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Domains of discontinuity of Anosov representations in flag manifolds and oriented flag manifolds

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Abstract

An infinite discrete subgroup of a Lie group acts on its homogeneous spaces. If the action is proper on an open subset, we call this subset a domain of discontinuity. In this thesis we investigate criteria when this happens, for some groups and spaces.

In the first part, we consider the action of an Anosov subgroup $\Gamma \subset G$ of a semi–simple Lie group on the associated flag manifolds. It is known that domains of discontinuity can be constructed from combinatorial objects called balanced ideals [KLP18]. For Δ–Anosov groups, we prove that every maximal and every cocompact domain of discontinuity arises from this construction, up to a few exceptions in low rank. In particular, this shows that some flag manifolds admit no cocompact domain of discontinuity. Applied to Hitchin representations, we determine exactly those flag manifolds which admit cocompact domains of discontinuity and give the number of different domains in the case of Grassmannians.

In the second part, we extend the theory of balanced ideals to the action of $\Gamma \subset G$ on oriented flag manifolds. These are quotients G/P, where P is a subgroup lying between a parabolic subgroup and its identity component. Under the condition that the limit curve of Γ lifts to some oriented flag manifold, we identify cocompact domains of discontinuity in oriented flag manifolds which we do not see in the unoriented setting. They even exist in some cases where in the unoriented flag manifold there are no cocompact domains at all. These include in particular domains in some oriented Grassmannians for Hitchin representations, which we also show to be nonempty.

As another application of the oriented setup, we give a new lower bound on the number of connected components of Δ–Anosov representations of a closed surface group into $\mathrm{SL}(n, \mathbb{R})$. We further use certain balanced ideals to construct a compactification of locally symmetric spaces arising from Anosov representations into $\mathrm{Sp}(2n, \mathbb{R})$. Finally, we discuss an approach to generalize the construction of domains of discontinuity to other homogeneous spaces.

Zusammenfassung

Eine unendliche diskrete Untergruppe einer Lie–Gruppe wirkt auf deren homogenen Räumen. Ist die Wirkung auf einer offenen Teilmenge eigentlich, dann nennen wir diese Teilmenge einen Diskontinuitätsbereich. In dieser Arbeit wollen wir für gewisse Gruppen und Räume Kriterien für die Existenz solcher Bereiche untersuchen.

Im ersten Teil betrachten wir die Wirkung einer Anosov–Untergruppe $\Gamma \subset G$ einer halbeinfachen Lie–Gruppe auf den zugeordneten Fahnenmannigfaltigkeiten. Es ist bekannt, dass Diskontinuitätsbereiche mithilfe von kombinatorischen Objekten, den ausgeglichenen Idealen, konstruiert werden können [KLP18]. Für Δ–Anosov–Gruppen beweisen wir, dass jeder maximale und jeder kokompakte Diskontinuitätsbereich aus dieser Konstruktion entsteht, bis auf ein paar Ausnahmen in niedrigem Rang. Insbesondere sehen wir, dass manche Fahnenmannigfaltigkeiten keine kokompakten Diskontinuitätsbereiche haben. Angewandt auf Hitchin–Darstellungen können wir exakt bestimmen, welche Fahnenmannigfaltigkeiten kokompakte Diskontinuitätsbereiche enthalten, und die Anzahl dieser Bereiche in den Grassmannschen Mannigfaltigkeiten bestimmen.

Im zweiten Teil erweitern wir die Theorie der ausgeglichenen Ideale auf die Wirkung von $\Gamma \subset G$ auf orientierten Fahnenmannigfaltigkeiten. Dies sind Quotienten G/P, wobei P eine Untergruppe ist, die zwischen einer Parabolischen und ihrer Identitätskomponente liegt. Unter der Bedingung, dass sich die Randkurve von Γ auf eine orientierte Fahnenmannigfaltigkeit hochheben lässt, finden wir kokompakte Diskontinuitätsbereiche, die in der unorientierten Theorie nicht auftauchen. Diese existieren sogar in manchen Fällen, in denen es in der unorientierten Fahnenmannigfaltigkeit gar keine kokompakten Bereiche gibt. Insbesondere schließt dies Diskontinuitätsbereiche für Hitchin–Darstellungen in orientierten Grassmannschen ein, von denen wir auch zeigen, dass sie nicht leer sind.

Als eine weitere Anwendung des orientierten Aufbaus geben wir eine neue untere Schranke an die Anzahl der Zusammenhangskomponenten von Δ–Anosov–Darstellungen von einer geschlossenen Flächengruppe in $\mathrm{SL}(n, \mathbb{R})$ an. Weiterhin benutzen wir bestimmte ausgeglichene Ideale zur Konstruktion einer Kompaktifizierung von lokalsymmetrischen Räumen, die von Anosov–Darstellungen in $\mathrm{Sp}(2n, \mathbb{R})$ stammen. Schließlich diskutieren wir noch einen Ansatz, um die Konstruktion von Diskontinuitätsbereichen auf andere homogene Räume zu erweitern.

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Contents

1 Introduction

1.1 Discrete subgroups of semi–simple Lie groups

Studying geometries always involves studying their symmetry transformations. In fact, since Felix Klein's Erlangen program [Kle72], it has become common to characterize properties of a geometry as those being invariant by a group of transformations G, which is generally a Lie group.

If we want to study objects inside a geometry described by G, or if we want to equip a manifold with a geometric structure modeled on it, then we are quickly led to consider subgroups of G, and in particular discrete subgroups. They come in two fundamentally different forms: The finite subgroups, for which one can restrict to compact groups G, and the infinite discrete subgroups, which are interesting for non–compact G, and whose most interesting features lie in their behaviour at infinity. We want to examine this second kind of subgroups of a semi–simple Lie group G and their actions on various spaces. Such a group G can for example be one of the classical matrix groups, like $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SO}_0(p,q)$, or $\mathrm{Sp}(2n,\mathbb{R})$.

To be able to speak about deformations of discrete subgroups, a useful viewpoint is to fix a finitely presented group Γ and consider the space $\mathrm{Hom}(\Gamma,G)$ of its group homomorphisms into G. We call these *representations* of Γ in G, and the space $\mathrm{Hom}(\Gamma,G)$ the *representation variety*. Indeed, if Γ is generated by k elements, then $\mathrm{Hom}(\Gamma,G)$ can be realized as a subset of G^k cut out by polynomial equations given by the relations of Γ. In particular, this gives $\mathrm{Hom}(\Gamma,G)$ a topology which does not depend on the choice of the k generators.

The *character variety*

$$\chi(\Gamma,G) \coloneqq \mathrm{Hom}(\Gamma,G)/\!\!/G \coloneqq \mathrm{Haus}(\mathrm{Hom}(\Gamma,G)/G)$$

is the space of representations up to conjugation by G, or more precisely its largest Hausdorff quotient[1]. Taking the largest Hausdorff quotient is in fact equivalent to restricting to the subset of reductive representations [Ric88; CLM18].

The character variety contains a subset $\chi_{\mathrm{disc}}(\Gamma,G) \subset \chi(\Gamma,G)$ of representations ρ which have a finite kernel and whose image $\rho(\Gamma)$ is discrete in G. Under some general assumptions, e.g. when Γ is a hyperbolic group, $\chi_{\mathrm{disc}}(\Gamma,G)$ is a closed subset of $\chi(\Gamma,G)$ [GM87b, Lemma 1.1]. Its elements describe all discrete subgroups of G which are isomorphic to a quotient of Γ by a finite group.

[1]For any topological space X, $\mathrm{Haus}(X)$ is the quotient of X by the following equivalence relation: $x,y \in X$ are equivalent if and only if $x \sim y$ for every equivalence relation $\sim$ which makes $X/\sim$ a Hausdorff space.

The most classical case is when $\Gamma = \pi_1 S$ is the fundamental group of a closed orientable surface S of genus $g \geq 2$, and $G = \mathrm{PSL}(2,\mathbb{R})$. Then the space $\chi(\Gamma, G)$ consists of $4g-3$ connected components, which can be distinguished by the Euler class of the plane bundle $\widetilde{S} \times_\rho \mathbb{R}^2 \to S$ associated to $\rho \in \mathrm{Hom}(\Gamma, G)$. This *Euler number* takes integer values in the interval $[2-2g, \dots, 2g-2]$ [Gol88]. $\chi_{\mathrm{disc}}(\Gamma, G)$ is the union of two of these connected components, namely those corresponding to Euler numbers $2-2g$ and $2g-2$.

Either of these components can be identified with Teichmüller space, the space of marked hyperbolic structures on S, by taking the quotient of the ρ–action on $\mathbb{H}^2$. Teichmüller space, and therefore each of these two components of $\chi(\pi_1 S, \mathrm{PSL}(2,\mathbb{R}))$, is homeomorphic to $\mathbb{R}^{6g-6}$. In this way, discrete injective representations into $\mathrm{PSL}(2,\mathbb{R})$ can be interpreted as holonomies of hyperbolic structures on S.

1.2 Anosov representations

If G is a semi–simple Lie group of higher rank, then $\chi_{\mathrm{disc}}(\Gamma, G)$ is generally not a union of connected components. Instead, it is a closed subspace whose structure is largely unknown.

In several cases, well–behaved subsets of $\chi_{\mathrm{disc}}(\Gamma, G)$ were found: For example, Hitchin identified a component of $\chi(\pi_1 S, \mathrm{PSL}(n,\mathbb{R}))$ [Hit87], now called the Hitchin component, which was later shown to consist of discrete injective representations [Lab06]. Further examples of such nice subsets of $\chi_{\mathrm{disc}}(\Gamma, G)$ are positive representations [FG06], maximal representations into Hermitian Lie groups [BIW10], or automorphism groups of convex divisible sets [Ben05].

The discovery of these notions was the starting point for what we call *higher Teichmüller theory* today, the study of discrete subgroups of higher rank Lie groups via techniques inspired from those of hyperbolic geometry. It is in this sense a higher rank analogue of classical *Teichmüller theory*, the theory of hyperbolic or conformal structures on surfaces.

Most of the aforementioned classes of representations are instances of the more general notion of *Anosov representations* $\chi_{\mathrm{Anosov}}(\Gamma, G) \subset \chi_{\mathrm{disc}}(\Gamma, G)$ from a hyperbolic group Γ [Gro87]. Anosov representations can be seen as a generalization of convex cocompact representations (see below) to Lie groups of higher rank. A first definition by Labourie [Lab06] was generalized to its current form by Guichard and Wienhard [GW12], but in recent years several equivalent characterizations were found [KLP17; GGKW17; BPS16; DGK17; an overview can be found in KLP17].

The following definition of Anosov representations from [BPS16] is one of the simplest. For concreteness, let us focus on the group $G = \mathrm{SL}(n,\mathbb{R})$ for now. The general case is similar, but requires more prerequisites from Lie theory (see Definition 2.1.9).

Definition 1.2.1. Let $\Delta = \{1, \dots, n-1\}$ and $\theta \subset \Delta$ non–empty. Let Γ be a finitely generated group. A representation $\rho\colon \Gamma \to \mathrm{SL}(n,\mathbb{R})$ is *θ–Anosov* if there are constants $C, c > 0$ such that

$$\frac{\sigma_i(\rho(\gamma))}{\sigma_{i+1}(\rho(\gamma))} \geq Ce^{c\,\ell(\gamma)} \qquad \forall \gamma \in \Gamma,\ i \in \theta,$$

where ℓ is the word length in any finite generating system for Γ and $\sigma_1(g) \geq \cdots \geq \sigma_n(g) \geq 0$ are the singular values of $g \in \mathrm{SL}(n, \mathbb{R})$.

One consequence of this definition is that Γ has to be a *hyperbolic group* [KLP14b; BPS16]. A hyperbolic group is a group whose Cayley graph for some (and therefore any) finite generating set is a δ–hyperbolic space for some $\delta > 0$, meaning that for every triangle, each of the sides is contained in a δ–neighborhood of the other two. A central feature of hyperbolic groups (or generally δ–hyperbolic spaces) is that we can associate to them a boundary at infinity $\partial_\infty\Gamma$ [Gro87]. The elements of $\partial_\infty\Gamma$ are equivalence classes of geodesic rays in the Cayley graph, with two rays being equivalent if they are at bounded distance from each other. There is also a natural topology on $\partial_\infty\Gamma$, given by pointwise convergence of the geodesic rays. In the case $\Gamma = \pi_1 S$ for a closed surface S, the boundary $\partial_\infty\Gamma$ is a circle and can be identified with the boundary at infinity of the hyperbolic plane e.g. in the disc model. If Γ is a free group, then $\partial_\infty\Gamma$ is a Cantor set. See [KB02] for a summary of basic facts on hyperbolic groups.

The θ–Anosov representations $\chi_{\theta\text{–Anosov}}(\Gamma, G)$ are a subset of $\chi_{\mathrm{disc}}(\Gamma, G)$ and an open set in $\chi(\Gamma, G)$. We can see them as the "nice" part of $\chi_{\mathrm{disc}}(\Gamma, G)$ of representations we know something about because we can study them by deformations.

A lot of information about a θ–Anosov representation is captured by its *boundary map* or *limit map*, that is a unique map

$$\xi \colon \partial_\infty\Gamma \to \mathcal{F}_\theta$$

into the *flag manifold* $\mathcal{F}_\theta$ which is continuous, ρ–equivariant, transverse and dynamics–preserving (see Section 2.1.5 for more details).

The (partial) flag manifold $\mathcal{F}_\theta$ for a non–empty subset $\theta \subset \Delta = \{1, \ldots, n-1\}$ is the space of sequences of subspaces

$$V^{i_1} \subset V^{i_2} \subset \cdots \subset V^{i_k} \subset \mathbb{R}^n, \qquad \dim V^{i_j} = i_j,\ \theta = \{i_1, \ldots, i_k\},$$

called *flags*. Special cases of flag manifolds are the *full flag manifold* $\mathcal{F}_\Delta$ and the *Grassmannians* $\mathrm{Gr}(k, n) = \mathcal{F}_{\{k\}}$ of k–dimensional subspaces of $\mathbb{R}^n$. The flag manifolds can be realized as homogeneous spaces G/P with $P \subset G$ a parabolic subgroup (a proper subgroup containing the group B of upper triangular matrices). The flag manifolds are compact spaces since the maximal compact subgruop $K = \mathrm{SO}(n) \subset G$ acts transitively on them.

While ξ is continuous, it is usually not a smooth map, but often has a rough, fractal structure. Two examples of limit curves of representations in $\chi_{\mathrm{Anosov}}(\pi_1 S, \mathrm{SL}(3, \mathbb{R}))$ are shown in Figure 1.1, one of which is C^1, while the other is not differentiable.

Flag manifolds and Anosov representations can be defined for general semi–simple Lie groups G. The role of Δ is then filled by the set of simple restricted roots of G.

There is a large variety of Anosov representations, each with their own distinctive features. The examples include:

- Let X be a negatively curved (i.e. rank 1) symmetric space. These are the hyperbolic spaces with isometry group $\mathrm{SO}_0(n, 1)$, complex and quaternionic hyperbolic spaces with isometry groups $\mathrm{SU}(n, 1)$ and $\mathrm{Sp}(n, 1)$, and an exceptional 16–dimensional space with isometry group a real form of F_4 [Hel79, X.6.2]. A discrete subgroup $\Gamma \subset \mathrm{Isom}(X)$

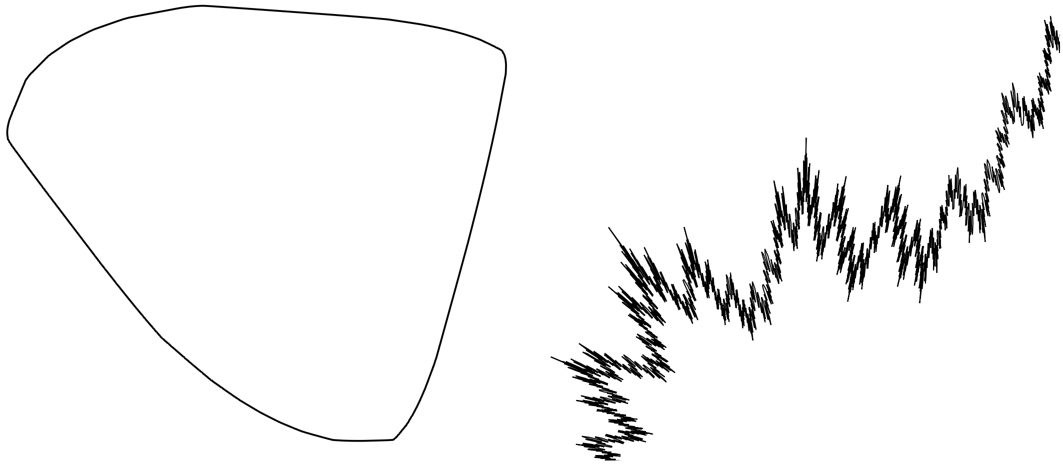


Figure 1.1: Two different examples of limit curves $\xi\colon S^1 \to \mathbb{R}\mathrm{P}^2$ of Anosov representations in $\chi_{\mathrm{Anosov}}(\pi_1 S, \mathrm{SL}(3,\mathbb{R}))$, shown in an affine chart for $\mathbb{R}\mathrm{P}^2$. The representation on the left is in the Hitchin component, while the right one is of Barbot type.

is called *convex cocompact* if it preserves a non–empty closed convex subset $C \subset X$ and the quotient $\Gamma\backslash C$ is compact. In rank 1, this property is equivalent to the inclusion being an Anosov representation [GW12, Theorem 1.8].

- If $\Gamma = \pi_1 S$ is the fundamental group of a closed surface S of genus at least 2, then $\chi(\pi_1 S, \mathrm{SL}(n,\mathbb{R}))$ for $n \geq 3$ has 3 connected components if n is odd and 6 if n is even [Hit87]. One respectively two of these components consist entirely of Δ–Anosov representations and are called *Hitchin components* [Lab06]. These are the components which contain representations of the form $\iota \circ \rho_0$, where $\rho_0\colon \pi_1 S \to \mathrm{SL}(2,\mathbb{R})$ is discrete and injective and $\iota\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$ is the irreducible representation, which is given by the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathrm{Sym}^{n-1}(\mathbb{R}^2) \cong \mathbb{R}^n$.
- If we compose a discrete injective representation $\rho_0\colon \pi_1 S \to \mathrm{SL}(2,\mathbb{R})$ with the reducible embedding $\iota'\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(3,\mathbb{R})$ (that is, let $\mathrm{SL}(2,\mathbb{R})$ act on a fixed 2–dimensional subspace), then the composition $\iota' \circ \rho_0$ is also Δ–Anosov, though it is not in the Hitchin component. Since the Anosov representations form an open set, small deformations of $\iota' \circ \rho_0$ are still Anosov, but in contrast to the Hitchin case, large continuous deformations may fail to even be discrete. These representations were studied by Barbot [Bar10] and a similar construction works for all $\mathrm{SL}(n,\mathbb{R})$ with odd n (see Section 4.6).
- More generally, if we start with an Anosov representation $\rho_0\colon \Gamma \to G$ and a (non–trivial) Lie group representation $\varphi\colon G \to G'$ into some other semi–simple Lie group G', the composition $\varphi \circ \rho_0$ will also be Anosov. Due to openness, small deformations of it are still Anosov. This is the primary way of producing Anosov representations in higher rank and all examples listed here are of this type. Exceptions are e.g. representations of reflection groups [DGK18; LM17].
- If G is a Hermitian Lie group and Γ a closed surface group, a topological invariant for representations in $\chi(\Gamma, G)$ generalizing the Euler number is the *Toledo invariant.*

Representations with maximal Toledo invariant are called maximal representations, and are also examples of Anosov representations [BIW10].

- If a subgroup $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ acts properly discontinuously and cocompactly on a strictly convex subset $\Omega \subset \mathbb{R}\mathrm{P}^{n-1}$, a so–called convex divisible set [Ben04], then the inclusion is $\{1, n-1\}$–Anosov by [GW12, Proposition 6.1].
- Another class of Δ–Anosov representations (which is a far more restrictive notion than the other kinds of Anosovness) of free groups Γ are those $\rho \colon \Gamma \to \mathrm{SL}(2n, \mathbb{R})$ which arise as a sum of discrete injective representations $\rho_1, \dots, \rho_n \colon \Gamma \to \mathrm{SL}(2, \mathbb{R})$ such that ρ_i uniformly dominates ρ_j for all $i \geq j$. This means that $\log \lambda_1(\rho_i(\gamma)) \geq c \log \lambda_1(\rho_j(\gamma))$ for some constant $c > 1$ and all $\gamma \in \Gamma$, where λ_1 denotes the highest eigenvalue. Then ρ is Δ–Anosov, as are small deformations [GGKW17, 7.1].

The fact that $\chi_{\mathrm{Anosov}}(\Gamma, G)$ is open and $\chi_{\mathrm{disc}}(\Gamma, G)$ is closed already implies that these sets are not equal in most higher rank situations. However, most known examples of representations in $\chi_{\mathrm{disc}}(\Gamma, G)$ are actually Anosov or limits of Anosov representations.

1.3 Domains of discontinuity

We can give Anosov representations an interpretation as holonomies of geometric structures on compact manifolds, similar to how every representation in $\chi_{\mathrm{disc}}(\pi_1 S, \mathrm{PSL}(2, \mathbb{R}))$ is associated to a hyperbolic structure on S. Particularly rich in higher rank are geometries modeled on flag manifolds. Not only do they capture a lot of information about the representation, but they also appear as the boundary at infinity of Riemannian symmetric spaces.

The action of an Anosov representation $\rho \in \mathrm{Hom}(\Gamma, G)$ on a flag manifold $\mathcal{F}$ is in general not proper, so $\Gamma \backslash \mathcal{F}$ will not be a manifold. However, in [GW12] Guichard and Wienhard described a way of removing a "bad set" from a suitable flag manifold such that ρ acts properly discontinuously and cocompactly on the complement. In other words, they constructed cocompact domains of discontinuity:

Definition 1.3.1. A *domain of discontinuity* $\Omega \subset X$ for ρ is a $\rho(\Gamma)$–invariant open subset of a G–space X such that the action $\Gamma \curvearrowright \Omega$ via ρ is proper. It is called *cocompact* if the quotient $\Gamma \backslash \Omega$ is compact.

Note that we require domains of discontinuity to be open subsets. In contrast, Danciger, Gueritaud and Kassel [DGK17; DGK18] and Zimmer [Zim17] recently proved that the Anosov property is equivalent to the existence of certain cocompact domains in $\mathbb{R}\mathrm{P}^n$ or $\mathbb{H}^{p,q}$. These domains are closed subsets.

A systematic construction of (open) domains of discontinuity for Anosov representations in flag manifolds was given by Kapovich, Leeb and Porti in [KLP18]. Given a θ–Anosov representation ρ which comes with a ρ–equivariant limit map $\xi \colon \partial_\infty \Gamma \to \mathcal{F}_\theta$, we want to find a cocompact domain of discontinuity in a flag manifold $\mathcal{F}_\eta$, which may be different from $\mathcal{F}_\theta$.

To construct such domains, [KLP18] use a combinatorial object called a *balanced ideal.* The finite set $G\backslash(\mathcal{F}_\theta \times \mathcal{F}_\eta)$ is naturally equipped with a partial order (the *Bruhat order*, which describes the inclusion relations of orbit closures) and with an involution. A subset $I \subset G\backslash(F_\theta \times \mathcal{F}_\eta)$ is called balanced ideal if it is an ideal with respect to the Bruhat order and every orbit of the involution has two elements, exactly one of which is contained in I (definitions of these notions can be found in Section 2.1). [KLP18] then prove that, for every balanced ideal I and every θ–Anosov representation with limit map ξ, the set

$$\Omega_{\rho,I} = \mathcal{F}_\eta \setminus \bigcup_{x \in \partial_\infty \Gamma} \{f \in \mathcal{F}_\eta \mid G(\xi(x), f) \in I\}$$

is a cocompact domain of discontinuity.

In this thesis, this idea will be extended in several directions.

1.4 Balanced ideals and maximal domains

For Δ–Anosov representations also the converse of the above construction is true. Namely, we will show in Chapter 3 that every cocompact domain of discontinuity in a flag manifold actually comes from a balanced ideal.

Theorem 1.4.1. *Let $\rho\colon \Gamma \to G$ be a Δ–Anosov representation and $\Omega \subset \mathcal{F}_\eta$ a cocompact domain of discontinuity for ρ in some flag manifold $\mathcal{F}_\eta$. Then there is a balanced ideal $I \subset G\backslash(\mathcal{F}_\Delta \times \mathcal{F}_\Delta)$ such that the lift of Ω to the full flag manifold $\mathcal{F}_\Delta$ is a union of connected components of $\Omega_{\rho,I}$.*

We can say more if the dimension of the bad set is not too big. To compute this dimension, we associate to a semi–simple Lie group G a number $\mathrm{mbic}(G)$ which gives a lower bound on the codimension of the set we have to remove for every limit point $x \in \partial_\infty\Gamma$. It increases with the rank of G, for example $\mathrm{mbic}(\mathrm{SL}(n, \mathbb{R})) = \lfloor (n + 1)/2 \rfloor$. The general definition is given in Definition 3.1.20. With this we get

Theorem 1.4.2. *Let $\rho\colon \Gamma \to G$ be a Δ–Anosov representation and assume* $\dim \partial_\infty\Gamma \leq \mathrm{mbic}(G) - 2$. *Then there is a 1:1 correspondence of balanced ideals in $G\backslash(\mathcal{F}_\Delta \times \mathcal{F}_\eta)$ and non–empty cocompact domains of discontinuity in $\mathcal{F}_\eta$.*

A key point for these theorems is that cocompact domains are maximal among all domains of discontinuity, at least if they are connected. We establish a correspondence between minimal fat ideals and maximal domains of discontinuity, even if they are not cocompact. This approach only works for Δ–Anosov representations: Section 3.1.6 shows an example of an Anosov, but not Δ–Anosov representation which admits infinitely many maximal domains of discontinuity. They are not cocompact.

If $G = \mathrm{SL}(n, \mathbb{R})$ or $G = \mathrm{SL}(n, \mathbb{C})$ we get a full description which flag manifolds admit cocompact domains of discontinuity, by combining Theorem 1.4.2 with a criterion for the existence of balanced ideals.

Theorem 1.4.3. *Let Γ be a hyperbolic group, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let $\rho\colon \Gamma \to \mathrm{SL}(n, \mathbb{K})$ be a Δ–Anosov representation. Choose integers $i_0, \dots, i_{k+1}$ with $0 = i_0 < i_1 < \cdots < i_{k+1} = n$. Denote by $\mathcal{F}$ the corresponding flag manifold*

$$\mathcal{F} = \{V^{i_1} \subset \cdots \subset V^{i_k} \subset \mathbb{K}^n \mid \dim_{\mathbb{K}} V^{i_j} = i_j\}.$$

Assume that

$$n \geq \begin{cases} 2 \dim \partial_\infty \Gamma + 3 & \text{if } \mathbb{K} = \mathbb{R} \\ 2\lfloor (\dim \partial_\infty \Gamma + 1)/2 \rfloor + 1 & \text{if } \mathbb{K} = \mathbb{C} \end{cases}$$

and let $\delta = |\{0 \leq j \leq k \mid i_{j+1} - i_j \text{ is odd}\}|$. Then

(i) If n is even, a non–empty cocompact open domain of discontinuity for $\Gamma \overset{\rho}{\curvearrowright} \mathcal{F}$ exists if and only if $\delta \geq 1$.

(ii) If n is odd, a non–empty cocompact open domain of discontinuity for $\Gamma \overset{\rho}{\curvearrowright} \mathcal{F}$ exists if and only if $\delta \geq 2$.

In particular, for surface group representations into $\mathrm{SL}(n, \mathbb{R})$ acting on Grassmannians we get:

Corollary 1.4.4. *Let $n \geq 5$ and let $\rho\colon \pi_1 S \to \mathrm{SL}(n, \mathbb{R})$ be a Δ–Anosov representation from the fundamental group of a surface S with or without boundary. Then the induced action $\pi_1 S \overset{\rho}{\curvearrowright} \mathrm{Gr}(k, n)$ on the Grassmannian of k–planes in $\mathbb{R}^n$ admits a non–empty cocompact domain of discontinuity if and only if n is even and k is odd. For $n = 6, 8, 10$, the number of different such domains is*

	$k=1$	$k=3$	$k=5$	$k=7$	$k=9$
$n=6$	1	2	1		
$n=8$	1	7	7	1	
$n=10$	1	42	2227	42	1

While we do not know a general formula for these numbers, they are obtained by counting balanced ideals. This is a combinatorial problem and needs no information about the representation except that it is Δ–Anosov. So we can use a computer program to enumerate all balanced ideals. Chapter 6 shows more results from this enumeration.

In low ranks, e.g. $\mathrm{SL}(n, \mathbb{R})$ with $n \leq 4$ if Γ is a surface group, the existence of cocompact domains of discontinuity depends on more information about the geometry of ρ, so we cannot make general lists like above in these cases. But for Hitchin representations, the cocompact domains of discontinuity are known in these low ranks. We will briefly discuss this in Section 3.2.2.

1.5 Domains of discontinuity in oriented flag manifolds

The goal in the second part of this thesis is to generalize the construction of cocompact domains of discontinuity from [KLP18] to actions on *oriented flag manifolds*. Before giving more details on what we mean by that, let us illustrate it with an example.

Let $\rho_0 \colon \Gamma \to \mathrm{SL}(3,\mathbb{R})$ be a discrete injective representation of a free group Γ whose image is contained in $\mathrm{SO}_0(2,1)$ and has no parabolic elements. Such a representation can be interpreted as the holonomy of a complete open finite type hyperbolic surface S. ρ_0 is Δ–Anosov and so is every nearby $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(3,\mathbb{R}))$. Let $\xi \colon \partial_\infty\Gamma \to \mathcal{F}_{\{1,2\}}$ be the limit map of ρ and $\xi^1 \colon \partial_\infty\Gamma \to \mathbb{R}\mathrm{P}^2$ and $\xi^2 \colon \partial_\infty\Gamma \to \mathrm{Gr}(2,3)$ its components. We consider the action of Γ on $\mathbb{R}\mathrm{P}^2$ defined by ρ. The maximal domain of discontinuity for this action is (by Corollary 3.1.15)

$$\Omega = \mathbb{R}\mathrm{P}^2 \setminus \bigcup_{x \in \partial_\infty\Gamma} \mathbb{P}(\xi^2(x)),$$

so it arises by removing a line from $\mathbb{R}\mathrm{P}^2$ for every element of $\partial_\infty\Gamma$. Since the boundary $\partial_\infty\Gamma$ of a free group Γ is a Cantor set, we remove a Cantor set of lines (Figure 1.2 left). Ω is not cocompact and in fact there are no cocompact domains of discontinuity for this action in $\mathbb{R}\mathrm{P}^2$ (see Lemma 3.1.16).

Surprisingly, a cocompact domain of discontinuity for this representation does exist in the space of oriented lines, which is simply the double cover S^2 of $\mathbb{R}\mathrm{P}^2$. This was first observed by Choi and can be found in [CG17]. We can describe the domain as follows. For every boundary point $x \in \partial_\infty\Gamma$, the limit map defines a great circle $\xi^2(x)$ in S^2 and two antipodal points on it (the lifts of $\xi^1(x)$). These two points split the great circle into two halves, and we remove from S^2 one of the closed halves, where the choices are made consistently so the half circles are all disjoint (Figure 1.2 right). As a result, we get two "caps" at the top and bottom and a countable number of "strips" joining them, one for every gap in the limit set. In the quotient, the caps become two copies of the surface S with open ends, and the strips become cylinders connecting the two copies. The resulting space is homeomorphic to the double of a surface with boundary and in particular compact.

As it turns out, this example is far from the only one where passing to a finite cover of the flag manifold leads to new cocompact domains of discontinuity. For instance, the above can be generalized to higher dimensions: Let $\rho \colon \Gamma \to \mathrm{PSL}(4n+3,\mathbb{R})$ be a small deformation of the composition of a convex cocompact representation into $\mathrm{PSL}(2,\mathbb{R})$ and the irreducible representation. Then there is a cocompact domain of discontinuity in S^{4n+2}. It is obtained by removing the spherical projectivizations of $2n+2$–dimensional (half–dimensional in S^{4n+2}) half–spaces.

The construction also works if Γ is a closed surface group and ρ is a Hitchin representation. In S^2 this will only give the two "caps", the quotients of which are homeomorphic to the closed surface. But in higher dimensional S^{4n+2} we get cocompact domains which do not exist in the unoriented case. This family of examples was independently found by Danciger, Guéritaud and Kassel.

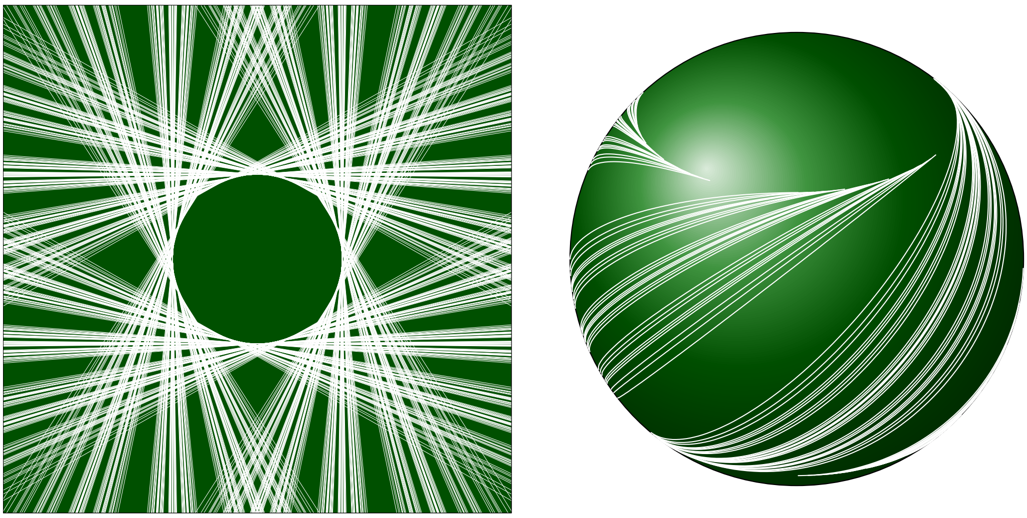


Figure 1.2: The maximal domain of discontinuity in $\mathbb{RP}^2$ (left) and the cocompact domain in S^2 (right).

We try to gain a better understanding of this behavior by generalizing the construction in [KLP18] to actions on oriented flag manifolds. The above examples will be special cases of this. An example of an oriented flag (for $\mathrm{SL}(n,\mathbb{R})$) is a usual flag with orientations assigned to some of these spaces. More generally, for a semi–simple Lie group G, we consider as oriented flag manifolds the homogeneous spaces G/P where P is a proper subgroup of G containing the identity component B_0 of a minimal parabolic subgroup. For technical reasons, we will restrict ourselves to linear groups G in this chapter.

Let us assume that the oriented flag manifold $\widehat{\mathcal{F}} = G/P$ is a finite cover of the unoriented flag manifold $\mathcal{F}_\theta$. Then we call a θ–Anosov representation *P–Anosov* if its limit map ξ lifts continuously and equivariantly to $\widehat{\mathcal{F}}$. There is in fact a unique maximal choice of such an oriented flag manifold $\widehat{\mathcal{F}}$ that one can lift ξ to (Proposition 4.2.4). To such a lift $\widehat{\xi}\colon \partial_\infty\Gamma \to \widehat{\mathcal{F}}$, we can associate its *transversality type* w_0, which is the G–orbit

$$w_0 = G(\widehat{\xi}(x), \widehat{\xi}(y)) \in G\backslash(\widehat{\mathcal{F}} \times \widehat{\mathcal{F}}), \qquad x, y \in \partial_\infty\Gamma,\ x \neq y$$

of a pair of distinct oriented limit flags (it is independent of x, y).

If $\widehat{\mathcal{F}}'$ is another oriented flag manifold, then as in the unoriented case, there is a partial order $\leq$ on $G\backslash(\widehat{\mathcal{F}} \times \widehat{\mathcal{F}}')$ describing orbit closures. Furthermore, the transversality type w_0 induces an order–reversing involution on $G\backslash(\widehat{\mathcal{F}} \times \widehat{\mathcal{F}}')$ (see Section 4.1.4 for the definition). This allows us to define the notion of a w_0–balanced ideal $I \subset G\backslash(\widehat{\mathcal{F}} \times \widehat{\mathcal{F}}')$: It is an ideal with respect to $\leq$ which is sent onto its complement by the involution w_0. Our main theorem of this chapter is then the following:

Theorem 1.5.1. *Let Γ be a non–elementary word–hyperbolic group and G a connected semi–simple, linear Lie group. Let $\rho\colon \Gamma \to G$ be a θ–Anosov representation with limit map $\xi\colon \partial_\infty\Gamma \to \mathcal{F}_\theta$ which admits a ρ–equivariant continuous lift $\widehat{\xi}\colon \partial_\infty\Gamma \to \widehat{\mathcal{F}}$ to some oriented flag manifold $\widehat{\mathcal{F}}$ finitely covering $\mathcal{F}_\theta$. Let w_0 be the transversality type of $\widehat{\xi}$ and*

$I \subset G\backslash(\widehat{\mathcal{F}} \times \widehat{\mathcal{F}}')$ a w_0–balanced ideal, where $\widehat{\mathcal{F}}'$ is another oriented flag manifold. Define $\mathcal{K} \subset \widehat{\mathcal{F}}'$ by

$$\mathcal{K} = \bigcup_{x \in \partial_\infty \Gamma} \{f \in \widehat{\mathcal{F}}' \mid G(\widehat{\xi}(x), f) \in I\}.$$

Then $\mathcal{K}$ is Γ–invariant and closed and Γ acts properly discontinuously and cocompactly on the domain $\Omega = \widehat{\mathcal{F}}' \setminus \mathcal{K}$.

If we take $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{F}}'$ to be unoriented flag manifolds (so in particular $\widehat{\mathcal{F}} = \mathcal{F}_\theta$), then we recover the original theorem from [KLP18].

We also develop a combinatorial description of the sets $G\backslash(\widehat{\mathcal{F}} \times \widehat{\mathcal{F}}')$ and the partial order and involution on them. While this can be done in the unoriented case by using quotients of the Weyl group $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$, our description of these *oriented relative positions* is in terms of what we call the *extended Weyl group* $\widetilde{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})_0$.

As an example, we apply Theorem 1.5.1 to the action of Hitchin representations on Grassmannians. If $\rho\colon \Gamma \to \mathrm{PSL}(n, \mathbb{R})$ is a Hitchin representation, it defines an action of Γ on the Grassmannian $\mathrm{Gr}(k, n)$. If n is even and k is odd, then there is a cocompact domain of discontinuity $\Omega \subset \mathrm{Gr}(k, n)$. For odd $n \geq 5$, no cocompact domains exist in any $\mathrm{Gr}(k, n)$ (by Corollary 1.4.4). However, we find oriented Grassmannians admitting cocompact domains of discontinuity in these cases. More precisely, there is a cocompact domain of discontinuity in the oriented Grassmannian $\mathrm{Gr}^+(k, n)$ whenever n is odd and $k(n + k + 2)/2$ is even (see Proposition 4.5.3).

1.6 Other results

While developing the theorems described in Section 1.4 and Section 1.5, we obtained some smaller independent results, which either emerged as a by–product or are still at an early stage of development and will be extended in the future. In the last two chapters we present and discuss these results.

1.6.1 Connected components of Anosov representations

The theory of oriented Anosov representations has another use concerning connected components of Anosov representations. It is based on the following proposition:

Proposition 1.6.1. *Let the oriented flag manifold $\widehat{\mathcal{F}} = G/P$ be a finite cover of $\mathcal{F}_\theta$. Then the set of P–Anosov representations is open and closed in the space of all θ–Anosov representations.*

Consequently, this notion can be used to distinguish connected components of Anosov representations: If two θ–Anosov representations lie in the same connected component, then both the maximal choice of oriented flag manifold $\widehat{\mathcal{F}}$ which their boundary maps can be lifted to and the transversality type of their boundary maps must agree (Corollary 4.2.11). We apply this fact in Section 4.6 to Δ–Anosov representations of closed surface groups Γ into $\mathrm{SL}(n, \mathbb{R})$ for odd n. Namely, we consider block embeddings constructed by composing a discrete injective representation into $\mathrm{SL}(2,\mathbb{R})$ with irreducible representations into $\mathrm{SL}(k,\mathbb{R})$ and $\mathrm{SL}(n-k,\mathbb{R})$. For different choices of block sizes, we show that these representations lie in different connected components of $\mathrm{Hom}_{\Delta\text{–Anosov}}(\Gamma,\mathrm{SL}(n,\mathbb{R}))$. Together with an observation by Thierry Barbot and Jaejeong Lee, which is explained in [KK16, Section 4.1], this leads to the following lower bound for the number of connected components:

Proposition 1.6.2. *Let $\Gamma = \pi_1 S$, where S is a closed surface of genus $g \geq 2$, and let n be odd. Then the space $\mathrm{Hom}_{\Delta\text{–Anosov}}(\Gamma,\mathrm{SL}(n,\mathbb{R}))$ has at least $2^{2g-1}(n-1)+1$ connected components.*

1.6.2 Domains of discontinuity in other homogeneous spaces

We can also look at the action of an Anosov representation $\rho\colon \Gamma \to G$ on G–homogeneous spaces G/H which are not flag manifolds. If ρ has a limit map into G/P and the double quotient $G\backslash(G/P \times G/H)$ is finite, it is possible to extend the notion of fat ideals and construct domains of discontinuity in G/H by a similar method. Since G/H is not compact, there is little hope of generalizing the cocompactness part as well.

The set $G\backslash(G/P \times G/H)$ again carries a partial order $\leq$, but in contrast to the case of flags or oriented flags, there is no natural order–reversing involution. We have to replace it with a relation which specifies which position an element of G/H can have towards a pair of transverse flags in G/P. We then call $I \subset G\backslash(G/P \times G/H)$ a fat ideal if it is an ideal with respect to $\leq$ and if every element of $G\backslash(G/P \times G/H)$ is either in I or related to something in I. With this definition, we get the following statement, which is analogous to the flag case:

Theorem 1.6.3. *Let $\rho\colon \Gamma \to G$ be an Anosov representation with limit map $\xi\colon \partial_\infty\Gamma \to G/P$ and let $I \subset G\backslash(G/P \times G/H)$ be a fat ideal. Then*

$$\Omega = G/H \setminus \bigcup_{z \in \partial_\infty \Gamma} \{x \in G/H \mid G(\xi(z), x) \in I\}$$

is a Γ–invariant open set and the ρ–action of Γ on Ω is proper.

This part is joint work with León Carvajales and is an ongoing project.

1.6.3 Enumerating balanced ideals

The set $G\backslash(\mathcal{F}_\theta \times \mathcal{F}_\eta)$ of relative positions between two flag manifolds $\mathcal{F}_\theta$ and $\mathcal{F}_\eta$ has a combinatorial description in terms of the Weyl group, which makes it possible to enumerate all balanced ideals using a computer program. Together with David Dumas, the author devised and implemented an algorithm to efficiently do this.

The table in Corollary 1.4.4 is a result of this, and in Chapter 6 we give some additional lists and tables in interesting cases. Our program can also be found online at `https://florianstecker.de/balancedideals/`.

1.6.4 Compactification of certain Hermitian symmetric spaces

An observation from studying the list of balanced ideals is the existence of a balanced ideal in $\mathrm{Sp}(2n,\mathbb{C})\backslash(\mathrm{Lag}(\mathbb{C}^{2n}) \times \mathrm{Lag}(\mathbb{C}^{2n}))$ for odd n. This allows us to construct a compactification for locally symmetric spaces associated to $\{\alpha_n\}$–Anosov representations, which is modeled on the bounded symmetric domain compactification of the symmetric space. In particular, this includes maximal representations into $\mathrm{Sp}(2n,\mathbb{R})$.

The symmetric space $X = \mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$, like any Hermitian symmetric space, can be realized as a bounded symmetric domain $D \subset \mathbb{C}^{n(n+1)/2}$ [Hel79, Theorem VIII.7.1]. Concretely, we can take as D the set of symmetric complex matrices Z such that $1 - \overline{Z}Z$ is positive definite. Its closure $\overline{D}$ in $\mathbb{C}^{n(n+1)/2}$ is the *bounded symmetric domain compactification* of X.

Theorem 1.6.4. *Let n be odd and $\rho\colon \Gamma \to \mathrm{Sp}(2n,\mathbb{R})$ be an $\{\alpha_n\}$–Anosov representation. Then there exists a subset $\widehat{D} \subset \mathbb{C}^{n(n+1)/2}$ such that $D \subset \widehat{D} \subset \overline{D}$ on which $\rho(\Gamma)$ acts properly discontinuously with compact quotient. This quotient $\Gamma\backslash\widehat{D}$ is a compactification of the locally symmetric space $\Gamma\backslash D$.*

1.7 Outline

We start in Section 2.1 with some basic definitions and fix notation for the rest of the thesis. The rest of Chapter 2 gives detailed proofs of some technical lemmas which will be needed later. None of this is original, but we need it in a slightly different setting than we could find in the literature. Section 2.2 investigates orbits and orbit closures of the $B_0 \times B_0$–action on G by left– and right– multiplication, following [BT72]. Section 2.3 treats expanding actions on compact homogeneous spaces. This is based on [KLP18].

Chapter 3 proves the results described in Section 1.4, that all cocompact domains of discontinuity for Δ–Anosov representations come from balanced ideals. The proofs of the main results are contained in Section 3.1, while Section 3.2 applies this to representations into the special linear group. This work was published on the arXiv as [Ste18].

Chapter 4 is concerned with the action of Anosov representations on oriented flag manifolds. First, Section 4.1 gives definitions such as oriented flag manifolds and oriented relative positions. Section 4.2 introduces an oriented version of the Anosov property and shows that these representations form a union of connected components inside the Anosov representations. Section 4.3 then proves the main result about cocompact domains of discontinuity in oriented flag manifolds. Section 4.4 is mainly of combinatorial nature and gives some examples of balanced ideals in this more general and flexible setting. Section 4.5 applies the results to Hitchin representations and generalized Schottky representations. Finally, Section 4.6 discusses how to get a lower bound for the number of connected components of Anosov representations out of this theory. Chapter 4 was joint work with Nicolaus Treib and was published on the arXiv as [ST18].

Chapter 5 generalizes the construction of domains of discontinuity to other homogeneous spaces, as outlined in Section 1.6.2. This part is joint work with León Carvajales and still an ongoing project.

Chapter 6 lists balanced ideals in some interesting cases. In Section 6.4 we use some of these balanced ideals to construct a compactification of certain locally symmetric spaces. Chapter 6 is also included in [Ste18].

2 Preliminaries and technical lemmas

We begin by introducing the basic notions of Lie theory, flag manifolds, and Anosov representations. Then we prove in detail two technical results, which are known, but which we need in a slightly different setting than how they can be found in the literature [BT72; KLP18], even though the arguments are very similar.

2.1 Basic definitions

2.1.1 Lie groups and flag manifolds

Let G be a connected semi–simple Lie group with finite center and $\mathfrak{g}$ its Lie algebra. Choose a maximal compact subgroup $K \subset G$ with Lie algebra $\mathfrak{k} \subset \mathfrak{g}$ and let $\mathfrak{p} = \mathfrak{k}^{\perp}$ be the orthogonal complement in $\mathfrak{g}$ with respect to the Killing form. Further choose a maximal subspace $\mathfrak{a} \subset \mathfrak{p}$ on which the Lie bracket vanishes. The dimension of $\mathfrak{a}$ is called the *rank* of G.

