Fakultät für Mathematik und Informatik
Ruprecht-Karls-Universität Heidelberg

Masterarbeit

Spektrale Netze und Fock-Goncharov Koordinaten

Name:	Evgenii Rogozinnikov
Matrikelnummer:	3263967
E-Mail-Adresse:	erogozinnikov@gmail.com
Betreuer:	Prof. Dr. Anna Wienhard
Datum der Abgabe:	14.09.2016

Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den 14.09.2016 ....

Zusammenfassung

Die Theorie der spektralen Netze wurde von D. Gaiotto, G. W. Moore and A. Neitzke [6–9] während ihrer Forschung über supersymmetrische Feldtheorie in der Physik entwickelt. Diese Konstruktion ist aber auch vom großen Interesse für die Differentialgeometrie, insbesondere für die Theorie der geometrischen Strukturen auf Flächen.

In dieser Masterarbeit wird die Konstruktion von Darstellungen der Fundamentalgruppe einer Fläche in eine Matrix-Lie-Gruppe mit Hilfe der Nicht-Abelisierungsabbildung spektraler Netze diskutiert. Insbesondere wird der Fall der kleinen spektralen Netze untersucht. Kleine spektrale Netze von Rang 2 und 3 werden besonders ausführlich studiert. Es wird der Zusammenhang zwischen spektralen Netzen und projektiver Geometrie analysiert. Wir zeigen, dass ein flacher Zusammenhang und eine Triangulation der Fläche uns die Familie der projektiven Invarianten liefern, und zwar Doppelverhältnisse für spektrale Netze von Rang 2 und Fock-Goncharov Koordinaten [4] für spektrale Netze von Rang 3. Wir untersuchen, inwiefern diese projektiven Invarianten den flachen Zusammenhang auf der Fläche bestimmen, und zeigen, dass spektrale Netze Koordinaten auf der Charaktervarietät der Fundamentalgruppe einer Fläche mit Werten in einer Matrix-Lie-Gruppe liefern. Insbesondere stimmen diese Koordinaten mit Doppelverhältnissen im Fall der spektralen Netze von Rang 2 und mit Fock-Goncharov Koordinaten in einigen Fällen von spektralen Netzen von Rang 3 überein. Zum Schluss führen wir andere Koordinaten ein, die ähnlich zu Fock-Goncharov Koordinaten sind, aber natürlicher in unserem Fall sind, weil sie mit den Homotopiekonstanten von geschlossenen Kurven auf der Fläche übereinstimmen. Wir untersuchen auch, wie sich diese Koordinaten ändern, wenn wir eine andere Triangulation der Fläche wählen.

Abstract

The theory of spectral networks was developed by D. Gaiotto, G. W. Moore and A. Neitzke [6–9] during their research of the theory of supersymmetry in physics. But this construction is also of interest in differential geometry, especially for the theory of geometric structures on surfaces.

In the present master thesis the construction of representations of the fundamental group of a surface in a matrix Lie group using the non abelianisation map and spectral networks is discussed. Especially, the case of small spectral networks is investigated. Small spectral networks of rank 2 and 3 are discussed in detail and the connection between spectral networks and projective geometry is analyzed. We show that a flat connection and a triangulation of the surface define a collection of projective invariants, namely, cross ratios for spectral networks of rank 2 and Fock-Goncharov coordinates [4] for spectral networks of rank 3. We explore to what extent these projective invariants define a flat connection on the surface and show that spectral networks yield coordinates on the character variety of the fundamental group of a surface with values in a matrix Lie group. In particular, these coordinates correspond to cross ratios in the case of spectral networks of rank 2 and to Fock-Goncharov coordinates in some cases of small spectral networks of rank 3. Finally, we define other coordinates, which are similar to Fock-Goncharov coordinates, but are more natural in our case because they agree with homotopy constants of closed curves on the surface. We also investigate how these coordinates change if we choose another triangulation of the surface.

Acknowledgements

I am using this opportunity to express my gratitude to my research supervisor Frau Professor Dr. Anna Wienhard for introducing me to the topic, motivation as well for the continuous support of my master's study.

I also show my gratitude to Herr Dr. Daniele Alessandrini for the useful comments, remarks and engagement through the learning process of this master's thesis. His guidance helped me in all the time of research and writing of this thesis.

Without their assistance and dedicated involvement in every step throughout the process, this master's thesis would have never been accomplished. I thank you very much for your support and understanding over these past two years. I could not have imagined having a better supervisor and mentor for my master's studies.

I am also very grateful to all of the Differential Geometry Research Group members and Faculty of Mathematics and Computer Science of the University of Heidelberg for their help and support.

Finally, I must express my very profound gratitude to my mother and to my spouse for providing me with unfailing support, patience and continuous encouragement throughout my years of study and through the process of researching and writing this thesis. This accomplishment would not have been possible without them. Thank you.

Evgenii Rogozinnikov

Contents

1 Introduction

1.1 Basic notions

We consider a closed orientable surface $\bar{S}$ of genus g with a finite collection $P = \{s_1, ..., s_n\},\ n \in \mathbb{N}$ of marked points. We also consider the surface $S = \bar{S} \setminus P$ and assume that the Euler characteristic of S is negative. We name the points in P the *punctures of* S. For each point $p \in S$ and $Q \subseteq P$ there is the natural group homomorphism $r_{Q,p}\colon \pi_1(S, p) \to \pi_1(S \cup Q, p)$ of fundamental groups of S and $S \cup Q$ with respect to the point $p \in S$. This homomorphism is surjective, because each curve on $\bar{S}$ can be homotopically deformed to the curve which does not contain any punctures.

Definition 1.1. *The closed curve $\gamma\colon [0,1] \to S$ with $\gamma(0) = \gamma(1) = p$ is called* ***peripheral*** *if there exist $s \in P$ such that $r_{\{s\},p}([\gamma])$ is the identity of $\pi_1(S \cup \{s\}, p)$.*

Remark 1.2. *Obviously, the property "to be peripheral" for a closed curve does not depend on the choice of base point and is an invariant by free homotopy in S.*

Further, we consider a Lie group G which is one of the following groups $GL(k,K)$, $SL(k,K)$ or $PGL(k,K)$ where K is the field $\mathbb{R}$ or $\mathbb{C}$, $k \in \mathbb{N}$. We denote by $\mathrm{Hom}^*(\pi_1(S), G)$ the space of all completely reducible representations from the fundamental group $\pi_1(S)$ of the surface S into the Lie group G. The Lie group G acts on $\mathrm{Hom}^*(\pi_1(S), G)$ by conjugation.

Definition 1.3. *The quotient space*

$$X(\pi_1(S), G) := \mathrm{Hom}^*(\pi_1(S), G)/G$$

is called ***character variety****.*

Remark 1.4. *We can define a topology on $X(\pi_1(S), G)$ in a following way. We choose $p \in S$ and some generators $[\gamma_1], ..., [\gamma_l]$ of $\pi_1(S, p)$. We consider a following map:*

$$\begin{array}{ccc} \mathrm{Hom}^*(\pi_1(S,p), G) & \to & G^l \\ \rho & \mapsto & (\rho([\gamma_1], ..., \rho([\gamma_l])) \end{array}$$

This map is injective, and we can use this map to define the topology on $\mathrm{Hom}^*(\pi_1(S,p),G)$ *as a subspace topology*

The topology on $X(\pi_1(S,p),G)$ *is defined as the quotient topology by the action of* G. *One can prove that this topology does not depend on the choices of* $p \in S$ *and of generators* $[\gamma_1],...,[\gamma_l]$. *So we get a topology on* $X(\pi_1(S),G)$.

We want to study $X(\pi_1(S),G)$ using the theory of flat connections on vector bundles.

We consider a vector bundle $\pi\colon E \to S$ over the surface S whose fiber E_p for each $p \in S$ is isomorphic to $\mathbb{C}^k$ with the isomorphism $\theta_p\colon E_p \to \mathbb{C}^k$. Further, we will also write only E for a vector bundle if the surface S and the projection map π is fixed and if this does not cause confusion.

We consider a connection ∇ on the vector bundle E. We denote the vector bundle E with a connection ∇ by (E,∇).

Definition 1.5. *A* ***flat bundle*** *is a vector bundle with a flat connection.*

For the curve $\gamma\colon [0,1] \to S$ we denote $T_\gamma\colon E_{\gamma(0)} \to E_{\gamma(1)}$ the parallel transport operator along γ.

Definition 1.6. *A subbundle* E' *over* S *of the vector bundle* E *over* S *is called* ***parallel*** *if for each* $x \in E'$ *and for each curve* $\gamma\colon [0,1] \to S$ *such that* $\gamma(0) = \pi(x)$

$$T_\gamma(x) \in E'.$$

Definition 1.7. *A vector bundle* E *over* S *is called* ***completely reducible*** *if for each parallel subbundle* E' *of* E *there is a parallel subbundle* E'' *over* S *such that* $E = E' \oplus E''$.

Proposition 1.8. *Let* S *a smooth manifold. There is a bijection* Ψ *between the set* $\mathrm{Hom}(\pi_1(S),GL(k,\mathbb{C}))/GL(k,\mathbb{C})$ *of all representations from the fundamental group* $\pi_1(S)$ *of* S *into* $GL(k,\mathbb{C})$ *modulo conjugation by* $GL(k,\mathbb{C})$ *and the set of all isomorphism classes of flat vector bundles over* S *of rank* k.

The restriction of Ψ *on the character variety* $X(\pi_1(S),GL(k,\mathbb{C}))$ *yield the bijection between* $X(\pi_1(S),GL(k,\mathbb{C}))$ *and the set of all isomorphic classes of flat completely reducible vector bundles over* S *of rank* k.

Proof. First, we fix the base point $p \in S$. For each loop $\gamma\colon [0,1] \to S$, $\gamma(0) = \gamma(1) = p$ the parallel transport T_γ is an element of $GL(k,\mathbb{C})$. Because the connection ∇ is flat, T_γ only depends on the homotopy class $[\gamma] \in \pi_1(S,p)$. So we can define a representation $\rho_{p,\nabla}\colon \pi_1(S,p) \to GL(k,\mathbb{C})$ as

$$\rho_{p,\nabla}([\gamma]) = \theta_p \circ T_\gamma \circ \theta_p^{-1}$$

for all $[\gamma] \in \pi_1(S)$.

If we consider another base point p', then the representation changes by conjugation with $\theta_{p'} T_\delta \theta_p^{-1} \in GL(k,\mathbb{C})$, where $\delta\colon [0,1] \to S$ a curve such that $\delta(0) = p$, $\delta(1) = p'$.

If we consider another vector bundle $\tilde{\pi}\colon \tilde{E} \to S$ bundle, which is isomorphic to E with an bundle isomorphism $L\colon E \to E'$, then the representation changes by conjugation with $\tilde{\theta}_p L_p \theta_p^{-1} \in GL(k,\mathbb{C})$, where $\tilde{\theta}_p\colon \tilde{E}_p \to \mathbb{C}^k$ a linear isomorphism.

If we consider another linear isomorphism $\theta'_p\colon E_p \to \mathbb{C}^k$, then the representation changes by conjugation with $\theta'_p \theta_p^{-1} \in GL(k,\mathbb{C})$.

So we have a well-defined map which take an isomorphic class of flat vector bundle over S and give us the element of $\mathrm{Hom}(\pi_1(S), GL(k,\mathbb{C}))/GL(k,\mathbb{C})$.

In particular, let E be a completely reducible. Each invariant subspace $V \subseteq \mathbb{C}^k$ of $\rho_{p,\nabla}$ generate by parallel transport along all curves on S the parallel subbundle E' of E. Therefore, there is a subbundle E'' of E such that $E = E' \oplus E''$. In particular, $E_p = E'_p \oplus E''_p$. Therefore, $\theta_p(E''_p)$ is complementary invariant spaces of $\rho_{p,\nabla}$ for V. So is $\rho_{p,\nabla}$ completely reducible representation.

If E is not completely reducible, then we have a parallel subbundle E' of E, which does not have a complementary parallel subbundle. If the corresponding representation $\rho_{p,\nabla}$ is completely reducible, then the set $\theta_p(E'_p)$ have a complementary invariant subspace $V \subseteq C^k$. So we can by $E''_p = \theta_p^{-1}(V)$ the parallel subbundle of E generate, which is obviously a parallel complement to E', what is impossible. Therefore, $\rho_{p,\nabla}$ is also not completely reducible.

We can do a converse. If a representation $\rho\colon \pi_1(S) \to GL(k,\mathbb{C})$ is given then we can construct a vector bundle E over S and a flat connection ∇ such that $\rho = \rho_{p,\nabla}$ for some $p \in S$.

We consider the universal covering $\tilde{S}$ of S with the projection map $pr\colon \tilde{S} \to S$ and consider the trivial vector bundle $\tilde{S} \times \mathbb{C}^k$ with the standard product connection, which is flat. The fundamental group $\pi_1(S,p)$ acts on $\tilde{S}$ by deck transformations $D_{[\gamma]}\colon \tilde{S} \to \tilde{S}$ for $[\gamma] \in \pi_1(S,p)$. It acts also on the fiber $\mathbb{C}^k$ by representation ρ. We consider the following diagonal action:

$$\begin{array}{rccc} \Delta\colon & \pi_1(S,p) & \to & \tilde{S} \times \mathbb{C}^k \\ & [\gamma] & \mapsto & \big((\tilde{q}, v) \mapsto (D_{[\gamma]}(\tilde{q}), \rho([\gamma^{-1}])v)\big) \end{array}$$

Further, we take the quotient $\tilde{S} \times \mathbb{C}^k/\Delta$. Because the action of $\pi_1(S,p)$ on $\tilde{S} \times \mathbb{C}^k$ is diagonal and S is homeomorphic to $\tilde{S}/\pi_1(S,p)$, the quotient is a vector bundle over S. Because the action of $\pi_1(S,p)$ on $\mathbb{C}^k$ is linear and the connection on $\tilde{S} \times \mathbb{C}^k$ is flat, the push forward connection on S is flat.

If we consider the representation $\rho^g = g\rho g^{-1}$ for some $g \in GL(k,\mathbb{C})$, then we get the bundle isomorphism

$$\begin{aligned} L\colon \quad \tilde{S} \times \mathbb{C}^k/\Delta &\to \tilde{S} \times \mathbb{C}^k/\Delta^d \\ [q,v] &\mapsto [q,g(v)] \end{aligned}$$

where Δ^g is the corresponding to ρ^g diagonal action. It is easy to see that this is a well-defined bundle isomorphism.

So we get the map which takes the isomorphic class of the representation $\rho\colon \pi_1(S) \to GL(k,\mathbb{C})$ and gives us the isomorphic class of flat vector bundles.

Moreover, it is easy to see that for all representations $\rho\colon \pi_1(S,p) \to GL(k,\mathbb{C})$ is $\rho_{p,\nabla_\rho} = \rho$ for all $p \in S$

We have also to show that for each vector bundle with a flat connection (E,∇) the constructed vector bundle $(\tilde{S} \times \mathbb{C}^k/\Delta, \nabla_{\rho_{p,\nabla}})$ is isomorphic to (E,∇).

First we define the bundle map $\tilde{L}\colon \tilde{S} \times \mathbb{C}^k \to E$. We fix $\tilde{p} \in pr^{-1}(p)$. Let $\tilde{q} \in \tilde{S}$, we consider the curve $\tilde{\gamma}\colon [0,1] \to \tilde{S}$ such that $\tilde{\gamma}(0) = \tilde{p}$, $\tilde{\gamma}(1) = \tilde{q}$. The corresponding curve $\gamma = pr \circ \tilde{\gamma}$ joins the points $p \in S$ and $q = pr(\tilde{q}) \in S$. We define $\tilde{L}(\tilde{q},v) = T_\gamma \theta_p^{-1}(v)$. Because $\tilde{S}$ is simply connected and the connection on E is flat, for fixed $\tilde{q}$ this construction does not depend on $\tilde{\gamma}$ and is smooth. Moreover, the diagram

$$\begin{array}{ccc} \tilde{S} \times \mathbb{C}^k & \xrightarrow{L} & E \\ \downarrow^{\tilde{\pi}} & & \downarrow^{\pi} \\ \tilde{S} & \xrightarrow{pr} & S \end{array}$$

commute. So we get the well-defined bundle map.

The map $\tilde{L}$ is also Δ-invariant. If we take $(\tilde{q},v),(\tilde{q}',v') \in \tilde{S} \times \mathbb{C}^k$ such that $(\tilde{q},v) = \Delta(\tilde{q}',v')$, then there is an element $[\beta] \in \pi_1(S,p)$ and a deck transformation D such that $\tilde{q}' = D(\tilde{q})$ such that $\Delta([\beta]) = (D, \rho([\beta^{-1}]))$ for $\rho = \rho_{p,\nabla}$. Moreover, if we fix the curves $\tilde{\gamma}$ and $\tilde{\gamma}'$, which join $\tilde{p}$ and $\tilde{q}$, resp. q' an consider the lift $\tilde{\beta}$ of β to $\tilde{S}$ with $\tilde{\beta}(0) = p$ and the lift $\tilde{\gamma}_1$ of γ such that $\tilde{\gamma}'(0) = \tilde{\beta}(1)$, then $\tilde{\beta} * \tilde{\gamma}_1 \equiv \tilde{\gamma}'$, because $D \circ \tilde{\gamma} \equiv \tilde{\gamma}_1$ and $\tilde{S}$ is simply connected.

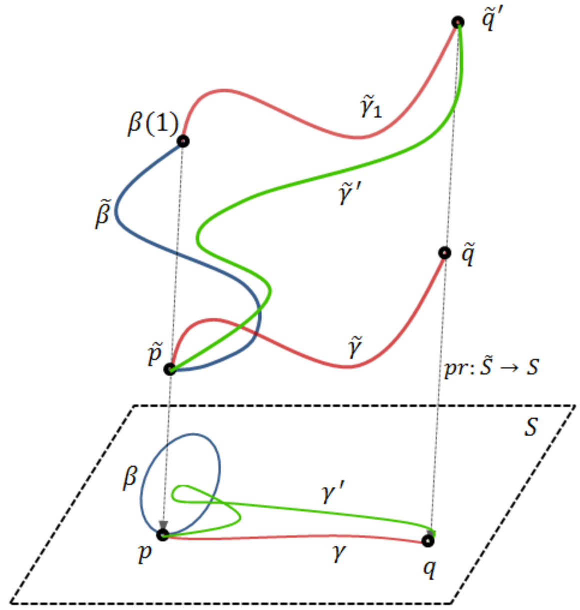

Fig. 1.1: Picture in $\tilde{S}$ and projection on S.

So we get

$$\tilde{L}(\tilde{q}', v') = \tilde{L}(\tilde{\gamma}'(1), v') = \tilde{L}(\tilde{\beta} * \tilde{\gamma}_1(1), \rho([\beta^{-1}])v) = T_{\beta*\gamma} \circ \theta_p^{-1}(\rho([\beta^{-1}])v) =$$

$$= T_{\beta*\gamma} \circ \theta_p^{-1} \circ \theta_p \circ T_\beta^{-1} \circ \theta_p^{-1}(v)) = T_{\beta*\gamma} \circ T_{\beta^{-1}} \circ \theta_p^{-1}(v) = T_\gamma \theta_p^{-1}(v) = \tilde{L}(\tilde{q}, v)$$

Therefore, there is the unique well-defined bundle map $L\colon \tilde{S} \times \mathbb{C}^k/\Delta \to E$ such that $\tilde{L} = L \circ \varepsilon$, where $\varepsilon\colon \tilde{S} \times \mathbb{C}^k \to \tilde{S} \times \mathbb{C}^k/\Delta$ the natural projection. Because $\tilde{S} \times \mathbb{C}^k/\Delta$ and E are vector bundles over the same manifold S and by construction L_p is a linear isomorphism for each $p \in S$, L is a bundle isomorphism.

So we have a bijection between $\mathrm{Hom}(\pi_1(S), GL(k, \mathbb{C})/GL(k, \mathbb{C})$ and the set of all isomorphic classes of flat vector bundles over S of rank k.

If we have the completely reducible representation ρ, then with this representation we construct the corresponding flat bundle (E, ∇) like above, which gives us the other representation $\rho_{p,\nabla}$. This representation is conjugate to ρ and is, therefore, completely reducible. How is proved above, (E, ∇) is also completely reducible. $\square$

Definition 1.9. *For $p \in S$ the representation $\rho_{p,\nabla}$ in the proof of the proposition 1.8 is called a* ***holonomy representation*** *of $\pi_1(S, p)$.*

Remark 1.10. *As we have seen in the proof of the proposition 1.8, if we change the base point $p \in S$, then the representation changes only by conjugation in*

$GL(k,\mathbb{C})$. So the element in $\mathrm{Hom}(\pi_1(S), GL(k,\mathbb{C}))/GL(k,\mathbb{C})$ only depends on the flat connection.

Remark 1.11. *In case $k = 1$ $GL(1,\mathbb{C}) \cong \mathbb{C}^*$ is an abelian group, therefore, the representation sends all commutators in $\pi_1(S)$ to identity, so we have a representation of the first homology group $H_1(S,\mathbb{Z})$ into $\mathbb{C}^*$:*

$$X(\pi_1(S), GL(1,\mathbb{C})) = \mathrm{Hom}(H_1(S,\mathbb{Z}), \mathbb{C}^*).$$

$H_1(S,\mathbb{Z})$ is the free $\mathbb{Z}$-module of dimension $l = 2g+n-1$, where g is the genus of S, n is the number of punctures. So we can take a basis $(e_1, ..., e_l)$ of $H_1(S,\mathbb{Z})$ over $\mathbb{Z}$. Then each representation $\rho \in \mathrm{Hom}(H_1(S,\mathbb{Z}), \mathbb{C}^*)$ is well-defined by the tuple $(\rho(e_1), ..., \rho(e_l)) \in (\mathbb{C}^*)^l$. The map $\rho \mapsto (\rho(e_1), ..., \rho(e_l))$ is bijective and it gives us an exact description of $X(\pi_1(S), GL(1,\mathbb{C}))$:

$$X(\pi_1(S), GL(1,\mathbb{C})) \cong (\mathbb{C}^*)^l$$

To describe the general case $X(\pi_1(S), GL(k,\mathbb{C}))$, we want to use the abelian case. To do this, we will find another surface Σ and a map

$$X(\pi_1(\Sigma), GL(1,\mathbb{C})) \to X(\pi_1(S), GL(k,\mathbb{C})) \tag{1.1}$$

Definition 1.12. *Each map, which satisfies (1.1), is called a **non-abelianisation map**.*

Remark 1.13. *In this work we want to describe some non-abelianisation maps that are finite-to-one and have an open dense image. This will give coordinates on character varieties.*

1.2 Decorated character variety

In this section we want to construct an extension of the character variety which we will use to define coordinates on the character variety.

First, we consider a representation $\rho \in \mathrm{Hom}^*(\pi_1(S), G)$, where G is one of Lie groups $GL(k,K)$, $SL(k,K)$ or $PGL(k,K)$, which satisfies the following conditions:

1. For each peripheral element $g \in \pi_1(S)$ the matrix $\rho(g)$ has an invariant flag $D(g) = (V_1(g), ..., V_k(g))$, where $\dim(V_i(g)) = i$, $i \in \{1, ..., k\}$. For one representation there can be a lot of choices of flags for each g. We can fix this choice by fixing of the following map:

$$\begin{array}{cccc} D\colon & \{g \in \pi_1(S) \mid g \text{ is peripheral}\} & \to & \{F \mid F \text{ is a flag of } K^k\} \\ & g & \mapsto & D(g) = (V_1(g), ..., V_k(g)) \end{array}.$$

The map D must satisfy the following properties:

a) If $g_1, g_2 \in \pi_1(S)$ are two peripheral elements conjugated by $h \in \pi_1(S)$, $hg_1h^{-1} = g_2$, then

$$\rho(h)(D(g_1)) = D(g_2)$$

b) For every $k \in \mathbb{Z} \setminus \{0\}$ and for every peripheral element $g \in \pi_1(S)$

$$D(g) = D(g^k)$$

By these properties, for every puncture, one have to choose only one flag, then the flags associated to the other peripheral elements going to the same punctures are determined. We call this map D **decoration** of ρ.

2. For each peripheral element $g \in \pi_1(S)$ the matrix $\rho(g)$ is conjugated to a matrix of the following form:

$$\mathrm{diag}(J_{\lambda_1,m_1}, ..., J_{\lambda_r,m_r}),$$

where all $m_i \in \mathbb{N}$, $m_1 + ... + m_r = k$, J_{λ_i,m_i} is a Jordan block $m_i \times m_i$ corresponding to the eigenvalue λ_i and all $\lambda_i \in \mathbb{C}^*$ are different.

We consider the set

$$\mathrm{Hom}^d(\pi_1(S), G) := \left\{ (\rho, D) \;\middle|\; \begin{array}{c} \rho \text{ is completely reducible and satisfies 1,2,} \\ D \text{ decoration of } \rho \end{array} \right\}$$

We call elements of $\mathrm{Hom}^d(\pi_1(S), G)$ **decorated representations**. We have a natural projection

$$\begin{array}{ccc} \mathrm{Hom}^d(\pi_1(S), G) & \to & \mathrm{Hom}^*(\pi_1(S), G) \\ (\rho, D) & \mapsto & \rho \end{array}.$$

The Lie group G acts by conjugations on representations and on flags of K^k.

Definition 1.14. *The quotient space*

$$X^d(\pi_1(S), G) := \mathrm{Hom}^d(\pi_1(S), G)/G$$

is called ***decorated character variety****. We denote by $[\rho, D]$ the element of the decorated character variety, which contains the decorated representation (ρ, D).*

Remark 1.15. *We also have a natural projection:*

$$\begin{array}{ccc} X^d(\pi_1(S), G) & \to & X(\pi_1(S), G) \\ [\rho, D] & \mapsto & [\rho] \end{array}.$$

One can prove that this projection is a finite-to-one map and has an open dense image.

We want to study the character variety using Fock-Goncharov coordinates, which we will define later. Fock-Goncharov coordinates are not defined on the decorated character variety but on a dense subset of it. To construct this subset, we need to fix an ideal triangulation $\mathcal{T}$ of S. If we choose two triangles with a common side, we get 4 peripheral curves [see fig. 1.2].

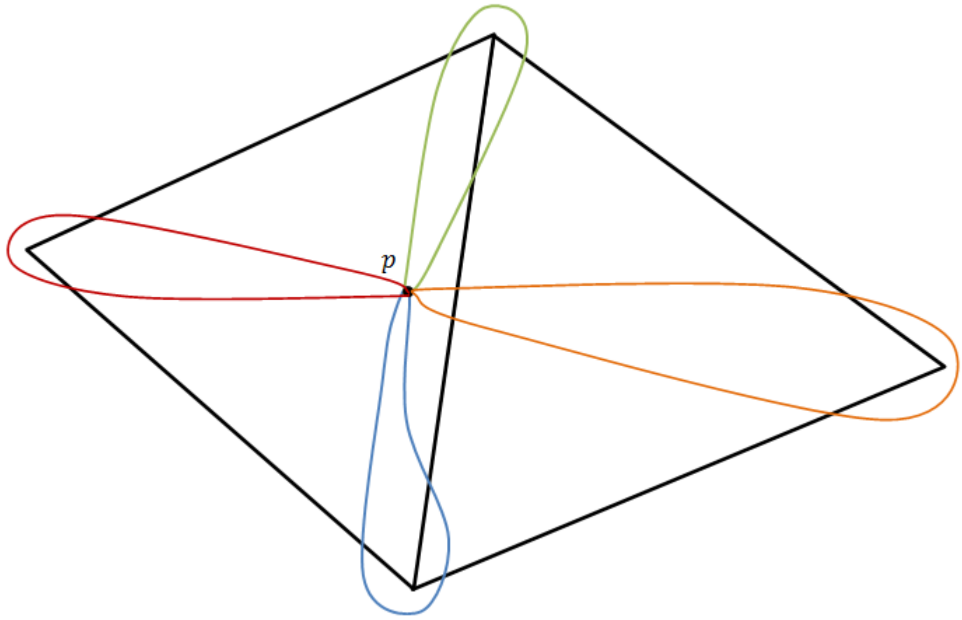

Fig. 1.2: Picture in S

Definition 1.16. *Two flags $F_1 = (V_1, ..., V_k)$ and $F_2 = (W_1, ..., W_k)$ of K^k are called* ***transversal*** *if*

$$\dim(V_l + W_m) = \min\{l + m, k\} \text{ for } l, m \in \{1, ..., k\}.$$

Definition 1.17. *Let $[\rho, D] \in X^d(\pi_1(S), G)$. We say that flags of $[\rho, D]$ are* ***transversal with respect to the triangulation*** *$\mathcal{T}$ if for each two triangles with a common side the corresponding 4 peripheral curves have pairwise transversal flags by decorated representation (ρ, D).*

Remark 1.18. *The property "to be transversal" for two flags is invariant by the action of G. Therefore, the definition above is correct.*

We denote by $X^d(\pi_1(S), G, \mathcal{T})$ the set of all decorated representation which are transversal with respect to the triangulation $\mathcal{T}$. This is an open dense subset of $X^d(\pi_1(S), G)$.

Remark 1.19. *In this thesis we describe* $X^d(\pi_1(S), PGL(2,\mathbb{C}), \mathcal{T})$ *and* $X^d(\pi_1(S), PGL(3,\mathbb{C}), \mathcal{T})$ *using spectral networks and Fock-Goncharov coordinates. Because of remark 1.15 and because* $X^d(\pi_1(S), G, \mathcal{T})$ *is open and dense in* $X^d(\pi_1(S), G)$ *this description will give us local coordinates on an open dense subset of* $X(\pi_1(S), PGL(2,\mathbb{C}))$ *and* $X(\pi_1(S), PGL(3,\mathbb{C}))$.

1.3 Regular homotopy

To construct the non-abelianisation map we will use branched coverings of S and some kind of path lifting rule. But in contrast to coverings, the path lifting to branched coverings is not homotopically invariant. So we need to modify the path lifting for branched coverings. To do this, we need first consider the regular paths instead of continuous paths.

Definition 1.20. *A path* $\gamma\colon [0,1] \to S$ *is called* ***regular*** *if* $\gamma \in C^1([a,b])$ *and* $\dot\gamma(t) \neq 0$ *for all* $t \in [0,1]$.

Definition 1.21. *We say that two regular paths* $\gamma_1, \gamma_2\colon [0,1] \to S$ *have the* ***same extremes*** *if* $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(1) = \gamma_2(1)$, $\dot\gamma_1(0) = \dot\gamma_2(0)$, $\dot\gamma_1(1) = \dot\gamma_2(1)$.

Definition 1.22. *Two regular paths with same extremes* $\gamma_1, \gamma_2\colon [0,1] \to S$ *are called* ***regular homotopic*** *if they are connected by a smooth homotopy* $H\colon [0,1] \times [0,1] \to S$ *such that*

$$H(t,0) = \gamma_1(t)$$
$$H(t,1) = \gamma_2(t)$$
$$H(0,s) = \gamma_1(0) = \gamma_2(0)$$
$$H(1,s) = \gamma_1(1) = \gamma_2(1)$$
$$\frac{\partial H}{\partial t}(0,s) = \dot\gamma_1(0) = \dot\gamma_2(0)$$
$$\frac{\partial H}{\partial t}(1,s) = \dot\gamma_1(1) = \dot\gamma_2(1)$$
$$\frac{\partial H}{\partial t}(t,s) \neq 0$$

for all $s, t \in [0,1]$.

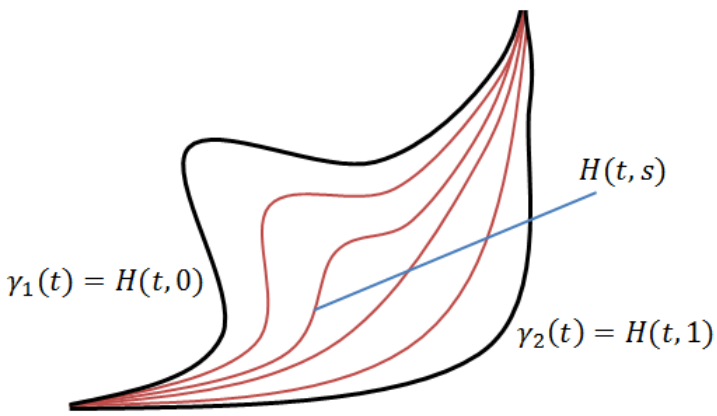

Fig. 1.3: Regular homotopy H of paths γ_1 and γ_2 with same extremes

If we have a regular path γ on S, we can lift it to the the unit tangent bundle US of S:

$$\begin{array}{rccc} \gamma^U: & [0,1] & \to & U \\ & t & \mapsto & \left(\gamma(t), \dfrac{\dot\gamma(t)}{\|\dot\gamma(t)\|}\right) \end{array}$$

Remark 1.23. *To define the unit tangent bundle of S, we need a Riemannian metric on S. But, obviously, the unit tangent bundle does not depend on the metric.*

Because for surfaces with punctures $\pi_1(US) = \mathbb{Z} \times \pi_1(S)$, the fundamental group of the unit tangent bundle fits in the following exact sequence:

$$0 \to \mathbb{Z} \to \pi_1(US) \to \pi_1(S) \to 0.$$

Definition 1.24. *For two regular paths γ_1, γ_2 with the same extremes which are homotopic in the usual sense, the number*

$$w(\gamma_1, \gamma_2) = \gamma_1^U (\gamma_2^U)^{-1} \in \mathbb{Z}$$

*is called **winding number**.*

Remark 1.25. *The winding number is unique defined up to a sign. If we choose the orientation of the surface S, we can define the sign of winding number. On the fig. 1.4 we can see how we can calculate the winding number of two curves by the standard orientation of the plane.*

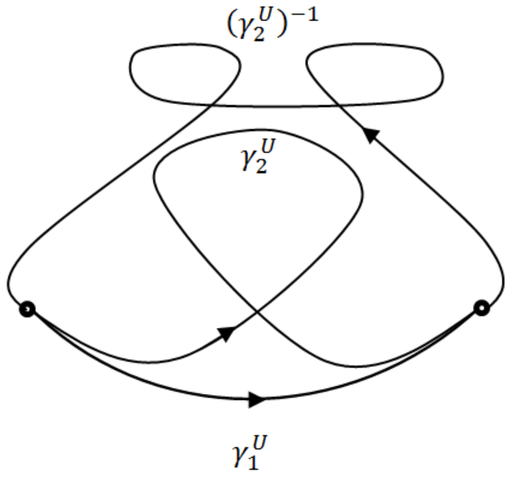

Fig. 1.4: Winding number $w(\gamma_1^U, \gamma_2^U) = -1$.

