A TWIST CONDITION FOR MAGNETIC FLOWS ON THE TWO-SPHERE

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Summary

The magnetic flow on the two-dimensional sphere S^2 is determined by a Riemannian metric g and a two-form σ, each on S^2. The triple (S^2, g, σ) is called a magnetic system. The goal of this thesis is to find a condition on the magnetic system so that the magnetic flow has infinitely many periodic orbits. We will use existing results that allow us to view the problem in a contact geometric context where the dynamics of the Reeb flow corresponds to the dynamics of the magnetic flow on different kinetic energy level sets. We will then construct a global surface of section that is an annulus and find a condition for which the first return map of the surface of section satisfies the requirements of the Poincaré-Birkhoff Theorem which then implies that the first return map has infinitely many fixed points and the Reeb flow therefore has an infinite number of periodic Reeb orbits.

Zusammenfassung

Der magnetische Fluss auf der zweidimensionalen Sphäre S^2 ist bestimmt durch eine Riemansche Metrik g und einer zwei-Form σ, jeweils auf S^2. Das Tripel (S^2, g, σ) wird auch magnetisches System genannt. Das Ziel dieser Arbeit ist eine Bedingung an das magnetischen System zu finden, für die der magnetische Fluss unendlich viele periodische Orbiten hat. Wir werden existierende Resultate verwenden, um das Problem in einem kontakgeometrischen Kontext zu betrachten. Dabei korrespondiert die Dynamik des Reeb Flusses mit dem des magnetischen Flusses auf unterschiedlichen Niveaumengen der kinetischen Energiefunktion . Wir werden dann einen Poincaré-Schnitt konstruieren, der ein Kreisring ist, und eine Bedingung finden, für welche die Poincaré-Abbildung die Voraussetzungen des Satzes von Poincaré-Birkhoff erfüllt, welcher dann impliziert, dass die Poincaré-Abbildung unendlich viele Fixpunkte und der Reeb Fluss unendlich viele periodische Orbiten besitzt.

Contents

1 Introduction

Since the end of the 18th century, when theoretical electromagnetism emerged, many physicists and mathematicians have been interested in making statements about the motion of a charged particle in an electromagnetic field.

The motion of a particle, constrained to a surface and subjected to a Lorentz force, is described by a differential equation (Newton's second law of motion). The set of solutions gives rise to a flow, which we call the magnetic flow.

In this thesis we restrict our studies to the magnetic flow on the two-sphere, which induces a flow on its tangent space TS^2 (We will usually refer to the induced flow on TS^2 as the magnetic flow). The magnetic flow depends on a Riemannian metric g and a two-form σ, each on S^2. The triple (S^2, g, σ) is called a magnetic system.

Another way of describing the magnetic flow is by studying the Hamiltonian flow of the manifold $(TS^2, \omega_\sigma := d\lambda - \pi^*\sigma)$, associated to the kinetic energy function

$$E : TS^2 \to \mathbb{R}, \quad E(x, v) = \frac{1}{2} g_x(v, v).$$

The one-form $\lambda \in \Omega^1(TS^2)$ corresponds to the tautological one-form on T^*S^2.

If $\sigma = 0$, the Hamiltonian flow describes the geodesic flow. For the geodesic flow on S^2, it was proven in [Banger, 1993] that there are infinitely many periodic orbits for every Riemannian metric.

In this thesis we want to find a condition on the magnetic system so that the magnetic flow has infinitely many periodic orbits. We will prove the following theorem:

Theorem. *Let $(S^2, g, \sigma = f \operatorname{vol}_g)$ be a magnetic system such that its magnetic strength f is positive everywhere. Assume that f has a minimum point p_- and a maximum point p_+. If*

$$\frac{1}{f^3}(p_+)\sqrt{\det \operatorname{Hess}^g_f(p_+)} \neq \frac{1}{f^3}(p_-)\sqrt{\det \operatorname{Hess}^g_f(p_-)},$$

where Hess^g_f is the Hessian in orthonormal coordinates according to g, then the magnetic flow has infinitely many periodic orbits with speed $m > 0$, for every m small enough.

The level sets $\Sigma_m := \{E = m\}$ are invariant under the Hamiltonian flow. The dynamics of the flow on (TS^2, ω_σ) restricted to Σ_m corresponds to the dynamics of the flow of the manifold (TS^2, ω_m) restricted to $SS^2 := \Sigma_1$, where $\omega_m := m d\lambda - \pi^*\sigma$. This enables us to study the dynamics on the different level sets Σ_m by studying the dynamics of the Hamiltonian flow corresponding to the one-parameter family of forms ω_m, restricted to SS^2.

In Section 3 we look at some properties of the low energy levels: that is Σ_m for m small. An important result from [Benedetti, 2014] is that for m small enough ω_m has a primitive of contact type, i.e. there is a contact form $\lambda_m \in \Omega^1(SS^2)$ such that $d\lambda_m = \omega_m$. Because the dynamics of the Hamiltonian flow on (SS^2, ω_m) is equivalent to that of the Reeb flow of (SS^2, λ_m), we can study the magnetic flow in the framework of contact geometry.

Based on the results of [Benedetti and Kang, 2018], in Section 4 we construct a global surface of section as an annulus in SS^2. In other words, we find an embedding $S : \mathbb{A} \to SS^2$ such that $S(\mathring{\mathbb{A}})$ is transversal to the Reeb flow, and the boundary $S(\partial\mathbb{A})$ is the support of periodic orbits of the Reeb flow.

In Section 5 we revisit the Poincaré-Birkhoff theorem which tells us that area preserving homeomorphisms of the annulus $F : \mathbb{A} \to \mathbb{A}$ have infinitely many fixed points if F satisfies the so called boundary twist condition.

We then find a condition on the magnetic system so that the first return map of our constructed global surface of section satisfies the boundary twist condition. As a consequence of the first return map being twist, it has infinitely many fixed points, meaning that the Reeb flow has infinitely many periodic Reeb orbits. As the dynamics of the Reeb flow of (SS^2, λ_m) corresponds to the dynamics of the magnetic flow on Σ_m, this directly implies that the magnetic flow has infinitely many periodic orbits.

2 Preliminaries

In this chapter we briefly introduce the mathematical objects and some theoretical basics which we need throughout this thesis.

2.1 General notation

Let $S^1 = \mathbb{Z}/2\pi$. If not otherwise mentioned, M is a closed orientable surface with a Riemannian metric g. Unless stated differently all objects are assumed to be smooth. We have the tangent bundle $\pi : TM \to M$, the metric g induces the Levi-Civita connection $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$. The canonical volume form on M induced by g is denoted by vol_g. We usually denote points in TM by (x, v), where $x \in M$ and $v \in T_xM$. We also have the Sasaki metric on TM and can work with it's corresponding Levi-Civita connection.

The metric g induces an isomorphism between TM and it's dual space T^*M.

$$\begin{aligned} \flat : TM &\to T^*M \\ (x, v) &\mapsto g_x(v, \cdot). \end{aligned}$$

Its inverse is denoted by $\sharp : T^*M \to TM$.

Let M_0 be a smooth manifold. The vector space of smooth k-forms on M_0 is denoted by $\Omega^k(M_0)$. If we have another smooth manifold M_1 and a map $\psi : M_0 \to M_1$, then we denote its differential at $p \in M_0$ by $T_p\psi : T_pM_0 \to T_{\psi(p)}M_1$.

For each $x \in M$, the metric g induces a norm $\|\cdot\|$ on T_xM that we can use to define a norm for vector fields and one-forms. Let $W \in \Gamma(M)$ and $\eta \in \Omega^1(M)$ then

$$\|W\|_{C^0} := \sup_{x \in M} \|W_x\|, \quad \|\eta\|_{C^0} := \sup_{x \in M} \sup_{v \in T_xM} \frac{|\eta_x(v)|}{\|v\|}.$$

A norm for two-forms can be defined similarly. For higher C^k norms used in this thesis we refer to the Appendix of [Benedetti and Kang, 2018]. A summary of the C^k estimates that are important for us can be found in Subsection 6.1.

2.2 Horizontal and vertical lifts

We briefly introduce horizontal and vertical lifts which we will occasionally use in this thesis.

The double tangent bundle $T\pi : TTM \to TM$ splits into a vertical and horizontal subbundle, denoted by $V(TM)$ and $H(TM)$ respectively, such that $H_e(TM) \oplus V_e(TM) = T_eTM$, for each $e \in TM$. The vertical bundle is fiberwise defined as $V_e(TM) := \ker T_e\pi$. The vertical lift at (x, v) is a map $L^V_{(x,v)} : T_xM \to T_{(x,v)}TM$, defined by $L^V_{(x,v)}(w) = \frac{d}{dt}|_{t=0}(v + tw)$. We have

$$T_{(x,v)}\pi(L^V_{(x,v)}(w)) = \frac{d}{dt}|_{t=0}\pi(v + tw) = \frac{d}{dt}|_{t=0}x = 0,$$

showing that $L^V_{(x,v)}(w)$ lies indeed in the vertical space.

By the explanation of the term 'connection' in [Jost, 2005] on page 104, using the Levi-Civita connection ∇, we can define the horizontal bundle the following way. Let $e \in TM$, then $V_e(TM) = \ker T_e\pi$. Let $x := \pi(e)$, $\mathcal{X} \in T_xM$ and $c : \mathbb{R} \to M$ be a curve such that $c(0) = x$ and $\dot{c}(0) = \mathcal{X}$. We can use ∇ to parallel transport e along c yielding a curve $\hat{e} : \mathbb{R} \to TM,\ t \mapsto \hat{e}(t)$, where $\hat{e}(0) = e$. This induces a map

$$
\begin{aligned}
L_e^H : T_xM &\to T_eTM \\
\mathcal{X} &\mapsto \frac{d}{dt}|_{t=0}\hat{e}(t),
\end{aligned}
$$

which is called the horizontal lift at e. Define $H_e(TM) := L_e^H(TM)$ to be the horizontal space at e and observe that

$$
T_e\pi L_e^H(\mathcal{X}) = \frac{d}{dt}|_{t=0}\pi(\hat{e}(t)) = \frac{d}{dt}|_{t=0}c(t) = \mathcal{X}.
$$

Hence, L_e^H is injective and $H_e(TM) \cap V_e(TM) = \{0\}$, implying that

$$
H_e(TM) \oplus V_e(TM) = T_eTM.
$$

2.3 The geodesic flow

The geodesic vector field $X \in \Gamma(TTM)$ is the generator of the geodesic flow ϕ^X, which is defined as the solution of the geodesic equation

$$
\nabla_v v = 0.
$$

Choose the Lagrangian $L(x, v) := \frac{1}{2}g_x(v, v)$. With help of the Legendre transformation

$$
\begin{aligned}
\mathcal{L} : TM &\to T^*M \\
(x, v) &\mapsto \frac{dL}{dv}|_{(x,v)},
\end{aligned}
$$

we can define the Hamiltonian $H : T^*M \to \mathbb{R}$

$$
H(p) := p(\mathcal{L}^{-1}(p)) - L(\mathcal{L}^{-1}(p)),
$$

The tautological one-form on T^*M is defined by $\lambda^* := p \circ T\pi$ for $p \in T^*M$.

Let ϕ^{X^*} be the Hamiltonian flow of the Hamiltonian vector field X^* of $d\lambda^*$ associated to the Hamiltonian H. Then, the geodesic flow on TM can be obtained via $\phi^X = \mathcal{L}^{-1}(\phi^{X^*})$.

We can use $\mathcal{L}$ to pull H and λ^* back to TM. Define

$$
\lambda := \mathcal{L}^*\lambda^*, \quad E := H \circ \mathcal{L}.
$$

Since $\mathcal{L}(x, v) = \frac{dL}{dv}|_{(x,v)} = g_x(v, .)$, for each $w \in T_{(x,v)}TM$

$$
\lambda_{(x,v)}(w) = \mathcal{L}(x, v)(T_{(x,v)}\pi(w)) = g_x(v, T_{(x,v)}\pi(w)).
$$

And the energy can be expressed as

$$\begin{aligned}E(x,v) = H \circ \mathcal{L}(x,v) &= \frac{\partial}{\partial v} L(x,v) - L(x,v) \\ &= g_x(v,v) - \frac{1}{2} g_x(v,v) = \frac{1}{2} g_x(v,v).\end{aligned}$$

Since $\mathcal{L}(x,y) = \flat(x,y)$, we can instead define λ as $\flat^* \lambda^*$.

Therefore, if λ is the pullback of the tautological one-form on T^*M under $\flat$, then the geodesic vector field X can be equivalently defined as the Hamiltonian vector field of $d\lambda$, associated to the kinetic energy function $E(x,v) = \frac{1}{2} g_x(v,v)$. The pair $(TM, d\lambda)$ is called the standard tangent bundle.

A detailed discussion about the geodesic flow can be found in [Geiges, 2008] Section 1.5.

2.4 The magnetic flow

Let $\sigma \in \Omega^2(M)$ and define the generalized Lorentz force $F : TM \to TM$ by the equation

$$g_x(F_x(v), w) = \sigma(v,w) \quad \forall v, w \in TM. \tag{1}$$

The magnetic flow is defined to be the solution of the equation

$$\nabla_v v = F_x(v).$$

As the geodesic flow, the magnetic flow is generated by a vector field, which we call the magnetic vector field $X^\sigma \in \Gamma(TTM)$.

On a local domain $U \subset M$ we can find a one-form $\theta \in \Omega^1(M)$, such that $d\theta = \sigma$. Choose a Lagrangian $L^\sigma(x,v) := \frac{1}{2} g_x(v,v) - \theta_x(v)$ on the domain U.

We define the Legendre transformation for L^σ and the corresponding Hamiltonian by

$$(x,v) \mapsto \frac{dL^\sigma}{dv}|_{(x,v)}, \quad H^\sigma(p) := p(\mathcal{L}_\sigma^{-1}) - L(\mathcal{L}_\sigma^{-1}).$$

Define the one-form on T^*M

$$\lambda_\sigma^* := \lambda^* - \pi^* \theta$$

And the symplectic form

$$\omega_\sigma^* := d\lambda_\sigma^* = d\lambda^* - \pi^* \sigma.$$

Like in the geodesic case, the magnetic flow corresponds to the flow of the Hamiltonian vector field of ω_σ^* associated to H^σ.

Once again, we can pull the objects back to TM.

$$\begin{aligned}H \circ \mathcal{L}_\sigma(x,v) &= \frac{\partial}{\partial v} L^\sigma(x,v) - L^\sigma(x,v) \\ &= g_x(v,v) - \theta_x(v) - \frac{1}{2} g_x(v,v) + \theta_x(v) = \frac{1}{2} g_x(v,v) = E(x,v).\end{aligned}$$

Therefore,

$$\omega_\sigma := \mathcal{L}_\sigma^* \omega_\sigma^* = \mathcal{L}_\sigma^*(\lambda^* - \pi^*\theta) = d\lambda - \pi^*\sigma.$$

The magnetic vector field X^σ is then given by the Hamiltonian vector field of ω_σ associated to the energy function E.

2.5 Magnetic systems

This subsection is based on Section 2.2 of [Benedetti, 2014]. As we have seen in the previous subsections, we have a one-form $\lambda \in \Omega^1(TM)$ defined as the pull-back under $\flat$ of the tautological one-form on T^*M that acts on vectors $w \in T_{(x,v)}TM$ by

$$\lambda_{(x,v)}(w) = g_{(x,v)}(v, T_{(x,v)}\pi(w)).$$

Its exterior differential $d\lambda$ is symplectic and the tuple $(TM, d\lambda)$ is a symplectic manifold, which we call the standard tangent bundle.

We have a symplectic form $\omega_\sigma := d\lambda - \pi^*\sigma$, where $\sigma \in \Omega^2(M)$. The tuple (TM, ω_σ) is called the twisted tangent bundle.

We keep the notation and denote the geodesic and magnetic vector field by X and X^σ respectively.