For any functional $\alpha \in \mathfrak{a}^*$ let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \ \forall H \in \mathfrak{a}\}$$

be the *restricted root space* and let $\Sigma = \{\alpha \in \mathfrak{a}^* \mid \mathfrak{g}_\alpha \neq 0\}$ be the set of *restricted roots.* Also choose a simple system $\Delta \subset \Sigma$ and let $\Sigma^{\pm} \subset \Sigma$ be the corresponding positive and negative roots. Note that in contrast with the complex case, Σ is in general not reduced, i.e. there can be $\alpha \in \Sigma$ with $2\alpha \in \Sigma$ (but no other positive multiples except 2 or 1/2). We denote by $\Sigma_0 \subset \Sigma$ the indivisible roots (the roots α with $\alpha/2 \notin \Sigma$ and let $\Sigma_0^{\pm} = \Sigma^{\pm} \cap \Sigma_0$. Let $\mathfrak{a}^+ = \{X \in \mathfrak{a} \mid \alpha(X) > 0 \,\forall \alpha \in \Delta\}$ and $\overline{\mathfrak{a}^+} = \{X \in \mathfrak{a} \mid \alpha(X) \geq 0 \,\forall \alpha \in \Delta\}$ be the *positive Weyl chamber* and its closure.

Note that any choice of the triple $(K, \mathfrak{a}, \Delta)$ is equivalent by conjugation in G (see [Hel79, Theorem 2.1] and [Kna02, Theorems 2.63, 6.51, 6.57]).

Define for $\varnothing \neq \theta \subset \Delta$ the Lie subalgebras

$$\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Sigma^-} \mathfrak{g}_\alpha, \quad \mathfrak{p}_\theta = \bigoplus_{\alpha \in \Sigma^+ \cup \mathrm{span}(\Delta \setminus \theta)} \mathfrak{g}_\alpha$$

and let $A, N, N^- \subset G$ be the connected Lie subgroups corresponding to $\mathfrak{a}$, $\mathfrak{n}$, and $\mathfrak{n}^-$. The exponential map of G restricts to diffeomorphisms from $\mathfrak{a}$ to A, $\mathfrak{n}$ to N and $\mathfrak{n}^-$ to N^-. Let $P_\theta = N_G(\mathfrak{p}_\theta)$ be the *parabolic subgroup* corresponding to θ. In particular, $\mathfrak{b} = \mathfrak{p}_\Delta$ is the *minimal parabolic subalgebra* and $B = P_\Delta$ the *minimal parabolic subgroup.* It decomposes as $B = Z_K(\mathfrak{a})AN$ via the Iwasawa decomposition. A parabolic subgroup or subalgebra is generally defined as a conjugate of the "standard parabolics" $\mathfrak{p}_\theta$ or P_θ, but when we say "parabolic" here, we will usually mean only $\mathfrak{p}_\theta$ or P_θ themselves.

The quotients

$$\mathcal{F}_\theta := G/P_\theta$$

are compact G–homogeneous spaces and are called (partial) flag manifolds.

The *Weyl group* of G is the finite group $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. It can be viewed as a group of linear isometries of $\mathfrak{a}$ equipped with the Killing form. A natural generating set of W is given by Δ, identifying every $\alpha \in \Delta$ with the orthogonal reflection along $\ker \alpha$. We will write $\ell(w)$ for the word length of $w \in W$ in this generating set. As (W, Δ) is a finite Coxeter system, there is a unique *longest element* $w_0 \in W$, which squares to the identity. The *opposition involution* is the map $\iota(w) = w_0 w w_0$ on W. It restricts to an involution $\iota \colon \Delta \to \Delta$ of the simple roots. Choosing any representative of w_0 in $N_K(\mathfrak{a})$, we have $w_0 N w_0^{-1} = N^-$.

A central feature of W for us is the Bruhat decomposition [Kna02, Theorem 7.40]

$$G = \bigsqcup_{w \in W} BwB.$$

Note that writing BwB, just like $w_0 N w_0^{-1}$ above, is a slight abuse of notation, since w is not an element of G, but a choice of representative in $N_K(\mathfrak{a})$ is required. However, this choice is irrelevant since $Z_K(\mathfrak{a}) \subset B$. We will often use these kinds of shorthands when dealing with the Weyl group. For example, for a subgroup $H \subset G$ containing $Z_K(\mathfrak{a})$ we will write $H \cap W \subset W$ instead of an ugly expression like $(H \cap N_K(\mathfrak{a}))/Z_K(\mathfrak{a})$.

Finally for this subsection, let

$$\mu \colon G \to \overline{\mathfrak{a}^+}$$

be the *Cartan projection* defined by the KAK–decomposition. That is, for every element $g \in G$ there are $k, \ell \in K$ and a unique $\mu(g) \in \overline{\mathfrak{a}^+}$ such that $g = k\, e^{\mu(g)}\, \ell$. k and ℓ are uniquely defined up to an element in the centralizer of $\mu(g)$.

2.1.2 Relative positions

Definition 2.1.1. For any pair of closed subgroups $H_1, H_2 \subset G$ we define a *relative position map*

$$\mathrm{pos} \colon G/H_1 \times G/H_2 \to H_1 \backslash G / H_2, \qquad ([g_1],[g_2]) \mapsto [g_1^{-1} g_2].$$

It captures all information about a pair in $G/H_1 \times G/H_2$ which is preserved by G. Likewise, we call the double quotient $H_1 \backslash G/H_2$ the *set of relative positions.*

Definition 2.1.2. If $H_1 \backslash G/H_2$ is a finite set, it carries a natural partial order given by

$$H_1 g H_2 \leq H_1 g' H_2 \quad \Leftrightarrow \quad H_1 g H_2 \subset \overline{H_1 g' H_2}.$$

Lemma 2.1.3. *The relation $\leq$ is a partial order.*

Proof. The only thing to show here is that $\overline{H_1 g H_2} = \overline{H_1 g' H_2}$ implies $H_1 g H_2 = H_1 g' H_2$. To see this, we prove that every orbit closure $\overline{H_1 g H_2}$ contains a unique relatively open orbit, which is $H_1 g H_2$. Note that every orbit $H_1 g H_2 \subset G$ is at least an immersed submanifold and therefore a countable union of embedded submanifolds. Embedded submanifolds are always locally closed, meaning they are the intersection of an open and a closed set. It is easy to see that a locally closed set is either nowhere dense or contains a non–empty open subset.

Since we assumed $H_1 \backslash G / H_2$ to be finite, $\overline{H_1 g H_2}$ consists of finitely many orbits and thus countably many locally closed sets, which are clearly also locally closed as subsets of $\overline{H_1 g H_2}$. By the Baire category theorem, these cannot all be nowhere dense, so some double coset $H_1 g' H_2 \subset \overline{H_1 g H_2}$ contains an open subset (open in $\overline{H_1 g H_2}$). Then $H_1 g' H_2$ must be open in $\overline{H_1 g H_2}$, and therefore intersect the dense subset $H_1 g H_2$. So $H_1 g H_2 = H_1 g' H_2$, and this is the unique open orbit in $\overline{H_1 g H_2}$. □

It is not hard to see that $\mathrm{pos}(x, y) \leq \mathrm{pos}(x', y')$ if and only if $G(x, y) \subset \overline{G(x', y')}$. The order can therefore be interpreted as measuring the "genericity" of the relative position of a pair. In other words: If we have sequences $(x_n) \in (G/H_1)^{\mathbb{N}}$ and $(y_n) \in (G/H_2)^{\mathbb{N}}$ converging to x and y with $\mathrm{pos}(x_n, y_n)$ constant, then $\mathrm{pos}(x, y) \leq \mathrm{pos}(x_n, y_n)$.

Definition 2.1.4. A pair $(x, y) \in G/H_1 \times G/H_2$ is *transverse* if $\mathrm{pos}(x, y)$ is maximal in this ordering.

In the case of two parabolic subgroups $P_\theta, P_\eta \subset G$, the set of relative positions is finite and has a combinatorial description via the Weyl group:

$$P_\theta \backslash G / P_\eta \cong \langle \Delta \backslash \theta \rangle \backslash W / \langle \Delta \backslash \eta \rangle.$$

Here $\langle A \rangle$ denotes the subgroup generated by $A \subset W$. The identification follows from the Bruhat decomposition together with the fact that $P_\theta \cap W = \langle \Delta \backslash \theta \rangle$. We will sometimes abbreviate the double quotient $\langle \Delta \backslash \theta \rangle \backslash W / \langle \Delta \backslash \eta \rangle$ as $W_{\theta,\eta}$.

For any $x \in \mathcal{F}_\theta$ and $w \in W_{\theta,\eta}$ we call the space

$$C_w(x) = \{y \in \mathcal{F}_\eta \mid \mathrm{pos}(x, y) = w\}$$

the *Bruhat cell* or *Schubert cell* of w relative to x (it is indeed a cell if $\eta = \theta = \Delta$, but not in general). The order described above is given by the *Bruhat order* on the Weyl group W, that is if $s_1 \dots s_k$ is a reduced expression in Δ for some representative in W of $w \in W_{\theta,\eta}$, then

$$\{w' \in W_{\theta,\eta} \mid w' \leq w\} = \{s_1^{i_1} \dots s_k^{i_k} \mid i_j \in \{0, 1\}\},$$

i.e. we get all lower elements by removing some number of letters. This classical fact actually follows from Proposition 2.2.13 proved below.

A consequence of this is that the position represented by w_0 is the unique maximal one, so two flags $f \in \mathcal{F}_\theta$ and $f' \in \mathcal{F}_\eta$ are transverse if and only if $\mathrm{pos}_{\theta,\eta}(f, f') = w_0$.

2.1.3 Ideals

In this section, let X be a finite set equipped with a partial order $\leq$ and an order–reversing involution $\sigma \colon X \to X$. The main example is of course $X = W_{\theta,\eta}$ with the Bruhat order $\leq$. If $\iota(\theta) = \theta$, then left–multiplication with w_0 descends to an order–reversing involution of $W_{\theta,\eta}$.

Definition 2.1.5. A subset $I \subset X$ is called an *ideal* if for every $x \in I$ and $y \in X$ with $y \leq x$, we have $y \in I$. Furthermore,

- I is called σ*–fat* if $x \notin I$ implies $\sigma(x) \in I$.
- I is called σ*–slim* if $x \in I$ implies $\sigma(x) \notin I$.
- I is called σ*–balanced* if it is σ–fat and σ–slim.

Observe that there can be no σ–balanced ideal if σ has a fixed point. Conversely, if σ has no fixed points, there will be balanced ideals by the following lemma. For the case of the Weyl group, this is proved in [KLP18, Proposition 3.29].

Lemma 2.1.6. *If σ has no fixed point, then every minimal σ–fat ideal and every maximal σ–slim ideal is σ–balanced.*

Proof. The two statements are equivalent by replacing an ideal I by $X \setminus \sigma(I)$. So assume that $I \subset X$ is a minimal σ–fat ideal which is not σ–balanced. Choose a maximal element $x \in I \cap \sigma(I) \neq \varnothing$ and let $I' = I \setminus \{x\}$. If I' is an ideal, it is clearly σ–fat, contradicting minimality of I. So I' is not an ideal. Then there exist $x_1 \leq x_2$ with $x_2 \in I'$ but $x_1 \notin I'$. So $x_1 = x$ since I is an ideal. Furthermore $\sigma(x_2) \leq \sigma(x_1) = \sigma(x)$ and $\sigma(x) \in I$, so $\sigma(x_2) \in I$ and therefore $x_2 \in I \cap \sigma(I)$. Since x is maximal in $I \cap \sigma(I)$ and $x \leq x_2$, this implies $x_2 = x = x_1$, a contradiction. $\square$

2.1.4 Proper actions

Let Γ be a Lie group acting smoothly on a manifold X.

Definition 2.1.7. The action of Γ on X is *proper* if the map

$$\Gamma \times X \to X \times X, \qquad (\gamma, x) \mapsto (\gamma x, x)$$

is a proper map, i.e. the preimage of every compact set is compact.

An equivalent characterization of properness is the following: The action is proper if for all sequences $(x_n) \in X^{\mathbb{N}}$ and $(\gamma_n) \in \Gamma^{\mathbb{N}}$ such that (x_n) and $(\gamma_n x_n)$ converge, a subsequence of (γ_n) converges.

If Γ is a discrete (i.e. 0–dimensional) group, this has a reformulation in terms of dynamically related points, given in [Fra05, Definition 1]:

Definition 2.1.8. Let Γ be discrete, $(\gamma_n) \in \Gamma^{\mathbb{N}}$ a divergent sequence (i.e. no element occurs infinitely many times) and $x, y \in X$. Then x *is dynamically related to* y *via* (γ_n), $x \overset{(\gamma_n)}{\sim} y$, if there is a sequence $(x_n) \in X^{\mathbb{N}}$ such that

$$x_n \to x \quad \text{and} \quad \gamma_n x_n \to y.$$

We say x and y are *dynamically related*, $x \sim y$, if they are dynamically related via any divergent sequence in Γ.

It is easily proved that the group action is proper if and only if there are no dynamically related points. Proper actions of discrete groups are often also called *properly discontinuous*.

The significance of proper actions comes from the fact that their orbit spaces have a simple local description: If Γ is discrete and acts properly, and $x \in X$, then a Γ–invariant neighborhood of x is Γ–equivariantly diffeomorphic to a neighborhood of the zero section in $\Gamma \times_{\Gamma_x} T_xX$ (for a proof see e.g. [GGK02, Theorem B.24]). Here Γ_x is the stabilizer at x, which is a finite group, and acts linearly on T_xX. If the action is free, i.e. all Γ_x are trivial, then this gives $\Gamma\backslash X$ the structure of a smooth manifold. If the action is not free, we still get at least an orbifold structure on $\Gamma\backslash X$.

2.1.5 Anosov representations

Let Γ be a Gromov hyperbolic group and G a semi–simple Lie group. Recall that a Gromov hyperbolic group comes equipped with a boundary $\partial_\infty\Gamma$. For example, if Γ is the fundamental group of a finite type surface, then $\partial_\infty\Gamma$ is a circle if the surface is closed and a Cantor set otherwise.

There exist several equivalent definitions of Anosov representations [KLP14b; KLP14a; GGKW17; BPS16]. One of them is the following, from [BPS16]:

Definition 2.1.9. Let $\theta \subset \Delta$ be non–empty and $\iota(\theta) = \theta$. A representation $\rho\colon \Gamma \to G$ is *θ–Anosov* (or *P_θ–Anosov*) if there are constants $C, c > 0$ such that

$$\alpha(\mu(\rho(\gamma))) \geq C|\gamma| - c \qquad \forall \alpha \in \theta, \gamma \in \Gamma,$$

where $|\cdot|$ is the word length in Γ with respect to any finite generating set.

Definition 2.1.10. Let Γ be a word hyperbolic group, $\partial_\infty\Gamma$ its Gromov boundary and $\xi\colon \partial_\infty\Gamma \to \mathcal{F}_\theta$ a map.

(i) A sequence $(g_n) \in G^{\mathbb{N}}$ is called *θ–divergent* (or *P_θ–divergent*) if

$$\alpha(\mu(g_n)) \to \infty \qquad \forall \alpha \in \theta.$$

A representation $\rho\colon \Gamma \to G$ is *θ–divergent* if for every divergent sequence $\gamma_n \to \infty$ in Γ its image $\rho(\gamma_n)$ is θ–divergent.

(ii) ξ is called *transverse* if for every pair $x \neq y \in \partial_\infty\Gamma$, the images $\xi(x), \xi(y) \in \mathcal{F}_\theta$ are transverse, i.e. $\operatorname{pos}(\xi(x), \xi(y)) = w_0$.

(iii) ξ is called *dynamics–preserving* if, for every element $\gamma \in \Gamma$ of infinite order, its unique attracting fixed point $\gamma^+ \in \partial_\infty\Gamma$ is mapped to an attracting fixed point of $\rho(\gamma)$.

The main fact we need about Anosov representations is that a θ–Anosov representation $\rho\colon \Gamma \to G$ admits a unique ρ–equivariant *limit map*

$$\xi\colon \partial_\infty\Gamma \to \mathcal{F}_\theta$$

which is continuous, transverse and dynamics–preserving [KLP14a; BPS16]. It can be constructed as follows: For $x \in \partial_\infty\Gamma$, take a diverging sequence $(\gamma_n) \in \Gamma^{\mathbb{N}}$ with attracting fixed point x. Then $(\rho(\gamma_n))$ as a sequence of maps $\mathcal{F}_\theta \to \mathcal{F}_\theta$ converges locally uniformly on an open dense subset to a constant map, the value of which is $\xi(x)$ (see Lemma 3.1.9).

Note that the limit map is also commonly used to define Anosov representations: Every θ–divergent representation of a hyperbolic group which admits a continuous, ρ–equivariant, transverse, dynamics–preserving limit map is θ–Anosov [GGKW17, Theorem 1.3].

Remark 2.1.11. There are many different conventions in use for the "type" of an Anosov representation, which are all equivalent. Besides the convention used here, which follows [GGKW17], it is also common to define the parabolic subalgebra as $\mathfrak{p}_\theta = \bigcup_{\alpha\in\Sigma^+\cup\mathrm{span}(\theta)} \mathfrak{g}_\alpha$, effectively interchanging the roles of θ and $\Delta\backslash\theta$ [GW12]. Furthermore, some authors do not require $\iota(\theta) = \theta$ and instead write (P^+, P^-)–Anosov or $(G, P^+ \cap P^-)$–Anosov, with P^-, P^+ being a pair of opposite parabolics. Kapovich–Leeb–Porti instead use terms like τ_{mod}–Anosov, τ_{mod}–CEA, τ_{mod}–RCA, τ_{mod}–URU or τ_{mod}–Morse, with τ_{mod} being the simplex in the boundary of symmetric space fixed by P_θ [KLP17]. Finally, some special cases have been given additional names like k–Anosov, projective Anosov, symplectic Anosov, etc.

2.2 The $B_0 \times B_0$–action on G

2.2.1 Extended Weyl group and the refined Bruhat decomposition

We will assume for this section that G is a linear group. Let $B \subset G$ be the minimal parabolic subgroup, and B_0 its identity component, i.e. the connected subgroup of $\mathfrak{g}$ with Lie algebra $\mathfrak{b} = \bigoplus_{\alpha\in\Sigma^+\cup\{0\}} \mathfrak{g}_\alpha$. We consider the action of $B_0 \times B_0$ on G by left– and right–multiplication. The goal of this section is to prove Proposition 2.2.4, which is a refinement of the Bruhat decomposition. It will be an important ingredient for our description of relative positions of oriented flags. The proof requires some rather technical preparations.

First of all, we define the groups

$$\widetilde{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})_0, \quad \overline{M} = Z_K(\mathfrak{a})/Z_K(\mathfrak{a})_0,$$

where $Z_K(\mathfrak{a})_0$ is the identity component of the centralizer of $\mathfrak{a}$ in the maximal compact subgroup K. We call $\widetilde{W}$ the extended Weyl group. If we write $\ell(w)$ for $w \in \widetilde{W}$ we mean the length $\ell(\pi(w))$ of its projection to W. See Section 4.4.2 for a description of $\widetilde{W}$, $\overline{M}$, B_0 etc. in the case $G = \mathrm{SL}(n, \mathbb{R})$.

Remark 2.2.1. Our attention in this section and Chapter 4 will be restricted to semi–simple Lie groups G such that the group $\overline{M}$ is finite abelian and consists entirely of involutions. This holds for all G which are linear, i.e. isomorphic to a closed subgroup of some $\mathrm{GL}(n, \mathbb{R})$ (see [Kna02, Theorem 7.53] and note that all connected linear Lie groups have a complexification). Also, every linear semi–simple Lie group has a finite center [Kna02, Proposition 7.9]. All our arguments work equally well for Lie groups which are not linear, as long as their center is finite and $\overline{M}$ is finite abelian and consists of involutions. These assumptions on $\overline{M}$ do not appear to be essential for our theory, but they significantly simplify several arguments, e.g. the statement and proof of Lemma 2.2.9.

As restricted root systems are not necessarily reduced, we will work with the set Σ_0 of indivisible roots, i.e. the roots $\alpha \in \Sigma$ such that $\alpha/2 \notin \Sigma$. For any $\alpha \in \Sigma_0^+ = \Sigma^+ \cap \Sigma_0$ let $\mathfrak{u}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. Then $\mathfrak{u}_\alpha$ is a subalgebra of $\mathfrak{g}$. Let $U_\alpha \subset G$ be the connected subgroup with Lie algebra $\mathfrak{u}_\alpha$.

For $\alpha, \beta \in \Sigma_0^+$ let $(\alpha, \beta) \subset \Sigma_0^+$ be the set of all indivisible roots which can be obtained as positive linear combinations of α and β. Then $[\mathfrak{u}_\alpha, \mathfrak{u}_\beta] \subset \bigoplus_{\gamma \in (\alpha,\beta)} \mathfrak{u}_\gamma$. For every $w \in W$ the set $\Psi_w = \Sigma_0^+ \cap w\Sigma_0^-$ has the property that $(\alpha, \beta) \subset \Psi_w$ for all $\alpha, \beta \in \Psi_w$. Let U_w be the connected subgroup of G with Lie algebra $\mathfrak{u}_w = \bigoplus_{\alpha \in \Psi_w} \mathfrak{u}_\alpha$.

Lemma 2.2.2. *Let $\Psi' \subset \Psi \subset \Sigma_0^+$ such that $(\alpha, \beta) \subset \Psi$ and $(\alpha, \gamma) \subset \Psi'$ for all $\alpha, \beta \in \Psi$ and $\gamma \in \Psi'$. Let $\mathfrak{u} = \bigoplus_{\alpha \in \Psi} \mathfrak{u}_\alpha$ and $\mathfrak{u}' = \bigoplus_{\alpha \in \Psi'} \mathfrak{u}_\alpha$ and let $U, U' \subset G$ be the corresponding connected subgroups. Let $\Psi \setminus \Psi' = \{\alpha_1, \dots, \alpha_n\}$, in arbitrary order. Then*

$$U = U' U_{\alpha_1} \cdots U_{\alpha_n}.$$

In particular, for $\Psi = \Psi_w$ and $\Psi' = \varnothing$, we have $U_w = \prod_{\alpha \in \Psi_w} U_\alpha$, where the product can be written in arbitrary order.

Proof. First note that the conditions ensure that $\mathfrak{u}, \mathfrak{u}' \subset \mathfrak{g}$ are subalgebras and that $\mathfrak{u}'$ is an ideal of $\mathfrak{u}$. We proceed by induction on $n = |\Psi \setminus \Psi'|$. The case $n = 0$ is trivial and for $n = 1$ the statement is shown in [Kna02, Lemma 7.97].

If $n \geq 1$, choose a longest root α_k among $\alpha_1, \dots, \alpha_n$ and let $\Psi'' = \Psi' \cup \{\alpha_k\}$. Then $(\alpha, \beta) \subset \Psi' \subset \Psi''$ for all $\alpha \in \Psi$ and $\beta \in \Psi''$, since every element of (α, α_k) will be longer than α_k and therefore in Ψ'. So $\mathfrak{u}'' = \bigoplus_{\alpha \in \Psi''} \mathfrak{u}_\alpha$ is an ideal of $\mathfrak{u}$. Let $U'' \subset G$ be its connected subgroup. Since $\mathfrak{u}'$ and $\mathfrak{u}''$ are ideals of $\mathfrak{u}$, U' and U'' are normal subgroups of U. Now let $g \in U$. By the induction hypothesis there are $g'' \in U''$, $g_- \in U_{\alpha_1} \cdots U_{\alpha_{k-1}}$ and $g_+ \in U_{\alpha_{k+1}} \cdots U_{\alpha_n}$ such that $g = g'' g_- g_+$. Since $U'' \subset U$ is normal, $g = g_- \overline{g}'' g_+$ for $\overline{g}'' = g_-^{-1} g'' g_- \in U''$. Since $\Psi' \subset \Psi''$ satisfy the assumptions of the Lemma for $n = 1$, we get $\overline{g}'' = g' g_0$ for some $g' \in U'$ and $g_0 \in U_{\alpha_k}$. Now set $\overline{g}' = g_- g' g_-^{-1} \in U'$, then $g = \overline{g}' g_- g_0 g_+ \in U' U_{\alpha_1} \cdots U_{\alpha_n}$, as required. □

Lemma 2.2.3. *Let $w \in N_K(\mathfrak{a})$. Then the map*

$$U_w \times B \to G, \quad (u, b) \mapsto uwb \tag{2.1}$$

is a smooth embedding with image BwB. The restriction of (2.1) to $U_w \times B_0$ maps onto $U_w w B_0 = B_0 w B_0$.

Proof. We get (2.1) as a composition

$$U_w \times B \xrightarrow{\mathrm{conj}_{w^{-1}} \times \mathrm{id}} w^{-1}U_w w \times B \hookrightarrow N^- \times B \to G \xrightarrow{L_w} G.$$

The first and last map are diffeomorphisms, the inclusion is a smooth embedding and the multiplication map $N^- \times B \to G$ is a diffeomorphism onto an open subset of G by [Kna02, Lemma 6.44, Proposition 7.83(e)]. So the composition is a smooth embedding. We only have to compute its image, i.e. that $U_w wB = BwB$.

To prove this, use the Iwasawa decomposition $B = NAZ_K(\mathfrak{a})$ to get $BwB = NwB$ and then write, using Lemma 2.2.2,

$$NwB = \Big(\prod_{\alpha \in \Psi_w} U_\alpha\Big)\Big(\prod_{\alpha \in \Sigma_0^+ \setminus \Psi_w} U_\alpha\Big) wB = U_w w \Big(\prod_{\alpha \in \Sigma_0^+ \setminus \Psi_w} w^{-1} U_\alpha w\Big) B.$$

For all $\alpha \in \Sigma_0^+ \setminus \Psi_w$ we have $w^{-1}\alpha \in \Sigma_0^+$, so $\mathrm{Ad}_{w^{-1}} \mathfrak{u}_\alpha = \mathfrak{u}_{w^{-1}\alpha} \subset \mathfrak{n}$ and $w^{-1}U_\alpha w \subset N \subset B$, so $BwB = U_w wB$. If we restrict (2.1) to the connected component $U_w \times B_0$, its image is $U_w w B_0$. The Iwasawa decomposition shows that $B_0 = NAZ_K(\mathfrak{a})_0$, so $B_0 w B_0 = NwB_0$ and this equals $U_w w B_0$ by the same argument as above. □

Proposition 2.2.4 (Refined Bruhat decomposition)**.** *G decomposes into the disjoint union*

$$G = \bigsqcup_{w \in \widetilde{W}} B_0 w B_0.$$

Proof. Let $\pi \colon \widetilde{W} \to W$ be the projection to the Weyl group. By the Bruhat decomposition [Kna02, Theorem 7.40], G decomposes disjointly into BwB for $w \in W$, so we only have to show that

$$BwB = \bigsqcup_{w' \in \pi^{-1}(w)} B_0 w' B_0. \tag{2.2}$$

Lemma 2.2.3 identifies BwB with $U_w \times B$, the connected components of which are the sets $U_w \times mB_0$ for $m \in \overline{M}$. These correspond via the map from Lemma 2.2.3 to the subsets $U_w wmB_0 = B_0 wmB_0 \subset BwB$. Also $\pi^{-1}(w) = \{wm \mid m \in \overline{M}\}$, proving (2.2). □

2.2.2 Orbit closures

We now turn to analyzing the closures of $B_0 \times B_0$–orbits in G. We call such orbits *refined Bruhat cells.* The main result of this section is Proposition 2.2.13, which gives a combinatorial description of closures of refined Bruhat cells. This part is similar to [BT72, Section 3], where Borel and Tits describe the left and right action of the Borel subgroup for an algebraic group G. Most of their arguments also work in our setting.

Let $\widetilde{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})_0$ as before and $\pi \colon \widetilde{W} \to W$ the projection to the Weyl group $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. As described in Section 2.1, Δ is realized as a generating set of W and we write ℓ for the word length with respect to Δ. We want to define a lift $\mathsf{v} \colon \Delta \to \widetilde{W}$ of this generating set so that $\mathsf{v}(\alpha) \in (P_{\Delta \setminus \{\alpha\}})_0$ for every $\alpha \in \Delta$.

The construction requires some Lie theory. If $\langle \cdot, \cdot \rangle$ is the Killing form on $\mathfrak{g}$ and $\Theta\colon \mathfrak{g} \to \mathfrak{g}$ the Cartan involution, which is 1 on $\mathfrak{k}$ and -1 on $\mathfrak{p}$, then $\|X\|^2 = -\langle X, \Theta X\rangle$ defines a norm on $\mathfrak{g}$. Its restriction to $\mathfrak{a}$ is just $\langle X, X\rangle$. For $\alpha \in \mathfrak{a}^*$ let $H_\alpha \in \mathfrak{a}$ be its dual with respect to $\langle \cdot, \cdot \rangle$, i.e. $\langle H_\alpha, X\rangle = \alpha(X)$ for all $X \in \mathfrak{a}$. We use the norm on $\mathfrak{a}^*$ defined by $\|\alpha\|^2 = \langle H_\alpha, H_\alpha\rangle$.

Definition 2.2.5. For every $\alpha \in \Delta$ choose a vector $E_\alpha \in \mathfrak{g}_\alpha$ such that $\|E_\alpha\|^2 = 2\|\alpha\|^{-2}$. Then we define

$$\mathsf{v}(\alpha) = \exp\left(\frac{\pi}{2}\left(E_\alpha + \Theta E_\alpha\right)\right).$$

By [Kna02, Proposition 6.52(c)] this is in $N_K(\mathfrak{a})$ and acts on $\mathfrak{a}$ as a reflection along the kernel of the simple root α. We will regard $\mathsf{v}(\alpha)$ as an element of $\widetilde{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})_0$.

Remarks 2.2.6.

(i) $\mathsf{v}(\alpha) \in \widetilde{W}$ is almost independent of the choice of E_α: If $\dim \mathfrak{g}_\alpha > 1$, then the set of admissible E_α is connected. Since $\widetilde{W}$ is discrete and $\mathsf{v}(\alpha)$ depends continuously on E_α this means that $\mathsf{v}(\alpha) \in \widetilde{W}$ is independent of E_α. In particular, we get the same $\mathsf{v}(\alpha)$ when substituting E_α by $-E_\alpha$, so $\mathsf{v}(\alpha) = \mathsf{v}(\alpha)^{-1}$. On the other hand, if $\dim \mathfrak{g}_\alpha = 1$ then $\mathsf{v}(\alpha)$ need not be of order 2, and there can be two different choices for $\mathsf{v}(\alpha)$, which are inverses of each other. If they do not coincide, $\mathsf{v}(\alpha)$ is of order 4 since $\mathsf{v}(\alpha)^2$ acts trivially on $\mathfrak{a}$ and is therefore contained in $\overline{M}$.

(ii) By our assumption of G being linear, $\overline{M}$ consists of involutions, and therefore $\mathsf{v}(\alpha)$ has order either 2 or 4. In the group $\mathrm{SL}(n, \mathbb{R})$ for example, $\mathsf{v}(\alpha)$ is of order 4 for all simple restricted roots α, while in $\mathrm{SO}_0(p, q)$, $p < q$, the image of the "last" simple root α_p is of order 2.

(iii) For every $\alpha \in \Delta$, we have $\pi(\mathsf{v}(\alpha)) = \alpha \in W$, so the projection of $\mathsf{v}(\Delta)$ to W is just the usual generating set Δ. In fact, $\mathsf{v}(\Delta)$ also generates the group $\widetilde{W}$ (this can be seen using Lemma 4.1.11 with $\theta = \Delta$).

Recall that $\Psi_w = \Sigma_0^+ \cap w\Sigma_0^-$.

Lemma 2.2.7. *Let $w_1, w_2 \in W$. Then*

$$\Psi_{w_1} \cap w_1(\Psi_{w_2}) = \varnothing, \quad \Psi_{w_1 w_2} \subset \Psi_{w_1} \cup w_1(\Psi_{w_2}), \quad |\Psi_{w_1}| = \ell(w_1).$$

Furthermore, $\Psi_w \subset \operatorname{span}\theta$ for $\theta \subset \Delta$ if and only if $w \in \langle\theta\rangle \subset W$.

Proof. First observe that $\alpha\Sigma_0^+ = \Sigma_0^+ \setminus \{\alpha\} \cup \{-\alpha\}$ and therefore $\Psi_\alpha = \{\alpha\}$ for any $\alpha \in \Delta$. The first two identities follow easily from the definition of Ψ_w and the inequality $|\Psi_w| \leq \ell(w)$ is a direct consequence of $\Psi_{w\alpha} \subset \Psi_w \cup w\Psi_\alpha$ for every $\alpha \in \Delta$.

We want to show that $|\Psi_w| = r$ implies $\ell(w) = r$ by induction on $r \in \mathbb{N}$. For $r = 0$ this follows from the fact that W acts freely on positive systems of roots. If $|\Psi_w| = r > 0$, then $\Delta \not\subset w\Sigma_0^+$, as otherwise $\Sigma_0^+ \subset w\Sigma_0^+$ and thus $\Psi_w = \varnothing$. So choose $\alpha \in \Delta \cap w\Sigma_0^- \subset \Psi_w$. Then

$$\alpha\Psi_{\alpha w} = \alpha\Sigma_0^+ \cap w\Sigma_0^- = (\Sigma_0^+ \setminus \{\alpha\} \cup \{-\alpha\}) \cap w\Sigma_0^- = \Psi_w \setminus \{\alpha\},$$

so $|\Psi_{\alpha w}| = r - 1$ and thus $\ell(\alpha w) = r - 1$ by the induction hypothesis. So $\ell(w) = r$.

To prove the remaining statement, note that the reflection along a root α maps every other root β into $\operatorname{span}(\alpha, \beta)$. So $\operatorname{span}(\theta)$ is invariant by every $w \in \langle\theta\rangle$. Assume the equivalence of $\Psi_w \subset \operatorname{span}\theta$ and $w \in \langle\theta\rangle$ was already proved for $w \in W$ and let $\ell(\alpha w) = \ell(w) + 1$. Then $\Psi_{\alpha w} = \Psi_\alpha \sqcup \alpha\Psi_w = \{\alpha\} \sqcup \alpha\Psi_w$ is contained in $\operatorname{span}\theta$ if and only if $\alpha \in \theta$ and $w \in \langle\theta\rangle$, proving what we wanted. ☐

The two cases $\operatorname{ord}(\mathsf{v}(\alpha)) = 4$ or $\operatorname{ord}(\mathsf{v}(\alpha)) = 2$ in Remark 2.2.6 are related to whether there is an associated embedded $\mathrm{SL}(2, \mathbb{R})$ or $\mathrm{PSL}(2, \mathbb{R})$:

Lemma 2.2.8. *Let $\alpha \in \Delta$ and E_α as in Definition 2.2.5. Then there is a Lie group homomorphism $\Phi\colon \mathrm{SL}(2, \mathbb{R}) \to G$, which is an immersion and satisfies*

(i) $\mathrm{d}_1\Phi\colon \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{g}$ *maps* $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ *to* E_α, $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ *to* $-\Theta E_\alpha$, *and* $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ *to* $2\|\alpha\|^{-2} H_\alpha$,

(ii) $\Phi\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = \mathsf{v}(\alpha)$,

(iii) Φ *is an isomorphism if* $\operatorname{ord}(\mathsf{v}(\alpha)) = 4$, *and* $\ker\Phi = \{\pm 1\}$ *if* $\operatorname{ord}(\mathsf{v}(\alpha)) = 2$.

Proof. $\mathrm{d}_1\Phi$ as defined in (i) is a monomorphism of Lie algebras [Kna02, Proposition 6.52], so it integrates to an immersive Lie group homomorphism $\widetilde{\Phi}\colon \widetilde{\mathrm{SL}(2, \mathbb{R})} \to G$. Since $\ker\widetilde{\Phi} \subset \widetilde{\mathrm{SL}(2, \mathbb{R})}$ is a normal subgroup and discrete and $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ is connected, conjugation actually fixes $\ker\widetilde{\Phi}$ pointwise, i.e.

$$\ker\widetilde{\Phi} \subset Z(\widetilde{\mathrm{SL}(2, \mathbb{R})}) = \exp(\mathbb{Z}X), \quad X = \left(\begin{smallmatrix} 0 & \pi \\ -\pi & 0 \end{smallmatrix}\right).$$

Now $\widetilde{\Phi}(\exp(kX)) = \exp(\mathrm{d}_1\Phi(kX)) = \exp(k\pi(E_\alpha + \Theta E_\alpha)) = \mathsf{v}(\alpha)^{2k}$. If $\operatorname{ord}(\mathsf{v}(\alpha)) = 2$, then $\ker\widetilde{\Phi} = \exp(\mathbb{Z}X)$ and if $\operatorname{ord}(\mathsf{v}(\alpha)) = 4$, then $\ker\widetilde{\Phi} = \exp(2\mathbb{Z}X)$. By Remark 2.2.6(i), these are the only possibilities. In any case, $\widetilde{\Phi}$ descends to a homomorphism Φ on $\mathrm{SL}(2, \mathbb{R}) = \widetilde{\mathrm{SL}(2, \mathbb{R})}/\exp(2\mathbb{Z}X)$, having the desired properties. ☐

We can now already understand the closures of double cosets containing one of the generators, and show how to decompose parabolic subgroups determined by one simple restricted root according to Proposition 2.2.4.

Lemma 2.2.9. *Let $\alpha \in \Delta$ and let $s = \mathsf{v}(\alpha) \in \widetilde{W}$. Then*

$$(P_{\Delta\setminus\{\alpha\}})_0 = B_0 \cup B_0 s B_0 \cup B_0 s^2 B_0 \cup B_0 s^3 B_0, \tag{2.3}$$

$$\overline{B_0 s B_0} = B_0 \cup B_0 s B_0 \cup B_0 s^2 B_0, \tag{2.4}$$

$$B_0 s B_0 s^{-1} B_0 = B_0 \cup B_0 s B_0 \cup B_0 s^{-1} B_0. \tag{2.5}$$

Note that these unions are disjoint unless s has order 2.

Proof. First note that $P_{\Delta\setminus\{\alpha\}} = B \cup BsB$. This follows from the Bruhat decomposition and the following argument: An element $w \in W$ is contained in $P_{\Delta\setminus\{\alpha\}} = N_G(\mathfrak{p}_{\Delta\setminus\{\alpha\}})$ if and only if $\mathrm{Ad}_w \mathfrak{p}_{\Delta\setminus\{\alpha\}} \subset \mathfrak{p}_{\Delta\setminus\{\alpha\}}$. This holds if and only if w preserves $\Sigma_0^+ \cup \operatorname{span}(\alpha)$, or equivalently $\Psi_w \subset \{\alpha\}$. By Lemma 2.2.7 this is true if and only if $w \in \{1, \alpha\}$.

We now distinguish two cases, depending on the dimension of $\mathfrak{u}_\alpha$. First assume that $\dim\mathfrak{u}_\alpha > 1$. In this case, $(P_{\Delta\setminus\{\alpha\}})_0 \cap B \subset (P_{\Delta\setminus\{\alpha\}})_0$ is a closed subgroup of codimension at least 2.

Therefore, its complement $(P_{\Delta\setminus\{\alpha\}})_0 \cap BsB$ in $(P_{\Delta\setminus\{\alpha\}})_0$ is connected, thus equal to B_0sB_0, which is a connected component of BsB by Lemma 2.2.3. This implies $(P_{\Delta\setminus\{\alpha\}})_0 \cap Z_K(\mathfrak{a}) = Z_K(\mathfrak{a})_0$, as otherwise there would be $m \in \overline{M} \setminus \{1\}$ with $B_0sB_0m = B_0smB_0 \subset (P_{\Delta\setminus\{\alpha\}})_0$, but this is disjoint from B_0sB_0 by Proposition 2.2.4. So $(P_{\Delta\setminus\{\alpha\}})_0 = B_0 \cup B_0sB_0$. Since $s \in \widetilde{W}$ must have order 2 in this case by Proposition 2.2.4, this is (2.3) as we wanted.

To see (2.4) and (2.5) in this case, we only have to prove that the inclusions $B_0sB_0 \subset \overline{B_0sB_0}$ and $B_0 \subset B_0sB_0s^{-1}B_0$ are strict: Using B_0–invariance from both sides, this will imply $\overline{B_0sB_0} = B_0sB_0s^{-1}B_0 = (P_{\Delta\setminus\{\alpha\}})_0 = B_0 \cup B_0sB_0$. And indeed, $B_0sB_0 = \overline{B_0sB_0}$ would imply that B_0sB_0 and B_0 are closed, so $B_0sB_0 \cup B_0$ would not be connected. Also, if $B_0 = B_0sB_0s^{-1}B_0$, then $sB_0s^{-1} \subset B_0$, so $\mathfrak{g}_{-\alpha} = \mathrm{Ad}_s\, \mathfrak{g}_\alpha \subset \mathrm{Ad}_s\, \mathfrak{b} \subset \mathfrak{b}$, a contradiction.

Now we consider the case $\dim \mathfrak{u}_\alpha = 1$. Then $P_{\Delta\setminus\{\alpha\}}/B_0$ is a compact 1–dimensional manifold, i.e. a disjoint union of circles. Denote by π the projection from $P_{\Delta\setminus\{\alpha\}}$ to the quotient. Let $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and let Φ be the map from Lemma 2.2.8. The map $\gamma\colon \mathbb{R} \to P_{\Delta\setminus\{\alpha\}}/B_0$ defined by $\gamma(t) = \pi(\Phi(\exp(te))s)$ is an injective smooth curve in this 1–manifold. This is because the map $\mathbb{R} \to U_\alpha,\ t \mapsto \Phi(\exp(te)) = \exp(tE_\alpha)$ is injective, and the map $U_\alpha \to P_{\Delta\setminus\{\alpha\}}/B_0,\ u \mapsto \pi(us)$ is injective as a consequence of Lemma 2.2.3. Therefore, its limits for $t \to \pm\infty$ exist and $\overline{\gamma(\mathbb{R})} = \gamma(\mathbb{R}) \cup \{\gamma(\pm\infty)\}$. Now by Lemma 2.2.3

$$B_0sB_0 = U_\alpha sB_0 = \exp(\mathfrak{u}_\alpha)sB_0 = \pi^{-1}(\pi(\exp(\mathfrak{u}_\alpha)s)) = \pi^{-1}(\gamma(\mathbb{R})),$$

so

$$\overline{B_0sB_0} = \pi^{-1}(\overline{\gamma(\mathbb{R})}) = B_0sB_0 \cup \pi^{-1}(\gamma(\infty)) \cup \pi^{-1}(\gamma(-\infty)).$$

To compute the limits, note that

$$\Phi\begin{pmatrix} |t|^{-1} & \mathrm{sgn}(t) \\ 0 & |t| \end{pmatrix} = \exp\left[\mathrm{d}_1\Phi\begin{pmatrix} -\log|t| & 0 \\ 0 & \log|t| \end{pmatrix}\right] \exp\left[\mathrm{d}_1\Phi\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}\right] \in B_0,$$

so

$$\begin{aligned}\lim_{t\to\pm\infty} \gamma(t) &= \lim_{t\to\pm\infty} \pi(\Phi(\exp(te)s)) = \lim_{t\to\pm\infty} \pi\left[\Phi\left[\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} |t|^{-1} & \mathrm{sgn}(t) \\ 0 & |t| \end{pmatrix}\right]\right] \\ &= \pi\left[\Phi\left[\lim_{t\to\pm\infty}\begin{pmatrix} -\mathrm{sgn}(t) & 0 \\ -|t|^{-1} & -\mathrm{sgn}(t) \end{pmatrix}\right]\right] = \pi(\Phi(\mp 1)) = \pi(s^{1\pm 1}).\end{aligned}$$

So $\overline{B_0sB_0} = B_0sB_0 \cup B_0 \cup B_0s^2B_0$, which is (2.4).

Since $B_0 \cup B_0sB_0 \cup B_0s^2B_0 \cup B_0s^3B_0 = \overline{B_0sB_0} \cup \overline{B_0s^3B_0} \subset P_{\Delta\setminus\{\alpha\}}$, $P_{\Delta\setminus\{\alpha\}}$ decomposes into the disjoint union

$$P_{\Delta\setminus\{\alpha\}} = \bigsqcup_{m\in\overline{M}} (B_0 \cup B_0sB_0)\, m = \bigsqcup_{m\in\langle s^2\rangle\setminus\overline{M}} (\overline{B_0sB_0} \cup \overline{B_0s^3B_0})\, m.$$

Therefore, $\overline{B_0sB_0} \cup \overline{B_0s^3B_0}$ is closed and open in $P_{\Delta\setminus\{\alpha\}}$, hence equal to $(P_{\Delta\setminus\{\alpha\}})_0$.

Finally, to prove (2.5), we claim that, for $t, \tau \in \mathbb{R}$,

$$\pi(\exp(tE_\alpha)s\exp(\tau E_\alpha)s^{-1}) = \begin{cases} \pi(1) & \text{if } \tau = 0, \\ \pi(\exp((t-\tau^{-1})E_\alpha)s^{\mathrm{sgn}(\tau)}) & \text{if } \tau \neq 0. \end{cases} \tag{2.6}$$

This then shows $\pi(U_\alpha s U_\alpha s^{-1}) = \pi(1) \cup \pi(U_\alpha s) \cup \pi(U_\alpha s^{-1})$ and therefore

$$B_0 s B_0 s^{-1} B_0 = U_\alpha s U_\alpha s^{-1} B_0 = B_0 \cup U_\alpha s B_0 \cup U_\alpha s^{-1} B_0 = B_0 \cup B_0 s B_0 \cup B_0 s^{-1} B_0.$$

The claim (2.6) is clear if $\tau = 0$, since $\exp(tE_\alpha) \in B_0$. So let $\tau \neq 0$. Then

$$\begin{aligned}\pi(e^{tE_\alpha} s e^{\tau E_\alpha} s^{-1}) &= \pi\left[\Phi\left[\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & \tau\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right]\right]\\ &= \pi\left[\Phi\left[\begin{pmatrix}1 & t-\tau^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & \operatorname{sgn}(\tau)\\ -\operatorname{sgn}(\tau) & 0\end{pmatrix}\begin{pmatrix}|\tau| & -\operatorname{sgn}(\tau)\\ 0 & |\tau|^{-1}\end{pmatrix}\right]\right]\\ &= \pi\left[\Phi\left[\begin{pmatrix}1 & t-\tau^{-1}\\ 0 & 1\end{pmatrix}\right] s^{\operatorname{sgn}(\tau)}\right] = \pi(\exp((t-\tau^{-1})E_\alpha) s^{\operatorname{sgn}(\tau)}). \qquad \square\end{aligned}$$

In preparation for the general case, the next three lemmas show how products of refined Bruhat cells behave.

