P,R_i)}_{g^0_i} T^{-(\varepsilon+\eta)(P,R_i)}_{g^1_i}\right).
$$

This converges to $\psi^{\varepsilon+\eta}_{PQ}$ as r goes to infinity. Thus, it suffices to show that $\mathrm{d}_G(\psi^{\varepsilon+\eta}_{\mathcal{C}_r}, \widehat{\psi}^{\eta,\varepsilon}_{\mathcal{C}_r})$ tends to 0 as r goes to infinity.

Using the triangle inequality, left-invariance and almost-right invariance of the metric d_G, we have

$$\begin{aligned}
&\mathrm{d}_G(\psi^{\varepsilon+\eta}_{\mathcal{C}_r}, \widehat{\psi}^{\eta,\varepsilon}_{\mathcal{C}_r}) \\
&\quad = \mathrm{d}_G\left(\psi^{\varepsilon+\eta}_{\mathcal{C}_r}, \psi^{\eta}_{PR}\prod_{i=1}^{n_r}\left(\left(T^{\varepsilon+\eta(P,R_i)}_{g_i^0}T^{-(\varepsilon+\eta)(P,R_i)}_{g_i^1}\right)\left(T^{\eta(P,R_i)}_{g_i^1}\psi^{\eta}_{R_iR_{i+1}}T^{-\eta(P,R_i)}_{g_{i+1}^0}\right)\right)\right) \\
&\quad \leq \mathrm{d}_G\left(\psi^{\varepsilon+\eta}_{\mathcal{C}_r}, \psi^{\eta}_{PR}\psi^{\varepsilon+\eta}_{\mathcal{C}_r}\right) \\
&\qquad + \mathrm{d}_G\left(\psi^{\eta}_{PR}\psi^{\varepsilon+\eta}_{\mathcal{C}_r}, \psi^{\eta}_{PR}\prod_{i=1}^{n_r}\left(\left(T^{\varepsilon+\eta(P,R_i)}_{g_i^0}T^{-(\varepsilon+\eta)(P,R_i)}_{g_i^1}\right)\left(T^{\eta(P,R_i)}_{g_i^1}\psi^{\eta}_{R_iR_{i+1}}T^{-\eta(P,R_i)}_{g_{i+1}^0}\right)\right)\right) \\
&\quad \leq \left\|\mathrm{Ad}_{\left(\psi^{\varepsilon+\eta}_{\mathcal{C}_r}\right)^{-1}}\right\| \mathrm{d}_G(\mathrm{Id}, \psi^{\eta}_{PR_1}) \\
&\qquad + \mathrm{d}_G\left(\prod_{i=2}^{n_r}\left(T^{\varepsilon+\eta(P,R_i)}_{g_i^0}T^{-(\varepsilon+\eta)(P,R_i)}_{g_i^1}\right),\right. \\
&\qquad\quad \left.\prod_{i=2}^{n_r}\left(\left(T^{\varepsilon+\eta(P,R_i)}_{g_i^0}T^{-(\varepsilon+\eta)(P,R_i)}_{g_i^1}\right)\left(T^{\eta(P,R_i)}_{g_i^1}\psi^{\eta}_{R_iR_{i+1}}T^{-\eta(P,R_i)}_{g_{i+1}^0}\right)\right)\right)
\end{aligned}$$

Iterating these estimates gives

$$\mathrm{d}_G(\psi^{\varepsilon+\eta}_{\mathcal{C}_r}, \widehat{\psi}^{\eta,\varepsilon}_{\mathcal{C}_r}) \quad \leq \sum_{i=0}^{n_r}\left\|\mathrm{Ad}_{\left(\psi^{\varepsilon+\eta}_{\mathcal{C}_r\setminus\{R_1,\ldots,R_i\}}\right)^{-1}}\right\| \mathrm{d}_G\left(\mathrm{Id}, \left(T^{\eta(P,R_i)}_{g_i^1}\psi^{\eta}_{R_iR_{i+1}}T^{-\eta(P,R_i)}_{g_{i+1}^0}\right)\right). \tag{A.2}$$