Definition 2.1. *The triple (M, g, σ) is called a* ***magnetic system****, where σ is the corresponding* ***magnetic form****. We call (M, g, σ) a* ***symplectic magnetic system*** *if the magnetic form is symplectic.*

Definition 2.2. *Because σ is a top-dimensional form, there is a unique function $f : M \to \mathbb{R}$, called the* ***magnetic strength****, such that $\sigma = f\,\mathrm{vol}_g$.*

Let $j : \mathbb{R} \times TM \to TM$ be the rotational flow going in positive direction with speed 2π. Then $j_{1/4} : TM \to TM$ is a map that rotates each fiber by $\frac{\pi}{2}$ in the positive sense. The Lorentz force can be expressed in terms of the magnetic strength by $F_x(v) = f(x)j_{1/4}(x, v)$. One can see this by taking an orthonormal frame $\{e_1, e_2\}$ and and plugging it into Equation 1.

$$g_x(f(x)j_x(e_1), e_2) = f(x) = f(x)vol_M(e_1, e_2) = \sigma(e_1, e_2).$$

Let V be the generator of the flow j, and fix the notation V for the rest of this thesis.

The vector field V can also be defined as the vertical lift of $j_{1/4}(x, v)$, while the geodesic vector field X can be defined via the horizontal lift of v. Indeed, write $\phi^X(t) = (x(t), v(t))$ then, $T_{(x(t),v(t))}\pi(X) = \frac{d}{dt}|_{t=0}\pi(x(t), v(t)) = v(t)$. Implying that $X_{(x,v)} = L^H_{(x,v)}(v)$.

As mentioned in [Benedetti, 2014] Chapter 1, we can decompose the magnetic field X^σ into its horizontal and vertical part $X^\sigma = X + fV$.

Because of $dE(X^\sigma) = \omega(X^\sigma, X^\sigma) = 0$, the level-sets of the kinetic energy function are invariant under the flow of X^σ, therefore we can study the restrictions of the magnetic field X^σ to the sets

$$\Sigma_m := \{\ (x, v) \in TM \mid E(x, v) = m\ \} \text{ for } m \geq 0.$$

Taking the scaling map $S_m : TM \to TM$, $(x, v) \mapsto (x, \frac{v}{m})$, we get

$$TS_m(X^\sigma|_{\Sigma_m}) = (mX + fV)|_{\Sigma_1} = mX_E^{\frac{\sigma}{m}}|_{\Sigma_1}.$$

This means that instead of studying the magnetic flow of the magnetic system (M, g, σ) on Σ_m, we can study the rescaled magnetic system $(M, g, \frac{\sigma}{m})$ on Σ_1. We abbreviate

$$\omega_m := m\sigma_{\frac{\sigma}{m}} = md\lambda - \pi^*\sigma, \quad X_m := mX_E^{\frac{\sigma}{m}} = mX + fV. \tag{2}$$

Note that scaling the magnetic field doesn't change its dynamics. Also note that

$$\omega_m \to \sigma \text{ and } X_m \to fV$$

as $m \to 0$, showing that the dynamics becomes very simple as m goes to zero.

3 The magnetic flow on low energy levels

From now on we restrict our studies to the symplectic magnetic system (S^2, g, σ). In this section we are going to discuss some results of [Benedetti, 2014] Chapter 4 and 6, and [Benedetti and Kang, 2018] Part I. that are going to help us understand the magnetic flow on low energy levels: that is, the magnetic flow on the level sets Σ_m for m small.

3.1 Contact geometric framework

The 2-form ω_m having a primitive that is contact on SS^2, for m small enough, is one of the main results of [Benedetti, 2014].

Let K be the Gaussian curvature on M. Define the curvature form $\sigma_g \in \Omega^2(M)$ by $\sigma_g := K vol_M$. By [Benedetti, 2014] Example 2.15, the corresponding S^1-connection form $\tau \in \Omega^1(SS^2)$ is then defined by $\tau(V) = 1$ and $d\tau = -\pi^*\sigma_g$. Recall that V is the generator of the 2π periodic flow that rotates each fiber of SS^2.

From [Benedetti, 2014] Chapter 4 we know that the 1-form $\lambda_m := m\lambda - \pi^*\beta + \tau$, where $d\beta = \sigma - \sigma_g$, is a primitive of ω_m and that there exists a $m_\beta > 0$ such that λ_m is a positive contact form on SS^2 for $m \in [0, m_\beta]$. Thus, for such m the function

$$h_m(x, v) := \lambda_m(X^m) = m^2 - \beta_x(v)m + f(x)$$

is positive and $R^m := \frac{1}{h_m}X^m = \frac{m}{h_m}X + \frac{f}{h_m}V$ is the Reeb vector field of λ_m.

This shows that to understand the dynamics of the flow of X^m, we can instead study the Reeb vector field R^m of λ_m.

The following lemma, which was taken from [Benedetti, 2014] Lemma 7.4, enables us to bring λ_m in a form that is easier to work with.

Lemma 3.1. *There exists a diffeomoprhism $F_m : SS^2 \to SS^2$ and a real function*

$$q_m : SS^2 \to \mathbb{R}$$

such that

$$F_m^*\lambda_m = e^{q_m}\lambda_0.$$

The map $[0, m_\beta) \to C^\infty(SS^2, \mathbb{R})$, $m \mapsto q_m$ is smooth and admits a Taylor expansion at $m = 0$

$$q_m = \frac{m^2}{2f} + o(m^2).$$

Proof. See [Benedetti, 2014] Lemma 7.4. □

Expanding $e^{q_m}\lambda_0$ at $m = 0$ implies

$$e^{q_m}\lambda_0 = (1 + \frac{m^2}{2f})\lambda_0 + o(m^2) = \frac{1}{H_m}\lambda_0 + o(m^2),$$

where $H_m := 1 - \dfrac{m^2}{2f}$.

Therefore, there exists a $m_1 \in [0, m_\beta]$ such that

$$F_m^* \lambda_m = \frac{1}{H_m} \lambda_0 + o(m^2).$$

for $m \in [0, m_1)$.

3.2 The Ginzburg function

Another important result from [Benedetti, 2014] Section 7 is going to help us find periodic orbits near critical non-degenerate points of the magnetic strength f. This will be important for us later to construct a surface of section whose boundaries have to be the support of periodic orbits.

We state Proposition 7.5 in [Benedetti, 2014].

Proposition 3.2. *There exists a smooth family of functions $m \mapsto S_m$, where $S_m : SS^2 \to \mathbb{R}$, such that*

1. *the critical points of S_m are the support of those periodic orbits of X^m which are close to a vertical fiber;*
2. *the following expansion at $m = 0$ holds:*

$$S_m = 2\pi + \frac{\pi}{f} m^2 + o(m^2).$$

Proof. See [Benedetti, 2014] Section 7.2. □

Its corollary ([Benedetti, 2014], Corollary 7.6) links the periodic orbits to non-degenerated critical points of the magnetic strength f. We use a slightly adapted version of the corollary's statement:

Corollary 3.3. *If $x \in S^2$ is a non-degenerate critical point of $f : S^2 \to \mathbb{R}$, then there exists a smooth family of curves $m \mapsto \gamma_m$, such that*

1. *γ_0 winds uniformly once around $S_x S^2$ in the positive sense;*
2. *the support of γ_m is a periodic orbit for X_m;*
3. *$\operatorname{dist}(x, \gamma_m(0)) = O(m)$.*

Proof. Because of (3.2) we have the expansion $S_m = 2\pi + \frac{\pi}{f} m^2 + o(m^2) = 2\pi + m^2 \widehat{S}_m$, where $\widehat{S}_m = \frac{\pi}{f} + o(m)$. S_m and $\widehat{S}_m$ have the same critical points and all points of $S_p S^2$ are critical points for $\widehat{S}_0$ if p is a critical point of f. Assuming that p is a non-degenerate critical point and that m is small enough, by applying the inverse function theorem one can find a critical point p_m of $\widehat{S}_m$ such that $\operatorname{dist}(p_m, S_p S^2) = O(m)$. Because the critical points of S_m correspond with periodic Reeb orbits, the corollary follows. See [Benedetti, 2014] Corollary 7.6 for more details. □

3.3 A suitable Darboux covering

For $m = 0$, the contact form λ_m and its Reeb vector field R^m reduce to

$$\lambda_0 = -\pi^*\beta + \tau, \quad R^0 = V.$$

Recall that V generates a rotational 2π periodic flow on the fibers of TS^2. For each $(x, v) \in SS^2$, the flow $\phi_s^V(x, v) : S^1 \times S_xS^2 \to S_xS^2$ simply rotates v by s degrees. Therefore, ϕ^V induces a free S^1 action on SS^2. Contact forms whose Reeb vector fields induce a S^1 action are called **Zoll forms** and are a central subject of study in [Benedetti and Kang, 2018]. We follow the procedure described in the beginning of Chapter 3 in [Benedetti and Kang, 2018] to find a suitable Darboux covering of (λ_0, SS^2).

Let $a > 0$, take the Euclidean metric g_{std} on $\mathbb{R}^2$ and denote by B and B' the ball of radius a and $\frac{a}{2}$ respectively.

Consider the tautological one-form $\bar{\lambda}_{std} = \frac{1}{2}(x_1 dx_2 - x_2 dx_1)$, where $(x_1, x_2) \in B$.

Using the trivial bundle $\pi_{std} : B \times S^1 \to B$, we can define the forms

$$\lambda_{std} := d\phi + \pi_{std}^*\bar{\lambda}_{std}, \quad \omega_{std} := d\lambda_{std} = \pi_{std}^* d\bar{\lambda}_{std}$$

on $B \times S^1$, where ϕ is the fiber coordinate in S^1. The Reeb vector field of λ_{std} is given by $R_{std} := \partial_\phi$ and also called the standard Reeb vector field.

Let $Z \subset SS^2$ be a finite set of points and $Q := \pi(Z)$. Consider S^1-equivariant embeddings

$$\mathfrak{D}_z : B \times S^1 \to SS^2, \quad \mathfrak{D}_z(0, 0) = z, \ \forall z \in Z,$$

and the corresponding embeddings

$$\mathfrak{d}_q : B \to S^2, \quad \mathfrak{d}_q(0) = q, \ \forall q \in Q$$

such that

$$\pi \circ \mathfrak{D}_z = \mathfrak{d}_{\pi(z)} \circ \pi_{std}, \ \forall z \in Z.$$

Because of SS^2 being compact, if a is small enough, we can assume that

D1 $S^2 = \bigcup_{q \in Q} \mathfrak{d}_q(B)$,

D2 $\exists C_{\mathfrak{D}} > 0, \ \|T\mathfrak{D}_z\|_{C^1} < C_{\mathfrak{D}}, \ \forall z \in Z$,

D3 $\mathfrak{D}_z^*\lambda_0 = \lambda_{std}$.

3.4 Weakly normalized forms

Lemma 3.1 enabled us to find a diffeomorphism $F_m : SS^2 \to SS^2$ such that

$$F_m^* \lambda_m = \frac{1}{H_m} \lambda_0 + o(m^2) \tag{3}$$

for m small enough and $H_m := 1 - \frac{m^2}{2f \circ \pi}$.

To construct a global surface of section of annulus type for the Reeb flow of $F_m^* \lambda_m$, we need to find two periodic Reeb orbits, to whose support we can send the two boundary components of the section. In Subsection 3.2 we already briefly discussed that periodic Reeb orbits can always be found near non-degenerate critical points of the magnetic strength f.

With help of some results from [Benedetti and Kang, 2018] Chapter 3, we find a diffeomorphism that maps the periodic Reeb orbits near non-degenerate critical points of f to the circle fibers of these points. Define a class of one-forms that have a periodic Reeb orbit winding uniformly around the S^1-fiber of some point in the positive sense.

Definition 3.4. *We call a contact form $\alpha \in \Omega^1(SS^2)$ **weakly normalized** at $q \in S^2$ if α has a periodic orbit winding uniformly once around $S_q S^2$ in the positive sense.*

One of our goals therefore is to find a diffeomorphism that pulls $F_m^* \lambda_m$ back to a form that is weakly normalized at a certain point. However, we don't want to lose the convenient expansion $\frac{1}{H_m} \lambda_0 + o(m^2)$ of $F_m^* \lambda_m$, meaning that the diffeomorphism we find needs to be 'small' enough.

To measure distances between one-forms we use the following norm, which was introduced in [Benedetti and Kang, 2018] Chapter 3.

Definition 3.5. *Let $\alpha \in \Omega^1(SS^2)$, the $C^{k,+}$-norm for 1-forms is defined by*

$$\|\alpha\|_{C^{k,+}} := \|\alpha\|_{C^k} + \|d\alpha\|_{C^k}.$$

Definition 3.6. *We call the contact form $\alpha_{H_m} := \frac{1}{H_m} \lambda_0$ the **unperturbed form**. And its Reeb vector field the $R_{\alpha_{H_m}}$ the **unperturbed Reeb vector field**.*

Define a set of families of forms that are $C^{1,+}$-close to the unperturbed form α_{H_m}.

$$\mathcal{B} := \{ \ \alpha_m \in \Omega^1(SS^2) \ | \exists M > 0, \forall m \in [0, M) : \|\alpha_{H_m} - \alpha_m\|_{C^{1,+}} = o(m^2) \ \}. \tag{4}$$

With $\alpha_m \in \Omega^1(SS^2)$ we actually mean a one-parameter family of forms in $\Omega^1(SS^2)$.

We will sometimes refer to forms in $\mathcal{B} - \{\alpha_{H_m}\}$ as **perturbed forms** and to their Reeb vector fields as **perturbed Reeb vector fields**.

Observe that $F_m^* \lambda_m \in \mathcal{B}$ and that every $\alpha_m \in \mathcal{B}$ has the expansion $\alpha_m = \frac{1}{H_m} \lambda_0 + o(m^2)$ at $m = 0$. The diffeomorphism we construct to weakly normalize a contact form is then required to leave the set $\mathcal{B}$ invariant, implying that it preserves the expansion.

Talking about forms in $\mathcal{B}$ being weakly normalized only makes sense for contact forms. However, Lemma 6.5 implies that every $\alpha_m \in \mathcal{B}$ is contact for m small enough.

Let $q \in S^2$, we denote the subset of forms in $\mathcal{B}$ that are weakly normalized at q by

$$\mathcal{B}_q := \{ \ \alpha_m \in \mathcal{B} \mid \exists M > 0, \forall m \in [0, M) : \ \alpha_m \text{ is weakly normalized at } q \ \}.$$

We also denote the subset of forms that are weakly normalized at multiple points $q_1, \ldots, q_k$ by $\mathcal{B}_{q_1,\ldots,q_k}$.

For the time being we only take care of the case when p is a non-degenerate critical point of f. Our goal then is to find, for any $\alpha_m \in \mathcal{B}$, a diffeomorphism $\psi : SS^2 \to SS^2$, such that the pullback $\psi^*\alpha_m$ lies in $\mathcal{B}_p$.

We do this in multiple steps. The following lemma shows that, if f has a non-degenerate critical point, then we can transform any $\alpha_m \in \mathcal{B}$ to a form that still is in $\mathcal{B}$ and has a periodic Reeb orbit that 'pierces' the fiber at this non-degenerate critical point. Later we will find another diffeomorphism that sends the complete orbit to the fiber it pierces.

Lemma 3.7. *Let $a_m \in \mathcal{B}$, $p \in S^2$ be a non-degenerate critical point of f and $m > 0$ small enough, then there exists a diffeomorphism $\Psi_{1,m}$ such that ${\Psi_{1,m}}^*\alpha_m$ lies in $\mathcal{B}$ and has a periodic orbit γ_m with period $T_m := 2\pi + O(m)$ and $\gamma_m(0) \in S_pS^2$.*

Proof. The proof is the local version of Lemma 3.5 in [Benedetti and Kang, 2018], with the only addition that it is ensured that $\alpha_m \in \mathcal{B}$.

We know from corollary 3.3 that there exists a periodic Reeb orbit $\hat{\gamma}_m$ of α_m such that

$$\text{dist}(\hat{\gamma}_m(0), \hat{x}) = O(m), \text{ for some } \hat{x} \in S_pS^2. \tag{5}$$

Fix $\hat{x}$ and ensure that m is small enough so that $\hat{\gamma}_m(0) \in \mathfrak{D}_{\hat{x}}(B' \times S^1)$.