Remark 1.26. *Winding number is antisymmetric:* $w(\gamma_1^U, \gamma_2^U) = -w(\gamma_2^U, \gamma_1^U)$.

The relation "to be regular homotopic" on the set of all regular paths on the surface S is an equivalence relation which we denote ρ.

The object which is very close to the fundamental group of the unit tangent bundle is the regular fundamental group:

Definition 1.27. *Let S is a surface, $p \in S$, $v \in T_pS$. The set*

$$\pi_1^{reg}(S, p, v) = \{\gamma \mid \gamma \text{ is a regular path on } S, \gamma(0) = \gamma(1) = p, \dot{\gamma}(0) = \dot{\gamma}(1) = v\}/\rho$$

on which the multiplication is defined as a concatenation of curves is called the ***regular fundamental group*** *of S with respect to $p \in S$ and $v \in T_pS$.*

Remark 1.28. *Up to regular homotopy the neutral element e of $\pi_1^{reg}(S, p, v)$ looks like on fig. 1.5.*

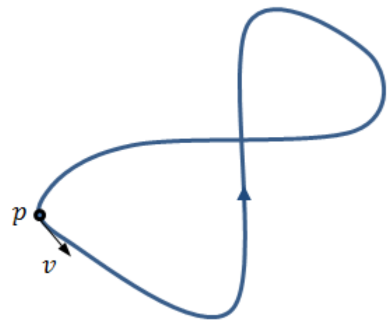

Fig. 1.5: Neutral element of $\pi_1^{reg}(S, p, v)$.

It is easy to see that the regular fundamental group, like the usual fundamental group, is well defined and for all $p, p' \in S$ and all $v \in T_pS$, $v' \in T_{p'}S$ the groups $\pi_1^{reg}(S, p, v)$ and $\pi_1^{reg}(S, p', v')$ are isomorphic. So we can consider the $\pi_1^{reg}(S)$ as the isomorphy class of $\pi_1^{reg}(S, p, v)$.

The connection between the fundamental group of the unit tangent bundle and the regular fundamental group of the surface is given by the following theorem (Smale) [3]:

Theorem 1.29 (Smale). *The map*

$$\begin{aligned} \theta\colon \quad \pi_1^{reg}(S, p, v) &\to \pi_1(US, (p, v)) \\ [\gamma] &\mapsto [(\gamma, \dot{\gamma})] \end{aligned}$$

is a group isomorphism.

Remark 1.30. *The non-trivial part in this theorem is to prove that for two homotopic curves in US the corresponding regular curves in S are regularly homotopic in S.*

With this theorem we have the following exact sequence:

$$0 \to \mathbb{Z} \to \pi_1^{reg}(S) \to \pi_1(S) \to 0.$$

In general, for closed surfaces this sequence does not split, but for surfaces with punctures it always splits.

We can take a quotient relative to the subgroup $\langle 2 \rangle \subseteq \mathbb{Z}$. So we get the other exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \pi_1^s(S) \to \pi_1(S) \to 0,$$

where $\pi_1^s(S) := \pi_1^{reg}(S)/\langle 2\rangle$. This sequence always splits also for closed surfaces, and a choice of spitting is equivalent to a spin structure.

We need the following construction of the semigroup ring, which is very similar to the well-known group ring construction in algebra.

Definition 1.31. *Let H be a semigroup, written multiplicatively, and let R be a ring. We consider $\tilde{R}[H]$ defined as the set of mappings $f\colon H \to R$ of finite support. This set has a natural structure of an R-module.*

To turn $\tilde{R}[H]$ into a ring, we define the product of f and g to be the mappings:

$$f \cdot g\colon x \mapsto \begin{cases} \sum_{uv=x} f(u)g(v), & \text{if } u, v \in H \text{ exist such that } x = uv \\ 0, & \text{otherwise} \end{cases}.$$

The summation is well defined because f and g are of finite support, and the ring axioms are readily verified.

The zero element $0_H \in H$ is an element of H such that $0_H u = u 0_H = 0_H$ for all $u \in H$. Obviously, if a zero element exist in H, then this is unique.

Further, we consider the ideal $\mathcal{O} = (0_H)$ where 0_H is the zero element of H if this exist (if H does not contain the zero element then $\mathcal{O} = \{0\}$). The ***semigroup ring*** *of H over R, which we will denote by $R[H]$, is the quotient ring $\tilde{R}[H]/\mathcal{O}$.*

Some variations in the notation and terminology are in use. In particular, the mappings such as $f\colon H \to R$ are sometimes written as what are called "formal linear combinations of elements of H, with coefficients in R":

$$\sum_{h \in H} f(h)h,$$

or simply

$$\sum_{h \in H} f_h h,$$

where the sum is always finite because for almost all $f(h) = f_h = 0$.

Now we are ready to define some important algebraic objects over the surface S. We use definitions from [1].

The set $REGPATHS(S)$ of all regular paths γ in S up to regular homotopy ρ supplemented with the formal symbol 0 have the following natural semigroup structure. The multiplication in $REGPATHS(S)$ is the concatenation if it is possible and 0 otherwise, we also define the left and right multiplication of an element with 0 as 0.

We can also consider the corresponding semigroup ring $\mathbb{Z}[REGPATHS(S)]$ as in definition 1.31. By construction, for all $p \in S$ and all $v \in T_pS$ we have natural inclusions

$$\pi_1^{reg}(S, p, v) \hookrightarrow REGPATHS(S) \hookrightarrow \mathbb{Z}[REGPATHS(S)].$$

Similarly we can construct the semigroup $PATHS(S)$ of all paths on S up to homotopy with the formal symbol 0 and then the semigroup ring $\mathbb{Z}[PATHS(S)]$. Also for all $p \in S$ we have natural inclusions

$$\pi_1(S, p) \hookrightarrow PATHS(S) \hookrightarrow \mathbb{Z}[PATHS(S)].$$

Moreover, because each regular path is continuous and the relation ρ of regular homotopy is finer then the relation of usual homotopy, we have the natural surjective projection (semigroup homomorphism) $\tau\colon REGPATHS(S) \to PATHS(S)$ which can be continued to the surjective ring homomorphism $\tau\colon \mathbb{Z}[REGPATHS(S)] \to \mathbb{Z}[PATHS(S)]$.

Further, we consider the ideal

$$\mathcal{I} = \left(\left\{ \gamma_1 - (-1)^{w(\gamma_1,\gamma_2)}\gamma_2 \;\middle|\; \begin{array}{c} \gamma_1, \gamma_2 \in REGPATHS(S) \\ \text{are homotopic and have same extremes} \end{array} \right\} \right).$$

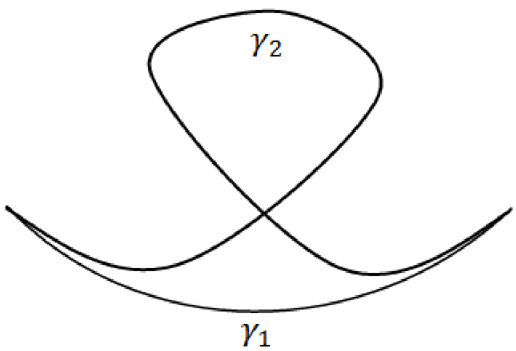

Fig. 1.6: Example of γ_1 and γ_2 such that $\gamma_1 - (-1)^{w(\gamma_1,\gamma_2)}\gamma_2 \in \mathcal{I}$.

Now we are ready to define homotopy path algebra of the surface. We use the definition from [1].

Definition 1.32. *The quotient ring*

$$HPA(S) = \mathbb{Z}[REGPATHS(S)]/\mathcal{I}$$

is called ***homotopy path algebra*** *of* S.

Because for $\gamma_1, \gamma_2 \in \pi^{reg}(S, p, v)$, $(p \in S,\, v \in T_pS)$ we have: $\gamma_1 - \gamma_2 \in \mathcal{I}$ if and only if $\gamma_1\gamma_2^{-1} \in \langle 2 \rangle$, the fundamental group with signs $\pi_1^s(S, p, v)$ of S can be included as a subset in the homotopy path algebra $HPA(S)$ of S.

We consider the natural projection $\pi\colon REGPATHS(S) \to HPA(S)$. Because we identify by $\mathcal{I}$ only elements with same extremes, so the tangent vectors at the extreme points of elements $\pi([\gamma])$ for $\gamma \in REGPATHS(S)$ are well defined. This gives us the possibility to give the following definition:

Definition 1.33. *Let* $V\colon S \to TS$ *be a non-zero tangent vector field. For every* $[\gamma] \in PATHS(S)$ *the set* $\pi(\tau^{-1}([\gamma]))$ *has two elements that agree with* V *at the extremes. A semigroup homomorphism* $\sigma\colon PATHS(S) \to HPA(S)$ *is called a* ***spin structure*** *with respect to the vector field* V *if* $\sigma([\gamma])$ *agree with* V *at the extremes and*

$$\sigma([\gamma]) \in \pi(\tau^{-1}([\gamma]))$$

for every $[\gamma] \in PATHS(S)$.

By construction of $HPA(S)$ for each element of $0 \neq [\gamma] \in PATHS(S)$ and for each non-zero tangent vector field $V\colon S \to TS$ the set $\pi(\tau^{-1}([\gamma])) \subseteq HPA(S)$ contains exactly two elements x and $-x$ which at extreme points agree with a vector field V. Moreover, the set $\pi(\tau^{-1}(PATHS(S)))$ generates $HPA(S)$. We consider a subring of $HPA(S)$ which is generated by elements of $\pi(REGPATHS(S))$, which agree with the vector field V at extreme points. We denote this subring by $HPA(S, V)$.

Using a spin structure σ we can construct a ring homomorphism $\sigma'\colon HPA(S, V) \to \mathbb{Z}[PATHS(S)]$ by its definition on generators: for each element $[\gamma] \in PATHS(S)$ and $x \in \pi(\tau^{-1}([\gamma])) \cap HPA(S, V)$ let

$$\sigma'(x) := \begin{cases} [\gamma], & \sigma([\gamma]) = x \\ -[\gamma], & \sigma([\gamma]) = -x \end{cases}$$

So we have $\sigma' \circ \sigma = id_{PATHS(S)}$. The map σ' is also called a spin structure with respect to the vector field V.

Also a spin structure σ with respect to the vector field V gives us a group homomorphism $\pi_1(S, p) \to \pi_1^s(S, p, V(p))$ for each $p \in S$ because for fixed $p \in S$ the group $\pi_1(S,p)$ is contained in $PATHS(S)$, $\pi_1^s(S,p,V(p))$ is contained in $HPA(S,V)$ and because of $\sigma([\gamma]) \in \pi(\tau^{-1}([\gamma])) \subseteq \pi_1^s(S,p,V(p))$ for every $[\gamma] \in \pi_1(S,p)$, we get $\sigma(\pi_1(S,p)) \subseteq \pi_1^s(S,p,V(p))$.

1.4 Non-abelianisation map

We consider a surface S with punctures, $\pi\colon \Sigma \to S$ is a branched covering. Let $p \in S$, $0 \neq v \in T_pS$. Let W be a non-zero vector field on Σ such that $D_q\pi(W(q)) = v$ for all $q \in \pi^{-1}(p)$.

We want to construct a non-abelianisation map

$$X(\pi_1(\Sigma), GL(1,\mathbb{C})) \to X(\pi_1(S), GL(k,\mathbb{C})).$$

We will do it using spectral networks, namely, the spectral network will give us a map $\pi_1^s(S,p,v) \to HPA(\Sigma, W)$, which actually is a path lifting rule form S to Σ.

If we take an element of $X(\pi_1(\Sigma), GL(1,\mathbb{C}))$, so by proposition 1.8 we get a flat connection on Σ. If we also fix spin structures on S and Σ, then we get a sequence which gives us the representation $\pi_1(S,p) \to GL(k,\mathbb{C})$:

$$\pi_1(S,p) \xrightarrow[\text{on } S]{\text{spin str.}} \pi_1^s(S,p,v) \underset{[*]}{\longrightarrow} HPA(\Sigma, W) \xrightarrow[\text{on } \Sigma]{\text{spin str.}}$$
$$\xrightarrow[\text{on } \Sigma]{\text{spin str.}} \mathbb{Z}(PATHS(\Sigma)) \xrightarrow[\text{on } \Sigma]{\text{flat conn.}} GL(k,\mathbb{C})$$

The equivalence class of this representation is an element of the character variety $X(\pi_1(S), GL(k,\mathbb{C}))$, so the definition of the non-abelianisation is complete, except for the path lifting rule [*], which we will define in the second chapter using spectral networks.

An equivalent way to construct this representation is using twisted connections. The usual flat connection ∇ can be considered as a map which takes an element $[\gamma] \in PATHS(S)$ and gives a linear map $T_\gamma\colon E_{\gamma(0)} \to E_{\gamma(1)}$ which we interpret as a parallel transport operator along γ. We can extend this map to the map from $\mathbb{Z}(PATHS(S))$ to the set $L(E) = \bigoplus_{p,q\in S} L(E_p, E_q)$ where $L(E_p, E_q)$ is the set of all linear maps from E_p to E_q. On $L(E)$ the multiplication can be defined in the following way: for $A \in L(E_p, E_q)$, $B \in L(E_r, E_s)$, $(p,q,r,s \in S)$ we define $BA \in L(E_p, E_s)$ the usual composition of two linear maps if $q = s$ and 0 otherwise. With this multiplication $L(E)$ becomes a ring and ∇ becomes a ring homomorphism from $\mathbb{Z}(PATHS(S))$ to $L(E)$.

Let $\sigma'\colon HPA(S,V) \to \mathbb{Z}[PATHS(S)]$ is a spin structure. The composition $\nabla \circ \sigma'$ gives us a ring homomorphism from $HPA(S,V)$ to $L(E)$.

Definition 1.34. *A ring homomorphism*

$$\begin{array}{rccc} \nabla^T \colon & HPA(S,V) & \to & L(E) \\ & x & \mapsto & A_x \end{array}$$

such that $A_x \in L(E_{\gamma(0)}, E_{\gamma(1)})$ *for all* $x = \pi([\gamma])$, $[\gamma] \in REGPATHS(S,V)$ *is called* ***twisted connection*** *on* E.

Remark 1.35. *The restriction of the twisted connection* $\nabla^T \colon HPA(S,V) \to L(E)$ *on* $\pi_1^s(S,p,V(p))$ *gives us a group homomorphism* $\rho_{\nabla^T} \colon \pi_1^s(S,p,V(p)) \to GL(E_p)$ *which is called* ***twisted representation***.

Remark 1.36. *A twisted connection can be also defined in the following way:*

Let $\pi_S \colon E \to S$ *be a vector bundle of rank* k *over* S, $\pi_{US} \colon E' \to US$ *a vector bundle of rank* k *over the unit tangent bundle* US *of* S, $\theta \colon E' \to E$ *a smooth map, such that for each* $(p,v) \in UT$ *the restriction of* θ *on the fiber* $E'_{(p,v)}$

$$\theta_{(p,v)} \colon E'_{(p,v)} \to E_p$$

is a linear isomorphism and the following diagram commutes:

$$\begin{array}{ccc} E' & \xrightarrow{\theta} & E \\ \pi_{US}\big\downarrow & & \big\downarrow\pi_S \\ US & \xrightarrow[\text{projection}]{\text{natural}} & S \end{array}$$

A fiber U_pS *of* US *over* $p \in S$ *is diffeomorphic to* S^1, *which fundamental group is isomorphic to* $\mathbb{Z}$. *We fix* $v \in U_pS$ *and an isomorphism* $\pi_1(U_pS, v) \to \mathbb{Z}$ *and choose a loop* δ_p *in* U_pS, *which corresponds to* 1 *by this isomorphism.*

A flat connection on E' *is called a twisted connection on* E *if for all* $p \in S$ *the parallel transport operator* $T_{\delta_p} = -id_{E'_{(p,v)}}$.

It is easy to see that this definition does not depend on the choice of δ_p *because the connection is flat, and does not depend on the choice of* $v \in U_pS$ *and of isomorphism* $\pi_1(U_pS, v) \to \mathbb{Z}$ *because* U_pS *is connected and* $\mathrm{Aut}(\mathbb{Z}) = \{\pm id_{\mathbb{Z}}\}$, *therefore,* $T_{\delta_p^{-1}} = -id_v^{-1} = -id_v$.

A twisted connection on E *defines a map* $PATHS(US) \to L(E')$, *which sends the homotopy class of a curve on* US *to the parallel transport operator along this curve. Because the connection is flat, this operator does not depend on a choice of this curve in a homotopy class.*

Using this definition we can define the twisted connection as in definition 1.34. First, we have an natural map

$$\begin{array}{rcl} REGPATHS(S) & \to & PATHS(US) \\ [\gamma] & \mapsto & [(\gamma, \dot\gamma)] \end{array}$$

For the element $[(\gamma, \dot\gamma)]$ *we get an parallel transport operator*

$$T'_\gamma \colon E'_{(\gamma(0),\dot\gamma(0))} \to E'_{(\gamma(1),\dot\gamma(1))}.$$

Using θ *we get a linear map*

$$T_\gamma = \theta_{(\gamma(1),\dot\gamma(1))} \circ T'_\gamma \circ \theta^{-1}_{(\gamma(0),\dot\gamma(0))} \colon E_{\gamma(0)} \to E_{\gamma(1)}.$$

So we have a map

$$\begin{array}{rcl} REGPATHS(S) & \to & L(E) \\ [\gamma] & \mapsto & T_\gamma \end{array}.$$

This map can be uniquely extended by $\mathbb{Z}$*-linearity to the map*

$$\mathbb{Z}[REGPATHS(S)] \to L(E).$$

And it is also easy to prove that the ideal $\mathcal{I}$ *goes by this map to* 0*. So we can take a quotient and get the well defined map* $HPA(S) \to L(E)$*. The restriction of this map on* $HPA(S,V)$ *gives us a twisted connection in sense of definition 1.34.*

With spin structures on S and twisted connection on Σ we get a sequence which gives us the representation $\pi_1(S,p) \to GL(k,\mathbb{C})$:

$$\pi_1(S,p) \xrightarrow[\text{on } S]{\text{spin str.}} \pi_1^s(S,p,v) \underset{[*]}{\longrightarrow} HPA(\Sigma, W) \xrightarrow[\text{on } \Sigma]{\text{twisted conn.}} GL(k,\mathbb{C}), \tag{1.2}$$

where $p \in S$, $v \in T_pS$ and W is a non-zero vector field on Σ such that $D_q\pi(W(q)) = v$ for all $q \in \pi^{-1}(p)$. The map $[*]$ from $\pi_1^s(S,p,v)$ to $HPA(\Sigma, W)$ is given by the spectral network.

Because the spin structure on Σ gives us the twisted connection on the vector bundle over Σ, further, by considering of spectral networks, we will assume that the twisted connection and corresponding twisted representation on Σ is given.

1.5 Spin structure associated with a vector field

Using a non-zero vector field V on S we can construct a spin structure. Because S has punctures, a non-zero vector field on S always exists. In this section we will assume that every regular curve γ on S agrees with V at extreme points.

Let $\gamma\colon [a,b] \to S$ be a regular curve. We consider a Riemanian metric g on S, then for each $p \in S$ we have the ortonormal basis $(e_1(p), e_2(p))$, such that $e_1(p) = V(p)/\|V(p)\|$ and $g(e_i(p), e_j(p)) = \delta_{ij}$ for all $p \in S$. Moreover, the vector field e_2 can be chosen so that for all $p \in S$ the basis $(e_1(p), e_2(p))$ is positive oriented with respect to the orientation of S. In this basis we have

$$\dot\gamma(t) = x(t)e_1(\gamma(t)) + y(t)e_2(\gamma(t)).$$

So we have the well-defined smooth map

$$\begin{array}{rccc} \gamma'\colon & [a,b] & \to & S^1 \\ & t & \mapsto & \dfrac{(x(t), y(t))}{\|\dot\gamma(t)\|} \end{array}$$

which is a curve in S^1 and $\gamma'(a) = \gamma'(b) = (1,0)$. This curve defines an element in the fundamental group $\pi_1(S^1, (1,0))$ which is isomorphic to $\mathbb{Z}$ by an isomorphism $\theta\colon \pi_1(S^1,(1,0)) \to \mathbb{Z}$. This isomorphism is unique if we assume that $\theta([\alpha]) = 1 \in \mathbb{Z}$ for $\alpha(t) = (\cos(t), \sin(t)), t \in [0, 2\pi]$.

Definition 1.37. *The number*

$$W(\gamma, V) := \theta([\gamma'])$$

*we call the **winding number** of γ with respect to the vector field V.*

Remark 1.38. *It is easy to see that for the definition of the winding number actually we do not need the vector field on the whole surface. It is enough to have the vector field along γ.*

Remark 1.39. *The regular homotopy $H\colon [a,b] \times [0,1] \to S$ of two regular curves $\gamma_1, \gamma_2\colon [a,b] \to S$ with the same extremes compatible with V induces the homotopy $H'\colon [a,b] \times [0,1] \to S^1$ of $\gamma_1'\colon [a,b] \to S^1$ and $\gamma_2'\colon [a,b] \to S^1$ by the formula*

$$H'(t,s) := \frac{(x(t,s), y(t,s))}{\|\frac{\partial H}{\partial t}(t,s)\|},$$

where

$$\frac{\partial H}{\partial t}(t,s) = x(t,s)e_1(H(t,s)) + y(t,s)e_2(H(t,s)).$$

Therefore, the winding number is invariant by regular homotopy.

Remark 1.40. *For two regular curves γ_1, γ_2 such that*

$$0 \neq [\gamma_1][\gamma_2] \in REGPATHS(S)$$

we have $W(\gamma_1 * \gamma_2, V) = W(\gamma_1, V) + W(\gamma_2, V)$.

Lemma 1.41. *For each $p \in S$ there exist a regular closed curve δ compatible with V, which is homotopic to $\{p\}$ and $W(\delta, V) = 1$.*

Proof. We choose a chart (U, x) of S such that $p \in U$, U open in S, contractible, $x\colon U \to \tilde{U} \subseteq \mathbb{R}^2$ is a homeomorphism, $\phi(p) = (0,0)$, $D_p x(V(p)) = \frac{\partial}{\partial x_1}$.
If we fix some $0 < \varepsilon < 1$, then we can assume that U is small enough that $D_p x(V(q)) = r(q)\frac{\partial}{\partial x_1} + s(q)\frac{\partial}{\partial x_2}$ with $|r(q) - 1| < \varepsilon$, $|s(q)| < \varepsilon$.
We consider a curve $\beta(t, \varepsilon) := (R\sin(t), -R\cos(t) + R), t \in [0, 2\pi]$ where $R = R(\varepsilon)$ is smooth in ε and small enough so that β is contained in $\tilde{U}$. Then the curve $\delta := x^{-1} \circ \beta$ is contained in U. Obviously, $\delta(t, \varepsilon)$ is homotopic to $\{p\}$ for each ε because U is contractible.

$$\dot{\beta}(t, \varepsilon) = (R\cos(t), R\sin(t)).$$

Because for ε small enough the vector field $Dx(V)$ is close to $\frac{\partial}{\partial x_1}$, the curve $\delta'(\cdot, \varepsilon)\colon [0, 2\pi] \to S^1$ is close to $\alpha(t) = (\cos(t), \sin(t))$. That means $\delta'(t, \varepsilon) = (X(t, \varepsilon), Y(t, \varepsilon))$, X, Y are smooth functions of two variables and

$$\lim_{\varepsilon \to 0} \delta'(t, \varepsilon) = \alpha'(t)$$

for all $t \in [0, 2\pi]$. So we have that α is homotopic to $\delta'(\cdot, \varepsilon)$ for all ε small enough. Therefore,

$$W(\delta, V) = \theta([\delta']) = \theta([\alpha]) = 1.$$

□

Remark 1.42. *Analogously we can prove that for all $n \in \mathbb{Z}$ there exist a regular closed curve δ_n compatible with V, which is homotopic to $\{p\}$ and $W(\delta_n, V) = n$.*

Lemma 1.43. *For each two regular curves $\gamma_1, \gamma_2\colon [a, b] \to S$ with the same extremes and which are homotopic in the usual sense we have*

$$W(\gamma_2, V) - W(\gamma_1, V) = w(\gamma_2, \gamma_1).$$

Proof. Two regular curves with the same extremes, which are homotopic, are regular homotopic if and only if their winding numbers agree. We consider the curve δ_w from remark 1.42 for $p := \gamma_1(a)$ and $w := w(\gamma_2, \gamma_1)$. Then

$$w(\gamma_1 * \delta_w, \gamma_1) = w(\gamma_2, \gamma_1) = w.$$

Therefore, $w(\gamma_1 * \delta_w, \gamma_2) = 0$. This means that γ_2 and $\gamma_1 * \delta_w$ are regular homotopic because δ is homotopic to p and therefore γ_2 is homotopic to $\gamma_1 * \delta_w$ and they have the same extremes.

Further, $W(\gamma_1 * \delta_w, V) = W(\gamma_1, V) + w$. Because the winding number is invariant by regular homotopy, we have $W(\gamma_1 * \delta_w, V) = W(\gamma_2, V)$. Therefore,

$$W(\gamma_2, V) - W(\gamma_1, V) = w(\gamma_2, \gamma_1).$$

$\square$

From lemma 1.43 follows that for two regular paths γ_1 and γ_2 with the same extremes belong to the same class of $HPA(S, V)$ if and only if the number $W(\gamma_2, V) - W(\gamma_1, V) = w(\gamma_2, \gamma_1)$ is even. For each continuous path $\gamma\colon [a, b] \to S$ we can always choose a regular path γ^* compatible with V which is homotopic to γ. Either $W(\gamma^*, V)$ or $W(\delta * \gamma^*, V)$ is even, where δ is a regular path from lemma 1.41 for $p = \gamma(0)$.

We define the spin structure $\sigma\colon PATHS(S) \to HPA(S, V)$ in the following way: $\sigma([\gamma])$ is the class of γ^* in $HPA(S, V)$ if $W(\gamma^*, V)$ is even, or the class of $\delta * \gamma^*$ in $HPA(S, V)$ otherwise. By this definition the element in $HPA(S, V)$ is well defined and does not depend on the choice of γ^* because $\pi(\tau^{-1}([\gamma])) \subseteq HPA(S, V)$ contains exactly two elements namely $\pi([\gamma^*])$ and $\pi([\delta * \gamma^*])$ and we choose the unique element of $HPA(S, V)$ which contains regular curves with even winding numbers with respect to V. Moreover, the map σ satisfies all conditions of the spin structure. We call this spin structure associated with the vector field V.

2 Small spectral networks

2.1 The definition of small spectral network

We consider a closed orientable surface $\bar{S}$ with punctures $P = \{s_1, ..., s_n\},\ n \in \mathbb{N}$ and we also consider the surface $S = \bar{S} \setminus P$ (as in the first chapter). Let $\pi\colon \bar{\Sigma} \to \bar{S}$ a $k : 1$ branched covering of $\bar{S}$. We consider also the corresponding restriction $\pi\colon \Sigma \to S$ where $P' = \pi^{-1}(P)$ is the set of punctures on $\Sigma = \bar{\Sigma} \setminus P'$. We assume that the branched covering is not ramified over the punctures. Therefore, we denote by B the set of branch points on S, and $B' = \pi^{-1}(B)$.

We assume that every branch point is simple. That means every branch point $b \in B$ has an open neighborhood U such that

$$\pi^{-1}(U) = \coprod_{i=1}^{k-1} V_i,$$

where V_i are open neighborhoods of one of the pre-images of b in Σ, $\pi|_{V_i}\colon V_i \to U$ is an homeomorphism for $i = 1, ..., k-2$, and $\pi|_{V_{k-1}}\colon V_{k-1} \to U$ looks like $z \to z^2$.

We assume also, that for each $s \in P$ an order on the fiber $\pi^{-1}(s) = \{s_i^{(1)}, ..., s_i^{(k)}\}$ of p is given: $s_i^{(1)} < s_i^{(2)} < ... < s_i^{(k)}$. This order on the fiber over s is an additional structure on the covering.

Now we are ready to define spectral networks. In this thesis we use the definition from [1]. This definition looks different from the one given in [6], but it is actually equivalent.

Definition 2.1. *A* ***small spectral network*** $\mathcal{W}$ ***of rank*** k *over* S *is a graph on* $\bar{\Sigma}$*, that means a finite collection of injective regular paths* $\{p_h\colon [-1,1] \to \bar{\Sigma}\}_{h\in H}$ *(*H *is a finite set of indices) satisfying the following conditions:*

1. $\pi(p_h(t)) = \pi(p_h(-t))\ \ \forall t \in [-1,1]$

2. $\pi(p_h(0) \in B$ *is a branch point*

3. $\pi(p_h(1)) = \pi(p_h(-1)) = s_j \in P$ *(for some* j*) is a puncture and* $p_h(-1) < p_h(1)$ *with respect to the order on* $\pi^{-1}(s_j)$

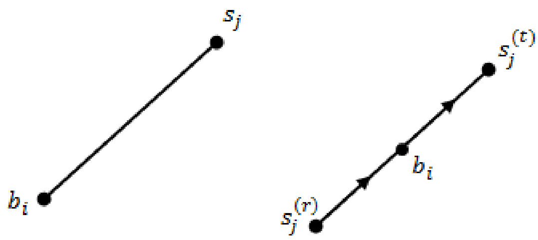

Fig. 2.1: The picture in $\bar{S}$ left, in $\bar{\Sigma}$ right.

4. A neighborhood of the branch points looks like

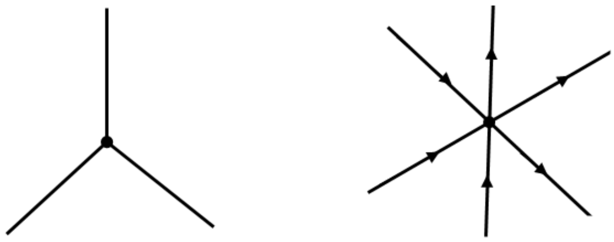

Fig. 2.2: The picture in $\bar{S}$ left, in $\bar{\Sigma}$ right.

5. Different paths p_h and $p_{h'}$ in $\bar{\Sigma}$ can meet only at a point $p_h(t) = p_{h'}(t')$ which is

a) a branch point (which means $t = t' = 0$),

b) a puncture (which means $t, t' \in \{-1, 1\}$),

or in following case:

c) $t \cdot t' > 0$. In this case we also need that the intersection is transverse and the path p_h does not intersect any line of the spectral network at the point $p_h(-t)$ and the path $p_{h'}$ does not intersect any line of the spectral network at the point $p_{h'}(-t')$.

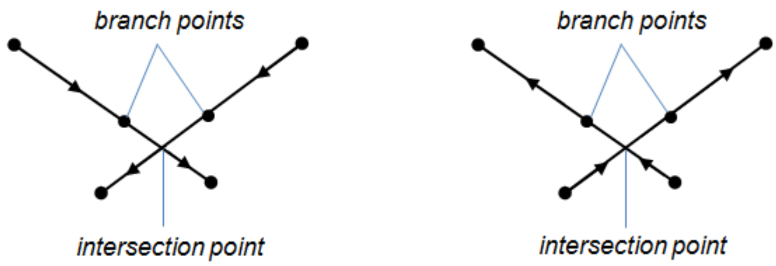

Fig. 2.3: The picture in $\bar{\Sigma}$.

We denote

$$W_\Sigma = \bigcup_{h\in H} p_h([-1,1]) \subseteq \bar{\Sigma},$$

$$W_S = \bigcup_{h\in H} \pi(p_h([-1,1])) \subseteq \bar{S}.$$

Remark 2.2. *For small spectral networks the following intersection of paths is forbidden:*

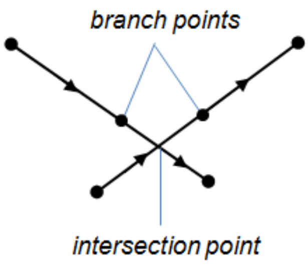

Fig. 2.4: The picture in $\bar{\Sigma}$.

In general spectral networks this intersection is allowed. It is called a joint. In this case every time there is a joint, one additional path must be added to the spectral network.

Proposition 2.3. *The intersection described in remark 2.2 and in definition 2.1 (5c) can not occur by spectral networks of rank 2.*

Proof. This fact follows directly from the axiom (5c) in definition 2.1.