By Lemma 5.2, $\mathrm{d}_G(\mathrm{Id}, \psi^{\varepsilon+\eta}_{\mathcal{C}})$ is uniformly bounded by a constant C_0 for every finite subset $\mathcal{C} \subset \mathcal{C}_{PQ}$, with C_0 depending on k and ρ. Hence, also the factors $\left\|\mathrm{Ad}_{\left(\psi^{\varepsilon+\eta}_{\mathcal{C}_r\setminus\{R_1,\ldots,R_i\}}\right)^{-1}}\right\|$ are uniformly bounded. Further, we have for every i, using that the norm d_G is almost-conjugation invariant,

$$\begin{aligned}
&\mathrm{d}_G\left(\mathrm{Id}, T^{\eta(P,R_i)}_{g_i^1}\psi^{\eta}_{R_iR_{i+1}}T^{-\eta(P,R_i)}_{g_{i+1}^0}\right) \\
&\quad \leq \left\|\mathrm{Ad}_{\left(T^{-\eta(P,R_i)}_{g_{i+1}^0}\right)}\right\| \mathrm{d}_G\left(\mathrm{Id}, T^{-\eta(P,R_i)}_{g_{i+1}^0}T^{\eta(P,R_i)}_{g_i^1}\psi^{\eta}_{R_iR_{i+1}}\right) \\
&\quad \leq \left\|\mathrm{Ad}_{\left(T^{-\eta(P,R_i)}_{g_{i+1}^0}\right)}\right\| \left(\mathrm{d}_G\left(\mathrm{Id}, T^{-\eta(P,R_i)}_{g_{i+1}^0}T^{\eta(P,R_i)}_{g_i^1}\right) + \mathrm{d}_G\left(\mathrm{Id}, \psi^{\eta}_{R_iR_{i+1}}\right)\right).
\end{aligned} \tag{A.3}$$

Combining (A.2) and (A.3), we obtain

$$\mathrm{d}_G(\psi^{\varepsilon+\eta}_{\mathcal{C}_r}, \widehat{\psi}^{\eta,\varepsilon}_{\mathcal{C}_r}) \leq C_0\sum_{i=0}^{n_r}\left\|\mathrm{Ad}_{\left(T^{-\eta(P,R_i)}_{g_{i+1}^0}\right)}\right\| \left(\mathrm{d}_G\left(\mathrm{Id}, T^{-\eta(P,R_i)}_{g_{i+1}^0}T^{\eta(P,R_i)}_{g_i^1}\right) + \mathrm{d}_G\left(\mathrm{Id}, \psi^{\eta}_{R_iR_{i+1}}\right)\right). \tag{A.4}$$

We estimate the terms separately. By definition, $T^{-\eta(P,R_i)}_{g_{i+1}^0} = m_{g_{i+1}^0}\exp(\eta(P,R_i))m^{-1}_{g_{i+1}^0}$, so

$$\mathrm{Ad}_{\left(T^{-\eta(P,R_i)}_{g_{i+1}^0}\right)} = \mathrm{Ad}_{m_{g_{i+1}^0}}\mathrm{Ad}_{\exp(\eta(P,R_i))}\mathrm{Ad}_{m^{-1}_{g_{i+1}^0}}.$$

The oriented geodesics g_{i+1}^0 lie in a compact set depending on k, hence there exists a bound on

$\left\|\mathrm{Ad}_{m_{g^0_{i+1}}}\right\|$ and $\left\|\mathrm{Ad}_{m^{-1}_{g^0_{i+1}}}\right\|$ depending on k and ρ only. By Lemma A.3 and Lemma 3.23, there exist constants C_1, B such that

$$\begin{aligned}\left\|\mathrm{Ad}_{\left(T^{-\eta(P,R_i)}_{g^0_{i+1}}\right)}\right\| &\leq \left\|\mathrm{Ad}_{m_{g^0_{i+1}}}\right\| \left\|\mathrm{Ad}_{\exp(\eta(P,R_i))}\right\| \left\|\mathrm{Ad}_{m^{-1}_{g^0_{i+1}}}\right\| \\ &\leq C_1 e^{\|\eta(P,R_i)\|_{\mathfrak{a}_\theta}} \\ &\leq C_1 e^{C\|\eta\|(r(R_i)+1)}. \end{aligned} \tag{A.5}$$