Lemma 2.2.10. *For any $w_1, w_2 \in \widetilde{W}$ with $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ we have $B_0 w_1 w_2 B_0 = B_0 w_1 B_0 w_2 B_0$.*

Proof. From Lemma 2.2.3 we obtain the identities $B_0 w_1 B_0 w_2 B_0 = U_{w_1} w_1 U_{w_2} w_2 B_0$ and $B_0 w_1 w_2 B_0 = U_{w_1 w_2} w_1 w_2 B_0$. We want to show $U_{w_1 w_2} = U_{w_1} w_1 U_{w_2} w_1^{-1}$. By Lemma 2.2.2 both sides can be written as products of U_α for some set of α. For the left hand side, the product is taken over all $\alpha \in \Psi_{w_1 w_2}$ while for the right hand side we need all $\alpha \in \Psi_{w_1} \cup w_1 \Psi_{w_2}$. But it follows from Lemma 2.2.7 that $\Psi_{w_1 w_2} = \Psi_{w_1} \cup w_1 \Psi_{w_2}$ if $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. $\square$

Lemma 2.2.11. *Let $w \in \widetilde{W}$ and $s = \mathsf{v}(\alpha) \in \widetilde{W}$ for some $\alpha \in \Delta$. Then $\ell(sw) = \ell(w) \pm 1$ and*

$$B_0 s B_0 w B_0 = \begin{cases} B_0 s w B_0 & \text{if } \ell(sw) = \ell(w) + 1,\\ B_0 w B_0 \cup B_0 s w B_0 \cup B_0 s^2 w B_0 & \text{if } \ell(sw) = \ell(w) - 1.\end{cases}$$

Proof. Clearly $|\ell(sw) - \ell(w)| \leq 1$ since $\pi(s) \in W$ is in the generating system, but also $\ell(sw) \neq \ell(w)$ by the property of Coxeter groups that only words with an even number of letters can represent the identity. If $\ell(sw) = \ell(w) + 1$, then the statement follows from Lemma 2.2.10. Assume $\ell(sw) = \ell(w) - 1$. Then $\ell(s^{-1}w) = \ell(s^{-2}sw) = \ell(w) - 1$ since $\ell(s^{-2}) = 0$. So $B_0 s B_0 s^{-1} w B_0 = B_0 w B_0$ by the first part and also $B_0 s B_0 s B_0 = B_0 s B_0 \cup B_0 s^2 B_0 \cup B_0 s^3 B_0$ by Lemma 2.2.9. We thus get

$$\begin{aligned}B_0 s B_0 w B_0 &= B_0 s B_0 s B_0 s^{-1} w B_0 = B_0 s B_0 s^{-1} w B_0 \cup B_0 s^2 B_0 s^{-1} w B_0 \cup B_0 s^3 B_0 s^{-1} w B_0\\ &= B_0 w B_0 \cup B_0 s w B_0 \cup B_0 s^2 w B_0,\end{aligned}$$

again using the first part of the lemma for the last equality. $\square$

Lemma 2.2.12. *Let $w \in \widetilde{W}$ and $s = \mathsf{v}(\alpha) \in \widetilde{W}$ for some $\alpha \in \Delta$. Then $\overline{B_0 s B_0 w B_0} = \overline{B_0 s B_0}\ \overline{B_0 w B_0}$ and $\overline{B_0 s B_0} B_0 w B_0 = B_0 w B_0 \cup B_0 s w B_0 \cup B_0 s^2 w B_0$.*

Proof. For the first part, note that $\overline{B_0 s B_0}\ \overline{B_0 w B_0} \subset \overline{B_0 s B_0 w B_0}$. We want to prove that $\overline{B_0 s B_0}\ \overline{B_0 w B_0}$ is closed. Consider the map

$$f \colon G/B_0 \to \mathcal{C}(G/B_0), \quad gB_0 \mapsto g\overline{B_0 w B_0}$$

where $\mathcal{C}(G/B_0)$ is the set of closed subsets of G/B_0. Since G/B_0 is a compact space and f is G–equivariant, the space $\mathcal{C}(G/B_0)$ is compact with the Hausdorff metric, f is continuous and for any closed subset $A \subset G/B_0$ the union $\bigcup_{x \in A} f(x)$ is closed (see e.g. Proposition 2.3.1, Lemma 2.3.8(i), and Lemma 2.3.8(ii)).

In particular, the union of all elements of $f(\overline{B_0sB_0}/B_0)$ is a closed subset of G/B_0, and so is its preimage in G. But this is just $\overline{B_0sB_0}\ \overline{B_0wB_0}$, which is therefore a closed set containing $B_0sB_0wB_0$, hence equal to $\overline{B_0sB_0wB_0}$.

For the second part, Lemma 2.2.9 implies that

$$\overline{B_0sB_0}B_0wB_0 = B_0wB_0 \cup B_0sB_0wB_0 \cup B_0s^2B_0wB_0$$

and in both cases of Lemma 2.2.11 this equals what we want. ☐

We now arrive at the following combinatorial description of closures of refined Bruhat cells.

Proposition 2.2.13. *Let $w \in \widetilde{W}$ and $\pi(w) = \alpha_1 \dots \alpha_k$ a reduced expression by simple root reflections for the projection $\pi(w) \in W$ of w to the Weyl group. Then $w = \mathsf{v}(\alpha_1)\dots\mathsf{v}(\alpha_k)m$ for some $m \in \overline{M}$. Let*

$$A_w = \{\mathsf{v}(\alpha_1)^{i_1} \dots \mathsf{v}(\alpha_k)^{i_k}\, m \mid i_1, \dots, i_k \in \{0,1,2\}\} \subset \widetilde{W}.$$

be the set of words that can be obtained by deleting or squaring some of the letters. Then

$$\overline{B_0wB_0} = \bigcup_{w' \in A_w} B_0w'B_0.$$

In particular, A_w does not depend on the choice of reduced word for $\pi(w)$.

Proof. First of all, since $\pi(\mathsf{v}(\alpha_1)\dots\mathsf{v}(\alpha_k)) = \alpha_1\dots\alpha_k = \pi(w)$, there exists $m \in \overline{M}$ such that $w = \mathsf{v}(\alpha_1)\dots\mathsf{v}(\alpha_k)m$.
We now prove the second statement by induction on $\ell(w)$. If $\ell(w) = 0$, then $w \in \overline{M}$, so $B_0wB_0 = wB_0$ is already closed, and $A_w = \{w\}$. Now let $\ell(w) > 0$ and assume the statement is already proven for all $\widetilde{w} \in \widetilde{W}$ with $\ell(\widetilde{w}) < \ell(w)$. Assume that $w = \mathsf{v}(\alpha_1)\dots\mathsf{v}(\alpha_k)m$ as above. Then we can write $w = s\widetilde{w}$ with $s = \mathsf{v}(\alpha_1)$ and $\ell(\widetilde{w}) = \ell(w) - 1$. Using Lemma 2.2.11 and Lemma 2.2.12 we get

$$\begin{aligned}\overline{B_0wB_0} = \overline{B_0s\widetilde{w}B_0} = \overline{B_0sB_0\widetilde{w}B_0} = \overline{B_0sB_0}\ \overline{B_0\widetilde{w}B_0} &= \bigcup_{w' \in A_{\widetilde{w}}} \overline{B_0sB_0}\, B_0w'B_0 \\ = \bigcup_{w' \in A_{\widetilde{w}}} B_0w'B_0 \cup B_0sw'B_0 \cup B_0s^2w'B_0 &= \bigcup_{w' \in A_w} B_0w'B_0.\end{aligned}$$ ☐

2.3 Group actions on compact homogeneous spaces

The goal of this section is to give a detailed proof of Proposition 2.3.11, which gives a sufficient criterion for cocompactness of a discrete group action on a compact homogeneous space. This is the equivalent of [KLP18, Proposition 5.30] in a slightly more general setting. All key arguments of that paper still work.

For a compact metric space Z let $\mathcal{C}(Z)$ be the space of closed subsets, equipped with the Hausdorff metric. The following fact will be useful later on (see for example [BH99, Lemma 5.31] for a proof).

Proposition 2.3.1. *The space $\mathcal{C}(Z)$, equipped with the Hausdorff metric d_H, is a compact metric space.*

Definition 2.3.2. Let Z be a metric space, g a homeomorphism of Z and Γ a group acting on Z by homeomorphisms.

(i) g is *expanding at* $z \in Z$ if there exists an open neighbourhood $z \in U \subset Z$ and a constant $c > 1$ (the *expansion factor*) such that

$$d(gx, gy) \geq c\, d(x, y)$$

for all $x, y \in U$.

(ii) Let $A \subset Z$ be a subset. The action of Γ on Z is *expanding at* A if for every $z \in A$ there is a $\gamma \in \Gamma$ which is expanding at z.

In [KLP18], expansion was used together with a new and more general definition of transverse expansion to prove cocompactness.

Definition 2.3.3 ([KLP18, Definition 5.28])**.** Let Z be a compact metric space, g a homeomorphism of Z and $\mathcal{Q}\colon \Lambda \to \mathcal{C}(Z)$ a map from any set Λ.
Then g is *expanding at $z \in Z$ transversely to $\mathcal{Q}$* if there is an open neighbourhood $z \in U$ and a constant $c > 1$ such that

$$d(gx, g\mathcal{Q}(\lambda)) \geq c\, d(x, \mathcal{Q}(\lambda))$$

for all $x \in U$ and all $\lambda \in \Lambda$ with $\mathcal{Q}(\lambda) \cap U \neq \varnothing$.

Lemma 2.3.4 ([KLP18, Remark 5.22])**.** *If the action of Γ on Z is expanding at a closed Γ–invariant subset $A \subset Z$ then it is arbitrarily strongly expanding, i.e. for every $z \in A$ and $c > 1$ there is a $\gamma \in \Gamma$ which is expanding at z with expansion factor c.*

Proof. If the action is expanding at $z \in A$ with some expansion factor, then it is expanding by the same factor in a neighbourhood of z. By covering A with finitely many such neighborhoods, we can assume that the action is expanding with a uniform expansion factor $C > 1$. Now let $z \in A$ and let $\gamma_1 \in \Gamma$ be expanding at z by the factor C. Let $\gamma_2 \in \Gamma$ be expanding at $\gamma_1 z \in A$ by C. Then $\gamma_2\gamma_1$ expands at z by C^2. Iterating this, we get an element $\gamma_n \cdots \gamma_1 \in \Gamma$ which is expanding with expansion factor $C^n \geq c$ at z. □

Let G be a Lie group and X, Y be compact G–homogeneous spaces. Fix Riemannian metrics on X, Y and a left–invariant Riemannian metric on G. Recall that smooth maps between manifolds are locally Lipschitz with respect to any Riemannian distances.

Lemma 2.3.5. *There exists a compact subset $S \subset G$ such that for every pair $(x, y) \in X^2$, there exists $s_{xy} \in S$ satisfying $s_{xy}x = y$.*

Proof. Fix a basepoint $x_0 \in X$ and let V be a precompact open neighborhood of the identity in G. Since $G \to X,\ g \mapsto gx_0$ is a submersion, Vx_0 is a neighborhood of x_0. Then by compactness there are finitely many $g_1, \dots, g_n \in G$ such that the sets $g_i V x_0$ cover X. So $S = \overline{g_1 V} \cup \cdots \cup \overline{g_n V}$ is a compact subset of G which maps x_0 to any point in X. The set SS^{-1} is compact and satisfies the desired transitivity property. □

Lemma 2.3.6. *There exists a constant $C > 0$ such that the following holds: For any two points $x, y \in X$, there exists $g \in G$ satisfying $gx = y$ and $d(1, g) \leq Cd(x, y)$.*

Proof. Assume by contradiction that there are sequences $(x_n), (y_n) \in X^{\mathbb{N}}$ such that every g_n sending x_n to y_n must satisfy $d(1, g_n) > nd(x_n, y_n)$. After taking subsequences, we have $x_n \to x,\ y_n \to y$. If $x \neq y$, we obtain in particular that $d(1, g_n) \to \infty$ for every choice of g_n sending x_n to y_n. But by Lemma 2.3.5, a compact subset of G already acts transitively on X, so g_n can be chosen such that $d(1, g_n)$ remains bounded. We are thus left with the case $x = y$. Since the map $G \to X,\ g \mapsto gx$ is a smooth submersion, there exists a local section at x: There is a neighborhood $x \in U$ and a smooth map $s : U \to G$ satisfying $s(x) = 1$ and $s(z)x = z$ for every $z \in U$. After shrinking U if necessary, s is C'–Lipschitz for some $C' > 0$. For large n, x_n and y_n are inside U, and we have $s(y_n)s(x_n)^{-1}x_n = y_n$. Since inversion in G is a smooth map and therefore C''–Lipschitz close to the identity, it follows that (after possibly shrinking U some more)

$$d\big(1, s(y_n)s(x_n)^{-1}\big) = d\big(s(y_n)^{-1}, s(x_n)^{-1}\big) \leq C'' d(s(y_n), s(x_n)) \leq C'C'' d(y_n, x_n),$$

a contradiction. □

Lemma 2.3.7. *Let $A \subset G$ be a compact set. Then there exists a constant $C > 0$ such that:*

- *The map $A \to X,\ g \mapsto gx$ is C–Lipschitz.*
- *For every $g \in A$, the diffeomorphism $X \to X,\ x \mapsto gx$ is C–Lipschitz.*

Proof. The map $G \times X \to X,\ (g, x) \mapsto gx$ is smooth and thus locally Lipschitz. Its restriction to the compact set $A \times X$ is therefore Lipschitz. This implies both parts of the claim. □

The following auxiliary lemma is a combination of the corresponding statements in Lemmas 7.1, 7.3 and 7.4 in [KLP18], transferred to our setting.

Lemma 2.3.8. *Let*

$$\mathcal{Q} \colon X \to \mathcal{C}(Y)$$

be a G–equivariant map. Then there are constants $L, D > 0$ such that

(i) $\mathcal{Q}$ is L–Lipschitz.

(ii) If $A \subset X$ is compact, then $\bigcup_{x \in A} \mathcal{Q}(x)$ is compact.

(iii) For all $x \in X$ and $y \in Y$ there exists $x' \in X$ such that $y \in \mathcal{Q}(x')$ and

$$d(x', x) \leq D\, d(y, \mathcal{Q}(x)).$$

(iv) If $\Lambda \subset X$ is compact with $\mathcal{Q}(\lambda) \cap \mathcal{Q}(\lambda') = \varnothing$ for all distinct $\lambda, \lambda' \in \Lambda$, then the map

$$\pi\colon \bigcup_{\lambda\in\Lambda} \mathcal{Q}(\lambda) \to \Lambda$$

mapping every point of $\mathcal{Q}(\lambda)$ to λ is a uniformly continuous fiber bundle (in the subspace topologies).

Proof.

(i) Let $x, y \in X$ be arbitrary. Using Lemma 2.3.6, we choose $g \in G$ such that $gx = y$ and $d(1, g) \le Cd(x, y)$. By equivariance, $g\mathcal{Q}(x) = \mathcal{Q}(y)$ holds. As $\operatorname{diam}(X)$ is finite, Lemma 2.3.7 implies that $d(gz, hz) \le C'd(g, h)$ for any $g, h \in \overline{B_{C\cdot\operatorname{diam}(X)}(1)}$ and $z \in Y$. Both constants C, C' do not depend on the choice of x and y. Therefore,

$$\begin{aligned} d_H(\mathcal{Q}(x), \mathcal{Q}(y)) &= \max\left\{\max_{a\in\mathcal{Q}(x)} d(a, g\mathcal{Q}(x)), \max_{gb\in g\mathcal{Q}(x)} d(\mathcal{Q}(x), gb)\right\} \\ &\le \max\left\{\max_{a\in\mathcal{Q}(x)} d(a, ga), \max_{gb\in g\mathcal{Q}(x)} d(b, gb)\right\} \le C'd(1, g) \le CC'd(x, y). \end{aligned}$$

(ii) Let (y_n) be a sequence in $\bigcup_{x\in A} \mathcal{Q}(x)$ and $x_n \in A$ such that $y_n \in \mathcal{Q}(x_n)$. Passing to a subsequence we can assume that $y_n \to y \in Y$ and $x_n \to x \in A$. But

$$d(y_n, \mathcal{Q}(x)) \le d_H(\mathcal{Q}(x_n), \mathcal{Q}(x)) \to 0$$

by (i), so $d(y, \mathcal{Q}(x)) = 0$, which means $y \in \mathcal{Q}(x)$ since $\mathcal{Q}(x)$ is closed.

(iii) Let $a \in \mathcal{Q}(x)$ be such that $d(y, \mathcal{Q}(x)) = d(y, a)$. By Lemma 2.3.6, there is an element g with $ga = y$ and $d(1, g) \le Cd(y, a)$. Moreover, since $\operatorname{diam}(Y)$ is finite, Lemma 2.3.7 implies that $d(x, gx) \le C'd(1, g)$. Therefore, gx is the point x' we were looking for.

(iv) We start by showing continuity of π. Assume that $y_n \in \mathcal{Q}(x_n)$, $y_n \to y \in \bigcup_{\lambda\in\Lambda} \mathcal{Q}(\lambda)$ and $\pi(y_n) = x_n \to x \in \Lambda$. We need to show that $\pi(y) = x$. Since $\mathcal{Q}$ is continuous, we have $\mathcal{Q}(x_n) \to \mathcal{Q}(x)$ in $\mathcal{C}(Y)$. Therefore, $d(y, \mathcal{Q}(x)) = 0$, so $y \in \mathcal{Q}(x)$ and $\pi(y) = x$ as $\mathcal{Q}(x)$ is closed. By compactness of $\bigcup_{\lambda\in\Lambda} \mathcal{Q}(\lambda)$ (according to (ii)), π is uniformly continuous.

Now we construct a local trivialization. Let $x \in \Lambda$ be a point, U a neighborhood of x in X, and $s : U \to G$ a smooth local section of the submersion $G \to X,\ g \mapsto gx$. Then the map

$$\begin{aligned} (\Lambda \cap U) \times \mathcal{Q}(x) &\to \bigcup_{\lambda\in\Lambda} \mathcal{Q}(\lambda) \\ (\lambda, y) &\mapsto s(\lambda)y \end{aligned}$$

is a homeomorphism onto its image, since its inverse is given by

$$y \mapsto (\pi(y), s(\pi(y))^{-1}y).$$

□

The following key lemma shows how expansion in X leads to expansion transverse to the map $\mathcal{Q}\colon X \to \mathcal{C}(Y)$ in Y (compare [KLP18, Lemma 7.5]).

Lemma 2.3.9. *Let $\mathcal{Q}\colon X \to \mathcal{C}(Y)$ be G–equivariant and $\Lambda \subset X$ compact with $\mathcal{Q}(\lambda) \cap \mathcal{Q}(\lambda') = \varnothing$ for all distinct $\lambda, \lambda' \in \Lambda$. Let $g \in G$ be expanding at $\lambda \in \Lambda$ with expansion factor $c > LD$, where L and D are the constants from Lemma 2.3.8. Then g is expanding at every $y \in \mathcal{Q}(\lambda)$ transversely to $\mathcal{Q}|_\Lambda$.*

Proof. We give a short outline of the proof before delving into the details. Let V be a neighborhood of the point $y \in \mathcal{Q}(\lambda)$, $y' \in V$ and $\lambda' \in \Lambda$ such that $\mathcal{Q}(\lambda') \cap V \neq \varnothing$. We want to choose $x \in X$ with $y' \in \mathcal{Q}(x)$ such that the following string of inequalities holds:

$$d(y', \mathcal{Q}(\lambda')) \leq d_H(\mathcal{Q}(x), \mathcal{Q}(\lambda')) \leq L\, d(x, \lambda') \overset{(*)}{\leq} c^{-1} L\, d(gx, g\lambda') \overset{(**)}{\leq} c^{-1} LD\, d(gy', \mathcal{Q}(g\lambda'))$$

The first two inequalities are true for any choice of x. For $(*)$, x and λ' need to be close to λ so g is expanding. For $(**)$, we need gx to be a "good" choice in the sense of Lemma 2.3.8(iii). Our task is to make sure that the choices of V and x can be made accordingly.

Since g is expanding, there is an open neighbourhood $U \subset X$ of λ with $d(gz, gz') \geq c\, d(z, z')$ for all $z, z' \in U$. We can assume that $U = B_\varepsilon(\lambda)$ for some $\varepsilon > 0$. Let π be the bundle map from Lemma 2.3.8(iv). Since π is uniformly continuous there is $\delta > 0$ with

$$d(\pi(y), \pi(y')) < \frac{\varepsilon}{2} \quad \text{whenever} \quad d(y, y') < \delta. \tag{2.7}$$

We can assume that $\delta \leq \frac{\varepsilon}{4\alpha\beta D}$ where α and β are Lipschitz constants for the action of g^{-1} on X and the action of g on Y.

Let $V = B_\delta(y)$ and let $\lambda' \in \Lambda$ with $\mathcal{Q}(\lambda') \cap V \neq \varnothing$ and $y' \in V$. Then by Lemma 2.3.8(iii) (applied to $g\lambda'$ and gy') there is gx such that $gy' \in \mathcal{Q}(gx)$ and thus $y' \in \mathcal{Q}(x)$ and

$$d(gx, g\lambda') \overset{(**)}{\leq} D\, d(gy', \mathcal{Q}(g\lambda')).$$

Next we want to show that $x, \lambda' \in U = B_\varepsilon(\lambda)$ by bounding $d(\lambda, \lambda')$ and $d(x, \lambda')$. First, since $\mathcal{Q}(\lambda')$ intersects V, there is a point $p \in \mathcal{Q}(\lambda')$ with $d(p, y) < \delta$. So $d(\lambda, \lambda') = d(\pi(y), \pi(p)) < \varepsilon/2$ by (2.7). Second,

$$d(x, \lambda') \leq \alpha\, d(gx, g\lambda') \leq \alpha D\, d(gy', g\mathcal{Q}(\lambda')) \leq \alpha\beta D\, d(y', \mathcal{Q}(\lambda')) \leq 2\alpha\beta D\delta \leq \varepsilon/2,$$

so $x \in U$ and also $\lambda' \in U$. This implies $(*)$. Therefore,

$$d(y', \mathcal{Q}(\lambda')) \leq c^{-1} LD\, d(gy', \mathcal{Q}(g\lambda')),$$

and $c^{-1}LD < 1$, so g is transversely expanding. □

Lemma 2.3.10. *Let Y be a compact metric space acted upon by a group Γ and $\Xi \subset Y$ a compact, Γ–invariant subset. Assume that for every $y \in \Xi$, there exists a neighborhood U_y, an element $\gamma \in \Gamma$ and a constant $c > 1$ such that*

$$d(\gamma y', \Xi) \geq c\, d(y', \Xi) \quad \forall y' \in U.$$

Then Γ acts cocompactly on $Y \setminus \Xi$.

Proof. By compactness of Ξ, we can find finitely many points $y_1, \dots, y_n \in Y$ such that their associated neighborhoods U_{y_i} cover Ξ. Moreover, there exists a $\delta > 0$ such that their union $\bigcup_i U_{y_i}$ contains the δ–neighborhood $N_\delta(\Xi)$. Let $c > 1$ be the minimal expansion factor of the corresponding elements γ_i. We will show that every orbit $\Gamma y,\ y \in Y \setminus \Xi$ has a representative in $Y \setminus N_\delta(\Xi)$. This will prove the lemma since $Y \setminus N_\delta(\Xi)$ is compact.
Indeed, if $y \in N_\delta(\Xi) \setminus \Xi$, there exists $i_0 \in \{1, \dots, n\}$ such that $y \in U_{y_{i_0}}$. Therefore, $d(\gamma_{i_0} y, \Xi) \geq c d(y, \Xi)$. If $d(\gamma_{i_0} y, \Xi) \geq \delta$, we are done. Else, we repeat the procedure until we obtain a point in the orbit which does not lie in $N_\delta(\Xi)$. □

After these preparations, we now turn to our goal for this section: A criterion for group actions to be cocompact. Let $\rho\colon \Gamma \to G$ be a representation of a discrete group into a Lie group G. This defines an action of Γ on any compact G–homogeneous spaces X and Y. Connecting expansion, transverse expansion and Lemma 2.3.10 yields the following useful result (compare [KLP18, Proposition 5.30]):

Proposition 2.3.11. *Let $\mathcal{Q}\colon X \to \mathcal{C}(Y)$ be G–equivariant and let $\Lambda \subset X$ be compact and Γ–invariant with $\mathcal{Q}(\lambda) \cap \mathcal{Q}(\lambda') = \varnothing$ for all distinct $\lambda, \lambda' \in \Lambda$. Also assume that the action of Γ on X is expanding at Λ. Then Γ acts cocompactly on $\Omega = Y \setminus \bigcup_{\lambda\in\Lambda} \mathcal{Q}(\lambda)$.*

Proof. Since Λ is closed and Γ–invariant, we know by Lemma 2.3.4 that the action of Γ is expanding arbitrarily strongly at every point $\lambda \in \Lambda$. Lemma 2.3.9 therefore shows that the action of Γ is expanding at every point of $\bigcup_{\lambda\in\Lambda} \mathcal{Q}(\lambda)$ transversely to $\mathcal{Q}$. We will show that this implies the prerequisites of Lemma 2.3.10 and thus the action on Ω is cocompact.
First of all, we observe that for any point $z \in Y$, we have

$$d\Big(z, \bigcup_{\lambda\in\Lambda} \mathcal{Q}(\lambda)\Big) = d(z, \mathcal{Q}(\lambda_z)) \tag{2.8}$$

for some $\lambda_z \in \Lambda$ (this follows from compactness of $\bigcup_{\lambda\in\Lambda} \mathcal{Q}(\lambda)$, Lemma 2.3.8(ii)). Now let $y \in \bigcup_{\lambda\in\Lambda} \mathcal{Q}(\lambda)$ and let a neighborhood $U \ni y$ and $\gamma \in \Gamma$ be chosen such that γ is c–expanding on U transversely to $\mathcal{Q}$. There exists $\varepsilon > 0$ satisfying

$$B_\varepsilon(\gamma y) \subset \gamma U.$$

Let $\delta > 0$ be sufficiently small such that

$$\gamma B_\delta(y) \subset B_{\varepsilon/2}(\gamma y).$$

For any point $y' \in B_\delta(y)$, let $\lambda_{\gamma y'}$ be chosen as in (2.8). Since $d(\gamma y', \gamma y) < \varepsilon/2$ and $\gamma y \in \bigcup_{\lambda\in\Lambda} \mathcal{Q}(\lambda)$, we necessarily have $\mathcal{Q}(\lambda_{\gamma y'}) \cap B_\varepsilon(\gamma y) \neq \varnothing$. Therefore,

$$\gamma^{-1}\mathcal{Q}(\lambda_{\gamma y'}) \cap U \neq \varnothing.$$

Since $y' \in B_\delta(y) \subset U$, transverse expansion now implies

$$d\Big(\gamma y', \bigcup_{\lambda\in\Lambda} \mathcal{Q}(\lambda)\Big) = d(\gamma y', \mathcal{Q}(\lambda_{\gamma y'})) \geq c d(y', \gamma^{-1}\mathcal{Q}(\lambda_{\gamma y'})) \geq c d\Big(y', \bigcup_{\lambda\in\Lambda} \mathcal{Q}(\lambda)\Big). \qquad \square$$

3 Balanced ideals and domains of discontinuity of Anosov representations

This chapter will prove the results described in Section 1.4, that all cocompact domains of discontinuity for Δ–Anosov representations come from balanced ideals. It is identical to [Ste18] up to minor changes like harmonizing the notation with the rest of this thesis.

3.1 Domains of discontinuity

In this section, we prove the main theorems, Theorem 3.1.17 and Theorem 3.1.23 which correspond to Theorem 1.4.1 and Theorem 1.4.2 in the introduction.

3.1.1 Divergent sequences

We first consider the behaviour of divergent sequences $(g_n) \in G^{\mathbb{N}}$ in the semi–simple Lie group G. Let $\theta, \eta \subset \Delta$ be non–empty subsets of the simple restricted roots and assume $\iota(\theta) = \theta$.

Recall that we call a sequence $(g_n) \in G^{\mathbb{N}}$ *θ–divergent* if $\alpha(\mu(g_n)) \to \infty$ for all $\alpha \in \theta$.

Definition 3.1.1. A θ–divergent sequence $(g_n) \in G^{\mathbb{N}}$ is *simply θ–divergent* if it has KAK–decompositions $g_n = k_n a_n \ell_n$ such that (k_n) and (ℓ_n) converge to some $k, \ell \in K$. Then $g^- = [\ell^{-1} w_0] \in \mathcal{F}_\theta$ and $g^+ = [k] \in \mathcal{F}_\theta$ are the *repelling and attracting limits* of this sequence.

Remark 3.1.2. By compactness of K, every θ–divergent sequence in G has a simply θ–divergent subsequence. The limits (g^-, g^+) do not depend on the choice of decomposition. If a simply θ–divergent sequence $(g_n) \in G^{\mathbb{N}}$ has limits $(g^-, g^+) \in \mathcal{F}_\theta^2$, then (g_n^{-1}) is also simply θ–divergent with limits (g^+, g^-).

In [KLP18], the following characterization of θ–divergent sequences is used.

Lemma 3.1.3. *Let $(g_n) \in G^{\mathbb{N}}$ be a simply θ–divergent sequence with limits $(g^-, g^+) \in \mathcal{F}_\theta^2$. Then*

$$g_n|_{C_{w_0}(g^-)} \to g^+$$

locally uniformly as functions from $\mathcal{F}_\theta$ to $\mathcal{F}_\theta$ (where g^+ is the constant function).

Proof. By assumption, we can write $g_n = k_n e^{A_n} \ell_n$ with $k_n \to k$ and $\ell_n \to \ell$ in K, $A_n \in \overline{\mathfrak{a}^+}$ and $\alpha(A_n) \to \infty$ for all $\alpha \in \theta$. Furthermore, $g^- = [\ell^{-1} w_0]$ and $g^+ = [k]$. Now let $(f_n) \in \mathcal{F}_\theta^{\mathbb{N}}$ be a sequence converging to $f \in C_{w_0}(g^-)$. Then $\ell_n f_n \to \ell f \in C_{w_0}([w_0])$. Since this is an open set, we can assume that $\ell_n f_n \in C_{w_0}([w_0])$ for all n. By the Langlands decomposition of P_θ [Kna02, Proposition 7.83] we can write $\ell_n f_n = [\exp(X_n)]$ with

$$X_n = \sum_\alpha X_n^\alpha \in \bigoplus_{\alpha \in \Sigma^- \setminus \operatorname{span}(\Delta \setminus \theta)} \mathfrak{g}_\alpha.$$

All of the roots α appearing in this sum are linear combinations of simple roots with only non–positive coefficients, and with at least one coefficient of a root in θ being negative. So $\alpha(A_n) \to -\infty$ for all such roots α, and therefore

$$g_n f_n = [k_n e^{A_n} e^{X_n}] = [k_n e^{A_n} e^{X_n} e^{-A_n}] = \left[k_n \exp\left(\sum_\alpha e^{\alpha(A_n)} X_n^\alpha \right) \right] \to [k] = g^+. \qquad \square$$

Lemma 3.1.4 is the key step of the proof of proper discontinuity in [KLP18, Proposition 6.5]. It states that flags in $\mathcal{F}_\eta$ can only be dynamically related (Definition 2.1.8) by the action of ρ if their relative positions satisfy the inequality (3.1). Lemma 3.1.6 is a converse to this statement in the case $\theta = \Delta$: It says that whenever two flags f, f' satisfy a relation like (3.1), then they are indeed dynamically related.

Lemma 3.1.4. *Let $(g_n) \in G^{\mathbb{N}}$ be a simply θ–divergent sequence and let $(g^-, g^+) \in \mathcal{F}_\theta^2$ be its limits. Let $f, f' \in \mathcal{F}_\eta$ be dynamically related via (g_n). Then*

$$\operatorname{pos}_{\theta,\eta}(g^+, f') \leq w_0 \operatorname{pos}_{\theta,\eta}(g^-, f). \tag{3.1}$$

Proof. As f, f' are dynamically related, there exists a sequence $f_n \in \mathcal{F}_\eta$ converging to f such that $g_n f_n \to f'$. We can write $f_n = h_n f$ for some sequence $h_n \in G$ converging to 1. Let $w = \operatorname{pos}_{\theta,\eta}(g^-, f)$. Then there exists $g \in G$ such that $g(g^-, f) = ([1], [w])$. Define $F = [g^{-1} w_0] \in \mathcal{F}_\theta$. We get the following relative positions:

$$\operatorname{pos}_{\theta,\theta}(g^-, F) = w_0, \qquad \operatorname{pos}_{\theta,\eta}(g^-, f) = w, \qquad \operatorname{pos}_{\theta,\eta}(F, f) = w_0 w.$$

Now $h_n F \to F$ and since F and almost all of the $h_n F$ are in $C_{w_0}(g^-)$, we get by Lemma 3.1.3 that $g_n h_n F \to g^+$. So

$$\operatorname{pos}(g^+, f') \leq \operatorname{pos}(g_n h_n F, g_n h_n f) = \operatorname{pos}(F, f) = w_0 w = w_0 \operatorname{pos}(g^-, f). \qquad \square$$

Lemma 3.1.5. *Let $(A_n) \in \overline{\mathfrak{a}^+}^{\mathbb{N}}$ be a Δ–divergent sequence and $n^+ \in N, n^- \in N^-$. Then there exists a sequence $(h_n) \in G^{\mathbb{N}}$ such that $h_n \to n^-$ and $e^{A_n} h_n e^{-A_n} \to n^+$.*

Proof. We can write $n^- = e^{X^-}$ for $X^- \in \mathfrak{n}^-$ and $n^+ = e^{X^+}$ with $X^+ \in \mathfrak{n}$. Let $H_n = X^- + e^{-\operatorname{ad} A_n} X^+$ and $h_n = e^{H_n}$. For all $\alpha \in \Sigma^+$ and $X_\alpha \in \mathfrak{g}_\alpha$ we know that $e^{-\operatorname{ad} A_n} X_\alpha = e^{-\alpha(A_n)} X_\alpha$ converges to 0. As X^+ is a linear combination of these, $H_n \to X^-$ and thus $h_n \to n^-$. On the other hand

$$e^{A_n} e^{H_n} e^{-A_n} = \exp(\operatorname{Ad}_{e^{A_n}} H_n) = \exp(e^{\operatorname{ad} A_n} H_n) = \exp(e^{\operatorname{ad} A_n} X^- + X^+)$$

which converges to $n^+ = e^{X^+}$ by a similar argument. $\square$

Lemma 3.1.6. *Let $(g_n) \in G^{\mathbb{N}}$ be a simply Δ–divergent sequence with limits $(g^-, g^+) \in \mathcal{F}_\Delta^2$. Let $f, f' \in \mathcal{F}_\eta$ and $w \in W_{\Delta,\eta}$ with*

$$\mathrm{pos}_{\Delta,\eta}(g^-, f) = w, \qquad \mathrm{pos}_{\Delta,\eta}(g^+, f') = w_0 w. \tag{3.2}$$

Then f is dynamically related to f' via (g_n).

Proof. Fix some representative in $N_K(\mathfrak{a})$ for w and w_0. Let $g_n = k_n e^{A_n} \ell_n$ be a KAK–decomposition, such that $(k_n), (\ell_n) \in K^{\mathbb{N}}$ converge to $k, \ell \in K$ and $\alpha(A_n) \to \infty$ for all $\alpha \in \Delta$. Then the limits can be written as

$$g^- = [\ell^{-1} w_0] \in \mathcal{F}_\Delta, \qquad g^+ = [k] \in \mathcal{F}_\Delta.$$

Because of (3.2) there exist $h, h' \in G$ with

$$g^- = [h] \in \mathcal{F}_\Delta, \qquad f = [hw] \in \mathcal{F}_\eta, \qquad g^+ = [h'] \in \mathcal{F}_\Delta, \qquad f' = [h' w_0 w] \in \mathcal{F}_\eta.$$

So $w_0^{-1} \ell h, k^{-1} h' \in B$, which means we can write $w_0^{-1} \ell h = nam$ and $k^{-1} h' = n'a'm'$ for some $n, n' \in N$, $a, a' \in A$ and $m, m' \in Z_K(\mathfrak{a})$. Consequently,

$$\begin{aligned} f &= [hw] = [\ell^{-1} w_0 namw] = [\ell^{-1}(w_0 n w_0^{-1}) w_0 w], \\ f' &= [h' w_0 w] = [kn'a'm' w_0 w] = [kn' w_0 w], \end{aligned}$$

since elements of A and $Z_K(\mathfrak{a})$ commute with Weyl group elements. By Lemma 3.1.5 there is a sequence $(h_n) \in G^{\mathbb{N}}$ with $h_n \to w_0 n w_0^{-1}$ and $e^{A_n} h_n e^{-A_n} \to n'$. Let $f_n = [\ell_n^{-1} h_n w_0 w] \in \mathcal{F}_\eta$. Then $f_n \to f$ and

$$g_n f_n = [k_n e^{A_n} h_n w_0 w] = [k_n e^{A_n} h_n e^{-A_n} w_0 w] \to [kn' w_0 w] = f'. \qquad \square$$

3.1.2 Limit sets

Now let Γ be a non–elementary hyperbolic group and $\rho \colon \Gamma \to G$ a representation.

Definition 3.1.7. Let $\theta \subset \Delta$ be non–empty and ι–invariant. The *limit set* of ρ is the set

$$\Lambda_{\rho,\theta} = \{g^+ \mid \exists (g_n) \in \rho(\Gamma)^{\mathbb{N}} \text{ simply } \theta\text{–divergent with limits } (g^-, g^+)\} \subset \mathcal{F}_\theta$$

and the set of *limit pairs* is

$$\Lambda_{\rho,\theta}^{[2]} = \{(g^-, g^+) \mid \exists (g_n) \in \rho(\Gamma)^{\mathbb{N}} \text{ simply } \theta\text{–divergent with limits } (g^-, g^+)\} \subset \mathcal{F}_\theta^2.$$

These limit sets are particularly well–behaved for Anosov representations. Namely, we have the following well–known facts:

Proposition 3.1.8. *If ρ is θ–Anosov with limit map $\xi \colon \partial_\infty \Gamma \to \mathcal{F}_\theta$, then*

$$\Lambda_{\rho,\theta} = \xi(\partial_\infty \Gamma), \qquad \Lambda_{\rho,\theta}^{[2]} = \xi(\partial_\infty \Gamma)^2.$$

The first part can be found e.g. in [GGKW17, Theorem 5.3(3)]. We give a detailed proof of the second part. We first need two short lemmas.

Lemma 3.1.9. *Let $\rho\colon \Gamma \to G$ be a θ–Anosov representation with limit map $\xi\colon \partial_\infty\Gamma \to \mathcal{F}_\theta$, $(\gamma_n) \in \Gamma^{\mathbb{N}}$ a diverging sequence and $(\gamma^-, \gamma^+) \in \partial_\infty\Gamma^2$ such that*

$$\gamma_n|_{\partial_\infty\Gamma\setminus\{\gamma^-\}} \to \gamma^+$$

locally uniformly. Then on $\mathcal{F}_\theta$ we also have the locally uniform convergence

$$\rho(\gamma_n)|_{C_{w_0}(\xi(\gamma^-))} \to \xi(\gamma^+). \tag{3.3}$$

Proof. By restricting to a subsequence we can assume that $\rho(\gamma_n)$ is simply θ–divergent with limits $(g^-, g^+) \in \mathcal{F}_\theta^2$, so $\rho(\gamma_n)|_{C_{w_0}(g^-)} \to g^+$ by Lemma 3.1.3. Since $\Lambda_{\rho,\theta} = \xi(\partial_\infty\Gamma)$, we have $g^- = \xi(x)$ for some $x \in \partial_\infty\Gamma$. Let $z \in \partial_\infty\Gamma \setminus \{\gamma^-, x\}$. Then $\xi(z) \in C_{w_0}(\xi(x))$, so $\rho(\gamma_n z) = \rho(\gamma_n)\xi(z) \to g^+$, but also $\gamma_n z \to \gamma^+$, so $g^+ = \xi(\gamma^+)$. The same argument applied to γ_n^{-1} instead of γ_n shows that $g^- = \xi(\gamma^-)$. Finally, since any subsequence of (γ_n) has a subsequence satisfying (3.3), it actually holds for the whole sequence (γ_n). ☐

Lemma 3.1.10. *Let $(\gamma^-, \gamma^+) \in \partial_\infty\Gamma^2$. Then there exists a divergent sequence $(\gamma_n) \in \Gamma^{\mathbb{N}}$ such that*

$$\gamma_n|_{\partial_\infty\Gamma\setminus\{\gamma^-\}} \to \gamma^+ \tag{3.4}$$

locally uniformly.

Proof. Fix a metric on $\partial_\infty\Gamma$. Let $P \subset \partial_\infty\Gamma^2$ be the set of fixed point pairs of infinite order elements of Γ. P is dense in $\partial_\infty\Gamma^2$ by [Gro87, 8.2.G], so we find a sequence $(\gamma_n) \in \Gamma^{\mathbb{N}}$ whose fixed point pairs (γ_n^-, γ_n^+) approach (γ^-, γ^+). Substituting each γ_n by a sufficiently high power, we can assume that γ_n maps the complement of $B_{1/n}(\gamma_n^-)$ into $B_{1/n}(\gamma_n^+)$ by [Gro87, 8.1.G]. Now let $x_n \to x$ be any convergent sequence in $\partial_\infty\Gamma$ with $x \neq \gamma^-$. Once n is large enough such that

$$d(x_n, x) \leq \tfrac{1}{3}d(x, \gamma^-), \quad d(\gamma_n^-, \gamma^-) \leq \tfrac{1}{3}d(x, \gamma^-), \quad 1/n \leq \tfrac{1}{3}d(x, \gamma^-),$$

then x_n lies outside of $B_{1/n}(\gamma_n^-)$, so $\gamma_n x_n \in B_{1/n}(\gamma_n^+)$ and therefore $\gamma_n x_n \to \gamma^+$. ☐

Proof of $\Lambda^{[2]}_{\rho,\theta} = \xi(\partial_\infty\Gamma)^2$. We first prove $\xi(\partial_\infty\Gamma)^2 \subset \Lambda^{[2]}_{\rho,\theta}$. Let $(\gamma^-, \gamma^+) \in \partial_\infty\Gamma^2$ and let $(\gamma_n) \in \Gamma^{\mathbb{N}}$ be a divergent sequence satisfying (3.4), which exists by Lemma 3.1.10. By Lemma 3.1.9 this implies

$$\rho(\gamma_n)|_{C_{w_0}(\xi(\gamma^-))} \to \xi(\gamma^+).$$

Let $\rho(\gamma_{n_k})$ be any simple subsequence of the θ–divergent sequence $(\rho(\gamma_n))$ and $(g^-, g^+) \in \mathcal{F}_\theta^2$ its limits. Then by Lemma 3.1.3

$$\rho(\gamma_{n_k})|_{C_{w_0}(g^-)} \to g^+,$$

so $g^+ = \xi(\gamma^+)$ since $C_{w_0}(\xi(\gamma^-)) \cap C_{w_0}(g^-) \neq \varnothing$. It is not hard to see that (3.4) implies $\gamma_n^{-1}|_{\partial_\infty\Gamma\setminus\gamma^+} \to \gamma^-$, and repeating the argument for this sequence shows that $g^- = \xi(\gamma^-)$. So $(\xi(\gamma^-), \xi(\gamma^+)) \in \Lambda^{[2]}_{\rho,\theta}$.

For the other direction, let $(g^-, g^+) \in \Lambda^{[2]}_{\rho,\theta}$. Then there exists a divergent sequence $(\gamma_n) \in \Gamma^{\mathbb{N}}$ such that $(\rho(\gamma_n))$ is simply θ–divergent with limits (g^-, g^+). Passing to a subsequence, we can also assume that it satisfies (3.4) for some pair $(\gamma^-, \gamma^+) \in \partial_\infty\Gamma^2$, since $\Gamma \curvearrowright \partial_\infty\Gamma$ is a convergence group action. As before, we can use Lemma 3.1.9 as well as Lemma 3.1.3 to show that $g^+ = \xi(\gamma^+)$, and $g^- = \xi(\gamma^-)$ by applying the same argument to the sequence of inverses. So $(g^-, g^+) \in \xi(\partial_\infty\Gamma)^2$. $\square$

3.1.3 Maximal domains of discontinuity

Recall that we call $\Omega \subset \mathcal{F}_\eta$ a domain of discontinuity if it is an open Γ–invariant subset on which Γ acts properly. In this section, we deal with maximal domains of discontinuity, i.e. those which are not contained in any strictly larger domain of discontinuity.

Definition 3.1.11. Let $\Omega \subset \mathcal{F}_\eta$, $\Lambda \subset \mathcal{F}_\theta$, and $I \subset W_{\theta,\eta}$. We define

$$\Omega(\Lambda, I) = \{x \in \mathcal{F}_\eta \mid \operatorname{pos}(\ell, x) \notin I \ \forall \ell \in \Lambda\} \subset \mathcal{F}_\eta$$
$$I(\Lambda, \Omega) = W_{\theta,\eta} \setminus \{\operatorname{pos}(\ell, x) \mid \ell \in \Lambda, x \in \Omega\} \subset W_{\theta,\eta}$$

Remark 3.1.12. It is easy to see that $I \subset I(\Lambda, \Omega(\Lambda, I))$ and $\Omega \subset \Omega(\Lambda, I(\Lambda, \Omega))$. If I is an ideal and Λ is closed, then $\Omega(\Lambda, I)$ is open by Lemma 2.3.8(ii).

For completeness, the following proposition is the properness argument from [KLP18]. It is a simple consequence of Lemma 3.1.4.

Proposition 3.1.13. *Let $\rho\colon \Gamma \to G$ be θ–Anosov with limit set $\Lambda = \Lambda_{\rho,\theta}$ and $I \subset W_{\theta,\eta}$ be a fat ideal. Then $\Omega = \Omega(\Lambda, I) \subset \mathcal{F}_\eta$ is a domain of discontinuity (but it could be empty).*

Proof. Assume that Ω is not a domain of discontinuity. Then there is a dynamical relation $f \sim f'$ via some sequence $(g_n) \in \rho(\Gamma)^{\mathbb{N}}$. By taking a subsequence we can assume that (g_n) is simply θ–divergent with limits $(g^-, g^+) \in \Lambda^{[2]}_{\rho,\theta} = \Lambda^2$ (Proposition 3.1.8). Since $f \in \Omega$ and $g^- \in \Lambda$, we have $\operatorname{pos}_{\theta,\eta}(g^-, f) \notin I$, so $w_0 \operatorname{pos}_{\theta,\eta}(g^-, f) \in I$ as I is fat. By Lemma 3.1.4 and since I is an ideal we get $\operatorname{pos}_{\theta,\eta}(g^+, f') \in I$. But this is impossible since $f' \in \Omega$. $\square$

Proposition 3.1.14. *Let $\rho\colon \Gamma \to G$ be Δ–Anosov with limit map $\xi\colon \partial_\infty\Gamma \to \mathcal{F}_\Delta$ and $\Lambda = \xi(\partial_\infty\Gamma)$. Let $\Omega \subset \mathcal{F}_\eta$ be a maximal domain of discontinuity of ρ. Then $I := I(\Lambda, \Omega) \subset W_{\Delta,\eta}$ is a fat ideal and $\Omega = \Omega(\Lambda, I)$.*

Proof. We first prove that I is an ideal. If not, there are $w' \leq w$ with $w \in I$ and $w' \notin I$. So there exist $\ell \in \Lambda$ and $x \in \Omega$ such that $\operatorname{pos}(\ell, x) = w'$, i.e. $x \in C_{w'}(\ell)$. But $C_w(\ell) \subset \mathcal{F}_\eta \setminus \Omega$ which is closed, so $x \in C_{w'}(\ell) \subset \overline{C_w(\ell)} \subset \mathcal{F}_\eta \setminus \Omega$, a contradiction. So I is an ideal.

If I was not fat, there would be a $w \in W_{\Delta,\eta}$ with $\operatorname{pos}(\ell, x) = w$ and $\operatorname{pos}(\ell', x') = w_0 w$ for some $\ell, \ell' \in \Lambda$ and $x, x' \in \Omega$. But $(\ell, \ell') \in \Lambda^{[2]}_{\rho,\Delta}$ by Proposition 3.1.8, so there is a simply Δ–divergent sequence $(\rho(\gamma_n)) \in \rho(\Gamma)^{\mathbb{N}}$ with limits (ℓ, ℓ'). So x and x' would be dynamically related by Lemma 3.1.6, a contradiction.