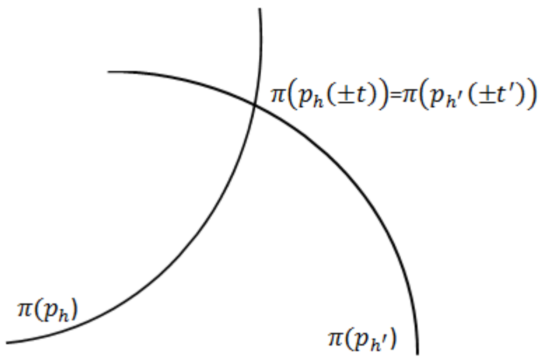

Fig. 2.5: The picture in S.

If we assume that there are two paths p_h and $p_{h'}$ such that there exist $t, t' \in (0,1) \setminus \{0\}$ and

$$\pi(p_h(t)) = \pi(p_h(-t)) = \pi(p_{h'}(t') = \pi(p_{h'}(t'),$$

then because $\pi\colon \Sigma \to S$ has exactly two sheets, there are only two possibilities:

$$p_h(t) = p_{h'}(t') \text{ and } p_h(-t) = p_{h'}(-t')$$

or

$$p_h(t) = p_{h'}(-t') \text{ and } p_h(-t) = p_{h'}(t').$$

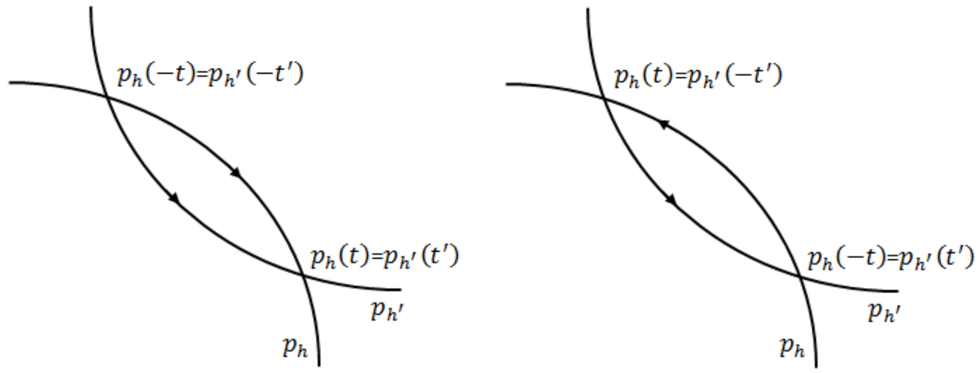

Fig. 2.6: The picture in Σ.

Both of them are prohibited by the axiom (5c) in the definition of the spectral network. □

Remark 2.4. *From the proposition 2.3 follows that all spectral networks of rank 2 are small.*

2.2 Path lifting using spectral network

Let $\pi\colon \bar{\Sigma} \to \bar{S}$ be a $k:1$ branched covering satisfying the conditions above and let $\mathcal{W}$ be a small spectral network of rank k on Σ. We want to construct a map $\pi_1^s(S,p,v) \to HPA(\Sigma, V')$, where $p \in S$, $v \in T_pS$, V' is a non-zero vector field on Σ such that $D_q\pi(W(q)) = v$ for all $q \in \pi^{-1}(p)$.

We consider a smooth path $\gamma\colon [0,1] \to S$ such that $[\gamma] \in \pi_1^s(S,p,v)$. To lift this path to Σ we split it in pieces $\gamma = \gamma_1 * \gamma_2 * ... * \gamma_n$ such that each γ_i intersects the spectral network at most once. Here and later with $*$ we denote the concatenation of smooth curves.

If γ_i does not intersect the spectral network, then we lift it in the usual way and get k lifts $\gamma_i^{(j)}$, $j \in \{1, ..., k\}$ in Σ which also do not intersect spectral network. We can construe regular paths $\gamma_i^{(j)}$ as elements of $HPS(\Sigma)$ (as it is done in the first chapter).

If γ_i intersects the spectral network, then there are only two standard lifts $\gamma_i^{(j)}$ and $\gamma_i^{(l)}$ of γ_i, which intersect a path of the spectral network. We denote this path p_h. In this case we have to add to the usual lifts a new path γ_i', like on fig. 2.7. We can also construe regular paths $\gamma_i^{(j)}$ and γ_i' as elements of $HPS(\Sigma)$ (as it is done in the first chapter).

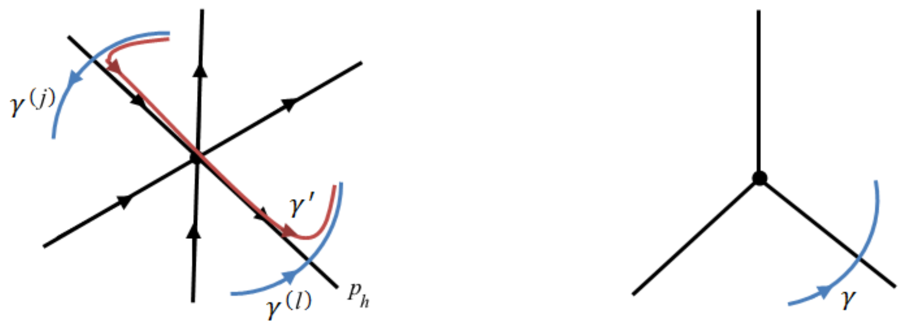

Fig. 2.7: The picture in $\bar{\Sigma}$ left, in $\bar{S}$ right.

Definition 2.5. *The* ***lift*** $\tilde{\gamma}$ *in* Σ *with respect to the spectral network* $\mathcal{W}$ *of the curve* γ *is the product of all lifts of* γ_i *in* $HPA(\Sigma)$*:*

$$\tilde{\gamma} = \Big(\sum_{i=1}^{k} \gamma_1^{(i)} + \gamma_1'\Big) \cdot ... \cdot \Big(\sum_{i=1}^{k} \gamma_n^{(i)} + \gamma_n'\Big),$$

where $\gamma_i' = 0$ *for pieces* γ_i *which do not intersect the spectral network.*

Lemma 2.6. *This path lifting rule is invariant by regular homotopy.*

Proof. To prove that we need to check the homotopic invariance of the path lifting in the following three cases (see fig.2.8) because up to regular homotopy we can always decompose every curve in a product of curves of these kinds.

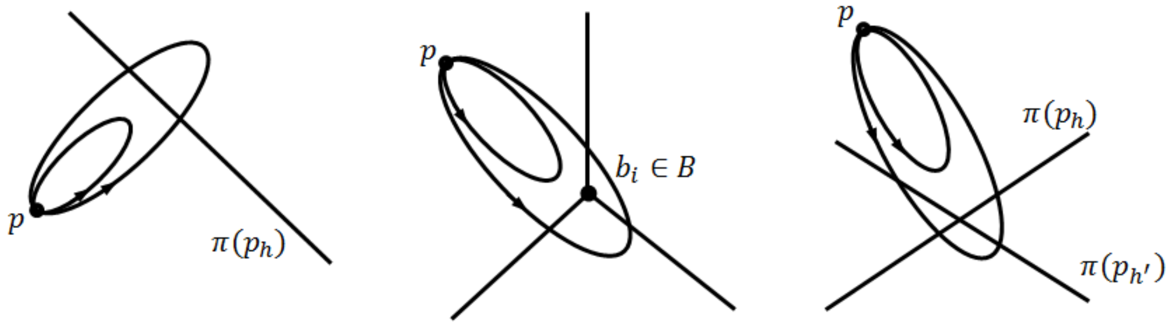

Fig. 2.8: The picture in $\bar{S}$.

Case 1.

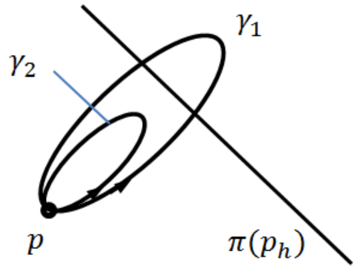

Fig. 2.9: The picture in $\bar{S}$.

We have to lift two curves γ_1 and γ_2 to Σ with the path lifting rule of the spectral network. Because γ_2 does not intersect the spectral network, its standard lifts do not do it as well. Therefore, we have the lift of γ_2 in $HPA(\Sigma)$:

$$\sum_{i=1}^{k} \gamma_2^{(i)}.$$

For γ_1 we also get k standard lifts but in this case exactly two of them intersect the spectral network [see fig. 2.10]:

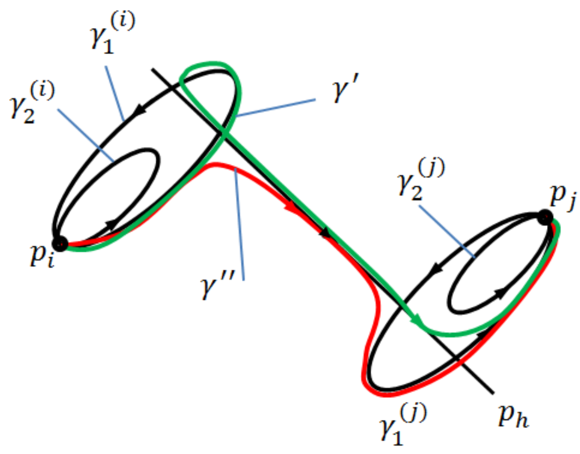

Fig. 2.10: The picture in $\bar{\Sigma}$.

Therefore, we have to add two new lifts γ' and γ''. So we have the lift in the $HPA(\Sigma)$:

$$\sum_{i=1}^{k} \gamma_1^{(i)} + \gamma' + \gamma''.$$

But in $HPA(\Sigma)$ we have that $\gamma' = -\gamma''$, because γ' and γ'' have the same extremes and $w(\gamma', \gamma'') = \pm 1$ (the sign depends on the orientation on the surface). Therefore, we get the lift in $HPA(\Sigma)$:

$$\sum_{i=1}^{k} \gamma_1^{(i)},$$

which agree with

$$\sum_{i=1}^{k} \gamma_2^{(i)}$$

in $HPA(\Sigma)$ because $\gamma_1^{(i)}$ is regularly homotopic to $\gamma_2^{(i)}$ for all $i \in \{1, ..., k\}$ as standard lifts of homotopic curves with the same base point.

Case 2.

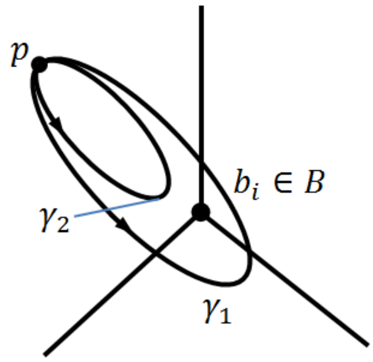

Fig. 2.11: The picture in $\bar{S}$.

We have to lift two curves γ_1 and γ_2 to Σ with the path lifting rule of the spectral network. Because γ_2 does not intersect the spectral network, its standard lifts do not do it as well. Therefore, we have the lift of γ_2 in $HPA(\Sigma)$:

$$\sum_{i=1}^{k} \gamma_2^{(i)}.$$

For γ_1 we also get k standard lifts but in this case exactly two of them intersect the spectral network [see fig. 2.12]:

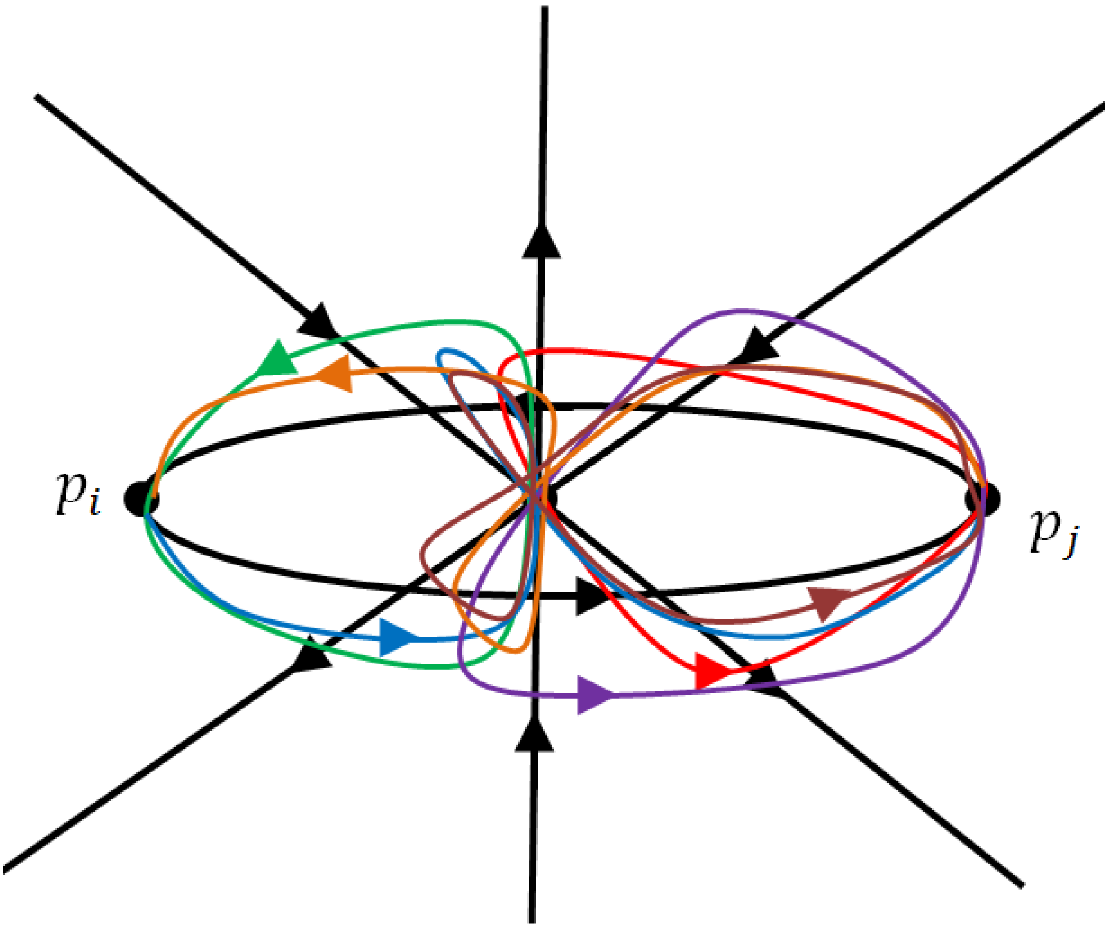

Fig. 2.12: The picture in $\bar{\Sigma}$.

Therefore, we have to add 6 new lifts. Schematically we can draw our lift as:

Fig. 2.13: The picture in $\bar{\Sigma}$.

So we can see that this lift agree with the lift of γ_2

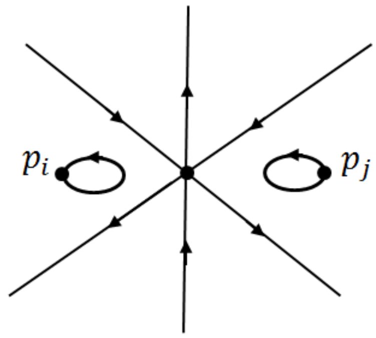

Fig. 2.14: The picture in $\bar{\Sigma}$.

Case 3.

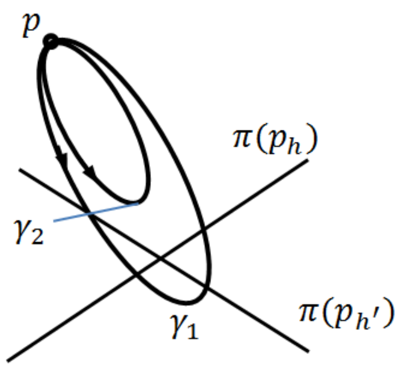

Fig. 2.15: The picture in $\bar{S}$.

Because of proposition 2.3 the covering Σ has at least 3 sheets. If we lift this picture to Σ, we can get three possible pictures, namely, two cases where p_h and $p_{h'}$ intersect each other, which distinguish by the position of the intersection point and the branch point. In the case 3.1. the branch point is located before the intersection point with respect to orientation of lines of spectral network. In the case 3.2. the branch point is located after the intersection point with respect to orientation of lines of spectral network. In the case 3.3. lines p_h and $p_{h'}$ do not intersect.

Case 3.1. We have exactly 3 standard lifts of γ_1 which intersect spectral network in the following position:

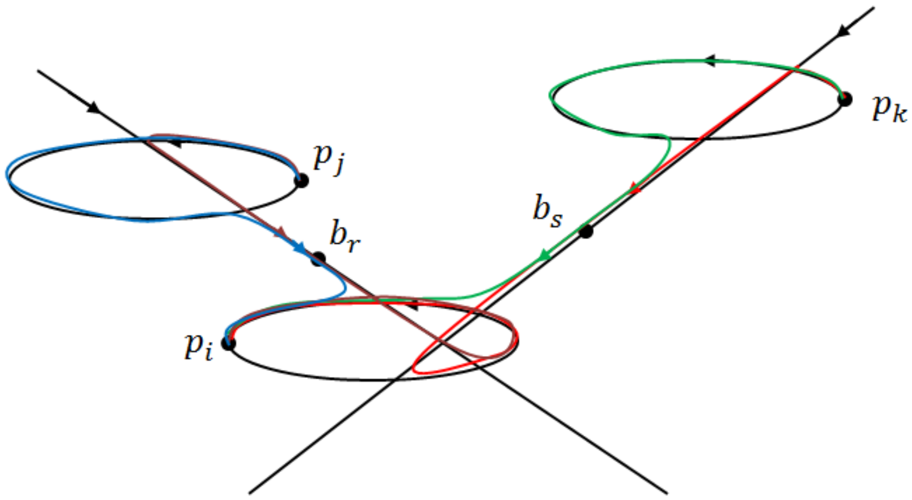

Fig. 2.16: The picture in $\bar{\Sigma}$.

Therefore, we have to add 4 new lifts. Schematically we can draw our lift as:

Fig. 2.17: The picture in $\bar{\Sigma}$.

So we can see that this lift agree with the lift of γ_2, which consist only of standard lifts.

Case 3.2. We have exactly 3 standard lifts of γ_1 which intersect spectral network in following position:

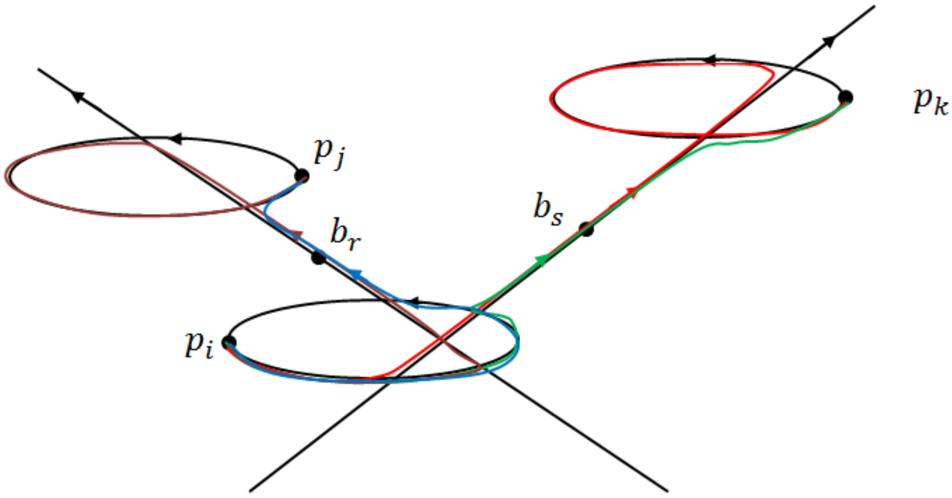

Fig. 2.18: The picture in $\bar{\Sigma}$.

Therefore, we have to add 4 new lifts. Schematically we can draw our lift as:

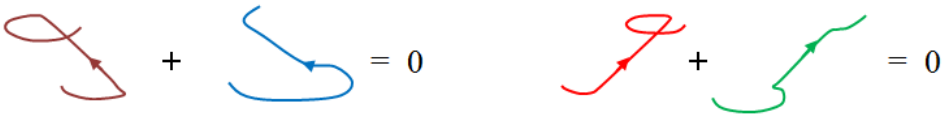

Fig. 2.19: The picture in $\bar{\Sigma}$.

So we can see that this lift agree with the lift of γ_2, which consist only of standard lifts.

Case 3.3. If the covering Σ has at least 4 sheets, then it can be possible that we have exactly 4 lifts of γ_1 which intersect two lines p_h and $p_{h'}$ of spectral networks but these lines p_h and $p_{h'}$ do not intersect each other. This case is similar to case 1 applied twice.

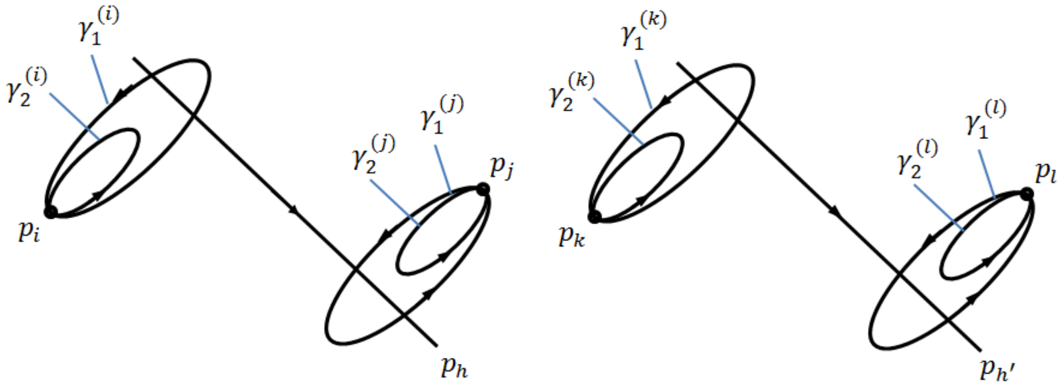

Fig. 2.20: The picture in $\bar{\Sigma}$.

□

Remark 2.7. *For the kind of intersection, which is forbidden in remark 2.2, the path lifting rule defined in this section is not homotopically invariant.*

2.3 Non-abelianisation map

Now we are ready to construct a non-abelianisation map using spectral networks. We consider a surface S with punctures $P = \{s_1, ..., s_n\}$, a $k:1$ branched covering $\pi\colon \bar{\Sigma} \to \bar{S}$. We take some element of the character variety $X(\pi_1(\Sigma), GL(1,\mathbb{C}))$ which yields us a line bundle $\pi_\Sigma\colon E \to \Sigma$ over Σ with a flat connection ∇'. We also fix a spectral network $\mathcal{W}$ on Σ.

Further, we fix $p \in S \setminus W_S$, $v \in T_pS$, a non-zero vector field V' on Σ such that $D_q\pi(V'(q)) = v$ for all $q \in \pi^{-1}(p)$

Using a spin structure on Σ, which we can construct using the vector field V' on Σ, and proposition 1.8 we get a flat twisted connection on E.

We construct a representation $\rho\colon \pi_1(S,p) \to GL(k,\mathbb{C})$. We consider:

$$V_p := \bigoplus_{i=1}^{k} E_{p_i},$$

where $\{p_1, \ldots p_k\} = \pi^{-1}(p)$, $E_p = \pi_\Sigma^{-1}(p_i)$. We have the natural basis of V_p, namely $(e_1, ..., e_k)$, where (e_i) is a basis of E_{p_i}.

If we consider a regular curve γ on S such that $[\gamma] \in \pi_1^s(S,p,v)$, we can lift it to Σ with respect to the spectral network $\mathcal{W}$. So we get an element $x \in HPA(\Sigma, V')$, which is by definition a finite sum of some curves γ_r on Σ, $r \in I$, I is a finite index set such that

$$\gamma_r(0), \gamma_r(1) \in \pi^{-1}(p) \tag{2.1}$$

and

$$x = \sum_{r \in I} \gamma_r.$$

We consider a vector $w = \sum_{i=1}^{k} a_i e_i \in V_p$, $i \in \{1, .., k\}$. Because of (2.1) the twisted connection gives us the element

$$\nabla'(x) = \sum_{r \in I} T_{\gamma_r} \in \bigoplus_{\tilde{p}, \tilde{q} \in \pi^{-1}(p)} L(E_{\tilde{p}}, E_{\tilde{q}}),$$

where T_{γ_r} are parallel transport operators along γ_r given by twisted connection on Σ.

For each γ_r such that $\gamma_r(0) = p_i$, $\gamma_r(1) = p_j$, $i, j \in \{1, ..., k\}$ we can consider $T_{\gamma_r} e_i = t_k e_j \in E_{p_j}$. We can extend T_{γ_r} to a linear map on V_p by the following rule:

$$T_{\gamma_r} e_l = \begin{cases} t_k e_j, & l = i \\ 0, & l \neq i \end{cases}$$

We define the map by $T_\gamma \colon V_p \to V_p$ by the rule $T_\gamma = \sum_{r \in I} T_{\gamma_k}$. By construction this map is linear and since the path lifting using spectral network is homotopically invariant and the connection on Σ is flat, T_γ depends only on the homotopy class of γ in $\pi_1(S, p, v)$.

So we get a map $\rho \colon \pi_1^s(S, p, v) \to GL(V_p)$ which is a group homomorphism. It means that we have a the twisted representation. After using of a spin structure on S we get an element in the character variety $X(\pi_1(S), GL(k, \mathbb{C}))$.

So we see that an element of the character variety $X(\pi_1(\Sigma), GL(1, \mathbb{C}))$, spin structures on S and Σ and a spectral network $\mathcal{W}$ yield an element of the character variety $X(\pi_1(S), GL(k, \mathbb{C}))$.

$$\pi_1(S, p) \xrightarrow[\text{on } S]{\text{spin str.}} \pi_1^s(S, p, v) \xrightarrow[\text{network}]{\text{spectral}} HPA(\Sigma, W) \xrightarrow[\text{on } \Sigma]{\text{twisted conn.}} GL(k, \mathbb{C})$$

So we get a non-abelianisation map.

2.4 Invariant flag

In this section we show that the parallel transport operator along a peripheral curve on S always has a natural invariant flag. This means that a spectral network over

S always yields a natural map $X(\pi_1(\Sigma), GL(1,\mathbb{C})) \to X^d(\pi_1(S), GL(k,\mathbb{C}))$. We will use this map later to define coordinates on the character variety $X(\pi_1(S), GL(k,\mathbb{C}))$.

We fix a base point p on S and consider a regular peripheral curve γ on S around a puncture $s \in P$ with $\gamma(0) = \gamma(1) = p$ [see fig. 2.21].

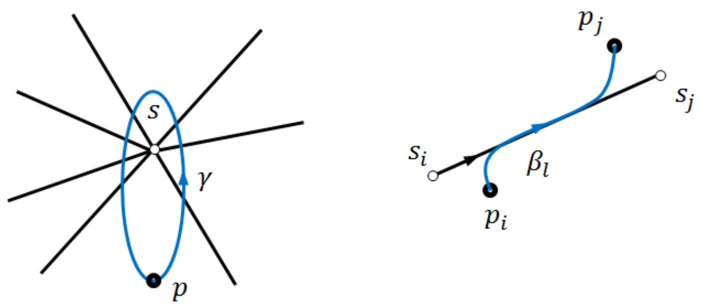

Fig. 2.21: The picture in S (left) and in Σ (right).

We lift this curve to Σ with respect to the spectral network. So we get k standard lifts $\gamma^{(i)}$, $i \in \{1, ..., k\}$, $\gamma^{(i)}$ goes around $s_i \in P'$ and some additional curves β_l, $l \in I$, I is an index set. We assume that the order of spectral network on punctures $s_1, ...s_k$ agree with the natural order on $\{1, ..., k\}$. That means that $s_i < s_j$ if and only if $i < j$.

By definition of the spectral network the paths of spectral networks on Σ go form a puncture with a smaller number to a puncture with a bigger number. Therefore, the parallel transport along each β_l gives us a linear map $T_{\beta_l}\colon E_{p_i} \to E_{p_j}$ with $i < j$.

We get the linear map $T_\gamma\colon V_p \to V_p$, whose matrix T^γ in the basis $(e_1, ..., e_k)$, where (e_i) is a basis of E_{p_i} have an lower triangular form:

$$T^\gamma = \begin{pmatrix} g_1 & 0 & \dots & 0 \\ * & g_2 & \dots & 0 \\ .. & \dots & \dots & \dots \\ * & * & \dots & g_k \end{pmatrix},$$

where g_i are homotopy constants of standard lifts $\gamma^{(i)}$. That means that the flag

$$F_\gamma = \begin{pmatrix} \langle e_k \rangle \\ \langle e_{k-1}, e_k \rangle \\ \dots \\ \langle e_1, ..., e_k \rangle \end{pmatrix}$$

is an invariant by T_γ. This flag depends on the spectral network and on the base point but not on the flat connection on Σ. In the next section we show how the matrix T^γ changes if we change the base point $p \in S$ and so we will see how this invariant flag changes if we change a base point.

So we have shown that the map defined in the section 2.3 is actually a map $X(\pi_1(\Sigma), GL(1,\mathbb{C})) \to X^d(\pi_1(S), GL(k,\mathbb{C}))$.

2.5 Properties of a non-abelianisation map given by spectral networks

Remark 2.8. *The non-abelianisation map described in the section 2.3 is a continuous finite-to-one map.*

There is a hypothesis that for each surface S with punctures a $k:1$ branched covering $\pi\colon \Sigma \to S$ exists such that

$$\dim(X(\pi_1(\Sigma), GL(1,\mathbb{C}))) = \dim(X(\pi_1(S), GL(k,\mathbb{C})))$$

If this is right, then the non-abelianisation map described in the section 2.3 has an open image.

It is possible to show that this map is a finite-to-one map. One can define symplectic structures on $X(\pi_1(\Sigma), GL(1,\mathbb{C}))$ and $X^d(\pi_1(S), GL(k,\mathbb{C}))$ and show that the map $X(\pi_1(\Sigma), GL(1,\mathbb{C})) \to X^d(\pi_1(S), GL(k,\mathbb{C}))$ from the section 2.4 is a local symplectomorphism (see [6]). This proves that this map is locally injective, and because the natural map $X^d(\pi_1(S), GL(k,\mathbb{C})) \to X(\pi_1(S), GL(k,\mathbb{C}))$ is a finite-to-one map, the non-abelianisation map $X(\pi_1(\Sigma), GL(1,\mathbb{C})) \to X(\pi_1(S), GL(k,\mathbb{C}))$ from the section 2.3 is a finite-to-one map.

In cases $k=2$ and $k=3$ it can be shown using Fock-Goncharov coordinates that this map has a dense image.

2.6 Change of the base point

Let $\pi\colon \bar{\Sigma} \to \bar{S}$ be a $n:1$ branched covering, $\pi_\Sigma\colon E \to \Sigma$ be a line bundle with a flat connection ∇ and let $\mathcal{W}$ be a small spectral network of rank $n \in \mathbb{N}$ on Σ.

We fix the point $p \in S$ and the tangent vector $v \in T_pS$. This gives us the representation of $\pi_1^s(S,p,v)$ in $GL(V_p)$ that means we have the group homomorphism $\Gamma_p\colon \pi_1^s(S,p,v) \to GL(V_p)$ where $\pi_1^s(S,p,v)$ is the fundamental group with signs of S with the base point p and tangent vector v. We denote by e the neutral element of this group. If two curves represent the same element of $\pi_1^s(S,p,v)$, we will denote this with the symbol $\equiv$.

The (twisted) flat connection on S (resp. on Σ) gives us for each regular path γ the parallel transport operator which we will denote T_γ. With respect to bases in start and finish points the operator T_γ has the matrix which we will denote $T^\gamma = (t_{ij}^\gamma)$.

In this paragraph we want to find out how this representation changes if we change the base point $p \in S$ and the corresponding tangent vector $v \in T_pS$.

We choose the other point $p' \in S$ and the tangent vector $v' \in T_{p'}S$ and choose two smooth curves $\xi\colon [0,1] \to S$ and $\eta\colon [0,1] \to S$ with $\xi(0) = \eta(1) = p$, $\xi(1) = \eta(0) = p'$, $\dot\xi(0) = \dot\eta(1) = v$, $\dot\xi(1) = \dot\eta(0) = v'$ and $\xi * \eta = \pm e \in \pi_1^s(S,p,v)$ [see fig. 2.22]. Then for each element $\alpha' \in \pi_1^s(S,p',v')$ we have $\alpha := \xi * \alpha' * \eta \in \pi_1^s(S,p,v)$. Then

$$T_\alpha = T_\eta T_{\alpha'} T_\xi.$$

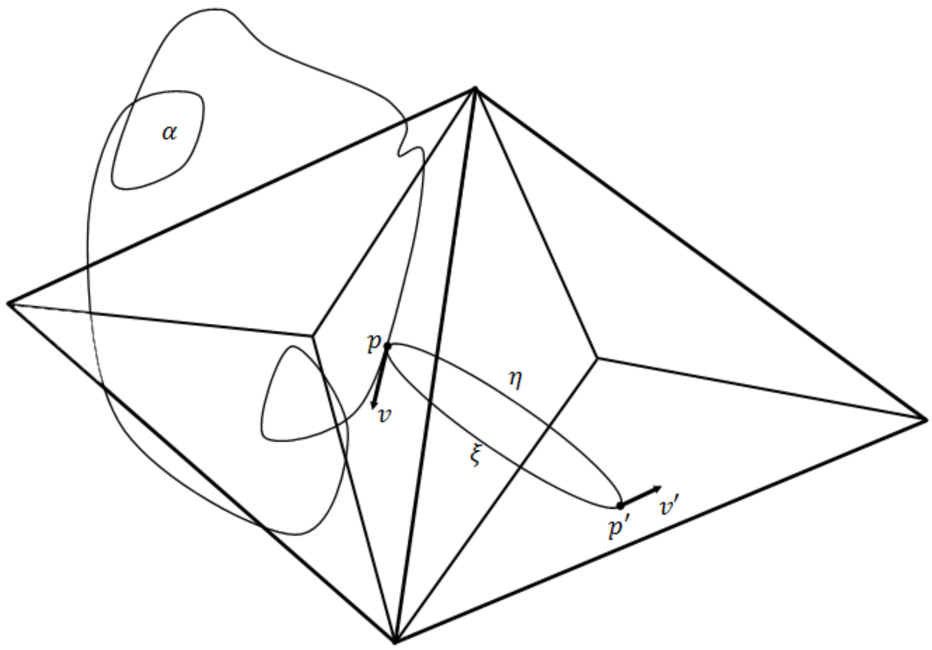

Fig. 2.22: The picture in S.