By definition of $\mathcal{C}_r$, all components between R_i and R_{i+1} have divergence radius at most $r+1$, so we can apply Lemma 5.8 and obtain constants C_2, A' such that

$$\mathrm{d}_G(\mathrm{Id}, \psi^\eta_{R_i R_{i+1}}) \leq C_2 e^{-A'r}. \tag{A.6}$$

Finally, with Proposition 4.18, there exists constants C_3, A_1 such that

$$\begin{aligned}\mathrm{d}_G\left(\mathrm{Id}, T^{-\eta(P,R_i)}_{g^0_{i+1}} T^{\eta(P,R_i)}_{g^1_i}\right) &= \mathrm{d}_G\left(T^{-\eta(P,R_i)}_{g^0_{i+1}}, T^{\eta(P,R_i)}_{g^1_i}\right) \\ &\leq C_3\left(\left(e^{\|\eta(P,R_i)\|}+1\right)\mathrm{d}(g^0_{i+1}, g^1_i)^{A_1}\right).\end{aligned}$$

Note that we can apply Proposition 4.18, since for r big enough, R_i and R_{i+1} are separated by wedges. Using the same techniques as in the proof of Lemma 4.7, together with Lemma 3.23 , we find that there exist constants C_4, C and A such that

$$\mathrm{d}_G\left(\mathrm{Id}, T^{-\eta(P,R_i)}_{g^0_{i+1}} T^{\eta(P,R_i)}_{g^1_i}\right) \leq C_4\left(e^{C\|\eta\|(r(R_i)+1)}+1\right)e^{-Ar}. \tag{A.7}$$

Note that here, the constants A and A' from (A.6) satisfy $A' = -(C\|\eta\| - A)$ (see Lemma 5.8) and C is as in (A.5), coming from Lemma 3.23. Combining the estimates from (A.4) to (A.7), and using that $r(R_i) \leq r$ for all $i = 1, \dots, n_r$, it follows that

$$\begin{aligned}&\mathrm{d}_G(\psi^{\varepsilon+\eta}_{\mathcal{C}_r}, \widehat{\psi}^{\eta,\varepsilon}_{\mathcal{C}_r}) \\ &\quad\leq C_0 C_1 \sum_{i=1}^{n_r} e^{C\|\eta\|(r(R_i)+1)}\left(\left(e^{C\|\eta\|(r(R_i)+1)}+1\right)e^{-Ar} + C_2 e^{-A'r}\right) \\ &\quad\leq C_0 C_1 \sum_{i=1}^{n_r} e^{C\|\eta\|(r+1)}\left(\left(e^{C\|\eta\|(r+1)}+1\right)e^{-Ar} + C_2 e^{-A'r}\right) \\ &\quad\leq C_0 C_1 n_r \left(e^{C\|\eta\|(r+1)}\left(\left(e^{C\|\eta\|(r+1)}+1\right)e^{-Ar} + C_2 e^{-A'r}\right)\right) \\ &\quad\leq C_3 n_r e^{2C\|\eta\|(r+1)-A'r}\end{aligned}$$

Thus, for $\|\eta\|$ small enough,

$$\mathrm{d}_G(\psi^{\varepsilon+\eta}_{\mathcal{C}_r}, \widehat{\psi}^{\eta,\varepsilon}_{\mathcal{C}_r}) \leq Cre^{-A''r}$$

for constants C and A'', where we additionally use that $n_r \leq Dr$ for some constant D by Lemma 3.11. It follows that $\psi^{\varepsilon+\eta}_{\mathcal{C}_r}$ and $\widehat{\psi}^{\eta,\varepsilon}_{\mathcal{C}_r}$ have the same limit as r tends to infinity, so $\widehat{\psi}^{\eta,\varepsilon}_{\mathcal{C}_r}$ converges to $\psi^{\varepsilon+\eta}_{PQ}$.