Now $\Omega \subset \Omega(\Lambda, I)$ and Proposition 3.1.13 says that I being fat implies $\Omega(\Lambda, I)$ is a domain of discontinuity. So by maximality $\Omega = \Omega(\Lambda, I)$. $\square$

Corollary 3.1.15. *Every maximal domain of discontinuity of a Δ–Anosov representation with limit set Λ is of the form $\Omega(\Lambda, I)$ for a minimal fat ideal $I \subset W_{\Delta,\eta}$.*

Proof. Let Ω be a maximal domain of discontinuity. Then by Proposition 3.1.14 $I(\Lambda,\Omega)$ is a fat ideal and $\Omega = \Omega(\Lambda, I(\Lambda,\Omega))$. In general, there could be other ideals generating the same domain. Let I be minimal among all fat ideals $\widetilde{I}$ with $\Omega(\Lambda,\widetilde{I}) = \Omega$. Then I is in fact minimal among all fat ideals, as otherwise there would be another fat ideal I' with $\Omega = \Omega(\Lambda, I) \subsetneq \Omega(\Lambda, I')$, contradicting maximality of Ω. □

This statement does not hold in general for θ–Anosov representations with $\theta \neq \Delta$. In Section 3.1.6 we will discuss an example of a reducible representation of a free group into $\mathrm{Sp}(4,\mathbb{R})$ which has uncountably many different maximal domains of discontinuity.

3.1.4 Cocompactness

The most important fact we need about cocompact domains of discontinuity is that they are essentially maximal. More precisely:

Lemma 3.1.16. *Let $\rho\colon \Gamma \to G$ be a representation and $\Omega \subset \mathcal{F}_\eta$ a cocompact domain of discontinuity. Then Ω is a union of connected components of a maximal domain of discontinuity.*

Proof. By Zorn's lemma Ω is contained in some maximal domain of discontinuity $\widetilde{\Omega} \in \mathcal{F}_\eta$ and it is an open subset. Then also $\Gamma\backslash\Omega \subset \Gamma\backslash\widetilde{\Omega}$, where $\Gamma\backslash\Omega$ is compact and $\Gamma\backslash\widetilde{\Omega}$ is Hausdorff. So $\Gamma\backslash\Omega$ is closed in $\Gamma\backslash\widetilde{\Omega}$ and therefore Ω is also closed in $\widetilde{\Omega}$. □

This immediately leads to our first main theorem. Let $\pi_\eta\colon \mathcal{F}_\Delta \to \mathcal{F}_\eta$ be the natural projection.

Theorem 3.1.17. *Let $\rho\colon \Gamma \to G$ be Δ–Anosov with limit map $\xi\colon \partial_\infty\Gamma \to \mathcal{F}_\Delta$ and let $\Omega \subset \mathcal{F}_\eta$ be a cocompact domain of discontinuity for ρ. Then there is a balanced ideal $\widetilde{I} \subset W$ such that $\pi_\eta^{-1}(\Omega)$ is a Γ–invariant union of connected components of $\Omega(\xi(\partial_\infty\Gamma),\widetilde{I}) \subset \mathcal{F}_\Delta$.*

Proof. The natural projection $\pi_\eta\colon \mathcal{F}_\Delta \to \mathcal{F}_\eta$ is smooth, G–equivariant and proper. This implies that $\widetilde{\Omega} = \pi_\eta^{-1}(\Omega)$ is also a cocompact domain of discontinuity. So by Lemma 3.1.16 there is a maximal domain of discontinuity $\widehat{\Omega} \subset \mathcal{F}_\Delta$ and $\widetilde{\Omega}$ is a union of connected components of $\widehat{\Omega}$. By Corollary 3.1.15 $\widehat{\Omega} = \Omega(\Lambda, I)$ for a minimal fat ideal $I \subset W$. But since the action of w_0 on W has no fixed points, every minimal fat ideal in W is balanced by Lemma 2.1.6. □

We know from [KLP18] that domains constructed from a balanced ideal are cocompact. The combination of the next two lemmas shows that if the domain is dense, the converse also holds. That is, if a domain constructed from a fat ideal is cocompact, then this ideal must be balanced.

Lemma 3.1.18. *Let $I \subset W_{\theta,\eta}$ be a fat ideal and Λ the limit set of a θ–Anosov representation ρ. Let*

$$D(\Lambda, I) = \{x \in \mathcal{F}_\eta \mid \exists \ell \neq \ell' \in \Lambda\colon\ \mathrm{pos}(\ell, x), \mathrm{pos}(\ell', x) \in I\}$$

and let $\Omega_0 \subset \Omega$ be a $\rho(\Gamma)$–invariant union of connected components of $\Omega := \Omega(\Lambda, I)$. Then Ω_0 can be cocompact only if $\overline{\Omega_0} \cap D(\Lambda, I) = \varnothing$.

Proof. Assume that Ω_0 is cocompact and $x \in \partial\Omega_0$. Take a sequence $(x_n) \in \Omega_0^{\mathbb{N}}$ with $x_n \to x$. Let $(h_n) \in G^{\mathbb{N}}$ be a sequence converging to the identity such that $x_n = h_n x$. By cocompactness, a subsequence of (x_n) converges in the quotient. Passing to this subsequence, there is $(g_n) \in \rho(\Gamma)^{\mathbb{N}}$ such that $g_n x_n \to x' \in \Omega_0$. Clearly $g_n \to \infty$ as otherwise a subsequence of $(g_n x_n)$ would converge to something in $\partial\Omega_0$. Passing to a subsequence another time we can also assume that (g_n) is simply θ–divergent with limits $(g^-, g^+) \in \Lambda^2$.

Now let $\ell \in \Lambda \backslash \{g^-\}$. Then $h_n \ell \to \ell$ and thus $g_n h_n \ell \to g^+$ by Lemma 3.1.3 since $\ell \in C_{w_0}(g^-)$ and this is an open set. So

$$\operatorname{pos}(g^+, x') \leq \operatorname{pos}(g_n h_n \ell, g_n h_n x) = \operatorname{pos}(\ell, x).$$

Since $x' \in \Omega(\Lambda, I)$ we know $\operatorname{pos}(g^+, x') \notin I$, so $\operatorname{pos}(\ell, x) \notin I$. This holds for every $\ell \in \Lambda \backslash \{g^-\}$, so $x \notin D(\Lambda, I)$. We have thus proved that $\partial\Omega_0 \cap D(\Lambda, I) = \varnothing$. Also $\Omega_0 \cap D(\Lambda, I) = \varnothing$ holds by definition. $\square$

Lemma 3.1.19. *In the setting of Lemma 3.1.18 an ideal $I \subset W_{\theta,\eta}$ is slim if and only if $D(\Lambda, I) = \varnothing$.*

Proof. First assume that I is not slim, i.e. there is $w \in I$ with $w_0 w \in I$. Let $\ell \neq \ell' \in \Lambda$. Since ℓ, ℓ' are transverse there is $g \in G$ such that $g\ell = [1]$ and $g\ell' = [w_0] \in \mathcal{F}_\theta$. Let $x = [g^{-1} w] \in \mathcal{F}_\eta$. Then

$$\operatorname{pos}_{\theta,\eta}(\ell, x) = \operatorname{pos}_{\theta,\eta}([1], [w]) = w \in I, \quad \operatorname{pos}_{\theta,\eta}(\ell', x) = \operatorname{pos}_{\theta,\eta}([w_0], [w]) = w_0 w \in I,$$

so $x \in D(\Lambda, I)$. Conversely, suppose that $x \in D(\Lambda, I)$. Then there are transverse $\ell, \ell' \in \Lambda \subset \mathcal{F}_\theta$ such that $\operatorname{pos}(\ell, x), \operatorname{pos}(\ell', x) \in I$. Let $g \in G$ with $g\ell = [1]$ and $g\ell' = [w_0]$. As for any flag, there exist $n \in N$ and $w \in W$ with $gx = [nw] \in \mathcal{F}_\eta$. Choose any Δ–divergent sequence $a_k = e^{A_k} \in A$ with $A_k \in \overline{\mathfrak{a}^+}$. Then $a_k^{-1} n a_k \to 1$. Now

$$\operatorname{pos}_{\theta,\eta}([w_0], [nw]) = \operatorname{pos}_{\theta,\eta}([w_0 w_0^{-1} a_k w_0], [nww^{-1} a_k w]) = \operatorname{pos}_{\theta,\eta}([w_0], [a_k^{-1} n a_k w])$$

and since $[a_k^{-1} n a_k w] \to [w]$ we get $\operatorname{pos}(\ell', x) = \operatorname{pos}([w_0], [nw]) \geq w_0 w$ and thus $w_0 w \in I$. But also $w = \operatorname{pos}([1], [nw]) = \operatorname{pos}(\ell, x) \in I$, so I is not slim. $\square$

3.1.5 Dimensions

If the domain Ω comes from a balanced ideal, the "bad set" $\mathcal{F}_\eta \setminus \Omega$ fibers over $\partial_\infty \Gamma$. The dimension of the fiber is bounded by the following quantity, depending only on G:

Definition 3.1.20. For a subset $A \subset \Sigma$ of the simple roots let

$$\dim A = \sum_{\alpha \in A} \dim \mathfrak{g}_\alpha.$$

The we can define the *minimal balanced ideal codimension* of G

$$\operatorname{mbic}(G) = \min_{\substack{w \in W \\ w_0 w \not\geq w}} \dim \Psi_w, \qquad \Psi_w = \Sigma^+ \cap w\Sigma^-.$$

Dumas and Sanders showed in [DS17, Theorem 4.1] that if the Weyl group W of G has no factors of type A_1, then $w_0 w \leq w$ for all $w \in W$ with $\ell(w) \leq 1$, and that the same is true for $\ell(w) \leq 2$ if W also has no factors of type A_2, A_3 or B_2. This implies $\operatorname{mbic}(G) \geq 2$ resp. $\operatorname{mbic}(G) \geq 3$ in these cases (and even higher lower bounds if the root spaces are more than one–dimensional, e.g. in the case of complex groups).

Example 3.1.21. For the special linear group we have

$$\operatorname{mbic}(\mathrm{SL}(n,\mathbb{R})) = \left\lfloor \frac{n+1}{2} \right\rfloor, \qquad \operatorname{mbic}(\mathrm{SL}(n,\mathbb{C})) = 2\left\lfloor \frac{n+1}{2} \right\rfloor.$$

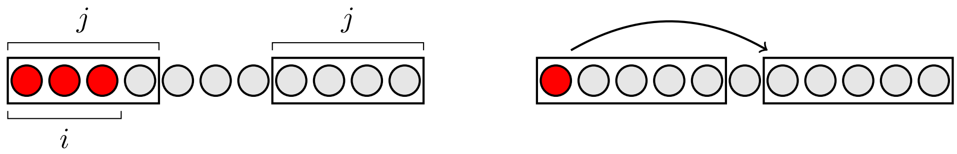
To see this, recall that the Weyl group of $\mathrm{SL}(n,\mathbb{R})$ can be identified with the symmetric group S_n with its standard generating set of adjacent transpositions. There is also a simple description of the Bruhat order on S_n: Define, for any permutation $w \in S_n$ and integers i, j

$$w[i,j] \coloneqq |\{a \leq i \mid w(a) \leq j\}|.$$

Then $w \leq w'$ if and only if $w[i,j] \geq w'[i,j]$ for all i,j [BB06, Theorem 2.1.5]. So $w \leq w_0 w$ if and only if

$$w[i,j] \geq w_0 w[i,j] = |\{a \leq i \mid w(a) > n - j\}|$$

for all i, j. Since every root space $\mathfrak{g}_\alpha$ is 1–dimensional and $|\Psi_w| = \ell(w)$, $\operatorname{mbic}(\mathrm{SL}(n,\mathbb{R}))$ is therefore the minimal word length an element $w \in S_n$ has to have such that there is a pair (i,j) not satisfying this inequality. We can express this problem in a nice graphical way: Suppose we have n balls in a row which we can permute. What is the minimal length of a permutation such that for some choice of i and j, if the first i balls were initially painted red, then after the permutation there are more red ones among the last j than among the first j?



The solution of this elementary combinatorial problem can be seen in the right picture. At least $\lfloor (n+1)/2 \rfloor$ adjacent transpositions are needed, and the minimum is obtained e.g. by choosing $i = 1$ and $j = \lfloor n/2 \rfloor$. The argument for $\mathrm{SL}(n,\mathbb{C})$ is the same except that the root spaces $\mathfrak{g}_\alpha$ are 2–dimensional.

Similarly to the nonemptiness proof in [GW12, Theorem 9.1], a bound on the dimension of the limit set can ensure that the domain is dense or connected.

Lemma 3.1.22. *Let $\rho\colon \Gamma \to G$ be Δ–Anosov with limit map $\xi\colon \partial_\infty \Gamma \to \mathcal{F}_\Delta$. Then*

(i) $\dim \partial_\infty \Gamma \leq \operatorname{mbic}(G)$.

(ii) If $I \subset W_{\Delta,\eta}$ is a balanced ideal and $\dim \partial_\infty \Gamma \leq \operatorname{mbic}(G) - 1$, *then $\Omega(\xi(\partial_\infty \Gamma), I) \subset \mathcal{F}_\eta$ is dense.*

(iii) If $I \subset W_{\Delta,\eta}$ is a balanced ideal and $\dim \partial_\infty \Gamma \leq \mathrm{mbic}(G) - 2$, then $\Omega(\xi(\partial_\infty\Gamma), I) \subset \mathcal{F}_\eta$ is connected.

Proof. We can assume that $\eta = \Delta$ in parts (ii) and (iii) as otherwise we could just lift to $\mathcal{F}_\Delta$. So let $I \subset W$ be a balanced ideal. We will calculate the covering dimension of

$$\mathcal{K} = \mathcal{F}_\eta \setminus \Omega(\xi(\partial_\infty\Gamma), I) = \bigcup_{x \in \partial_\infty \Gamma} \bigcup_{w \in I} C_w(\xi(x)).$$

Since I is balanced, $\mathcal{K}$ is a continuous fiber bundle over $\partial_\infty\Gamma$ with fiber $\bigcup_{w\in I} C_w([1])$ by Lemma 2.3.8. Since the dimension can be calculated in local trivializations and the fiber is a CW–complex, $\dim \mathcal{K} = \dim \partial_\infty\Gamma + \dim \bigcup_{w\in I} C_w([1])$ [Mor77, Theorem 2]. To bound the latter dimension, we use that $\dim C_w([1]) = \dim \Psi_w$ and that all $w \in I$ satisfy $w_0 w \not\leq w$. Furthermore, it is easy to check that $\dim \Psi_w = \dim \mathcal{F}_\Delta - \dim \Psi_{w_0 w}$ for every $w \in W$. So we get the estimate

$$\dim \bigcup_{w\in I} C_w([1]) = \max_{w \in I} \dim \Psi_w \leq \max_{w_0 w \not\leq w} \dim \Psi_w = \dim \mathcal{F}_\Delta - \mathrm{mbic}(G). \tag{3.5}$$

Now if we assume $\dim \partial_\infty \Gamma \leq \mathrm{mbic}(G) - 1$, then $\dim \mathcal{K} \leq \dim \mathcal{F}_\Delta - 1$. So Ω must be dense, as otherwise $\mathcal{K}$ would contain an open subset and therefore $\dim \mathcal{K} = \dim \mathcal{F}_\Delta$. This proves (ii).

For part (iii), we can use Alexander duality [Hat10, Theorem 3.44]: For a compact set K of a closed manifold M, there is an isomorphism $H_i(M, M \setminus K; \mathbb{Z}) \cong \check{H}^{n-i}(K; \mathbb{Z})$ for every i. Since $\dim \mathcal{K} \leq \dim \mathcal{F}_\Delta - 2$, and every Čech cohomology group above the covering dimension vanishes, we have $H_0(\mathcal{F}_\Delta, \Omega; \mathbb{Z}) = \check{H}^n(\mathcal{K}; \mathbb{Z}) = 0$ and $H_1(\mathcal{F}_\Delta, \Omega; \mathbb{Z}) = \check{H}^{n-1}(\mathcal{K}; \mathbb{Z}) = 0$. So by the long exact sequence of the pair $(\mathcal{F}_\Delta, \Omega)$ there is an isomorphsim $H_0(\Omega; \mathbb{Z}) \cong H_0(\mathcal{F}_\Delta; \mathbb{Z})$, i.e. Ω is connected.

Finally, for part (i), we just need a balanced ideal which gives equality in (3.5). This always exists: Take $w' \in W$ such that $w_0 w' \not\leq w'$ and which realizes the maximum. Then the ideal generated by w' is slim and can therefore be extended to a balanced ideal I by Lemma 2.1.6 with $\max_{w\in I} \dim \Psi_w = \dim \Psi_{w'} = \max_{w_0 w \not\leq w} \dim \Psi_w$. The corresponding $\mathcal{K}$ then satisfies $\dim \mathcal{F}_\Delta \geq \dim \mathcal{K} = \dim \partial_\infty \Gamma + \dim \mathcal{F}_\Delta - \mathrm{mbic}(G)$, so $\dim \partial_\infty\Gamma \leq \mathrm{mbic}(G)$. □

Theorem 3.1.23. *Let $\rho\colon \Gamma \to G$ be Δ–Anosov with limit map $\xi\colon \partial_\infty\Gamma \to \mathcal{F}_\Delta$ and $\Lambda = \xi(\partial_\infty\Gamma)$. Assume that $\dim \partial_\infty\Gamma \leq \mathrm{mbic}(G) - 2$. Then every non–empty cocompact domain of discontinuity in $\mathcal{F}_\eta$ is dense and connected and there is a bijection*

$$\{\textit{balanced ideals in } W_{\Delta,\eta}\} \leftrightarrow \{\textit{non–empty cocompact domains of discontinuity in } \mathcal{F}_\eta\}$$

given by $I \mapsto \Omega(\Lambda, I)$ and $\Omega \mapsto I(\Lambda, \Omega)$.

Proof. Let Ω be a non–empty cocompact domain of discontinuity. By Theorem 3.1.17 $\pi_\eta^{-1}(\Omega)$ is a union of connected components of $\Omega(\Lambda, \widetilde{I})$ for some balanced ideal $\widetilde{I} \subset W$. But by Lemma 3.1.22 $\Omega(\Lambda, \widetilde{I})$ is dense and connected. So $\pi_\eta^{-1}(\Omega) = \Omega(\Lambda, \widetilde{I})$ and Ω is dense and connected.

We have to prove that both maps are well–defined and inverses of each other. If Ω is a non–empty cocompact domain of discontinuity, then it is a union of connected components of some maximal domain $\widetilde{\Omega} \subset \mathcal{F}_\eta$ by Lemma 3.1.16. Since Ω is dense it equals $\widetilde{\Omega}$ and is maximal itself. So by Proposition 3.1.14 $I(\Lambda, \Omega)$ is a fat ideal and $\Omega = \Omega(\Lambda, I(\Lambda, \Omega))$. Since $\Omega(\Lambda, I(\Lambda, \Omega))$ is dense and cocompact, Lemma 3.1.18 shows that $D(\Lambda, I(\Lambda, \Omega)) = \varnothing$, and by Lemma 3.1.19 this is equivalent to $I(\Lambda, \Omega)$ being slim, so $I(\Lambda, \Omega)$ is balanced.

Conversely, if $I \subset W_{\Delta,\eta}$ is a balanced ideal, then $\Omega(\Lambda, I)$ is a cocompact domain of discontinuity by the main theorem of [KLP18]. It is dense and thus non–empty by Lemma 3.1.22(ii). By the above, $I(\Lambda, \Omega(\Lambda, I))$ is then a balanced ideal, and since $I \subset I(\Lambda, \Omega(\Lambda, I))$, they must be equal. $\square$

3.1.6 A representation into $\mathrm{Sp}(4, \mathbb{R})$ with infinitely many maximal domains

In this section, we describe an example of a representation (of a free group into $\mathrm{Sp}(4, \mathbb{R})$) which is Anosov (but not Δ–Anosov) and where the analogue of Corollary 3.1.15 does not hold, i.e. there are maximal domains of discontinuity which do not come from a balanced ideal. In fact, it will admit infinitely many maximal domains of discontinuity, which are however not cocompact. It is unclear whether the cocompact domains for general Anosov representations can still be classified using balanced ideals.

Let $\Gamma = F_m$ be a free group in m generators and $\rho_0 \colon \Gamma \to \mathrm{SL}(2, \mathbb{R})$ the holonomy of a compact hyperbolic surface with boundary. Such a representation is Anosov with a limit map $\xi_0 \colon \partial_\infty \Gamma \to \mathbb{RP}^1$ whose image $\Lambda_0 \coloneqq \xi_0(\partial_\infty \Gamma)$ is a Cantor set. We will now consider the representation $\rho = \iota \circ \rho_0$ into $\mathrm{Sp}(4, \mathbb{R})$, where

$$\iota \colon \mathrm{SL}(2, \mathbb{R}) \to \mathrm{Sp}(4, \mathbb{R}), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a\mathbf{1} & b\mathbf{1} \\ c\mathbf{1} & d\mathbf{1} \end{pmatrix},$$

with $\mathbf{1}$ being the 2×2 identity matrix. Here we chose the symplectic form $\omega = \left(\begin{smallmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{smallmatrix}\right)$. Then ρ is $\{\alpha_2\}$–Anosov (where α_2 is the simple root mapping a diagonal matrix to twice its lowest positive eigenvalue), but not Δ–Anosov. Therefore, it carries a limit map $\xi \colon \partial_\infty \Gamma \to \mathrm{Lag}(\mathbb{R}^4)$ to the manifold of Lagrangian subspaces.

The space $\mathrm{Lag}(\mathbb{R}^4)$ admits the following (non–injective) parametrization:

$$\begin{aligned} \Theta \colon \mathbb{RP}^1 \times \mathbb{RP}^1 \times \mathbb{RP}^1 &\to \mathrm{Lag}(\mathbb{R}^4) \\ \left(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right], \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right], \left[\begin{smallmatrix} e \\ f \end{smallmatrix}\right]\right) &\mapsto \left\langle \begin{pmatrix} ae \\ af \\ be \\ bf \end{pmatrix}, \begin{pmatrix} cf \\ -ce \\ df \\ -de \end{pmatrix} \right\rangle. \end{aligned}$$

Θ is a smooth surjective map and the non–injectivity is precisely given by

$$\Theta(p, p, r) = \Theta(p, p, r') \quad \text{and} \quad \Theta(p, q, r) = \Theta(q, p, Rr) \qquad \forall p, q, r, r' \in \mathbb{RP}^1$$

with $R = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

The action of $A \in \mathrm{SL}(2,\mathbb{R})$ on $\mathrm{Lag}(\mathbb{R}^4)$ through ι is just

$$\iota(A)\Theta(p,q,r) = \Theta(Ap, Aq, r).$$

This results in the following simple description of the dynamical relations by the ρ–action on $\mathrm{Lag}(\mathbb{R}^4)$:

Lemma 3.1.24. *Let $\sim$ be the dynamical relation on $\mathrm{Lag}(\mathbb{R}^4)$ by the action of $\rho(\Gamma)$. Then for all $x, y \in \Lambda_0$ and $p, q, r \in \mathbb{RP}^1$ we have*

$$\Theta(p,q,r) \sim \Theta(y,y,r) \quad \textit{and} \quad \Theta(p,x,r) \sim \Theta(y,q,r). \tag{3.6}$$

These are all dynamical relations.

Proof. Assume that $\Theta(p,q,r) \sim \Theta(p',q',r')$ via a sequence $(\rho(\gamma_n)) \in \rho(\Gamma)^{\mathbb{N}}$. Passing to subsequences, we can assume that $(\rho_0(\gamma_n)) \in \mathrm{SL}(2,\mathbb{R})^{\mathbb{N}}$ is simply divergent with limits $(x,y) \in \Lambda_0$ and that there are sequences $(p_n), (q_n), (r_n) \in (\mathbb{RP}^1)^{\mathbb{N}}$ such that

$$p_n \to \widetilde{p}, \quad \rho_0(\gamma_n)p_n \to \widetilde{p}', \quad q_n \to \widetilde{q}, \quad \rho_0(\gamma_n)q_n \to \widetilde{q}', \quad r_n \to \widetilde{r},$$
$$\Theta(\widetilde{p},\widetilde{q},\widetilde{r}) = \Theta(p,q,r), \quad \Theta(\widetilde{p}',\widetilde{q}',\widetilde{r}) = \Theta(p',q',r').$$

So $\widetilde{p} \sim \widetilde{p}'$ and $\widetilde{q} \sim \widetilde{q}'$ via $(\rho_0(\gamma_n))$. This either means that $\widetilde{p} = \widetilde{q} = x$ or $\widetilde{p}' = \widetilde{q}' = y$, in which case the relation is of the first type in (3.6), or that $(\widetilde{p},\widetilde{q}') = (x,y)$ or $(\widetilde{q},\widetilde{p}') = (x,y)$, which is of the second type.

Conversely, let $y \in \Lambda_0$ and $p, q \in \mathbb{RP}^1$. Since $|\Lambda_0| \geq 3$ we find $x \in \Lambda_0 \setminus \{p,q\}$, and by Lemma 3.1.10 there is a sequence $(g_n) \in \rho_0(\Gamma)^{\mathbb{N}}$ which is simply divergent with limits (x,y). Then $p \overset{(g_n)}{\sim} y$ and $q \overset{(g_n)}{\sim} y$, which proves the first relation in (3.6). For the second relation let $x, y \in \Lambda_0$ and $p, q, r \in \mathbb{RP}^1$. If $x = p$ or $y = q$ then it follows from the first relation. Otherwise, take a simply divergent sequence $(g_n) \in \rho_0(\Gamma)^{\mathbb{N}}$ with limits (x,y). Then $p \overset{(g_n)}{\sim} y$ and $q \overset{(g_n^{-1})}{\sim} x$. This shows the second relation in (3.6). $\square$

Proposition 3.1.25. *Let $A \subset \mathbb{RP}^1$ be a minimal closed subset such that $A \cup RA = \mathbb{RP}^1$. Then*

$$\Omega_A = \mathrm{Lag}(\mathbb{R}^4) \setminus \{\Theta(p,q,r) \mid p \in \Lambda_0, q \in \mathbb{RP}^1, r \in A\}$$

is a maximal domain of discontinuity for ρ.

Proof. Ω_A is open since $(\mathbb{RP}^1)^3$ is compact and Θ therefore is a closed map.

Assume that there was a dynamical relation within Ω_A. Then following (3.6) it would either be of the form $\Theta(p,q,r) \sim \Theta(x,x,r)$ or $\Theta(p,x,r) \sim \Theta(y,q,r)$ with $x, y \in \Lambda_0$ and $p, q, r \in \mathbb{RP}^1$. In the first case, $\Theta(x,x,r)$ is independent of r, so we can assume $r \in A$, and $\Theta(x,x,r)$ can thus not be in Ω_A. In the second case, $\Theta(p,x,r) = \Theta(x,p,Rr)$ can be in Ω_A only if $r \notin RA$ and $\Theta(y,q,r) \in \Omega_A$ implies $r \notin A$. But by assumption both can not hold at the same time.

Finally, assume that Ω_A was not maximal, i.e. there was another domain of discontinuity $\Omega \subset \mathrm{Lag}(\mathbb{R}^4)$ with $\Omega_A \subsetneq \Omega$. Let

$$A' = \{r \in \mathbb{RP}^1 \mid \forall x \in \Lambda_0, q \in \mathbb{RP}^1 \colon \Theta(x,q,r) \notin \Omega\}.$$

Then $A' \subsetneq A$ and A' is closed. Since A is minimal among closed sets with $A \cup RA = \mathbb{R}\mathrm{P}^1$, there has to exist some $r \in \mathbb{R}\mathrm{P}^1 \setminus (A' \cup RA')$. But since $r, Rr \notin A'$ then there are $x, y \in \Lambda_0$ and $p, q \in \mathbb{R}\mathrm{P}^1$ with $\Theta(y, q, r), \Theta(x, p, Rr) \in \Omega$. But these are dynamically related by Lemma 3.1.24, a contradiction. □

Through the accidental isomorphism $\mathrm{PSp}(4, \mathbb{R}) \cong \mathrm{SO}_0(2, 3)$ the space $\mathrm{Lag}(\mathbb{R}^4)$ can be identified with the space of isotropic lines in $\mathbb{R}^{2,3}$. The form of signature $(2, 3)$ restricts to a Lorentzian metric on this space, which is why it is also called the $(2 + 1)$ Einstein universe. A detailed explanation of its geometry can be found in [Bar+08, Section 5].

We can use this to visualize Θ and the construction of Ω_A above: The limit set $\xi(\partial_\infty\Gamma) \subset \mathrm{Lag}(\mathbb{R}^4)$ is a Cantor set on the line $\{\Theta(x, x, *) \mid x \in \mathbb{R}\mathrm{P}^1\}$. If we take two different points on this line, described by $x, y \in \mathbb{R}\mathrm{P}^1$, their light cones intersect in the circle $\{\Theta(x, y, r) \mid r \in \mathbb{R}\mathrm{P}^1\}$, where r acts as a global angle coordinate. If $x, y \in \Lambda_0$ then every point on the future pointing light ray emanating from x in a direction r is dynamically related to every point on the past pointing light ray from y in direction r (the red and blue lines in Figure 3.1). So by choosing the set $A \subset \mathbb{R}\mathrm{P}^1$, we decide for every angle whether to take out from our domain all the future or all the past pointing light rays emanating from the points in the limit set in this direction.

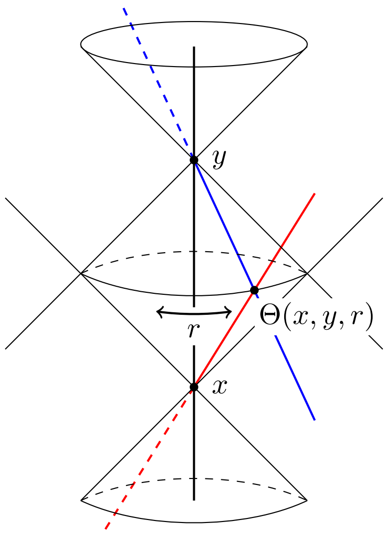


Figure 3.1: The parametrization Θ interpreted by intersecting light cones in $\mathrm{Lag}(\mathbb{R}^4)$. The vertical line is the set of points $\Theta(x, x, *)$ containing the limit set.

3.2 Representations into $\mathrm{SL}(n, \mathbb{R})$ or $\mathrm{SL}(n, \mathbb{C})$

3.2.1 Balanced ideals

The question for which $\eta \subset \Delta$ there exists a balanced ideal in $W_{\Delta,\eta}$ is only combinatorial. For $G = \mathrm{SL}(n, \mathbb{K})$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ the answer is given by the following proposition. Theorem 1.4.3 and its corollaries then immediately follow using Theorem 3.1.23.

A maximal compact subgroup of $\mathrm{SL}(n,\mathbb{R})$ is $K = \mathrm{SO}(n)$ and for $G = \mathrm{SL}(n,\mathbb{C})$ we can choose $K = \mathrm{SU}(n)$. In either case, a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{so}(n)^\perp$ resp. $\mathfrak{su}(n)^\perp$ are the traceless real diagonal matrices, and a simple system of restricted roots is given by $\{\alpha_i = \lambda_i - \lambda_{i+1}\}$, where $\lambda_i \colon \mathfrak{a} \to \mathbb{R}$ maps to the i–th diagonal entry.

Proposition 3.2.1. *Let $\eta = \{\alpha_{i_1}, \dots, \alpha_{i_k}\} \subset \Delta$ be a subset of the simple roots of* $\mathrm{SL}(n,\mathbb{K})$, *with $0 = i_0 < i_1 < \cdots < i_k < i_{k+1} = n$. Let*

$$\delta = |\{0 \le j \le k \mid i_{j+1} - i_j \text{ is odd}\}|.$$

If n is even, a balanced ideal exists in $W_{\Delta,\eta}$ if and only if $\delta \ge 1$. If n is odd, a balanced ideal exists in $W_{\Delta,\eta}$ if and only if $\delta \ge 2$.

Proof. A balanced ideal exists if and only if the action of w_0 on $W_{\Delta,\eta}$ by left–multiplication fixes no element of $W_{\Delta,\eta}$ (see Lemma 2.1.6). This means that $ww_0w^{-1} \notin \langle \Delta \backslash \eta \rangle$ for any $w \in W$. The Weyl group of $\mathrm{SL}(n,\mathbb{K})$ can be identified with the symmetric group S_n with its generators Δ being the adjacent transpositions. Assume first that n is even. Then w_0 is the order–reversing permutation and its conjugates are precisely the fixed point free involutions in S_n. So the existence of balanced ideals is equivalent to every involution in $\langle \Delta \backslash \eta \rangle$ having a fixed point.

Now observe that $\langle \Delta \backslash \eta \rangle$ is a product of symmetric groups, namely $\langle \Delta \backslash \eta \rangle \cong \prod_{j=0}^k S_{i_{j+1}-i_j}$, and that there are fixed point free involutions in S_k if and only if k is even. So a balanced ideal exists iff at least one of the $i_{j+1} - i_j$ is odd, i.e. $\delta \ge 1$.

The same argument works if n is odd, except that the conjugates of w_0 are then involutions in S_n with precisely one fixed point (every involution has at least one), and so we need $\delta \ge 2$ to have none of these in $\langle \Delta \backslash \eta \rangle$. ∎

For the action on Grassmannians, Proposition 3.2.1 specializes to the following simple condition: A balanced ideal exists in $W_{\Delta,\{\alpha_k\}}$ if and only if n is even and k is odd. We can enumerate all balanced ideals for $n \le 10$ using a computer and obtain the following number of balanced ideals in $W_{\Delta,\{\alpha_k\}}$:

	$k=1$	$k=3$	$k=5$	$k=7$	$k=9$
$n=2$	1				
$n=4$	1	1			
$n=6$	1	2	1		
$n=8$	1	7	7	1	
$n=10$	1	42	2227	42	1

In particular, a cocompact domain of discontinuity in projective space $\mathbb{R}\mathrm{P}^{n-1}$ or $\mathbb{C}\mathrm{P}^{n-1}$ exists if and only if n is even. Interestingly, these are also precisely the dimensions which admit complex Schottky groups by [Can08].

3.2.2 Hitchin representations

Let Γ be the fundamental group of a closed surface. As mentioned in Section 1.2, a *Hitchin representation* $\rho\colon \Gamma \to \mathrm{SL}(n,\mathbb{R})$ is a representation which can be continuously deformed to a representation of the form $\iota \circ \rho_0$ where $\rho_0\colon \Gamma \to \mathrm{SL}(2,\mathbb{R})$ is discrete and injective and $\iota\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$ is the irreducible representation. Hitchin representations are Δ–Anosov [Lab06].

Theorem 3.1.23 together with Example 3.1.21 shows that if $n \geq 5$ then the cocompact domains of discontinuity of a Hitchin representation in any flag manifold $\mathcal{F}_\eta$ are in 1:1 correspondence with the balanced ideals in $W_{\Delta,\eta}$. These were discussed in Section 3.2.1.

For completeness, let us also have a look at the cases $n \in \{2,3,4\}$.

In $\mathrm{SL}(2,\mathbb{R})$ the Hitchin representations are just the discrete injective representations. The only flag manifold is $\mathbb{RP}^1$, and since the limit maps $\xi\colon \partial_\infty\Gamma \to \mathbb{RP}^1$ of Hitchin representations are homeomorphisms, there can be no non–empty domain of discontinuity in $\mathbb{RP}^1$.

In the case of $\mathrm{SL}(3,\mathbb{R})$ there is only a single balanced ideal $I \subset W$. By Theorem 3.1.17 the lift of any cocompact domain of discontinuity to the full flag manifold $\mathcal{F}_\Delta$ must be a union of connected components of the corresponding domain, which is

$$\Omega(\Lambda, I) = \{f \in \mathcal{F}_\Delta \mid \forall x \in \partial_\infty\Gamma\colon f^1 \neq \xi^1(x) \wedge f^2 \neq \xi^2(x)\}.$$

It is known (see [CG93]) that for any Hitchin representation ρ into $\mathrm{SL}(3,\mathbb{R})$ there exists a properly convex open domain $D \subset \mathbb{RP}^2$ on which ρ acts properly discontinuously and cocompactly. The image of the limit map of ρ are then the flags consisting of a point on ∂D and the tangent line of D through this point. The domain $\Omega(\Lambda, I) \subset \mathcal{F}_\Delta$ therefore splits into three connected components: One of them (shown in red in Figure 3.2) consists of flags (i.e. a point and a line through it in $\mathbb{RP}^2$) with the point inside D. The second component (blue in Figure 3.2) are flags whose line avoids D, and the third (green) consists of flags whose line goes through D but with the point being outside.

Only the red component descends to a domain in $\mathbb{RP}^2$, and only the blue one to $\mathrm{Gr}(2,3)$, each forming the unique cocompact domain of discontinuity in these manifolds. The cocompact domains in the full flag manifold are any unions of one or more of the three components.

Finally, let's have a look at $\mathrm{SL}(4,\mathbb{R})$. There are ten balanced ideals in W in this case (see Section 6.1.3), corresponding to ten maximal domains of discontinuity in $\mathcal{F}_\Delta$. By dimension arguments as in the proof of Lemma 3.1.22 all of these domains are dense and eight of them are connected. The other two domains are

$$\Omega_1 = \{f \in \mathcal{F}_\Delta \mid \forall x \in \partial_\infty\Gamma\colon f^1 \not\subset \xi^2(x)\}, \quad \Omega_2 = \{f \in \mathcal{F}_\Delta \mid \forall x \in \partial_\infty\Gamma\colon \xi^2(x) \not\subset f^3\}.$$

The topology of these domains does not change when the representation is continuously deformed, and from the Fuchsian case we can easily see that Ω_1 and Ω_2 each have two connected components, all of which are lifts of domains in $\mathbb{RP}^3$ or $\mathrm{Gr}(3,4)$, respectively. The quotient of one of the components in $\mathbb{RP}^3$ describes a convex foliated projective structure on the unit tangent bundle of S [GW08].

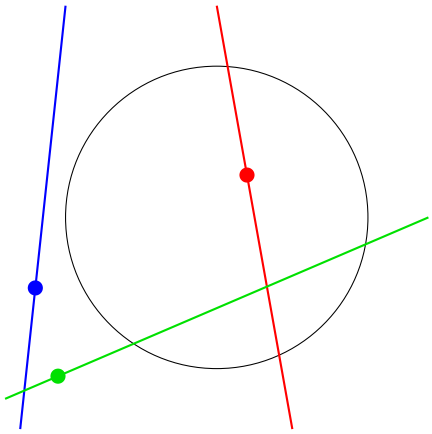


Figure 3.2: The three connected components of the maximal domain of discontinuity in $\mathcal{F}_\Delta$ for a Hitchin representation $\rho\colon \Gamma \to \mathrm{SL}(3,\mathbb{R})$. One exemplary flag out of every component is shown, appearing as a point on a line in $\mathbb{R}\mathrm{P}^2$.

Out of the other 8 cocompact domains in $\mathcal{F}_\Delta$, one descends to the partial flag manifolds $\mathcal{F}_{1,2}$, $\mathcal{F}_{2,3}$ and $\mathcal{F}_{1,3}$, each. Counting all possible combinations of connected components separately, we have 14 different non–empty cocompact domains of discontinuity in $\mathcal{F}_\Delta$.

4 Domains of discontinuity in oriented flag manifolds

In this chapter, we will prove the results described in Section 1.5 concerning the action of Anosov representations on oriented flag manifolds.

Other than before, we will assume for this chapter that G is a linear group. This will be relevant to ensure that $\overline{M}$ is abelian and all its nontrivial elements have order 2 (see Section 2.2). Furthermore, to avoid confusion when dealing with different kinds of quotient spaces, we introduce a convention: equivalence classes of quotients of W or $\widetilde{W}$ will be written with double brackets, i.e. $[\![w]\!]$.

4.1 Oriented relative positions

4.1.1 Oriented flag manifolds

Let B be the minimal parabolic subgroup as defined in Section 2.1.1 and B_0 its identity component. Note that a proper closed subgroup $B_0 \subset P \subsetneq G$ containing B_0 has a parabolic Lie algebra and is thus a union of connected components of a parabolic subgroup.

Definition 4.1.1. Let $B_0 \subset P \subsetneq G$ be a proper closed subgroup containing B_0. We call such a group *(standard) oriented parabolic subgroup* and the quotient G/P an *oriented flag manifold.*

Example 4.1.2. Let $G = \mathrm{SL}(n, \mathbb{R})$ be the special linear group. Then B_0 is the set of upper triangular matrices with positive diagonal entries $\lambda_1, \ldots, \lambda_n$. The space G/B_0 can be identified with the space of complete oriented flags, i.e. complete flags with a choice of orientation in every dimension. An example of a closed subgroup $B_0 \subsetneq P \subsetneq G$ is the group of upper triangular matrices where λ_1 and λ_2 are allowed to be negative, while the remaining entries are positive. The space G/P identifies with the space of complete flags with a choice of orientation on every component except the 1-dimensional one. In this way, all partial flag manifolds with a choice of orientation on a subset of the components of the flags can be obtained. However, we can also consider e.g. the group $P' = \langle B_0, -1 \rangle$ if n is even. Its corresponding oriented flag manifold G/P' is the space of complete oriented flags up to simultaneously changing the orientation on every odd-dimensional component.

The parabolic subgroups of G are parametrized by proper subsets θ of Δ. We want a similar description for oriented parabolics. To define this, recall the lift $\mathsf{v} \colon \Delta \to \widetilde{W}$ we defined in Section 2.2.2:

Definition 4.1.3. For every $\alpha \in \Delta$ choose a vector $E_\alpha \in \mathfrak{g}_\alpha$ such that $\|E_\alpha\|^2 = 2\|\alpha\|^{-2}$. Then
$$\mathsf{v}(\alpha) = \exp\left(\frac{\pi}{2}\left(E_\alpha + \Theta E_\alpha\right)\right)$$
is in $N_K(\mathfrak{a})$. Regarded as an element of $\widetilde{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})_0$, a different choice of E_α can only yield the same $\mathsf{v}(\alpha)$ or its inverse.

Now we can define our objects parametrizing oriented parabolic subgroups. Recall we defined $\overline{M} = Z_K(\mathfrak{a})/Z_K(\mathfrak{a})_0$.

Definition 4.1.4. Let $\varnothing \neq \theta \subset \Delta$ and $\overline{M}_\theta = \langle \mathsf{v}(\Delta\backslash\theta)\rangle \cap \overline{M}$. Let $\overline{M}_\theta \subset E \subset \overline{M}$ be a subgroup. Then we call the group $R = \langle \mathsf{v}(\Delta\backslash\theta), E\rangle \subset \widetilde{W}$ an *oriented parabolic type.*

Remarks 4.1.5.

(i) This definition does not depend on the choices involved in v (see Remark 2.2.6(i)).

(ii) For every oriented parabolic type R, there is a unique pair (θ, E) with $\varnothing \neq \theta \subset \Delta$, $\overline{M}_\theta \subset E \subset \overline{M}$, and $R = \langle \mathsf{v}(\Delta\backslash\theta), E\rangle$. In fact, using Lemma 4.1.8 below, we can recover θ and E from R by
$$R \cap \overline{M} = \langle \mathsf{v}(\Delta\backslash\theta)\rangle E \cap \overline{M} = \overline{M}_\theta E = E$$
and
$$\pi(R) \cap \Delta = \pi(\langle \mathsf{v}(\Delta\backslash\theta)\rangle) \cap \Delta = \langle \Delta\backslash\theta\rangle \cap \Delta = \Delta\backslash\theta$$
where π is the projection from $\widetilde{W}$ to W.

Proposition 4.1.6. *The map*
$$\{\textit{oriented parabolic types}\} \to \{\textit{oriented parabolic subgroups}\}$$
mapping R to $P_R = B_0 R B_0$ is a bijection. Its inverse maps P to $P \cap \widetilde{W}$. We will call $P \cap \widetilde{W}$ the type *of P.*

Definition 4.1.7. Let P_R be the oriented parabolic of type $R = \langle \mathsf{v}(\Delta\backslash\theta), E\rangle$. Then we write
$$\mathcal{F}_R = G/P_R, \qquad \mathcal{F}_\theta = G/P_\theta$$
for the associated oriented and unoriented flag manifolds. $\mathcal{F}_R$ is a finite cover of $\mathcal{F}_\theta$.

The remainder of Section 4.1.1 is a proof of Proposition 4.1.6. We first need a few lemmas.

Lemma 4.1.8. *Let $\alpha \in \Delta$ and $w \in \widetilde{W}$ such that $\pi(w)$ and α are commuting elements of W. Then $w\,\mathsf{v}(\alpha)\,w^{-1} \in \{\mathsf{v}(\alpha), \mathsf{v}(\alpha)^{-1}\} \subset \widetilde{W}$. In particular, this holds for any $w \in \overline{M}$. As a consequence, for any $\theta \subset \Delta$ and any subgroup $E \subset \overline{M}$*
$$\langle \mathsf{v}(\theta), E\rangle = \langle \mathsf{v}(\theta)\rangle\, E = E\,\langle \mathsf{v}(\theta)\rangle.$$

Proof. We compute, using that Ad_w commutes with the Cartan involution,

$$w\,\mathsf{v}(\alpha)\,w^{-1} = \exp\left(\tfrac{\pi}{2}\left(\mathrm{Ad}_w E_\alpha + \Theta\,\mathrm{Ad}_w E_\alpha\right)\right).$$

Since Ad_w preserves $\|\cdot\|$ and the root α is preserved by w this just corresponds to a different choice of $E_\alpha \in \mathfrak{g}_\alpha$ in the definition of $\mathsf{v}(\alpha)$, so $w\,\mathsf{v}(\alpha)\,w^{-1}$ must be either $\mathsf{v}(\alpha)$ or $\mathsf{v}(\alpha)^{-1}$ by Remark 2.2.6(i). So in particular $m\,\langle \mathsf{v}(\theta)\rangle\, m^{-1} \subset \langle \mathsf{v}(\theta)\rangle$ for any $m \in \overline{M}$ and $\theta \subset \Delta$, which shows the second statement. □

Lemma 4.1.9. *Let R, S be oriented parabolic types and $w \in \widetilde{W}$. Then*

$$B_0 R B_0 w B_0 S B_0 = B_0 R w S B_0.$$

Proof. Let $R = \langle \mathsf{v}(\Delta\backslash\theta), E\rangle$. We first prove $B_0 w' B_0 w B_0 \subset B_0 R w B_0$ for all $w \in \widetilde{W}$ and $w' \in R$ by induction on $\ell(w')$. If $\ell(w') = 0$, then $w' \in \overline{M}$, so $B_0 w' B_0 w B_0 = B_0 w' w B_0 \subset B_0 R w B_0$. If $\ell(w') > 0$ then we can find $\alpha \in \Delta\backslash\theta$ and $s = \mathsf{v}(\alpha)$ with $w' = w'' s$ and $\ell(w') = \ell(w'') + 1$. So by Lemma 2.2.11

$$\begin{aligned} B_0 w' B_0 w B_0 &= B_0 w'' s B_0 w B_0 = B_0 w'' B_0 s B_0 w B_0 \\ &\subset B_0 w'' B_0 w B_0 \cup B_0 w'' B_0 s w B_0 \cup B_0 w'' B_0 s^2 w B_0, \end{aligned}$$

which is in $B_0 R w B_0$ by the induction hypothesis, since $s, w'' \in R$. So $B_0 R B_0 w B_0 \subset B_0 R w B_0$. By the same argument $B_0 S B_0 w^{-1} B_0 \subset B_0 S w^{-1} B_0$, and thus $B_0 w B_0 S B_0 \subset B_0 w S B_0$. Together, this shows the lemma. □

Lemma 4.1.10. *If R is an oriented parabolic type, then $B_0 R B_0$ is a closed subgroup of G.*

Proof. Closedness follows from Proposition 2.2.13 as $A_w \subset R$ for every $w \in R$. This is because we can write $w = w'm$ with $w' \in \langle \mathsf{v}(\Delta\backslash\theta)\rangle$ and $m \in E$, and then $A_w = A_{w'} m \subset R$. To see that $B_0 R B_0$ is a subgroup we take $w, w' \in R$ and need to prove that $B_0 w B_0 w' B_0 \subset B_0 R B_0$. But this follows from Lemma 4.1.9 (with $S = 1$). □

Lemma 4.1.11. *Let $\theta \subset \Delta$ be non–empty. Then $P_\theta \cap \widetilde{W} = \langle \mathsf{v}(\Delta\backslash\theta), \overline{M}\rangle$ and $(P_\theta)_0 \cap \widetilde{W} = \langle \mathsf{v}(\Delta\backslash\theta)\rangle$.*

Proof. Since P_θ is B–invariant from both sides, it is a union of Bruhat cells, so $P_\theta \cap \widetilde{W} = \pi^{-1}(P_\theta \cap W)$. Recall that $P_\theta = N_G(\mathfrak{p}_\theta)$, so $w \in W$ is in P_θ if and only if $\mathrm{Ad}_w\, \mathfrak{p}_\theta \subset \mathfrak{p}_\theta$. This holds if and only if w preserves $\Sigma_0^+ \cup \mathrm{span}(\Delta\backslash\theta)$. A simple computation shows that this is equivalent to $\Psi_w \subset \mathrm{span}(\Delta\backslash\theta)$, which in turn is equivalent to $w \in \langle \Delta\backslash\theta\rangle \subset W$ by Lemma 2.2.7. This proves the first equality.