We want to find out how the maps T_ξ and T_η look like. The goal of this paragraph is to proof the following proposition:

Proposition 2.9. *There are bases of E_p and E'_p such that the matrix T^ξ and T^η of the parallel transport operator T_ξ and T_η in these bases looks like $T^\xi = (t_{ji}^\xi)$,*

$T^\eta = (t^\eta_{ji})$ *where*

$$t^\xi_{ji} = \frac{1}{l_j \nu_{ji}} \sum_{\beta \in \Xi_{ji}} \tilde{b},$$

$$t^\eta_{ji} = \frac{1}{k_i \nu_{ji}} \sum_{\beta \in H_{ji}} \tilde{b}$$

(the sense of all coefficients will be defined later).

Proof. To prove this, we have to lift ξ and η to Σ with respect to the spectral network. We will prove the proposition for ξ, the proof for η is analog.

The lift of ξ is an element $\hat{\xi} \in HPA(\Sigma)$ which we can write

$$\hat{\xi} = \sum_{\beta \in \Xi} \beta$$

where Ξ is the set of curves on Σ such that $\beta(0) \in \pi^{-1}(p)$, $\beta(1) \in \pi^{-1}(p)$, $\dot{\beta}(0) = (D_{\beta(0)}\pi)^{-1}(v)$ $\dot{\beta}(1) = (D_{\beta(1)}\pi)^{-1}(v')$ for all $\beta \in \Xi$. The maps $(D_{\beta(0)}\pi)^{-1}$ and $(D_{\beta(1)}\pi)^{-1}$ are well-defined because π is a diffeomorphism in a neighborhood of all $r \in \pi^{-1}(p) =: \{p_1, ..., p_n\}$ and $r' \in \pi^{-1}(p') =: \{p'_1, ..., p'_n\}$. We denote also $v_i = (D_{p_i}\pi)^{-1}(v)$ and $v'_i = (D_{p'_i}\pi)^{-1}(v')$.

Further, for each p_i the set Ξ always contains the standard lift (without respect to spectral network) with the start point p_i and end point p'_i which we denote ξ_i (we can always number the points of $\pi^{-1}(p')$ so that this is satisfied).

The same applies for η. We have the lift $\hat{\eta} \in HPA(S)$ and the set H of lifts and n standard lifts η_i with $\eta_i(0) = p'_i$. Moreover, because of $\xi * \eta = \pm e$ we have $\eta_i(1) = p_i$.

Because

$$V_p = \bigoplus_{i=1}^{n} E_{p_i}$$

the choice of the basis (e_i) of each E_{p_i} gives us a basis of V_p. The same applies to the point p': (e'_i) is the basis of $E_{p'_i}$, and $(e'_1, ..., e'_n)$ is the basis of $V_{p'}$. With respect to these two bases the map T_ξ is represented by the matrix $T^\xi = (t^\xi_{ij})_{i,j=1,...,n}$.

We denote

$$\Xi_{ji} = \{\beta \in \Xi \mid \beta(0) = p_i, \beta(1) = p'_j\}.$$

Then each $\beta \in \Xi_{ji}$ gives us the number (one dimensional matrix) b such that $T_\beta(e_i) = b e'_j$, and, obviously, the element t^ξ_{ji} of T^ξ is exactly the sum of all this b for all $\beta \in \Xi_{ji}$:

$$t^\xi_{ji} = \sum_{\beta \in \Xi_{ji}} b.$$

Therefore, we need to compute the numbers b. For vector bundles with flat connection is much easier to compute these numbers for closed curves. Hence, we can complement each $\beta \in \Xi_{ji}$ with the fixed smooth curve with start point p'_j and end point p_i.

First of all, we can go from p'_j to p_j along η_j and then from p_j we have to go to p_i. For this reason, we choose for each pare (p_i, p_j) the smooth curve $\gamma_{ij} \colon [0,1] \to \Sigma$ such that $\gamma(0) = p_i, \gamma(1) = p_j, \dot{\gamma}(0) = v_i, \dot{\gamma}(1) = v'_j$. We denote

$$T_{\gamma_{ij}} e_i = \nu_{ij} e_j,\ T_{\xi_i} e_i = k_i e'_i,\ T_{\eta_i} e'_i = l_i e_i.$$

The numbers k_i, l_i, ν_{ij} are given by flat connection ∇' on Σ. For simplicity, we assume that γ_{ii} is the trivial curve then $\nu_{ii} = 1$ for all $i = 1, ..., n$.

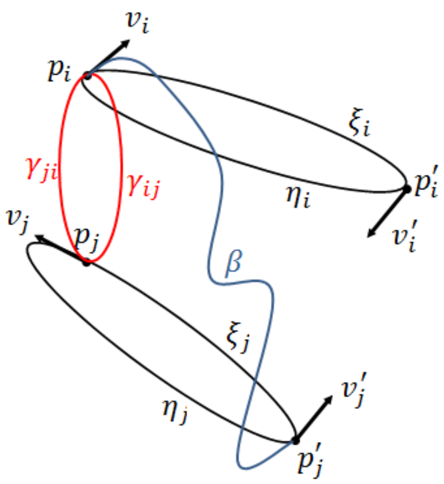

Fig. 2.23: The picture in Σ.

We consider the curve $\tilde{\beta} = \beta * \eta_j * \gamma_{ji}$. For this curve we can compute the number $\tilde{b}$ such that $T_{\tilde{\beta}} e_i = \tilde{b} e_i$ and then because of $\tilde{b} = b l_j \nu_{ji}$ we have

$$b = \frac{\tilde{b}}{l_j \nu_{ji}}.$$

Therefore,

$$t^{\xi}_{ji} = \frac{1}{l_j \nu_{ji}} \sum_{\beta \in \Xi_{ji}} \tilde{b}. \tag{2.2}$$

The same applies for η:

$$t^{\eta}_{ji} = \frac{1}{k_i \nu_{ji}} \sum_{\beta \in H_{ji}} \tilde{b}.$$

$\square$

Remark 2.10. *Because of independent choice of the bases of* T_pS *and* $T_{p'}S$ *we can always choose the basis* $(e'_1, ..., e'_n)$ *of* $T_{p'}S$ *so that* $k_i = 1$ *or* $l_i = 1$. *For this reason, we can assume* $e'_i = T_{\xi_i} e_i$ *or* $e'_i = T^{-1}_{\eta_i} e_i$.

Moreover, because $\xi * \eta = \pm e$, *if* $k_i = 1$, *then* $l_i = \pm 1$ *and*

$$e'_i = T_{\xi_i} e_i \text{ and } e'_i = \pm T^{-1}_{\eta_i} e_i.$$

Remark 2.11. *If* Ξ *contains only standard lift of* ξ *and* H *contains only standard lift of* η *we have*

$$t^{\xi}_{ii} = k_i, t^{\eta}_{ii} = l_i, t^{\xi}_{ji} = t^{\eta}_{ji} = 0 (i \neq j).$$

This case we have if ξ *and* η *do not intersect the spectral network. With remark 2.10 we can assume in this case that* $T^{\xi} = 1_n = \pm T^{\eta}$.

Now we can find out how the elements of the matrix $T^{\alpha'} = (t^{\alpha'}_{ij})$ of operator $T_{\alpha'} \colon V_{p'} \to V_{p'}$ look like.

We denote $\hat{\alpha}' \in HPA(\Sigma)$ the lift of α' in Σ with respect to spectral network. Then

$$\hat{\alpha}' = \sum_{\beta \in A'} \beta$$

where A' is the set of all lifts of α'. Let $A'_{ji} \subseteq A'$ is the set of all lifts β of α' such that $\beta(0) = p'_i, \beta(1) = p'_j$.

Corollary 2.12.

$$t^{\alpha'}_{ji} = \frac{k_j l_i \nu_{ij}}{d_{ji}} \sum_{\beta \in A'_{ji}} \tilde{b},$$

where $T_{\delta_{ji}} e_i = d_{ji} e_i$ *and* $\delta_{ji} = \gamma_{ij} * \xi_j * \gamma'_{ji} * \eta_i$

Proof. We know that

$$t^{\alpha'}_{ji} = \sum_{\beta \in A'_{ji}} b$$

where $T_\beta e'_i = b e'_j$ for $\beta \in A'_{ji}$.

If also for each pare (p'_i, p'_j) the smooth curve $\gamma'_{ij} \colon [0, 1] \to \Sigma$ is chosen such that $\gamma'(0) = p'_i, \gamma'(1) = p'_j, \dot{\gamma}'(0) = v'_i, \dot{\gamma}'(1) = v'_j$ [see fig. 2.24] and

$$T_{\gamma'_{ij}} e'_i = \nu'_{ij} e'_j,$$

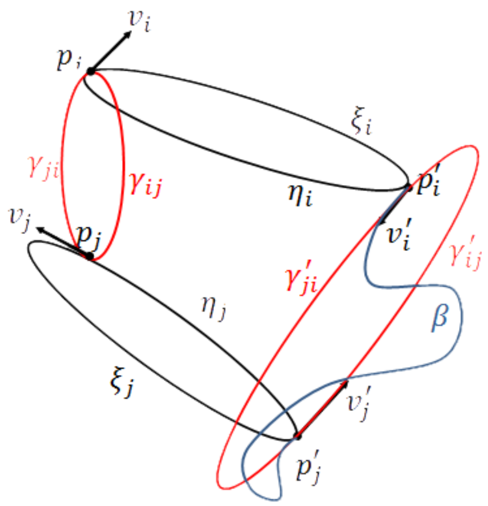

Fig. 2.24: The picture in Σ.

then we can write $\tilde{\beta} := \beta * \gamma'_{ji}$ and $T_{\tilde{\beta}} e'_i = \tilde{b} e'_j$ where $\tilde{b} = b\nu'_{ji}$. Therefore,

$$t_{ji}^{\alpha'} = \frac{1}{\nu'_{ji}} \sum_{\beta \in A'_{ji}} \tilde{b}.$$

We want to eliminate ν'_{ji}. We denote $\delta_{ji} = \gamma_{ij} * \xi_j * \gamma'_{ji} * \eta_i$ then $d_{ji} = \nu_{ij} k_j \nu'_{ji} l_i$ where $T_{\delta_{ji}} e_i = d_{ji} e_i$. Therefore,

$$\nu'_{ji} = \frac{d_{ji}}{k_j l_i \nu_{ij}}$$

$$t_{ji}^{\alpha'} = \frac{k_j l_i \nu_{ij}}{d_{ji}} \sum_{\beta \in A'_{ji}} \tilde{b}.$$

$\square$

2.7 Fock-Goncharov coordinates

In this section we introduce Fock-Goncharov coordinates, which parameterize the space $\mathcal{P}_3^n$ of pairs of convex n-gons in $\mathbb{RP}^2$, one inscribed into the other, and considered up to the action of $PSL(3, \mathbb{R})$ [4].

We consider the pair of convex n-gons in $\mathbb{RP}^2$, one inscribed into the other and cut the inscribed polygon into triangles and mark two distinct points on every edge of the triangulation except the edges of the polygon. Mark also one point inside each triangle. The following theorem proved in [4] gives us the canonical bijection between $\mathcal{P}_3^n$ and assignments of positive real numbers to the marked points.

Theorem 2.13. *There exists a canonical bijective correspondence between the space $\mathcal{P}_3^n$ and assignments of positive real numbers to the marked points.*

Definition 2.14. *The assignments of numbers to the marked points from the theorem 2.13 are called* ***Fock-Goncharov coordinates*** *on $\mathcal{P}_3^n$.*

We show how this canonical bijection works. First, we consider the case of $\mathcal{P}_3^3$. We consider the pair of triangles as a collection of three lines l_A, l_B, l_C in $\mathbb{RP}^2$ with a point p_A, p_B, p_C on each of lines. On each line l_i ($i \in \{A, B, C\}$) we have two points p_{ij}, p_{ik} of intersection with two other lines l_j and l_k ($\{j,k\} = \{A,B,C\} \setminus \{i\}$) [see fig. 2.25]. Moreover, for each line l_i we have two point p_j, p_k ($\{j,k\} = \{A,B,C\}\setminus\{i\}$) which do not lie on this line. These two points define us the line l_{jk} which intersects l_i in the point $p_{i,jk}$. So we get 4 points of each line.

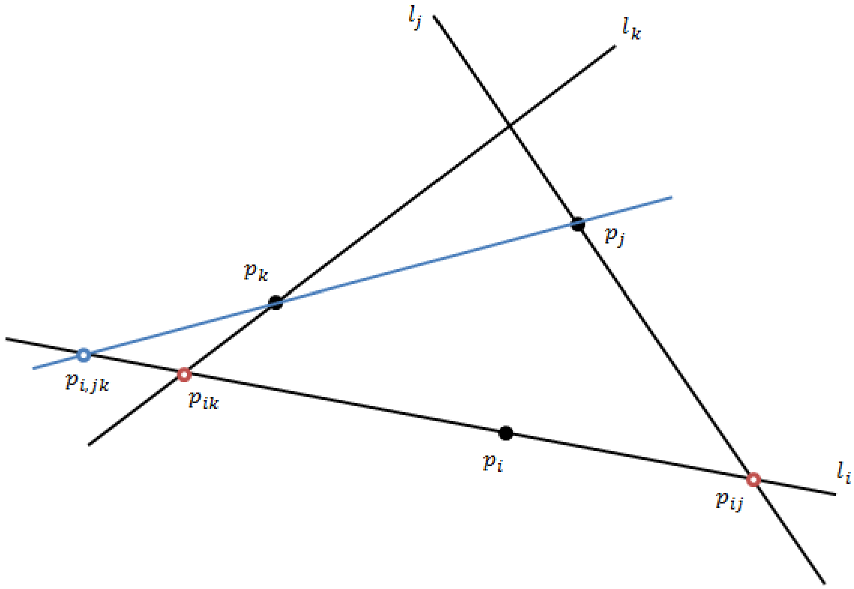

Fig. 2.25: Picture in $\mathbb{RP}^2$.

The coordinate $\mathcal{X}_{ABC}$ which we also denote by X if the triangle is fixed is defined as a cross ratio

$$\mathcal{X}_{ABC} = X = -[p_i, p_{i,jk}, p_{ij}, p_{ik}],$$

where $\{i,j,k\} = \{A,B,C\}$. Because the cross ratio is a projective invariant, this does not depend on the choice of $i \in \{A,B,C\}$ and on the choice of the affine chart.

Further, we consider the case of $\mathcal{P}_3^4$. In this case we have to choose one of two possible triangulation of the quadrangle. With respect to this triangulation two coordinates $\mathcal{X}_{ABC}$ and $\mathcal{X}_{DBC}$ which we also denote by Y if the frangulation is fixed. We also define the coordinate $\mathcal{Z}_{BC}$ which we also denote by Z if the triangulation is fixed as a cross ratio

$$\mathcal{Z}_{BC} = Z = [p_{CB}, p_{C,AB}, p_C, p_{C,DB}],$$

where p_{CB} which is an intersection point of lines l_C and l_B, $p_{C,AB}$ which is an intersection point of lines l_C and $\overline{p_A p_B}$ and $p_{C,DB}$ which is an intersection point of lines l_C and $\overline{p_D p_B}$ (see fig. 2.26). Because the cross ratio is a projective invariant, this does not depend on the choice of the affine chart.

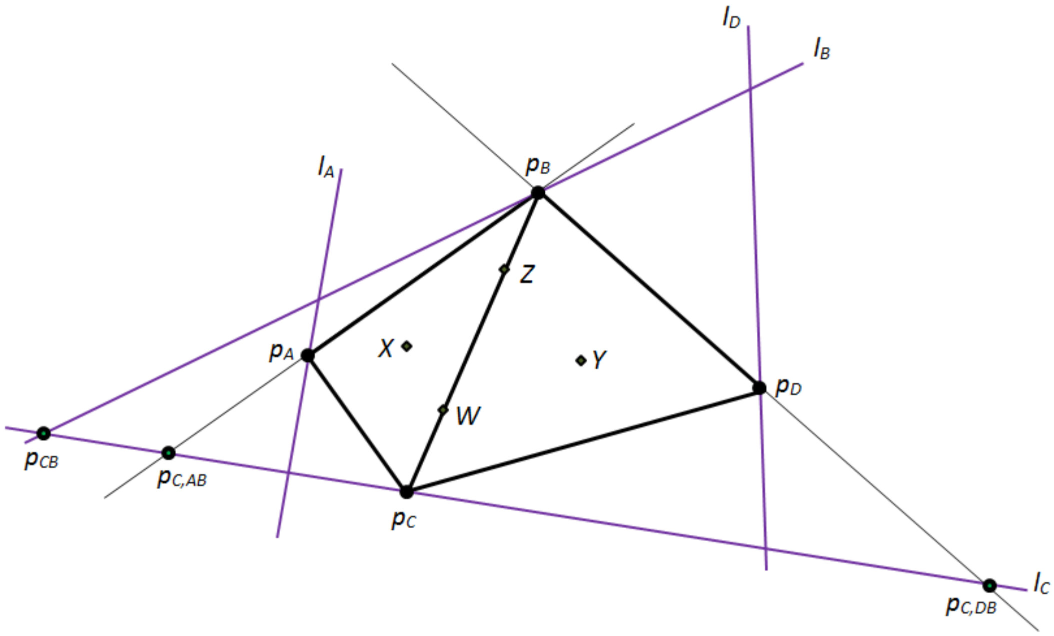

Fig. 2.26: Picture in $\mathbb{RP}^2$.

In the same way the coordinate $\mathcal{Z}_{CB}$ which we also denote by W is defined:

$$\mathcal{Z}_{CB} = W = [p_{BC}, p_{B,DC}, p_B, p_{B,AC}],$$

where p_{BC} which is an intersection point of lines l_B and l_C, $p_{B,AC}$ which is an intersection point of lines l_B and $\overline{p_A p_C}$ and $p_{B,DC}$ which is an intersection point of lines l_B and $\overline{p_D p_C}$.

Finally, in general case, if we have the pair of n-gons in $\mathbb{RP}^2$, one inscribed into the other, we cut the inscribed polygon into triangles. For each triangle of this triangulation the $\mathcal{X}$-coordinate is defined, for each pair of triangles with a common side two $\mathcal{Z}$-coordinates are defined. So we get the collection of Fock-Goncharov coordinates described in the theorem 2.13.

Now we want to describe, how Fock-Goncharov coordinates change if we change the triangulation. We consider the simplest case if change the triangulation only in one quadrangle [see fig. 2.27]. This transformation is called flip. We can always get the general case as a sequence of flips.

In [4] the formulas are given, which express how coordinates X', Y', Z', W' depend on X, Y, Z, W:

$$X' = \frac{1+Z}{XZ(1+W)}, \quad Y' = \frac{1+W}{YW(1+Z)},$$
$$Z' = X\frac{1+W+WY+WYZ}{1+Z+ZX+ZXW},$$
$$W' = Y\frac{1+Z+ZX+ZXW}{1+W+WY+WYZ}.$$

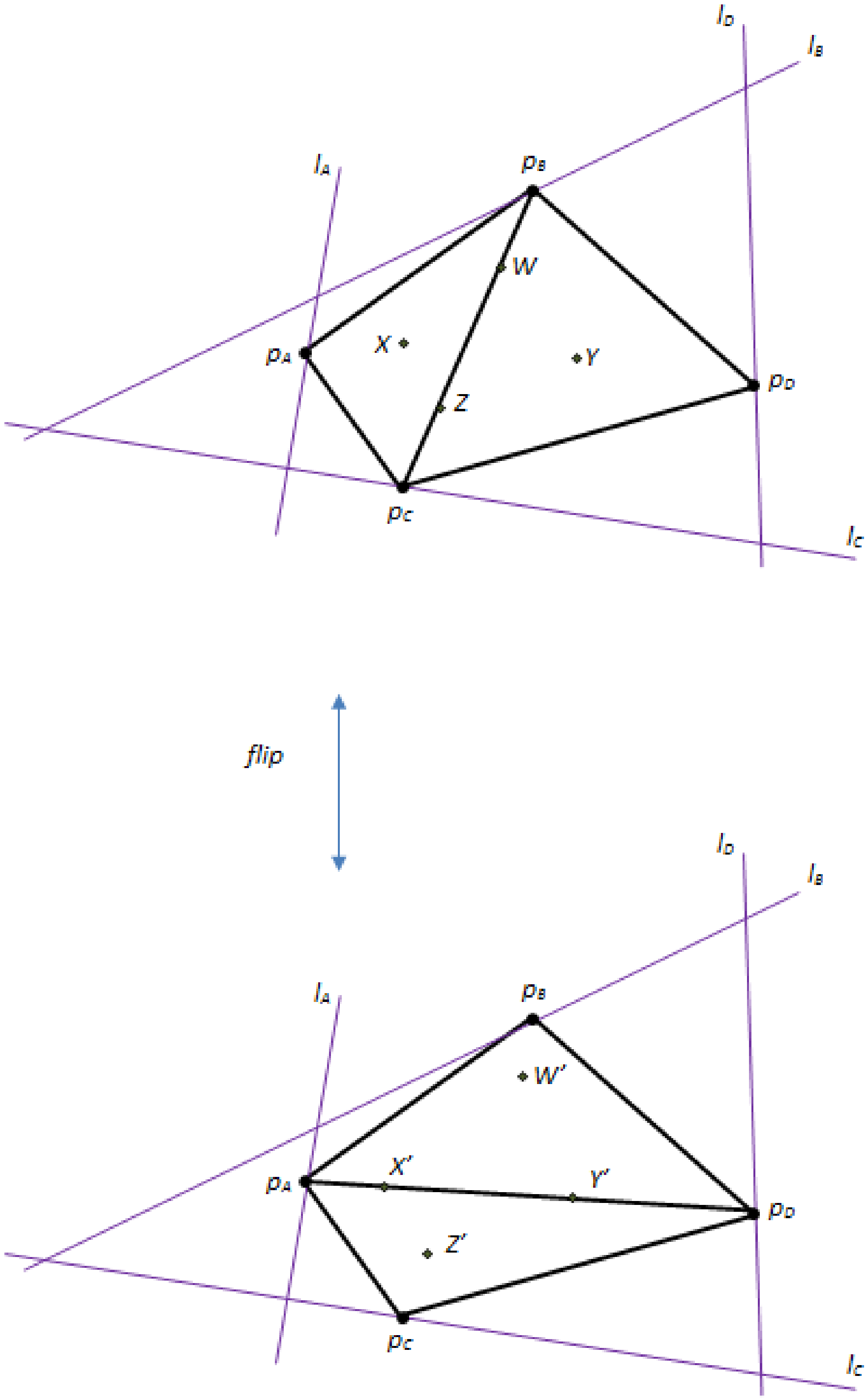

Fig. 2.27: Flip. Picture in $\mathbb{RP}^2$.

2.8 Fock-Goncharov coordinates and invariant flag

We consider the surface S with punctures $P = \{s_1, ..., s_n\}$, an ideal triangulation $\mathcal{T}$ of S and an element $[\rho, D] \in X^d(\pi_1(S), G, \mathcal{T})$ like in the section 1.2, where G is one of the following groups $GL(3, K)$, $SL(3, K)$, $PGL(3, K)$, $K \in \{\mathbb{R}, \mathbb{C}\}$. If we fix $p \in S$, then we can choose a representation $\rho\colon \pi_1(S, p) \to G$ of the class $[\rho, D]$.

For the matrix $\rho([\gamma])$, where γ is a peripheral closed curve, an invariant flag $F_\rho(\gamma) = (V_1(\gamma), V_2(\gamma), V_3(\gamma))$ is fixed, where $\dim(V_i(\gamma)) = i$, $i \in \{1, 2, 3\}$.

Because flags of representation are transversal, for two triangles ABC and DBC we get two triangles $p_A p_B p_C$ and $p_D p_B p_C$ in $K\mathbb{P}^2$ with the common side $p_B p_C$, which are inscribed in the 4-gon generated by lines l_A, l_B, l_C, l_D.

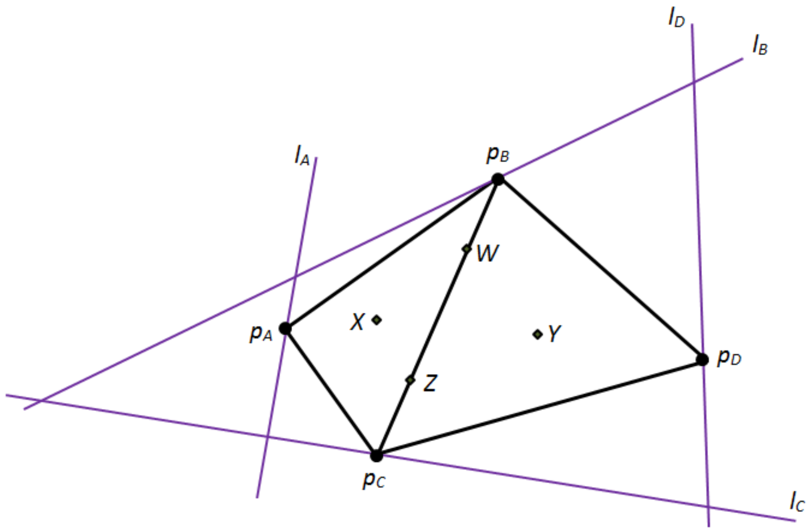

Fig. 2.28: The picture in $K\mathbb{P}^2$.

We can define the corresponding Fock-Goncharov coordinates X, Y, Z, W as in the previous section, but in this case all points have coordinates in K and, therefore, X, Y, Z, W can also take values in K. Moreover, these coordinates do not depend on the choice of representation in the class $[\rho, D]$ because they are projective invariants and, therefore, are invariant by conjugation of corresponding matrices. So we see that an element $[\rho, D]$ of the decorated character variety $X^d(\pi_1(S), G, \mathcal{T})$ yields us a collection of Fock-Goncharov coordinates.

Definition 2.15. *We say that for the ideal triangulation of S a collection of Fock-Goncharov coordinates is defined if for each pair of triangles (T_1, T_2) with a* *common side a tuple* $(X, Y, Z, W) \in K^4$ *is defined so that the corresponding tuple for* (T_2, T_1) *is* (Y, X, W, Z).

We will see later that an ideal triangulation of S and a corresponding for this triangulation collection of Fock-Goncharov coordinates define a representation of $\pi_1(S)$ but only into the group $PGL(3, K)$.

2.9 Free group representation

We consider a surface S with punctures. Let $\mathcal{T}$ be an ideal triangulation of S. We number its elements by $\{1, ..., r\} \subseteq \mathbb{N}$.

$$\mathcal{T} = \{T_i \mid i \in \{1, ..., r\}\}.$$

The natural order on the set $\{1, ..., r\}$ induce a total order ϱ on triangles on $\mathcal{T}$. This order ϱ is an additional structure on $\mathcal{T}$.

Let C be the set of all sides of triangulation. We generate a free group F_C by this set. Sometimes we will identify elements of F_C with words over $C \cup C^{-1}$. Moreover, for each element $b \in F_C$ there is the unique shortest word over $C \cup C^{-1}$ which represents b.

We also consider a point $p \in S$ which does not lie on lines of triangulation. We want to construct a representation $\chi\colon \pi_1(S, p) \to F_C$.

To do this, we consider a loop $\gamma\colon [0, 1] \to S$, $\gamma(0) = \gamma(1) = p$. We say that γ changes the triangle by $t = t_0$ if $\gamma(t_0)$ is contained on the line of triangulation and there exist $\varepsilon > 0$ such that for all $t \in (t_0 - \varepsilon, t_0)$ and for all $t' \in (t_0, t_0 + \varepsilon)$ $\gamma(t)$ and $\gamma(t')$ lie in different triangles.

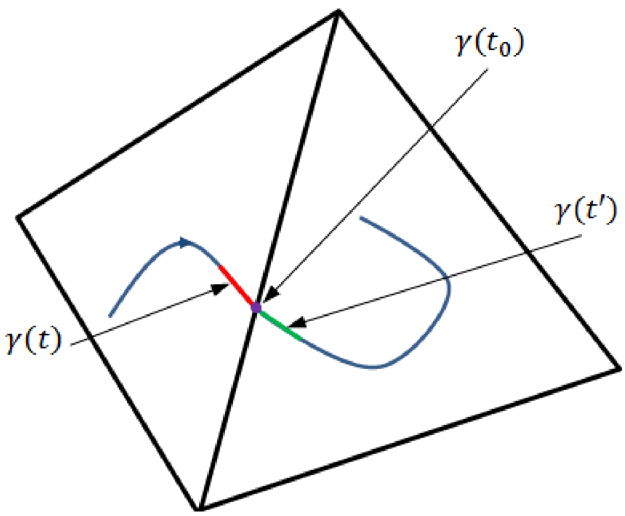

Fig. 2.29: The picture in S.

Without loss of generality we can always assume that γ changes the triangle at all points t_0 such that $\gamma(t_0)$ lies on a line of triangulation, because in $[\gamma] \in \pi_1(S,p)$ there always exist such curve.

We assume, that γ changes the triangle by following values of parameter $t_0, ..., t_r$. So we get a sequence of pairs of triangles (T_{i_0}, T_{i_1}),...,(T_{i_r}, T_{i_0}) with a common sides $c_0, ..., c_r \in C$. With this sequence we can associate an element

$$c_\gamma = c_r^{\varepsilon_0} \cdot ... \cdot c_0^{\varepsilon_0} \in F_C,$$

where $\varepsilon_l = 1$ if $i_l < i_{l+1}$ and $\varepsilon_l = -1$ otherwise.

It is easy to see that $c_{\gamma_1 * \gamma_2} = c_{\gamma_2} c_{\gamma_1}$ for two loops γ_1, γ_2. To show that the map

$$\begin{array}{rccc} \chi\colon & \pi_1(S,p) & \to & F_C \\ & [\gamma] & \mapsto & c_\gamma \end{array}$$

is an injective representation, we need the following proposition:

Proposition 2.16. *Let $p \in S$ and $[\gamma] \in \pi_1(S,p)$.*

1. *The element c_γ depends only on the homotopy class $[\gamma]$.*
2. *If $[\gamma] \neq [\delta]$, then $c_\gamma \neq c_\delta$.*

Proof. The triangulation $\mathcal{T}$ of S induces a triangulation $\mathcal{T}'$ on the universal covering $\pi\colon S' \to S$. We also choose a point $p' \in \pi^{-1}(p)$.

With the triangulation $\mathcal{T}'$ we can associate a directed graph Γ, whose vertices are triangles of $\mathcal{T}'$ and edges are common sides of triangles which are oriented consistent to the order on $\mathcal{T}$. That means that for each two triangles $T', T'' \in \mathcal{T}'$ with a common side the corresponding edge starts in a triangle T' if $\pi(T') = T_i$, $\pi(T'') = T_j$ and $i < j$ [see fig. 2.30]. Because S' is connected, the graph Γ is also connected.

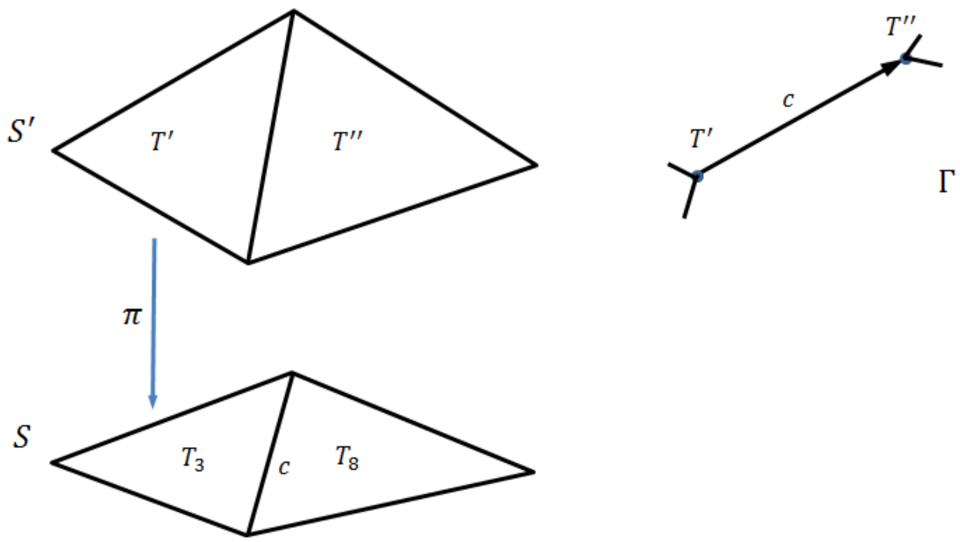

Fig. 2.30: Orientation in Γ.