Let $(x_m, \phi_m) \in B \times S^1$, such that

$$\mathfrak{D}_{\hat{x}}((x_m, \phi_m)) = \hat{\gamma}_m(0).$$

Let $\tilde{\phi}_m \in (\pi/2, \pi/2)$ be a lift of ϕ_m to $\mathbb{R}$, such that $|(x_m, \phi_m)| = O(m)$ by (). Consider the function

$$K_m(x) := \mu \cdot (\tilde{\phi}_m + g_{std}(x, ix_m)),$$

where $\mu : B \to [0, 1]$ is a cut-off function that equals 1 on B' and 0 outside a neighborhood of B' contained in the inner of B. By Section 2.3 in [Geiges, 2008], a contact vector field Y^m of $(\lambda_{std}, B \times S^1)$ can be defined via the contact Hamiltonian $K_m \circ \pi_{std}$ and the equations

$$\begin{aligned}
&\lambda_{std}(Y^m) = K_m \circ \pi_{std}, \\
&\iota_{Y^m} d\lambda_{std} = d(K_m \circ \pi_{std})(R_{\lambda_{std}})\lambda_{std} - d(K_m \circ \pi_{std}).
\end{aligned}$$

Because $R_{\lambda_{std}}$ lies in the kernel of π_{std} the second equation simplifies to

$$\iota_{Y_m} d\lambda_{std} = -d(K_m \circ \pi_{std}).$$

Y^m is uniquely defined by those equations. Let ϕ^{Y^m} be the flow generated by Y^m. By Cartan's 'magic' formula we have

$$\frac{d}{dt}\phi_t^{Y^m *}\lambda_{std} = \mathcal{L}_{Y^m}\lambda_{std} = d(\lambda_{std}(Y^m)) + \iota_{Y^m} d\lambda_{std} = 0,$$

where the last equality followed from Y^m satisfying the contact Hamiltonian equations above. This implies, together with $\phi_0^{Y^m*}\lambda_{std} = \lambda_{std}$, that $\phi_t^{Y^m*}\lambda_{std} = \lambda_{std}$ for all $t \in \mathbb{R}$.

Furthermore, observe that $K_m(x_m) = \tilde{\phi}_m = \lambda_{std}((x_m, \phi_m))$ implying that $(tx_m, t\phi_m)$ is a flow line of $\phi_t^{Y^m}$.

Hence, $\phi_1^{Y^m}$ is a strict contactomorphism of $(\lambda_{std}, B \times S^1)$, sends 0 to (x_m, ϕ_m) and equals $\mathrm{id}_{B\times S^1}$ outside a neighborhood of B'. Since $\left\|(x_m, \tilde{\phi}_m)\right\| = O(m)$,

$$\|Y^m\|_{C^k} = O(m), \text{ for every } k \in \mathbb{N}. \tag{6}$$

Because of $\left\|\phi_1^{Y^m}(x) - x\right\| = \left\|\int_0^1 Y^m(\phi_t(x))dt\right\| = O(m)$, we have $\left\|\phi_1^{Y^m} - id_{B\times S^1}\right\|_{C^1} = O(m)$.

In the coordinates of $B \times S^1$, by expanding at $m = 0$ and using the form

$$1/H_m(x) = 1 + m^2/2f + o(m^2),$$

we directly get $1/H_m(\phi_1^{Y^m}(x)) = 1/H_m(x) + o(m^2)$. Because of (6), the higher derivatives of $\phi_1^{Y^m}$ are all bounded and won't affect the order of the $o(m^2)$ term. Therefore, together with the fact that $\phi_1^{Y^m}$ is a contactomorphism, we have

$$\left\|\frac{1}{H_m}\lambda_{std} - \frac{1}{H_m \circ \phi_1^{Y_m}}\phi_1^{Y_m*}\lambda_{std}\right\|_{C^{1,+}} = o(m^2). \tag{7}$$

Now, define the global contactomorphism $\Psi_{1,m} : SS^2 \to SS^2$:

$$\Psi_{1,m}(x) := \begin{cases} \mathfrak{D}_{\hat{x}} \circ \phi_1^{Y^m} \circ \mathfrak{D}_{\hat{x}}^{-1}(x), & \text{if } x \in \mathfrak{D}_{\hat{x}}(B \times S^1). \\ \mathrm{id}_{SS^2}(x), & \text{if } x \notin \mathfrak{D}_{\hat{x}}(B \times S^1). \end{cases}$$

Set $\gamma_m := \Psi_{1,m}^{-1} \circ \hat{\gamma}_m$, then $\gamma_m(0) \in S_pS^2$ and $\gamma_m$is a periodic Reeb orbit of $\Psi_{1,m}^*\alpha_m$.

Equation 7, together with (**D2**), implies $\left\|\alpha_m - \psi_{m,1}^*\alpha_m\right\|_{C^{1,+}} = o(m^2)$. □

We want to build a diffeomorphism that takes the periodic Reeb Orbit we get via Lemma 3.7 and maps it to the fiber S_pS^2. In the following lemma we see that the unperturbed form α_{H_m} already has a periodic Reeb orbit winding around S_pS^2. Our plan is to later send the Reeb orbit from Lemma 3.12 to the periodic orbit of α_{H_m} winding around S_pS. We can then use Lemma 6.5 to estimate the C^2 difference between both orbits which helps us ensure that the diffeomorphism we later build leaves $\mathcal{B}$ invariant.

Lemma 3.8. *Let p be a critical point of f then, α_{H_m} has a periodic orbit that winds uniformly around S_pS^2 in the positive sense.*

Proof. By Lemma 7.14 in [Benedetti, 2014] the Reeb vector field $R^{\alpha_{H_m}}$ of $H_m\lambda_0$ splits as follows:

$$R^{\alpha_{H_m}} = L^H(X_{H_m}) + H_mV,$$

where X_{H_m} is the Hamiltonian vector field on (S^2, σ) associated to the Hamiltonian H_m. Because $dH_m(p) = 0$, we have $\sigma_p(X_{H_m}(p), .) = 0$.

By assumption, $\sigma = f\,\mathrm{vol}_g$ does nowhere vanish. Hence, $X_{H_m}(p) = 0$. Therefore,

$$R^{\lambda_0} = H_m V,$$

where V was the generator of the flow rotating each fiber in the positive sense. $\square$

If p is a non-degenerate critical point of f then, after applying Lemma 3.7, we can assume that $\alpha_m \in \mathcal{B}$ has a periodic Reeb orbit intersecting S_pS^2. Furthermore, we know that in this case α_{H_m} has a periodic orbit winding around S_pS. We can make the following estimates about the distance between both orbits and the corresponding Reeb vector fields along them.

Lemma 3.9. *Let $\alpha_m \in \mathcal{B}$ with a periodic Reeb orbit γ_m with period $T_m = 2\pi + O(m)$ such that $\gamma_m(0) \in S_pS^2$ and then $\gamma_m \subset B \times S^1$ for m small enough. Furthermore, if we write $(x_m, \theta_m) : \mathbb{R} \to B \times \mathbb{R}$ for the lift of $\mathfrak{D}_p^{-1} \circ \gamma_m$ with $(x_m(0), \theta_m(0)) = 0$, then*

$$\|(x_m(t), \theta_m(t) - H_m(p)t)\| = o(m^2), \qquad \left\|(\dot{x}_m, \dot{\theta}_m - H_m(p))\right\|_{C^1} = o(m^2) \tag{8}$$

Proof. Since $\gamma_m(0) \in S_pS^2$, $\gamma_m(t) \in \mathfrak{D}_p(B \times S^1)$ for $t \in [0, t_0]$, for some $t_0 \in [0, T_m]$. Then, $\gamma_{p,m}(t) := \mathfrak{D}_p^{-1} \circ \gamma_m(t)$ is well defined for $t \in [0, t_0]$ small.

Let γ_{H_m} be the periodic Reeb orbit of α_{H_m} with $\gamma_{H_m}(0) = \gamma_m(0)$, Set

$$\begin{aligned} \alpha_{p,m} &= {\mathfrak{D}_p}^* \alpha_m, & \gamma_{p,m} &= {\mathfrak{D}_p}^{-1} \circ \gamma_m, \\ \alpha_{p,H_m} &= {\mathfrak{D}_p}^* \alpha_{H_m}, & \gamma_{p,H_m} &= {\mathfrak{D}_p}^{-1} \circ \gamma_{H_m}. \end{aligned}$$

and let $R_{\alpha_{p,m}}$ and $R_{\alpha_{p,H_m}}$ denote the respective Reeb vector fields. We calculate

$$\begin{aligned} &\left\|R_{\alpha_{p,m}}(\gamma_{p,m}) - R_{\alpha_{p,H_m}}(\gamma_{p,H_m})\right\|_{C^0} \\ &\leq \left\|R_{\alpha_{p,m}}(\gamma_{p,m}) - R_{\alpha_{p,H_m}}(\gamma_{p,m})\right\|_{C^0} + \left\|R_{\alpha_{p,m}}(\gamma_{p,H_m}) - R_{\alpha_{p,H_m}}(\gamma_{p,H_m})\right\|_{C^0} \\ &\leq \left\|R_{\alpha_{p,m}} - R_{\alpha_{p,H_m}}\right\|_{C^0} + \left\|R_{\alpha_{p,H_m}}\right\|_{C^1} \|\gamma_{p,m} - \gamma_{p,H_m}\|_{C^0}. \end{aligned} \tag{9}$$

Therefore,

$$\begin{aligned} \|\gamma_{p,m} - \gamma_{p,H_m}\|_{C^0} &\leq \int_0^t \left\|R_{\alpha_{p,m}}(\gamma_{p,m}) - R_{\alpha_{p,H_m}}(\gamma_{p,H_m})\right\|_{C^0} ds \\ &\leq t\left\|R_{\alpha_{p,m}} - R_{\alpha_{p,H_m}}\right\|_{C^0} + \int_0^t \left\|R_{\alpha_{p,H_m}}\right\|_{C^1} \|\gamma_{p,m} - \gamma_{p,H_m}\|_{C^0} ds. \end{aligned}$$

By applying the integral form of Gronwall's lemma we get

$$\|\gamma_{p,m} - \gamma_{p,H_m}\|_{C^0} \leq t\left\|R_{\alpha_{p,m}} - R_{\alpha_{p,H_m}}\right\|_{C^1} \exp\bigl(t\|R_{p,H_m}\|_{C^1}\bigr). \tag{10}$$

And because of Lemma 6.5 and **D2**, there exists $A > 0$

$$\left\|R_{\alpha_{p,m}} - R_{\alpha_{p,H_m}}\right\|_{C^1} < A\|\alpha_m - \alpha_{H_m}\|_{C^{1,+}} = o(m^2). \tag{11}$$

Combining Equation 10 and 11 yields

$$\|(x_m(t), \theta_m(t) - H_m(p)t)\| = \|\gamma_{p,m}(t) - \gamma_{p,H_m}(t)\| = o(m^2) \exp\Bigl(t\left\|TR_{\alpha_{p,H_m}}\right\|_{C^1}\Bigr). \tag{12}$$

Since $\gamma_m(t)$ has period $T_m = 2\pi + O(m)$, and

$$\|x_m(t)\| \leq \|(x_m(t), \theta_m(t) - H_m(p)t)\| = o(m^2)\exp\big(t\|TR_{p,H_m}\|_{C^1}\big),$$

we can achieve that $\gamma_m \subset B$ by making m small enough, therefore $\gamma_{p,m}(t)$ is defined for all t and

$$\|\gamma_{p,m}(t) - \gamma_{p,H_m}(t)\| = o(m^2)\exp\big(T_m\|TR_{p,H_m}\|_{C^1}\big) = o(m^2). \tag{13}$$

This shows the first equation at (8).

For the second equation, use Equation 13 and 11 and calculate

$$\begin{aligned}\left\|\frac{d}{dt}\gamma_{p,m} - \frac{d}{dt}\gamma_{p,H_m}\right\|_{C^0} &= \big\|R_{\alpha_{p,m}}(\gamma_{p,m}) - R_{\alpha_{p,H_m}}(\gamma_{p,H_m})\big\|_{C^0} \\ &\leq \big\|R_{\alpha_{p,m}} - R_{\alpha_{p,H_m}}\big\|_{C^0} + \big\|TR_{\alpha_{p,H_m}}\big\|_{C^1}\|\gamma_{p,m} - \gamma_{p,H_m}\|_{C^0} \\ &= o(m^2).\end{aligned} \tag{14}$$

Doing similar estimates as in Equation 9 for the C^1-norm we get

$$\begin{aligned}&\big\|R_{\alpha_{p,m}}(\gamma_{p,m}) - R_{\alpha_{p,H_m}}(\gamma_{p,H_m})\big\|_{C^1} \\ &\leq \big\|R_{\alpha_{p,m}}(\gamma_{p,m}) - R_{\alpha_{p,H_m}}(\gamma_{p,m})\big\|_{C^1}(1 + \|\dot{\gamma}_m\|_{C^0}) + \big\|R_{\alpha_{p,m}}(\gamma_{p,H_m}) - R_{\alpha_{p,H_m}}(\gamma_{p,H_m})\big\|_{C^1} \\ &\leq \big\|R_{\alpha_{p,m}} - R_{\alpha_{p,H_m}}\big\|_{C^1} + \|R_{p,H_m}\|_{C^2}\|\gamma_{p,m} - \gamma_{p,H_m}\|_{C^1}.\end{aligned} \tag{15}$$

By Equation 14 and the previous C^0 estimate we have $\|\gamma_{p,m} - \gamma_{p,H_m}\|_{C^1} = o(m^2)$. This, together with Equation 11 and Equation 15, implies

$$\big\|R_{\alpha_{p,m}}(\gamma_{p,m}) - R_{\alpha_{p,H_m}}(\gamma_{p,H_m})\big\|_{C^1} = o(m^2).$$

Hence,

$$\Big\|(\dot{x}_m, \dot{\theta}_m - H_m(p))\Big\|_{C^1} = \big\|R_{\alpha_{p,m}}(\gamma_{p,m}) - R_{\alpha_{p,H_m}}(\gamma_{p,H_m})\big\|_{C^1} = o(m^2),$$

showing the second equation at (8). □

To build the desired diffemorphism, we are going to reparameterize the orbits of our interest so that they have period 2π.

Definition 3.10. *Let be p a non-degenerate critical point of f, $\alpha_m \in \mathcal{B}$ and γ_m a periodic Reeb orbit of α_m such that $\gamma_m(0) \in S_pS^2$.*

Define the 2π periodic reparameterized Reeb orbit

$$\gamma_m^{rep} : S^1 \to SS^2,\ t \mapsto \gamma_m(\frac{T_m}{2\pi}t),$$

and the trivial periodic Reeb orbit of λ_0 winding around S_pS^2 and starting at $\gamma_m(0)$

$$\bar{\gamma}_m : S^1 \to SS^2,\ t \mapsto \phi_t^{\lambda_0}(\gamma_m(0)).$$

Then, $\bar{\gamma}_m$ is a 2π-periodic reparameterized Reeb orbit of α_{H_m}.

The following lemma gives an estimate of the C^2 difference of the 2π-periodic reparameterized Reeb orbits we just defined.

Lemma 3.11. *Let $\alpha_m \in \mathcal{B}$ with a periodic Reeb orbit γ_m such that $\gamma_m(0) \in S_pS^2$ and $\gamma_m \subset B \times S^1$. Let*

$$\gamma^{rep}_{p,m}(t) := \mathfrak{D}_p^{-1}(\gamma^{rep}_m(t)), \quad \bar{\gamma}_{p,m}(t) := \mathfrak{D}_p^{-1}(\bar{\gamma}_m(t)),$$

then

$$\left\|\gamma^{rep}_{p,m} - \bar{\gamma}_{p,m}\right\|_{C^2} = o(m^2).$$

Proof. By plugging T_m into the first equation at (8) of Lemma 3.9 we get

$$|2\pi - T_m H_m(p)| = o(m^2). \tag{16}$$

Now, for the second order derivative, estimate

$$\begin{aligned}
\left\|\frac{d}{dt}\gamma^{rep}_{p,m} - \frac{d}{dt}\bar{\gamma}_{p,m}\right\|_{C^1} &= \left\|\frac{T_m}{2\pi}(\dot{x}_m, \dot{\theta}_m)_{rep} - (0,1)\right\|_{C^1} \\
&\le \frac{T_m}{2\pi}\left\|(\dot{x}_m, \dot{\theta}_m - H_m(p))_{rep}\right\|_{C^1} + \left|\frac{T_m H_m(p)}{2\pi} - 1\right| \\
\text{(By Equation 8)} \quad &= o(m^2) + \left|\frac{T_m H_m(p)}{2\pi} - 1\right| \\
&= o(m^2) + \frac{1}{2\pi}|T_m H_m(p) - 2\pi| \\
\text{(By Equation 16)} \quad &= o(m^2)
\end{aligned}$$

□

In the next lemma, we construct a diffeomorphism that maps the periodic orbit that we obtained via Lemma 3.9, to the fiber S_pS^2.