As a consequence,we have

$$\begin{aligned}
{\varphi'}^{\varepsilon}_{PQ} &= \lim_{r\to\infty} {\varphi'}^{\varepsilon}_{\mathcal{C}_r} \\
&= \lim_{r\to\infty} \widehat{\psi}^{\eta,\varepsilon}_{\mathcal{C}_r} T^{(\varepsilon+\eta)(P,Q)}_{g^0_Q} \left(\varphi^{\eta}_{PQ}\right)^{-1} \\
&= \psi^{\varepsilon+\eta}_{PQ} T^{(\varepsilon+\eta)(P,Q)}_{g^0_Q} \left(\varphi^{\eta}_{PQ}\right)^{-1} \\
&= \varphi^{\varepsilon+\eta}_{PQ} \left(\varphi^{\eta}_{PQ}\right)^{-1},
\end{aligned}$$

which finishes the proof.

□

A.3. Convergence of a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in $\pi_1(S)$

In the proof of Theorem 6.1, we look at a non-isolated oriented leaf g of the lamination $\tilde{\lambda}$ and a sequence $(\gamma_n)_{n\in\mathbb{N}}$ such that the sequence of leaves $(\gamma_n \cdot g)_{n\in\mathbb{N}}$ converges to g. We want to show that, as a sequence in $\pi_1(S)$, $(\gamma_n)_{n\in\mathbb{N}}$ converges to $g^+ \in \partial_\infty \tilde{S} \cong \partial\pi_1(S)$.

Lemma A.5. *Let g be an oriented geodesic in $\tilde{S}$ with endpoints g^+ and g^-, and $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty}(\gamma_n \cdot g^+) = g^+$ and $\lim_{n\to\infty}(\gamma_n \cdot g^-) = g^-$. Then $\lim_{n\to\infty}\gamma_n = g^+$ in the sense that the sequence $(\gamma_n)_{n\in\mathbb{N}}$ and the quasi-geodesic ray defining g^+ have bounded distance. Likewise, $\lim_{n\to\infty}\gamma_n^{-1} = g-$.*

Proof. The proof consists of several steps. In the first step, we show that the axes of the elements γ_n converge to g. In the second step, we show that for a fixed basepoint o on g, the sequence $(\gamma_n \cdot o)_{n\in\mathbb{N}}$ is a conical sequence in the sense that it lies within a K-neighborhood of the geodesic g for some $K > 0$. It then follows that $\lim_{n\to\infty}\gamma_n = g^+$. Lastly, we conclude that also $\lim_{n\to\infty}\gamma_n^{-1} = g^-$.

For $n \in \mathbb{N}$, let t_n be the translation length of γ_n. Then t_n goes to infinity as n tends to ∞. To see this, let o be a base point on g, and let y_n be the point on the axis of γ_n that is closest to g. If g and the axis of γ_n intersect, then y_n lies on g. Assume for a contradiction that $t_n < t$ for some $t < \infty$. Then, by definition of the translation length, $t_n = \mathrm{d}(y_n, \gamma_n \cdot y_n)$, so

$$\mathrm{d}(o, \gamma_n \cdot o) \leq \mathrm{d}(o, y_n) + \mathrm{d}(y_n, \gamma_n \cdot y_n) + \mathrm{d}(\gamma_n \cdot y_n, \gamma_n \cdot o) = t_n + 2\mathrm{d}(o, y_n).$$

If $\mathrm{d}(o, y_n)$ is unbounded, then the axes of the elements γ_n collapse to a point in the boundary. Since we know that $\gamma_n \cdot g$ converges to g, it follows that the translation lengths t_n converge to 0. Thus, there is a compact neighborhood of o containing infinitely many elements $\gamma_n \cdot o$, contradicting the fact that $\pi_1(S)$ acts properly discontinuously. Hence, $\mathrm{d}(o, y_n) < C$ for some C, so $\mathrm{d}(o, \gamma_n \cdot o) < t + 2C$ for infinitely many n, again contradicting the fact that $\pi_1(S)$ acts properly discontinuously. It follows that the sequence $(t_n)_{n\in\mathbb{N}}$ is unbounded.