By uniformization theorem the universal covering of S is homeomorphic to $\mathbb{R}^2$ because S is not compact. Each line of triangulation on S' divide $\mathbb{R}^2$ in two connected components. For the graph Γ that means that, if we remove one edge, then the graph fall in two components. That means that Γ is a tree. Moreover, Γ can be construed as a subgraph of a Cayley graph $\Gamma(F_C, C)$ if we identify $1 \in F_C$ with $T' \in \mathcal{T}'$ such that $p' \in T'$. Because F_C is free generated by C, the graph $\Gamma(F_C, C)$ is also a tree.

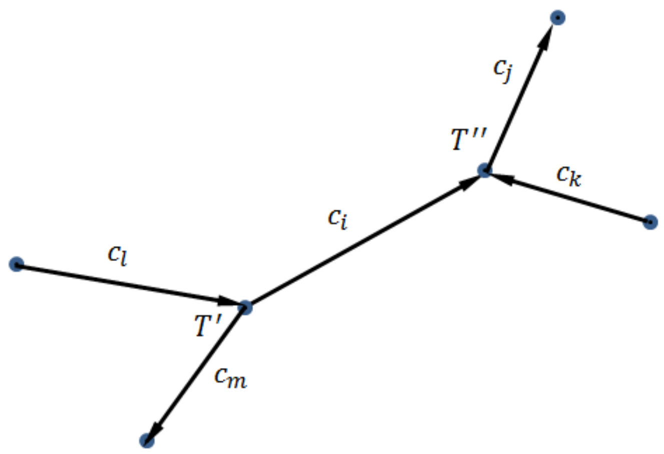

Fig. 2.31: Graph Γ.

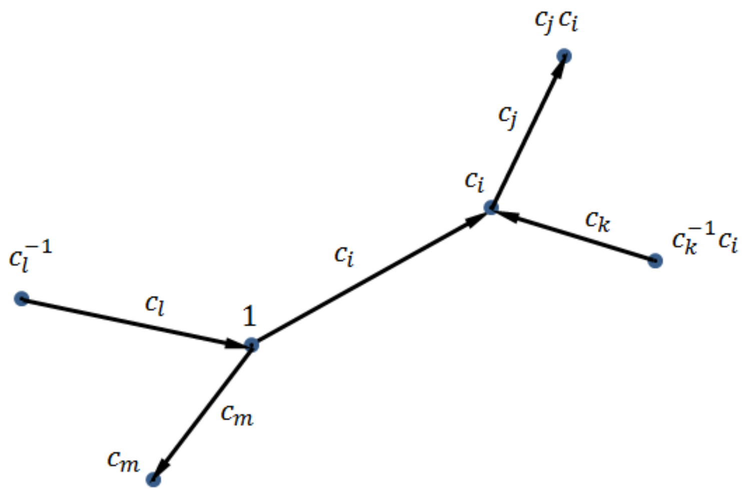

Fig. 2.32: Grapth $\Gamma(F_C, C)$.

We have to prove that $c_\gamma = 1$ for each loop γ on S with $\gamma(0) = \gamma(1) = p$ which is null homotopic. We have a unique lift γ' on S' such that $\gamma'(0) = \gamma'(1) = p'$. For γ' we can construct a path in Γ. We start in a vertex $T' \in \mathcal{T}'$ such that $p' \in T'$ and every time if γ' changes the triangle we goes along an edge, witch corresponds to the side of triangulation $\mathcal{T}'$, which γ' intersects. The path finishes also in the triangle T' because $p' \in T'$. That means that the corresponding finish vertex in the Cayley graph $\Gamma(F_C, C)$ is identity. Therefore, $c_\gamma = 1$. This proves (1).

To prove (2) we also use the universal covering $\pi\colon S' \to S$. Let $\gamma\colon [0,1] \to S$ be a loop and $c_\gamma = 1$. The corresponding to γ path in Γ is trivial. Therefore, the lift of γ to S' starts and finishes in the same triangle in S'. This means that the lift of γ is a loop in S'. Because S' is simply connected, the lift of γ is trivial and so γ is also trivial. This proves (2). $\square$

Definition 2.17. *We call the representation*

$$\begin{array}{cccc} \chi\colon & \pi_1(S,p) & \hookrightarrow & F_C \\ & [\gamma] & \mapsto & c_\gamma \end{array}$$

free group representation *with respect to the order ϱ on $\mathcal{T}$.*

Remark 2.18. *We have shown that the fundamental group can be construed as a subgroup of a free group. By Nielsen–Schreier theorem every subgroup of a free group is free. So we get a well known fact that the fundamental group of a surface with punctures is free (namely, of rank $2g+n-1$).*

Remark 2.19. *The correspondence $\gamma \mapsto c_\gamma$ can be extended in a similar way for each curve which starts in p and finishes in a point q which does not lie on the line of triangulation. This correspondence is also invariant by homotopy. If we consider all possible curves γ with this condition and generate elements $c_\gamma \in F_C$, we get a subset of F_C, we denote this subset $F_C(S,p)$.*

By construction, for all words $b \in F_C(S,p)$ all prefixes of b are contained in $F_C(S,p)$ because we can always restrict the corresponding path γ on the appropriate subinterval. Therefore, the corresponding to this subset subgraph of $\Gamma(F_C,C)$ is connected. In particular, this is a tree because $\Gamma(F_C,C)$ is it.

For this semigroup we can define the following map. We consider the universal covering $\pi\colon S' \to S$ of S and corresponding triangulation $\mathcal{T}'$. We denote by P' the set of all vertices of triangulation of $\mathcal{T}'$. We choose a lift $p' \in \pi^{-1}(p)$. This lies in a triangle T_0' on S' with vertices $s_1', s_2', s_3' \in P'$.

We consider two triangles T_1' and T_2' of triangulation $\mathcal{T}'$ with vertices $r_1', r_2', r_3' \in P'$ and $r_4', r_2', r_3' \in P'$ with a common side $c' = r_2'r_3'$. We consider $c = \pi(c')$ and define a map $f_c\colon \{r_1', r_2', r_3', r_4'\} \to \{r_1', r_2', r_3', r_4'\}$: $f_c(r_1') = r_4'$, $f_c(r_2') = r_3'$, $f_c(r_3') = r_2'$, $f_c(r_4') = r_1'$. Obviously, $f_c = f_c^{-1}$.

If we choose $s' \in P'$, then there exist a path γ' in S', which starts in p' and finishes in a triangle T', witch has s' as a vertex. For this $\pi(\gamma')$ we have a corresponding path in Cayley-Graph $\Gamma(F_C,C)$ with start point in 1, which corresponds to an element in $c_\gamma = c_m^{\varepsilon_m}...c_1^{\varepsilon_1} \in F_C(S,p)$, where $\varepsilon_i \in \{-1,1\}$. We define the action

$$\begin{array}{cccc} \psi\colon & F_C(S,p) \times \{s_1', s_2', s_3'\} & \to & P' \\ & (c_\gamma, s_i') & \mapsto & f_{c_m} \circ ... \circ f_{c_1}(s_i') \end{array}$$

Remark 2.20. *By construction, because* $f_{c_i} \circ f_{c_i} = id$*, this map is well defined (does not depend on representation of* c_γ *by a word over* $C \cup C^{-1}$*). Because* S' *is connected, this map is surjective. This map is also injective in the second variable if the first variable is fixed.*

Moreover, because the graph Γ *is a tree, for every* $s' \in P'$ *there exists the shortest path in* Γ *form* T_0' *to a triangle which has* s' *as a vertex. For this shortest path in* Γ *there exist unique up to homotopy curve* γ' *on* S'*, which starts in* p' *and finishes in a triangle whose vertex* s' *is.*

This path corresponds to a path in the subgraph of the Cayley-Graph $\Gamma(F_C, C)$ *which generates by* $F_C(S,p)$ *witch starts in* 1*. We consider the corresponding for this path shortest word* $b \in F_C(S,p)$*. It is uniquely defined and* $s' \in \psi(b, \{s_1', s_2', s_3'\})$*. So we can define the inverse map* $\psi' \colon P' \to F_C(S,p) \times \{s_1', s_2', s_3'\}$*, which gives us the unique shortest word* $b \in F_C(S,p)$ *and the unique vertex* s_i'*,* $i \in \{1,2,3\}$ *such that* $\psi(b, s_i') = s'$

We will use this map further when we construct a representation of $\pi_1(S,p)$ *using Fock-Goncharov coordinates.*

3 Examples of small spectral networks

3.1 Spectral network of rank 2

In this example we consider a spectral network of rank 2 over the surface S. We want to find the eigenvectors of parallel transport operators along peripheral curves on S.

We consider the surface S, a 2:1 branched covering $\pi\colon \Sigma \to S$ and the line bundle $\pi_\Sigma\colon E \to \Sigma$ over Σ with a flat connection. We assume that the spectral network $\mathcal{W}$ over S is given, which induces a twisted representation $\pi_1^s(S, p, v) \to GL(2, \mathbb{C})$ for $p \in S$, $v \in T_pS$ as in the chapter 2.3.

For each branch point b on S we have exactly three punctures, which are connected with b by lines of the spectral network. In this way the spectral network $\mathcal{W}$ yields an ideal triangulation of S with the property that each triangle on S contains exactly one branch point like on the fig. 3.1. In Σ we have to glue along red sides.

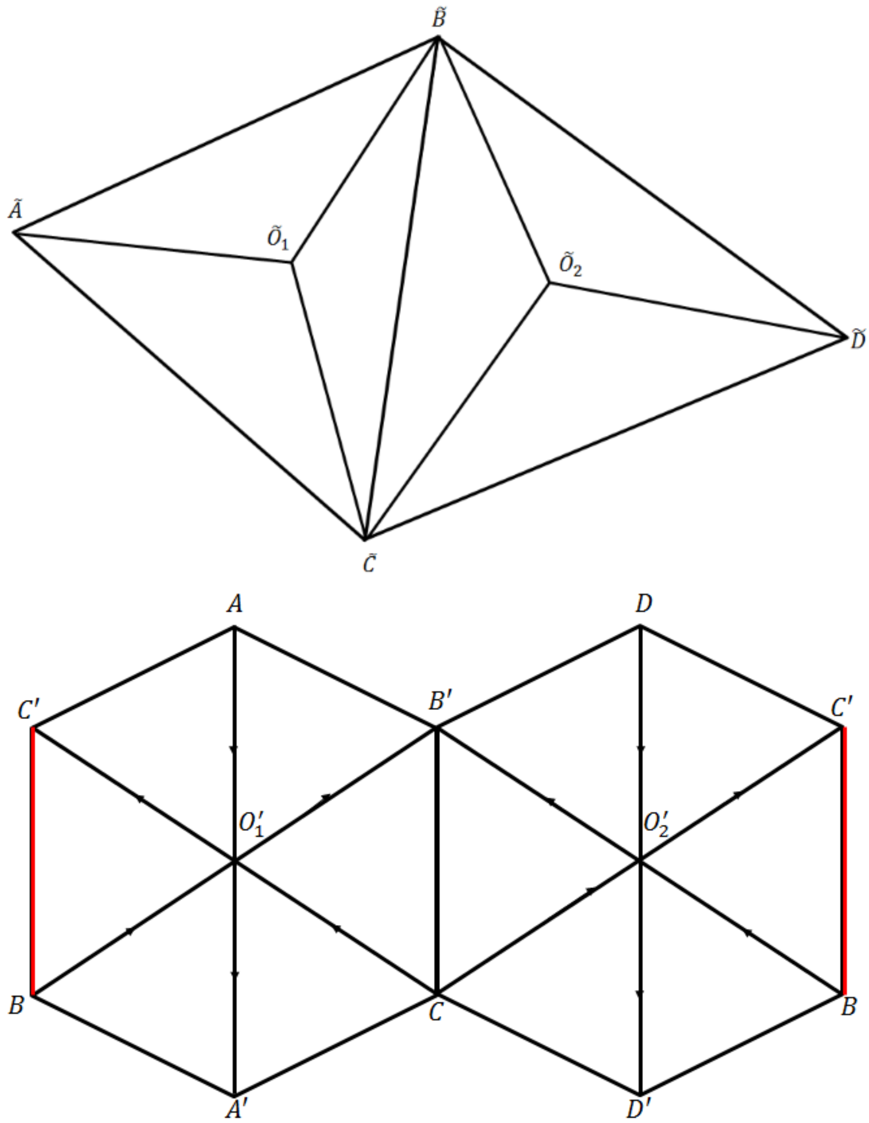

Fig. 3.1: Spectral network of rank 2. Picture in S above, picture in Σ below.

We choose two triangles with vertices $\tilde{A}, \tilde{B}, \tilde{C}$ and $\tilde{D}, \tilde{B}, \tilde{C}$ in S. In each triangle we have one branch point $\tilde{O}_1$ and $\tilde{O}_2$. The lifted to Σ points we denote $O_1 = \pi^{-1}(\tilde{O}_1)$, $O_2 = \pi^{-1}(\tilde{O}_2)$, $\{A, A'\} = \pi^{-1}(\tilde{A})$, $\{B, B'\} = \pi^{-1}(\tilde{B})$, $\{C, C'\} = \pi^{-1}(\tilde{C})$, $\{D, D'\} = \pi^{-1}(\tilde{D})$. We assume the following order on the spectral network: $A < A', B < B', C < C', D < D'$.

Further, we choose points $p, p' \in S \setminus B$ such that p lies in the triangle $\tilde{O}_1\tilde{B}\tilde{C}$, $v \in T_pS$, $v' \in T_{p'}S$. We also choose curves ξ, η in S such that $\xi(0) = \eta(1) = p$, $\xi(1) = \eta(0) = p'$, $\dot{\xi}(0) = \dot{\eta}(1) = v$, $\dot{\xi}(1) = \dot{\eta}(0) = v'$ and $\xi * \eta \equiv -e$.

In Σ we consider $\{p_1, p_2\} = \pi^{-1}(p)$, $v_i = D_{p_i}\pi^{-1}(v)$, $i \in \{1,2\}$ and choose curves γ_{12} and γ_{21} such that $\gamma_{12}(0) = \gamma_{21}(1) = p_1$, $\gamma_{21}(0) = \gamma_{12}(1) = p_2$, $\dot{\gamma}_{12}(0) = \dot{\gamma}_{21}(1) = v_1$, $\dot{\gamma}_{12}(1) = \dot{\gamma}_{21}(0) = v_2$ and $\gamma_{12} * \gamma_{21} \equiv -e$.

Now we calculate the matrix T^ξ for two special chosen curves ξ.

Case 1: p and p' lie in the triangle $\tilde{A}\tilde{B}\tilde{C}$ [see fig. 3.2].

Because the curves ξ and η intersect the spectral network only once, we have $\xi * \eta \equiv -e$. By lifting of ξ to Σ we get three curves: two standard lifts ξ_1 and ξ_2 and one curve β which starts in p_2 and finishes in p_1'. Therefore,

$$T^\xi = \begin{pmatrix} k_1 & b \\ 0 & k_2 \end{pmatrix}$$

where $T_\beta(e_2) = be_1'$.

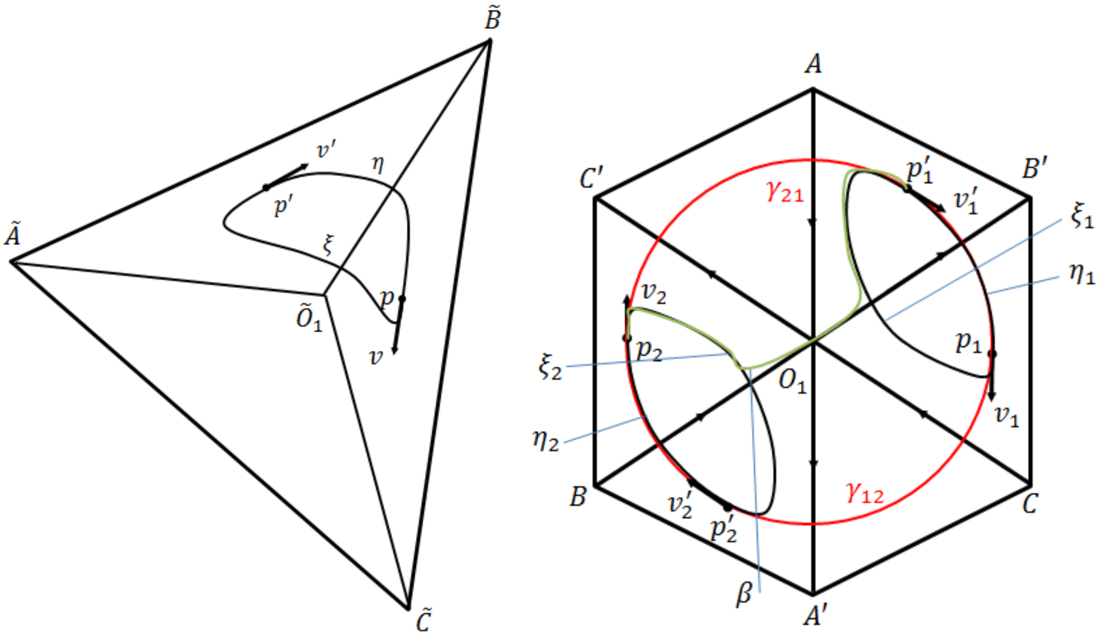

Fig. 3.2: Spectral network of rank 2. Picture in S left, picture in Σ right.

Because $\beta * \eta_1 * \gamma_{12} \equiv -e$, we have with (2.2):

$$b = -\frac{1}{l_1 \nu_{12}} = -k_1 \nu_{21}.$$

With remark 2.10 we can assume $k_1 = k_2 = -l_1 = -l_2 = 1$, therefore,

$$T^\xi = \begin{pmatrix} k_1 & -k_2\nu_{21} \\ 0 & k_2 \end{pmatrix} = [k_i = 1] = \begin{pmatrix} 1 & -\nu_{21} \\ 0 & 1 \end{pmatrix}. \tag{3.1}$$

Case 2: p and p' lie in the different adjacent triangles [see fig. 3.3].

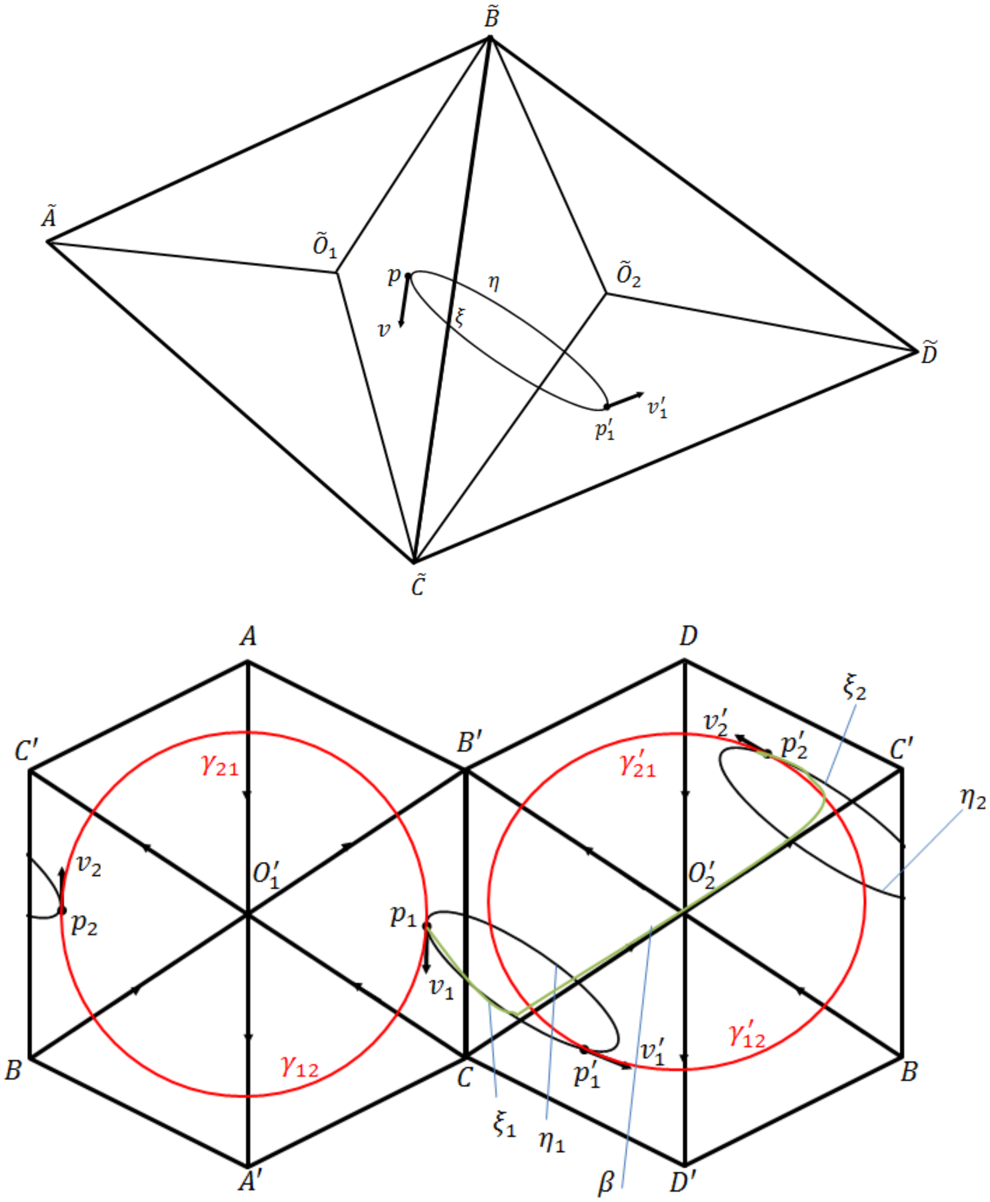

Fig. 3.3: Spectral network of rank 2. Picture in S above, picture in Σ below.

Because of $\xi * \eta \equiv -e$, by lifting of ξ to Σ we get three curves: two standard lifts ξ_1 and ξ_2 and one curve β which starts in p_1 and finishes in p_2'. Therefore,

$$T^{\xi} = \begin{pmatrix} k_1 & 0 \\ b & k_2 \end{pmatrix}$$

where $T_{\beta}(e_1) = be_2'$.

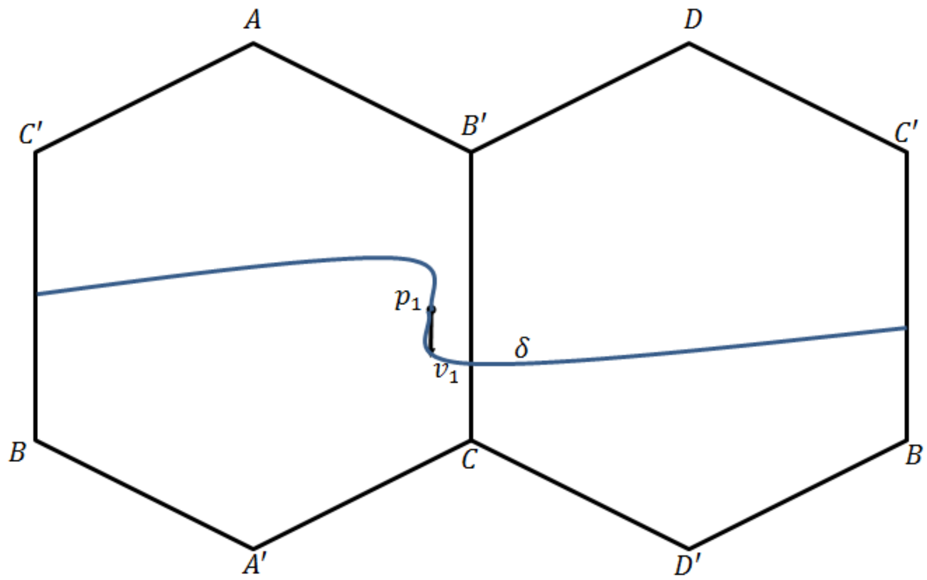

Fig. 3.4: Spectral network of rank 2. Picture in Σ. The curve δ.

Because $\beta * \eta_2 * \gamma_{21} \equiv -\delta$ [see fig. 3.3 and fig. 3.4], we have with (2.2):

$$b = -\frac{d}{l_2 \nu_{21}} = -dk_2\nu_{12}.$$

where $T_\delta e_1 = de_1$. With remark 2.10 we can assume $k_1 = k_2 = -l_1 = -l_2 = 1$, therefore,

$$T^\xi = \begin{pmatrix} k_1 & 0 \\ -dk_2\nu_{12} & k_2 \end{pmatrix} = [k_i = 1] = \begin{pmatrix} 1 & 0 \\ -d\nu_{12} & 1 \end{pmatrix}. \tag{3.2}$$

Now we are ready to calculate the eigenvectors of peripheral curves on S. We consider now four peripheral curves $\alpha_A, \alpha_B, \alpha_C, \alpha_D$ [see fig. 3.5].

The parallel transport operator each of these curves always has one eigenvector which is independent on the spectral network and depends only on the surface Σ.

Curve α_B. For this curve the eigenvector is easy to see. In the basis (e_1, e_2) it is

$$v_B = e_1$$

and the matrix of parallel transport operator along α_B in this basis has a form

$$T^{\alpha_B} = \begin{pmatrix} b' & * \\ 0 & * \end{pmatrix},$$

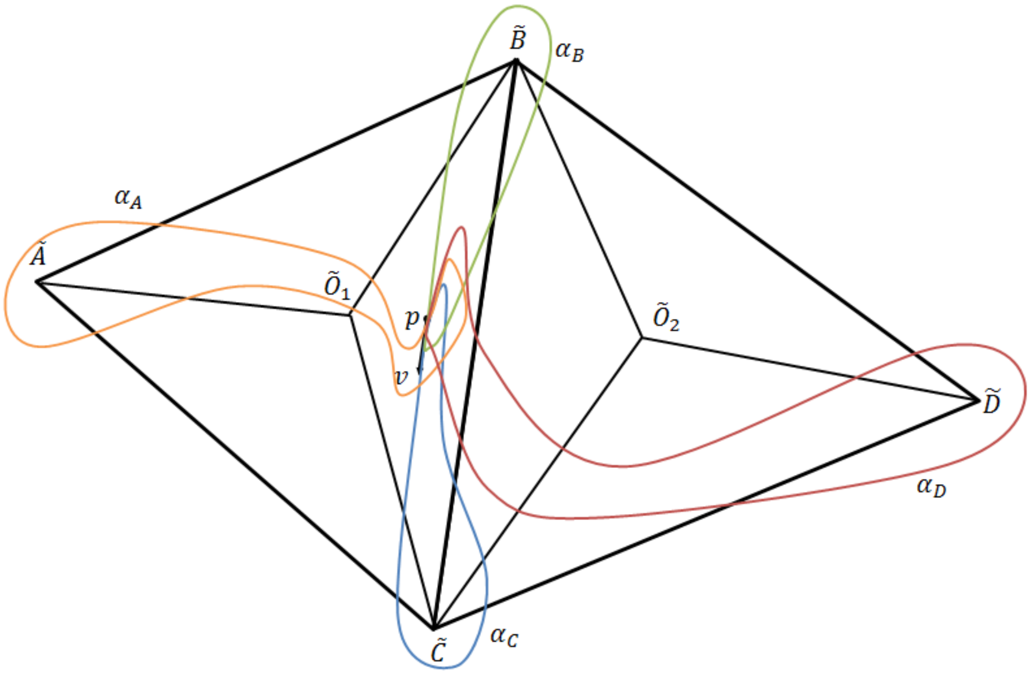

Fig. 3.5: The picture in S.

where b' is the representation of $\alpha_{1,B}$ in $\mathbb{C}^*$.

Curve α_C. For this curve the eigenvector is also easy to see. In the basis (e_1, e_2) it is

$$v_C = e_2$$

$$T^{\alpha_C} = \begin{pmatrix} * & 0 \\ * & c' \end{pmatrix},$$

where c' is the representation of $\alpha_{2,C}$ in $\mathbb{C}^*$.

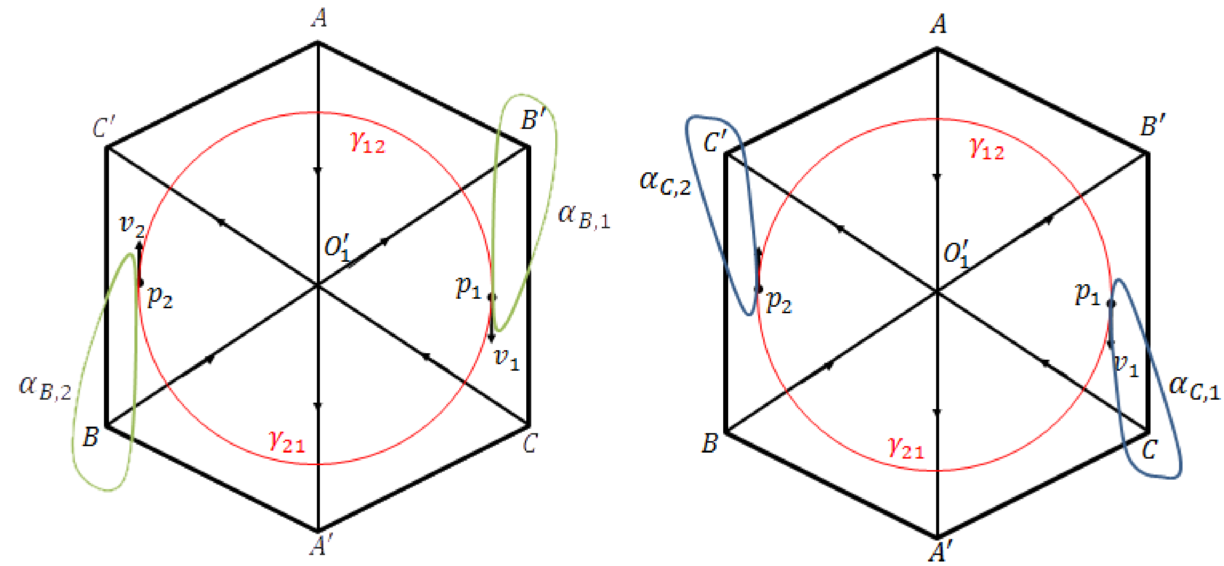

Fig. 3.6: The picture in Σ.

Curve α_A. To determine the eigenvector for α_A we apply the case 1. We change the base point and consider the curve β [see fig. 3.7].

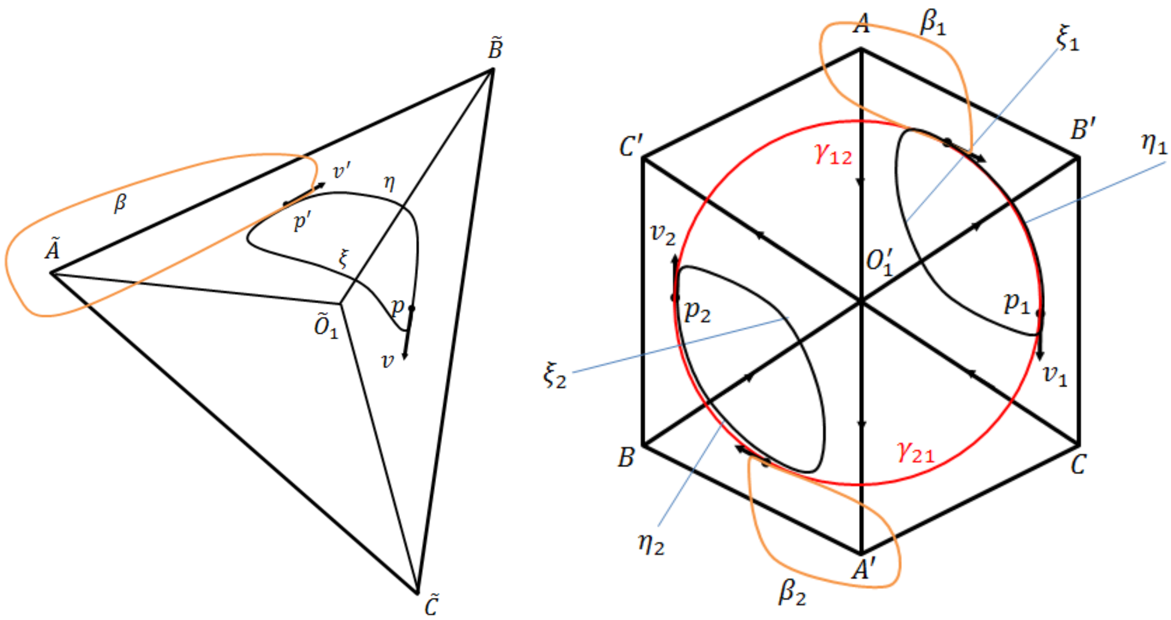

Fig. 3.7: The picture in S (left) and in Σ (right).

The eigenvector of T_β is e_2'. Because $\alpha_A \equiv -\xi * \beta * \eta$ we have

$$T_{\alpha_A} = T_\xi^{-1} T_\beta T_\xi.$$

Using (3.1) we get the eigenvector of T_{α_A} is

$$v_A = T_\xi^{-1} e_2' = \nu_{21} e_1 + e_2.$$

Curve α_D. To determine the eigenvector for α_D we apply the case 2 and choose the other point p' like in case 2, choose the curve β' [see fig. 3.8]. For $\gamma_{12}', \gamma_{21}'$ we assume $T_{\gamma_{ij}'} e_i' = \nu_{ij}' e_j'$. The eigenvector of $T_{\beta'}$ is $v_{\beta'} = e_1'$.