Lemma 3.12. *Let $\alpha_m \in \mathcal{B}$ and $p \in S^2$ be a non-degenerate critical point of f. For $m > 0$ small enough, there exists a diffeomorphism Ψ_m such that $\Psi_m^*\alpha_m \in \mathcal{B}_p$.*

Proof. This proof is a slightly adapted version of the proof of Lemma 3.6 in [Benedetti and Kang, 2018].

Let $\Psi_{1,m}$ be the contactomorphism from Lemma 3.7, and set

$$\alpha_{\Psi_{1,m}} := \Psi_{1,m}^*\alpha_m.$$

Let γ_m be the Reeb orbit of α_{ψ_m} such that $\gamma_m(0) \in S_pS^2$. By Lemma 3.9 we can assume that $\gamma_m \subset \mathfrak{D}_p(B \times S^1)$ for m small enough.

Consider the $\gamma^{rep}_{m,p}(t) := \mathfrak{D}_p^{-1}(\gamma^{rep}_m(t))$ on $B' \times S^1$. and write $(x^{rep}_m(t), \theta^{rep}(t)) = \gamma^{rep}_{m,p}(t)$. By making m even smaller we can ensure that $x_m(t) \in B'$.

Let $(\hat{x}^{rep}_m, \hat{\theta}^{rep}) : \mathbb{R} \to B \times \mathbb{R}$ be the lift of $(x^{rep}_m, \theta^{rep})$ such that $(\hat{x}^{rep}_m(0), \hat{\theta}^{rep}(0)) = 0$. Consider the diffeomorphism

$$\begin{aligned}&\hat{\psi}_m : B \times \mathbb{R} \to B \times \mathbb{R}\\ &\qquad (x,s) \mapsto (x + K(\|x\|)\hat{x}^{rep}_m(s), s + K(\|x\|)(\hat{\theta}^{rep}_m(s) - s)),\end{aligned}$$

where $K : [0,a] \to [0,1]$ is a cut-off function that equals 1 on $[0, a/2]$ and 0 on $[\frac{3}{4}a, a]$. The 2π periodic diffeomorphism $\hat{\psi}_m$ is the lift of a diffeomorphism $\psi_m : B \times S^1 \to B \times S^1$.

From Lemma 3.11 we know that $\|\gamma^{rep}_{p,m} - \bar{\gamma}_{p,m}\|_{C^2} = o(m^2)$, implying that

$$\begin{aligned}\|\psi_m - id_{B\times S^1}\|_{C^2} &\leq \|K \cdot (x^{rep}_m, \theta^{rep}_m - id_{S^1})\|_{C^2}\\ &\leq \|K\|_{C^2}\left\|\gamma^{rep}_{p,m} - \bar{\gamma}_{p,m}\right\|_{C^2} = o(m^2),\end{aligned} \tag{17}$$

. On $B \times S^1$, Equation 17 and Lemma 6.4 imply

$$\left\|\psi^*_m(\mathfrak{D}^*_p \alpha_{\Psi_{1,m}}) - \mathfrak{D}^*_p \alpha_{\Psi_{1,m}}\right\|_{C^{1,+}} = o(m^2). \tag{18}$$

Define the global diffeomorphism $\Psi_{2,m} : SS^2 \to SS^2$:

$$\Psi_{2,m}(x) := \begin{cases} \mathfrak{D}_p \circ \psi_m \circ \mathfrak{D}^{-1}_p(x), & \text{if } x \in \mathfrak{D}_p(B \times S^1).\\ \mathrm{id}_{SS^2}(x), & \text{if } x \notin \mathfrak{D}_p(B \times S^1).\end{cases}$$

Then, by Equation 18 and the Darboux property **D2** together with Lemma 6.2, there exists an $A > 0$ such that

$$\left\|\Psi^*_{2,m}\alpha_{\Psi_{1,m}} - \alpha_{\Psi_{1,m}}\right\|_{C^{1,+}} \leq A\left\|\psi^*_m(\mathfrak{D}^*_p\alpha_{\Psi_{1,m}}) - \mathfrak{D}^*_p\alpha_{\Psi_{1,m}}\right\|_{C^{1,+}} = o(m^2).$$

This means that $\Psi^*_{2,m}\alpha_{\Psi_{1,m}} \in \mathcal{B}$.

By construction, $\Psi^2_m \circ \gamma_0 = \gamma^{rep}_m$ hence, $\Psi^*_{2,m}\alpha_{\Psi_{1,m}} \in \mathcal{B}_p$ and $\Psi_m := \Psi_{2,m} \circ \Psi_{1,m}$ is the desired diffeomorphism. ☐

Corollary 3.13. *Let $p_1, \ldots, p_k$ be critical non-degenerate points of f. For $m > 0$ small enough, there exists a diffeomorphism ψ_m such that $\psi^*_m \alpha_m \in \mathcal{B}_{p_1,\ldots,p_k}$.*

Proof. Observe that the diffeomorphism from Lemma 3.12 equals the identity outside a small $O(m)$ neighborhood of the non-degenerate critical point the lemma is applied to. Because non-degenerate critical points are isolated, we can choose m small enough, so that no critical point lies in the neighborhood of another one. By applying Lemma 3.12 multiple times we get diffeomorphisms $\psi^1_m, \ldots, \psi^k_m$ for each point so that $(\psi^1_m \circ \ldots \circ \psi^k_m)^* \alpha_m$ is weakly normalized at $p_1, \ldots, p_k$. Set $\psi_m := \psi^1_m \circ \ldots \circ \psi^k_m$. ☐

4 A global surface of section

From now on, let p_+, p_- be the North and South Pole of S^2, with respect to some global polar coordinates. Assume that f has a local minimum at p_{min} and local maximum at p_{max}. Morse's lemma famously implies that non-degenerate critical points are isolated. Therefore, we can choose coordinates of S^2 such that $p_+ = p_{max}$ and $p_- = p_{min}$.

For the remainder of this thesis, we assume that f has a local maximum at p_+ and a local minimum at p_-. Denote by γ_+ and γ_- the periodic Reeb orbits of ϕ^{λ_0} which wind uniformly, in positive direction, around $S_{p_+}S^2$ and $S_{p_-}S^2$ respectively.

At first, we will construct a global surface of section for the Reeb flow ϕ^{λ_0} and show that this surface is also a global surface of section for the Reeb flow of forms $\alpha_m \in \mathcal{B}_{p_+,p_-}$ for m small enough.

As the definition of a global surface of section, we use a slightly modified version of Definition 3.11 in [Benedetti and Kang, 2018].

Definition 4.1. *Let ϕ be the Reeb flow of some contact form on SS^2 and N a compact surface. An embedding $\bar{S} : N \to SS^2$ is a **global surface of section** for ϕ if the following properties hold:*

(i) The surface $\bar{S}(\mathring{N})$ is transverse to the flow ϕ and $\bar{S}(\partial N)$ is the support of a finite collection of periodic orbits of ϕ.

(ii) For each $z \in SS^2 \setminus \bar{S}(\partial N)$, there is a $t_- < 0 < t_+$ such that $\phi_{t_-}(z), \phi_{t_+}(z)$ lie in $\bar{S}(\mathring{N})$.

*Associated to the global surface of section, there is a **first return time** and **first return map**. The definitions are as follows:*

First return time:

$$\bar{\tau} : \bar{S}(\mathring{N}) \to \mathbb{R}, \quad \bar{\tau}(q) := \inf\{\ t > 0 \mid \phi_t(q) \in \bar{S}(\mathring{N})\ \},$$

First return map:

$$\bar{P} : \bar{S}(\mathring{N}) \to \bar{S}(\mathring{N}), \quad \bar{P}(q) := \phi_{\bar{\tau}(q)}(q).$$

Let ϕ^η be the Reeb flow of some contact form $\eta \in \Omega^1(SS^2)$. Notice that the requirement for the global surface of section having a boundary is necessary.

Indeed, assume we had a compact surface $N_0 \subset SS^2$ that is transverse to ϕ^η. Then, $d\eta$ has no kernel on N_0 and therefore is a volume form on N_0. By Stokes theorem

$$\int_{N_0} d\eta = \int_{\partial N_0} \eta.$$

Now, if $\partial N_0 = \emptyset$, the right hand side vanishes, thus a surface of section without boundary is not possible.

Later we will see that in the case of ϕ being the flow of a form in $\mathcal{B}_{p_+,p_-}$ the first return time and first return map extend to the boundary, which is important because twist maps need to be defined on the boundary.

4.1 Construction of the surface of section

Based on the definition in [Milnor, 1965] on page 22 we define:

Definition 4.2 (Index of a vector field). *Let Y be a vector field on S^2 with a singularity at some point $p \in S^2$. In local coordinates $U \subset \mathbb{R}^2$, choose a small ball $\mathcal{B} \subset U$ centered at p and consider the map*

$$\bar{v}_p : \partial\mathcal{B} \to S^1, \quad x \mapsto \frac{Y_x}{\|Y_x\|}.$$

Define the index at p by $\mathrm{index}_Y(p) := \deg(\bar{v}_p)$*, where* deg *means the degree of continuous S^1 maps.*

Now, define the annulus $\mathbb{A} := [0,1] \times S^1$ and the following sets.

$$\mathfrak{C}_+ := [0,a) \times S^1, \quad \mathfrak{C}_- := (1-a,1] \times S^1, \quad \mathfrak{C} := \mathfrak{C}_+ \cup \mathfrak{C}_-, \tag{19}$$

where a was the radius of B, which was defined in Section 3.3. Choose $(r,\theta) \in [0,1] \times S^1$ as the coordinates of $\mathbb{A}$ and orient $\mathbb{A}$ according to $dr \wedge d\theta > 0$. Orient $\mathfrak{C}$ similarly.

Let $\hat{S}^2 := S^2 - \{p_+, p_-\}$ and $z_+ \in S_{p_+}S^2$, $z_- \in S_{p_-}S^2$. Furthermore, let $\mathfrak{P} : \mathbb{A} \to S^2$ be a polar coordinate map with the following properties:

(i) $\mathfrak{P}(0,\theta) = p_+$, $\mathfrak{P}(1,\theta) = p_-$, for all $\theta \in S^1$;

(ii) The unit normalized radial vector field $\partial_\theta : \mathring{\mathbb{A}} \to SS^2$ induced by $\mathfrak{P}$ has singularities at p_+ and p_- with index 1 each.

(iii) ∂_θ extends to a map $\hat{\partial}_\theta : \mathbb{A} \to SS^2$, with $\hat{\partial}_\theta(0,\theta) = \phi_\theta^{\lambda_0}(z_+)$, $\hat{\partial}_\theta(1,\theta) = \phi_{-\theta}^{\lambda_0}(z_-)$.

Denote the restriction of $\mathfrak{P}$ to $\mathring{\mathbb{A}}$ by $\mathring{\mathfrak{P}}$.

The map $\mathfrak{P}$ is already a surface of section for the Reeb flow of λ_0. However, we want to have local coordinates on a collar region of $\partial\mathbb{A}$, for which the pullback of λ_0, under the surface of section embedding, has a standard coordinate expression.

Choose a local chart $\varphi_{z_+} : B \times S^1 \to SS^2$ with

$$\varphi_{z_+}(0,\phi) = \phi_\phi^{\lambda_0}(z_+), \quad \varphi_{z_+}(re^{i\theta},\theta) = \hat{\partial}_\theta(y_+(r,\theta)).$$

Here, $(r,\theta) \in \mathring{\mathfrak{C}}_+$ and $y_+(r,\theta) := \mathfrak{P}^{-1} \circ \pi \circ \varphi_{z_+}(re^{i\theta},\theta)$.

By Darboux, we can assume that we have coordinates on B such that $\varphi_{z_+}^*\pi^*\sigma = \pi_{std}^*\omega_{std}$. Contrary to the contact Darboux charts we defined in subsection 3.3, φ_{z_+} does not pull λ_0 back to the standard form. Instead, we have $\varphi_{z_+}^*\lambda_0 = \pi_{std}^*\eta + d\phi$, where $\phi \in S^1$ is the fiber coordinate and η an one-form with $\pi_{std}^*d\eta = \pi_{std}^*\omega_{std}$.

Then, $d(\pi_{std}^*\lambda_{std} - \pi_{std}^*\eta) = 0$ and by Poincaré's lemma there is a function $h : B \to \mathbb{R}$, with $h(0) = 0$ and $dh = \pi_{std}^*\lambda_{std} - \pi_{std}^*\eta$.

Set $\varphi_h(x,\phi)=\varphi_{z_+}(x,\phi+h)$, then

$$\varphi_h^*\lambda_0=\pi_{std}^*\eta+d(\phi+h)=\pi_{std}^*\lambda_{std}.$$

Since $h(0)=0$, we have $\varphi_h(0,\phi)=\varphi_{z_+}(0,\phi)=\hat{\partial}_\theta(0,\phi)$, and because of the way φ_{z_+} was chosen, we can assume , after ensuring that a (See (19)) is small enough, that

$$g(\varphi_h(re^{i\theta},\theta),\hat{\partial}_\theta(y_h(r,\theta)))>0,$$

where $y_h(r,\theta):=\mathfrak{P}^{-1}\circ\pi\circ\varphi_h(re^{i\theta},\theta)$ and $(r,\theta)\in\mathring{\mathfrak{C}}_+$.

A similar Darboux chart can be constructed for z_- as the base point. In this case, we start from a chart $\varphi_{z_-}:B\times S^1\to SS^2$ with

$$\varphi_{z_-}(0,\phi)=\phi_\phi^{\lambda_0}(z_-),\quad \varphi_{z_-}((1-r)e^{-i\theta},-\theta)=\hat{\partial}_\theta(y_-(r,\theta)),$$

where $(r,\theta)\in\mathring{\mathfrak{C}}_-$ and $y_-(r,\theta):=\mathfrak{P}^{-1}\circ\pi\circ\varphi_{z_-}((1-r)e^{-i\theta},-\theta)$. Analogously to the case where z_+ was the base point, we can do the same procedure to construct a Darboux chart that coincides with φ_{z_-} on the boundary.

Consider charts $\mathfrak{D}_{z_+}$, $\mathfrak{D}_{z_-}$ with corresponding charts $\mathfrak{d}_{p_+}$, $\mathfrak{d}_{p_-}$ of S^2 such that

$$\pi\circ\mathfrak{D}_{z_\pm}=\mathfrak{d}_{p_\pm}\circ\pi_{std}.$$

By the above discussion, we can assume that for $\mathfrak{D}_{z_+}$, $\mathfrak{D}_{z_-}$ we have

$$\begin{aligned} &g(\mathfrak{D}_{z_+}(re^{i\theta},\theta),(\hat{\partial}_\theta\circ\mathring{\mathfrak{P}}^{-1}\circ\mathfrak{d}_{z_+})(re^{i\theta}))>0,\\ &g(\mathfrak{D}_{z_-}((1-r)e^{-i\theta},-\theta),(\hat{\partial}_\theta\circ\mathring{\mathfrak{P}}^{-1}\circ\mathfrak{d}_{z_-})((1-r)e^{-i\theta}))>0.\end{aligned}\tag{20}$$

on $\mathfrak{C}_+$ and $\mathfrak{C}_-$ respectively. Furthermore, we can assume that

$$\mathfrak{D}_{z_+}(0,\theta)=\hat{\partial}_\theta(0,\theta),\quad \mathfrak{D}_{z_-}(0,-\theta)=\hat{\partial}_\theta(1,\theta).\tag{21}$$

Define the embedding $S_{\mathfrak{C}}:\mathfrak{C}\to SS^2$

$$S_{\mathfrak{C}}(r,\theta)=\begin{cases}\mathfrak{D}_{z_+}(re^{\theta i},\theta) & \text{if } r\in[0,a),\\ \mathfrak{D}_{z_-}((1-r)e^{-\theta i},-\theta) & \text{if } r\in(1-a,1].\end{cases}$$

and $\mathfrak{i}_{\mathfrak{C}}:\mathring{\mathfrak{C}}\to SS^2$:

$$\mathfrak{i}_{\mathring{\mathfrak{C}}}(r,\theta)=\begin{cases}\mathfrak{d}_{p_+}(re^{i\theta}), & \text{if } r\in(0,a)\\ \mathfrak{d}_{p_-}((1-r)e^{-i\theta}), & \text{if } r\in(1-a,1).\end{cases}$$

Notice that $S_{\mathfrak{C}}\circ\mathfrak{i}_{\mathring{\mathfrak{C}}}^{-1}$ is a local section of SS^2 with singularities at p_+ and p_-, each of index 1.