Let γ_n^+ and γ_n^- be the attracting and repelling fixed points of γ_n in $\partial_\infty \tilde{S}$, respectively. Up to a subsequence, (γ_n^+, γ_n^-) converges to a pair $(x^+, x^-) \in \left(\partial_\infty \tilde{S}\right)^2$. Since by assumption, $\lim_{n\to\infty} \gamma_n \cdot g = g^+$, and the translations lengths do not go to 0, it follows that $x^+ = g^+$. Further, $x^- = g^-$, because if we had $x^- \neq g^-$, then, since the translations lengths t_n go to infinity, $\lim_{n\to\infty} \gamma_n \cdot g^- = g^+$, which is a contradiction. Hence, it follows that $\lim_{n\to\infty}(\gamma_n^+, \gamma_n^-) = (g^+, g^-)$, so the axes of γ_n converge to g. This finishes the first step of the proof.

For $n \in \mathbb{N}$, let d_n be the distance between g and the axis of γ_n. By computations in the upper half plane model, one sees that the fact that $\gamma_n \cdot g$ converges to g gives an explicit asymptotic relation between the d_n and the translation length t_n, namely $d_n < e^{-t_n}$ for large n. If the axis of γ_n and g intersect with angle ϕ_n, then similar arguments show that $\phi_n < e^{-t_n}$. Using this asymptotic relation, we can compute $\gamma_n \cdot o$ and see that it lies in a K-neighborhood of the geodesic g for some K. Thus, the sequence $(\gamma_n \cdot o)_{n\in\mathbb{N}}$ is conical.

Let $(\alpha_n)_{n\in\mathbb{N}}$ be a geodesic in (the Cayley graph of) $\pi_1(S)$ with limit g^+. Then the sequence $(\alpha_n \cdot o)_{n\in\mathbb{N}}$ lies on a geodesic ray in $\tilde{S}$ with limit g^+. Since the sequence $(\gamma_n \cdot o)_{n\in\mathbb{N}}$ is conical, for all $n \in \mathbb{N}$, there exists $m_n \in \mathbb{N}$ such that $\mathrm{d}(\gamma_n \cdot o, \alpha_{m_n} \cdot o) \leq K'$ for some K'. Up to taking a subsequence, we can assume that m_n is strictly increasing in n. As the orbit map $\pi_1(S) \to \tilde{S}$ is a quasi-isometry, it follows that $\mathrm{d}(\gamma_n, \alpha_{m_n}) \leq K''$ for some K'', where now the distance is in $\pi_1(S)$. Since two sequences in $\pi_1(S)$ that remain within bounded distance define the same point in the boundary, it follows that $\lim_{n\to\infty} \gamma_n = \lim_{m\to\infty} \alpha_m = g^+$.

To conclude the proof, note that for every n, γ_n^{-1} and γ_n have the same axis and the same translation length. This implies that the sequence of geodesics $(\gamma_n^{-1} \cdot g)_{n\in\mathbb{N}}$ converges to g, and with the same arguments as above, we conclude that $((\gamma_n)^{-1} \cdot o)_{n\in\mathbb{N}}$ is conical and that $\lim_{n\to\infty} \gamma_n^{-1} = g^-$. □

A.4. Proof of Lemma 8.11

In Section 8.1, we define full-stretching maps $\widehat{T}_g^H$. For a distance estimate between two full-stretching maps (Lemma 8.12), we need the auxiliary Lemma 8.11. In this section, we prove Lemma 8.11. We recall the statement for the reader's convenience. For the notation, we refer back to Section 8.1.

Lemma A.6 (Lemma 8.11). *Let $R \in \mathcal{C}_{gh}$ be a pinched inner component. Then there exists an element $m_{g_R^0 g_R^1} \in \mathrm{SL}(n, \mathbb{R})$ sending the line splitting for g_R^1 to the line splitting for g_R^0, and constants $C, A > 0$ depending on $\tilde{k}$ and ρ such that*

$$\mathrm{d}_{\mathrm{SL}(n,\mathbb{R})}(\mathrm{Id}, m_{g_R^0 g_R^1}) \leq Ce^{-Ar(R)}.$$

Proof. As in the proof of Proposition 4.18, up to conjugating ρ by an element in $\mathrm{SL}(n, \mathbb{R})$, we can assume that the splitting associated to g_R^0 is the standard splitting given by the basis vectors $e_1, \ldots, e_n$. In this case, $m_{g_R^0 g_R^1}$ is an element sending the splitting given by g_R^1 to the standard line splitting.