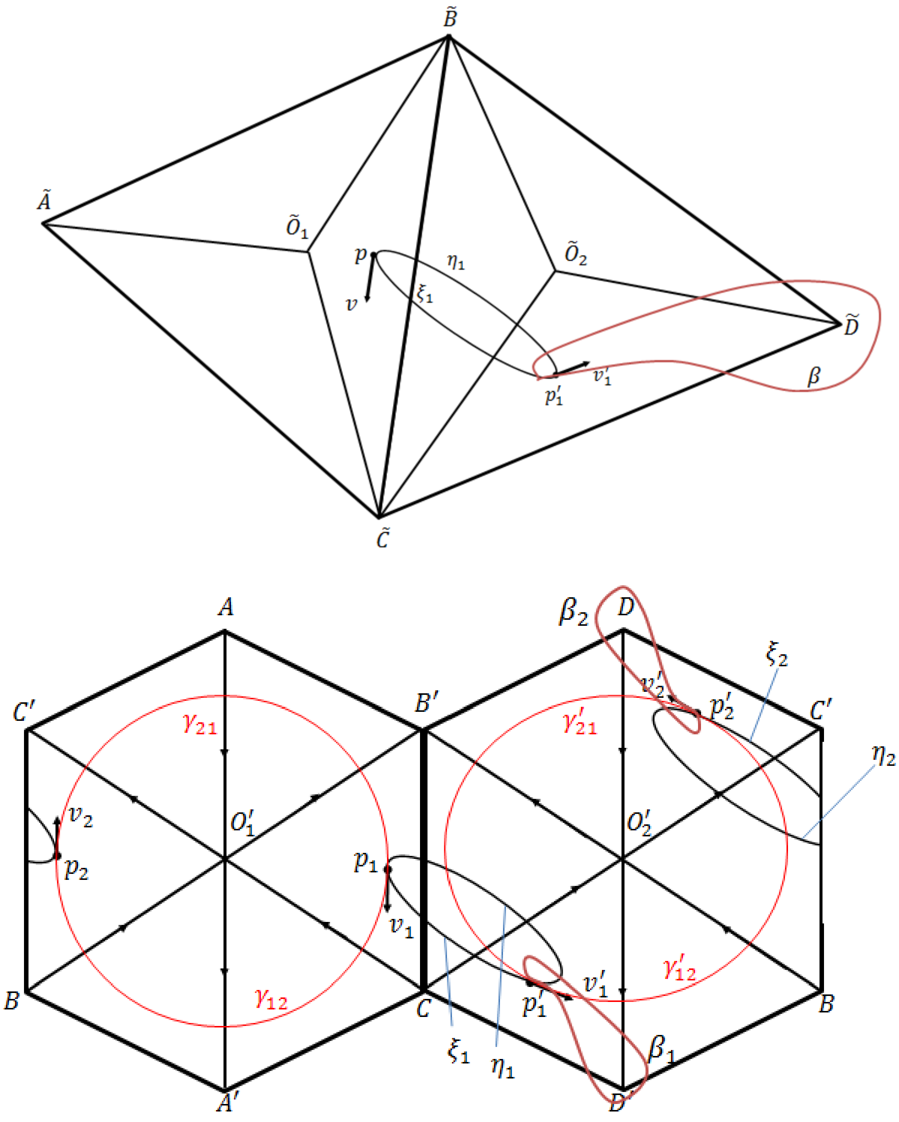

Fig. 3.8: The picture in S (above) and in Σ (below).

We use (3.2), then the eigenvector of T_{α_D} is

$$v_D = T_\xi^{-1} e_1' = e_1 + d\nu_{12} e_2.$$

3.2 Spectral networks of rank 2 and cross ratios.

As we have seen in the previous paragraph, the ideal triangulation of the surface with punctures yields the collection of eigenvectors in $\mathbb{C}^2$ of the parallel transport along peripheral curves. Namely, for each pair of triangles with a common side we have four eigenvectors.

The projectivisation yields for each eigenvector a point in $\mathbb{CP}^1$. So we get four point in $\mathbb{CP}^1$. We can consider the cross ratio of these points.

We choose the affine chart with respect to the basis $(e_1, e_1 + e_2)$. Then the eigenvectors v_B, v_C, v_A, v_D correspond to complex numbers x_B, x_C, x_A, x_D:

$$x_i = \frac{v_i^1}{v_i^1 + v_i^2}.$$

$$x_B = 1,\ x_C = 0,\ x_A = \frac{\nu_{21}}{1+\nu_{21}},\ x_D = \frac{1}{d\nu_{12}+1},$$

where d is the holonomy along the curve δ. We consider the cross ratio

$$[x_A, x_D, x_B, x_C] = \frac{\left(\dfrac{1}{d\nu_{12}+1} - 1\right)\dfrac{\nu_{21}}{1+\nu_{21}}}{\dfrac{1}{d\nu_{12}+1}\left(\dfrac{\nu_{21}}{1+\nu_{21}} - 1\right)} = \frac{d\nu_{12}\nu_{21}}{-1} = d$$

The constant d depends only on the flat connection ∇' on Σ and does not depend on the spectral network.

We can see that an ideal triangulation of S and a flat connection on Σ define the collection of cross ratios on S. Conversely, if an ideal triangulation on S and the collection of numbers (cross ratios) for each pair of triangles with common side on S with respect to this triangulation are given, we get the numbers d for each pair of triangles with common side. We can construct a representation $\rho\colon \pi_1(S,p) \to PGL(2,\mathbb{C})$, $p \in S$, p do not lie on lines of triangulation in the following way.

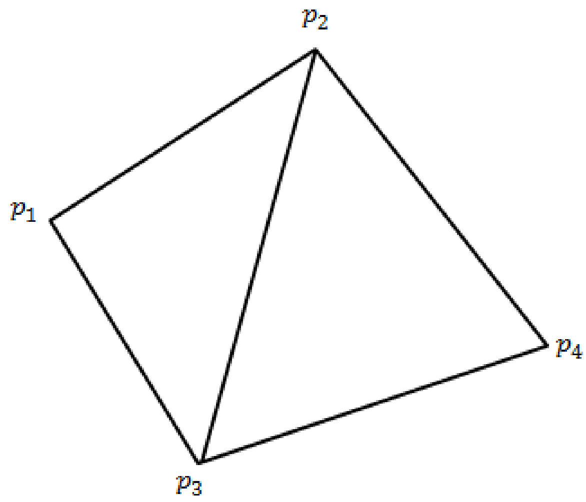

Fig. 3.9: The picture in S.

First, we fix a ideal triangulation $\mathcal{T}$ of S and some total order ϱ on triangles of this triangulation. Further, we choose the pair of triangles with vertices $p_1, p_2, p_4, p_3 \in S$ and the common side $c = p_2p_3$. For this pair of triangles the number d is defined. We associate p_i (for $i = 1, 2, 3, 4$) with points $x_1 = [0 : 1]$, $x_2 = [1 : 1]$, $x_3 = [1 : 0]$, $x_4 = [d : 1]$ in $\mathbb{CP}^1$. So we have a unique Möbius transformation ϕ, which x_1, x_2, x_3 sends to x_4, x_3, x_2. This transformation corresponds to an unique element $M \in PGL(2, \mathbb{C})$. We also fix the number ε, which is equal to 1 if the triangle $p_1p_2p_3$ is smaller then $p_4p_3p_2$ with respect to ϱ, and is equal to -1 otherwise.

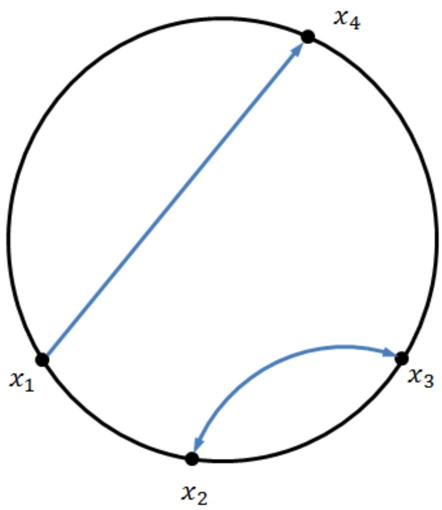

Fig. 3.10: The picture in $\mathbb{PC}^1$. Action of ϕ.

We consider a group homomorphism $\kappa\colon F_C \to PGL(2,\mathbb{C})$, which is defined on generators of F_C in a following way $\kappa(c) := M^{\varepsilon}$ for all sides c of triangulation $\mathcal{T}$.

Further, we consider the universal covering $\pi\colon S' \to S$ of S and corresponding triangulation $\mathcal{T}'$. We denote by P' the set of all vertices of triangulation of $\mathcal{T}'$. We want to construct a function $f\colon P' \to \mathbb{CP}^1$. First, we choose a lift $p' \in \pi^{-1}(p)$. This lies in a triangle T_0 on S' with vertices s_1', s_2', s_3'. We define f for these three points in the following way: $f(s_1') := [0:1]$, $f(s_2') := [1:1]$, $f(s_3') := [1:0]$. We use the map ψ from the section 2.9 and assume that the order ϱ agree with the order ϱ in the section 2.9. If we choose $s' \in P'$, then by remark 2.20 there exist a unique shortest word $b \in F_C(S,p)$ and unique $i \in \{1,2,3\}$ such that $\psi(b, s_i) = s'$. Because b and i are unique, we define $f(s') := \kappa(b)(f(s_i))$.

Now we are ready to construct a representation $\rho\colon \pi_1(S,p) \to PGL(2,\mathbb{C})$. We take an element $g \in \pi_1(S,p)$. We can consider this element as a deck transformation of S', which also acts on P' and $g(s_i') = r_i' \in P'$, $i \in \{1,2,3\}$. There exist a unique element $\rho(g) \in PSL(2,\mathbb{C})$ such that $\rho(f(s_i)) = f(r_i)$. By this rule the map $\rho\colon \pi_1(S,p) \to PGL(2,\mathbb{C})$ is well defined, and it is easy to see that this is a group homomorphism, so we get a representation.

Remark 3.1. *Because the order ϱ on $\mathcal{T}$ and choice of sign of ε are consistent, the constructed representation $\rho\colon \pi_1(S,p) \to PGL(2,\mathbb{C})$ does not depend on this order. Moreover, this representation depends only on a triangulation $\mathcal{T}$ and on cross ratios, which are given with respect to this triangulation.*

3.3 Spectral network of rank 2 over the sphere with three punctures

The simplest interesting case of example 3.1 is a spectral network of rank 2 over the sphere with three punctures. In this case the spectral network on S consist only of two triangles with vertices in punctures $\tilde{A}, \tilde{B}, \tilde{C}$, so we need to identify the point $\tilde{A}$ with the point $\tilde{D}$, the edge $\tilde{A}\tilde{B}$ with the edge $\tilde{D}\tilde{B}$ and the edge $\tilde{A}\tilde{C}$ with the edge $\tilde{D}\tilde{C}$ (see figures of example 3.1).

Thus we consider $S = S^2 \backslash \{\tilde{A}, \tilde{B}, \tilde{C}\}$. On S we can consider the ideal triangulation with vertices in $\tilde{A}, \tilde{B}, \tilde{C}$. We get two triangles. In each triangle we have one branch point $\tilde{O}_1$ and $\tilde{O}_2$. We denote the points lifted to Σ by $O_1 = \pi^{-1}(\tilde{O}_1)$, $O_2 = \pi^{-1}(\tilde{O}_2)$, $\{A, A'\} = \pi^{-1}(\tilde{A})$, $\{B, B'\} = \pi^{-1}(\tilde{B})$, $\{C, C'\} = \pi^{-1}(\tilde{C})$. We assume the following order on the spectral network $A < A', B < B', C < C'$.

We chose a base point p with a base vector in the triangle $\tilde{O}_1\tilde{B}\tilde{C}$. We want to describe the parallel transport operators for curves $\alpha_B, \alpha_C, \alpha_{1A}, \alpha_{2A}$ around the point B, C and A. For A we consider two different curves (see figure 3.11).

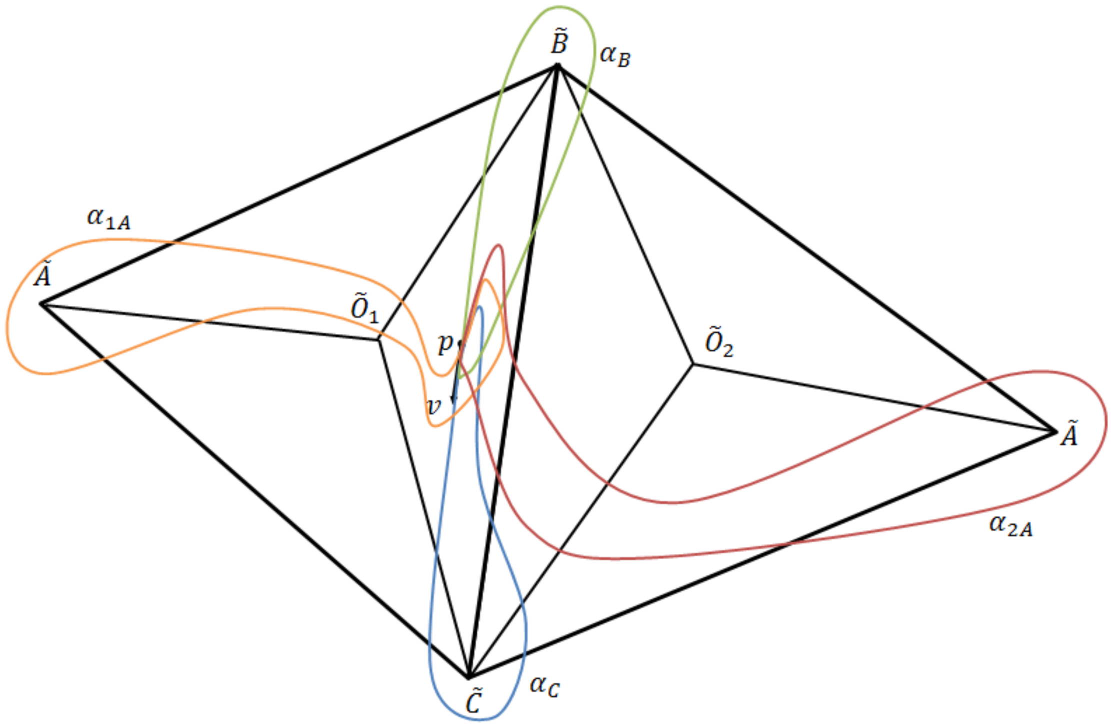

Fig. 3.11: The picture in S.

In Σ we choose the basis (e_1) in p_1 and (e_2) in p_2 and curves γ_{12} and γ_{21}. In our case $\gamma_{12} * \gamma_{21} \equiv -e \in \pi(S, p_1, v_1)$. Therefore, $\nu_{12}\nu_{21} = -1$. The value of one of these constants can be freely chosen. For each point A, A', B, B', C, C' the flat connection ∇' on Σ defines constants a, a', b, b', c, c' which characterize the parallel transport along the corresponding peripheral curve around each of these points in the positive sense.

Curve α_B. We denote $T^{\alpha_B} = (t_{ij})$ (only for this case). The lift of the curve α_B gives us 4 curves. We have one curve α_{1B} around B' with start and end in p_1, therefore,

$$t_{11} = -b'.$$

We have one curve α_{2B} around B with start and end in p_2, therefore,

$$t_{22} = -b.$$

And finally, we have two curves β_1, β_2 with start in p_2 and finish in p_1. To compute the corresponding numbers for this curves we have to complete these curves with γ_{12}. So we have

$$t_{12} = \frac{-b' + a'bb'c}{\nu_{12}}.$$

We do not have curves from p_1 to p_2, therefore,

$$t_{21} = 0.$$

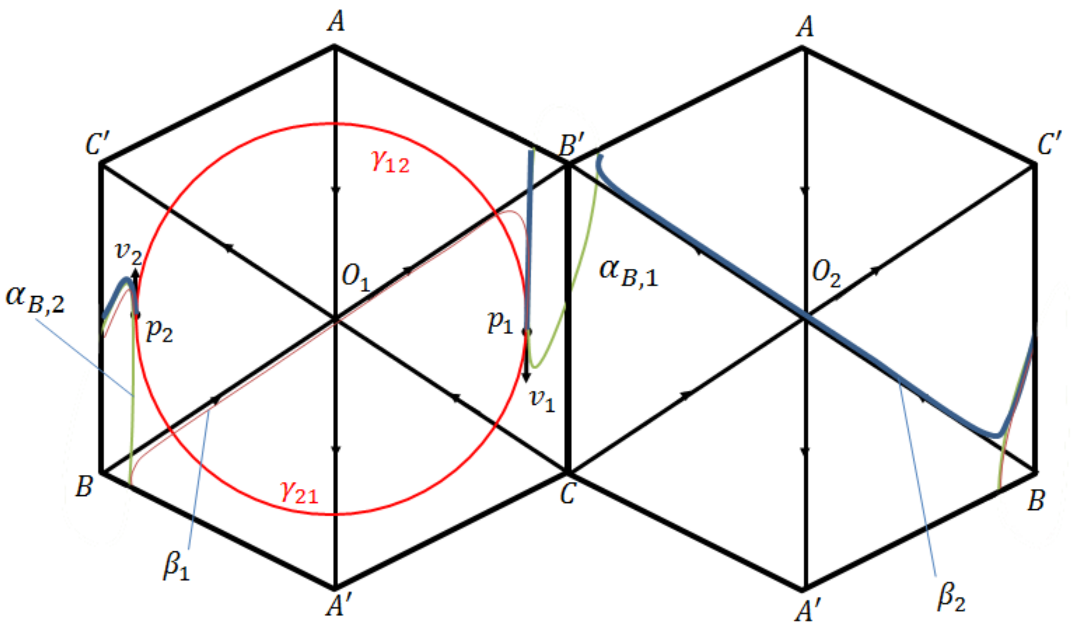

Fig. 3.12: The picture in Σ.

So we have the matrix:

$$T^{\alpha_B} = \begin{pmatrix} -b' & \frac{b'-a'bb'c}{\nu_{12}} \\ 0 & -b \end{pmatrix}$$

Curve α_C. Similarly to the curve α_B:

$$T^{\alpha_C} = \begin{pmatrix} -c & 0 \\ \frac{c'-ab'cc'}{\nu_{21}} & -c' \end{pmatrix}$$

Curve α_{1A}. To find out how the parallel transport operator for the curve $\alpha_1 A$ looks like, we use the case 1 and we choose the new base point p' in the triangle O_1CA', and two curves ξ and η which connect p and p'. Further, we consider the curve β (like figure 3.7 of example 3.1). Then $\alpha_{1A} \equiv -\xi * \beta * \eta$ and $\xi * \eta \equiv -e$. Then $T_\eta = -T_\xi^{-1}$ and

$$T_{\alpha_{1A}} = T_\xi^{-1} T_\beta T_\xi.$$

To compute the matrix T^β of the parallel transport operator T_β in basis (e_1', e_2') we choose the curves $\gamma_{12}', \gamma_{21}'$ such that $\gamma_{12}' * \gamma_{21}' \equiv -e$ and then the computing of T^β is similar to cases 1 and 2. So we write:

$$T^\beta = \begin{pmatrix} -a & 0 \\ \frac{a - aa'bc'}{\nu_{21}'} & -a' \end{pmatrix}$$

where $T_{\gamma_{ij}'} e_i' = \nu_{ij}' e_j'$, $i, j \in \{1, 2\}$ and $\nu_{12}' \nu_{21}' = -1$.

We also need to compute T^ξ. We use (3.1) from example 3.1.

$$T^\xi = \begin{pmatrix} 1 & -\nu_{21} \\ 0 & 1 \end{pmatrix}$$

We also need to express ν_{ij}' in terms of ν_{ij}, k_i. To do this we note that

$$\xi_1 * \gamma_{12}' * \eta_2 * \gamma_{21} \equiv e.$$

Therefore,

$$1 = k_1 \nu_{12}' l_2 \nu_{21} = -k_1 \nu_{12}' k_2^{-1} \nu_{21} = k_1 \nu_{12}' k_2^{-1} \nu_{12}^{-1},$$

$$\nu_{12}' = \frac{k_2}{k_1} \nu_{12} = [k_i = 1] = \nu_{12}$$

and also

$$\nu_{21}' = \frac{k_1}{k_2} \nu_{21} = [k_i = 1] = \nu_{21}.$$

Now we can compute $T^{\alpha_{1A}}$:

$$T^{\alpha_{1A}} = (T^\xi)^{-1} T^\beta T^\xi = \begin{pmatrix} -aa'bc' & -\nu_{21}(a' - aa'bc') \\ \frac{a - aa'bc'}{\nu_{21}} & -a - a' + aa'bc' \end{pmatrix}$$

Curve α_{2A}. Here we use the case 2 of example 3.1 (see figure 3.8). The calculation is very similar to the curve α_{1A}, therefore, we write the answer:

$$T^{\alpha_{2A}} = (T^\xi)^{-1} T^\beta T^\xi = \begin{pmatrix} 1 & 0 \\ -d\nu_{12} & 1 \end{pmatrix}^{-1} \begin{pmatrix} -a' & \frac{a' - aa'bc'}{d\nu_{12}} \\ 0 & -a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -d\nu_{12} & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} -a - a' + aa'b'c & \frac{a - aa'b'c}{d\nu_{12}} \\ -d\nu_{12}(a' - aa'b'c) & -aa'b'c \end{pmatrix}.$$

In this case we have $d = ab'c'$, therefore,

$$T^{\alpha_{2A}} = \begin{pmatrix} -a - a' + aa'b'c & \frac{aa'bc - a(a')^2bb'c^2}{\nu_{12}} \\ -\nu_{12}(aa'b'c' - a^2a'(b')^2cc') & -aa'b'c \end{pmatrix}.$$

We can summarize the results of this section in the following proposition.

Proposition 3.2. *The representation of the fundamental group of the sphere with three punctures to $GL(2,\mathbb{C})$ given by a spectral network of rank 2 up to conjugation is generated by following matrices:*

$$T^{\alpha_B} = \begin{pmatrix} -b' & \frac{b' - a'bb'c}{\nu_{12}} \\ 0 & -b \end{pmatrix},$$

$$T^{\alpha_C} = \begin{pmatrix} -c & 0 \\ \frac{c' - ab'cc'}{\nu_{21}} & -c' \end{pmatrix},$$

$$T^{\alpha_{1A}} = \begin{pmatrix} -aa'bc' & -\nu_{21}(a' - aa'bc') \\ \frac{a - aa'bc'}{\nu_{21}} & -a - a' + aa'bc' \end{pmatrix},$$

$$T^{\alpha_{2A}} = \begin{pmatrix} -a - a' + aa'b'c & \frac{aa'bc - a(a')^2bb'c^2}{\nu_{12}} \\ -\nu_{12}(aa'b'c' - a^2a'(b')^2cc') & -aa'b'c \end{pmatrix},$$

where α_B, α_C, α_{1A}, α_{2A} are peripheral curves [see fig. 3.11]; a, b, c, a', b', c' are complex constants, which satisfy the condition $aa'bb'cc' = 1$ and are given by the spectral network; ν_{12}, ν_{21} are complex constants, which satisfy the condition $\nu_{12}\nu_{21} = -1$, and the value of one of these constants can be freely chosen.

3.4 Small spectral network of rank 3.

In this example we consider a special case of a spectral network of rank 3 over the surface S.

We consider the surface S, a $3:1$ branched covering $\pi\colon \Sigma \to S$ and a line bundle $\pi_\Sigma\colon E \to \Sigma$ over Σ with a flat connection. We consider an ideal triangulation of S and assume that there is a spectral network $\mathcal{W}$ on S with the following properties: for each triangle of the triangulation there are exactly three branch points, which are contained in this triangle and the pictures of this triangle in S and its lift in Σ look like on the fig. 3.13, 3.14, 3.15.

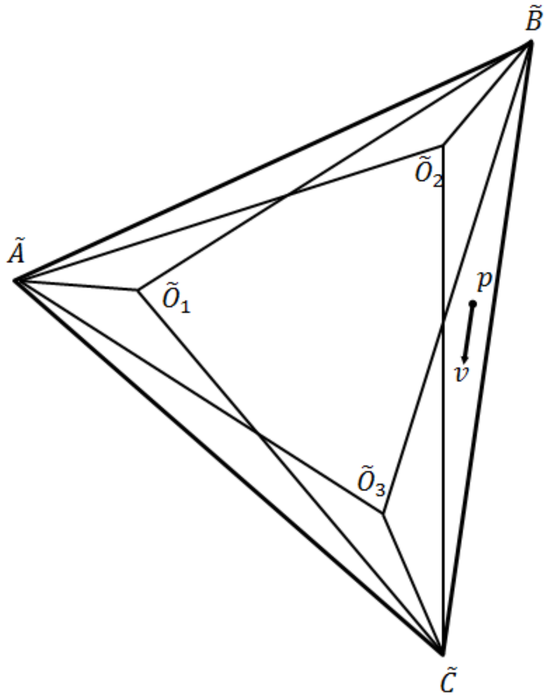

Fig. 3.13: The picture in S.

The spectral network $\mathcal{W}$ induces a twisted representation $\pi_1^s(S, p, v) \to GL(3, \mathbb{C})$ for $p \in S$, $v \in T_pS$ as in the chapter 2.3.

We chose a triangle $\tilde{A}\tilde{B}\tilde{C}$ in S. This triangle has three branch point $\tilde{O}_1, \tilde{O}_2, \tilde{O}_3$ [see fig. 3.13].

We denote the points lifted to Σ by $O_i = \pi^{-1}(\tilde{O}_i)$ for $i = 1, 2, 3$, $\{A, A', A''\} = \pi^{-1}(\tilde{A})$, $\{B, B', B''\} = \pi^{-1}(\tilde{B})$, $\{C, C', C''\} = \pi^{-1}(\tilde{C})$. We assume the following order on the spectral network: $A < A' < A''$, $B < B' < B''$, $C < C' < C''$. We also choose point $p \in S \setminus B$, $v \in T_pS$. $\pi^{-1}(p) = \{p_1, p_2, p_3\}$. We denote by e_i the basis of E_{p_i} .

We can draw the covering Σ in two ways. On the fig. 3.14 three sheets of the covering are drawn which we glue along the dotted lines of the same color.

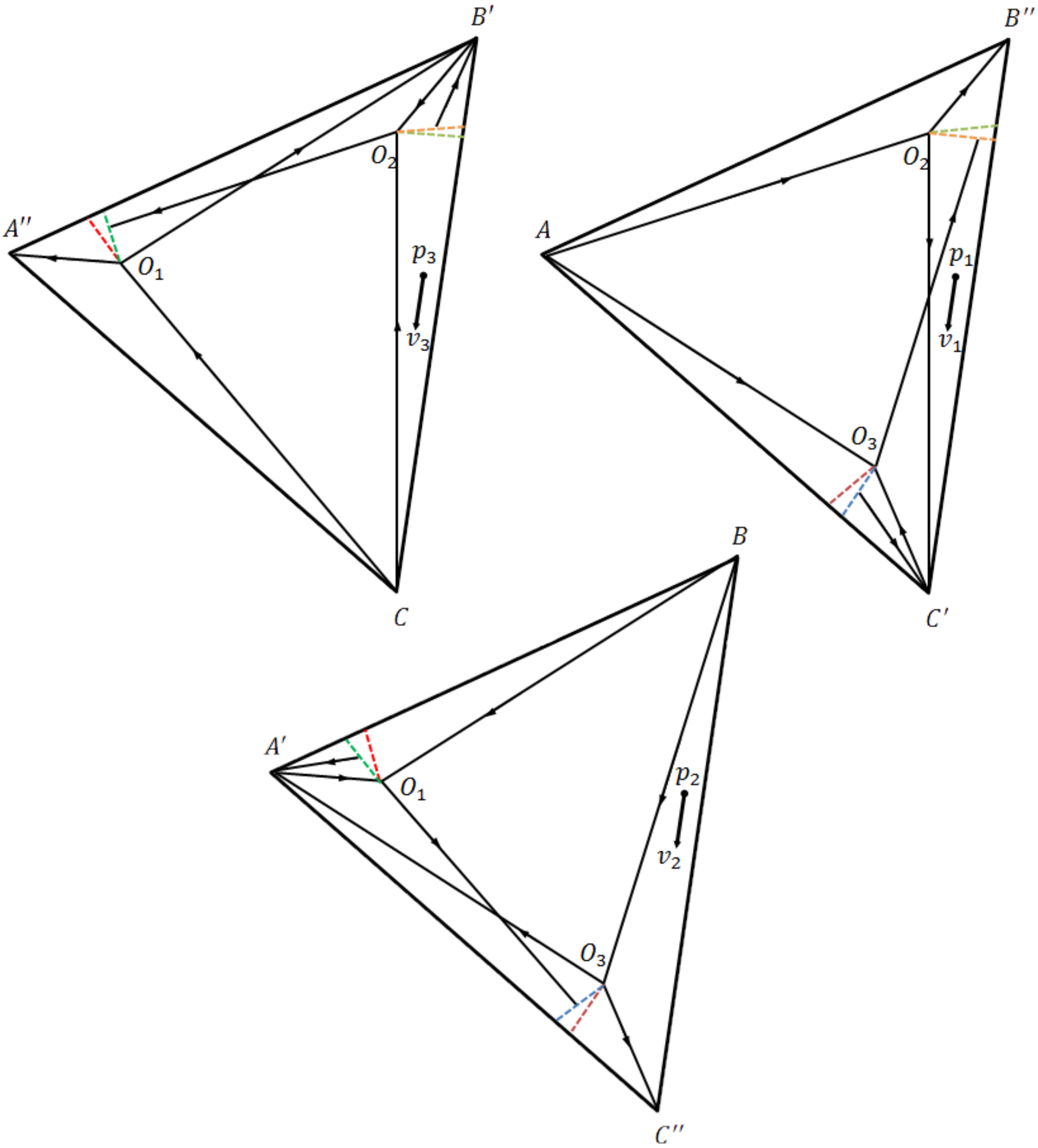

Fig. 3.14: The picture in Σ.

We can also draw the flat picture of the covering Σ which we can use further [see fig. 3.15].

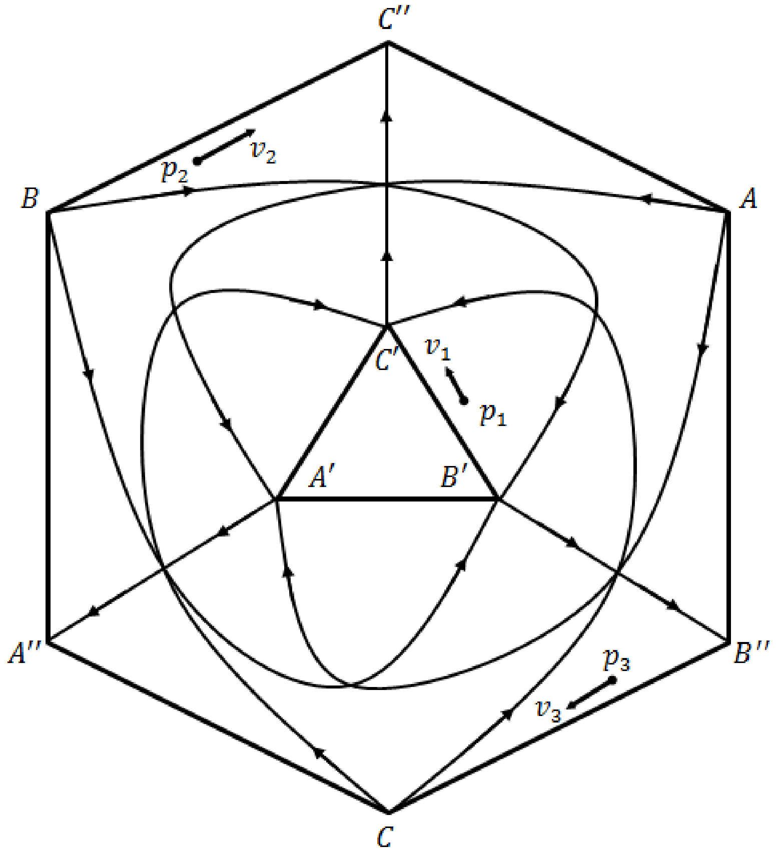

Fig. 3.15: The picture in Σ.

As in the case of a spectral network of rank 2 we want to study the parallel transport along peripheral curves α_A, α_B, α_C around the points $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ namely for each parallel transport operator we want to find the invariant flag. To do this we lift these curves in Σ.

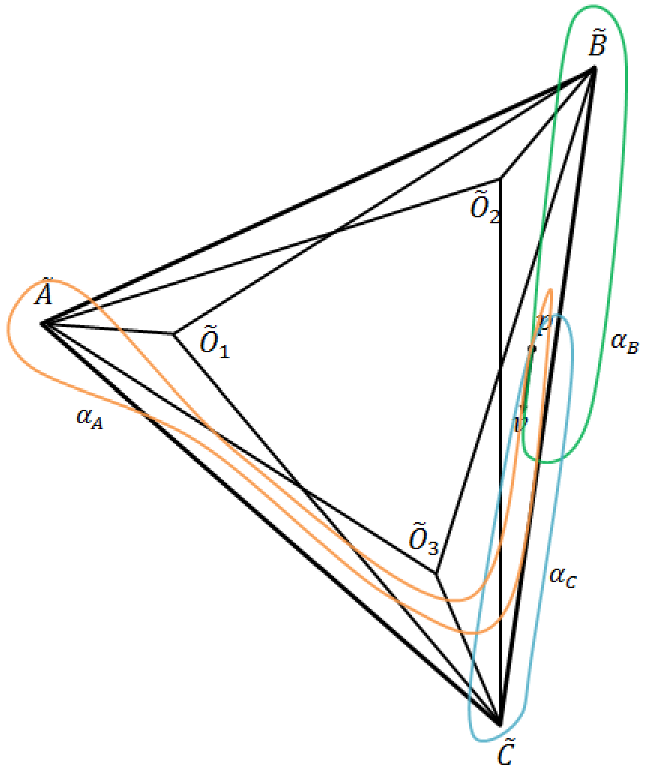

Fig. 3.16: The picture in S.