Because of (20), for all $x\in\mathfrak{i}_{\mathring{\mathfrak{C}}}(\mathring{\mathfrak{C}})$

$$g(S_{\mathfrak{C}}\circ\mathfrak{i}_{\mathring{\mathfrak{C}}}^{-1}(x),\ \hat{\partial}_\theta\circ\mathring{\mathfrak{P}}^{-1}(x))>0.$$

Choose compact neighborhoods $K_{p_+} \subset \mathfrak{i}_{\mathring{\mathfrak{C}}}(\mathfrak{C}_+)$, $K_{p_-} \subset \mathfrak{i}_{\mathring{\mathfrak{C}}}(\mathfrak{C}_-)$ and $K'_{p_+} \subset \mathring{K}_{p_+}$, $K'_{p_-} \subset \mathring{K}_{p_-}$ of p_+ and p_- respectively. Let $\mu : S^2 \to [0,1]$ be a bump function, such that $\mu = 1$ on $K'_{p_+} \cup K'_{p_-}$ and $\mu = 0$ outside of $K_{p_+} \cup K_{p_-}$. Define $\hat{S} : \hat{S}^2 \to SS^2$

$$\hat{S}(x) := \frac{\mu(x) \cdot (S_{\mathfrak{C}} \circ \mathfrak{i}_{\mathring{\mathfrak{C}}}^{-1})(x) + (1 - \mu(x)) \cdot (\hat{\partial}_\theta \circ \mathring{\mathfrak{P}}^{-1})(x)}{\left\| \mu(x) \cdot (S_{\mathfrak{C}} \circ \mathfrak{i}_{\mathring{\mathfrak{C}}}^{-1})(x) + (1 - \mu(x)) \cdot (\hat{\partial}_\theta \circ \mathring{\mathfrak{P}}^{-1})(x) \right\|}.$$

Then, $\hat{S}$ is a section of SS^2 that vanishes only at p_- and p_+.

The section $\hat{S} \circ \mathring{\mathfrak{P}}$ extends to an embedding $S' : \mathbb{A} \to SS^2$ such that on the boundary

$$S'(0, \theta'_+(\theta)) = \mathfrak{D}_{z_+}(0, \theta) = \phi^{\lambda_0}_\theta(z_+), \quad S'(1, -\theta'_-(\theta)) = \mathfrak{D}_{z_-}(0, -\theta) = \phi^{\lambda_0}_{-\theta}(z_-),$$

where $\theta'_\pm : S^1 \to S^1$ are diffeomorphisms of the circle. Using a bump function, one can define a diffeomorphism $\xi : \mathbb{A} \to \mathbb{A}$ such that $\xi(0, \theta) = (0, \theta'_+(\theta))$ and $\xi(0, \theta) = (0, \theta'_-(\theta))$. Then, $S := S' \circ \xi : \mathbb{A} \to SS^2$ is an embedding of the annulus such that

$$S(0, \theta) = \mathfrak{D}_{z_+}(0, \theta) = \phi^{\lambda_0}_\theta(z_+), \quad S(1, -\theta) = \mathfrak{D}_{z_-}(0, -\theta) = \phi^{\lambda_0}_{-\theta}(z_-). \tag{22}$$

Choose collar neighborhoods $\mathcal{N}_+$ of $\partial^+\mathbb{A}$ and $\mathcal{N}_-$ of $\partial^-\mathbb{A}$ with $\mathfrak{P}(\mathring{\mathcal{N}}_+) \subset K'_{p_+}$, $\mathfrak{P}(\mathring{\mathcal{N}}_-) \subset K'_{p_-}$ and set $\mathcal{N} = \mathcal{N}_+ \cup \mathcal{N}_-$. There exist coordinates (r, θ) on $\mathcal{N}$ such that $(0, \theta)$ and $(1, \theta)$ are the original coordinates of $\mathbb{A}$ winding around the boundary and the pullback $S^*\lambda_0$ has the coordinate expression

$$\begin{aligned} S^*\lambda_{0|\mathcal{N}_+} &= (\frac{1}{2}r^2 + 1)d\theta, \\ S^*\lambda_{0|\mathcal{N}_-} &= -(\frac{1}{2}(1-r)^2 + 1)d\theta. \end{aligned}$$

In these coordinates, S can be written as

$$S(r, \theta)_{|\mathcal{N}_+} = \mathfrak{D}_{z_+}(re^{i\theta}, \theta), \quad S(r, \theta)_{|\mathcal{N}_-} = \mathfrak{D}_{z_-}((1-r)e^{-i\theta}, -\theta).$$

(After extending $\mathfrak{P}^{-1} \circ \mathfrak{i}_{\mathring{\mathfrak{C}}}$ to the boundary, the coordinates on $\mathcal{N}$ are provided by $\xi^{-1} \circ \mathfrak{P}^{-1} \circ \mathfrak{i}_{\mathring{\mathfrak{C}}}$)

Let $x \in S^2$, observe that for $z \in S_xS^2$, $\phi^{\lambda_0}(z)$ simply winds around S_xS^2 in the positive sense. If $x \notin \{p_+, p_-\}$, then $\phi^{\lambda_0}(z)$ intersects $S(\mathring{\mathbb{A}})$, as S on $\mathring{\mathbb{A}}$ was defined to be a section of SS^2. If $x \in \{p_+, p_-\}$ then, by Equation 22, the flow line $\phi^{\lambda_0}(z)$ is supported by $S(\partial\mathbb{A})$. Therefore, S is a global surface of section for ϕ^{λ_0}.

4.2 Extending weakly normalized Reeb vector fields

The following discussion is based on [Benedetti and Kang, 2018] Chapter 3.4. We have an annulus $\mathbb{A} = [0,1] \times S^1$ and a global surface of section $S : \mathbb{A} \to SS^2$. Set

$$\partial^+\mathbb{A} := \{0\} \times S^1, \quad \partial^-\mathbb{A} := \{1\} \times S^1.$$

By combining S with the Reeb flow of λ_0 we get the map

$$\begin{aligned} \Xi : \mathbb{A} \times S^1 &\to SS^2 \\ (q, s) &\mapsto \phi^{\lambda_0}_s(S(q)). \end{aligned} \tag{23}$$

In coordinates (r,θ,s) on $\mathcal{N}\times S^1$ the map Ξ can be expressed as

$$\Xi_{|\mathcal{N}} := \begin{cases} \mathfrak{D}_{z_+}(re^{i\theta},\theta+s) & \text{if } (r,\theta)\in\mathcal{N}_+ \\ \mathfrak{D}_{z_-}((1-r)e^{-i\theta},-\theta+s) & \text{if } (r,\theta)\in\mathcal{N}_-. \end{cases} \tag{24}$$

Observe that Ξ is not injective only on the boundary, so by restricting Ξ to the interior we get a diffeomorphism

$$\mathring{\Xi} : \mathring{\mathbb{A}}\times S^1 \to \pi^{-1}(S^2-\{p_+,p_-\}).$$

We have $T\mathring{\Xi}(\partial_s) = \frac{\partial}{\partial s}\mathring{\Xi} = \dot{\phi}^{\lambda_0} = R_{\lambda_0}$, showing that the Reeb vector field $R_{\mathring{\Xi}^*\lambda_0}$ of $\mathring{\Xi}^*\lambda_0$ equals $\partial_{s|\mathring{\mathbb{A}}\times S^1}$. Therefore, ∂_s is a smooth extension of the Reeb vector field $R_{\mathring{\lambda}_0}$. We have the local coordinate expressions

$$\Xi^*\lambda_{0|\mathcal{N}_+\times S^1} = (1+\frac{r^2}{2})d\theta + ds, \quad \Xi^* d\lambda_{0|\mathcal{N}_+\times S^1} = r dr d\theta, \tag{25}$$

$$\Xi^*\lambda_{0|\mathcal{N}_-\times S^1} = -(1+\frac{(1-r)^2}{2})d\theta + ds, \quad \Xi^* d\lambda_{0|\mathcal{N}_-\times S^1} = (1-r) dr d\theta, . \tag{26}$$

From the coordinate expression we directly see that

$$\Xi^*\lambda_0(\partial_s) = 1 \quad \iota_{\partial_s}(\Xi^* d\lambda_0) = 0 \quad {\phi^{\partial_s}}^*(\Xi^*\lambda_0) = \Xi^*\lambda_0. \tag{27}$$

We show that the Reeb vector fields of forms, that are weakly normalized at p_+ and p_-, extend to the boundary in a similar manner.

Fix some $\alpha_m \in \mathcal{B}_{p_+,p_-}$ and denote the pullbacks of α_m and the undisturbed form α_{H_m} by

$$\beta_m := \Xi^*\alpha_m, \quad \beta_{H_m} := \Xi^*\alpha_{H_m}$$

respectively. The forms restricted to $\mathring{\mathbb{A}}\times S^1$ are the pullbacks under $\mathring{\Xi}$

$$\mathring{\beta}_m = \mathring{\Xi}^*\alpha_m \quad \mathring{\beta}_{H_m} = \mathring{\Xi}^*\alpha_{H_m}.$$

Both forms are contact since $\mathring{\Xi}$ is a diffeomorphism.

The following proposition is a slightly adapted version of Proposition 3.10 in [Benedetti and Kang, 2018].

Proposition 4.3. *The Reeb vector field $R_{\mathring{\beta}_m}$ of $\mathring{\beta}_m$ extends to a vector field R_{β_m} on $\mathbb{A}\times S^1$ such that R_{β_m} is tangent to $\partial\mathbb{A}\times S^1$ and*

$$\beta_m(R_{\beta_m}) = 1, \quad \iota_{R_{\beta_m}} d\beta_m = 0, \quad {\phi^{\beta_m}}^*\beta_m = \beta_m. \tag{28}$$

Since $\alpha_{H_m}\in\mathcal{B}$, the Reeb vector field $R_{\mathring{\beta}_{H_m}}$ of $\mathring{\beta}_{H_m}$ extends similarly to a vector field $R_{\beta_{H_m}}$ and we have

$$\left\| R_{\beta_m} - R_{\beta_{H_m}} \right\|_{C^0} = o(m^2).$$

Proof. In the coordinates $(x,\phi)\in B\times S^1$, given by $\mathfrak{D}_{p_z}$, we have the following local description of R_{α_m}.

$$R_{\alpha_m} = R^x_{\alpha_m} + R^\phi_{\alpha_m}\partial_\phi. \tag{29}$$

Because α_m is weakly normalized at p_+, the Reeb vector field is tangent to $S_{p_+}S^2$, thus $R^x_{\alpha_m}(0,\phi)=0$. By Lemma 6.6, $R^x_{\alpha_m}$ can be written as $R^x_{\alpha_m}(x,\phi)=W_{\alpha_m}(x,\phi)x$, where W_{α_m} is a matrix valued function with $\|W_{\alpha_m}\|_{C^0}\leq \left\|R^x_{\alpha_m}\right\|_{C^1}$.

We use W_{α_m} to write $R^x_{\alpha_m}$ in polar coordinates as

$$\begin{aligned}R^x_{\alpha_m}(re^{i\phi},\phi) &= g_{std}(W_{\alpha_m}(re^{i\theta},\phi)re^{i\theta},e^{i\theta})\partial_r + g_{std}(W_{\alpha_m}(re^{i\theta},\phi)re^{i\theta},ie^{i\theta})\frac{\partial_\theta}{r}\\ &= g_{std}(W_{\alpha_m}(re^{i\theta},\phi)re^{i\theta},e^{i\theta})\partial_r + g_{std}(W_{\alpha_m}(re^{i\theta},\phi)e^{i\theta},ie^{i\theta})\partial_\theta.\end{aligned}$$

Using the functions

$$\begin{aligned}R^r_{\alpha_m}(r,\theta,\phi) &:= g_{std}(W_{\alpha_m}(re^{i\theta},\phi)re^{i\theta},e^{i\theta}),\\ R^\theta_{\alpha_m}(r,\theta,\phi) &:= g_{std}(W_{\alpha_m}(re^{i\theta},\phi)e^{i\theta},ie^{i\theta}).\end{aligned}$$

we have $R_{\alpha_m}=R^r_{\alpha_m}\partial_r+R^\theta_{\alpha_m}\partial_\theta+R^\phi_{\alpha_m}\partial_\phi$ on $B\setminus\{0\}\times S^1$.

The functions $R^r_{\alpha_m}\circ\Xi_+, R^\theta_{\alpha_m}\circ\Xi_+$ are defined on $\mathcal{N}_+$ with

$$\max\{\left\|R^r_{\alpha_m}\circ\Xi_+\right\|_{C^0},\left\|R^\theta_{\alpha_m}\circ\Xi_+\right\|_{C^0}\}\leq\left\|R^x_{\alpha_m}\right\|_{C^1}.$$

Because of

$$T\Xi_+(\partial_r)=\partial_r,\quad T\Xi_+(\partial_\theta)=\partial_\theta+\partial_\phi,\quad T\Xi_+(\partial_s)=\partial_\phi,$$

we can define R_{β_m} on $\mathcal{N}_+\times S^1$ as

$$R_{\beta_m}:=(R^r_{\alpha_m}\circ\Xi_+)\partial_r+(R^\theta_{\alpha_m}\circ\Xi_+)\partial_\theta+(R^\phi_{\alpha_m}\circ\Xi_+-R^\theta_{\alpha_m}\circ\Xi_+)\partial_s. \tag{30}$$

Observe that

$$R^r_{\alpha_m}(0,\theta,\phi)=0, \tag{31}$$

indicating that $R_{\beta_m|\partial^+\mathbb{A}}$ is tangent to $\partial^+\mathbb{A}$. The same procedure works analogously for defining R_{β_m} on $\mathcal{N}_-\times S^1$, with $R_{\beta_m|\partial^-\mathbb{A}}$ being tangent to $\partial^-\mathbb{A}$. On $(\mathbb{A}-\mathcal{N}\times S^1)$ we set $R_{\beta_m}:=R_{\beta_m^\circ}$. The vector field R_{β_m} is the desired extension of $R_{\beta_m^\circ}$.

By definition $\beta_m(R_{\beta_m^\circ})=1$ and $\iota_{R_{\beta_m^\circ}}d\beta_m=0$. Because R_{β_m} is a smooth extension of $R_{\beta_m^\circ}$, the first two identities at (28) hold as well. This means that the lie derivative

$\mathcal{L}_{R_{\beta_m}}\beta_m=\iota_{R_{\beta_m}}d\beta_m+d(\beta_m(R_{\beta_m}))$ vanishes, implying that ${\phi^{\beta_m}}^*\beta_m=\beta_m$.

Now, we take care of the C^0 closeness to $R_{\beta_{H_m}}$.

At first, estimate $\left\|R_{\beta_m}-R_{\beta_{H_m}}\right\|_{C^0}$ on $\mathcal{N}\times S^1$. Similarly to (29), we can write the Reeb vector field of α_{H_m} near $S_{p_+}S^2$ in the coordinates $(x,\phi)\in B\times S^1$, given by $\mathfrak{D}_{z_+}$.

$$R_{\alpha_{H_m}}=R^x_{\alpha_{H_m}}+R^\phi_{\alpha_{H_m}}\partial_\phi.$$

We can again define functions in polar coordinates

$$\begin{aligned}R^r_{\alpha_{H_m}}(r,\theta,\phi) &:= g_{std}(W_{\alpha_{H_m}}(re^{i\theta},\phi)re^{i\theta},e^{i\theta}),\\ R^\theta_{\alpha_{H_m}}(r,\theta,\phi) &:= g_{std}(W_{\alpha_{H_m}}(re^{i\theta},\phi)e^{i\theta},ie^{i\theta})\end{aligned}$$

such that we obtain a description in polar coordinates $R_{\alpha_{H_m}} = R^r_{\alpha_{H_m}}\partial_r + R^\theta_{\alpha_{H_m}}\partial_\theta + R^\phi_{\alpha_{H_m}}\partial_\phi$ on $B\setminus\{0\}\times S^1$.