The proof consists of two steps. In the first step, we relate $\mathrm{d}_{\mathrm{SL}(n,\mathbb{R})}(\cdot, \cdot)$ to a matrix norm. In the second step, we construct $m_{g_R^0 g_R^1}$ and estimate its distance to Id with respect to this matrix norm.

Let $\|\cdot\|_1$ be the maximal column sum norm on $\mathrm{Mat}_{n\times n}(\mathbb{R})$ which is defined as $\|A\|_1 := \max_{j=1,\ldots,n} \sum_{i=1}^n |a_{ij}|$ for $A = (a_{ij})_{i,j} \in \mathrm{Mat}_{n\times n}(\mathbb{R})$. Let V be a normal neighborhood of the identity Id in $\mathrm{SL}(n, \mathbb{R})$. Then there exists a neighborhood $U \subset \mathfrak{sl}(n, \mathbb{R}) = T_{\mathrm{Id}}\mathrm{SL}(n, \mathbb{R})$ of $0 \in \mathfrak{g}$ such that $\exp|_U : U \to V$ is an isomorphism and geodesics in G are images of straight lines in V. For every $m \in V \subset \mathrm{SL}(n, \mathbb{R})$, we have $\mathrm{d}_{\mathrm{SL}(n,\mathbb{R})}(\mathrm{Id}, m) = \left\|\exp|_U^{-1}(m)\right\|_{\mathfrak{sl}(n,\mathbb{R})}$. The Lie group exponential is smooth, so in particular locally Lipschitz. By equivalence of norms, since $\mathfrak{sl}(n, \mathbb{R}) \cong \mathrm{Mat}_{n\times n}(\mathbb{R})$, there exists a constant $C_1 > 0$ such that

$$\mathrm{d}_{\mathrm{SL}(n,\mathbb{R})}(\mathrm{Id}, m) = \left\|(\exp|_U)^{-1}(\mathrm{Id}) - \exp|_U^{-1}(m)\right\|_{\mathfrak{sl}(n,\mathbb{R})} \leq C_1 \|\mathrm{Id} - m\|_1.$$

This finishes the first step.

For g_R^0 and g_R^1 sufficiently close, $m_{g_R^0 g_R^1}$ is close to Id, and we can assume that it lies in a normal neighborhood V. Hence, it is sufficient to show that $\left\|\mathrm{Id} - m_{g_R^0 g_R^1}\right\|_1 \leq Ce^{-Ar(R)}$ for constants $C, A > 0$ depending on $\tilde{k}$ and ρ.

Let m' be the matrix with columns $a_1, \ldots, a_n$, where $a_1, \ldots, a_n$ is a basis adapted to the line splitting from (8.3), chosen such that the angle $\measuredangle(e_i, a_i)$ is minimal and $\|a_i\|_2 = 1$. Then $m' \notin \mathrm{SL}(n, \mathbb{R})$ in general, we only know $m' \in \mathrm{GL}(n, \mathbb{R})$. Let m be the matrix with columns $a_1, \ldots, a_{n-1}, \frac{1}{\det(m')}a_n$, so m agrees with m' on the first $n-1$ columns and the last column is scaled by the inverse of the determinant of m'. Then m maps the standard line splitting to the line splitting for g_R^1 as in (8.3) and is an element of $\mathrm{SL}(n, \mathbb{R})$. Set $m_{g_R^0 g_R^1} := m^{-1}$. By left-invariance of the distance and with the considerations from above

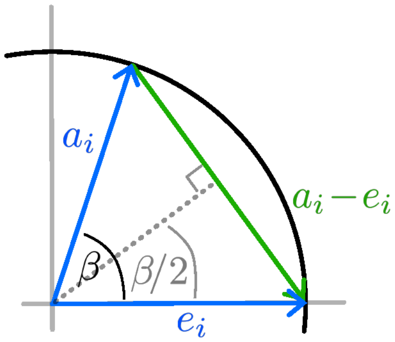


Figure A.1.: If $\|a_i\|_2 = 1$ and $\beta := \measuredangle(e_i, a_i)$, then the distance $\|a_i - e_i\|_2$ is given by $2\sin\left(\frac{\beta}{2}\right)$.