Curve α_B**.** Because of $B < B' < B''$ vector e_3 is the eigenvector of T_{α_B} and the space $\langle e_3, e_1 \rangle$ is invariant 2-space of T_{α_B}. So we have the invariant flag

$$F_B := \begin{pmatrix} \langle e_3 \rangle \\ \langle e_3, e_1 \rangle \\ V_p \end{pmatrix}.$$

Curve α_C**.** Because of $C < C' < C''$ vector e_2 is the eigenvector of T_{α_B} and the space $\langle e_2, e_1 \rangle$ is invariant 2-space of T_{α_B}. So we have the invariant flag

$$F_C := \begin{pmatrix} \langle e_2 \rangle \\ \langle e_2, e_1 \rangle \\ V_p \end{pmatrix}$$

Curve α_A**.** To find the invariant flag of T_{α_A}, we choose the other base point p', tangent vector v' (like on the fig. 3.17). $\pi^{-1}(p') = \{p'_1, p'_2, p'_3\}$. The basis of $E_{p'_i}$ we denote e'_i $(i = 1, 2, 3)$. We chose also two curves ξ, η which connect p and p'. The corresponding standard lifts are $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$.

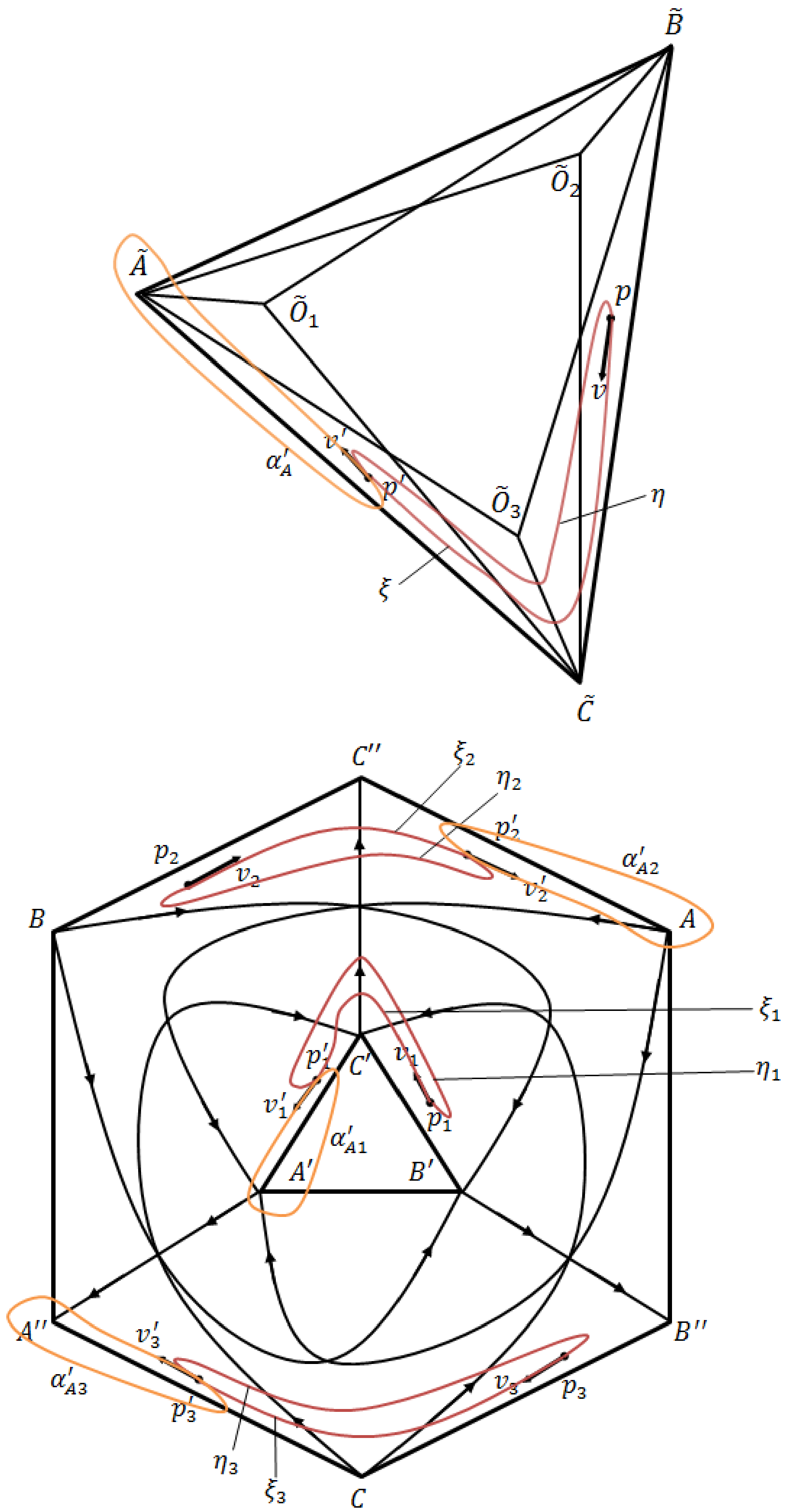

Fig. 3.17: The picture in S (up), in Σ (down).

First of all, we consider the curve α'_A. The invariant flag of the operator $T_{\alpha'_A}$ is easy to find. The eigenvector is e'_3, the invariant 2-space is $\langle e'_3, e'_1 \rangle$.

Because $\alpha_A \equiv -\xi * \alpha'_A * \eta$ we have $T_{\alpha_A} = T_\eta T_{\alpha'_A} T_\xi$. Moreover, because $\xi * \eta \equiv -e$ we have $T_\eta T_\xi = -id$. Therefore, to find the invariant flag of T_{α_A} we need to calculate the matrix T^ξ of the operator T_ξ in bases (e_1, e_2, e_3) and (e'_1, e'_2, e'_3). To do this we chose 6 curves γ_{ij} $(i, j = 1, 2, 3, i \neq j)$ which connect p_i and p_j [see fig. 3.18] so that $\gamma_{31} * \gamma_{12} = \gamma_{32}$ and $\gamma_{21} * \gamma_{13} \equiv -\gamma_{23}$ and denote $T_{\gamma_{ij}} e_i = \nu_{ij} e_j$. Because of $\gamma_{ij} * \gamma_{ji} \equiv -e$ we have $\nu_{ij}\nu_{ji} = -1$. Because of $\gamma_{31} * \gamma_{12} = \gamma_{32}$ we have $\nu_{31}\nu_{12} = \nu_{32}$.

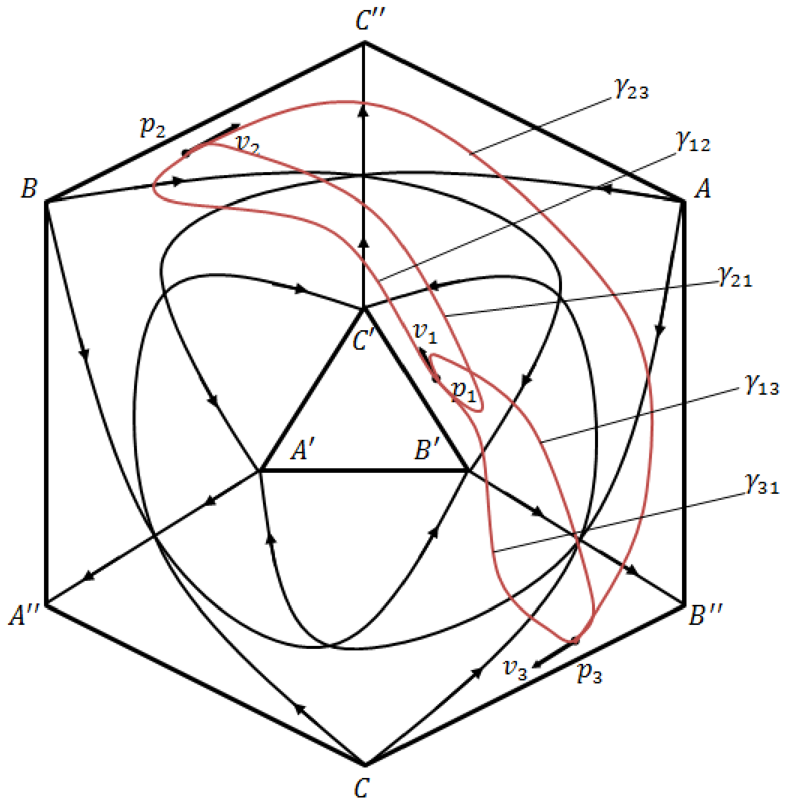

Fig. 3.18: The picture in Σ.

There is only one (standard) lift ξ_2 of ξ with the base point p_2, therefore, $t^\xi_{22} = k_2$, $t^\xi_{12} = t^\xi_{32} = 0$.

There are two lifts ξ_1 (standard lift) and β_1 [see fig. 3.19] of ξ with the base point p_2. Because ξ_1 ends in p_1, we have $t^\xi_{22} = k_2$. Because β_1 ends in p_2 and $\beta * \eta_1 * \gamma_{12} \equiv e$ we apply (2.2):

$$t^\xi_{21} = \frac{1}{l_2 \nu_{21}}, t^\xi_{31} = 0.$$

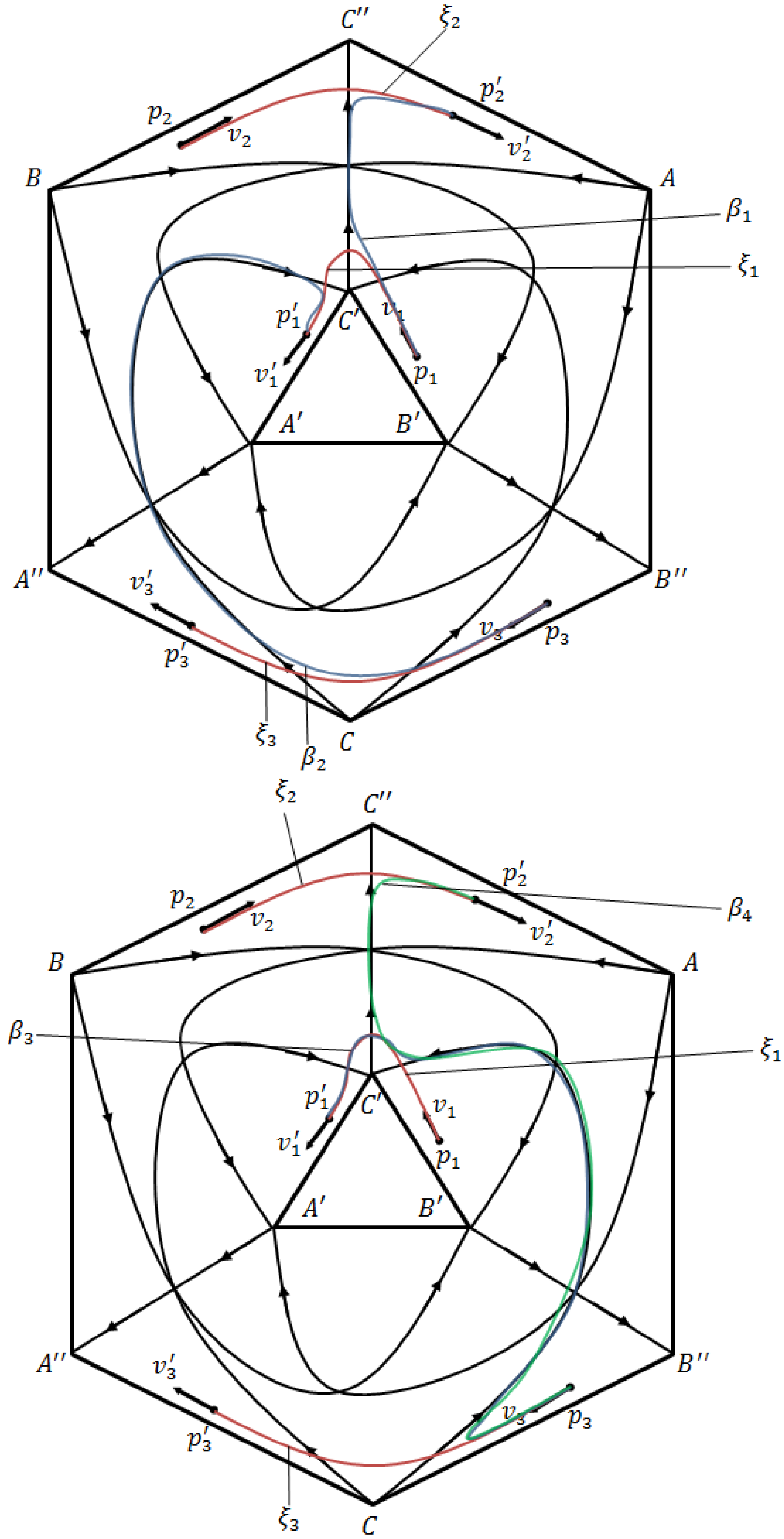

Fig. 3.19: The picture in Σ.

There are 4 lifts of ξ with start in p_3: standard lift ξ_3, two lifts β_2, β_3 which ends in p_1 and one lift β_4 which ends in p_2 [see fig. 3.19]. Because $\beta_2 * \eta_1 * \gamma_{13} \equiv \delta^{-1}$ and $\beta_3 * \eta_1 * \gamma_{13} \equiv e$ we have with (2.2):

$$t^{\xi}_{13} = \frac{1 - d^{-1}}{l_1 \nu_{13}}$$

Because $\beta_4 * \eta_2 * \gamma_{23} \equiv e$ we have with (2.2):

$$t^{\xi}_{23} = \frac{1}{l_2 \nu_{23}}$$

Therefore,

$$T^{\xi} = \begin{pmatrix} k_1 & 0 & \frac{1+d^{-1}}{l_1 \nu_{13}} \\ \frac{1}{l_2 \nu_{21}} & k_2 & \frac{1}{l_2 \nu_{23}} \\ 0 & 0 & k_3 \end{pmatrix}.$$

With remark 2.10 we assume $k_i = -l_i = 1$ and apply $\nu_{ij} = -\nu_{ji}^{-1}$ and $\nu_{31}\nu_{12} = \nu_{32}$. Moreover, we can choose $\nu_{31} = \nu_{12} = \nu_{32} = 1$. Therefore,

$$T^{\xi} = \begin{pmatrix} 1 & 0 & (1 + d^{-1})\nu_{31} \\ \nu_{12} & 1 & \nu_{32} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 + d^{-1} \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.3}$$

Then

$$(T^{\xi})^{-1} = \begin{pmatrix} 1 & 0 & -d^{-1} - 1 \\ -1 & 1 & d^{-1} \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.4}$$

Therefore, the invariant flag of T_{α_A} is

$$F_A := T_{\xi}^{-1} \begin{pmatrix} \langle e'_3 \rangle \\ \langle e'_3, e'_1 \rangle \\ V_{p'} \end{pmatrix} =$$

$$= \begin{pmatrix} \langle -(d^{-1} + 1)\nu_{31} e_1 + \nu_{32} d^{-1} e_2 + e_3 \rangle \\ \langle -(d^{-1} + 1)\nu_{31} e_1 + \nu_{32} d^{-1} e_2 + e_3, e_1 - \nu_{12} e_2 \rangle \\ V_p \end{pmatrix} =$$

$$= \begin{pmatrix} \langle -(d^{-1} + 1) e_1 + d^{-1} e_2 + e_3 \rangle \\ \langle -(d^{-1} + 1) e_1 + d^{-1} e_2 + e_3, e_1 - e_2 \rangle \\ V_p \end{pmatrix}.$$

So for each triangle of the triangulation of S we get exactly one non-trivial curve in Σ. We have seen by studying of the spectral network of rank 2 that one triangle does not give us non-trivial curves in Σ, but if we consider two triangles with the common side together, we get a non-trivial curve in Σ.

Now we consider the same situation but for the spectral network of rank 3. On fig. 3.20 we can see the picture in S and on fig. 3.21 we can see the corresponding picture in Σ, we glue sides of the same color.

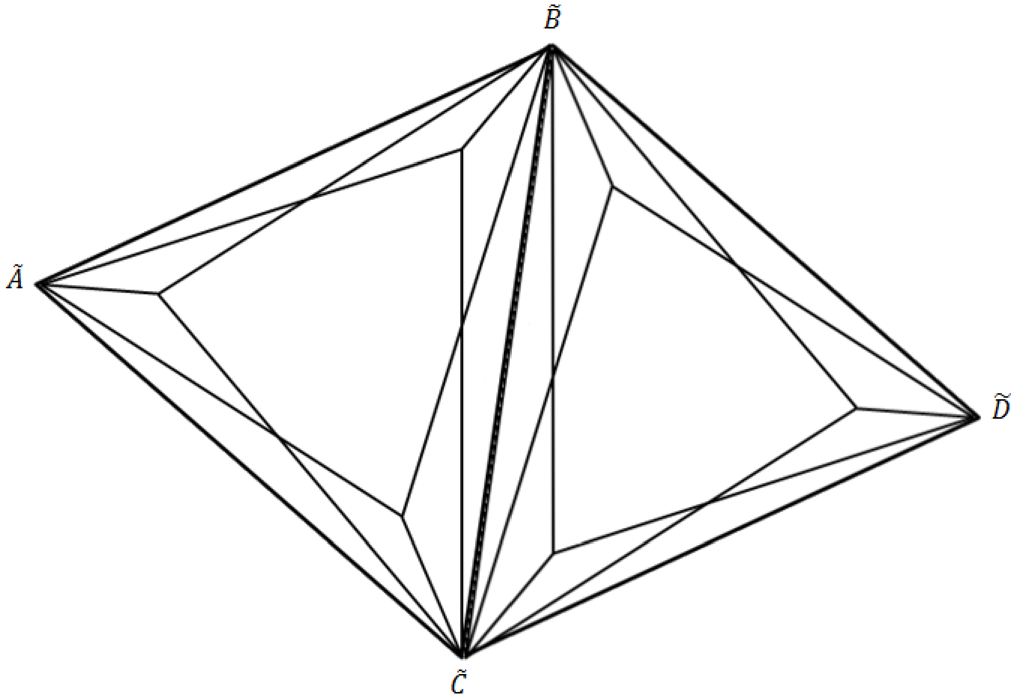

Fig. 3.20: The picture in S.

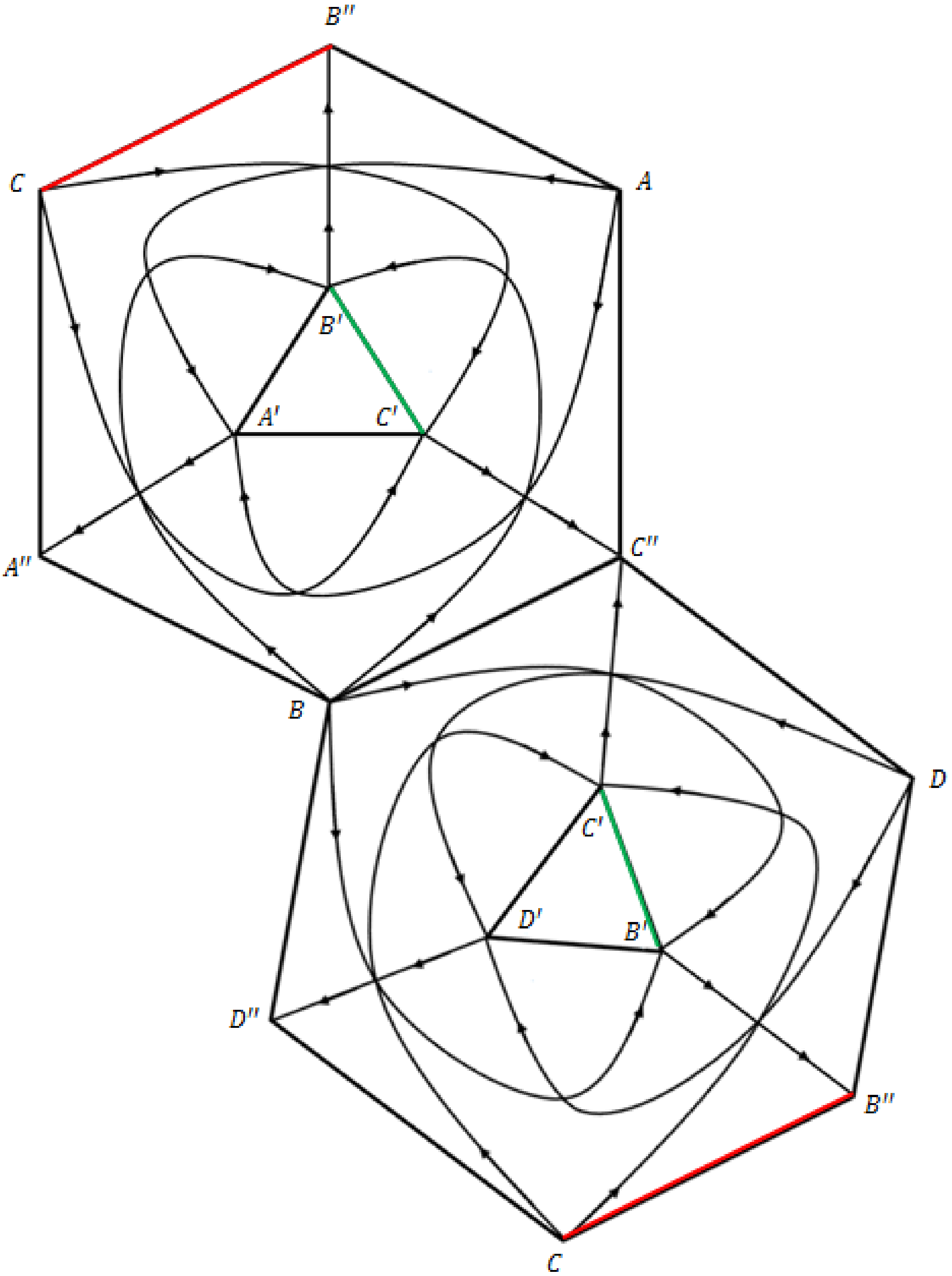

Fig. 3.21: The picture in Σ.

On fig. 3.22 we can see four independent in curves: δ, δ', ζ_B and ζ_C. Each of these curves gives us the number which defines the parallel transport along the curve. We denote these numbers d, d', z_B and z_C.

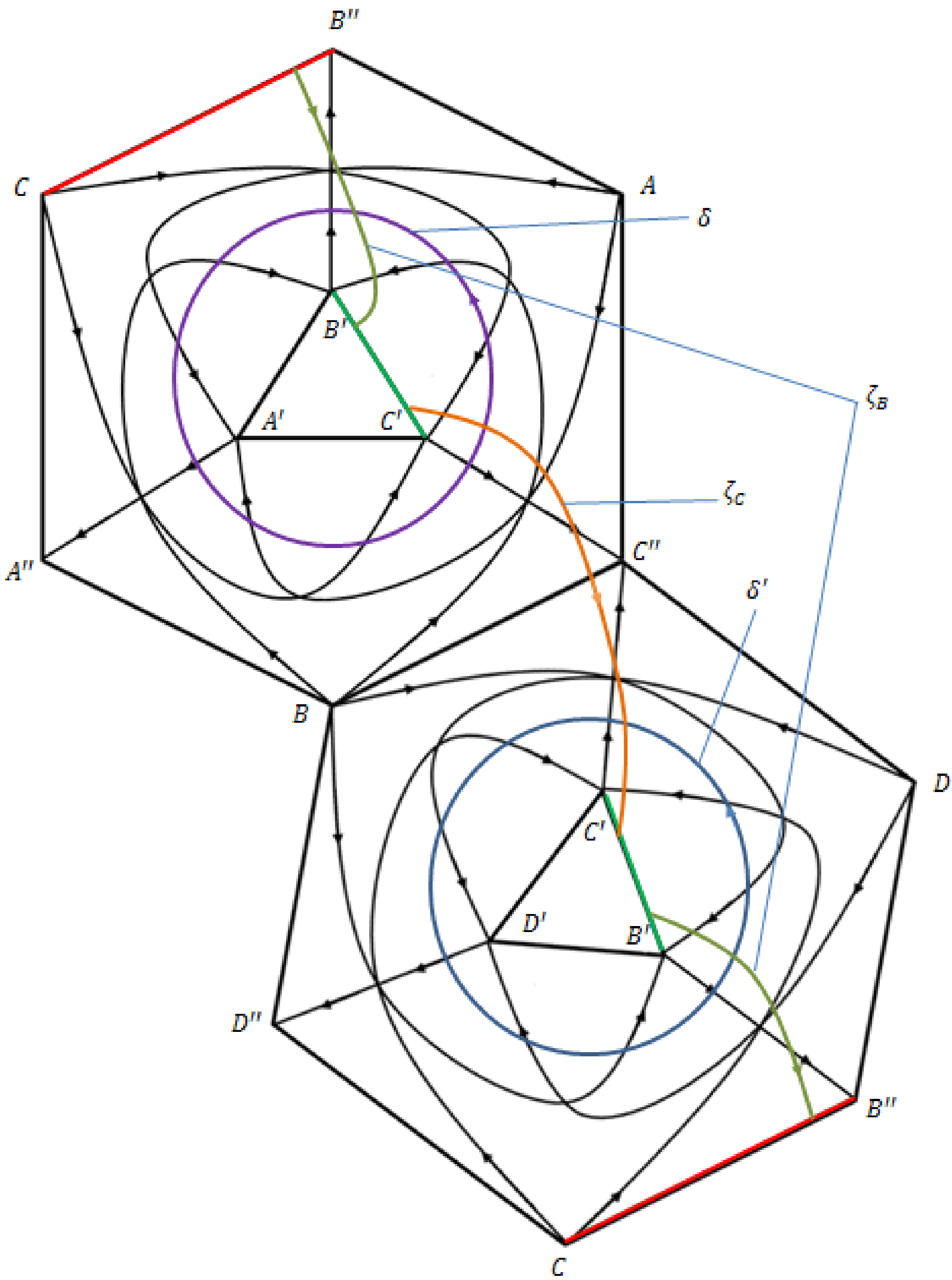

Fig. 3.22: The picture in Σ.

We can also draw three-dimensional pictures of Σ (without lines of spectral network):

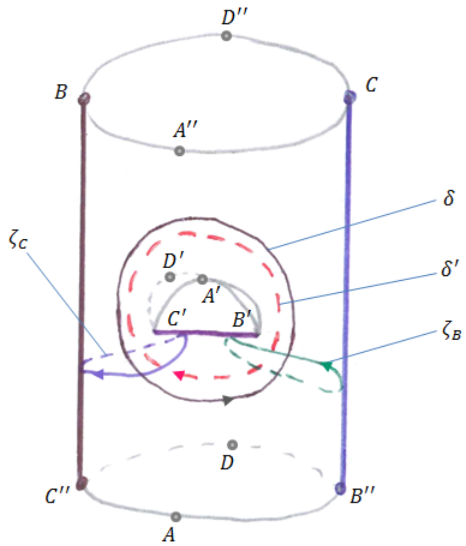

Fig. 3.23: The picture in Σ.

We can make a different picture of the same surface:

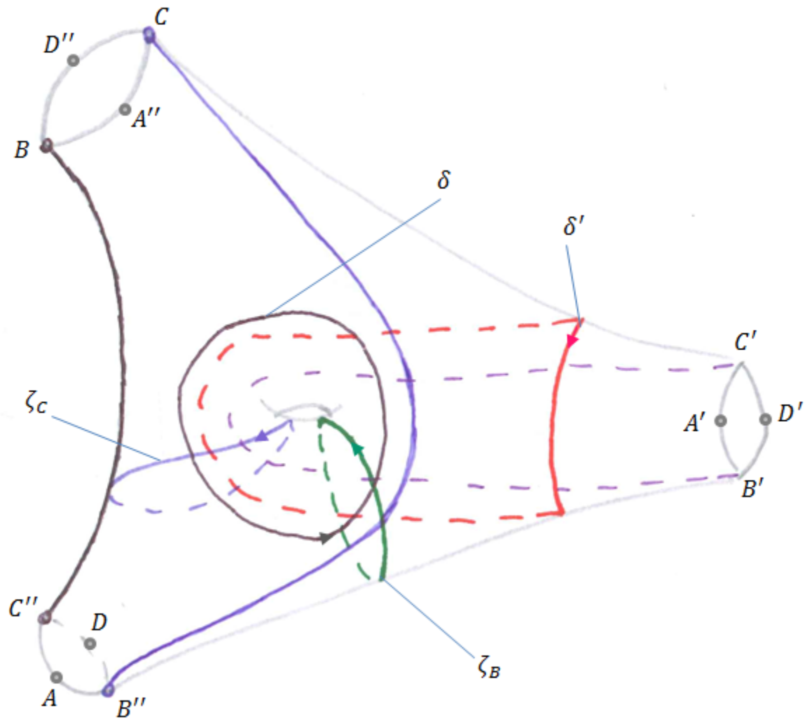

Fig. 3.24: The picture in Σ.

For the points $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ we have also found the invariant flags F_A, F_B, F_C. Now we want to find the invariant flag corresponding to the point $\tilde{D}$.

To do this, we choose the base point p like on the fig. 3.25, choose a peripheral curve α_D and calculate the invariant flag of the parallel transport along the curve α_D in the basis (e_1, e_2, e_3) of E_p.

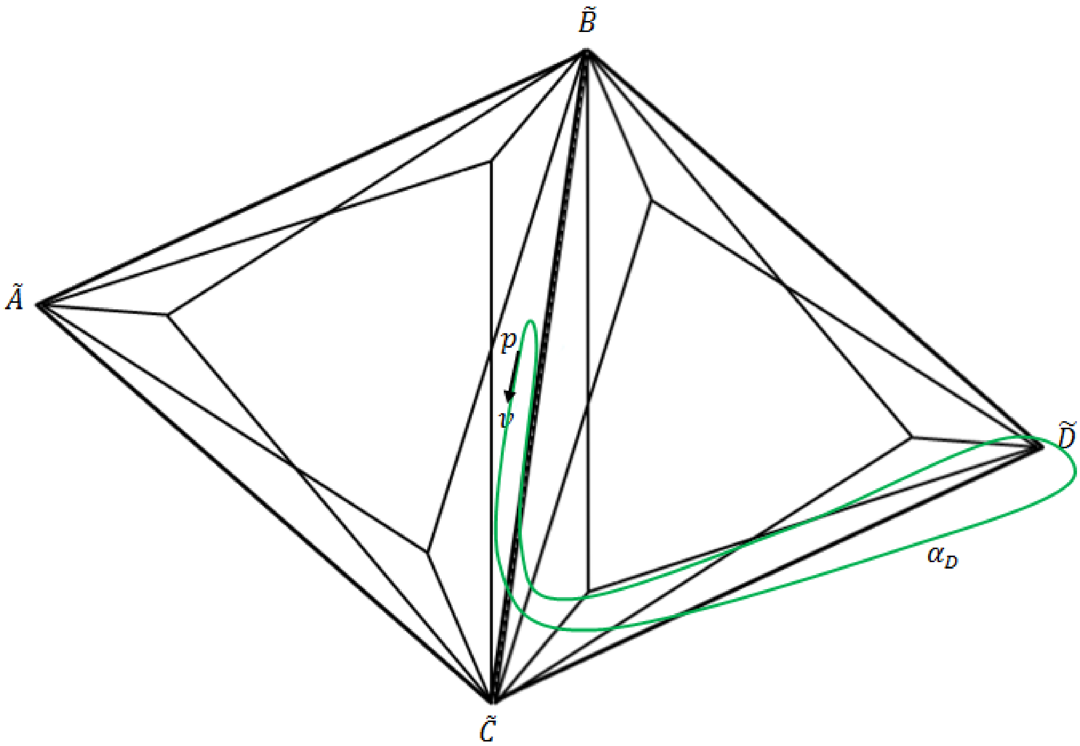

Fig. 3.25: The picture in S.

First of all, we change the base point. We choose the point p' in the triangle $\tilde{D}\tilde{B}\tilde{C}$ and curves ξ and η which connect p and p' and $\xi * \eta \equiv e$ [see fig. 3.26] We also choose the curve α'_D such that $\alpha_D = \xi * \alpha'_D * \eta$.