Again, by Lemma 6.6, there exists a matrix valued function W such that

$$W(x,\phi)x = R^x_{\alpha_m}(x) - R^x_{\alpha_{H_m}}(x)$$

and

$$\|W\|_{C^0} < \left\|R^x_{\alpha_m} - R^x_{\alpha_{H_m}}\right\|_{C^1} \leq \left\|R_{\alpha_m} - R_{\alpha_{H_m}}\right\|_{C^1} = o(m^2).$$

Furthermore, observe that $W = W_{\alpha_m} - W_{\alpha_{H_m}}$. Therefore,

$$\begin{aligned}\left\|R^r_{\alpha_m} - R^r_{\alpha_{H_m}}\right\|_{C^0} &= \left\|g_{std}((W_{\alpha_m} - W_{\alpha_{H_m}})(re^{i\theta},\phi)re^{i\theta}, e^{i\theta})\right\|_{C^0}\\ &\leq \|W\|_{C^0} = o(m^2).\end{aligned}$$

For the same reason, we have $\left\|R^\theta_{\alpha_m} - R^\theta_{\alpha_{H_m}}\right\|_{C^0} = o(m^2)$. For the terms in the ϕ direction we have

$$\left\|R^\phi_{\alpha_m} - R^\phi_{\alpha_{H_m}}\right\|_{C^0} \leq \left\|R_{\alpha_m} - R_{\alpha_{H_m}}\right\|_{C^0} = o(m^2).$$

Hence

$$\begin{aligned}\left\|R_{\beta_m} - R_{\beta_{H_m}}\right\|_{C^0} \leq 3\max\Big\{&\left\|R^r_{\alpha_m}\circ\Xi_+ - R^r_{\alpha_{H_m}}\circ\Xi_+\right\|_{C^0},\\ &\left\|R^\theta_{\alpha_m}\circ\Xi_+ - R^\theta_{\alpha_{H_m}}\circ\Xi_+\right\|_{C^0},\\ &\left\|R^\phi_{\alpha_m}\circ\Xi_+ - R^\phi_{\alpha_{H_M}}\circ\Xi_+ + R^\theta_{\alpha_m}\circ\Xi_+ - R^\theta_{\alpha_{H_m}}\circ\Xi_+\right\|_{C^0}\Big\}\\ \leq 3\max\Big\{&\left\|R^r_{\alpha_m} - R^r_{\alpha_{H_m}}\right\|_{C^0}, \left\|R^\theta_{\alpha_m} - R^\theta_{\alpha_{H_m}}\right\|_{C^0},\\ &\left\|R^\phi_{\alpha_m} - R^\phi_{\alpha_{H_M}}\right\|_{C^0} + \left\|R^\theta_{\alpha_m} - R^\theta_{\alpha_{H_m}}\right\|_{C^0}\Big\} = o(m^2).\end{aligned}$$

The estimation of the C^0 distance between the vector fields near $S_{p_-}S^2$ can be done analogously. Consequently $\left\|R_{\beta_m} - R_{\beta_{H_m}}\right\|_{C^0} = o(m^2)$ on $\mathcal{N}\times S^1$.

On $\mathring{\mathbb{A}}\times S^1$ we have

$$\begin{aligned}\left\|R_{\mathring{\beta}_m} - R_{\mathring{\beta}_{H_m}}\right\|_{C^0} &\leq C_0\left\|\mathring{\beta}_m - \mathring{\beta}_{H_m}\right\|_{C^0}\\ &= C_0\left\|\mathring{\Xi}^*(\alpha_m - \alpha_{H_m})\right\|_{C^0}\\ &\leq C_0\left\|T\mathring{\Xi}\right\|_{C^0}\|\alpha_m - \alpha_{H_m}\|_{C^0} = o(m^2).\end{aligned}$$

We conclude that $\left\|R_{\beta_m} - R_{\beta_{H_m}}\right\|_{C^0} = o(m^2)$. $\square$

4.3 Expanding the magnetic strength

Let $\omega \in \Omega^2(S^2)$ be a symplectic form, $h : S^2 \to \mathbb{R}$ a function and $x \in S^2$ a critical point of h. $\mathrm{Hess}_h^\omega(x)$ is the Hessian matrix of h at x in the coordinates of a Darboux Chart for ω, i.e. let $\psi : B \to S^2$ be a coordinate chart such that $\psi^*\omega = \omega_{std}$. Then $\mathrm{Hess}_h^\omega(x) := \mathrm{Hess}_{h\circ\psi^{-1}}(\psi(x))$. This clearly depends on the choice of ψ, however, the determinant $\det \mathrm{Hess}_h^\omega(x)$ is independent of the choice of Darboux coordinate, since for any other Darboux chart $\widetilde{\psi} : B \to S^2$, the derivative of the coordinate change $T_{\widetilde{\psi}(x)}(\psi \circ \widetilde{\psi}^{-1})$ lies in the symplectic group $\mathrm{Sp}(2)$. Hence,

$$\begin{aligned}
\det \mathrm{Hess}_h^\omega(x) &= \det \mathrm{Hess}_{h\circ\psi}(\psi^{-1}(x)) \\
&= \det\Big((T_{\widetilde{\psi}(x)}(\psi \circ \widetilde{\psi}^{-1}))^{-1} \,\mathrm{Hess}_{h\circ\psi}(\psi^{-1}(x)) T_{\widetilde{\psi}(x)}(\psi \circ \widetilde{\psi}^{-1})\Big) \\
&= \det \mathrm{Hess}_{h\circ\widetilde{\psi}}(\widetilde{\psi}^{-1}(x))
\end{aligned}$$

By Lemma 4.3, we know that the Reeb vector fields R_{β_m} are tangent to the boundary $\partial\mathbb{A}$. The following lemma will be useful to calculate the Reeb vector fields $R_{\beta_m|\partial\mathbb{A}}$. This will be important to later determine the return map of the global surface of section.

Lemma 4.4. *In the local coordinates of $\mathfrak{d}_{p_\pm} : B \times S^1 \to SS^2$, let $c_\pm := \sqrt{\det \mathrm{Hess}^\sigma_{1/f}(p_\pm)}$, then H_m has the expansion*

$$H_m(x) = 1 - \frac{m^2}{2}\Big(\frac{1}{f(p_\pm)} \pm \frac{c_\pm}{2}(x_1^2 + x_2^2) + o(|x|^2)\Big)$$

at $x = 0$.

Proof. In the coordinates of $\mathfrak{d}_{p_+}$, by expanding $1/f$ at $x = 0$, we get

$$1/f(x) = 1/f(p_+) + \frac{1}{2}x^T \,\mathrm{Hess}_{1/f}(0)x + o(|x|^2). \tag{32}$$

Because p_+ is a local minimum of $1/f$, there is a rotation $A \in SO_2(\mathbb{R})$, such that

$$\mathrm{Hess}_{1/f\circ A}(0) := A^T \,\mathrm{Hess}_{1/f}(0)A = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}, \text{ for some } a, b > 0. \tag{33}$$

Let $s : s^{-1}(B) \to B \subset \mathbb{R}^2$ be the symplectic transformation

$$s : (x_1, x_2) \mapsto (\sqrt{\frac{b}{a}}x_1, \sqrt{\frac{a}{b}}x_2).$$

Then, in the coordinates of $s \circ A \circ D_{p_+}$, the expansion at (32) becomes

$$1/f(x) = 1/f(p_+) + \frac{1}{2}ab(x_1^2 + x_2^2) + o(|x|^2).$$

And $c_+ := \sqrt{\det \mathrm{Hess}_{1/f}(p_+)} = ab$. The same works for p_-, the change of sign results from the eigenvalues of the Hessian matrix at (33) being negative in this case. ◻

Remark 4.5. *The numbers $c_\pm = \sqrt{\det \mathrm{Hess}_{1/f}(p_\pm)}$ do not depend on the choice of Darboux coordinates of $(S^2, f d\lambda_0)$ since the derivatives of the transition functions lie in the symplectic group* $\mathrm{Sp}(2)$.

4.4 The unperturbed Reeb vector field

Because $\beta_{H_m} = \Xi^*\lambda_{H_m} = \frac{1}{H_m}\Xi^*\lambda_0$, we have $\beta_{H_0} = \Xi^*\lambda_0$ and we can write

$$\beta_{H_m} = \frac{1}{H_m}\beta_{H_0} \tag{34}$$

, where we abused the notation and mean $H_m = H_m \circ \Xi$.

Due to Proposition 4.3, the Reeb vector field $R_{\mathring{\beta}_{H_m}}$ extends to a vector field R_β on $\mathbb{A} \times S^1$ satisfying

$$\beta_{H_m}(R_{\beta_{H_m}}) = 1, \quad \iota_{R_{\beta_{H_m}}} d\beta_{H_m} = 0, \quad {\phi^{\beta_{H_m}}}^* \beta_{H_m} = \beta_{H_m}. \tag{35}$$

To see that S is a surface of section for the Reeb flow of λ_{H_m} and to later calculate the first return map, we find a description of $R_{\mathring{\beta}_{H_m}}$ close to the boundaries $\partial\mathbb{A}$ in the coordinates $(r, \theta, s) \in \mathbb{A} \times S^1$.

Proposition 4.6. *Write the Reeb vector field $R_{\beta_{H_m}}$ in the coordinates $(r, \theta, s) \in \mathbb{A} \times S^1$.*

$$R_{\beta_{H_m}} = R^r_{\beta_{H_m}} \partial_r + R^\theta_{\beta_{H_m}} \partial_\theta + R^s_{\beta_{H_m}} \partial_s,$$

for some functions $R^r_{\beta_{H_m}}, R^\theta_{\beta_{H_m}}, R^s_{\beta_{H_m}} : \mathbb{A} \times S^1 \to \mathbb{R}$. Because $R_{\beta_{H_m}}$ is tangent to $\partial\mathbb{A}$ we know that $R^r_{\beta_{H_m}} = 0$ on $\partial\mathbb{A}$.

If $r = 0$ we have:

$$R^\theta_{\beta_{H_m}} = -\frac{m^2 c_+}{2}, \quad R^s_{\beta_{H_m}} = 1 + \frac{m^2 c_+}{2} - \frac{m^2}{2f(p_+)}.$$

Similarly, if $r = 1$:

$$R^\theta_{\beta_{H_m}} = -\frac{m^2 c_-}{2}, \quad R^s_{\beta_{H_m}} = 1 - \frac{m^2 c_-}{2} - \frac{m^2}{2f(p_-)},$$

where $c_\pm = \sqrt{\det \mathrm{Hess}^\sigma_{1/f}(p_\pm)}$, which was defined in Lemma 4.4.

Proof. Because of the equations at (35) and Equation 34, we have $1 = \beta_{H_m}(R_{\beta_{H_m}}) = \frac{1}{H_m}\beta_{H_0}(R_{\beta_{H_m}})$. Therefore,

$$\beta_{H_0}(R_{\beta_{H_m}}) = H_m. \tag{36}$$

Furthermore

$$\begin{aligned} 0 = \iota_{R_{\beta_{H_m}}} d\beta_{H_m} &= \iota_{R_{\beta_{H_m}}}(-\frac{dH_m}{H_m^2}\beta_{H_0} + \frac{1}{H_m} d\beta_{H_0}) \\ \iff 0 &= \iota_{R_{\beta_{H_m}}}(-\frac{dH_m}{H_m}\beta_{H_0} + d\beta_{H_0}). \end{aligned} \tag{37}$$

And

$$
\begin{aligned}
0 &= \iota_{R_{\beta_{H_m}}}(-\frac{dH_m}{H_m}\beta_{H_0} + d\beta_{H_0}) \\
&= -\frac{dH_m(R_{\beta_{H_m}})}{H_m}\beta_{H_0} + \frac{dH_m}{H_m}\beta_{H_0}(R_{\beta_{H_m}}) + \iota_{R_{\beta_{H_m}}} d\beta_{H_0} \\
\text{(By Equation 36)} &= -\frac{dH_m(R_{\beta_{H_m}})}{H_m}\beta_{H_0} + dH_m + \iota_{R_{\beta_{H_m}}} d\beta_{H_0}. \qquad (38)
\end{aligned}
$$

In local coordinates on $\mathcal{N} \times S^1$ we can write

$$
R_{\beta_{H_m}} = R^r_{\beta_{H_m}}\partial_r + R^\theta_{\beta_{H_m}}\partial_\theta + R^s_{\beta_{H_m}}\partial_s,
$$

for some functions $R^r_{\beta_{H_m}}, R^\theta_{\beta_{H_m}}, R^s_{\beta_{H_m}} : \mathbb{A} \times S^1 \to \mathbb{R}$.

At first, assume that r near 0. Because of Lemma 4.4, we know that

$$
H_m = 1 - \frac{m^2}{2f(p_+)} - \frac{m^2c_+}{4}r^2 + m^2o(r^2).
$$

Therefore, keeping in mind that H_m is smooth, we have

$$
dH_m = -\frac{m^2c_+r}{2}dr + m^2o(r). \qquad (39)
$$

Recall that $\beta_{H_m} = \Xi^*\lambda_0$, thus by Equation 25 we have

$$
\iota_{R_{\beta_{H_m}}} d\beta_{H_0} = r(R^r_{\beta_{H_m}}d\theta - R^\theta_{\beta_{H_m}}dr).
$$

Continuing from (38), together with Equation 39, we calculate

$$
\begin{aligned}
0 &= -\frac{dH_m(R_{\beta_{H_m}})}{H_m}\beta_{H_0} + dH_m + \iota_{R_{\beta_{H_m}}} d\beta_{H_0} \\
&= -\frac{-m^2c_+rR^r_{\beta_{H_m}} - 2m^2o(r)}{2H_m}((1+\frac{r}{2}^2)d\theta + ds) - \frac{m^2c_+r}{2}dr + m^2o(r) \qquad (40) \\
&\quad + r(R^r_{\beta_{H_m}}d\theta - R^\theta_{\beta_{H_m}}dr).
\end{aligned}
$$

Dividing by r and reordering results in

$$
\begin{aligned}
0 = &-\frac{-m^2c_+R^r_{\beta_{H_m}} + 2m^2o(1)}{2H_m}((1+\frac{r}{2}^2)d\theta + ds) + R^r_{\beta_{H_m}}d\theta \\
&- (\frac{m^2c_+}{2} + R^\theta_{\beta_{H_m}})dr + m^2o(1).
\end{aligned}
$$

By comparing the basis covector dr, we get

$$
\frac{m^2c_+}{2} + R^\theta_{\beta_{H_m}} + m^2O(r) = 0 \implies R^\theta_{\beta_{H_m}} = -\frac{m^2c_+}{2} - m^2O(r).
$$

Hence, $R^{\theta}_{\beta_{H_m}} = -\frac{m^2 c_+}{2}$ at $r = 0$.

By Equation 25 and Equation 36 we have $R^s_{\beta_{H_m}} + R^{\theta}_{\beta_{H_m}} = H_m$ at $r = 0$. Therefore,

$$R^s_{\beta_{H_m}} = H_m - R^{\theta}_{\beta_{H_m}} = 1 + \frac{m^2 c_+}{2} - \frac{m^2}{2f(p_+)}.$$

This proves the statement for $r = 0$.