For the second one, note that $\mathsf{v}(\alpha) \in (P_\alpha)_0 \subset (P_\theta)_0$ by Lemma 2.2.9 for every $\alpha \in \Delta\backslash\theta$, so $\langle \mathsf{v}(\Delta\backslash\theta)\rangle \subset (P_\theta)_0 \cap \widetilde{W}$. By Lemma 4.1.10, $P_{\langle \mathsf{v}(\Delta\backslash\theta)\rangle}$ is a closed subgroup of G and $P_{\langle \mathsf{v}(\Delta\backslash\theta)\rangle} \subset (P_\theta)_0$. But by the preceding paragraph, $P_\theta = P_{\langle \mathsf{v}(\Delta\backslash\theta), \overline{M}\rangle} = P_{\langle \mathsf{v}(\Delta\backslash\theta)\rangle}\overline{M}$ is a union of finitely many copies of $P_{\langle \mathsf{v}(\Delta\backslash\theta)\rangle}$. This is only possible if $P_{\langle \mathsf{v}(\Delta\backslash\theta)\rangle} = (P_\theta)_0$. □

Proof of Proposition 4.1.6. Lemma 4.1.10 shows that $P_R = B_0 R B_0$ is a closed subgroup containing B_0 for every oriented parabolic type R. On the other hand, the Lie algebra of such a subgroup P contains $\mathfrak{b}$ and is therefore of the form $\mathfrak{p}_\theta$ for some $\theta \subset \Delta$ [Kna02, Proposition 7.76]. So $(P_\theta)_0 \subset P \subset P_\theta$ and, by Lemma 4.1.11, $\langle \mathsf{v}(\Delta\backslash\theta)\rangle \subset P \cap \widetilde{W} \subset \langle \mathsf{v}(\Delta\backslash\theta)\rangle\overline{M}$. Let $E = P \cap \overline{M}$. Then $\overline{M}_\theta \subset E \subset \overline{M}$ and $P \cap \widetilde{W} = \langle \mathsf{v}(\Delta\backslash\theta)\rangle E$ is an oriented parabolic type.

So the maps in both directions are well–defined. It is clear by Proposition 2.2.4 that they are inverses of each other. □

4.1.2 Relative positions

Let P_R and P_S be the oriented parabolic subgroups of types $R = \langle \mathsf{v}(\Delta \backslash \theta), E\rangle$ and $S = \langle \mathsf{v}(\Delta\backslash\eta), F\rangle$ and let $\mathcal{F}_R$, $\mathcal{F}_S$ be the oriented flag manifolds. The general definition of relative positions (Definition 2.1.1) specializes to:

Definition 4.1.12. The *set of relative positions* is the quotient

$$\widetilde{W}_{R,S} = P_R\backslash G/P_S \cong G\backslash(\mathcal{F}_R \times \mathcal{F}_S),$$

and the map

$$\mathrm{pos}_{R,S} \colon \mathcal{F}_R \times \mathcal{F}_S \to \widetilde{W}_{R,S}$$

is called the *relative position map.*

Example 4.1.13. Consider the group $G = \mathrm{SL}(2,\mathbb{R})$ and $R = S = \{1\}$, so that both $\mathcal{F}_R$ and $\mathcal{F}_S$ are identified with S^1, the space of oriented lines in $\mathbb{R}^2$. Then there are two 2–dimensional and two 1–dimensional G–orbits in the space $S^1 \times S^1$. The 2–dimensional orbits consist of all transverse pairs (v, w) defining a positively or negatively oriented basis of $\mathbb{R}^2$. The 1–dimensional orbits consist of pairs $(v, \pm v)$.

As in the unoriented case, the relative positions admit a combinatorial description in the framework of the preceding sections. This is the main reason why we consider the parabolic types as subgroups of $\widetilde{W}$, and the reason for the notation $\widetilde{W}_{R,S}$. When we write $\widetilde{W}_{R,S}$ in the following, we will usually regard it as $R\backslash\widetilde{W}/S$ and we will write double brackets $[\![\,\cdot\,]\!]$ for equivalence classes in these quotients.

Proposition 4.1.14. *The map*

$$R\backslash\widetilde{W}/S \to P_R\backslash G/P_S$$

induced by the inclusion of $N_K(\mathfrak{a})$ into G is a bijection. In particular, $\widetilde{W}_{R,S}$ is a finite set.

Proof. It is clear from the definitions that the map is well–defined. To see that it is surjective, let $P_R g P_S \in P_R\backslash G/P_S$. By Proposition 2.2.4 $g \in B_0 w B_0$ for some $w \in \widetilde{W}$. Then $[\![w]\!] \in R\backslash\widetilde{W}/S$ maps to $[g]$.

To prove injectivity, let $w, w' \in \widetilde{W}$ with $P_R w P_S = P_R w' P_S$. Since $P_R = B_0 R B_0$ and $P_S = B_0 S B_0$ by Lemma 4.1.10, we can write $w' \in P_R w P_S = B_0 R B_0 w B_0 S B_0$. By Lemma 4.1.9, $B_0 R B_0 w B_0 S B_0 = B_0 R w S B_0$, and by Proposition 2.2.4 this implies $w' \in RwS$, proving injectivity. ☐

4.1.3 The Bruhat order

Again, let P_R and P_S be the oriented parabolic subgroups of types $R = \langle \mathsf{v}(\Delta\backslash\theta), E\rangle$ and $S = \langle \mathsf{v}(\Delta\backslash\eta), F\rangle$ and let $\mathcal{F}_R$, $\mathcal{F}_S$ be the oriented flag manifolds.

Recall that there is a partial order on $\widetilde{W}_{R,S} = P_R \backslash G / P_S$ given by inclusion relations of orbit closures (Definition 2.1.2), which we call the *Bruhat order*. If we have sequences of flags $f_n \to f \in \mathcal{F}_R$ and $f'_n \to f' \in \mathcal{F}_S$ with $\mathrm{pos}_{R,S}(f_n, f'_n)$ constant, then

$$\mathrm{pos}_{R,S}(f, f') \leq \mathrm{pos}_{R,S}(f_n, f'_n).$$

The Bruhat order thus encodes the "genericity" of a pair of flags.

The following lemma shows how the Bruhat order on $\widetilde{W}$ relates to that on the quotients $\widetilde{W}_{R,S} = R \backslash \widetilde{W} / S$.

Lemma 4.1.15. *Let $R \subset R'$ and $S \subset S'$ be oriented parabolic types. In this lemma, we write $[\![w]\!]$ for the equivalence class of $w \in \widetilde{W}$ in $\widetilde{W}_{R,S}$ and $[\![w]\!]'$ for its equivalence class in $\widetilde{W}_{R',S'}$. Then for every $w_1, w_2 \in \widetilde{W}$*

(i) If $[\![w_1]\!] \leq [\![w_2]\!]$, then $[\![w_1]\!]' \leq [\![w_2]\!]'$.

(ii) If $[\![w_1]\!]' \leq [\![w_2]\!]'$, then there exists $w_3 \in \widetilde{W}$ with $[\![w_3]\!]' = [\![w_2]\!]'$ and $[\![w_1]\!] \leq [\![w_3]\!]$.

(iii) If $[\![w_1]\!]' \leq [\![w_2]\!]'$, then there exists $w_3 \in \widetilde{W}$ with $[\![w_3]\!]' = [\![w_1]\!]'$ and $[\![w_3]\!] \leq [\![w_2]\!]$.

Proof. If $[\![w_1]\!] \leq [\![w_2]\!]$ then $w_1 \in \overline{P_R w_2 P_S} \subset \overline{P_{R'} w_2 P_{S'}}$. As the last term is $P_{R'}$–left and $P_{S'}$–right invariant and closed, this implies $\overline{P_{R'} w_1 P_{S'}} \subset \overline{P_{R'} w_2 P_{S'}}$, hence (i).

The assumption in (ii) is equivalent to $w_1 \in \overline{P_{R'} w_2 P_{S'}}$. By Lemma 4.1.9 and Lemma 4.1.10, $P_{R'} w_2 P_{S'} = B_0 R' w_2 S' B_0$, so there exist $r \in R'$ and $s \in S'$ such that $w_1 \in \overline{B_0 r w_2 s B_0}$. So $w_3 = r w_2 s$ satisfies the properties we want.

In part (iii), as $[\![w_1]\!]' \leq [\![w_2]\!]'$ there is a sequence $(g_n) \in G^{\mathbb{N}}$ such that $[g_n] \to [1]$ in $\mathcal{F}_{R'}$ and $[g_n w_2] \to [w_1]$ in $\mathcal{F}_{S'}$. Passing to a subsequence, we can also assume that $[g_n] \to f_1 \in \mathcal{F}_R$ and $[g_n w_2] \to f_2 \in \mathcal{F}_S$. Let $[\![w_3]\!] = \mathrm{pos}_{R,S}(f_1, f_2)$. Then $[\![w_3]\!] \leq [\![w_2]\!]$ and $[\![w_3]\!]' = \mathrm{pos}_{R',S'}(\pi_{R'}(f_1), \pi_{S'}(f_2)) = \mathrm{pos}_{R',S'}([1], [w_1]) = [\![w_1]\!]'$. ☐

The Bruhat order on $\widetilde{W}$ is defined by orbit closures of the $B_0 \times B_0$–action. Proposition 2.2.13 describes this in combinatorial terms. Combining this with Lemma 4.1.15 allows us to also describe the Bruhat order on $\widetilde{W}_{R,S}$ combinatorially. Essentially, we get everything lower than $[\![w]\!]$ in the Bruhat order by deleting or squaring letters in a suitable reduced word for w.

Proposition 4.1.16. *For any $w \in \widetilde{W}$ choose $\alpha_1, \dots, \alpha_k \in \Delta$ and $m \in \overline{M}$ such that $w = \mathsf{v}(\alpha_1) \dots \mathsf{v}(\alpha_k)\, m$ and that this is a reduced word, meaning $k = \ell(m)$. Then define*

$$A_w = \{\mathsf{v}(\alpha_1)^{i_1} \cdots \mathsf{v}(\alpha_k)^{i_k} m \mid i_1, \dots, i_k \in \{0,1,2\}\} \subset \widetilde{W}.$$

This is independent of the choice of reduced word. The Bruhat order on $\widetilde{W}$ is given by

$$w' \leq w \quad \Leftrightarrow \quad w' \in A_w. \tag{4.1}$$

On $\widetilde{W}_{R,S}$ it is given by

$$[\![w']\!] \leq [\![w]\!] \quad \Leftrightarrow \quad w' \in R A_w S \quad \Leftrightarrow \quad w' \in \bigcup_{r \in R,\, s \in S} A_{rws}. \tag{4.2}$$

Proof. The well–definedness of A_w and (4.1) hold by Proposition 2.2.13. For the general case (4.2), if $w' \in A_{rws}$ or $w' \in r^{-1}A_w s^{-1}$ for some $r \in R$, $s \in S$, then $w' \leq rws$ or $rw's \leq w$ in $\widetilde{W}$, respectively. By Lemma 4.1.15(i) both inequalities imply $[\![w']\!] \leq [\![w]\!]$. Conversely, if $[\![w']\!] \leq [\![w]\!]$, then $w' \leq rws$ and $r'w's' \leq w$ for some $r, r' \in R$ and $s, s' \in S$ by Lemma 4.1.15(ii) and Lemma 4.1.15(iii), so $w' \in A_{rws}$ and $w' \in r^{-1}A_w s^{-1}$. □

The following characterization of the Bruhat order on $\widetilde{W}$ will also be useful later:

Lemma 4.1.17. *Let $w, w' \in \widetilde{W}$ with $\ell(w') = \ell(w) + 1$. Let*

$$Q = \{w\,\mathsf{v}(\alpha)^{\pm 1}\,w^{-1} \mid w \in \widetilde{W}, \alpha \in \Delta\} \subset \widetilde{W}$$

be the set of conjugates of the standard generators or their inverses. Then

$$w \leq w' \iff \exists q \in Q \colon w = qw'.$$

Proof. The implication '⇒' follows from Proposition 2.2.13 by choosing q of the form

$$q = \mathsf{v}(\alpha_1)\dots\mathsf{v}(\alpha_{i-1})\mathsf{v}(\alpha_i)^{\pm 1}\mathsf{v}(\alpha_{i-1})^{-1}\dots\mathsf{v}(\alpha_1)^{-1}$$

for some i. For the other direction, assume that $w = qw'$ and write

$$w' = \mathsf{v}(\alpha_1)\dots\mathsf{v}(\alpha_k)\,m$$

for some $\alpha_1, \dots, \alpha_k \in \Delta$ with $k = \ell(w')$ and $m \in \overline{M}$. Then $\pi(w') = \alpha_1 \dots \alpha_k$ and by the strong exchange property of Coxeter groups [BB06, Theorem 1.4.3]

$$\pi(w) = \pi(q)\pi(w') = \alpha_1 \dots \widehat{\alpha_i} \dots \alpha_k$$

for some i, so $\pi(q) = (\alpha_1 \dots \alpha_{i-1})\alpha_i(\alpha_1 \dots \alpha_{i-1})^{-1}$. Set $c = \mathsf{v}(\alpha_1)\dots\mathsf{v}(\alpha_{i-1}) \in \widetilde{W}$. Then $c^{-1}qc \in \pi^{-1}(\alpha_i) \cap Q = \{\mathsf{v}(\alpha_i)^{\pm 1}\}$ by Lemma 4.1.8. So

$$w = qw' = c\,\mathsf{v}(\alpha_i)^{\pm 1}c^{-1}w' = \mathsf{v}(\alpha_1)\dots\mathsf{v}(\alpha_{i-1})\mathsf{v}(\alpha_i)^{1\pm 1}\mathsf{v}(\alpha_{i+1})\dots\mathsf{v}(\alpha_k)\,m \leq w',$$

where the inequality at the end follows by Proposition 2.2.13. □

The following lemmas will be useful when calculating with relative positions.

Lemma 4.1.18.

(i) $Z_K(\mathfrak{a})$ normalizes the subgroups P_R, P_S. Consequently, the (finite abelian) group $\overline{M}/E$ acts on $\mathcal{F}_R$ by right multiplication, and this action is simply transitive on fibers. The analogous statement holds for $\overline{M}/F$ acting on $\mathcal{F}_S$.
Furthermore, $\overline{M}/E$ acts on $\widetilde{W}_{R,S}$ by left multiplication and $\overline{M}/F$ acts on $\widetilde{W}_{R,S}$ by right multiplication. Both of these actions preserve the Bruhat order.

(ii) For any $f_1 \in \mathcal{F}_R$, $f_2 \in \mathcal{F}_S$, $m_1 \in \overline{M}/E$, $m_2 \in \overline{M}/F$, right multiplication by m_1 and m_2 has the following effect on relative positions:

$$\mathrm{pos}_{R,S}(R_{m_1}(f_1), R_{m_2}(f_2)) = m_1^{-1}\,\mathrm{pos}_{R,S}(f_1, f_2)m_2$$

Proof.

(i) It follows e.g. from Lemma 4.1.8 that $\overline{M}$ normalizes $R = \langle \mathsf{v}(\Delta\backslash\theta), E\rangle$ and $S = \langle \mathsf{v}(\Delta\backslash\eta), F\rangle$. Furthermore, $Z_K(\mathfrak{a})$ normalizes B_0 and thus also $P_R = B_0RB_0$ and $P_S = B_0SB_0$. This implies that the actions of $\overline{M}$ on $\mathcal{F}_R$ and $\mathcal{F}_S$ by right multiplication and on $\widetilde{W}_{R,S}$ by left and right multiplication are well–defined. Since E resp. F acts trivially, we obtain the induced actions of $\overline{M}/E$ resp. $\overline{M}/F$. The action of $\overline{M}/E$ on $\mathcal{F}_R$ is simply transitive on each fiber (over $\mathcal{F}_\theta$) since $\overline{M}\cap R = E$ (see Remark 4.1.5(ii)); in the same way, $\overline{M}/F$ acts simply transitively on fibers of $\mathcal{F}_S$.

The actions on $\widetilde{W}_{R,S}$ preserve the Bruhat order since, for $m, m' \in \overline{M}$,

$$\begin{aligned}\llbracket w\rrbracket \le \llbracket w'\rrbracket \Leftrightarrow \overline{P_RwP_S} \subset \overline{P_Rw'P_S} &\Leftrightarrow m\overline{P_RwP_S}m' \subset m\overline{P_Rw'P_S}m' \\ &\Leftrightarrow \overline{P_Rmwm'P_S} \subset \overline{P_Rmw'm'P_S} \Leftrightarrow \llbracket mwm'\rrbracket \le \llbracket mw'm'\rrbracket.\end{aligned}$$

(ii) Let $\mathrm{pos}_{R,S}(f_1, f_2) = \llbracket w\rrbracket \in \widetilde{W}_{R,S}$. This means that there exists some $g \in G$ such that $g(f_1, f_2) = ([1], [w])$. It follows that

$$m_1^{-1}g(R_{m_1}(f_1), R_{m_2}(f_2)) = m_1^{-1}([m_1], [wm_2]) = ([1], [m_1^{-1}wm_2]).$$

So we obtain $\mathrm{pos}_{R,S}(R_{m_1}(f_1), R_{m_2}(f_2)) = \llbracket m_1^{-1}wm_2\rrbracket$. □

Corollary 4.1.19. *Let $f \in \mathcal{F}_R$, $w \in \widetilde{W}$ and $m \in \overline{M}$. Then we have*

$$R_m(C^{R,S}_{\llbracket w\rrbracket}(f)) = C^{R,S}_{\llbracket wm\rrbracket}(f) = C^{R,S}_{\llbracket w\rrbracket}(R_{wmw^{-1}}(f)).$$

Proof. From the previous lemma we obtain

$$\mathrm{pos}_{R,S}(f, R_{m^{-1}}(f')) = \llbracket w\rrbracket \Leftrightarrow \mathrm{pos}_{R,S}(f, f') = \llbracket wm\rrbracket \Leftrightarrow \mathrm{pos}_{R,S}(R_{wmw^{-1}}(f), f') = \llbracket w\rrbracket. \quad \square$$

We close this section with an inequality for relative positions that will play an important role in the proof of cocompactness of domains of discontinuity in Section 4.3.2. It can be read as a triangle inequality if the position w_0 is a transverse one.

Lemma 4.1.20. *Let $w_0, w_1, w_2 \in \widetilde{W}$ with $w_0Rw_0^{-1} = R$. Assume there are $f_1, f_2 \in \mathcal{F}_R$ and $f_3 \in \mathcal{F}_S$ such that*

$$\begin{aligned}\mathrm{pos}_{R,R}(f_1, f_2) &= \llbracket w_0\rrbracket \in \widetilde{W}_{R,R},\\ \mathrm{pos}_{R,S}(f_1, f_3) &= \llbracket w_1\rrbracket \in \widetilde{W}_{R,S},\\ \mathrm{pos}_{R,S}(f_2, f_3) &= \llbracket w_2\rrbracket \in \widetilde{W}_{R,S}.\end{aligned}$$

Then

$$\llbracket w_1\rrbracket \ge \llbracket w_0w_2\rrbracket$$

in $\widetilde{W}_{R,S}$.

Proof. Using the G–action on pairs, we can assume that $(f_1, f_2) = ([1], [w_0])$. Then, since $\mathrm{pos}_{R,S}(f_2, f_3) = \mathrm{pos}_{R,S}([w_0], f_3) = [\![w_2]\!]$, Lemma 4.1.27 implies that f_3 has a representative in G of the form $w_0 u r w_2$ for some $u \in N$ and $r \in R$. We want to find elements $g_n \in G$ such that

$$g_n(f_1, f_3) = g_n([1], [w_0 u r w_2]) \xrightarrow{n\to\infty} ([1], [w_0 w_2]).$$

Let $(A_n) \in \overline{\mathfrak{a}^+}^{\mathbb{N}}$ be a sequence with $\alpha(A_n) \to \infty$ for all $\alpha \in \Delta$ and $g_n = w_0 r^{-1} e^{-A_n} w_0^{-1}$. Then $g_n \in P_R$ since A and R are normalized by w_0. Observe that $w_2^{-1} r^{-1} e^{A_n} r w_2 \in A \subset P_S$, since A is normalized by all of $\widetilde{W}$. Then g_n stabilizes $[1] \in \mathcal{F}_R$, and we calculate

$$\begin{aligned} g_n[w_0 u r w_2] &= [(w_0 r^{-1} e^{-A_n} w_0^{-1}) w_0 u r w_2 (w_2^{-1} r^{-1} e^{A_n} r w_2)] \\ &= [w_0 r^{-1} e^{-A_n} u e^{A_n} r w_2] \xrightarrow{n\to\infty} [w_0 w_2], \end{aligned}$$

where we used that $e^{-A_n} u e^{A_n} \xrightarrow{n\to\infty} 1$. □

4.1.4 Transverse positions

Recall that we call a position transverse if it is maximal in the Bruhat order. Let $T \subset \widetilde{W}$ and $T_{R,S} \subset \widetilde{W}_{R,S}$ be the set of transverse positions. The following lemma shows that the transverse positions are given by the lifts of the longest element w_0 of the Weyl group.

Lemma 4.1.21. *Let $\pi \colon \widetilde{W} \to W$ and $\pi_{R,S} \colon \widetilde{W} \to \widetilde{W}_{R,S}$ be the canonical projections. Then $T_{R,S} = \pi_{R,S}(\pi^{-1}(w_0))$.*

Proof. First consider the case $R = S = \{1\}$, i.e. the Bruhat order on $\widetilde{W}$. By Proposition 4.1.16, if $w \leq w'$, then $\ell(w) \leq \ell(w')$ with equality only if $w = w'$. This immediately implies that every lift of w_0 is maximal. Conversely, if $w \in \widetilde{W}$ is maximal, then so is $\pi(w)$ (e.g. by Lemma 4.1.15(i)). But w_0 is the unique maximal element in the Bruhat order on W [BB06, Proposition 2.3.1].

For general R, S, let $w \in \pi^{-1}(w_0)$ and assume $[\![w']\!] \geq [\![w]\!]$ in $\widetilde{W}_{R,S}$. Then $[\![w']\!] = [\![w'']\!]$ and for some $w'' \in \widetilde{W}$ with $w'' \geq w$ by Lemma 4.1.15(ii). Since w is maximal, $w'' = w$ and $[\![w']\!] = [\![w]\!]$, so $[\![w]\!]$ is maximal. On the other hand, if $[\![w]\!] \in T_{R,S}$ and $w \in \widetilde{W}$ is a maximal representative of the equivalence class $[\![w]\!]$, then for every $w' \geq w$, Lemma 4.1.15(i) shows that $[\![w']\!] \geq [\![w]\!]$, so $[\![w']\!] = [\![w]\!]$ and even $w' = w$. So w is maximal in $\widetilde{W}$ and thus $[\![w]\!] = \pi_{R,S}(w) \in \pi_{R,S}(\pi^{-1}(w_0))$. □

In contrast to the setting of unoriented flags, there can be multiple transverse positions. Indeed, there are always $|\overline{M}|$ transverse positions in $\widetilde{W}$. However, the situation is less obvious in double quotients $\widetilde{W}_{R,S}$: In the example in Section 4.4.5, there is a unique transverse position although the projection to unoriented relative positions is nontrivial.

Now we consider the action of transverse elements $w_0 \in T \subset \widetilde{W}$ on the set of relative positions $\widetilde{W}_{R,S}$. We want to identify those w_0 acting as order–reversing involutions. There are in general multiple choices which depend on the oriented parabolic type R. From now on we also assume that $\iota(\theta) = \theta$, where $R = \langle \mathsf{v}(\Delta \backslash \theta), E \rangle$.

Lemma 4.1.22. *Let $w_0 \in T \subset \widetilde{W}$. Then for any $w, w' \in \widetilde{W}$, $w \leq w'$ implies $w_0 w' \leq w_0 w$. If $w_0 E w_0^{-1} = E$ and $w_0^2 \in E$, then w_0 acts as an order–reversing involution on $\widetilde{W}_{R,S}$.*

Proof. Assume $w \leq w'$. Then by Proposition 2.2.13 there exists a sequence

$$w = w_1 \leq \cdots \leq w_k = w'$$

with $\ell(w_{i+1}) = \ell(w_i) + 1$. So by Lemma 4.1.17 $w_i = q_i w_{i+1}$ for some $q_i \in Q$. Therefore,

$$w_0 w_{i+1} = w_0 q_i^{-1} w_i = q_i' w_0 w_i$$

for $q_i' = w_0 q_i^{-1} w_0^{-1} \in Q$. By [BB06, Corollary 2.3.3], $\ell(w_0 w) = \ell(w_0) - \ell(w)$ for any $w \in \widetilde{W}$, so

$$\ell(w_0 w_i) = \ell(w_0) - \ell(w_i) = \ell(w_0) - \ell(w_{i+1}) + 1 = \ell(w_0 w_{i+1}) + 1,$$

so $w_0 w' = w_0 w_k \leq \cdots \leq w_0 w_1 = w_0 w$ by the same lemma.

We now show that w_0 normalizes R if it normalizes E, and thus the action of w_0 on $\widetilde{W}_{R,S} = R \backslash \widetilde{W} / S$ by left–multiplication is well–defined. The induced action of w_0 on reduced roots is given by ι and θ is ι–invariant. Moreover, Remark 2.2.6(i) implies that for every $\alpha \in \Delta \backslash \theta$, we have $w_0 \mathsf{v}(\alpha) w_0^{-1} = \mathsf{v}(\iota(\alpha))$ or $w_0 \mathsf{v}(\alpha) w_0^{-1} = \mathsf{v}(\iota(\alpha))^{-1}$. Therefore, $\langle \mathsf{v}(\Delta \backslash \theta) \rangle$ is normalized by w_0. Since $R = \langle \mathsf{v}(\Delta \backslash \theta), E \rangle$, the same is true for R.

If in addition we have $w_0^2 \in E$, the induced action on $\widetilde{W}_{R,S}$ is an involution. It is an easy consequence of Lemma 4.1.15 that the action on this quotient still reverses the order. □

Remark 4.1.23. For $w_0 \in T$ the condition $w_0 R w_0^{-1} = R$ is equivalent to $\iota(\theta) = \theta$ and $w_0 E w_0^{-1} = E$. Moreover, $w_0^2 \in E$ is equivalent to $w_0^2 \in R$.

Example 4.1.24. Consider $G = \mathrm{SL}(3, \mathbb{R})$ with its maximal compact $K = \mathrm{SO}(3, \mathbb{R})$ and $\mathfrak{a} = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & -\lambda_1 - \lambda_2 \end{pmatrix} \mid \lambda_1, \lambda_2 \in \mathbb{R} \right\}$. The extended Weyl group $\widetilde{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})_0$ consists of all permutation matrices A with determinant 1 – i.e. all matrices with exactly one ± 1 entry per line and row and all other entries 0, such that $\det(A) = 1$. The transverse positions are

$$\begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}, \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}, \begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix}, \begin{pmatrix} & & 1 \\ & 1 & \\ -1 & & \end{pmatrix}.$$

The first two of these are actually involutions in $\widetilde{W}$, so the condition $w_0^2 \in E$ is empty. The condition $w_0 E w_0^{-1} = E$ is a symmetry condition on E, similar to the condition $\iota(\theta) = \theta$. It does not depend on the choice of lift: Any other w_0' is of the form $w_0' = w_0 m$ for some $m \in \overline{M}$, and $\overline{M}$ is abelian. In Proposition 4.2.5, we will show that we can always assume it to hold in our setting.
The last two are not involutions in $\widetilde{W}$. The smallest possible choice of E containing their square is

$$E = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \right\}.$$

The existence of an involution w_0 on $\widetilde{W}_{R,S}$ allows us to define the notions fat, slim and balanced for an ideal $I \subset \widetilde{W}_{R,S}$ (see Definition 2.1.5). They play a crucial role in the description of properly discontinuous and cocompact group actions of oriented flag manifolds in Section 4.3.1 and Section 4.3.2.

Definition 4.1.25. Let $I \subset \widetilde{W}_{R,S}$ be an ideal and $w_0 \in T \subset \widetilde{W}$ satisfying $w_0 E w_0^{-1} = E$ and $w_0^2 \in E$. Then

- I is called w_0*–fat* if $x \notin I$ implies $w_0 x \in I$.
- I is called w_0*–slim* if $x \in I$ implies $w_0 x \notin I$.
- I is called w_0*–balanced* if it is fat and slim.

4.1.5 Refined Schubert strata

Definition 4.1.26. Let $f \in \mathcal{F}_R$ and $[\![w]\!] \in \widetilde{W}_{R,S}$. Then we call the set

$$C^{R,S}_{[\![w]\!]}(f) := \{f' \in \mathcal{F}_S \mid \operatorname{pos}_{R,S}(f,f') = [\![w]\!]\}$$

of flags at position $[\![w]\!]$ with respect to f a *refined Schubert stratum*. We sometimes omit the superscript R, S if it is clear from the context.

Every refined Schubert stratum admits the following simple (but in general not injective) parametrization by R and the unipotent subgroup $N \subset G$:

Lemma 4.1.27. *Let $w \in \widetilde{W}$. Then*

$$C^{R,S}_{[\![w]\!]}([1]) = NR[w] \subset \mathcal{F}_S.$$

Proof. Let $g \in G$ such that $[g] \in C^{R,S}_{[\![w]\!]}([1])$. Then by Lemma 4.1.9 we have

$$g \in P_R w P_S = B_0 R w P_S.$$

Using the Iwasawa decomposition $B_0 = NAZ_K(\mathfrak{a})_0$ and the fact that both A and $Z_K(\mathfrak{a})$ are normalized by $N_K(\mathfrak{a})$, this implies

$$B_0 R w P_S = NAZ_K(\mathfrak{a})_0 R w P_S = NRwAZ_K(\mathfrak{a})_0 P_S = NRwP_S. \qquad \square$$

If $R = S$ and $[\![w_0]\!] \in \widetilde{W}_{R,R}$ is a transverse position, then $C_{[\![w_0]\!]}(f)$ is a cell, parametrized as follows.

Lemma 4.1.28. *Assume that $\iota(\theta) = \theta$ and let $w_0 \in T$ such that $w_0 E w_0^{-1} = E$. Define*

$$\mathfrak{n}^-_\theta = \bigoplus_{\alpha \in \Sigma^- \setminus \operatorname{span}(\Delta \setminus \theta)} \mathfrak{g}_\alpha.$$

Then the map

$$\varphi \colon \mathfrak{n}^-_\theta \to C^{R,R}_{[\![w_0]\!]}([w_0^{-1}]), \qquad X \mapsto [e^X]$$

is a diffeomorphism.

Proof. Let $N_\theta^- \subset G$ be the connected subgroup with Lie algebra $\mathfrak{n}_\theta^-$. As a subgroup of N^- its exponential map $\mathfrak{n}_\theta^- \to N_\theta^-$ is a diffeomorphism. So it suffices to show that the projection map $\widetilde{\varphi}\colon N_\theta^- \to \mathcal{F}_R$ is a diffeomorphism onto $C_{[\![w_0]\!]}([w_0^{-1}])$.

First we verify that $C_{[\![w_0]\!]}([w_0^{-1}]) = \{[n] \mid n \in N_\theta^-\}$. We have

$$\begin{aligned} C_{[\![w_0]\!]}([w_0^{-1}]) &= \{f \in \mathcal{F}_R \mid \mathrm{pos}_{R,R}([1], w_0 f) = [\![w_0]\!]\} \\ &= \{f \in \mathcal{F}_R \mid \exists p \in G\colon [1] = [p],\ w_0 f = [p w_0]\} = \{[w_0^{-1} p w_0] \mid p \in P_R\} \end{aligned}$$

and $w_0 N_\theta^- w_0^{-1} \subset (P_{\iota(\theta)})_0 \subset P_R$, so it remains to show that $w_0^{-1} P_R w_0 \subset N_\theta^- P_R$. As a consequence of the Langlands decomposition [Kna02, Propositions 7.82(a) and 7.83(d)] we can write $P_\theta = N_\theta^+ Z_G(\mathfrak{a}_\theta)$ with $N_\theta^+ = w_0 N_{\iota(\theta)}^- w_0^{-1} = w_0 N_\theta^- w_0^{-1}$ and $\mathfrak{a}_\theta = \bigcap_{\beta \in \Delta \setminus \theta} \ker \beta$. Now Ad_{w_0} preserves $\mathfrak{a}_\theta$ and thus $w_0^{-1} Z_G(\mathfrak{a}_\theta) w_0 = Z_G(\mathfrak{a}_\theta)$, hence $w_0^{-1} P_\theta w_0 = N_\theta^- Z_G(\mathfrak{a}_\theta)$. As N_θ^- is connected, we even get $w_0^{-1}(P_\theta)_0 w_0 = N_\theta^- Z_G(\mathfrak{a}_\theta)_0$ and therefore

$$w_0^{-1} P_R w_0 = w_0^{-1}(P_\theta)_0 E w_0 = w_0^{-1}(P_\theta)_0 w_0 E = N_\theta^- Z_G(\mathfrak{a}_\theta)_0 E \subset N_\theta^- P_R.$$

To prove injectivity of $\widetilde{\varphi}$, let $n, n' \in N_\theta^-$ with $[n] = [n']$. Then $n^{-1} n' \in N_\theta^- \cap P_R = \{1\}$ by [Kna02, Proposition 7.83(e)], so $\widetilde{\varphi}$ is injective. To see that $\widetilde{\varphi}$ is regular, we observe that $\mathfrak{n}_\theta^-$ is composed of the root spaces of roots in $\Sigma^- \setminus \mathrm{span}(\Delta\backslash\theta)$ while $\mathfrak{p}_\theta$ has the root spaces $\Sigma^+ \cup \mathrm{span}(\Delta\backslash\theta)$. So $\mathfrak{g} = \mathfrak{n}_\theta^- \oplus \mathfrak{p}_\theta$ and $D_1\widetilde{\varphi}\colon \mathfrak{n}_\theta^- \to \mathfrak{g}/\mathfrak{p}_\theta$ is an isomorphism. By equivariance we then see that $\widetilde{\varphi}$ is a diffeomorphism onto its image. □

Using ideals in $\widetilde{W}_{R,S}$ and refined Schubert strata, we can define the following map $\mathcal{Q}$ assigning to each flag in $\mathcal{F}_R$ a subset of $\mathcal{F}_S$. It is the centerpiece of our construction of domains of discontinuity in Section 4.3.

Lemma 4.1.29. *Let $\mathcal{F}_R$ and $\mathcal{F}_S$ be two oriented flag manifolds, and let $I \subset \widetilde{W}_{R,S}$ be an ideal. Then the map*

$$\begin{aligned} \mathcal{Q}\colon \mathcal{F}_R &\to \mathcal{P}(\mathcal{F}_S) \\ f &\mapsto \bigcup_{[\![w]\!] \in I} C_{[\![w]\!]}(f) \end{aligned}$$

is G–equivariant with image in $\mathcal{C}(\mathcal{F}_S)$, the set of closed subsets of $\mathcal{F}_S$.

Proof. Observe that for any element $g \in G$ satisfying $[g] = f \in \mathcal{F}_R$ and any relative position $[\![w]\!] \in \widetilde{W}_{R,S}$, we have $C_{[\![w]\!]}(f) = g C_{[\![w]\!]}([1])$; in other words, the map $f \mapsto C_{[\![w]\!]}(f)$ from $\mathcal{F}_R$ to subsets of $\mathcal{F}_S$ is equivariant. By definition of the Bruhat order on $\widetilde{W}_{R,S}$, the closure of $C_{[\![w]\!]}(f)$ is given by

$$\overline{C_{[\![w]\!]}(f)} = \bigcup_{[\![w']\!] \leq [\![w]\!]} C_{[\![w']\!]}(f).$$

In particular, if $I \subset \widetilde{W}_{R,S}$ is an ideal, then $\bigcup_{[\![w]\!] \in I} C_{[\![w]\!]}(f)$ is closed for any $f \in \mathcal{F}_R$. □

4.2 P_R–Anosov representations

Let P_R be the oriented parabolic of type $R = \langle \mathsf{v}(\Delta\backslash\theta), E\rangle$, with $\iota(\theta) = \theta$. Moreover, let $w_0 \in T \subset \widetilde{W}$ be a transverse position. Let Γ be a finitely generated group and $\rho\colon \Gamma \to G$ a representation.

Recall that a representation $\rho\colon \Gamma \to G$ is P_θ–Anosov if Γ is word hyperbolic, ρ is P_θ–divergent and there is a continuous, transverse, dynamics–preserving, ρ–equivariant map $\xi\colon \partial_\infty\Gamma \to \mathcal{F}_\theta$ called limit map or boundary map.

We extend the notion of an Anosov representation to oriented parabolic subgroups by requiring that the limit map lifts to the corresponding oriented flag manifold.

Definition 4.2.1. Assume that Γ is non–elementary. The representation $\rho\colon \Gamma \to G$ is *P_R–Anosov* if it is P_θ–Anosov with limit map $\xi\colon \partial_\infty\Gamma \to \mathcal{F}_\theta$ and there is a continuous, ρ–equivariant lift $\widehat{\xi}\colon \partial_\infty\Gamma \to \mathcal{F}_R$ of ξ. Such a map $\widehat{\xi}$ will be called a limit map or boundary map of ρ as a P_R–Anosov representation. The relative position $\mathrm{pos}_{R,R}(\widehat{\xi}(x), \widehat{\xi}(y))$ for $x \neq y \in \partial_\infty\Gamma$ is its *transversality type*.

We should verify that the transversality type is in fact well–defined.

Lemma 4.2.2. *The relative position* $\mathrm{pos}_{R,R}(\widehat{\xi}(x), \widehat{\xi}(y))$ *in the above definition does not depend on the choice of x and y.*

Proof. By [Gro87, 8.2.I], there exists a dense orbit in $\partial_\infty\Gamma \times \partial_\infty\Gamma$. By equivariance of $\widehat{\xi}$, the relative position $[\![w_0]\!]$ of pairs in this orbit is constant. It is a transverse position because this orbit contains (only) pairs of distinct points. An arbitrary pair (x, y) of distinct points in $\partial_\infty\Gamma$ can be approximated by pairs in the dense orbit, so by continuity of $\widehat{\xi}$, we have

$$\mathrm{pos}_{R,R}(\widehat{\xi}(x), \widehat{\xi}(y)) \leq [\![w_0]\!].$$

But $\mathrm{pos}_{R,R}(\widehat{\xi}(x), \widehat{\xi}(y))$ is a transverse position, thus equality holds by Lemma 4.1.21. $\square$

Remarks 4.2.3.

(i) The definition (apart from the transversality type) makes sense for elementary hyperbolic groups, but it is not a very interesting notion in this case: The boundary has at most two points. Consequently, after restricting to a finite index subgroup, the boundary map always lifts to the maximally oriented setting. Moreover, after restricting to the subgroup preserving the boundary pointwise, the lifted boundary map holds no additional information.

(ii) In the oriented setting, the boundary map $\widehat{\xi} : \partial_\infty\Gamma \to \mathcal{F}_R$ is not unique: For any element $[m] \in \overline{M}/E$, the map $R_m \circ \widehat{\xi}$ is also continuous and equivariant. This gives all possible boundary maps in $\mathcal{F}_R$:
Since the unoriented boundary map $\xi : \partial_\infty\Gamma \to \mathcal{F}_\theta$ is unique [GW12, Lemma 3.3], an oriented boundary map $\widehat{\xi}'$ must be a lift of it. But if $\widehat{\xi}'$ agrees with $R_m \circ \widehat{\xi}$ at a single point, it must agree everywhere by equivariance and continuity since any orbit is dense in $\partial_\infty\Gamma$ [KB02, Proposition 4.2]. If the transversality type of $\widehat{\xi}$ was $[\![w_0]\!]$, then that of $R_m \circ \widehat{\xi}$ is $[\![m^{-1}w_0 m]\!]$ by Lemma 4.1.18.

The oriented flag manifold $\mathcal{F}_R$ in Definition 4.2.1 which is the target of the lift $\widehat{\xi}$ is not unique. However, there is a unique maximal choice of such a $\mathcal{F}_R$ (or equivalently, minimal choice of R), similar to the fact that an Anosov representation admits a unique minimal choice of θ such that it is P_θ–Anosov.

Proposition 4.2.4. *Let $\rho\colon \Gamma \to G$ be P_θ–Anosov. Then there is a unique minimal choice of E such that $\overline{M}_\theta \subset E \subset \overline{M}$ and ρ is P_R–Anosov, where $R = \langle \mathsf{v}(\Delta\backslash\theta), E\rangle$.*

Proof. Assume that there are two different choices E_1 and E_2 such that ρ is both P_{R_1}– and P_{R_2}–Anosov, where $R_i = \langle \mathsf{v}(\Delta\backslash\theta), E_i\rangle$. Let $E_3 = E_1 \cap E_2$. We will show that ρ is also P_{R_3}–Anosov. To do so, we have to construct a boundary map into $\mathcal{F}_{R_3}$ from the boundary maps into $\mathcal{F}_{R_1}$ and $\mathcal{F}_{R_2}$.

Let $\xi\colon \partial_\infty\Gamma \to \mathcal{F}_\theta$ be the boundary map of ρ as a P_θ–Anosov representation, and let $\xi_1\colon \partial_\infty\Gamma \to \mathcal{F}_{R_1}$, $\xi_2\colon \partial_\infty\Gamma \to \mathcal{F}_{R_2}$ be the two lifts we are given. Fix a point $x \in \partial_\infty\Gamma$, and let $F_x \in \mathcal{F}_{R_3}$ be a lift of $\xi(x) \in \mathcal{F}_\theta$. Denote by $\pi_1\colon \mathcal{F}_{R_3} \to \mathcal{F}_{R_1}$ and $\pi_2\colon \mathcal{F}_{R_3} \to \mathcal{F}_{R_2}$ the two projections. After right–multiplying ξ_1 with an element of $\overline{M}/E_1$ and ξ_2 with an element of $\overline{M}/E_2$, we may assume that $\pi_i(F_x) = \xi_i(x)$ for $i = 1, 2$. Set $\xi_3(x) := F_x$ and observe the following general property:

For every point $y \in \partial_\infty\Gamma$, there is at most one flag $F_y \in \mathcal{F}_{R_3}$ satisfying $\pi_i(F_y) = \xi_i(y)$ for $i = 1, 2$. Indeed, if gP_{R_3} and hP_{R_3} satisfy $gP_{R_i} = hP_{R_i}$ for $i = 1, 2$, there are elements $p_i \in P_{R_i}$ such that $g = hp_1 = hp_2$. This implies that $h^{-1}g \in P_{R_1} \cap P_{R_2} = P_{R_3}$.

By equivariance of ξ_1, ξ_2 and uniqueness of lifts to $\mathcal{F}_{R_3}$, we can extend ξ_3 equivariantly to a map $\xi_3\colon \Gamma x \to \mathcal{F}_{R_3}$. It is a lift of both $\xi_1|_{\Gamma x}$ and $\xi_2|_{\Gamma x}$. Recall that the orbit Γx is dense in $\partial_\infty\Gamma$ for any choice of x ([KB02, Proposition 4.2]). Using the corresponding properties of ξ_1 and ξ_2, we now show that this map is continuous and extends continuously to all of $\partial_\infty\Gamma$. Let $x_n \in \Gamma x$ and assume that $x_n \to x_\infty \in \partial_\infty\Gamma$. Then $\xi_i(x_n) \to \xi_i(x_\infty)$ for $i = 1, 2$. Therefore, there exist $m_n \in \overline{M}/(E_1 \cap E_2)$ such that $\xi_3(x_n)m_n$ converges in $\mathcal{F}_{R_3}$. By injectivity of the map

$$\overline{M}/E_1 \cap E_2 \to \overline{M}/E_1 \times \overline{M}/E_2$$

and convergence of $\xi_i(x_n)$, $i = 1, 2$, m_n must eventually be constant. Thus the limit $\xi_3(x_\infty) := \lim_{n\to\infty} \xi_3(x_n)$ exists and is the unique lift of $\xi_1(x_\infty), \xi_2(x_\infty)$ to $\mathcal{F}_{R_3}$. □

The following proposition shows that given a P_R–Anosov representation ρ of transversality type $[\![w_0]\!]$, we may always assume that R is stable under conjugation by w_0. This appeared as an assumption in Section 4.1.4 and plays a role later on when showing that balanced ideals give rise to cocompact domains of discontinuity.

Proposition 4.2.5. *Let $w_0 \in T$ and $E' = E \cap w_0 E w_0^{-1}$, and let $R = \langle \mathsf{v}(\Delta\backslash\theta), E\rangle$ and $R' = \langle \mathsf{v}(\Delta\backslash\theta), E'\rangle$. Assume that $\rho\colon \Gamma \to G$ is P_R–Anosov with a limit map $\widehat{\xi}\colon \partial_\infty\Gamma \to \mathcal{F}_R$ of transversality type $[\![w_0]\!] \in \widetilde{W}_{R,R}$. Then ρ is $P_{R'}$–Anosov.*

Proof. Let $x \neq z \in \partial_\infty\Gamma$, and consider the images $\widehat{\xi}(x), \widehat{\xi}(z) \in \mathcal{F}_R$. We claim that there is a unique lift $\eta_x(z) \in \mathcal{F}_{R'}$ satisfying

- $\eta_x(z)$ projects to $\widehat{\xi}(z)$.
- $\mathrm{pos}_{R,R'}(\widehat{\xi}(x), \eta_x(z)) = [\![w_0]\!]$.