Because of symmetry we can calculate the invariant flag of α'_D in basis (e'_1, e'_2, e'_3) at the point p using the invariant flag of α_A

$$F_D := \begin{pmatrix} \langle -(d'+1)\nu'_{31}e'_1 + \nu'_{32}d'e'_2 + e'_3 \rangle \\ \langle -(d'+1)\nu'_{31}e'_1 + \nu'_{32}d'e'_2 + e'_3, e'_1 - \nu'_{12}e'_2 \rangle \\ V_{p'} \end{pmatrix}.$$

Because ξ and η do not intersect spectral network, we can identify using remark 2.11 the basis (e_1, e_2, e_3) at the point p with the basis (e'_1, e'_2, e'_3) at the

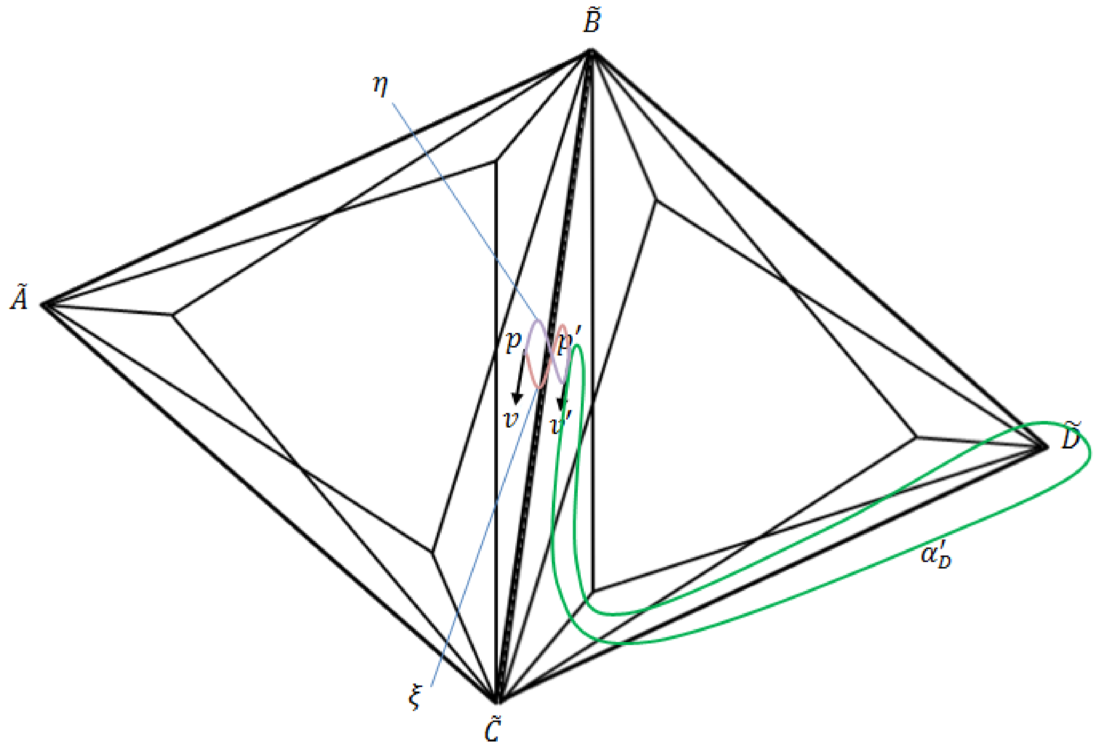

Fig. 3.26: The picture in S.

point p' by parallel transport along ξ, so we only have to calculate ν'_{ij}. To do this, we have to consider the corresponding curves γ'_{ij} such that $T_{\gamma'_{ij}} e'_i = \nu'_{ij} e'_j$.

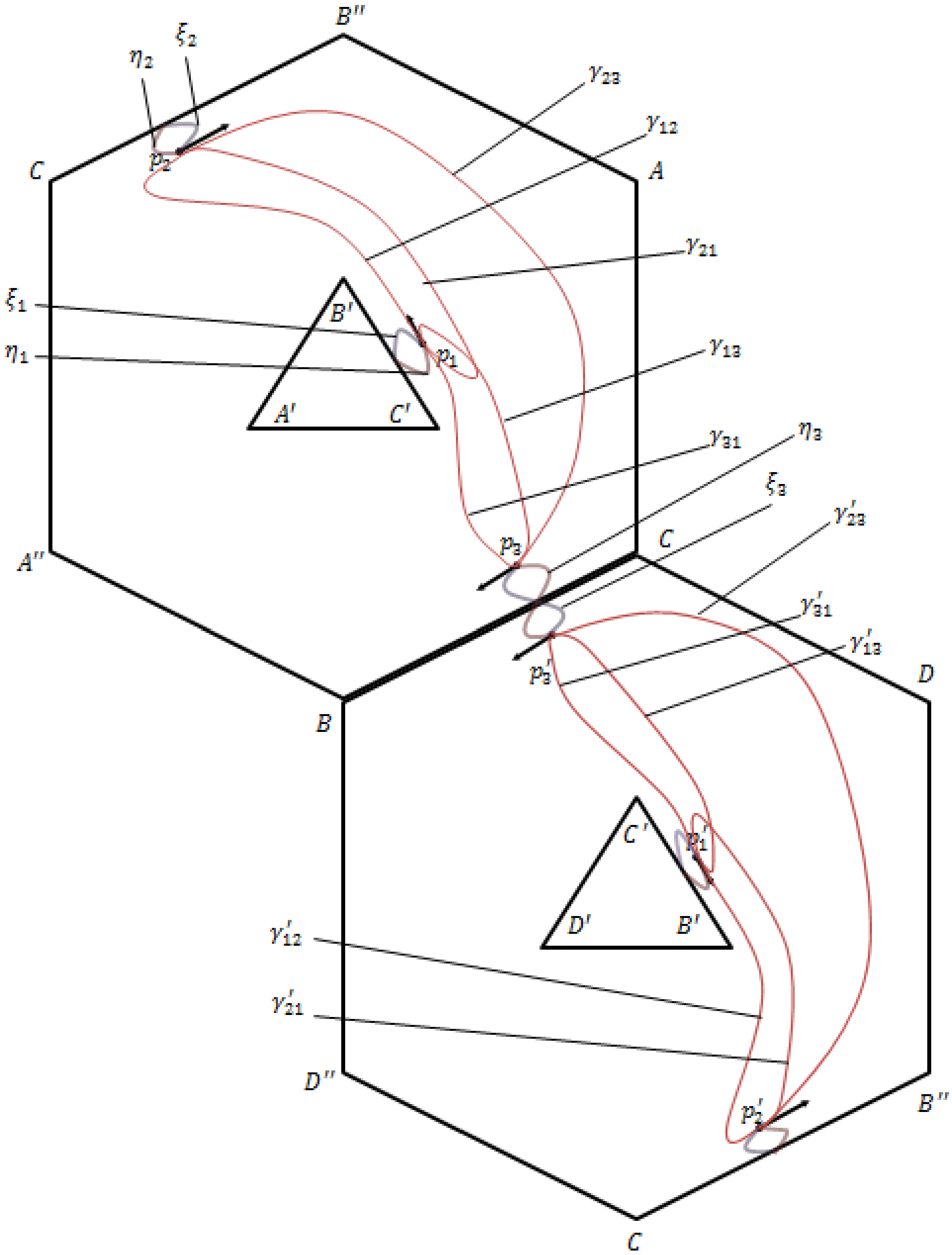

Fig. 3.27: The picture in Σ.

On the fig. 3.27 one can see that $\gamma'_{31} * \eta_1 * \gamma_{13} * \xi_3 \equiv \zeta_C$, therefore,

$$\nu'_{31} = \frac{z_C}{\nu_{13}} = -z_C \nu_{31} = -z_C.$$

Analogously, $\gamma'_{12} * \eta_2 * \gamma_{21} * \xi_1 \equiv \zeta_B$ and

$$\nu'_{12} = \frac{z_B}{\nu_{21}} = -z_B \nu_{12} = -z_B.$$

And also $\nu'_{32} = \nu'_{31}\nu'_{12} = z_B z_C$. So we get:

$$F_D := \begin{pmatrix} \langle (d'+1)z_C e_1 + z_B z_C d' e_2 + e_3 \rangle \\ \langle (d'+1)z_C e_1 + z_B z_C d' e_2 + e_3, e_1 + z_B e_2 \rangle \\ V_p \end{pmatrix}$$

3.5 Small spectral networks of rank 3 and Fock-Goncharov coordinates.

As we have seen in the previous paragraph, the ideal triangulation of the surface with punctures yields the collection of invariant flags in $\mathbb{C}^3$ of the parallel transport along curves which go around punctures. Namely, for each triangle we have three punctures and so we get three invariant flags and for each pair of triangles with common side we have four invariant flags.

The projectivisation yields for each invariant flag a line and a point on this line in $\mathbb{CP}^2$. So for two triangles $\tilde{A}\tilde{B}\tilde{C}$ and $\tilde{D}\tilde{B}\tilde{C}$ of the triangulation of S with the common side $\tilde{B}\tilde{C}$ we have four lines l_A, l_B, l_C and l_D and four points $p_A \in l_A$, $p_B \in l_B$, $p_C \in l_C$, $p_D \in l_D$ in $\mathbb{CP}^2$ (see fig. 3.28), which gives us four Fock-Goncharov coordinates X, Y, Z and W [4], which we have discussed in the section 2.7.

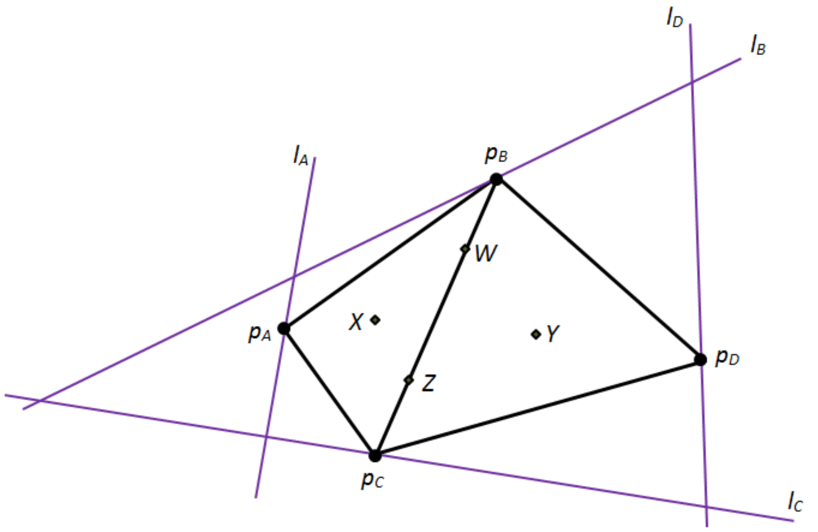

Fig. 3.28: The picture in $\mathbb{CP}^2$.

First of all, we calculate the homogeneous coordinates of the points $p_i \in \mathbb{CP}^2$, $i \in \{A, B, C, D\}$:

$$p_A = [-d^{-1} - 1 : d^{-1} : 1],$$

$$p_B = [0 : 0 : 1],$$

$$p_C = [0 : 1 : 0],$$

$$p_D = [(d' + 1)z_C : z_B z_C d' : 1],$$

Further, we determine the lines $l_i \subseteq \mathbb{CP}^2$, $i \in \{A, B, C, D\}$.

$$l_A = \langle [-d^{-1} - 1 : d^{-1} : 1], [1 : -1 : 0] \rangle,$$

$$l_B = \langle [0 : 0 : 1], [1 : 0 : 0] \rangle,$$

$$l_C = \langle [0 : 1 : 0], [1 : 0 : 0] \rangle,$$

$$l_D = \langle [(d' + 1)z_C : z_B z_C d' : 1], [1 : z_B : 0] \rangle.$$

If we fix three lines corresponding to one of the triangles $\tilde{A}\tilde{B}\tilde{C}$ or $\tilde{D}\tilde{B}\tilde{C}$, then on each line l_i ($i \in \{A, B, C\}$) we have two points p_{ij}, p_{ik} of intersection with two other lines l_j and l_k ($\{j, k\} = \{A, B, C\} \setminus \{i\}$) [see fig. 2.25]. Moreover, for each line l_i we have two point p_j, p_k ($\{j, k\} = \{A, B, C\} \setminus \{i\}$) which do not lie on this line. These two points define us the line l_{jk} which intersects l_i in the point $p_{i,jk}$. So we get 4 points of each line.

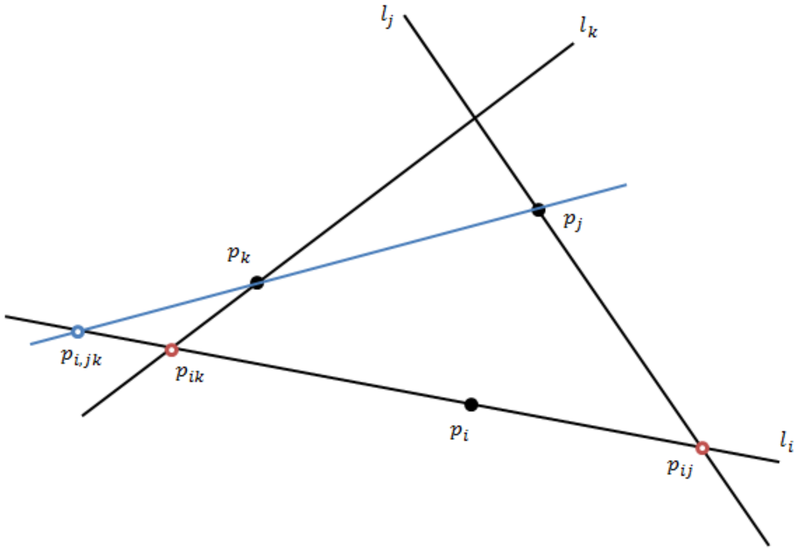

Fig. 3.29: The picture in $\mathbb{CP}^2$.

The coordinate X is defined as the cross ratio of 4 points p_i, p_{ij}, p_{ik}, $p_{i,jk}$ ($\{i,j,k\} = \{A,B,C\}$). Because the cross ratio is a projective invariant, this does not depend on the choice of $i \in \{A,B,C\}$ and the choice of the affine chart. We calculate this number for $i = C$ in the affine chart with respect to the basis $(e_2, e_3, e_1+e_2+e_3)$.

We get $p_C = (1,0)$ and the line l_B is defined by equation $(0,1)t,\ t \in \mathbb{R}$ and line l_C is defined by equation $(1,0)t,\ t \in \mathbb{R}$. Therefore, $p_{CB} = (0,0)$. With respect to this chart l_A is the infinite line, which intersects l_C in infinite point p_{CA}. The line l_{AB} is defined by the equation $(0,1) + t(d^{-1}, 1),\ t \in \mathbb{R}$ which intersects l_C in the point $p_{C,AB} = (-d^{-1}, 0)$. So we get the cross ration

$$X = -[p_C, p_{C,AB}, p_{CA}, p_{CB}] = -\frac{0-1}{0+d^{-1}} = d. \tag{3.5}$$

Because of symmetry we can get the coordinate Y as the cross ratio for the points p_B, $p_{B,CD}$, p_{BD} and p_{BC} from the equation (3.5):

$$Y = -[p_B, p_{B,CD}, p_{BD}, p_{BC}] = d'.$$

The coordinate Z is defined as a cross ratio of the points p_C, p_{CB} which is an intersection point of lines l_C and l_B, $p_{C,AB}$ which is an intersection point of lines l_C and $\overline{p_A p_B}$ and $p_{C,DB}$ which is an intersection point of lines l_C and $\overline{p_D p_B}$ (see fig. 3.30).

To calculate Z, we use the other affine chart with respect to the basis (e_1, e_3, e_2). In this chart p_B is the infinite point, l_B is the infinite line, $p_C = (0,0)$, l_C is defined by equation $(1,0)t,\ t \in \mathbb{R}$, $p_A = (-d-1, d)$, $p_D = (z_B z_C d')^{-1}(z_C(d'+1), 1)$.

In the chosen affine chart we have p_{CB} is a infinite point,

$$p_{C,AB} = (-d-1, 0),$$
$$p_{C,DB} = \left(\frac{d'+1}{z_B d'}, 0\right).$$

So we can calculate Z:

$$Z = [p_{CB}, p_{C,AB}, p_C, p_{C,DB}] = 1 + \frac{1+d'}{(1+d)d' z_B}.$$

Analogously we get the coordinate W

$$W = [p_{BC}, p_{B,DC}, p_B, p_{B,AC}] = 1 + \frac{1+d}{(1+d')d z_C}.$$

We can see that an ideal triangulation of S and a flat connection on Σ define the collection of Fock-Goncharov coordinates on S. Conversely, if an ideal triangulation

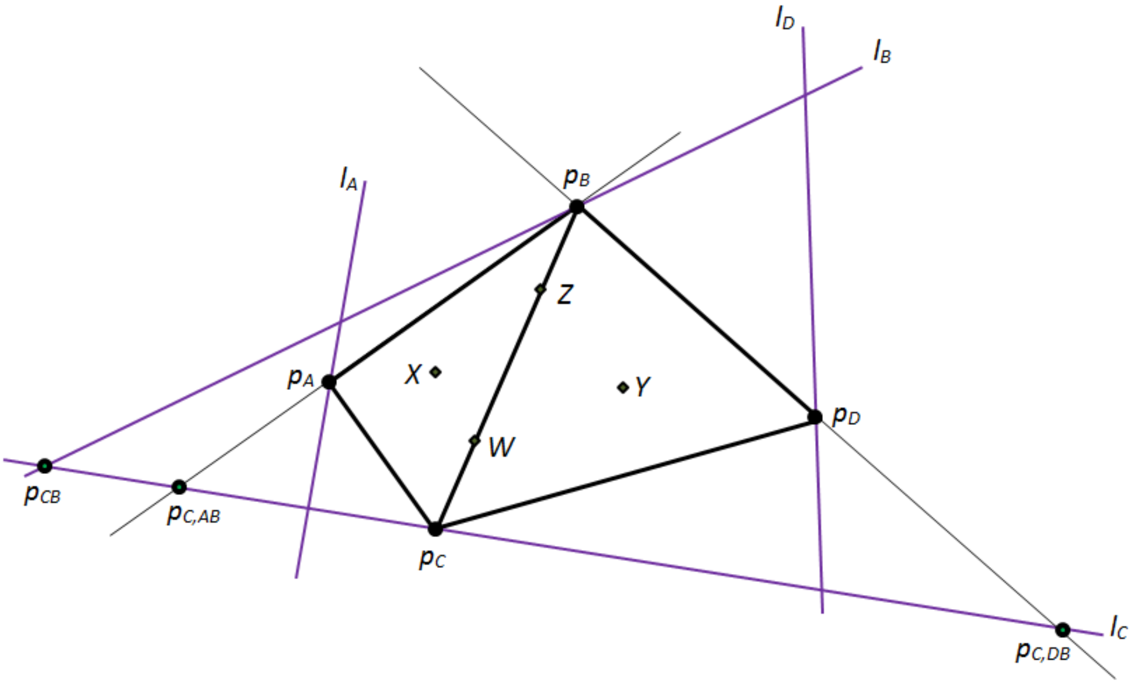

Fig. 3.30: The picture in $\mathbb{CP}^2$.

$\mathcal{T}$ on S and a collection of Fock-Goncharov coordinates with respect to this triangulation are given, we get the numbers d, d', z_B, z_C for each pair of triangles with common side. We can construct a representation $\rho\colon \pi_1(S,p) \to PGL(3,\mathbb{C})$, $p \in S$, p do not lie on lines of triangulation in the following way.

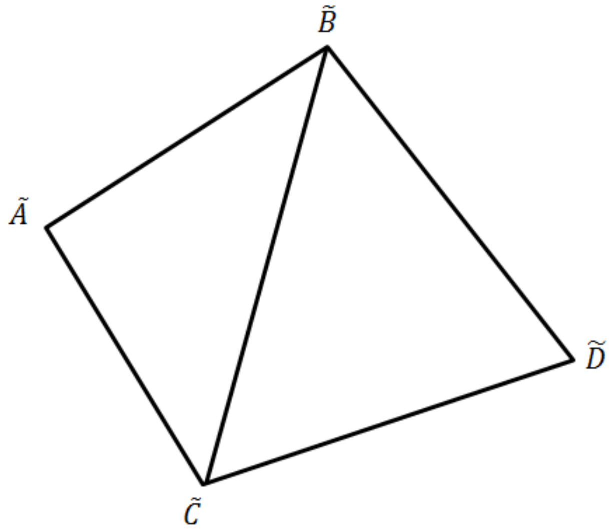

Fig. 3.31: The picture in S.

First, we fix some total order ϱ on triangles of this triangulation and choose a pair of triangles with vertices $\tilde{A}, \tilde{B}, \tilde{D}, \tilde{C} \in S$ and the common side $c = \tilde{B}\tilde{C}$. For this pair of triangles the numbers d, d', z_B, z_C are defined by Fock-Goncharov coordinates. We assume that by projectivization invariant flags F_i for $i \in \{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\}$ go to points of $\mathbb{CP}^2$:

$$p_A = [-d^{-1} - 1 : d^{-1} : 1],$$

$$p_B = [0 : 0 : 1],$$

$$p_C = [0 : 1 : 0],$$

$$p_D = [(d' + 1)z_C : z_B z_C d' : 1],$$

and lines:

$$l_A = \langle [-d^{-1} - 1 : d^{-1} : 1], [1 : -1 : 0] \rangle,$$

$$l_B = \langle [0 : 0 : 1], [1 : 0 : 0] \rangle,$$

$$l_C = \langle [0 : 1 : 0], [1 : 0 : 0] \rangle,$$

$$l_D = \langle [(d' + 1)z_C : z_B z_C d' : 1], [1 : z_B : 0] \rangle.$$

We have a unique projective transformation ϕ, witch sends p_A to p_D, l_A to l_D and fix other two points and lines. This transformation corresponds to a unique element $M \in PGL(3, \mathbb{C})$. In this way we get a bijective correspondence between the set of all pairs of triangles on S with a common side and $PGL(3, \mathbb{C})$. We also fix the number ε, which is equal to 1 if the triangle $\tilde{A}\tilde{B}\tilde{C}$ is smaller then $\tilde{D}\tilde{B}\tilde{C}$, and is equal to -1 otherwise.

We consider a group homomorphism $\kappa\colon F_C \to PGL(3,\mathbb{C})$, which is defined on generators of F_C in a following way $\kappa(c) := M^{\varepsilon}$ for all sides c of triangulation $\mathcal{T}$.

Further, we consider the universal covering $\pi\colon S' \to S$ of S and corresponding triangulation $\mathcal{T}'$. We denote by P' the set of all vertices of triangulation of $\mathcal{T}'$. We want to construct two function f_1 and f_2, such that f_2 sends each element $s' \in P'$ to a line $f_2(s')$ in $\mathbb{CP}^2$ and f_1 sends s' to a point $f_1(s') \in f_2(s')$.

First, we choose a lift $p' \in \pi^{-1}(p)$. This lies in a triangle T_0 on S' with vertices s_1', s_2', s_3'. We define f_1 and f_2 for this three points in a following way:

$$f_1(s_1') := [1:0:1],\ f_1(s_2') := [0:1:1],\ f_1(s_3') := [1:1:0],$$

$$f_2(s_1') := \langle [1:0:1],[0:0:1]\rangle,$$

$$f_2(s_2') := \langle [0:1:1],[0:0:1]\rangle,$$

$$f_2(s_3') := \langle [1:1:0],[1:0:0]\rangle.$$

We use the map ψ from the section 2.9 and assume that the order ϱ agree with the order ϱ in the section 2.9. If we choose $s' \in P'$, then by remark 2.20 there exist a unique shortest word $b \in F_C(S,p)$ and unique $i \in \{1,2,3\}$ such that $\psi(b, s_i) = s'$. Because b and i are unique, we define $f_j(s') := \kappa(b)(f_j(s_i))$, $j \in \{1,2\}$.

Now we are ready to construct a representation $\rho\colon \pi_1(S,p) \to PGL(3,\mathbb{C})$. We take an element $g \in \pi_1(S,p)$. We can consider this element as a deck transformation of S', which also acts on P' and $g(s_i') = r_i' \in P'$, $i \in \{1,2,3\}$. There exist a unique element $\rho(g) \in PGL(3,\mathbb{C})$ such that $\rho(f_j(s_i')) = f_j(r_i')$ for all $i \in \{1,2,3\}$, $j \in \{1,2\}$. By this rule the map $\rho\colon \pi_1(S,p) \to PGL(3,\mathbb{C})$ is well defined, and it is easy to see that this is a group homomorphism, so we get a representation.

Remark 3.3. *Because the order ϱ on $\mathcal{T}$ and choice of sign of ε are consistent, the constructed representation $\rho\colon \pi_1(S,p) \to PGL(3,\mathbb{C})$ does not depend on this order. Moreover, this representation depends only on a triangulation $\mathcal{T}$ and on Fock-Goncharov coordinates, which are given with respect to this triangulation.*

3.6 Other coordinates associated with spectral networks of rank 3

How we have seen, if we fix two triangles $\tilde{A}\tilde{B}\tilde{C}$ and $\tilde{D}\tilde{B}\tilde{C}$ with a common side, then four Fock-Goncharov coordinates are defined by homotopy constants d, d', z_B, z_C.

Two coordinates X and Y agree with constants d and d'. Two other coordinates Z and W depend on d, d' and z_B resp. z_C. Now we are going to give an interpretation of z_B and z_C which is similar to Fock-Goncharov coordinates. Instead of two points $p_{C,AB}$ and $p_{C,DB}$ we consider two points p_{CA} and p_{CD} which are points of intersection l_C and l_A resp. l_D (see fig. 3.32). The new U-coordinate we define as

$$U = -[p_C, p_{CB}, p_{CA}, p_{CD}]$$

Analogously, we define V-coordinate:

$$V = -[p_B, p_{BC}, p_{BD}, p_{BA}]$$

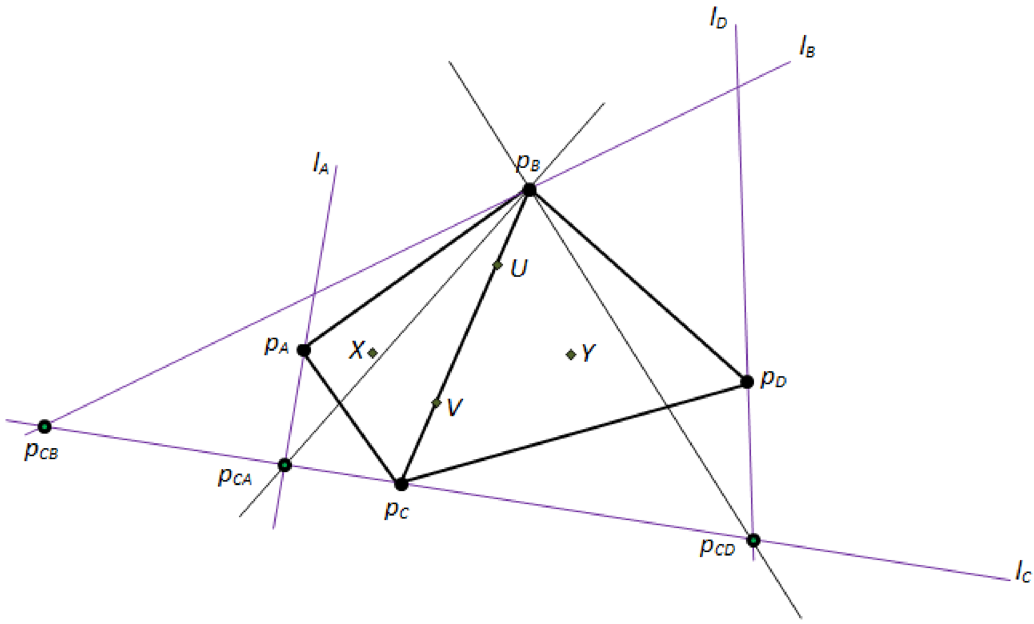

Fig. 3.32: Picture in $\mathbb{CP}^2$.

Now we calculate these coordinates for our pair of quadrangle given by spectral network. In the chart with respect to the basis (e_1, e_3, e_2) the point $p_{CA} = (-1, 0)$, p_{CB} is infinite point, $p_C = (0, 0)$, $p_{CD} = (z_B^{-1}, 0)$. Therefore,

$$U = z_B.$$

Analogously, we get

$$V = z_C.$$

We can see that these new coordinates determine Fock-Goncharov coordinates and vice versa, but in our case these coordinates are more natural because they correspond to homotopy classes of curves on Σ.

We also want to see, how this new coordinates change if we change the triangulation. Like in the case of Fock-Goncharov coordinates we consider only the case of a flip [see fig. 3.33].

A simple affine geometric calculation gives us transformation formulas of coordinates:

$$X' = \frac{(1+U)V}{(1+V)Y}, \quad Y' = \frac{(1+V)U}{(1+U)X},$$
$$U' = \frac{Y + V(1+Y+U)}{(X+U(1+X+V))Y}, \quad V' = \frac{X + U(1+X+V)}{(Y+V(1+Y+U))X}.$$

Remark 3.4. *These changes of coordinates are positive. This means that the minus sign does not appear in transformation formulas, as in the original Fock-Goncharov coordinates.*

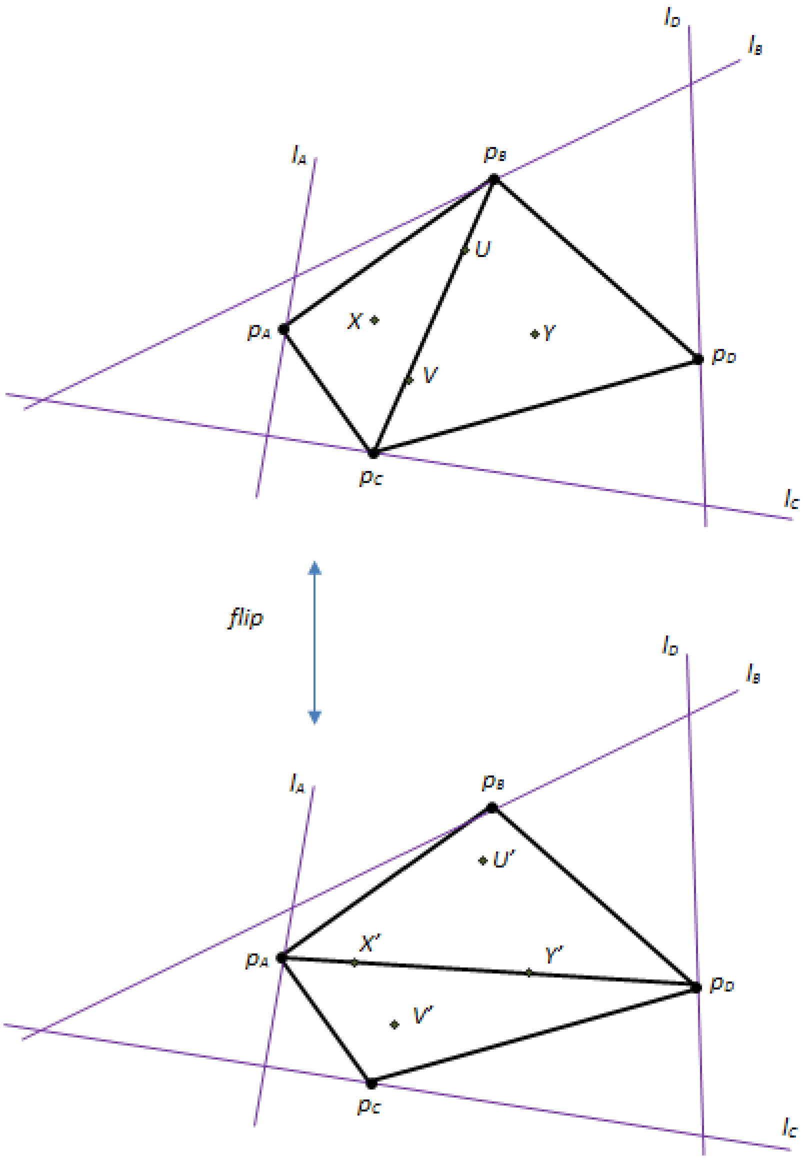

Fig. 3.33: Flip. Picture in $\mathbb{CP}^2$.

4 Bibliography

[1] Alessandrini, D., *Introduction to Spectral Networks*, mini-course at University of Illinois at Urbana-Champaign, 2014.

http://gear.math.illinois.edu/events/Archive-GeometryGroupsandDynamicsGEARSeminar.html

[2] Brambila-Paz, L., Newstead, P., Thomas, R.P.W., Garcia-Prada, O., *Moduli Spaces (London Mathematical Society Lecture Note Series)*, Cambrige University Press, Cambrige, 2014.

[3] Cucker, F., Wong, R.S.C., *The Collected Papers of Stephen Smale - Volume 1*, World Scientific Publishing Co. Pte. Ltd., Singapore, 2000.

[4] Fock, V.V, Goncharov, A.B., *Moduli spaces of convex projective structures on surfaces*, Advances in Math., Vol. 208 (2007), pp. 249-273.

[5] Fock, V.V, Goncharov, A.B., *Moduli spaces of local systems and higher Teichmüller theory*, Publications Mathématiques de l'IHÉS, Vol. 103 (2006), pp. 1-221.

[6] Gaiotto, D., Moore, G.W., Neitzke, A., *Spectral networks*, Annales Henri Poincaré, Vol. 14, No. 7 (2013), pp. 1643-1731.

[7] Gaiotto, D., Moore, G.W., Neitzke, A., *Spectral Networks and Snakes*, Annales Henri Poincaré, Vol. 15, No. 1 (2014), pp. 61-141.

[8] Hollands, L., Neitzke, A., *Spectral networks and Fenchel-Nielsen coordinates*, Letters in Mathematical Physics, Vol. 106.6 (2016), pp. 811-877.

[9] Moore, G.W., *Lecture notes for Felix Klein lectures*, 2012.

[10] Penner, R.C., *Decorated Teichmüller Theory, The QGM Master Class Series*, 2012, European Mathematical Society, Zürich, 2012.