Now, assume r is close to 1. Similar to the previous case, by Lemma 4.4, we get

$$H_m = 1 - \frac{m^2}{2f(p_-)} + \frac{m^2 c_-}{4}(1-r)^2 + m^2 o((1-r)^2),$$

$$dH_m = -\frac{m^2 c_-}{2}(1-r)dr + m^2 o(1-r).$$

Doing almost the same calculations as before, using the equations at 26

$$\begin{aligned}
0 &= -\frac{dH_m(R_{\beta_{H_m}})}{H_m}\beta_{H_0} + dH_m + \iota_{R_{\beta_{H_m}}} d\beta_{H_0} \\
&= -\frac{-m^2 c_-(1-r)R^r_{\beta_{H_m}} - 2m^2 o(1-r)}{2H_m}\left(-(1 + \frac{(1-r)}{2}^2)d\theta + ds\right) \\
&\quad - \frac{m^2 c_-}{2}(1-r)dr + m^2 o(1-r) + (1-r)(R_r d\theta - R_\theta dr).
\end{aligned} \tag{41}$$

reordering and dividing by $(1-r)$ results in

$$\begin{aligned}
0 &= -\frac{-m^2 c_- R^r_{\beta_{H_m}} + 2m^2 O(1-r)}{2H_m}\left(-(1 + R_r + \frac{(1-r)}{2}^2)d\theta + ds\right) \\
&\quad - (\frac{m^2 c_-}{2} + R_\theta)dr + m^2 O(r).
\end{aligned}$$

By comparing the covector dr again, we have $R_\theta = -\frac{m^2 c_-}{2}$. Equation 36 and Equation 26 then imply

$$R^s_{\beta_{H_m}} = 1 - \frac{m^2 c_-}{2} - \frac{m^2}{2f(p_-)}.$$

This concludes the proof for the case $r = 1$.

□

Corollary 4.7. *We have* $\left\|R_{\beta_{H_m}} - \partial_s\right\|_{C^0} = o(m)$ *and the embedding* S *is a global surface of section for the Reeb flow of* α_{H_m}.

Proof. Using the equations from Proposition 4.6, on the boundary $\partial\mathbb{A} \times S^1$ we have

$$\begin{aligned}\left\|R_{\beta_{H_m}} - \partial_s\right\|_{C^0} &\leq 3\max\{\left\|R^r_{\beta_{H_m}}\right\|_{C^0}, \left\|R^\theta_{\beta_{H_m}}\right\|_{C^0}, \left\|R^s_{\beta_{H_m}} - 1\right\|_{C^0}\} \\ &\leq 3\max\{0, |\frac{c_\pm m^2}{2}|, |\frac{c_\pm m^2}{2} - \frac{m^2}{2f(p_\pm)}|\} \leq o(m).\end{aligned}$$

On $\mathring{\mathbb{A}} \times S^1$, with help of Lemma 6.5 and Lemma 6.2, we can estimate

$$\left\|R_{\beta_{H_m}} - \partial_s\right\|_{C^0} \leq C\|\alpha_{H_m} - \lambda_0\|_{C^{0,+}} = o(m) \tag{42}$$

for some $C > 0$. Thus, $\left\|R_{\beta_{H_m}} - \partial_s\right\|_{C^0} = o(m)$ on $\mathbb{A} \times S^1$. This indicates that

$$ds(R_{\beta_{H_m}}) = 1 + o(m).$$

Therefore, $\phi^{\beta_{H_m}}$ is transverse to $\mathbb{A} \times \{0\}$ for m small enough. Because Ξ is diffeomorph on $\mathring{\mathbb{A}} \times \{0\}$, the flow $\phi^{\lambda_{H_M}} = \Xi_{|\mathring{\mathbb{A}}\times\{0\}} \circ \phi^{\beta_{H_m}}$ is transversal to $\Xi(\mathring{\mathbb{A}} \times \{0\}) = S(\mathring{A})$, showing that S is global a surface of section for the Reeb flow of λ_{H_m}. □

4.5 The perturbed Reeb vector field

Take again an arbitrary $\alpha_m \in \mathcal{B}_{p_+,p_-}$ and its pullback $\beta_m = \Xi^*\alpha_m$ to $\mathbb{A} \times S^1$.

Corollary 4.8. *We have* $\|R_{\beta_m} - \partial_s\|_{C^0} = o(m)$ *and the embedding* S *is a global surface of section for the Reeb flow of* α_m *for* m *small enough.*

Proof. By Proposition 4.3, $\left\|R_{\beta_m} - R_{\beta_{H_m}}\right\|_{C^0} = o(m^2)$. Together with Corollary 4.7 we have

$$\begin{aligned}\|R_{\beta_m} - \partial_s\| &= \left\|R_{\beta_m} - R_{\beta_{H_m}} + R_{\beta_{H_m}} - \partial_s\right\| \\ &\leq \left\|R_{\beta_m} - R_{\beta_{H_m}}\right\| + \left\|R_{\beta_{H_m}} - \partial_s\right\| = o(m).\end{aligned}$$

Similar to the proof of Corollary 4.7, S is a global a surface of section as a consequence of

$$ds(R_{\beta_m}) = 1 + o(m).$$

□

As for the unperturbed Reeb vector field $R_{\beta_{H_m}}$, we have a similar local description of the perturbed Reeb vector field R_{β_m} on the boundaries of $\mathbb{A} \times S^1$.

Corollary 4.9. *Write the Reeb vector field R_{β_m} in the coordinates $(r, \theta, s) \in \mathbb{A} \times S^1$*

$$R_{\beta_m} = R^r_{\beta_m} \partial_r + R^\theta_{\beta_m} \partial_\theta + R^s_{\beta_m} \partial_s,$$

for some functions $R^r_{\beta_m}, R^\theta_{\beta_m}, R^s_{\beta_m} : \mathbb{A} \times S^1 \to \mathbb{R}$. Because R_{β_m} is tangent to $\partial \mathbb{A}$ we know that $R^r_{\beta_m} = 0$ on $\partial \mathbb{A}$. If $r = 0$, we have:

$$R^\theta_{\beta_m} = -\frac{m^2 c_+}{2} + o(m^2), \quad R^s_{\beta_m} = 1 + \frac{m^2 c_+}{2} - \frac{m^2}{2f(p_+)} + o(m^2).$$

Similarly, if $r = 1$:

$$R^\theta_{\beta_m} = -\frac{m^2 c_-}{2} + o(m^2), \quad R^s_{\beta_m} = 1 - \frac{m^2 c_-}{2} - \frac{m^2}{2f(p_-)} + o(m^2).$$

Here, $c_\pm = \sqrt{\det \mathrm{Hess}^\sigma_{1/f}(p_\pm)}$, which was defined in Lemma 4.4.

Proof. From Proposition 4.3 we know that $\|R_{\beta_m} - R_{\beta_m}\|_{C^0} = o(m^2)$. Together with Proposition 4.6, which gave local description of R_{β_m} on the boundary of $\partial^+\mathbb{A} \times S^1$, we directly deduce

$$\begin{aligned} R^\theta_{\beta_m} &= \frac{m^2 c_+}{2} + o(m^2), \\ R^s_{\beta_m} &= 1 + \frac{m^2 c_+}{2} - \frac{m^2}{2f(p_+)} + o(m^2). \end{aligned}$$

The same argument works for the vector field on $\partial^-\mathbb{A} \times S^1$. □

4.6 The first return map

Knowing that R_{β_m} extends to the boundary, we can define a first return time and first return map that both extend to the boundary as well. The first return time $\tau_m(x) : \mathbb{A} \to \mathbb{R}$ is given by

$$\tau_m(x) = \inf\{\ t > 0 \mid \phi^{\beta_m}_t(x) \in \mathbb{A} \times \{0\}\ \}, \tag{43}$$

and the first return map $P_m : \mathbb{A} \to \mathbb{A}$ is

$$P_m(x) = \phi^{\beta_m}_{\tau_m}(x). \tag{44}$$

In the next Section we will use Proposition 4.9 to calculate the first return time and the first return map on the boundary $\partial\mathbb{A}$. We then compare the first return maps behavior on the two different boundary components $\partial^+\mathbb{A}$ and $\partial^-\mathbb{A}$ to check under what conditions P_m satisfies the boundary twist conditions, which we will discuss in the beginning of the next Section.

5 A Twist Condition for the magnetic flow

In this final section we are going to show that the first return map $P_m : \mathbb{A} \to \mathbb{A}$, which we defined in the previous section, is twist under a suitable condition on f.

As introduction and motivation on why we take great interest in P_m being twist, we briefly introduce the Poincaré-Birkhoff theorem and the consequences it carries for the Reeb flow corresponding to P_m.

5.1 The Poincaré-Birkhoff theorem

This subsection is based on [Le Calvez, 2011] Chapter 2.

Let $F : \mathbb{A} \to \mathbb{A}$ be a homeomorphism of the annulus $\mathbb{A} = [0,1] \times S^1$, that leaves the boundaries $\partial\mathbb{A}$ invariant. The universal cover is given by $\tilde{\mathbb{A}} := [0,1] \times \mathbb{R}$. Consider the translation $T : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}, (r,\theta) \mapsto (r, \theta + 2\pi)$ and the lift $\tilde{F} : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ of F to $\tilde{\mathbb{A}}$. Let $\tilde{F}_1 : \tilde{\mathbb{A}} \to [0,1]$ and $\tilde{F}_2 : \tilde{\mathbb{A}} \to \mathbb{R}$ functions such that $\tilde{F} = (\tilde{F}_1, \tilde{F}_2)$. We state the Poincaré-Birkhoff theorem, based on how it was formulated at [Le Calvez, 2011] Theorem 1.

Theorem 5.1. *If the following conditions hold:*

(i) $\tilde{F}^* dr \wedge d\theta = dr \wedge d\theta$.

(ii) For every $\theta \in \mathbb{R}$, *one has* $\tilde{F}_2(0,\theta) < \theta < \tilde{F}_2(1,\theta)$,

then $\tilde{F}$ *has at least two fixed points with different* T *orbit, i.e. if* $z_{1,2} \in \tilde{\mathbb{A}}$ *are the two fixed points, then there is no* $k \in \mathbb{Z}$ *such that* $T^k(z_1) = z_2$.

The following discussion about why the theorem actually implies that an infinite number of fixed points exist was taken from [Le Calvez, 2011] Chapter 2.

Assume that $z \in \mathbb{A}$ is a fixed point of F^q for some $q \in \mathbb{Z}_{\geq 1}$. The fixed point z is also called a **periodic point** of F and the smallest $q \in \mathbb{Z}_{\geq 1}$ such that $z = F^q(z)$ is called the period of z. Consider a lift $\tilde{z} \in \tilde{\mathbb{A}}$ of z, then there exists a $p \in \mathbb{Z}$ such that $\tilde{F}^q(\tilde{z}) = T^p(\tilde{z})$.

If we take a different lift $\hat{z} := T^r(\tilde{z})$ of z, then $\tilde{F}^q(\hat{z}) = T^r \circ \tilde{F}^q(\tilde{z}) = T^{r+p}(\tilde{z}) = T^p(\hat{z})$. Therefore, p is independent of the choice of the lift of z. The number p/q is called the **rotation number** of z. By choosing another lift $T^r \circ \tilde{F}$ of F we get

$$(T^r \circ \tilde{F})^q = T^{rq} \circ \tilde{F}^q = T^{qr+p},$$

with rotation number $p/q + r$ instead of p/q.

We state the corollary about the existence of infinitely many fixed points. This is a slightly adapted version of [Le Calvez, 2011] Corollary 2.

Corollary 5.2. *Let F be a homeomorphism of $\mathbb{A}$ leaving the boundaries $\partial\mathbb{A}$ invariant, and $\tilde{F} : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ the lift of F to $\tilde{\mathbb{A}}$. Assume that $\tilde{F}$ has the following properties:*

(i) $\tilde{F}^* dr \wedge d\theta = dr \wedge d\theta$.

(ii) There exist two numbers $\rho_0 < \rho_1 \in \mathbb{R}$ such that $\tilde{F}_2(0,\theta) \leq \theta + \rho_0 \leq \theta + \rho_1 \leq \tilde{F}_2(1,\theta)$.

Then, every reduced rational number $\rho = p/q$, that lies in the open interval (ρ_0, ρ_1), is the rotation number of a periodic orbit of F with period q.

Proof. Let $\rho = p/q$ be some reduced rational number in the open interval bounded by ρ_0 and ρ_1. The map $\tilde{F}^q \circ T^{-p}$ is another lift of F. Let $p : \tilde{\mathbb{A}} \to \mathbb{R}$ the projections onto the second component. For every $\theta \in \mathbb{R}$ we have

$$p \circ \tilde{F}^q \circ T^{-p}(0,\theta) \leq \theta - p + q\rho_0 < \theta$$
$$p \circ \tilde{F}^q \circ T^{-p}(1,\theta) \geq \theta - p + q\rho_1 > \theta.$$

Therefore $\tilde{F}^q \circ T^{-p}$ satisfies the requirements of the Poincaré-Birkhoff theorem [5.1], which ensures the existence of a fixed point $z \in \tilde{\mathbb{A}}$ of $\tilde{F}^q \circ T^{-p}$ implying $\tilde{F}^q(z) = T^p(z)$. Therefore z is a periodic point of F with rotation number p/q. □

This immediately implies that the homeomorphism F in Corollary 5.2 has an infinite number of periodic points.

Remark 5.3. *The second requirement (ii) of Corollary 5.2 is also called the **boundary twist condition**.*

Remark 5.4. *Corollary 5.2 is actually still valid if $\tilde{F}$ satisfies the requirements of Theorem 5.1 instead. For details see Chapter 2 in [Le Calvez, 2011].*

Remark 5.5. *Homeomorphisms of the annulus $\mathbb{A}$ that satisfy the requirements of the Poincarè-Birkhoff Theorem 5.1 or its corollary, are also referred to as **area preserving twist maps**. For convenience we sometimes simply refer to such maps as being twist.*

Going back to the first return map P_m. We go through all the requirements of an area preserving twist map. At first, observe that P_m is a diffeomorphism since it was defined via the flow ϕ^{β_m}. Furthermore, by Equation 28 we have ${\phi^{\beta_m}}^* d\beta_m = d\beta_m$ and $\iota_{R_{\beta_m}} = 0$. Because the flow ϕ^{β_m} of R_{β_m} is transverse to $\mathbb{A} \subset \mathbb{A} \times \{0\}$ and $d\beta_m$ only having a 1-dimensional kernel distribution, the restriction $d\beta_{m|\mathbb{A}}$ is an area form on $\mathbb{A}$. This means that P_m is an area preserving diffeomorphism.

The rest of this section will deal with the remaining boundary twist condition (ii) of Corollary 5.2.

5.2 The boundary twist condition of the first return map

Because of Proposition 4.3, the vector field R_{β_m} is tangent to $\partial\mathbb{A}\times S^1$ and on $\partial\mathbb{A}\times S^1$ it can be written in the coordinates $(\theta, s)\in\partial\mathbb{A}\times S^1$.

$$R_{\beta_m|\partial\mathbb{A}\times S^1} = R^\theta_{\beta_m}\partial_\theta + R^s_{\beta_m}\partial_s,$$

for functions $R^\theta_{\beta_m}, R^s_{\beta_m} : \partial\mathbb{A}\times S^1 \to \mathbb{R}$.

By Corollary 4.9, on $\partial^+\mathbb{A}$ we have

$$R^\theta_{\beta_m} = -\frac{m^2c_+}{2} + o(m^2), \quad R^s_{\beta_m} = 1 + \frac{m^2c_+}{2} - \frac{m^2}{2f(p_+)} + o(m^2).$$

On $\partial^-\mathbb{A}$:

$$R^\theta_{\beta_m} = -\frac{m^2c_-}{2} + o(m^2), \quad R^s_{\beta_m} = 1 - \frac{m^2c_-}{2} - \frac{m^2}{2f(p_-)} + o(m^2).$$

Recall that $c_\pm = \sqrt{\det \mathrm{Hess}^\sigma_{1/f}(p_\pm)}$.