To show this, let us first fix a good representative in G for $\widehat{\xi}(z)$: Since $\operatorname{pos}_{R,R}(\widehat{\xi}(x), \widehat{\xi}(z)) = [\![w_0]\!]$, there exists $h \in G$ such that

$$h(\widehat{\xi}(x), \widehat{\xi}(z)) = ([1], [w_0]).$$

Then $h^{-1}w_0 =: g \in G$ represents the flag $\widehat{\xi}(z)$ and also satisfies

$$\operatorname{pos}_{R,R'}(\widehat{\xi}(x), [g]) = [\![w_0]\!].$$

Any lift of $\widehat{\xi}(z)$ into $\mathcal{F}_{R'}$ can be written as $[gm] \in \mathcal{F}_{R'}$ for some $m \in E$. By Lemma 4.1.18, we have $\operatorname{pos}_{R,R'}(\widehat{\xi}(x), [gm]) = [\![w_0 m]\!]$. We claim that $[\![w_0 m]\!] = [\![w_0]\!] \in \widetilde{W}_{R,R'}$ implies that $m \in E'$ and therefore $[gm] = [g] \in \mathcal{F}_{R'}$, proving uniqueness of $\eta_x(z)$. Indeed, if $w_0 m = r w_0 r'$ for some $r \in R, r' \in R'$, we obtain

$$m = w_0^{-1} r w_0 r' \in w_0^{-1} R w_0 \cdot R' \subset w_0^{-1} R w_0.$$

Since

$$E \cap w_0^{-1} R w_0 = E \cap w_0^{-1} E w_0 = E',$$

it follows that $m \in E'$ and the lift $[g] \in \mathcal{F}_{R'}$ is unique.
This defines a map

$$\eta_x \colon \partial_\infty \Gamma \setminus \{x\} \to \mathcal{F}_{R'}$$

which is continuous since $\widehat{\xi}$ is continuous. We will show that it is independent of the choice of x, i.e. if $y \neq z$ is another point, we have $\eta_x(z) = \eta_y(z)$. Let $\gamma \in \Gamma$ be an element of infinite order with fixed points $\gamma^\pm \in \partial_\infty \Gamma$ such that $x \neq \gamma^-$ and $y \neq \gamma^-$. Then we have

$$[\![w_0]\!] = \operatorname{pos}_{R,R'}(\widehat{\xi}(x), \eta_x(\gamma^-)) = \operatorname{pos}_{R,R'}(\rho(\gamma)^n \widehat{\xi}(x), \rho(\gamma)^n \eta_x(\gamma^-))$$

for every $n \in \mathbb{N}$. Moreover, $\rho(\gamma)^n \widehat{\xi}(x) \to \widehat{\xi}(\gamma^+)$ and $\rho(\gamma)^n \eta_x(\gamma^-)$ is a lift of $\widehat{\xi}(\gamma^-)$. For every subsequence n_k such that $\rho(\gamma)^{n_k} \eta_x(\gamma^-)$ is constant, it follows that

$$\operatorname{pos}_{R,R'}(\widehat{\xi}(\gamma^+), \rho(\gamma)^{n_k} \eta_x(\gamma^-)) \leq [\![w_0]\!]. \tag{4.3}$$

But as $\operatorname{pos}_{R,R}(\widehat{\xi}(\gamma^+), \widehat{\xi}(\gamma^-)) = [\![w_0]\!]$, the position in (4.3) must be a transverse one, thus equality holds by Lemma 4.1.21. As seen before, this uniquely determines $\rho(\gamma)^{n_k} \eta_x(\gamma^-)$ among the lifts of $\widehat{\xi}(\gamma^-)$. Since the same holds for any subsequence n_k such that $\rho(\gamma)^{n_k} \eta_x(\gamma^-)$ is constant, $\rho(\gamma)$ fixes $\eta_x(\gamma^-)$ and we obtain

$$\operatorname{pos}_{R,R'}(\widehat{\xi}(\gamma^+), \eta_x(\gamma^-)) = [\![w_0]\!].$$

Applying the same argument to $y \neq \gamma^-$ shows that $\eta_x(\gamma^-) = \eta_y(\gamma^-)$.
Therefore, η_x and η_y are continuous functions on $\partial_\infty \Gamma \setminus \{x, y\}$ which agree on the dense subset of poles, hence they agree everywhere. We denote by

$$\eta \colon \partial_\infty \Gamma \to \mathcal{F}_{R'}$$

the continuous function defined by $\eta(y) = \eta_x(y)$ for any choice of $x \neq y$. It is ρ–equivariant because $\eta(\gamma y) = \eta_{\gamma x}(\gamma y) \in \mathcal{F}_{R'}$ is the lift of $\widehat{\xi}(\gamma y)$ defined by

$$\operatorname{pos}_{R,R'}(\widehat{\xi}(\gamma x), \eta_{\gamma x}(\gamma y)) = \operatorname{pos}_{R,R'}(\rho(\gamma)\widehat{\xi}(x), \eta_{\gamma x}(\gamma y)) = [\![w_0]\!],$$

which is $\rho(\gamma)\eta_x(y) = \rho(\gamma)\eta(y)$. $\square$

Remark 4.2.6. It is worth noting that the independence of $\eta_x(z)$ of the point x simplifies greatly if $\partial_\infty\Gamma$ is connected: If x and y can be connected by a path x_t in $\partial_\infty\Gamma$, we consider the lifts $\eta_{x_t}(z)$ along the path. They need to be constant by continuity of $\widehat{\xi}$, so $\eta_x(z)$ and $\eta_y(z)$ agree.

Example 4.2.7. Let us illustrate Proposition 4.2.5 with an example. Let $G = \mathrm{SL}(n,\mathbb{R})$ and $\rho : \Gamma \to G$ a representation which is P_θ–Anosov with $\theta = \{\alpha_1, \alpha_{n-1}\}$, so that we have a boundary map $\xi : \partial_\infty\Gamma \to \mathcal{F}_{1,n-1}$ into the space of partial flags comprising a line and a hyperplane. Assume that ρ is P_R–Anosov, where $R = \langle \mathsf{v}(\Delta\backslash\theta), \mathsf{v}(\alpha_{n-1})^2\rangle$. Then there is a boundary map $\widehat{\xi}$ into the space $\mathcal{F}_R$ of flags comprising an oriented line and an unoriented hyperplane. Let $x, z \in \partial_\infty\Gamma$ be two points as in the proof of the proposition. We can fix an orientation on $\widehat{\xi}(z)^{(n-1)}$ by requiring that $(\widehat{\xi}(x)^{(1)}, \widehat{\xi}(z)^{(n-1)})$, written in this order, induces the standard orientation on $\mathbb{R}^n$ (or the opposite orientation, depending on which element $w_0 \in \widetilde{W}$ we chose to represent the transversality type $[\![w_0]\!]$ of $\widehat{\xi}$). Doing so for all points $z \in \partial_\infty\Gamma$ extends the boundary map to a map into the space $\mathcal{F}_{R'}$ of flags comprising an oriented line and an oriented hyperplane.

As a consequence of the previous two propositions, the minimal oriented parabolic type associated to a P_θ–Anosov representation automatically has certain properties.

Proposition 4.2.8. *Let $\varnothing \neq \theta \subset \Delta$ be stable under ι, $R = \langle \mathsf{v}(\Delta\backslash\theta), E\rangle$ an oriented parabolic type, $w_0 \in T$ with $w_0 E w_0^{-1} = E$, and $\rho\colon \Gamma \to G$ be P_R–Anosov with transversality type $[\![w_0]\!] \in \widetilde{W}_{R,R}$. Then $w_0^2 \in E$.*

Proof. Let $x \neq y \in \partial_\infty\Gamma$ be two points in the boundary. Then $[\![w_0]\!] = \mathrm{pos}_{R,R}(\widehat{\xi}(x), \widehat{\xi}(y))$ where $\widehat{\xi}\colon \partial_\infty\Gamma \to F_R$ is the limit map. Then $\mathrm{pos}_{R,R}(\widehat{\xi}(y), \widehat{\xi}(x)) = [\![w_0^{-1}]\!]$. As observed in the proof of Lemma 4.2.2, there is a dense orbit in $\partial_\infty\Gamma \times \partial_\infty\Gamma$; let (a,b) be an element of this orbit. Since we can approximate both (x,y) and (y,x) by this orbit, continuity of $\widehat{\xi}$ implies that $[\![w_0]\!] \leq \mathrm{pos}_{R,R}(\widehat{\xi}(a), \widehat{\xi}(b))$ and $[\![w_0^{-1}]\!] \leq \mathrm{pos}_{R,R}(\widehat{\xi}(a), \widehat{\xi}(b))$. All of these are transverse positions, thus equality must hold in both cases by Lemma 4.1.21 and we conclude $[\![w_0]\!] = [\![w_0^{-1}]\!] \in \widetilde{W}_{R,R}$. Since $w_0 R w_0^{-1} = R$, this implies $w_0^2 \in R \cap \overline{M} = E$. ☐

The final part of this chapter is aimed at distinguishing connected components of Anosov representations by comparing the possible lifts of the limit map.

Proposition 4.2.9. *The set of P_R–Anosov representations is open and closed in the space of P_θ–Anosov representations $\mathrm{Hom}_{P_\theta\text{–}Anosov}(\Gamma, G) \subset \mathrm{Hom}(\Gamma, G)$.*

To prove this proposition, we will make use of the following technical lemma. Choose an auxiliary Riemannian metric on $\mathcal{F}_\theta$ and equip $\mathcal{F}_R$ with the metric which makes the finite covering $\pi_R\colon \mathcal{F}_R \to \mathcal{F}_\theta$ a local isometry.

Lemma 4.2.10. *Let $\widehat{\xi}\colon \partial_\infty\Gamma \to \mathcal{F}_R$ be a limit map of an P_R–Anosov representation and $\xi = \pi_R \circ \widehat{\xi}$. Then there exists $\delta > 0$ such that for every $x \in \partial_\infty\Gamma$*

(i) $\pi_R^{-1}(B_\delta(\xi(x))) = \bigsqcup_{[m]\in\overline{M}/E} B_\delta(R_m(\widehat{\xi}(x)))$, *and π_R maps any of these components isometrically to $B_\delta(\xi(x))$,*

(ii) and the set $R_m(B_\delta(\widehat{\xi}(x))) \subset \mathcal{F}_R$ intersects $\widehat{\xi}(\partial_\infty\Gamma)$ if and only if $[m] = 1 \in \overline{M}/E$.

Proof. By compactness of $\mathcal{F}_\theta$ there is an $\varepsilon > 0$ such that, for every $f \in \mathcal{F}_\theta$, the preimage of $B_\varepsilon(f)$ under π_R is the disjoint union of ε–balls around the preimages of x. Together with the choice of metric on $\mathcal{F}_R$, this shows (i) for any $\delta \leq \varepsilon$.

Now for every $x \in \partial_\infty\Gamma$ define

$$\mathcal{R}_x = \xi(\widehat{\xi}^{-1}(\mathcal{F}_R \setminus B_\varepsilon(\widehat{\xi}(x)))), \qquad \delta_x = \min\{\varepsilon,\ \tfrac{1}{2}d(\xi(x), \mathcal{R}_x)\}.$$

This is positive since $\mathcal{R}_x \subset \mathcal{F}_\theta$ is closed and ξ is injective. By compactness there is a finite collection $x_1, \ldots, x_m \in \partial_\infty\Gamma$ such that the sets $B_{\delta_{x_i}}(\xi(x_i)) \subset \mathcal{F}_\theta$ cover $\xi(\partial_\infty\Gamma)$. Let $\delta = \min_i \delta_{x_i}$. Then $U = B_\delta(\xi(x)) \subset B_\varepsilon(\xi(x))$ for every $x \in \partial_\infty\Gamma$, so $\pi_R^{-1}(U)$ decomposes into disjoint δ–balls as in (i). One of these is $V = B_\delta(\widehat{\xi}(x))$, and it is indeed the only one intersecting $\widehat{\xi}(\partial_\infty\Gamma)$:

If $y \in \partial_\infty\Gamma$ with $\widehat{\xi}(y) \in \pi_R^{-1}(U)$, then $\xi(y) \in U = B_\delta(\xi(x)) \subset B_{2\delta_{x_i}}(\xi(x_i))$ for some i. So

$$d(\xi(x_i), \xi(y)) < 2\delta_{x_i} \leq d(\xi(x_i), \mathcal{R}_{x_i}),$$

thus $\xi(y) \notin \mathcal{R}_{x_i}$ or equivalently $\widehat{\xi}(y) \in B_\varepsilon(\widehat{\xi}(x_i))$. So $\widehat{\xi}(y) \in \pi_R^{-1}(U) \cap B_\varepsilon(\widehat{\xi}(x_i))$, which is exactly V. $\square$

Proof of Proposition 4.2.9. To show openness, let ρ_0 be a P_R–Anosov representation with limit map $\widehat{\xi}_0 \colon \partial_\infty\Gamma \to \mathcal{F}_R$ and $\xi_0 = \pi_R \circ \widehat{\xi}_0$. Let δ be the constant from Lemma 4.2.10 for $\widehat{\xi}_0$. Choose $x_1, \ldots, x_k \in \partial_\infty\Gamma$ such that $B_{\delta/4}(\xi_0(x_i))$ cover $\xi_0(\partial_\infty\Gamma)$ and let $U_i = B_{\delta/2}(\xi_0(x_i))$ and $V_i = B_{\delta/2}(\widehat{\xi}_0(x_i))$. For every i we get a local section $s_i \colon U_i \to V_i$. If U_i and U_j intersect, then s_i and s_j coincide on the intersection, since $U_i \cup U_j$ is contained in a δ–ball, of which only a single lift can intersect $\widehat{\xi}_0(\partial_\infty\Gamma)$, so V_i and V_j both have to be contained in this lift. Therefore, the s_i combine to a smooth section $s \colon \bigcup_i U_i \to \bigcup_i V_i$.

For every $\rho_1 \in \mathrm{Hom}_{P_\theta\text{–Anosov}}(\Gamma, G)$ which is close enough to ρ_0, there is a path $\rho_t \in \mathrm{Hom}_{P_\theta\text{–Anosov}}(\Gamma, G)$ connecting ρ_0 and ρ_1 such that $d_{C^0}(\xi_t, \xi_0) < \delta/4$ for every $t \in [0,1]$. This is because $\mathrm{Hom}_{P_\theta\text{–Anosov}}(\Gamma, G)$ is open and the limit map depends continuously on the representation [GW12, Theorem 5.13]. Then for every $x \in \partial_\infty\Gamma$ there is an i such that

$$d(\xi_1(x), \xi_0(x_i)) \leq d_{C^0}(\xi_1, \xi_0) + d(\xi_0(x), \xi_0(x_i)) < \delta/2,$$

hence $\xi_1(x) \in U_i$. So $\xi_1(\partial_\infty\Gamma) \subset \bigcup_i U_i$ and we can define $\widehat{\xi}_1 = s \circ \xi_1$. This is a continuous lift of ξ_1. Note that also $\widehat{\xi}_0 = s \circ \xi_0$ and that we can equally define $\widehat{\xi}_t = s \circ \xi_t$ for every $t \in [0,1]$.

To show ρ_1–equivariance of $\widehat{\xi}_1$, let $\gamma \in \Gamma$, $x \in \partial_\infty\Gamma$ and consider the curves

$$\alpha(t) = \rho_t(\gamma)^{-1}\widehat{\xi}_t(\gamma x), \quad \beta(t) = \widehat{\xi}_t(x).$$

They are continuous and $\pi_R(\alpha(t)) = \rho_t(\gamma)^{-1}\pi_R(\widehat{\xi}_t(\gamma x)) = \xi_t(x) = \pi_R(\beta(t))$. Also $\alpha(0) = \rho_0(\gamma)^{-1}\widehat{\xi}_0(\gamma x) = \widehat{\xi}_0(x) = \beta(0)$ by ρ_0–equivariance of ξ_0. Therefore, the curves α and β coincide, so in particular $\widehat{\xi}_1$ is ρ_1–equivariant.

For closedness, let ρ_n be a sequence of P_R–Anosov representations with limit maps $\widehat{\xi}_n$ converging to the P_θ–Anosov representation ρ. Then the unoriented limit maps $\xi_n = \pi_R \circ \widehat{\xi}_n$

converge uniformly to ξ, the limit map of ρ. Let $\gamma \in \Gamma$ be an element of infinite order and $\gamma^-, \gamma^+ \in \partial_\infty\Gamma$ its poles. Since π_R is a finite covering, up to taking a subsequence, we can assume that $\widehat{\xi}_n(\gamma^+)$ converges to a point we call $\widehat{\xi}(\gamma^+)$. First, we are going to show that there is a neighborhood of γ^+ in $\partial_\infty\Gamma$ on which the maps $\widehat{\xi}_n$ converge uniformly to some limit.

As ρ is Anosov, the points $\xi(\gamma^-), \xi(\gamma^+) \in \mathcal{F}_\theta$ are transverse fixed points of $\rho(\gamma)$. Since $\xi_n(\gamma^\pm) \to \xi(\gamma^\pm)$, we can find an $\varepsilon > 0$ such that all elements of $\overline{B_\varepsilon(\xi(\gamma^+))}$ are transverse to $\xi_n(\gamma^-)$ for sufficiently large n. In particular, $\rho_n(\gamma^k)$ restricted to this ball converges uniformly to (the constant function with value) $\xi_n(\gamma^+)$ as $k \to \infty$. After shrinking ε, the preimage $\pi_R^{-1}(B_\varepsilon(\xi(\gamma^+)))$ is a union of finitely many disjoint copies of $B_\varepsilon(\xi(\gamma^+))$:

$$\pi_R^{-1}(B_\varepsilon(\xi(\gamma^+))) = \bigsqcup_{[m]\in\overline{M}/E} B_\varepsilon(\widehat{\xi}(\gamma^+))m$$

For n large, $\widehat{\xi}_n(\gamma^+) \in B_\varepsilon(\widehat{\xi}(\gamma^+))$. Furthermore, for large k, $\rho_n(\gamma^k)$ maps $B_\varepsilon(\xi(\gamma^+))$ into itself. Since $\rho_n(\gamma^k)B_\varepsilon(\widehat{\xi}(\gamma^+)) \subset \mathcal{F}_R$ is connected and contains $\widehat{\xi}_n(\gamma^+)$, it must be inside $B_\varepsilon(\widehat{\xi}(\gamma^+))$. So as $\rho_n(\gamma^k)|_{B_\varepsilon(\xi(\gamma^+))} \to \xi_n(\gamma^+)$ uniformly for $k \to \infty$, when seen as maps on $\mathcal{F}_R$, $\rho_n(\gamma^k)|_{B_\varepsilon(\widehat{\xi}(\gamma^+))}$ also converges uniformly to $\widehat{\xi}_n(\gamma^+)$. Now choose $\delta > 0$ such that $\xi(B_\delta(\gamma^+)) \subset B_{\varepsilon/2}(\xi(\gamma^+))$ and $\gamma^- \notin B_\delta(\gamma^+)$. We claim that then $\widehat{\xi}_n|_{B_\delta(\gamma^+)}$ converges uniformly to a lift of $\xi|_{B_\delta(\gamma^+)}$.

To see this, let n be large enough so that $d_{C^0}(\xi_n, \xi) < \varepsilon/2$. Then $\xi_n(B_\delta(\gamma^+)) \subset B_\varepsilon(\xi(\gamma^+))$. Let $y \in \widehat{\xi}_n(B_\delta(\gamma^+))$ be any point, and let $m \in \overline{M}$ be chosen such that $y \in B_\varepsilon(\widehat{\xi}(\gamma^+))m$. It follows that $\rho_n(\gamma^k)(y) \xrightarrow{k\to\infty} \widehat{\xi}_n(\gamma^+)m$. So by $\rho(\Gamma)$–invariance and closedness of $\widehat{\xi}_n(\partial_\infty\Gamma)$, $\widehat{\xi}_n(\gamma^+)m \in \widehat{\xi}_n(\partial_\infty\Gamma)$, so $[m] = 1 \in \overline{M}/E$ by transversality. Thus for all sufficiently large n, the image of $\widehat{\xi}_n|_{B_\delta(\gamma^+)}$ is entirely contained in $B_\varepsilon(\widehat{\xi}(\gamma^+))$, so we can use the local section $s : B_\varepsilon(\xi(\gamma^+)) \to B_\varepsilon(\widehat{\xi}(\gamma^+))$ to write $\widehat{\xi}_n|_{B_\delta(\gamma^+)} = s \circ \xi_n|_{B_\delta(\gamma^+)}$. This proves the stated uniform convergence on $B_\delta(\gamma^+)$.

Now we use local uniform convergence at γ^+ to obtain uniform convergence everywhere. For any point $y \in \partial_\infty\Gamma \setminus \{\gamma^-\}$ and any neighborhood $U \ni y$ whose closure does not contain γ^-, there is an integer $k(U)$ such that $\gamma^k(U) \subset B_\delta(\gamma^+)$ for all $k \geq k(U)$ [KB02, Theorem 4.3]. Then, for $z \in U$,

$$\widehat{\xi}_n(z) = \rho_n(\gamma)^{-k(U)}\widehat{\xi}_n(\gamma^{k(U)}z) \xrightarrow{n\to\infty} \rho(\gamma)^{-k(U)}\widehat{\xi}(\gamma^{k(U)}z),$$

so we get local uniform convergence on $\partial_\infty\Gamma \setminus \{\gamma^-\}$. Similarly, since $\widehat{\xi}_n = \rho_n(\zeta)^{-1} \circ \widehat{\xi} \circ \zeta$ for some $\zeta \in \Gamma$ with $\zeta\gamma^- \neq \gamma^-$, $\widehat{\xi}_n$ also converges uniformly in a neighborhood γ^-. So the maps $\widehat{\xi}_n$ converge uniformly to a limit $\widehat{\xi}$, which is continuous and equivariant. $\square$

From the previous proposition, we obtain the following two criteria to distinguish connected components of Anosov representations.

Corollary 4.2.11. *Let $\rho, \rho' \colon \Gamma \to G$ be P_θ–Anosov. Furthermore, let $R, R' \subset \widetilde{W}$ be the minimal oriented parabolic types such that ρ is P_R–Anosov and ρ' is $P_{R'}$–Anosov (see Proposition 4.2.4). Assume that ρ and ρ' lie in the same connected component of $\mathrm{Hom}_{P_\theta\text{–}Anosov}(\Gamma, G)$. Then the types R and R' agree. Furthermore, if $\widehat{\xi}, \widehat{\xi}' \colon \partial_\infty\Gamma \to \mathcal{F}_R$ are limit maps of ρ, ρ' of transversality types $[\![w_0]\!], [\![w_0']\!] \in \widetilde{W}_{R,R}$, then $[\![w_0]\!], [\![w_0']\!]$ are conjugate by an element of $\overline{M}$.*

Proof. By Proposition 4.2.9, ρ is also $P_{R'}$–Anosov and ρ' is P_R–Anosov. If R and R' were not equal, either R' would not be minimal for ρ' or R would not be minimal for ρ.
By Remark 4.2.3(ii), the transversality type of any limit map $\xi_\rho\colon \partial_\infty\Gamma \to \mathcal{F}_R$ of ρ is conjugate to $[\![w_0]\!]$ by an element of $\overline{M}$. By (the proof of) Proposition 4.2.9, ρ also admits a limit map of transversality type $[\![w_0']\!]$, so they must be conjugate by an element of $\overline{M}$. $\square$

4.3 Domains of discontinuity

In this section, we extend the machinery developed in [KLP18] to the setting of oriented flag manifolds (Definition 4.1.1). More specifically, we show that their description of cocompact domains of discontinuity for the action of Anosov representations on flag manifolds can be applied with some adjustments to oriented flag manifolds. Our main result is the following theorem, which is analogous to [KLP18, Theorem 7.14]:

Theorem 4.3.1. *Let Γ be a non–elementary word hyperbolic group and G a connected, semi–simple, linear Lie group. Furthermore, let $R, S \subset \widetilde{W}$ be oriented parabolic types and $w_0 \in T \subset \widetilde{W}$ a transverse position such that $w_0 R w_0^{-1} = R$ and $w_0^2 \in R$.*
Let $\rho\colon \Gamma \to G$ be a P_R-Anosov representation and $\xi : \partial_\infty\Gamma \to \mathcal{F}_R$ a limit map of transversality type $[\![w_0]\!] \in \widetilde{W}_{R,R}$. Assume that $I \subset \widetilde{W}_{R,S}$ is a w_0–balanced ideal, and define $\mathcal{K} \subset \mathcal{F}_S$ as

$$\mathcal{K} = \bigcup_{x\in\partial_\infty\Gamma} \bigcup_{[\![w]\!]\in I} C_{[\![w]\!]}(\xi(x)).$$

Then $\mathcal{K}$ is Γ–invariant and closed, and Γ acts properly discontinuously and cocompactly on the domain $\Omega = \mathcal{F}_S \setminus \mathcal{K}$.

We first observe that this theorem implies Theorem 1.5.1: By Proposition 4.2.5, when starting with a P_R–Anosov representation such that $w_0 R w_0^{-1} \neq R$, then it is actually $P_{R'}$–Anosov, with $R' = R \cap w_0 R w_0^{-1}$. Then the conditions $w_0 R' w_0^{-1} = R'$ and by Proposition 4.2.8 also $w_0^2 \in R$ are automatically satisfied. Also, if a balanced ideal is invariant by R from the left and S from the right, then it is also invariant by R' and gives a balanced ideal in $\widetilde{W}_{R',S}$.

A large part of the work required to prove this version, namely extending the Bruhat order to the extended Weyl group $\widetilde{W}$, was already done in Section 4.1 and Section 2.2. We prove proper discontinuity and cocompactness of the action of Γ on Ω separately in the following two subsections (Theorems 4.3.7 and 4.3.11). The part about cocompactness follows [KLP18] in all key arguments. Since oriented flag manifolds are not as established and well–studied as their unoriented counterparts, we reprove all the required technical lemmas in the setting of compact G-homogeneous spaces X, Y and G–equivariant maps between X and $\mathcal{C}(Y)$, the space of closed subsets of Y.

4.3.1 Proper discontinuity

Let P_R, P_S be oriented parabolic subgroups of types $R = \langle \mathsf{v}(\Delta\backslash\theta), E\rangle$ and $S = \langle \mathsf{v}(\Delta\backslash\eta), F\rangle$. Furthermore, let $w_0 \in T \subset \widetilde{W}$ be a transverse position. We assume that $\iota(\theta) = \theta$, $w_0 E w_0^{-1} = E$ and $w_0^2 \in E$, so that w_0 acts involutively on $\widetilde{W}_{R,S}$ (see Section 4.1.4).

The following definition of w_0–related limits is an oriented version of the one used for contracting sequences in [KLP18, Definition 6.1] (see also Lemma 3.1.3). The idea goes back to the study of discrete quasiconformal groups in [GM87a]. Apart from the dependence on the choice of w_0, we will see later that pairs of such limits are not unique in this setting (Lemma 4.3.5).

Definition 4.3.2. Let $(g_n) \in G^{\mathbb{N}}$ be a diverging sequence. A pair $F^-, F^+ \in \mathcal{F}_R$ is called a pair of w_0*–related limits* of the sequence (g_n) if

$$g_n|_{C_{[\![w_0]\!]}(F^-)} \xrightarrow{n\to\infty} F^+$$

locally uniformly.

Recall also the definition of dynamical relation (Definition 2.1.8): Two flags $F, F' \in \mathcal{F}_S$ are dynamically related if there are sequences $(g_n) \in G^{\mathbb{N}}$ and $F_n \in \mathcal{F}_S$ with $F_n \to F$ and $g_nF_n \to F'$. Using similar arguments as in the unoriented case (see Lemma 3.1.4), we can prove the following useful relative position inequality.

Lemma 4.3.3. *Let $(g_n) \in G^{\mathbb{N}}$ be a sequence admitting a pair $F^{\pm} \in \mathcal{F}_R$ of w_0–related limits. Assume that $F, F' \in \mathcal{F}_S$ are dynamically related via (g_n). Then*

$$\mathrm{pos}_{R,S}(F^+, F') \leq w_0 \,\mathrm{pos}_{R,S}(F^-, F).$$

Proof. Let $(F_n) \in \mathcal{F}_S^{\mathbb{N}}$ be a sequence such that $F_n \to F$ and $g_nF_n \to F'$. We pick elements $h_n \in G$ satisfying $F_n = h_nF$ and $h_n \to 1$. Writing $[\![w]\!] = \mathrm{pos}_{R,S}(F^-, F)$, it follows that there exists some $g \in G$ such that $g(F^-, F) = ([1], [w])$. Define $f \in \mathcal{F}_R$ as $f = [g^{-1}w_0]$, so that we obtain the following relative positions:

- $\mathrm{pos}_{R,R}(F^-, f) = [\![w_0]\!]$
- $\mathrm{pos}_{R,S}(F^-, F) = [\![w]\!]$
- $\mathrm{pos}_{R,S}(f, F) = \mathrm{pos}_{R,S}([w_0], [w]) = [\![w_0w]\!]$

In other words, f is chosen such that $\mathrm{pos}_{R,R}(F^-, f) = [\![w_0]\!]$ and $\mathrm{pos}_{R,S}(f, F)$ is as small as possible. Then, since $h_nf \to f$, f lies in the open set $C_{[\![w_0]\!]}(F^-)$ and $F^{\pm}$ are w_0–related limits, it follows that $g_nh_nf \to F^+$. Finally, observe that

$$\mathrm{pos}_{R,S}(g_nh_nf, g_nh_nF) = \mathrm{pos}_{R,S}(f, F)$$

is constant. We thus obtain the following inequalities:

$$\mathrm{pos}_{R,S}(F^+, F') \leq \mathrm{pos}_{R,S}(g_nh_nf, g_nh_nF) = \mathrm{pos}_{R,S}(f, F) = [\![w_0w]\!] \qquad \square$$

One consequence of this inequality is that being w_0–related limits is a symmetric condition.

Lemma 4.3.4 ([KLP18, (6.7)])**.** *If (F^-, F^+) is a pair of w_0–related limits in $\mathcal{F}_R$ of a sequence (g_n) then (F^+, F^-) is a pair of w_0–related limits of (g_n^{-1}).*

Proof. Let $F_n \to F$ be a convergent sequence in $C_{[\![w_0]\!]}(F^+) \subset \mathcal{F}_R$ and $g_{n_k}^{-1}F_{n_k} \to F' \in \mathcal{F}_R$ a convergent subsequence of $g_n^{-1}F_n$. This means F is dynamically related to F' via $(g_{n_k}^{-1})$ or equivalently F' is dynamically related to F via (g_{n_k}). So by Lemma 4.3.3 (with $S = R$)

$$[\![w_0]\!] = \operatorname{pos}_{R,R}(F^+, F) \leq w_0 \operatorname{pos}_{R,R}(F^-, F').$$

Since $[\![w_0]\!]$ is maximal in the Bruhat order, this implies that $w_0 \operatorname{pos}_{R,R}(F^-, F') = [\![w_0]\!]$ by Lemma 4.1.15. As w_0 induces an involution on $\widetilde{W}_{R,R}$, we obtain $\operatorname{pos}_{R,R}(F^-, F') = [\![1]\!]$, i.e. $g_{n_k}F_{n_k} \to F' = F^-$. By the same argument every subsequence of $g_n^{-1}F_n$ accumulates at F^- and thus $g_n^{-1}F_n \to F^-$, which shows that (F^+, F^-) are w_0–related limits of (g_n^{-1}). □

In the unoriented case, a P_θ–divergent sequence admits subsequences with unique attracting limits in $\mathcal{F}_\theta$. In the oriented case, however, this uniqueness is lost and all lifts of such a limit will be attracting on an open set.

Lemma 4.3.5. *Let $(g_n) \in G^{\mathbb{N}}$ be a P_θ–divergent sequence. Then there is a subsequence (g_{n_k}) admitting $|\overline{M}/E|$ pairs of w_0–related limits in $\mathcal{F}_R$. More precisely, the action*

$$\overline{M}/E \times \mathcal{F}_R^2 \to \mathcal{F}_R^2, \quad ([m], (F^-, F^+)) \mapsto (R_{w_0 m w_0^{-1}}(F^-), R_m(F^+)) \tag{4.4}$$

is simply transitive on the pairs of w_0–related limits of (g_{n_k}).

Proof. Observe that since $w_0 E w_0^{-1} = E$, conjugation by w_0 defines an action on $\overline{M}/E$. In other words, the choice of the representative $m \in \overline{M}$ in (4.4) does not matter.
Let us first prove that $\overline{M}/E$ acts simply transitively on the w_0–related limits of (g_{n_k}), assuming such limits exist. We know from Corollary 4.1.19 that

$$C_{[\![w_0]\!]}(R_{w_0 m w_0^{-1}}(F^-)) = C_{[\![w_0 m]\!]}(F^-) = R_m(C_{[\![w_0]\!]}(F^-)).$$

Because of this and since left and right multiplication commute, (4.4) restricts to an action on w_0–related limits of (g_{n_k}). It is free by the definition of E and $\mathcal{F}_R$. For transitivity, let $F^\pm$ and $F'^\pm$ be two w_0–related limit pairs for (g_{n_k}). Then

$$\bigcup_{[m] \in \overline{M}/E} R_m(C_{[\![w_0]\!]}(F^-)) = \bigcup_{[m] \in \overline{M}/E} C_{[\![w_0 m]\!]}(F^-)$$

is dense in $\mathcal{F}_R$ since its closure is all of $\mathcal{F}_R$ by Proposition 4.1.16. So for some $[m] \in \overline{M}/E$, $R_m(C_{[\![w_0]\!]}(F^-))$ must intersect the open set $C_{[\![w_0]\!]}(F'^-)$. On this intersection, g_{n_k} converges locally uniformly to $R_m(F^+)$ and F'^+, so $F'^+ = R_m(F^+)$. By Lemma 4.3.4, (F'^+, F'^-) and $(R_m(F^+), R_{w_0 m w_0^{-1}}(F^-))$ are w_0–related limits for the sequence $(g_{n_k}^{-1})$, so on $C_{[\![w_0]\!]}(F'^+) = C_{[\![w_0]\!]}(R_m(F^+))$ it locally uniformly converges to both F'^- and $R_{w_0 m w_0^{-1}}(F^-)$, so $F'^- = R_{w_0 m w_0^{-1}}(F^-)$.

What is left to show is the existence of w_0–related limits. This is done by an argument similar to [GKW15, Lemma 4.7]. Decompose the sequence g_n as $g_n = k_n e^{A_n} \ell_n$ with $k_n, \ell_n \in K$ and $A_n \in \overline{\mathfrak{a}^+}$. After taking a subsequence, we can assume that $k_n \to k$ and $\ell_n \to \ell$. We want to show that $F^- = [\ell^{-1} w_0^{-1}] \in \mathcal{F}_R$ and $F^+ = [k] \in \mathcal{F}_R$ are w_0–related limits of (g_n). We use the following characterization of locally uniform convergence: For every sequence $F_n \to F$

converging inside $C_{[\![w_0]\!]}(F^-)$ we want to show that g_nF_n converges to F^+. For sufficiently large n the sequence ℓ_nF_n will be inside $C_{[\![w_0]\!]}(\ell F^-) = C_{[\![w_0]\!]}([w_0^{-1}])$, so by Lemma 4.1.28 we can write $\ell_nF_n = [e^{X_n}]$ with

$$X_n \in \bigoplus_{\alpha\in\Sigma^-\setminus\mathrm{span}(\Delta\setminus\theta)} \mathfrak{g}_\alpha$$

converging to some X from the same space. So

$$g_nF_n = [k_ne^{A_n}e^{X_n}] = [k_ne^{A_n}e^{X_n}e^{-A_n}] = [k_n\exp(\mathrm{Ad}_{e^{A_n}}X_n)] = [k_n\exp(e^{\mathrm{ad}\,A_n}X_n)].$$

If we decompose $X_n = \sum_\alpha X_n^\alpha$ into root spaces then

$$e^{\mathrm{ad}\,A_n}X_n = \sum_{\alpha\in\Sigma^-\setminus\mathrm{span}(\Delta\setminus\theta)} e^{\alpha(A_n)}X_n^\alpha.$$

Now every $\alpha \in \Sigma^- \setminus \mathrm{span}(\Delta\setminus\theta)$ can be written as a linear combination of simple roots with non–positive coefficients and with the coefficient of at least one simple root $\beta \in \theta$ being strictly negative. As $\beta(A_n) \to \infty$ by P_θ–divergence, $\alpha(A_n)$ must converge to $-\infty$ and therefore $e^{\mathrm{ad}\,A_n}X_n$ goes to 0. This implies $g_nF_n \to [k] = F^+$, so $g_n|_{C_{[\![w_0]\!]}(F^-)} \to F^+$ locally uniformly. $\square$

Let Γ be a non–elementary word hyperbolic group and G a connected, semi–simple, linear Lie group (see Section 2.1 for some remarks on these assumptions).

Lemma 4.3.6. *Let $\rho : \Gamma \to G$ be a P_R–Anosov representation and let $\xi : \partial_\infty\Gamma \to \mathcal{F}_R$ be a continuous, equivariant limit map of transversality type $[\![w_0]\!]$. Then every sequence (γ_n) of distinct elements admits a subsequence (γ_{n_k}) and points $x, y \in \partial_\infty\Gamma$ such that $\gamma_{n_k}|_{\partial_\infty\Gamma\setminus\{x\}} \to y$ locally uniformly. Moreover, for any such subsequence, $(\xi(x), \xi(y))$ is a pair of w_0–related limits for the sequence $(\rho(\gamma_{n_k}))$.*

Proof. The first property is simply the fact that Γ acts as a convergence group on $\partial_\infty\Gamma$ [Bow99, Lemma 1.11].

To simplify notation, we assume from now on that $\gamma_n|_{\partial_\infty\Gamma\setminus\{x\}} \to y$ locally uniformly. By Lemma 4.3.5, there exists a subsequence $(\rho(\gamma_{n_k}))$ with w_0–related limits $F^\pm \in \mathcal{F}_R$. Then F^- is a lift of the unique repelling limit $\pi(F^-) \in \mathcal{F}_\theta$, and we have $\pi(F^-) \in \pi(\xi(\partial_\infty\Gamma))$ (see the description of the boundary map in [GGKW17, Theorem 5.3]). By right–multiplying F^- with an element $m \in \overline{M}/E$ and F^+ with $w_0^{-1}mw_0$ if necessary, we may assume that $F^- = \xi(x)$ for some $x \in \partial_\infty\Gamma$ (see Lemma 4.3.5). For any $x \neq z \in \partial_\infty\Gamma$, we have $\mathrm{pos}_{R,R}(F^-, \xi(z)) = \mathrm{pos}_{R,R}(\xi(x), \xi(z)) = [\![w_0]\!]$. Since the w_0–related attracting limit $F^+ \in \mathcal{F}_R$ is characterized by

$$\rho(\gamma_{n_k})|_{C_{[\![w_0]\!]}(F^-)} \xrightarrow{n\to\infty} F^+,$$

it follows that $\rho(\gamma_{n_k})(\xi(z)) \xrightarrow{n\to\infty} F^+$. As

$$\rho(\gamma_{n_k})(\xi(z)) = \xi(\gamma_{n_k}z) \xrightarrow{n\to\infty} \xi(y),$$

we obtain $\xi(y) = F^+$.

The same reasoning also shows that any subsequence (γ_{n_k}) of (γ_n) has a further subsequence $(\gamma_{n_{k_l}})$ such that $(\xi(x), \xi(y))$ is a pair of w_0–related limits of $(\rho(\gamma_{n_{k_l}}))$. So it is in fact a pair of w_0–related limits of the whole sequence $(\rho(\gamma_n))$. $\square$

Recall from Section 4.1.3 and Section 4.1.4 that a subset $I \subset \widetilde{W}_{R,S}$ is an ideal if $[\![w]\!] \in I$ and $[\![w']\!] \leq [\![w]\!]$ implies $[\![w']\!] \in I$, and it is w_0–fat if $[\![w]\!] \notin I$ implies $[\![w_0 w]\!] \in I$.

Theorem 4.3.7. *Let $\rho : \Gamma \to G$ be an P_R–Anosov representation and let $\xi : \partial_\infty \Gamma \to \mathcal{F}_R$ be a limit map of transversality type $[\![w_0]\!]$. Assume that $I \subset \widetilde{W}_{R,S}$ is a w_0–fat ideal, and define $\mathcal{K} \subset \mathcal{F}_S$ as*

$$\mathcal{K} = \bigcup_{x \in \partial_\infty \Gamma} \bigcup_{[\![w]\!] \in I} C_{[\![w]\!]}(\xi(x)).$$

Then $\mathcal{K}$ is Γ–invariant and closed, and Γ acts properly discontinuously on the domain $\Omega = \mathcal{F}_S \setminus \mathcal{K}$.

Proof. Γ–invariance and closedness of $\mathcal{K}$ follows from Lemma 4.1.29 and Lemma 2.3.8(ii).

Assume that the action of Γ on Ω is not proper. Then there exist $F, F' \in \Omega$ which are dynamically related by some sequence $(\rho(\gamma_n))$. This sequence is P_θ–divergent and by Lemma 4.3.6, a subsequence admits a pair of w_0–related limits of the form $\xi(x^\pm)$, where $x^\pm \in \partial_\infty \Gamma$. So Lemma 4.3.3 shows that

$$\mathrm{pos}_{R,S}(\xi(x^+), F') \leq w_0 \, \mathrm{pos}_{R,S}(\xi(x^-), F). \tag{4.5}$$

Since $F, F' \notin \mathcal{K}$, neither $\mathrm{pos}_{R,S}(\xi(x^+), F')$ nor $\mathrm{pos}_{R,S}(\xi(x^-), F)$ can be in I. As I is w_0–fat, this implies in particular $w_0 \, \mathrm{pos}_{R,S}(\xi(x^-), F) \in I$. But I is an ideal, so (4.5) implies $\mathrm{pos}_{R,S}(\xi(x^+), F') \in I$, a contradiction. $\square$

4.3.2 Cocompactness

We now come to the cocompactness part of Theorem 4.3.1. Owing to the fact that we want to apply everything to oriented flag manifolds, our setup here is more general than in [KLP18]. Nevertheless, all the key arguments from that paper still work. This includes in particular the idea of using expansion to prove cocompactness. The connection between (convex) cocompactness and expansion at the limit set was originally observed for Kleinian groups in [Sul85].

We included a detailed discussion of these arguments in Section 2.3 and just state the result here. The following notion of an expanding action was introduced by Sullivan in [Sul85, §9].

Definition 4.3.8. Let Z be a metric space, g a homeomorphism of Z and Γ a group acting on Z by homeomorphisms.

(i) g is *expanding at* $z \in Z$ if there exists an open neighbourhood $z \in U \subset Z$ and a constant $c > 1$ (the expansion factor) such that

$$d(gx, gy) \geq c\, d(x, y)$$

for all $x, y \in U$.

(ii) Let $A \subset Z$ be a subset. The action of Γ on Z is *expanding at* A if for every $z \in A$ there is a $\gamma \in \Gamma$ which is expanding at z.

Recall that for any compact metric space Z, we denote by $\mathcal{C}(Z)$ the set of compact subsets of Z. The key proposition will be the following:

Proposition 4.3.9 (Proposition 2.3.11)**.** *Let $\rho\colon \Gamma \to G$ be a representation of a discrete group Γ into a Lie group G, X and Y compact G–homogeneous spaces and $\mathcal{Q}\colon X \to \mathcal{C}(Y)$ a G–equivariant map. Let $\Lambda \subset X$ be compact and Γ–invariant such that the action of Γ on X is expanding at Λ. Assume further that $\mathcal{Q}(\lambda) \cap \mathcal{Q}(\lambda') = \varnothing$ for all distinct $\lambda, \lambda' \in \Lambda$. Then Γ acts cocompactly on $\Omega = Y \setminus \bigcup_{\lambda\in\Lambda} \mathcal{Q}(\lambda)$.*

We apply this to the setting of Anosov representations and oriented flag manifolds, to get the main result of this section. Let Γ, G, R, S and w_0 and I be as in Theorem 4.3.1. We need to show is that the action of a P_R–Anosov representation is expanding at the image of its limit map. This follows easily from the analogous statement in the unoriented setting (see [KLP17, Theorem 1.1(ii)]), but for convenience we also give a direct proof here.

Proposition 4.3.10. *Let $\rho : \Gamma \to G$ be a P_R–Anosov representation and $\xi : \partial_\infty\Gamma \to \mathcal{F}_R$ a limit map of transversality type $[\![w_0]\!] \in \widetilde{W}_{R,R}$. Then the action of Γ on $\mathcal{F}_R$ is expanding at $\xi(\partial_\infty\Gamma)$.*

Proof. First note that expansion does not depend on the choice of Riemannian metric on $\mathcal{F}_R$. To get rid of some constants, we will use a K–invariant metric for this proof.

Fix $x \in \partial_\infty\Gamma$. Since Γ is a non–elementary hyperbolic group, x is a conical limit point, meaning there is a sequence $(\gamma_n) \in \Gamma^{\mathbb{N}}$ and distinct points $b, c \in \partial_\infty\Gamma$ such that $\gamma_n|_{\partial_\infty\Gamma\setminus\{x\}} \to b$ locally uniformly and $\gamma_n x \to c$ (see [Bow99, Proposition 1.13] and [Tuk98, Theorem 1A]). We will show that $\rho(\gamma_n)$ is expanding at $\xi(x)$ for (some) large enough n.

Decompose $\rho(\gamma_n^{-1}) = k_n a_n \ell_n$ with $k_n, \ell_n \in K$ and $a_n = e^{A_n} \in \exp(\overline{\mathfrak{a}^+})$. After replacing (γ_n) by a subsequence, we may assume that k_n converges to k and ℓ_n converges to ℓ. As in the proof of Lemma 4.3.5 we see that $([\ell^{-1}w_0^{-1}], [k])$ is a pair of w_0–related limits for $(\rho(\gamma_n^{-1}))$. By Lemma 4.3.6 and Lemma 4.3.4, $(\xi(b), \xi(x))$ are also w_0–related limits for $(\rho(\gamma_n^{-1}))$. Using the action from Lemma 4.3.5 and possibly modifying the KAK–decomposition accordingly by some $m \in Z_K(\mathfrak{a})$, we can thus assume that $\xi(b) = [\ell^{-1}w_0^{-1}]$ and $\xi(x) = [k]$.

Since $b \neq c$, $\xi(c)$ is contained in $C_{[\![w_0]\!]}(\xi(b))$. This is an open set, so we can choose $\delta > 0$ such that $B_{2\delta}(\xi(c)) \subset C_{[\![w_0]\!]}(\xi(b))$. As a first step, we prove that a_n is contracting at

$$\mathcal{C} = \mathcal{F}_R \setminus N_\delta(\mathcal{F}_R \setminus C_{[\![w_0]\!]}([w_0^{-1}])).$$

By Lemma 4.1.28, $X \mapsto [e^X]$ is a diffeomorphism from $\mathfrak{n}_\theta^-$ to $C_{[\![w_0]\!]}([w_0^{-1}])$. Choose a scalar product on $\mathfrak{n}_\theta^-$ which makes the root spaces orthogonal. It defines a Riemannian metric on $\mathfrak{n}_\theta^-$ and therefore also on $C_{[\![w_0]\!]}([w_0^{-1}])$. On the compact subset $\mathcal{C} \subset C_{[\![w_0]\!]}([w_0^{-1}])$, it is comparable to the K–invariant metric on $\mathcal{F}_R$ up to a constant C. Now $a_n[\exp(X)] = [a_n \exp(X) a_n^{-1}] = [\exp(e^{\operatorname{ad} A_n} X)]$ for every $X \in \mathfrak{n}_\theta^-$, so the action of a_n on $C_{[\![w_0]\!]}([w_0^{-1}])$ translates to the linear action of $e^{\operatorname{ad} A_n}$ on $\mathfrak{n}_\theta^-$. So for every $z \in \mathcal{C}$

$$\|D_z a_n\| \leq C \|e^{\operatorname{ad} A_n}\|_{\mathfrak{n}_\theta^-} \leq C e^{-\min\{\alpha(A_n) \mid \alpha \in \Delta\setminus\theta\}}.$$

Here $\|\cdot\|_{\mathfrak{n}_\theta^-}$ is the operator norm with respect to the chosen norm on $\mathfrak{n}_\theta^-$. The second inequality holds since every root $\beta \in \Sigma^- \backslash \langle \Delta \backslash \theta \rangle$ has a strictly negative coefficient for at least one $\alpha \in \theta$ and nonpositive coefficients for all $\alpha' \in \Delta$, so $\beta(A_n) \leq -\alpha(A_n)$.