we have

$$\mathrm{d}_{\mathrm{SL}(n,\mathbb{R})}(\mathrm{Id}, m_{g_R^0 g_R^1}) = \mathrm{d}_{\mathrm{SL}(n,\mathbb{R})}(\mathrm{Id}, m) \leq C_1 \|\mathrm{Id} - m\|_1 \leq C_1 \left(\left\|\mathrm{Id} - m'\right\|_1 + \left\|m' - m\right\|_1\right).$$

We start with estimating $\|\mathrm{Id} - m'\|_1$. On $\mathbb{R}P^{n-1}$, consider the distance

$$\mathrm{d}_{\mathbb{R}P^{n-1}}(\langle v\rangle, \langle w\rangle) := |\sin\measuredangle(v, w)|.$$

Then, by definition of a_j, we have

$$\mathrm{d}_{\mathbb{R}P^{n-1}}(\ell_j(g_R^0), \ell_i(g_R^1)) = \mathrm{d}_{\mathbb{R}P^{n-1}}(\langle e_i\rangle, \langle a_i\rangle) = |\sin\measuredangle(e_i, a_i)|.$$

On $\mathbb{R}^n$, consider the norms $\|x\|_1 := \sum_{i=1}^n |x_i|$ and $\|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$. Since all norms on $\mathbb{R}^n$ are equivalent, there exists some $C > 0$ such that

$$\left\|\mathrm{Id} - m'\right\|_1 = \max_{i=1,\dots,n} \|e_i - a_i\|_1 \leq C \max_{i=1,\dots,n} \|e_i - a_i\|_2 . \tag{A.8}$$

By definition of a_i, basic trigonometry (Figure A.1) shows that

$$\|e_i - a_i\|_2 = 2\left|\sin\frac{\measuredangle(e_i, a_i)}{2}\right| = |\sin\measuredangle(e_i, a_i)| \left|\cos\frac{\measuredangle(e_i, a_i)}{2}\right|^{-1}.$$

By construction, $\measuredangle(e_i, a_i) \leq \frac{\pi}{2}$, so $0 \leq \frac{\measuredangle(e_i,a_i)}{2} \leq \frac{\pi}{4}$, which ensures that $\left|\cos\frac{\measuredangle(e_i,a_i)}{2}\right|^{-1}$ is bounded. In total, this shows that there is some constant $C > 0$ such that

$$\|e_i - a_i\|_2 \leq C|\sin\measuredangle(e_i, a_i)| = C\mathrm{d}_{\mathbb{R}P^{n-1}}(\ell_i(g_R^0), \ell_i(g_R^1) \leq C' e^{-Ar(R)}$$

by Lemma 8.10. Together with (A.8), this shows that $\|\mathrm{Id} - m'\|_1 \leq Ce^{-Ar(R)}$ for constants

$C, A > 0$. Further, we have by definition of $\left\|\cdot\right\|_1$, m and m',

$$\left\|m' - m\right\|_1 = \left\|a_n - \frac{1}{\det(m')}a_n\right\|_1 = \left|1 - \frac{1}{\det(m')}\right| \left\|a_n\right\|_1 \leq \left|1 - \frac{1}{\det(m')}\right|,$$

because $\left\|a_n\right\|_2 = 1$ implies that $\left\|a_n\right\|_1 \leq 1$. It is left to show that

$$\left|1 - \frac{1}{\det(m')}\right| = \left|\frac{\det(m') - 1}{\det(m')}\right| \leq Ce^{-Ar(R)}.$$

Since the function $f(y) := \frac{y}{y+1}$ is in $\mathcal{O}(y)$ for $y \to 0$, we need to show that $|\det(m') - 1| \leq Ce^{-Ar(R)}$, where we put $y = \det(m') - 1$. The determinant function, as a map from $\mathrm{Mat}_{n\times n}$ to $\mathbb{R}$ is polynomial, so locally Lipschitz. Hence, if we are close enough to the identity,

$$|\det(m') - 1| \leq C\left\|m' - \mathrm{Id}\right\|_1 = C'e^{-Ar(R)}$$

as we have seen above. After possibly adapting the constants, this finishes the proof. $\square$

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