Let $\tilde{\phi}^{\beta_m}$ be the lift of the Reeb flow ϕ^{β_m} to $\tilde{\mathbb{A}}$ such that $\tilde{\phi}^{\beta_m}_0(0) = 0$. By integrating the Reeb vector field along the boundaries for initial values $(\theta_0, s_0)\in\partial^+\tilde{\mathbb{A}}\times\mathbb{R}$, we get

$$\begin{aligned}&\tilde{\phi}^{\beta_m}_{t\,|\partial^+\tilde{\mathbb{A}}}(\theta_0, s_0)\\ &= (-t(\frac{m^2c_+}{2} + o(m^2)) + \theta_0,\ t(1+\frac{m^2c_+}{2} - \frac{m^2}{2f(p_+)} + o(m^2)) + s_0).\end{aligned} \tag{45}$$

If $(\theta_0, s_0)\in\partial^-\tilde{\mathbb{A}}\times\mathbb{R}$, we have

$$\begin{aligned}&\tilde{\phi}^{\beta_m}_{t\,|\partial^-\tilde{\mathbb{A}}}(\theta_0, s_0)\\ &= (-t(\frac{m^2c_-}{2} + o(m^2)) + \theta_0,\ t(1-\frac{m^2c_-}{2} - \frac{m^2}{2f(p_-)} + o(m^2)) + s_0),\end{aligned} \tag{46}$$

We can now determine the first return time and first return map on the boundary.

Let $\theta_0\in\partial\tilde{\mathbb{A}}$. By definition, the first return time for θ_0 is the smallest $\tau_0 > 0$, such that $\tilde{\phi}^{\beta_m}_{\tau_0\,|\partial\tilde{\mathbb{A}}}(\theta_0, 0) = (\theta_1, 2\pi)$, for some $\theta_1\in\partial\mathbb{A}$. Assuming that $\theta_0\in\partial^+\tilde{\mathbb{A}}$ and using the local description 45 of the flow, the return time τ_0 is determined by

$$\begin{aligned}2\pi &= \tau_0(1+\frac{m^2c_+}{2} - \frac{m^2}{2f(p_+)} + o(m^2))\\ \iff \tau_0 &= \frac{4\pi}{2+m^2c_+ - m^2/f(p_+) + o(m^2)},\end{aligned} \tag{47}$$

and θ_1 is given by

$$\begin{aligned}\theta_1 &= -\tau_0(\frac{m^2c_+}{2} + o(m^2)) + \theta_0\\ &= \frac{2\pi m^2c_+}{2+m^2c_+ - m^2/f(p_+) + o(m^2)} + o(m^2) + \theta_0\end{aligned} \tag{48}$$

Define the restrictions $\tau_m^+, \tau_m^- := \tau_{m|\partial^+\mathbb{A}}, \tau_{m|\partial^-\mathbb{A}}$ of the first return time of β_m defined at (43).

Then, by the calculations above,

$$\tau_m^+(\theta) = \frac{4\pi}{2 + m^2 c_+ - m^2/f(p_+) + o(m^2)}.$$

and similarly

$$\tau_m^-(\theta) = \frac{4\pi}{2 - m^2 c_- - m^2/f(p_-) + o(m^2)}.$$

Let $\tilde{P}_m : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ be the lift of the first return map P_m of β_m. By Equation 48, the restrictions $\tilde{P}_m^+, \tilde{P}_m^- := \tilde{P}_{m|\partial^+\mathbb{A}}, \tilde{P}_{m|\partial^-\mathbb{A}}$ can be described as follows:

$$\begin{aligned} \tilde{P}_m^+(\theta) &= \frac{2\pi m^2 c_+ + o(m^2)}{2 + m^2 c_+ - m^2/f(p_+) + o(m^2)} + \theta \\ &= \pi m^2 c_+ + o(m^2) + \theta \\ \\ \tilde{P}_m^-(\theta) &= \frac{2\pi m^2 c_- + o(m^2)}{2 - m^2 c_- - m^2/f(p_-) + o(m^2)} + \theta \\ &= \pi m^2 c_- + o(m^2) + \theta. \end{aligned} \tag{49}$$

We can now state a condition for the first return time being twist.

Proposition 5.6. *If $c_- \neq c_+$, where $c_- := \sqrt{\det \mathrm{Hess}^\sigma_{1/f}(p_-)}$ and $c_+ := \sqrt{\det \mathrm{Hess}^\sigma_{1/f}(p_+)}$, then the first return map $P_m : \mathbb{A} \to \mathbb{A}$ of the Reeb flow ϕ^{β_m} is an area preserving twist map for m small enough.*

Proof. We already mentioned why P_m is area preserving and a diffeomorphism and only need to show that P_m satisfies the boundary twist condition (ii) in Corollary 5.2.

Because of the equations at (49), for the lift $\tilde{P}_m$ we have

$$\tilde{P}_m(0,\theta) = (0, \pi m^2 c_+ + o(m^2) + \theta), \qquad \tilde{P}_m(1,\theta) = (1, \pi m^2 c_- + o(m^2) + \theta).$$

Set

$$\begin{aligned} \rho_0 &:= \frac{3}{4}\pi m^2 c_- + \frac{1}{4}\pi m^2 c_+, \\ \rho_1 &:= \frac{3}{4}\pi m^2 c_+ + \frac{1}{4}\pi m^2 c_-. \end{aligned}$$

Let $p_2 : \tilde{\mathbb{A}} \to \mathbb{R}, (r,\theta) \mapsto \theta$. W.l.o.g. $c_- < c_+$, then $\rho_0 < \rho_1$ and for m small enough we have

$$\begin{aligned} p_2 \circ \tilde{P}_m(1,\theta) &= \pi m^2 c_- + o(m^2) + \theta \\ &< \rho_0 + \theta < \rho_1 + \theta \\ &< \pi m^2 c_+ + o(m^2) + \theta = p_2 \circ \tilde{P}_m(0,\theta). \end{aligned}$$

□

As an immediate consequence, it follows that the Reeb flow of β_m has infinitely many periodic orbits if m is small enough.

5.3 Reformulated conditions on the Hessian

We reformulate the condition on the Hessian in Proposition 5.6. We compute in coordinates around $p_\pm$:

$$\begin{aligned}
\operatorname{Hess}^{\sigma}_{1/f}(p_\pm) &= (\frac{\partial^2}{\partial x_i x_j}|_{p_\pm}(1/f))_{i,j} \\
&= (\frac{2}{f^3}(p_\pm)\frac{\partial}{\partial x_i}|_{p_\pm} f - \frac{1}{f^2}(p_\pm)\frac{\partial^2}{\partial x_i \partial x_j}|_{p_\pm}(f))_{i,j} \\
&= -\frac{1}{f^2}(p_\pm)(\frac{\partial^2}{\partial x_i \partial x_j}|_{p_\pm}(f))_{i,j} \quad (\frac{\partial}{\partial x_i}|_{p_\pm} f = 0, \text{ since is } p_\pm \text{ is critical}) \\
&= -\frac{1}{f^2(p_\pm)} \operatorname{Hess}^{\sigma}_f(p_\pm).
\end{aligned}$$

Consequently

$$\sqrt{\det \operatorname{Hess}^{\sigma}_{1/f}(p_\pm)} = \frac{1}{f^2(p_\pm)} \sqrt{\det \operatorname{Hess}^{\sigma}_f(p_\pm)}. \tag{50}$$

Furthermore, the Hessian can be calculated according to Darboux coordinates of the volume form vol_g which is induced by the Riemannian metric g on S^2. Speaking about Darboux coordinates makes sense since vol_g is symplectic on S^2. In other words, let $\tilde{\psi} : V \subset \mathbb{R}^2 \to U \subset S^2$ be a diffeomorphism such that $\tilde{\psi}^* \mathrm{vol}_g = \omega_{std}$, then we are interested in the Hessian of $f \circ \tilde{\psi}$. Observe that such diffeomorphisms are exactly those whose induced basis vectors $\partial_{x_1}, \partial_{x_2}$ on $TU \subset TS^2$ form an orthonormal frame with respect to the metric g.

We will use the notation

$$\operatorname{Hess}^g := \operatorname{Hess}^{\mathrm{vol}_g}$$

for Hessians in such coordinates.

Assume that $p_\pm \in U$ and $\psi(0) = p_\pm$. Since $\sigma = f\,\mathrm{vol}_g$, we have $T_0\mathfrak{d}_\pm = 1/f(p_\pm)T_0\tilde{\psi}$, implying that $f(p_\pm)\mathbb{1} = T_0(\mathfrak{d}_\pm^{-1} \circ \tilde{\psi})$ which enables us to calculate

$$\begin{aligned}
\operatorname{Hess}^g_f(p_\pm) &= \operatorname{Hess}_{f\circ\tilde{\psi}}(0) \\
&= T_0^T(\mathfrak{d}_\pm^{-1} \circ \tilde{\psi}) \operatorname{Hess}_{f\circ\mathfrak{d}_\pm}(0) T_0(\mathfrak{d}_\pm^{-1} \circ \tilde{\psi}) \\
&= f^2(p_\pm) \operatorname{Hess}_{f\circ\mathfrak{d}_\pm^{-1}}(0) \\
&= f^2(p_\pm) \operatorname{Hess}^{\sigma}_f(p_\pm)
\end{aligned}$$

Hence,

$$\sqrt{\det \operatorname{Hess}^{\sigma}_f(p_\pm)} = \frac{1}{f}(p_\pm)\sqrt{\det \operatorname{Hess}^g_f(p_\pm)}.$$

Together with Equation 50 we get

$$\sqrt{\det \operatorname{Hess}^{\sigma}_{1/f}(p_\pm)} = \frac{1}{f^3}(p_\pm)\sqrt{\det \operatorname{Hess}^g_f(p_\pm)}.$$

5.4 Consequences for the magnetic flow

The magnetic flow was defined as the Hamiltonian flow ϕ^{X_m} of the symplectic twisted bundle $(TS^2, \omega = d\lambda - \pi^*\sigma)$. Recall that $\sigma = f\,\mathrm{vol}_g$. We have seen that the dynamics of the magnetic flow on the level sets Σ_m can be studied through the Reeb vector field of the contact manifold (SS^2, λ_m), where $d\lambda_m = \omega_m = m d\lambda - \pi^*\sigma$.

By Lemma 3.1, there exists a diffeomorphism $F : SS^2 \to SS^2$ such that $F^*\lambda_m = \frac{1}{H_m}\lambda_0 + o(m^2)$ for m small enough.

Set $\hat{\lambda}_m := F^*\lambda_m$ then, $\hat{\lambda}_m \in \mathcal{B}$ (See (4)). In other words, $\hat{\lambda}_m$ is $C^{1,+}$-close to the unperturbed form $\lambda_{H_m} = \frac{1}{H_m}\lambda_0$.

Assuming that p_+, p_- are non-degenerate maximum and minimum points respectively. By choosing suitable spherical coordinates we can assume that p_+ is the North-Pole and p_- is the South-Pole. In Subsection 3.4 about weakly normalized forms, we saw that there is a diffeomorphism Ψ_m such that ${\Psi_m}^*\hat{\lambda}_m$ is weakly normalized (See Definition 3.4). Set $\lambda_{\Psi_m} := {\Psi_m}^*\hat{\lambda}_m$

In Section 4 we constructed a global surface of section $S : \mathbb{A} \to SS^2$ for the Reeb flow of the unperturbed form λ_{H_m}. By Corollary 4.8, we know that S is a global section of surface for the Reeb flow $\phi^{\lambda_{\Psi_m}}$ of λ_{Ψ_m} for m small enough.

With the help of Corollary 4.9, it's possible to express the first return time and first return map, corresponding to $\phi^{\lambda_{\Psi_m}}$ and its surface of section, in local coordinates on the boundary. In this last section we used the description in local coordinates to see that the first return map is twist for m small enough, if $\frac{1}{f^3}(p_+)\sqrt{\det \mathrm{Hess}^g_f(p_+)} \neq \frac{1}{f^3}(p_-)\sqrt{\det \mathrm{Hess}^g_f(p_-)}$.

Then, Corollary 5.2 implies that the Reeb flow $\phi^{\lambda_{\Psi_m}}$ has infinitely many periodic orbits. As the dynamics of the Reeb flow is the same with that of the magnetic flow ϕ^{X^m}, restricted on the level set Σ_m, the magnetic flow has an infinite number of periodic orbits if m is small enough. In particular the magnetic flow has infinitely many periodic orbits in general (Without specifying a kinetic energy level set). We summarize these results in the following theorem.

Theorem 5.7. *Let (S^2, g, σ) be a magnetic system such that its magnetic strength f is positive everywhere. Assume that f has a minimum point p_- and a maximum point p_+. If*

$$\frac{1}{f^3}(p_+)\sqrt{\det \mathrm{Hess}^g_f(p_+)} \neq \frac{1}{f^3}(p_-)\sqrt{\det \mathrm{Hess}^g_f(p_-)},$$

where Hess^g_f is the Hessian in orthonormal coordinates according to g, then the magnetic flow has an infinite number of periodic orbits with speed m, for every m small enough.

6 Appendix

6.1 C^k norms and estimates

For the uniform C^k norms used in this thesis we refer to the appendix A of [Benedetti and Kang, 2018]. We only summarize a few lemmas that are important for us.

Definition 6.1. *Let $h \in \{1,2\}$, $k \in \{0,1\}$, and define the numbers $B_{h,k}(\|T\psi\|)$*

$$\begin{aligned} B_{1,0}(\|T\psi\|) &= \|T\psi\|_{C^0}, & B_{1,1}(\|T\psi\|) &= \|T\psi\|_{C^0} + \|T\psi\|_{C^1}, \\ B_{2,0}(\|T\psi\|) &= \|T\psi\|^2_{C^0}, & B_{2,1}(\|T\psi\|) &= \|T\psi\|^2_{C^0} + \|T\psi\|_{C^0}\|T\psi\|_{C^1}. \end{aligned}$$

Lemma A.1 in [Benedetti and Kang, 2018] states the following

Lemma 6.2. *For any $\vartheta \in \Omega^h(M_1)$ with $h \in \{1,2\}$ and any diffeomorphism $\psi : M_0 \to M_1$ we have*

$$\|\psi^*\vartheta\|_{C^k} \leq B_{h,k}(\|T\psi\|)\|\vartheta\|_{C^1}.$$

Remark 6.3. *This statement generalizes to C^k norms and differential forms of higher degree. For more details see lemma A.1 in [Benedetti and Kang, 2018].*

Lemma A.4 in [Benedetti and Kang, 2018] states the following

Lemma 6.4. *Let $B \subset R^2$ be a closed ball. For every $k \in \mathbb{N}$ and $\delta_0 > 0$, there exists $\delta_1 > 0$ such that, if $\psi_1, \psi_2 : B \times S^1 \to B \times S^1$ are smooth maps and $h \in \mathbb{N}$, then*

$$\|\psi_2 - \psi_1\|_{C^{k+1}} \leq \delta_1 \implies \|\psi_2^*\eta - \psi_1^*\eta\|_{C^k} \leq \delta_0\|\eta\|_{C^{k+1}}.$$

The following lemma is taken from Lemma A.6 in [Benedetti and Kang, 2018] and is extremely useful to estimate the C^k distance between Reeb vector fields.

Lemma 6.5. *Let $k \in \mathbb{N}$ and let M_0 be compact manifold of dimension $2n+1$ with contact form α_0. There exists a constant $A_k > 0$ such that, if α is a one-form on M_0 with $\|\alpha - \alpha_0\|_{C^{0,+}} < \delta$, then α is a contact form and there holds*

$$\|R_\alpha - R_{\alpha_0}\|_{C^k} \leq A_k\|\alpha - \alpha_0\|_{C^{k,+}}.$$

Proof. See Lemma A.6 in [Benedetti and Kang, 2018]. □

6.2 Vanishing functions as matrix valued functions

Lemma 6.6. *Let $U \subset \mathbb{R}^2$ an open neigbourhood of 0, $f : U \to \mathbb{R}^2$ a smooth map such that $f(0) = 0$. Then there exists a matrix valued function $W : U \to \mathbb{R}^{2\times 2}$ such that $f(x) = W_x x$ and $\|W\|_{C^0} < \|f\|_{C^1}$.*

Proof. Write $(f_1(x), f_2(x)) = f(x) \in \mathbb{R}^2$, $(x_1, x_2) = x$ and define

$$W_x := \begin{pmatrix} \int_0^1 \partial_1 f_1(x_1 u, x_2 u) du & \int_0^1 \partial_2 f_1(x_1 u, x_2 u) du \\ \int_0^1 \partial_1 f_2(x_1 u, x_2 u) du & \int_0^1 \partial_2 f_2(x_1 u, x_2 u) du \end{pmatrix}$$

and check that $W_x x = f(x)$. From the definition it follows that $\|W\|_{C^0} \leq \|f\|_{C^1}$. $\square$

7 References

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