Let n be large enough such that $\|D_z a_n\| < 1$ for all $z \in \mathcal{C}$. By a standard argument on the lengths of curves, this implies that every $z \in \mathcal{C}$ has a small neighborhood and a constant $\kappa < 1$ such that $d(a_n v, a_n w) \leq \kappa\, d(v,w)$ for all v, w in this neighborhood. Clearly, this means that a_n^{-1} is expanding on $a_n\mathcal{C}$. As our metric is K–invariant, $\rho(\gamma_n) = \ell_n^{-1} a_n^{-1} k_n^{-1}$ is thus expanding at $k_n a_n \mathcal{C} = \rho(\gamma_n^{-1})\ell_n^{-1}\mathcal{C}$.

But since $\ell_n \xi(\gamma_n x) \to \ell\, \xi(c)$ we can, by taking n large enough, assume that $\ell_n\xi(\gamma_n x) \in B_\delta(\ell\,\xi(c))$. Since $B_{2\delta}(\ell\,\xi(c)) \subset C_{[\![w_0]\!]}([w_0^{-1}])$, we have $B_\delta(\ell\,\xi(c)) \subset \mathcal{C}$. This shows $\xi(x) \in \rho(\gamma_n^{-1})\ell_n^{-1}\mathcal{C}$, so $\rho(\gamma_n)$ is expanding at that point, which is what we wanted to show. $\square$

Theorem 4.3.11. *Let $\rho : \Gamma \to G$ be an P_R–Anosov representation and $\xi : \partial_\infty\Gamma \to \mathcal{F}_R$ a limit map of transversality type $[\![w_0]\!] \in \widetilde{W}_{R,R}$. Assume that $I \subset \widetilde{W}_{R,S}$ is a w_0–slim ideal, and define the set $\mathcal{K} \subset \mathcal{F}_S$ as*

$$\mathcal{K} = \bigcup_{x \in \partial_\infty\Gamma} \bigcup_{[\![w]\!] \in I} C_{[\![w]\!]}(\xi(x)).$$

Then $\mathcal{K}$ is Γ–invariant and closed, and Γ acts cocompactly on the domain $\Omega = \mathcal{F}_S \setminus \mathcal{K}$.

Proof. Γ–invariance and closedness of $\mathcal{K}$ follows from Lemma 4.1.29 and Lemma 2.3.8(ii). As discussed in Lemma 4.1.29, the map

$$\begin{aligned} \mathcal{Q} \colon \mathcal{F}_R &\to \mathcal{C}(\mathcal{F}_S) \\ f &\mapsto \bigcup_{[\![w]\!] \in I} C_{[\![w]\!]}(f) \end{aligned}$$

is G–equivariant. Moreover, $\xi(\partial_\infty\Gamma)$ is compact and the action of Γ on $\mathcal{F}_R$ is expanding at $\xi(\partial_\infty\Gamma)$ since ρ is P_R–Anosov (Proposition 4.3.10). If we can show that $\mathcal{Q}(\xi(x)) \cap \mathcal{Q}(\xi(x')) = \varnothing$ for $x \neq x'$, cocompactness will follow from Proposition 4.3.9.
Let $y' \in \mathcal{Q}(\xi(x'))$ be any point. Since ξ has transversality type $[\![w_0]\!]$ and by definition of $\mathcal{Q}$, we have the relative positions

$$\begin{aligned} \mathrm{pos}_{R,R}(\xi(x), \xi(x')) &= [\![w_0]\!], \\ \mathrm{pos}_{R,S}(\xi(x'), y') &=: [\![w]\!] \in I. \end{aligned}$$

By Lemma 4.1.20, this implies that $\mathrm{pos}_{R,S}(\xi(x), y') \geq [\![w_0 w]\!]$. Since $[\![w_0 w]\!] \notin I$ by w_0–slimness and I is an ideal, $y' \notin \mathcal{Q}(\xi(x))$. $\square$

4.4 Examples of balanced ideals

In this section, we will describe explicitly the Bruhat order on $\widetilde{W}$ and the possible balanced ideals for the group $G = \mathrm{SL}(3,\mathbb{R})$. These examples already show how passing from W to $\widetilde{W}$ vastly increases the number of balanced ideals and therefore the possibilities to build cocompact domains of discontinuity. We have no classification of all balanced ideals, so explicit examples of balanced ideals will be restricted to low dimensions and some special cases in higher dimension.

4.4.1 Reduction to $R = \{1, w_0^2\}$ and $S = \{1\}$

In applications, we are usually given a fixed representation $\rho\colon \Gamma \to G$. If it is Anosov, then by Proposition 4.2.4, there is a unique minimal oriented parabolic type $R = \langle \mathsf{v}(\Delta\backslash\theta), E\rangle$ for ρ. If we also fix one of the possible lifts of the boundary map to $\mathcal{F}_R$, then we get a transversality type $[\![w_0]\!] \in \widetilde{W}_{R,R}$. To apply Theorem 4.3.1 and find cocompact domains of discontinuity in a given flag manifold $\mathcal{F}_S$, we need to look for w_0–balanced ideals in $\widetilde{W}_{R,S}$. Note that this notion of w_0–balanced only depends on the equivalence class $[\![w_0]\!]$.

To enumerate all balanced ideals as we want to do in this section, a different approach is more convenient: We first determine the set $T \subset \widetilde{W}$ of transverse positions. Then, we want to list all w_0–balanced ideals in $\widetilde{W}_{R,S}$ for $w_0 \in T$ and all possible oriented parabolic types $R = \langle \mathsf{v}(\Delta\backslash\theta), E\rangle$ and $S = \langle \mathsf{v}(\Delta\backslash\eta), F\rangle$. For this to be well–defined, w_0 must act as an involution on $\widetilde{W}_{R,S}$, which by Section 4.1.4 happens if $\iota(\theta) = \theta$, $w_0 E w_0^{-1} = E$ and $w_0^2 \in E$.

Note that the smallest E satisfying these conditions is $E = \{1, w_0^2\}$. The following lemma implies that when listing all possible w_0–balanced ideals, one can restrict to the minimal choice $R = \{1, w_0^2\}$ and $S = \{1\}$.

Lemma 4.4.1. *Let R and S be oriented parabolic types as above, and consider the projection $\pi\colon \widetilde{W}_{\{1,w_0^2\},\{1\}} \to \widetilde{W}_{R,S}$. Assume that $I \subset \widetilde{W}_{R,S}$ is a w_0–balanced ideal. Then $\pi^{-1}(I) \subset \widetilde{W}_{\{1,w_0^2\},\{1\}}$ is a w_0–balanced ideal as well.*

Proof. By Lemma 4.1.15(i), $\pi^{-1}(I)$ is again an ideal. Let $[\![w]\!] \in \widetilde{W}_{\{1,w_0^2\},\{1\}}$, and recall that w_0 acts by left multiplication on both $\widetilde{W}_{\{1,w_0^2\},\{1\}}$ and $\widetilde{W}_{R,S}$, satisfying

$$\pi([\![w_0 w]\!]) = w_0 \pi([\![w]\!]).$$

Therefore, we obtain the following equivalences:

$$[\![w]\!] \in \pi^{-1}(I) \Leftrightarrow \pi([\![w]\!]) \in I \Leftrightarrow w_0\pi([\![w]\!]) \notin I \Leftrightarrow [\![w_0 w]\!] \notin \pi^{-1}(I) \qquad \square$$

By this lemma, every w_0–balanced ideal in $\widetilde{W}_{R,S}$ is obtained by projecting a R–left invariant and S–right invariant w_0–balanced ideal of $\widetilde{W}_{\{1,w_0^2\},\{1\}}$. We can further reduce the number of w_0 we have to consider by observing that choices of w_0 conjugate by an element in $\overline{M}$ lead to essentially the same balanced ideals:

Lemma 4.4.2. *Let $I \subset \widetilde{W}_{\{1,w_0^2\},\{1\}}$ be a w_0–balanced ideal and $m \in \overline{M}$. Then mI is a mw_0m^{-1}–balanced ideal.*

Proof. mI is again an ideal by Lemma 4.1.18(i). It is mw_0m^{-1}–balanced because

$$[\![w]\!] \in mI \Leftrightarrow [\![m^{-1}w]\!] \in I \Leftrightarrow w_0[\![m^{-1}w]\!] \notin I \Leftrightarrow (mw_0m^{-1})[\![w]\!] \notin mI. \qquad \square$$

Given a w_0–balanced ideal I and an element $m \in \overline{M}$ such that $mw_0m^{-1} = w_0$, mI is again w_0–balanced. The cocompact domains obtained via Theorem 4.3.1 for I and mI are in general different. In contrast to this, the action of $\overline{M}$ by right–multiplication is easy to describe: An ideal I is w_0–balanced if and only if Im is. Moreover, by Lemma 4.1.18, the domain will simply change by global right–multiplication with m, i.e. by changing some orientations.

4.4.2 The extended Weyl group of $\mathrm{SL}(n,\mathbb{R})$

Let $G = \mathrm{SL}(n,\mathbb{R})$ with maximal compact $K = \mathrm{SO}(n,\mathbb{R})$ and $\mathfrak{a} \subset \mathfrak{sl}(n,\mathbb{R})$ the set of diagonal matrices with trace 0. Then $\Sigma = \{\lambda_i - \lambda_j \mid i \neq j\} \subset \mathfrak{a}^*$, where $\lambda_i \colon \mathfrak{a} \to \mathbb{R}$ is the i–th diagonal entry. Choose the simple system Δ consisting of all roots $\alpha_i := \lambda_i - \lambda_{i+1}$ with $i \leq i \leq n-1$. Then B_0 is the subgroup of upper triangular matrices with positive diagonal. The group $Z_K(\mathfrak{a})$ is the group of diagonal matrices with ± 1 entries and $\det = 1$. Its identity component is trivial, so $\overline{M} = Z_K(\mathfrak{a})$. The extended Weyl group $\widetilde{W} = N_K(\mathfrak{a})$ consists of all permutation matrices with determinant 1 – i.e. all matrices with exactly one ± 1 entry per line and row and all other entries 0, such that $\det = 1$.

A generating set $\mathsf{v}(\Delta)$ in the sense of Definition 2.2.5 is given by

$$\mathsf{v}(\alpha_i) = \begin{pmatrix} I_{i-1} & & & \\ & & -1 & \\ & 1 & & \\ & & & I_{n-i-1} \end{pmatrix}.$$

The transverse positions $T \subset \widetilde{W}$ are antidiagonal matrices with ± 1 entries. The number of -1 entries has to be even if n is equal to 0 or 1 mod 4, and odd otherwise. In one formula, it has the same parity as $(n-1)n/2$.

The group $\overline{M}$ is generated by diagonal matrices with exactly two -1 entries and the remaining entries $+1$. Conjugating $w_0 \in T$ by such an element negates the two lines and the two columns corresponding to the minus signs. This yields the following standard representatives for equivalence classes in T under conjugation by $\overline{M}$:

(i) If n is odd, the $(n-1)/2$–block in the upper right corner can be normalized to have $+1$–entries.

(ii) If n is even, the $(n-2)/2$–block in the upper right corner can be normalized to have $+1$–entries.

If $w_0, w_0' \in T$ of this form are different, they are not conjugate by an element of $\overline{M}$.

4.4.3 Balanced ideals for $\mathrm{SL}(3,\mathbb{R})$

Let $G = \mathrm{SL}(3,\mathbb{R})$ and $\Delta = \{\alpha_1, \alpha_2\}$ the set of simple roots, viewed as their associated reflections. These generate the Weyl group

$$W = \langle \alpha_1, \alpha_2 \mid \alpha_1^2 = \alpha_2^2 = (\alpha_1\alpha_2)^3 = 1 \rangle = \{1, \alpha_1, \alpha_2, \alpha_1\alpha_2, \alpha_2\alpha_1, \alpha_1\alpha_2\alpha_1\}.$$

The Bruhat order on W is just the order by word length and there is a unique longest element $w_0 = \alpha_1\alpha_2\alpha_1$ which acts on W from the left, reversing the order (see Figure 4.1).

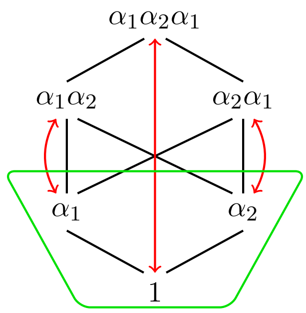


Figure 4.1: The Weyl group of $\mathrm{SL}(3,\mathbb{R})$. The black lines indicate the Bruhat order, in the sense that a line going downward from x to y means that x covers y in the Bruhat order. The red arrows show the involution induced by w_0. The subset surrounded by the green box is the only balanced ideal.

There is only one w_0–balanced ideal in this case, which is indicated by the green box in Figure 4.1.

Since $|\overline{M}| = 4$, each of the 6 elements of W has 4 preimages in $\widetilde{W}$, corresponding to different signs in the permutation matrix. The Bruhat order on $\widetilde{W}$ can be determined using Proposition 4.1.16 and is shown in Figure 4.2. See Section 4.4.4 for a geometric interpretation in terms of oriented flags.

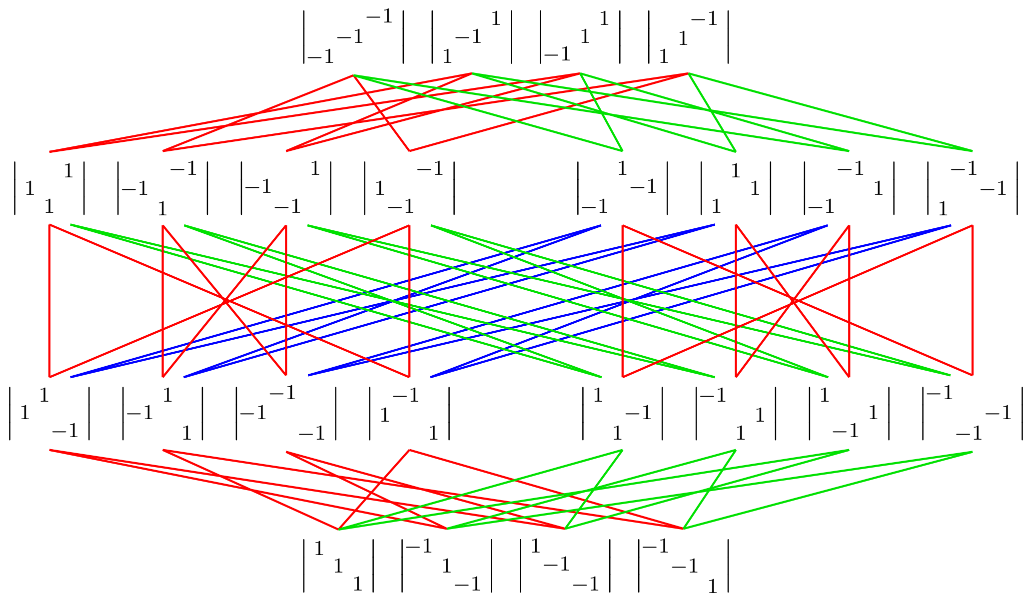


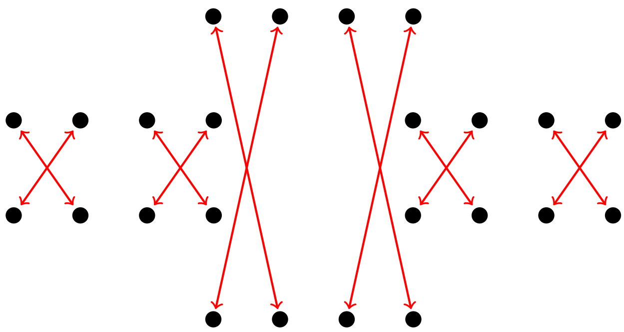
Figure 4.2: The Bruhat order on the extended Weyl group of $\mathrm{SL}(3,\mathbb{R})$. Different colors are used purely for better visibility.

To find balanced ideals, we first list the possible transverse positions $w_0 \in T$. These are

$$\begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}, \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}, \begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix}, \begin{pmatrix} & & 1 \\ & 1 & \\ -1 & & \end{pmatrix}.$$

The first two of these are conjugate by $\overline{M}$, as are the last two. So we have to distinguish two cases.

(i) $w_0 = \left(\begin{smallmatrix} & & 1 \\ & -1 & \\ 1 & & \end{smallmatrix}\right)$. Since $w_0^2 = 1$, the minimal choice of E (or R) is the trivial group, so $\widetilde{W}_{R,S} = \widetilde{W}_{\{1\},\{1\}} = \widetilde{W}$. The involution induced by w_0 acts on $\widetilde{W}$ in the following way (each dot represents the corresponding matrix from Figure 4.2):



We obtain the following balanced ideals:

a) The lift of the unoriented balanced ideal contains all relative positions in the bottom half of the picture.

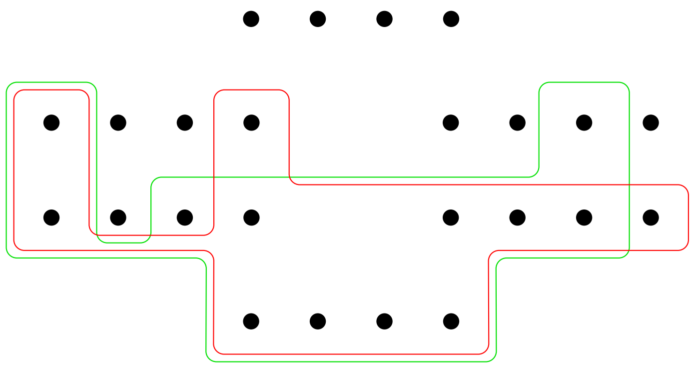
b) Ideals containing two positions from the third level and everything below these two positions in the Bruhat order. The possible pairs of positions from the third level that can be chosen are (ideals which are equivalent by right-multiplication by $\overline{M}$ are in curly brackets):

$$\{(1,4),(2,3)\},\{(5,8),(6,7)\}$$

as well as

$$\{(1,7),(2,8),(3,6),(4,5)\},\{(1,8),(2,7),(3,5),(4,6)\}.$$

In the following picture, we drew the examples $(1,4)$ in red and $(1,7)$ in green.



c) Ideals containing one relative position from the third level and everything except its w_0-image from the second level and below. There are 8 balanced ideals of this type, determined by the element on the third level. Right multiplication by $\overline{M}$ identfies the first 4 and the last 4 ideals.

In total, we find 21 w_0–balanced ideals, which form 7 equivalence classes with respect to right–multiplication with $\overline{M}$. Let us emphasize again that the balanced ideals in (b) and (c) are not lifts of balanced ideals from the unoriented setting. If a representation satisfies the prerequisites of Theorem 4.3.1, we therefore obtain new cocompact domains of discontinuity in oriented flag manifolds. For example, we will apply this to Hitchin representations in Section 4.5.1.

(ii) $w_0 = \begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix}$. Since $w_0^2 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$ is nontrivial, the minimal choice of R is $\{1, w_0^2\}$ and we consider $R\backslash\widetilde{W}$. By Lemma 4.1.15 we get the Bruhat order on $R\backslash\widetilde{W}$ as the projection of Figure 4.2. It is shown in Figure 4.3 alongside the action of w_0 on $R\backslash\widetilde{W}$.

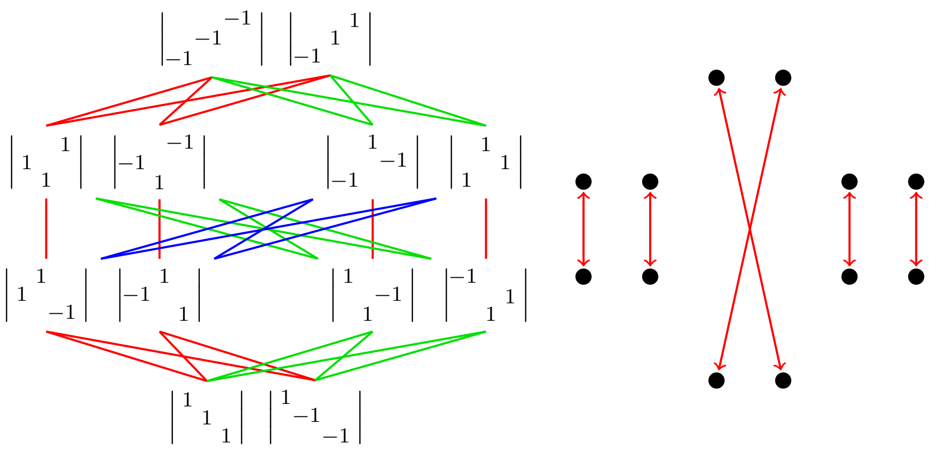


Figure 4.3: The Bruhat order on $R\backslash\widetilde{W}$ and the involution given by w_0. Different colors are used purely for better visibility.

In this case, the only balanced ideal is the lift of the unoriented one, and we do not obtain any new cocompact domains of discontinuity in oriented flag manifolds.

4.4.4 Geometric interpretation of relative positions for $\mathrm{SL}(3,\mathbb{R})$

In order to give a hands–on description of the various relative positions we saw in the previous subsection, we need notions of direct sums and intersections that take orientations into account. These notions appeared already in [Gui05], where they are used to describe curves of flags. First of all, let us fix some notation for oriented subspaces.

Definition 4.4.3. Let $A, B \subset \mathbb{R}^n$ be oriented subspaces. Then we denote by $-A$ the subspace A with the opposite orientation. If A and B agree as oriented subspaces, we write $A \overset{+}{=} B$.

Now let $A, B \subset \mathbb{R}^n$ be oriented subspaces. If they are (unoriented) transverse, taking a positive basis of A and extending it by a positive basis of B yields a basis of $A \oplus B$. Declaring this basis to be positive defines an orientation on $A \oplus B$. The orientation on the direct sum depends on the order we write the two subspaces in,

$$A \oplus B \overset{+}{=} (-1)^{\dim(A)\dim(B)} B \oplus A.$$

The case of intersections is slightly more difficult. Assume that $A, B \subset \mathbb{R}^n$ are oriented subspaces such that $A + B = \mathbb{R}^n$, and fix a standard orientation on $\mathbb{R}^n$. Let $A' \subset A$ be a subspace complementary to $A \cap B$ and analogously $B' \subset B$ a subspace complementary to $A \cap B$. We fix orientations on these two subspaces by requiring that

$$A' \oplus B \overset{+}{=} \mathbb{R}^n$$

and

$$A \oplus B' \overset{+}{=} \mathbb{R}^n.$$

Then there is a unique orientation on $A \cap B$ satisfying

$$A' \oplus (A \cap B) \oplus B' \overset{+}{=} \mathbb{R}^n.$$

This is the induced orientation on the intersection. Since the set of subspaces of A complementary to $A \cap B$ can be identified with $\mathrm{Hom}(A', A \cap B)$ and is therefore (simply) connected, the result does not depend on the choice of A', and analogously does not depend on the choice of B'. Like the oriented sum, it depends on the order we write the two subspaces in,

$$B \cap A \overset{+}{=} (-1)^{\operatorname{codim}(A)\operatorname{codim}(B)} A \cap B.$$

With this terminology at hand, consider the oriented relative positions shown in Figure 4.2. Let $f \in G/B_0$ be a reference complete oriented flag. We denote by $f^{(k)}$ the k–dimensional part of the flag f. Let $w = \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix} \in \widetilde{W}$. Then the refined Schubert stratum of flags at position w with respect to f

$$C_w(f) = \{F \in G/B_0 \mid f^{(2)} \oplus F^{(1)} \overset{+}{=} -\mathbb{R}^3,\ f^{(1)} \oplus F^{(2)} \overset{+}{=} -\mathbb{R}^3\}.$$

The other three positions of the highest level are characterized by the other choices of the two signs. Similarly, for the position $w' = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} \in \widetilde{W}$, we obtain

$$C_{w'}(f) = \{F \in G/B_0 \mid f^{(1)} \oplus F^{(1)} \overset{\pm}{=} f^{(2)},\ f^{(2)} \cap F^{(2)} \overset{\pm}{=} F^{(1)}\},$$

and the other three positions are characterized by the other choices of the two signs. Similar descriptions hold for the remaining oriented relative positions.

4.4.5 A simple example in odd dimension: Halfspaces in spheres

As the previous subsection demonstrated, calculating the most general relative positions and the Bruhat order gets out of hand very quickly as one increases the dimension. For example, in $\mathrm{SL}(5,\mathbb{R})$, there are 120 unoriented relative positions between complete flags and 1920 oriented relative positions between complete oriented flags. For practical reasons, it thus makes sense to restrict to more special cases, i.e. to consider relative positions $\widetilde{W}_{R,S}$ for bigger R, S than strictly necessary.

Let $G = \mathrm{SL}(2n+1,\mathbb{R})$ and $\theta = \{\alpha_n, \alpha_{n+1}\}$, $\eta = \{\alpha_1\}$, so that $\mathcal{F}_\theta$ is the space of partial flags consisting of the dimension n and $n+1$ parts, and $\mathcal{F}_\eta$ is $\mathbb{R}\mathrm{P}^{2n}$. Furthermore, let

$$E = \langle \overline{M}_\theta, \mathsf{v}(\alpha_n)^2 \mathsf{v}(\alpha_{n+1})^2 \rangle = \{m \in \overline{M} \mid m_{n+1,n+1} = +1\},$$

$F = \overline{M}_\eta$, $R = \langle \mathsf{v}(\Delta\backslash\theta), E\rangle$, and $S = \langle \mathsf{v}(\eta), F\rangle$. Then $\mathcal{F}_R$ is the space of oriented partial flags consisting of oriented n– and $(n+1)$–dimensional subspaces up to changing both orientations simultaneously, and $\mathcal{F}_S$ is S^{2n}, the space of oriented lines on $\mathbb{R}^{2n+1}$. Choose w_0 to be antidiagonal with -1 as the middle entry. The remaining entries are irrelevant for this example; if $2n+1 \in 4\mathbb{Z}+3$, there should be an odd number of minus signs, if $2n+1 \in 4\mathbb{Z}+1$, it has to be even.

The Bruhat order on the space $\widetilde{W}_{R,S}$ of relative positions as well as the involution w_0 are shown in Figure 4.4. We only need to keep track of the first column of the matrix representative since we right quotient by S. The left quotient by R then reduces the possible relative positions further.

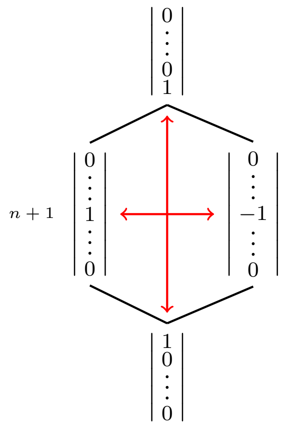


Figure 4.4: Oriented relative positions between $\mathcal{F}_R$ and $\mathcal{F}_S$

There are thus two balanced ideals, determined by choosing one of the two middle positions. This is in contrast to the unoriented case, where the two middle positions coincide and are a fixed point of the involution. In the case $n = 3$, the lifts of these two balanced ideals to $\widetilde{W}$ are among the 21 w_0–balanced ideals described in Section 4.4.3. In the notation of case (i)b there, these are the ideals $(1,4)$ and $(2,3)$. In particular, the balanced ideal indicated in red in Section 4.4.3 is the lift of the balanced ideal we obtain here in $\widetilde{W}_{R,S}$ by choosing the first of the middle positions.

The geometric description of the relative positions is as follows. Let $f \in \mathcal{F}_R$ be a reference flag. Then $[\![w]\!] = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ and $[\![w']\!] = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ correspond to

$$C_{[\![w]\!]}(f) = \{F \in S^{2n} \mid F \notin f^{(n+1)}\}, \qquad C_{[\![w']\!]}(f) = \{F \in S^{2n} \mid F \in f^{(n)}\},$$

and $[\![w'']\!] = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ corresponds to

$$C_{[\![w'']\!]}(f) = \{F \in S^{2n} \mid f^{(n)} \oplus F \overset{\pm}{=} f^{(n+1)}\}.$$

This can be rephrased slightly: The codimension 1 subspace $f^{(n)} \subset f^{(n+1)}$ decomposes $f^{(n+1)}$ into two half–spaces, and we say that an oriented line $l \subset f^{(n+1)}$ is in the positive half–space if $f^{(n)} \oplus l \overset{\pm}{=} f^{(n+1)}$. This is invariant under simultaneously changing the orientations of both $f^{(n)}$ and $f^{(n+1)}$ and therefore well–defined. Then $C_{[\![w']\!]}(f)$ is the spherical projectivization of the positive half of $f^{(n+1)}$.

The half great circles in Figure 1.2 in the introduction are an example of this construction, and the associated cocompact domain of discontinuity in S^2 for a convex cocompact representation $\rho\colon F_k \to \mathrm{SO}_0(2,1)$ corresponds to this balanced ideal. The higher–dimensional cases apply to Hitchin representations and generalized Schottky representations into $\mathrm{PSL}(4n+3,\mathbb{R})$, yielding cocompact domains of discontinuity in S^{4n+2} (see Section 4.5.1 and Section 4.5.2). The latter case includes in particular the motivating example from the introduction.

4.5 Examples of representations admitting cocompact domains of discontinuity

4.5.1 Hitchin representations

Let $\Gamma = \pi_1(S)$ be the fundamental group of a closed surface S of genus at least 2. Recall from Section 1.2 that a representation $\rho\colon \Gamma \to \mathrm{PSL}(n,\mathbb{R})$ is called *Hitchin representation* if it is in the same connected component as a representation of the form $\iota \circ \rho_0$, with $\rho_0\colon \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ discrete and injective and $\iota\colon \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$ the irreducible representation. Every Hitchin representation is B–Anosov [Lab06].

To find out if the limit map of a Hitchin representation lifts to an oriented flag manifold, let us first take a closer look at the irreducible representation. The standard Euclidean scalar product on $\mathbb{R}^2$ induces a scalar product on the symmetric product $\mathrm{Sym}^{n-1}\mathbb{R}^2$ by restricting the induced scalar product on the tensor power to symmetric tensors. Let $X = \binom{1}{0}$ and $Y = \binom{0}{1}$ be the standard orthonormal basis of $\mathbb{R}^2$. Then $e_i = \sqrt{\binom{n-1}{i-1}} X^{n-i}Y^{i-1}$ for $1 \leq i \leq n$ is an orthonormal basis of $\mathrm{Sym}^{n-1}\mathbb{R}^2$ and provides an identification $\mathbb{R}^n \cong \mathrm{Sym}^{n-1}\mathbb{R}^2$. For $A \in \mathrm{SL}(2,\mathbb{R})$ let $\iota(A) \in \mathrm{SL}(n,\mathbb{R})$ be the induced action on $\mathrm{Sym}^{n-1}\mathbb{R}^2$ in this basis. The homomorphism

$$\iota\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R}),$$

defined this way is the (up to conjugation) unique irreducible representation. It maps -1 to $(-1)^{n-1}$ and is therefore also well–defined as a map $\iota\colon \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$. The induced action on $\mathrm{Sym}^{n-1}\mathbb{R}^2$ preserves the scalar product described above, so $\iota(\mathrm{PSO}(2)) \subset \mathrm{PSO}(n)$.

It is easy to see that ι maps diagonal matrices to diagonal matrices. It also maps upper triangular matrices into $B_0 \subset \mathrm{PSL}(n,\mathbb{R})$ (that is, upper triangular matrices with the diagonal entries either all positive or all negative). Therefore, ι induces a smooth equivariant map

$$\varphi\colon \mathbb{R}\mathrm{P}^1 \to \mathcal{F}_{\{1\}}$$

between the complete oriented flag manifolds of $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(n,\mathbb{R})$.

Proposition 4.5.1. *Let $\rho\colon \Gamma \to \mathrm{PSL}(n,\mathbb{R})$ be a Hitchin representation. Then its limit map $\xi\colon \partial_\infty\Gamma \to \mathcal{F}_{\overline{M}} = G/B$ lifts to the fully oriented flag manifold $\mathcal{F}_{\{1\}} = G/B_0$ with transversality type*

$$w_0 = \begin{pmatrix} & & & \iddots \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}.$$

So all Hitchin representations are B_0–Anosov.

Proof. Since the B_0–Anosov representations are a union of connected components of B–Anosov representations by Proposition 4.2.9, we can assume that $\rho = \iota \circ \rho_0$ is Fuchsian.

Let $\xi_0\colon \partial_\infty\Gamma \to \mathbb{R}\mathrm{P}^1$ be the limit map of ρ_0 and $\pi\colon \mathcal{F}_{\{1\}} \to \mathcal{F}_{\overline{M}}$ the projection forgetting all orientations. Then the limit map $\xi\colon \partial_\infty\Gamma \to \mathcal{F}_{\overline{M}}$ of ρ is just $\pi \circ \varphi \circ \xi_0$ (this is the unique continuous and dynamics–preserving map, see [GGKW17, Remark 2.32b]). So $\widehat{\xi} = \varphi \circ \xi_0$ is a continuous and ρ–equivariant lift to $\mathcal{F}_{\{1\}}$. To calculate the transversality type, let $x, y \in \partial_\infty\Gamma$ with $\xi_0(x) = [1]$ and $\xi_0(y) = [w] \in \mathbb{R}\mathrm{P}^1$, where $w \in \mathrm{PSL}(2,\mathbb{R})$ is the anti–diagonal matrix with ± 1 entries. Then, since $\iota(w) = w_0$,

$$\mathrm{pos}(\widehat{\xi}(x), \widehat{\xi}(y)) = \mathrm{pos}(\varphi([1]), \varphi([w])) = \mathrm{pos}([1], [\iota(w)]) = w_0 \in \widetilde{W}. \qquad \square$$

Remark 4.5.2. Note that Hitchin representations map into $\mathrm{PSL}(n,\mathbb{R})$ and not $\mathrm{SL}(n,\mathbb{R})$. If n is even, the fully oriented flag manifold $\mathcal{F}_{\{1\}}$ in $\mathrm{PSL}(n,\mathbb{R})$ is the space of flags $f^{(1)} \subset \cdots \subset f^{(n-1)}$ with a choice of orientation on every part, but up to simultaneously reversing the orientation in every odd dimension (the action of -1). While we could lift ρ to $\mathrm{SL}(n,\mathbb{R})$, its limit map would still only lift to $\mathcal{F}^{\mathrm{SL}(n,\mathbb{R})}_{\{\pm 1\}} = \mathcal{F}^{\mathrm{PSL}(n,\mathbb{R})}_{\{1\}}$ and not give us any extra information.

Now that we know that Hitchin representations are B_0–Anosov, we can apply Theorem 4.3.7 and Theorem 4.3.11. For every w_0–balanced ideal in $\widetilde{W}$ we get a cocompact domain of discontinuity in the oriented flag manifold $\mathcal{F}_{\{1\}}$ of $\mathrm{PSL}(n,\mathbb{R})$. These include lifts of the domains in unoriented flag manifolds constructed in [KLP18], but also some new examples.

There are 21 different such w_0–balanced ideals if $n = 3$ (see Section 4.4.3) and already 4732 of them if $n = 4$, which makes it infeasible to list all of them here. However, it is not difficult to state in which oriented Grassmannians these domains exist.

Proposition 4.5.3. *Let $n \geq 3$ and $\rho\colon \Gamma \to \mathrm{PSL}(n,\mathbb{R})$ be a Hitchin representation. Assume that either*

(i) n is even and k is odd, or

(ii) n is odd and $k(n+k+2)/2$ is odd.

Then there exists a nonempty, open Γ–invariant subset $\Omega \subset \mathrm{Gr}^+(k,n)$ of the Grassmannian of oriented k–subspaces of $\mathbb{R}^n$, such that the action of Γ on Ω is properly discontinuous and cocompact.

Remarks 4.5.4.

(i) The domain Ω is not unique, unless n is even and $k \in \{1, n-1\}$.

(ii) In case (i) of Proposition 4.5.3, there is a cocompact domain of discontinuity also in the unoriented Grassmannian, and Ω is just the lift of one of these. The domains in case (ii) are new (see Section 3.2.2)

Proof. In the light of Theorem 4.3.1 it suffices to show that there is a w_0–balanced ideal in the set $\widetilde{W}/S$ where $S = \langle \mathsf{v}(\Delta \setminus \{\alpha_k\})\rangle$. A w_0–balanced ideal exists if and only if the action of w_0 on $\widetilde{W}/S$ has no fixed points (see Lemma 4.1.22 and Lemma 2.1.6).

To see that w_0 has no fixed points on $\widetilde{W}/S$, observe that every equivalence class in $\widetilde{W}/S$ has a representative whose first k columns are either the standard basis vectors $e_{i_1}, \dots e_{i_k}$ or $-e_{i_1}, e_{i_2}, \dots, e_{i_k}$, with $1 \leq i_1 < \dots < i_k \leq n$. So we can identify $\widetilde{W}/S$ with the set

$$\{\pm 1\} \times \{k\text{–element subsets of } \{1, \dots, n\}\}.$$

The action of w_0 on this is given by

$$(\varepsilon, \{i_1, \dots, i_k\}) \quad \mapsto \quad ((-1)^{k(k-1)/2 + \sum_j (i_j+1)}\varepsilon, \{n+1-i_k, \dots, n+1-i_1\}).$$

Only looking at the second factor, this can have no fixed points if n is even and k is odd, showing case (i). Otherwise, to get a fixed point it is necessary that $i_j + i_{k+1-j} = n+1$ for all $j \leq k$. But then

$$\frac{k(k-1)}{2} + \sum_{j=1}^{k} (i_j + 1) = \frac{k(k-1)}{2} + \frac{k}{2}(n+3) = \frac{k(n+k+2)}{2},$$

so w_0 fixes these elements if and only if $k(n+k+2)/2$ is even. Note that this number is always even if n and k are both even, which is why assuming n odd in case (ii) does not weaken the statement.

It remains to show that every $\Omega \in \mathrm{Gr}^+(k,n)$ constructed from a balanced ideal $I \subset \widetilde{W}/S$ is nonempty. Consider the lifts $\Omega' \subset \mathcal{F}_{\{1\}}$ of Ω and $I' \subset \widetilde{W}$ of I. Then Ω' is the domain in $\mathcal{F}_{\{1\}}$ given by I'. We will show that $\mathcal{F}_{\{1\}} \setminus \Omega'$ has covering codimension[1] at least 1, so Ω' must be nonempty. Similar arguments were used in [GW12] to prove the nonemptiness of certain domains. See [Nag83] for more background on dimension theory. In this case, we could also use the dimension of $\mathcal{F}_{\{1\}} \setminus \Omega'$ as a CW-complex, but the present approach has the benefit of generalizing to word hyperbolic groups with more complicated boundaries.

By Lemma 2.3.8 and the proof of Theorem 4.3.11, $\mathcal{F}_{\{1\}} \setminus \Omega'$ is homeomorphic to a fiber bundle over $\partial_\infty\Gamma \cong S^1$ with fiber $\bigcup_{[\![w]\!] \in I} C_{[\![w]\!]}([1])$. The covering dimension is invariant under homeomorphisms and has the following locality property: If a metric space is decomposed into open sets of dimension (at most) k, then the whole space is (at most) k–dimensional[2]. Therefore, the dimension of this fiber bundle equals the dimension of a local trivialization, that is, the dimension of the product $\mathbb{R} \times \bigcup_{[\![w]\!] \in I} C_{[\![w]\!]}([1])$. By [Mor77, Theorem 2], the dimension of a product is the sum of the dimensions of the factors whenever one of the factors is a CW complex[3]. Thus

$$\dim(\mathcal{F}_{\{1\}} \setminus \Omega') = 1 + \max_{w \in I'} \dim C_w([1]) = 1 + \max_{w \in I'} \ell(w).$$

If we know that $\ell(w) \leq \ell(w_0) - 2$ for every $w \in I'$, then, since $\dim \mathcal{F}_{\{1\}} = \ell(w_0)$, the codimension of $\mathcal{F}_{\{1\}} \setminus \Omega'$ is at least 1, so $\Omega \neq \varnothing$.

For $k < n$, if we write $w_k = \mathsf{v}(\alpha_1)\mathsf{v}(\alpha_2)\cdots\mathsf{v}(\alpha_k)$, and $\widetilde{w_k} = \mathsf{v}(\alpha_2)\cdots\mathsf{v}(\alpha_k)$, then by direct calculation, one verifies that

$$w_0 = w_{n-1}\cdots w_1$$

and

$$w_0\mathsf{v}(\alpha_k) = w_{n-1}\cdots w_{k+1}\widetilde{w_k}w_{k-1}\cdots w_1$$

(see Section 4.4.2 for an explicit description of $\widetilde{W}$). These are reduced expressions in the $\mathsf{v}(\alpha_i)$. So if $n \geq 3$, then $\mathsf{v}(\alpha_k) \leq w_0\mathsf{v}(\alpha_k)$ by Proposition 4.1.16. Therefore, the balanced ideal $I' \subset \widetilde{W}$ cannot contain $w_0\mathsf{v}(\alpha_k)$ and thus no element of length $\ell(w_0) - 1$. $\square$

A special case of such cocompact domains of discontinuity for Hitchin representations $\rho\colon \Gamma \to \mathrm{PSL}(4n+3,\mathbb{R})$ is described by the balanced ideals in Section 4.4.5: Let $\widehat{\xi}\colon \partial_\infty\Gamma \to \mathcal{F}_{\{1\}}$ be the boundary map of ρ, with image in complete oriented flags in $\mathbb{R}^{4n+3}$. Then the domain in S^{4n+2} is obtained by removing the spherical projectivizations of the positive halves of $\widehat{\xi}(x)^{(2n+2)}$, $x \in \partial_\infty\Gamma$. Note that in the case of $\mathrm{PSL}(3,\mathbb{R})$, the result is not very interesting: Consider the base case $\partial_\infty\Gamma \xrightarrow{\rho_0} \mathrm{PSL}(2,\mathbb{R}) \xrightarrow{\iota} \mathrm{PSL}(3,\mathbb{R})$, where ρ_0 is Fuchsian and ι is the irreducible representation. Since the limit set of ρ_0 is the full circle, the domain simply consists of two disjoint disks, and the quotient is two disjoint copies of the surface S (compare Figure 1.2). In higher dimension however, the domain is always connected and dense in S^{4n+2}.

[1] The (Lebesgue) covering dimension of a topological space X is the smallest number n such that every open cover of X admits a refinement with the property that each point of X is contained in at most $n+1$ of its elements.

[2] This follows from the equivalence of covering dimension and strong inductive dimension ([Nag83, Theorem II.7]) and locality of the strong inductive dimension.

[3] In that paper, Katětov–Smirnov covering dimension is used, which coincides with (Lebesgue) covering dimension for normal spaces.

4.5.2 Generalized Schottky representations

In [BT18], generalized Schottky groups in $G = \mathrm{PSL}(n, \mathbb{R})$ are introduced. The construction relies on the existence of a *partial cyclic order* on the space $G/B_0 = \mathcal{F}_{\{1\}}$, which is an oriented version of Fock–Goncharov triple positivity [FG06] and Labourie's 3–hyperconvexity [Lab06]. We will give a brief overview of generalized Schottky groups and their properties before showing how they fit into our framework. More details and proofs can be found in [BT17] and [BT18].

For all odd dimensions, the partial cyclic order on $\mathcal{F}_{\{1\}}$ is given as follows. Recall that we defined oriented direct sums in Section 4.4.4.

Definition 4.5.5. Let $G = \mathrm{PSL}(2n+1, \mathbb{R})$ and $F_1, F_2, F_3 \in \mathcal{F}_{\{1\}}$ be complete oriented flags. Then the triple (F_1, F_2, F_3) is *positive* or *increasing* if

$$F_1^{(i_1)} \oplus F_2^{(i_2)} \oplus F_3^{(i_3)} \overset{+}{=} \mathbb{R}^{2n+1}$$

for every choice of $i_1, i_2, i_3 \geq 0$ such that $i_1 + i_2 + i_3 = 2n + 1$.

Note that positivity of a triple includes in particular the condition $F_i^{(i_1)} \oplus F_j^{(i_2)} \overset{+}{=} \mathbb{R}^{2n+1}$ for $i < j$ and $i_1 + i_2 = 2n + 1$, which we call *oriented transversality* of F_i and F_j. In terms of relative positions, this means that $\mathrm{pos}(F_i, F_j) = w_0$, where w_0 is the transversality type of Hitchin representations (see Proposition 4.5.1). Having this partial cyclic order allows to define intervals in $\mathcal{F}_{\{1\}}$:

Definition 4.5.6. Let (F_1, F_3) be an oriented transverse pair of complete oriented flags. Then the *interval* between F_1 and F_3 is given by

$$(\!(F_1, F_3)\!) = \{F_2 \mid (F_1, F_2, F_3) \text{ is increasing}\}.$$

Consider a *cycle* $(F_1, \ldots, F_{4k})$, that is, a tuple such that (F_i, F_j, F_k) is increasing for any $i < j < k$. This defines the $2k$ intervals

$$I_i = (\!(F_{2i-1}, F_{2i})\!), \ 1 \leq i \leq 2k.$$

We say that a transformation $g \in \mathrm{PSL}(2n + 1, \mathbb{R})$ *pairs* two intervals I_i and I_j, $i \neq j$, if

$$g(F_{2i-1}) = F_{2j} \text{ and } g(F_{2i}) = F_{2j-1}.$$

Now pick k generators $g_1, \ldots, g_k \in \mathrm{PSL}(2n+1, \mathbb{R})$ pairing all of the intervals in some way.

Definition 4.5.7. A *purely hyperbolic generalized Schottky group in* $\mathrm{PSL}(2n + 1, \mathbb{R})$ is the group $\langle g_1, \ldots, g_k\rangle \subset \mathrm{PSL}(2n + 1, \mathbb{R})$, where the generators g_i are constructed as above. The associated representation $\rho\colon F_k \to \mathrm{PSL}(2n + 1, \mathbb{R})$ is called a *purely hyperbolic generalized Schottky representation.*

In this definition, "purely hyperbolic" refers to the fact that all of the endpoints of intervals are distinct. This assumption is important to obtain contraction properties and provides a link to Anosov representations. It is an easy consequence of the Ping–Pong–Lemma that all generalized Schottky representations are faithful.

A map between partially cyclically ordered sets is called *increasing* if it maps every increasing triple to an increasing triple. Moreover, the abstract group $\Gamma = F_k$ is identified with a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ by picking a model Schottky group in $\mathrm{PSL}(2,\mathbb{R})$ admitting the same combinatorial setup of intervals and generators as in $\mathrm{PSL}(2n+1,\mathbb{R})$. This yields a homeomorphism between $\partial_\infty\Gamma$ and a Cantor set in $\partial\mathbb{H}^2$.
The following result allows us to apply our theory of domains of discontinuity to purely hyperbolic generalized Schottky representations.

Theorem 4.5.8 ([BT18]). *Let $\rho\colon \Gamma = F_k \to \mathrm{PSL}(2n+1,\mathbb{R})$ be a purely hyperbolic generalized Schottky representation. Then ρ is B_0–Anosov. Moreover, the boundary map $\widehat{\xi}\colon \partial_\infty\Gamma \to \mathcal{F}_{\{1\}}$ is increasing.*

As observed above, the definition of the partial cyclic order on $\mathcal{F}_{\{1\}}$ implies that the transversality type w_0 of $\widehat{\xi}$ is the same as for Hitchin representations. Consequently, the same balanced ideals can be used to obtain domains of discontinuity. In particular, the balanced ideals from Section 4.4.5 yield cocompact domains in S^{4n+2} for purely hyperbolic generalized Schottky representations $\rho\colon \Gamma \to \mathrm{PSL}(4n+3,\mathbb{R})$. The motivating example from the introduction is a special case of this: Every convex cocompact representation $\rho_0\colon \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ admits a Schottky presentation, and the composition $\Gamma \xrightarrow{\rho_0} \mathrm{PSL}(2,\mathbb{R}) \xrightarrow{\iota} \mathrm{PSL}(3,\mathbb{R})$ with the irreducible representation ι is a purely hyperbolic generalized Schottky representation acting properly discontinuously and cocompactly on the domain shown in Figure 1.2.

4.6 Block embeddings of irreducible representations of $\mathrm{SL}(2,\mathbb{R})$

Let n be odd and $k \leq n$. Let $\iota_k\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(k,\mathbb{R})$ be the irreducible representation (see Section 4.5.1). Then we define

$$b_k\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R}), \quad A \mapsto \begin{pmatrix} \iota_k(A) & 0 \\ 0 & \iota_{n-k}(A) \end{pmatrix}.$$

Let Γ be the fundamental group of a closed surface of genus at least 2 and $\rho\colon \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ a Fuchsian (i.e. discrete and faithful) representation. Let $\overline{\rho}\colon \Gamma \to \mathrm{SL}(2,\mathbb{R})$ be a lift of ρ. We get every other lift of ρ as $\overline{\rho}^\varepsilon$, where $\varepsilon\colon \Gamma \to \mathbb{Z}/2\mathbb{Z}$ is a group homomorphism and $\overline{\rho}^\varepsilon(\gamma) = (-1)^{\varepsilon(\gamma)}\overline{\rho}(\gamma)$.

In this section, we will consider representations $\rho_k^\varepsilon = b_k \circ \overline{\rho}^\varepsilon$ obtained by composing a Fuchsian representation with b_k. Our main result is the following proposition and its corollary: For different choices of k and ε, the representations ρ_k^ε land in different connected components of Anosov representations.

Proposition 4.6.1. *The representation ρ_k^ε is B–Anosov and there exists $w_k \in T$ such that its limit map lifts to a continuous, equivariant map into $\mathcal{F}_{\{1,w_k^2\}}$ of transversality type $[\![w_k]\!]$. Thus ρ_k^ε is $P_{\{1,w_k^2\}}$–Anosov. The choice $R = \{1, w_k^2\}$ is minimal in the sense of Proposition 4.2.4.*

Futhermore, w_k is up to conjugation by elements of $\overline{M}$ given by

$$w_k = \begin{pmatrix} & & & J \\ & & \delta & \\ & K & & \\ L & & & \end{pmatrix}$$

with

$$\delta = \begin{cases} (-1)^{(k-1)/2}, & k \text{ odd} \\ (-1)^{(n-k-1)/2}, & k \text{ even} \end{cases}$$

and $J \in \mathrm{GL}(\frac{n-1}{2}, \mathbb{R})$, $K \in \mathrm{GL}(q-1, \mathbb{R})$, and $L \in \mathrm{GL}(\frac{Q-q+1}{2}, \mathbb{R})$ denoting blocks of the form

$$J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \quad K = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ \iddots & & & \end{pmatrix} \quad L = (-1)^{Q-1} \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$$

where $q = \min(k, n-k)$ and $Q = \max(k, n-k)$.

Proof. To simplify the description of the limit map, we will first modify the block embedding. For any $\lambda > 1$ the map b_k maps $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ to

$$g_\lambda = \begin{pmatrix} \lambda^{k-1} & & & & & & \\ & \lambda^{k-3} & & & & & \\ & & \ddots & & & & \\ & & & \lambda^{1-k} & & & \\ & & & & \lambda^{n-k-1} & & \\ & & & & & \ddots & \\ & & & & & & \lambda^{k+1-n} \end{pmatrix}.$$

Let $z \in \mathrm{SO}(n)$ be the permutation matrix (or its negative) such that the entries of $z g_\lambda z^{-1}$ are in decreasing order, and consider

$$\rho' = z \rho_k^\varepsilon z^{-1} = \iota \circ \overline{\rho}^\varepsilon,$$

where ι is the composition of b_k and conjugation by z. The representation ρ' is B–Anosov if and only ρ_k^ε is, and, since $\mathrm{SO}(n) \subset \mathrm{SL}(n, \mathbb{R})$ is connected, ρ' then lies in the same component of $\mathrm{Hom}_{B\text{–Anosov}}(\Gamma, \mathrm{SL}(n, \mathbb{R}))$ as ρ_k^ε. So we can consider ρ' instead of ρ_k^ε.

We first show that ρ' is B–Anosov. By [BPS16, Theorem 8.4], it suffices to show that there exist positive constants c, d such that for every $\alpha \in \Delta$ and every element $\gamma \in \Gamma$, we have

$$\alpha(\mu_n(\rho'(\gamma))) \geq c|\gamma| - d, \tag{4.6}$$

where μ_n is the Cartan projection in $\mathrm{SL}(n, \mathbb{R})$ and $|\cdot|$ denotes the word length in Γ. It follows from the description in Section 4.5.1 that ι maps $\mathrm{SO}(2)$ into $\mathrm{SO}(n)$, and it maps

$\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$ to $zg_\lambda z^{-1}$. Let α_0 denote the (unique) simple root for $\mathrm{SL}(2,\mathbb{R})$ and α_i the i–th simple root for $\mathrm{SL}(n,\mathbb{R})$. Then by the above, for $h \in \mathrm{SL}(2,\mathbb{R})$,

$$\alpha_i(\mu_n(\iota(h))) = \begin{cases} \frac{1}{2}\alpha_0(\mu_2(h)) & \text{if } \frac{n+1}{2} - q \leq i \leq \frac{n-1}{2} + q, \\ \alpha_0(\mu_2(h)) & \text{otherwise} \end{cases}$$

Since $\overline{\rho}^\varepsilon$ is Fuchsian and therefore Anosov, there are positive constants c_0, d_0 such that

$$\alpha_0(\mu_2(\overline{\rho}^\varepsilon(\gamma))) \geq c_0|\gamma| - d_0 \qquad \forall \gamma \in \Gamma.$$

This implies (4.6) with $c = c_0/2$ and $d = d_0/2$, so ρ' is B–Anosov.

The map ι maps $B_0^{\mathrm{SL}(2,\mathbb{R})}$ into $B_0^{\mathrm{SL}(n,\mathbb{R})}$ and -1 to some diagonal matrix $m = \iota(-1) \in \overline{M}$ with ± 1 entries. So $\iota(B) \subset B_0 \cup mB_0 = P_{\{1,m\}}$ and ι therefore induces smooth maps

$$\varphi \colon \mathbb{R}\mathrm{P}^1 \to \mathcal{F}_{\{1,m\}} = G/P_{\{1,m\}}, \qquad \psi \colon S^1 \to \mathcal{F}_{\{1\}} = G/B_0$$

which are ι–equivariant. Let $\pi \colon \mathcal{F}_{\{1,m\}} \to \mathcal{F}_{\overline{M}}$ be the projection which forgets orientations, and let $\overline{\xi} \colon \partial_\infty\Gamma \to \mathbb{R}\mathrm{P}^1$ be the limit map of $\overline{\rho}^\varepsilon$, a homeomorphism (which does not depend on ε). Then the curve $\xi = \pi \circ \varphi \circ \overline{\xi} \colon \partial_\infty\Gamma \to \mathcal{F}_{\overline{M}}$ is ρ'–equivariant and continuous. The definition of z ensures that it is also dynamics–preserving. So by [GGKW17, Remark 2.32b] ξ is the limit map of ρ', and $\widehat{\xi} = \varphi \circ \overline{\xi}$ is a continuous and equivariant lift to $\mathcal{F}_{\{1,m\}}$.

We now show that ξ does not lift to $\mathcal{F}_{\{1\}}^{\mathrm{SL}(n,\mathbb{R})}$. Write $\pi' \colon \mathcal{F}_{\{1\}} \to \mathcal{F}_{\overline{M}}$ and $p \colon S^1 \to \mathbb{R}\mathrm{P}^1$ for the projections. Then $\pi \circ \varphi \circ p = \pi' \circ \psi$. Now assume that ξ lifts to $\mathcal{F}_{\{1\}}$. Then the curve $\xi \circ \overline{\xi}^{-1} = \pi \circ \varphi \colon \mathbb{R}\mathrm{P}^1 \to \mathcal{F}_{\overline{M}}$ also lifts to some curve $\widehat{\varphi} \colon \mathbb{R}\mathrm{P}^1 \to \mathcal{F}_{\{1\}}$, i.e. $\pi' \circ \widehat{\varphi} = \pi \circ \varphi$. So

$$\pi' \circ \widehat{\varphi} \circ p = \pi \circ \varphi \circ p = \pi' \circ \psi.$$

By right–multiplication with an element of $\overline{M}$ we can assume that $\widehat{\varphi}([1]) = [1]$. So $\widehat{\varphi}(p([1])) = [1] = \psi([1])$, and uniqueness of lifts implies that $\widehat{\varphi} \circ p = \psi$. But $p([1]) = p([-1])$, so then $[1] = \psi([1]) = \psi([-1]) = [m] \in \mathcal{F}_{\{1\}}$, which is false since either k or $n-k$ has to be even and therefore $m \in \overline{M} \setminus \{1\}$.

To calculate the transversality type, let $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$. Then $\iota(w) \in \widetilde{W}$ and we can easily compute the relative position of $\widehat{\xi}$ at the points $x = \overline{\xi}^{-1}([1])$ and $y = \overline{\xi}^{-1}([w])$. It is

$$\mathrm{pos}(\widehat{\xi}(x), \widehat{\xi}(y)) = \mathrm{pos}(\varphi([1]), \varphi([w])) = \mathrm{pos}([1], [\iota(w)]) = [\![\iota(w)]\!].$$

Now $\iota_k(w)$ and $\iota_{n-k}(w)$ are antidiagonal, with alternating ± 1 entries and starting with $+1$ in the upper right corner. Conjugation by z interlaces the two blocks in the following way: The resulting matrix is antidiagonal, the middle entry is assigned to the odd–sized block and going towards the corners from there, entries are assigned alternatingly to the two blocks for as long as possible. Combined with the remarks on conjugation by $\overline{M}$ at the beginning of Section 4.4.2 and careful bookkeeping, this proves the claim about the transversality type $[\![w_k]\!]$. Since $\iota(w)^2 = \iota(-1) = m$, we have $w_k^2 = m$ (recall from Remark 4.2.3 (ii) that w_k is well–defined up to conjugation with $\overline{M}$, which does not change the square since $\overline{M}$ is abelian). $\square$

Corollary 4.6.2. *Let n be odd, $0 \leq k_1 \leq k_2 \leq \frac{n-1}{2}$ and $\rho_{k_1}^{\varepsilon_1}, \rho_{k_2}^{\varepsilon_2}$ be as in the previous proposition. If $\rho_{k_1}^{\varepsilon_1}$ and $\rho_{k_2}^{\varepsilon_2}$ are in the same connected component of $\mathrm{Hom}_{B\text{–Anosov}}(\Gamma, \mathrm{SL}(n,\mathbb{R}))$, then $k_1 = k_2$ and either $k_1 = k_2 = 0$ or $\varepsilon_1 = \varepsilon_2$.*

As a consequence, $\mathrm{Hom}_{B\text{–Anosov}}(\Gamma, \mathrm{SL}(n,\mathbb{R}))$ has at least $2^{2g-1}(n-1)+1$ components.

Proof. We saw before that ρ_k^ε is $P_{\{1,w_k^2\}}$–Anosov, with a limit map of transversality type $[\![w_k]\!]$, and that this is the minimal oriented parabolic for which ρ_k^ε is Anosov. By Corollary 4.2.11, if $\rho_{k_1}^{\varepsilon_1}$ and $\rho_{k_2}^{\varepsilon_2}$ were in the same connected component, then $[\![w_{k_1}]\!]$ and $[\![w_{k_2}]\!]$ would be conjugate by $\overline{M}$, which only occurs when if $k_1 = k_2$ by the discussion at the beginning of Section 4.4.2.

Now assume that $k_1 = k_2 = k \neq 0$ but $\varepsilon_1(\gamma) \neq \varepsilon_2(\gamma)$ for some $\gamma \in \Gamma$. Then $\overline{\rho}^{\varepsilon_1}(\gamma) = -\overline{\rho}^{\varepsilon_2}(\gamma)$, so one of them, say $\overline{\rho}^{\varepsilon_1}(\gamma)$, has two negative eigenvalues while the eigenvalues of $\overline{\rho}^{\varepsilon_2}(\gamma)$ are both positive. Then $\rho_k^{\varepsilon_1}(\gamma)$ has k (if k is even) or $n-k$ (if k is odd) negative eigenvalues, while $\rho_k^{\varepsilon_2}(\gamma)$ has only positive eigenvalues. But since $\rho(\gamma)$ has only real non–zero eigenvalues for every B–Anosov representation ρ, there can be no continuous path from $\rho_k^{\varepsilon_1}(\gamma)$ to $\rho_k^{\varepsilon_2}(\gamma)$ in this case.

In summary, we have $\frac{n-1}{2}$ different possible non–zero values for k and 2^{2g} different choices for ε (its values on the generators of Γ), giving $2^{2g-1}(n-1)$ connected components, plus the Hitchin component, $k = 0$. □

5 Domains of discontinuity in other homogeneous spaces

In this section, we consider the action of P–Anosov representations on the larger class of homogeneous spaces G/H with finitely many P–orbits. Some parts of the construction of domains of discontinuity generalize to this situation.

Let G be a connected semi–simple Lie group with finite center, $P = P_\theta \subset G$ a parabolic subgroup with $\iota(\theta) = \theta$, and $H \subset G$ any closed subgroup such that the double quotient $P\backslash G/H$ is finite. Recall that $P\backslash G/H$ then carries a natural partial order $\leq$ (Definition 2.1.2). Let $w_0 \in W$ be the longest element of the Weyl group.

Examples for such homogeneous spaces are the *symmetric spaces* associated to G, i.e. the spaces G/H where H is a union of connected components of the fixed point set of an involutive automorphism $\sigma\colon G \to G$. These spaces have finitely many B–orbits [Wol74] and the set $B\backslash G/H$ admits a Lie theoretic description [Mat79]. Note that if G/H is a *Riemannian symmetric space*, i.e. if H is compact, then $|B\backslash G/H| = 1$ and in fact every discrete subgroup $\Gamma \subset G$ acts properly discontinuously on G/H.

In contrast to the case of flag manifolds, there is no natural order–reversing involution available in this more general setting. Instead, in order to generalize the notion of fat ideal, we introduce the following symmetric relation:

Definition 5.1.1. We write $\xi \leftrightarrow \xi'$ for $\xi, \xi' \in P\backslash G/H$ if there exist $f, F \in G/P$ and $x \in G/H$ such that

$$\mathrm{pos}(f, F) = w_0, \qquad \mathrm{pos}(f, x) = \xi, \qquad \mathrm{pos}(F, x) = \xi'. \tag{5.1}$$

This relation also has a simple description in terms of double cosets:

Lemma 5.1.2. *Let $g, g' \in G$ be representatives of $\xi, \xi' \in P\backslash G/H$. Then $\xi \leftrightarrow \xi'$ if and only if $PgH \subset Pw_0Pg'H$.*

Proof. Using the transitive G–action on transverse pairs of flags, we see that $\xi \leftrightarrow \xi'$ if and only if there exists $x \in G/H$ such that $\mathrm{pos}([1], x) = \xi$ and $\mathrm{pos}([w_0], x) = \xi'$. Now this is equivalent to the existence of $p, p' \in P$ with $x = [pg]$ and $x = [w_0p'g']$ (choosing any representative in G for w_0), or in other words $g \in Pw_0Pg'H$. Since $Pw_0Pg'H$ is a union of double cosets, it then contains all of PgH. □

Lemma 5.1.3. *Let $\xi, \xi', \xi'' \in P\backslash G/H$ with $\xi \leftrightarrow \xi'$ and $\xi'' \geq \xi'$. Then $\xi \leftrightarrow \xi''$.*

Proof. We use Lemma 5.1.2. Let $g, g', g'' \in G$ represent ξ, ξ', ξ''. Then $Pg'H \subset Pw_0PgH$ and $Pg'H \subset \overline{Pg''H}$, so $Pw_0PgH \cap \overline{Pg''H} \neq \varnothing$. But note that Pw_0PgH is open, since it is a union of copies of the oben subset $Pw_0P \subset G$. So Pw_0PgH must in fact intersect, and therefore contain, $Pg''H$. $\square$

Example 5.1.4. If $H = Q$ is another parabolic subgroup and $\xi, \xi' \in P\backslash G/Q$, then $\xi \leftrightarrow \xi'$ if and only if $\xi' \geq w_0\xi$.

Proof. It is easy to show that $\xi \leftrightarrow w_0\xi$ for any $\xi \in P\backslash G/Q$: Just represent ξ by some $w \in W$ and consider the triple $[1], [w_0] \in G/P$ and $[w] \in G/Q$. Lemma 5.1.3 then shows that also $\xi \leftrightarrow \xi'$ for every $\xi' \geq w_0\xi$.

The converse is just an "unoriented" version of Lemma 4.1.20, but let us repeat the proof here: Suppose $\xi \leftrightarrow \xi'$ and represent ξ, ξ' by $w, w' \in W$. Then there exist $f, F \in G/P$ and $x \in G/Q$ with

$$\operatorname{pos}(f, F) = w_0, \qquad \operatorname{pos}(f, x) = w, \qquad \operatorname{pos}(F, x) = w'.$$

We can assume that $f = [1]$ and $F = [w_0]$. Then we can write $x = [urw]$ with $u \in N$ and $r \in R := P \cap W = \langle \Delta\backslash\theta \rangle$ (this follows e.g. from Lemma 4.1.27).

Now let $(A_n) \in \overline{\mathfrak{a}^+}^{\mathbb{N}}$ be a sequence with $\alpha(A_n) \to \infty$ for all $\alpha \in \Delta$ and let $g_n = e^{-A_n} \in G$. Then $g_n f = f$ and $g_n F = F$, but

$$g_n x = [e^{-A_n} urw] = [e^{-A_n} urwe^{\operatorname{Ad}_{rw}^{-1} A_n}] = [e^{-A_n} ue^{A_n} rw] \to [rw]$$

and therefore

$$w_0 w = \operatorname{pos}([w_0], [w]) = \operatorname{pos}([w_0 r^{-1}], [w]) = \operatorname{pos}([w_0], [rw]) \leq \operatorname{pos}(F, x) = w'. \qquad \square$$

There is an analogue of Lemma 3.1.4 in this general setting, which is as follows:

Lemma 5.1.5. *Let $(g_n) \in G^{\mathbb{N}}$ be a simply θ–divergent sequence and let $(g^-, g^+) \in \mathcal{F}_\theta^2$ be its limits. Let $x, y \in G/H$ be dynamically related via (g_n). Then*

$$\xi \leftrightarrow \operatorname{pos}(g^-, x) \Rightarrow \operatorname{pos}(g^+, y) \leq \xi \qquad \forall \xi \in P\backslash G/H. \tag{5.2}$$

Proof. Let $\xi \in P\backslash G/H$ with $\xi \leftrightarrow \operatorname{pos}(g^-, x)$. Then since the G–action on pairs of a fixed relative position is transitive, there exists $F \in G/P$ such that the triple (g^-, F, x) satisfies (5.1), i.e.

$$\operatorname{pos}(F, x) = \xi, \qquad \operatorname{pos}(g^-, F) = w_0.$$

Since x, y are dynamically related, there is a sequence $(x_n) \in (G/H)^{\mathbb{N}}$ converging to x such that $g_n x_n \to y$. We can write $x_n = h_n x$ for some sequence $h_n \to 1$ in G. Then $g_n h_n F \to g^+$ by Lemma 3.1.3, so

$$\operatorname{pos}(g^+, y) \leq \operatorname{pos}(g_n h_n F, g_n h_n x) = \operatorname{pos}(F, x) = \xi. \qquad \square$$

This allows us to construct domains of discontinuity in an analogous way.

Definition 5.1.6. Let $I \subset P\backslash G/H$ be an ideal with respect to $\leq$. Then we call I *fat* if for every $\xi \notin I$ there exists $\xi' \in I$ with $\xi \leftrightarrow \xi'$.

We do not define a notion of slim ideals because in contrast to the case of flag manifolds, the space G/H is usually not compact in this generalized setting, and there is therefore no hope of proving cocompactness for any of the domains of discontinuity. It remains to be seen if there is any meaningful generalization of the theory of slim ideals to this setting beyond cocompactness.

Theorem 5.1.7. *Let $\rho\colon \Gamma \to G$ be P–Anosov representation with limit map $\xi\colon \partial_\infty\Gamma \to G/P$ and $I \subset P\backslash G/H$ a fat ideal. Then*

$$\Omega = G/H \setminus \bigcup_{z \in \partial_\infty \Gamma} \{x \in G/H \mid \mathrm{pos}(\xi(z), x) \in I\}$$

is an open and Γ–invariant set and the action $\Gamma \curvearrowright \Omega$ via ρ is proper.

Proof. Ω is clearly Γ–invariant. Since I is an ideal, the set $\{x \in G/H \mid \mathrm{pos}(f, x) \in I\}$ is closed for every $f \in G/P$. Then Lemma 2.3.8(ii) shows that Ω is open.

To show properness, assume that $x, y \in \Omega$ are dynamically related. Then they are dynamically related by a simply θ–divergent sequence $(g_n) \in \rho(\Gamma)^{\mathbb{N}}$ with limits $(g^-, g^+) \in \xi(\partial_\infty\Gamma)^2$ (see Section 3.1). So by Lemma 5.1.5 we have

$$\xi \leftrightarrow \mathrm{pos}(g^-, x) \Rightarrow \mathrm{pos}(g^+, y) \leq \xi \qquad \forall \xi \in P\backslash G/H.$$

Now $\mathrm{pos}(g^-, x) \notin I$ as $x \in \Omega$, so $\xi' \leftrightarrow \mathrm{pos}(g^-, x)$ for some $\xi' \in I$ since I is a fat ideal. So $\mathrm{pos}(g^+, y) \leq \xi'$ and therefore $\mathrm{pos}(g^+, y) \in I$, which is a contradiction to the assumption that $y \in \Omega$. □

Note that, just as in the case of flag manifolds, the domain Ω from Theorem 5.1.7 could be empty. Once again, one could use dimension arguments to show non–emptiness in many cases. But one thing we can show in general is that there is always a non–trivial fat ideal.

Lemma 5.1.8. *$\xi \in P\backslash G/H$ is maximal if and only if*

$$C_\xi(x) := \{f \in G/P \mid \mathrm{pos}(f, x) = \xi\}$$

is open for any (or every) $x \in G/H$.

Proof. Clearly ξ is maximal if and only if $P\backslash G/H \setminus \{\xi\}$ is an ideal. Now by the definition of $\leq$ this happens if and only if the union of all double cosets except the one corresponding to ξ is closed, i.e. if this double coset (let's call it PgH) is an open subset of G. This is open if and only if $Hg^{-1}P$ is open, the projection to G/P of which is precisely $C_\xi([1])$. But by equivariance $C_\xi([1])$ is open if and only if $C_\xi(x)$ is open for any $x \in G/H$. □

Lemma 5.1.9. *Assume $|P\backslash G/H| > 1$. If $\xi \in P\backslash G/H$ is maximal, then $\xi \leftrightarrow \xi'$ for some non–maximal $\xi' \in P\backslash G/H$.*

Proof. Assume ξ is maximal and $\xi \not\leftrightarrow \xi'$ for every non–maximal ξ'. Then choose $F \in G/P$ and $x \in G/H$ with $\mathrm{pos}(F, x)$ not maximal (there have to be non–maximal positions since $|P\backslash G/H| > 1$ and G is connected). Now if $f \in G/P$ is transverse to F, then $\mathrm{pos}(f, x) \neq \xi$, as otherwise $\xi \leftrightarrow \mathrm{pos}(F, x)$. So

$$C_{w_0}(F) \subset \{f \in G/P \mid \mathrm{pos}(f, x) \neq \xi\} = G/P \setminus C_\xi(x).$$

But the set $C_{w_0}(F)$ of flags transverse to F is dense in G/P and $C_\xi(x)$ is open by Lemma 5.1.8, so this is a contradiction. $\square$

The following is a simple consequence of Theorem 5.1.7 and Lemma 5.1.9.

Corollary 5.1.10. *Assume $|P\backslash G/H| > 1$. Then $I = \{\xi \in P\backslash G/H \mid \xi$ not maximal$\}$ is a fat ideal.*

Therefore,

$$\Omega = \{x \in G/H \mid \mathrm{pos}(\xi(z), x) \text{ is maximal } \forall z \in \partial_\infty\Gamma\}$$

is a domain of discontinuity for any P–Anosov representation $\rho\colon \Gamma \to G$ with limit map $\xi\colon \partial_\infty\Gamma \to G/P$.

Example 5.1.11. Let $G = \mathrm{SL}(3, \mathbb{R})$ and $H = \mathrm{S}(\mathrm{GL}(2) \times \mathrm{GL}(1))$ be the subgroup stabilizing the plane $\langle e_1, e_2\rangle$ and the transverse line $\langle e_3\rangle$. Then G/H can be interpreted as the space of all transverse (plane,line)–pairs in $\mathbb{R}^3$. The set $B\backslash G/H$ of relative positions between such a pair and a full flag has 6 elements, shown in Figure 5.1.

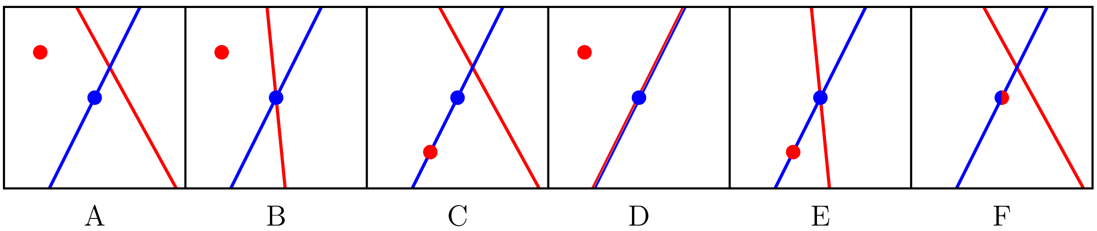


Figure 5.1: The relative positions between a transverse line and plane and a full flag in $\mathbb{R}^3$, shown in the projective plane $\mathbb{RP}^2$.

The ordering $\leq$ on $B\backslash G/H$, shown in Figure 5.2, has a single maximum (A), but three different minimal elements (D, E, F). Furthermore, $D \leftrightarrow F$ and $E \leftrightarrow E$, and so by Lemma 5.1.3 everything above D is related to everything above F, and everything above E is pairwise related by $\leftrightarrow$. This describes the whole relation $\leftrightarrow$ in this case.

Therefore, the two minimal fat ideals in $B\backslash G/H$ are $\{D, E\}$ and $\{E, F\}$. So for an Anosov representation $\rho\colon \Gamma \to \mathrm{SL}(3, \mathbb{R})$ we get the domains of discontinuity using Theorem 5.1.7:

$$\Omega_{\{D,E\}} = \{(L, P) \mid \forall x \in \partial_\infty\Gamma\colon P \neq \xi^2(x) \wedge (L \not\subset \xi^2(x) \vee \xi^1(x) \not\subset P)\},$$

$$\Omega_{\{E,F\}} = \{(L, P) \mid \forall x \in \partial_\infty\Gamma\colon L \neq \xi^1(x) \wedge (L \not\subset \xi^2(x) \vee \xi^1(x) \not\subset P)\}.$$

Note on the other hand that the action on G/H as a whole can not be proper: If $\gamma \in \Gamma$ has infinite order, and $\gamma^-, \gamma^+ \in G/B$ are the repelling and attracting fixed flags of $\rho(\gamma)$, then

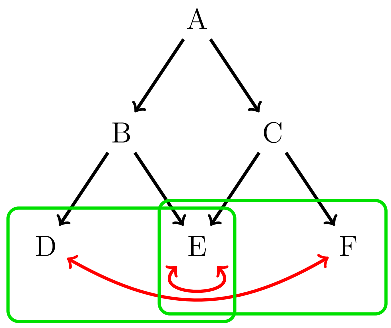


Figure 5.2: The order $\leq$ and part of the relation $\leftrightarrow$ on $B\backslash G/H$ and the two minimal fat ideals.

the line $(\gamma^-)^2 \cap (\gamma^+)^2$ and the plane $(\gamma^-)^1 \oplus (\gamma^+)^1$ are fixed by $\rho(\gamma)$ and are transverse, so they define an element in G/H. But if the action was proper, infinite order elements could not have fixed points.

It is unclear whether the domains of discontinuity constructed this way are maximal or in any other way special. It would also be good to have a combinatorial description of $P\backslash G/H$ and the relations $\leq$ and $\leftrightarrow$ on it, at least in some special cases like if G/H is a symmetric space. This is an ongoing project with León Carvajales and we hope to be able to answer these questions soon.

6 Lists of balanced ideals

The description of cocompact domains using balanced ideals often reduces finding them to enumerating balanced ideals, which is a purely combinatorial problem. This means we can use a computer to do it. This section shows the resulting lists and numbers in some potentially interesting cases. The program used to compute them was written by the author together with David Dumas. It can be found online at `https://florianstecker.de/balancedideals/`.

6.1 Balanced ideals in A_n

Assume the Coxeter system (W, Δ) defined by the Weyl group and the restricted roots of G is of type A_n. For example, G could be the group $\mathrm{SL}(n+1, \mathbb{R})$ or $\mathrm{SL}(n+1, \mathbb{C})$. We write $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ in such a way that $\alpha_i \alpha_{i+1} \in W$ is an element of order 3. The following tables show all balanced ideals in W. For every balanced ideal I, it shows the subset of Δ it is left– and right–invariant by. This means that for $\theta, \eta \subset \Delta$ the balanced ideals in $W_{\theta,\eta}$ are precisely those in W whose left–invariance includes $\Delta \backslash \theta$ and right–invariance includes $\Delta \backslash \eta$. We also show the (real resp. complex) dimension of the set we have to take out of the $\mathcal{F}_\Delta$ for every limit point, and a minimal set of elements of W generating I as an ideal.

6.1.1 Balanced ideals in A_1

left–invariance	**right–invariance**	**dimension**	**generators**
$\varnothing$	$\varnothing$	0	1

The corresponding cocompact domain of discontinuity for an Anosov representation $\Gamma \to \mathrm{SL}(2, \mathbb{R})$ with limit set $\Lambda \subset \mathbb{RP}^1$ is just the classical $\mathbb{RP}^1 \setminus \Lambda$.

6.1.2 Balanced ideals in A_2

left–invariance	**right–invariance**	**dimension**	**generators**
$\varnothing$	$\varnothing$	1	α_1, α_2

For two full flags $f, F \in \mathcal{F}_\Delta$ in $\mathbb{R}^3$, $\mathrm{pos}(f, F) = \alpha_1$ if and only if $f^1 = F^1$ but $f^2 \neq F^2$ and $\mathrm{pos}(f, F) = \alpha_2$ if and only if $f^2 = F^2$ but $f^1 \neq F^1$. The ideal $I = \{1, \alpha_1, \alpha_2\}$ generated by these two elements also contains the identity element in W, corresponding to the position $f = F$. Applied to a Δ–Anosov representation $\rho\colon \Gamma \to \mathrm{SL}(3,\mathbb{R})$ with limit map $\xi\colon \partial_\infty\Gamma \to \mathcal{F}_\Delta$, this means that

$$\begin{aligned}\Omega &= \{f \in \mathcal{F}_\Delta \mid \mathrm{pos}(\xi(x), f) \notin I \;\forall x \in \partial_\infty\Gamma\} \\ &= \{f \in \mathcal{F}_\Delta \mid f^1 \neq \xi^1(x) \wedge f^2 \neq \xi^2(x) \;\forall x \in \partial_\infty\Gamma\}\end{aligned}$$

is a cocompact domain of discontinuity in the full flag manifold $\mathcal{F}_\Delta$.

6.1.3 Balanced ideals in A_3

left–invariance	right–invariance	dimension	generators
$\{\alpha_1,\alpha_3\}$	$\{\alpha_1,\alpha_2\}$	4	$\alpha_3\alpha_1\alpha_2\alpha_1$
$\{\alpha_1,\alpha_3\}$	$\{\alpha_2,\alpha_3\}$	4	$\alpha_1\alpha_2\alpha_3\alpha_2$
$\{\alpha_2\}$	$\varnothing$	3	$\alpha_1\alpha_2\alpha_1,\ \alpha_2\alpha_1\alpha_3,\ \alpha_2\alpha_3\alpha_2$
$\varnothing$	$\{\alpha_1\}$	3	$\alpha_3\alpha_2\alpha_1,\ \alpha_1\alpha_2\alpha_1,\ \alpha_2\alpha_1\alpha_3$
$\varnothing$	$\{\alpha_2\}$	3	$\alpha_1\alpha_3\alpha_2,\ \alpha_1\alpha_2\alpha_1,\ \alpha_2\alpha_3\alpha_2$
$\varnothing$	$\{\alpha_3\}$	3	$\alpha_1\alpha_2\alpha_3,\ \alpha_2\alpha_1\alpha_3,\ \alpha_2\alpha_3\alpha_2$
$\varnothing$	$\varnothing$	3	$\alpha_3\alpha_2\alpha_1,\ \alpha_1\alpha_3\alpha_2,\ \alpha_1\alpha_2\alpha_3$
$\varnothing$	$\varnothing$	3	$\alpha_3\alpha_2\alpha_1,\ \alpha_1\alpha_2\alpha_3,\ \alpha_2\alpha_1\alpha_3$
$\varnothing$	$\varnothing$	3	$\alpha_3\alpha_2\alpha_1,\ \alpha_1\alpha_3\alpha_2,\ \alpha_1\alpha_2\alpha_1,\ \alpha_2\alpha_3$
$\varnothing$	$\varnothing$	3	$\alpha_1\alpha_3\alpha_2,\ \alpha_1\alpha_2\alpha_3,\ \alpha_2\alpha_3\alpha_2,\ \alpha_2\alpha_1$

For example, the first line means that there is a balanced ideal in

$$W_{\{\alpha_2\},\{\alpha_3\}} = \langle\Delta\backslash\alpha_2\rangle\backslash W/\langle\Delta\backslash\alpha_3\rangle.$$

Therefore, every $\{\alpha_2\}$–Anosov representation into $\mathrm{SL}(4,\mathbb{R})$ with limit map $\xi\colon \partial_\infty\Gamma \to \mathcal{F}_{\{\alpha_2\}} = \mathrm{Gr}(2,4)$ has a cocompact domain of discontinuity in $\mathcal{F}_{\{\alpha_3\}} = \mathrm{Gr}(3,4)$. The generating element $\alpha_3\alpha_1\alpha_2\alpha_1$ corresponds to a relative position of two full flags f, F with $\dim(f^2 \cap F^3) = 2$, so the domain is

$$\Omega = \{H \in \mathrm{Gr}(3,4) \mid \dim(\xi^2(x) \cap H) < 2 \;\forall x \in \partial_\infty\Gamma\}.$$

6.1.4 The number of balanced ideals in A_4

There are 4608 balanced ideals in W, so we cannot list them all. Instead, the following table shows just how many balanced ideals exist in $W_{\theta,\eta}$ for any choice of $\theta, \eta \subset \Delta$ with $\iota(\theta) = \theta$. The rows correspond to different values of θ (for example **14** stands for $\theta = \{\alpha_1, \alpha_4\}$) while the columns correspond to η.

$\theta \backslash \eta$	**1234**	**123**	**134**	**124**	**234**	**12**	**13**	**14**	**23**	**24**	**34**	**1**	**2**	**3**	**4**
1234	4608	35	57	57	35	2	0	3	0	0	2	0	0	0	0
14	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	12	2	5	5	2	1	0	2	0	0	1	0	0	0	0

One feature stands out: There is only a single balanced ideal in $W_{\{\alpha_1,\alpha_4\},\Delta}$, and it has no right–invariances at all. In fact, we have the same situation generally in $W_{\{\alpha_1,\alpha_n\},\Delta}$ if W is of type A_n. For the case of A_3 compare the third row of the above table and recall that $W_{\theta,\Delta} = \langle \Delta \backslash \theta \rangle \backslash W$. For an $\{\alpha_1, \alpha_n\}$–Anosov representation into $\mathrm{SL}(n+1, \mathbb{R})$ with limit map $\xi \colon \partial_\infty \Gamma \to \mathcal{F}_{1,n}$, this corresponds to the cocompact domain

$$\Omega = \mathcal{F}_\Delta \setminus \{F \in \mathcal{F}_\Delta \mid \exists x \in \partial_\infty\Gamma, i \leq n \colon \xi^{(1)}(x) \subset F^{(i)} \subset \xi^{(n)}(x)\}$$

which was also constructed in [GW12, 10.2.3] using the adjoint representation.

6.2 $\{\alpha_1, \dots, \alpha_{p-1}\}$–Anosov representations into $\mathrm{SO}_0(p,q)$

Guichard and Wienhard recently identified an interesting class of surface group representations they call Θ–positive representations [GW18]. This includes a family of representations into $\mathrm{SO}_0(p,q)$ with $p < q$ which they conjecture to be a union of connected components and to be θ–Anosov with $\theta = \{\alpha_1, \dots, \alpha_{p-1}\}$. Here we ordered the simple roots such that non–consecutive ones commute and $\alpha_{p-1}\alpha_p$ has order 4. If this conjecture is true, balanced ideals in $W_{\theta,\eta}$ induce cocompact domains of discontinuity of these representations. Similarly to the table in Section 3.2.1, the following table shows the number of balanced ideals in $W_{\theta,\eta}$ for $\eta = \{\alpha_k\}$, i.e. corresponding to domains in Grassmannians $\mathrm{Is}_k(\mathbb{R}^{p,q})$ of isotropic k–subspaces.

p \ k	1	2	3	4	5	6	7
2	0	1					
3	0	1	1				
4	0	1	2	2			
5	0	1	7	14	3		
6	0	1	42	616	131	7	
7	0	1	429	303742	853168	8137	21

There is always a unique balanced ideal for $\eta = \{\alpha_2\}$. It corresponds to the cocompact domain of discontinuity

$$\Omega = \{V \in \mathrm{Is}_2(\mathbb{R}^{p,q}) \mid V \perp \xi^{(i)}(x) \Rightarrow V \cap \xi^{(i)}(x) = 0 \ \forall x \in \partial_\infty\Gamma \ \forall i \leq p-1\}$$

in the space of isotropic planes.

6.3 $\{\alpha_1, \alpha_2\}$–Anosov representations into F_4

There is another exceptional family of Θ–positive representations, which are conjectured to be $\{\alpha_1, \alpha_2\}$–Anosov in a group G with Weyl group of type F_4. This table shows the number of balanced ideals in $W_{\{\alpha_1,\alpha_2\},\eta}$ for different choices of η. Again, **134** is a shorthand for $\{\alpha_1, \alpha_3, \alpha_4\}$.

θ \ η	**1234**	**123**	**134**	**124**	**234**	**12**	**13**	**14**	**23**	**24**	**34**	**1**	**2**	**3**	**4**
12	1270	182	140	66	44	16	18	5	14	6	4	1	2	2	0

6.4 Symplectic Anosov representations into $\mathrm{Sp}(4n+2,\mathbb{R})$

In this section, we will prove Theorem 1.6.4, i.e. we construct a compactification for locally symmetric spaces modeled on the bounded symmetric domain compactification. We first recall some facts on this compactification, which can be found in [Hel79] and [Sat80].

Every Hermitian symmetric space can be realized as a bounded symmetric domain in some $\mathbb{C}^N$. That is an open, connected and bounded subset $D \subset \mathbb{C}^N$ such that for every point $x \in D$ there is an involutive holomorphic diffeomorphism from D to itself which has x as an isolated fixed point. Concretely, to get the symmetric space $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$ we can consider the bounded symmetric domain

$$D = \{Z \in \mathrm{Sym}(n,\mathbb{C}) \mid 1 - \overline{Z}Z \text{ is positive definite}\} \subset \mathbb{C}^{n(n+1)/2}.$$

The group of holomorphic diffeomorphisms of D is isomorphic to $\mathrm{Sp}(2n,\mathbb{R})$ and acts with stabilizer $\mathrm{U}(n)$. We compactify the symmetric space by taking the closure $\overline{D}$ in $\mathbb{C}^{n(n+1)/2}$. This is the *bounded symmetric domain compactification* of $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$.

Instead of working with bounded symmetric domains, we will use an equivalent model of $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$, the *Siegel space*. Let ω be a symplectic form on $\mathbb{R}^{2n}$ and $\omega_{\mathbb{C}}$ its complexification on $\mathbb{C}^{2n}$. Together with the real structure this defines an (indefinite) Hermitian form

$$h(v,w) := \frac{i}{2}\omega_{\mathbb{C}}(\overline{v},w) \qquad \forall v,w \in \mathbb{C}^{2n}.$$

The Siegel space is the subspace $\mathcal{H}_{n0} \subset \mathrm{Lag}(\mathbb{C}^{2n})$ of complex Lagrangians L such that $h|_{L\times L}$ is positive definite.

The correspondence between these models uses the Cayley transform: Regard the symmetric complex matrices $\mathrm{Sym}(n,\mathbb{C})$ as a subset of $\mathrm{Lag}(\mathbb{C}^{2n})$ by mapping $X \in \mathrm{Sym}(n,\mathbb{C})$ to $\{(Xv,v) \mid v \in \mathbb{C}^n\} \in \mathrm{Lag}(\mathbb{C}^{2n})$. The Cayley transform on $\mathrm{Lag}(\mathbb{C}^{2n})$ is just the action of the matrix

$$\frac{e^{i\pi/4}}{\sqrt{2}}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{C}).$$

It maps D to $\mathcal{H}_{n0}$ and $\overline{D}$ to $\overline{\mathcal{H}_{n0}}$, establishing an equivalence of these compactifications.

More generally, let $\mathcal{H}_{pq} \subset \mathrm{Lag}(\mathbb{C}^{2n})$ be the set of Lagrangians such that h restricted to them has signature (p,q), meaning that there is an orthogonal basis with p vectors of positive norm and q vectors of negative norm (and possibly null vectors). Then

$$\mathrm{Lag}(\mathbb{C}^{2n}) = \bigsqcup_{\substack{0\le p,q\le n \\ p+q\le n}} \mathcal{H}_{pq},$$

and the $\mathcal{H}_{pq}$ are precisely the orbits of the action of $\mathrm{Sp}(2n,\mathbb{R}) \subset \mathrm{Sp}(2n,\mathbb{C})$ on $\mathrm{Lag}(\mathbb{C}^{2n})$. Furthermore, the map

$$\mathcal{H}_{pq} \to \mathrm{Is}_{n-p-q}(\mathbb{R}^{2n}), \quad L \mapsto L \cap \overline{L}$$

makes every $\mathcal{H}_{pq}$ a fiber bundle over the isotropic $(n-p-q)$–subspaces with fiber the semi–Riemannian symmetric space $\mathrm{Sp}(2p+2q,\mathbb{R})/\mathrm{U}(p,q)$. In particular, this means $\mathcal{H}_{n0} = \mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$. A more detailed explanation can be found in [Wie16].

Now let $\rho\colon \Gamma \to \mathrm{Sp}(2n,\mathbb{R})$ be an $\{\alpha_n\}$–Anosov representation and $\xi\colon \Gamma \to \mathcal{F}_{\{\alpha_n\}} = \mathrm{Lag}(\mathbb{R}^{2n})$ its limit map. Important examples of these representations are maximal representations from a surface group $\Gamma = \pi_1 S$, if either S is a closed surface, or an open surface where the boundary elements map to Shilov hyperbolic elements of $\mathrm{Sp}(2n,\mathbb{R})$. The following theorem implies Theorem 1.6.4:

Theorem 6.4.1. *If n is odd, then there exists a balanced ideal $I \subset W_{\{\alpha_n\},\{\alpha_n\}}$. Therefore,*

$$\widehat{X} := \overline{\mathcal{H}_{n0}} \cap \Omega(\xi(\partial_\infty\Gamma), I) = \{L \in \overline{\mathcal{H}_{n0}} \mid \dim_{\mathbb{C}} L \cap \xi(x)^{\mathbb{C}} < n/2 \ \forall x \in \partial_\infty\Gamma\}$$

is Γ–invariant and the quotient $\Gamma\backslash\widehat{X}$ is a compactification of the locally symmetric space $\Gamma\backslash\mathcal{H}_{n0} = \Gamma\backslash\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$.

Proof. The Weyl group W of $\mathrm{Sp}(2n,\mathbb{C})$ can be identified with the group of permutations π of $\{-n,\dots,-1,1,\dots,n\}$ with $\pi(-i) = -\pi(i)$ for all i. In this identification, the generator α_n negates n and $-n$ keeping everything else fixed, while α_k for $k \neq n$ exchanges k with $k+1$ and $-k$ with $-k-1$. The longest element w_0 negates everything.

Denote by $[k] \in W_{\{\alpha_n\},\{\alpha_n\}} = \langle\alpha_1,\dots,\alpha_{n-1}\rangle\backslash W/\langle\alpha_1,\dots,\alpha_{n-1}\rangle$ the equivalence class of permutations which map exactly k positive numbers to positive ones. Then $w_0[k] = [n-k]$ and $[k] \leq [\ell]$ in the Bruhat order if and only if $k \geq \ell$. Furthermore, $\mathrm{pos}(L,L') = [k]$ for two Lagrangians $L, L' \in \mathrm{Lag}(\mathbb{C}^{2n})$ if and only if $\dim_{\mathbb{C}}(L \cap L') = k$.

If n is odd, then $I = \{[k] \mid k > n/2\}$ is a balanced ideal, so

$$\Omega := \Omega(\xi(\partial_\infty\Gamma), I) = \{L \in \mathrm{Lag}(\mathbb{C}^{2n}) \mid \dim_{\mathbb{C}}(L \cap \xi(x)^{\mathbb{C}}) < n/2 \ \forall x \in \partial_\infty\Gamma\}$$

is a cocompact domain of discontinuity for our $\{\alpha_n\}$–Anosov representation ρ by [KLP18]. Here ρ is regarded as a representation into $\mathrm{Sp}(2n,\mathbb{C})$. But because ρ maps into $\mathrm{Sp}(2n,\mathbb{R})$, it preserves $\mathcal{H}_{pq}$ and therefore also $\widehat{X}$. The quotient $\Gamma\backslash\widehat{X}$ is a closed subset of $\Gamma\backslash\Omega$ and thus also compact. ☐

Note that this is just a special case of a general principle to construct compactifications for locally symmetric spaces arising from Anosov representations described in [GKW15] and [KL18]. However, it is a particularly interesting one as this compactification is modeled on the bounded symmetric domain compactification for $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$.

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