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Poisson geometry in Evolutionary Game Theory

Slightly revised version

of the master thesis in Physics

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To the memory of Alberto Greggio

Abstract

In this work we show that zero-sum replicator evolutionary games in presence of an interior fixpoint admit a Hamiltonian description with respect to a cubic Poisson structure on the simplex. In the first chapter Poisson manifolds are studied, with particular focus on the methods of Poisson reduction. Via a reduction procedure we derive in the second chapter a stratified Poisson structure for the simplex. In the third chapter we introduce normal games in a population dynamics setting and the notions of stable Nash strategy and evolutionarily stable strategy. This concept implies an underlying dynamics for the evolution of the average strategy of the population, modeled with the replicator equation. Simple examples are discussed, but the focus is mainly geometrical: the main result proves the Hamiltonian character of the replicator vector field with respect to the derived Poisson structure.

In dieser Arbeit zeigen wir, dass Nullsummen-Replikator-Evolutionsspiele bei Vorhandensein eines inneren Fixpunktes eine Hamiltonsche Beschreibung in Bezug auf eine kubische Poisson-Struktur auf dem Simplex zulassen. Im ersten Kapitel werden Poisson-Mannigfaltigkeiten untersucht, mit besonderem Schwerpunkt auf den Methoden der Poisson-Reduktion. Über ein Reduktionsverfahren leiten wir im zweiten Kapitel eine geschichtete Poisson-Struktur für den Simplex ab. Im dritten Kapitel stellen wir sowohl normale Spiele in einer populationsdynamischen Umgebung als auch die Begriffe der stabilen Nash-Strategie und der evolutionär stabilen Strategie vor. Dieses Konzept geht von einer zugrunde liegenden Dynamik für die Entwicklung der durchschnittlichen Strategie der Population aus, die mit der Replikatorgleichung modelliert wird. Einfache Beispiele werden diskutiert, aber der Schwerpunkt liegt hauptsächlich auf der Geometrie: Das Hauptergebnis beweist den Hamilton-Charakter des Replikator-Vektorfeldes in Bezug auf die abgeleitete Poisson-Struktur.

Contents

List of Symbols

Throughout this work the meaning of the employed symbols shall always be clear from the context, and the same symbol may have different meanings in different contexts. Nevertheless here is a list of the most common symbols and their most common meanings.

$\alpha, \beta, \omega, \ldots$	differential forms $\in \Omega^k(M)$, p. 3
$\mathbb{C}P(n)$	complex projective space, p. 34
ω	symplectic form $\in \Omega^2(M)$, p. 4
π	Poisson bivector $\in \nu^2(M)$, p. 2
$X, Y, Z, \ldots$	vector fields $\in \tau(M) \equiv \tau^1(M) \equiv \nu^1(M)$, p. 3
X_f	Hamiltonian vector field of Hamiltonian function f , p. 10
$[\cdot, \cdot]$	Lie bracket, p. 69
$\Theta_t(p), \Theta^p(t)$	flow of a vector field, p. 74
$\flat : \tau(M) \to \Omega(M)$	flat homomorphism, p. 71
$\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$	exterior derivative, p. 69
$\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$	interior product with the vector field X, p. 69
$\mathscr{L}_X : \tau^k_l(M) \to \tau^k_l(M)$	Lie derivative along the vector field X, p. 75
$\sharp : \Omega(M) \to \tau(M)$	sharp homomorphism, p. 71
$\pi : M \to {}^M/_G$	canonical projection on quotient space, p. 26
$\{\cdot, \cdot\}$	Poisson bracket, p. 4
(H)	conjugacy class of a subgroup $H \subset G$, p. 27
$C^\infty(M)$	space of smooth functions $f : M \to \mathbb{R}$, p. 4
Δ^n	standard n-dimensional simplex, p. 31
$\nu^k(M)$	space of multi-vector fields, p. 70
$\Omega^k(M)$	space of k-forms, p. 3
T_pM, T^*_pM	tangent and cotangent spaces to M at p, p. 69
$\tau^k_l(M)$	space of (k, l) tensor fields over M, p. 3
G_p	isotropy subgroup, p. 25
$M_{(H)}$	orbit type submanifold, p. 27
M_H	isotropy type submanifold, p. 27

Chapter 1

Poisson Manifolds

This chapter is devoted to the theory of Poisson Manifolds. The fundamental idea is to endow a smooth manifold with an additional structure arising from an antisymmetric *biderivation*, or equivalently an antisymmetric $(2, 0)$ tensor field, called *bivector field*, fulfilling some additional condition. Think for analogy of a Riemannian structure, arising from a symmetric, positive-definite $(0, 2)$ tensor field; and of a symplectic structure, arising from a closed nondegenerate 2-form. The resemblance with the Riemannian case is just shallow, while we will see that there exists a strict connection between symplectic and Poisson manifolds.

The goal of this chapter is to develop the concepts required to derive in the next chapter a Poisson structure for the standard simplex, the domain of the replicator dynamical system.

We follow mainly [LM87], [Vai94] and [DZ05]. For general differential geometry concepts we refer to [Lee12].

1.1 Poisson structures

Notation 1.1. If M is a smooth manifold, $\tau_l^k(M)$ denotes the space of (k, l) tensor fields on M. For $(1, 0)$ fields, i.e. vector fields, we just write $\tau(M)$. The space of k-forms is denoted $\Omega^k(M)$, and we write just $\Omega(M)$ for 1-forms. All other employed differential geometry concepts are introduced in Appendix (A).

Definition 1.2. A *Poisson algebra* $(V, K, \circ, \{\cdot, \cdot\})$ is a vector space V over a field K endowed with two bilinear operations $\circ : V \times V \to V$ and $\{\cdot, \cdot\} : V \times V \to V$ such that

- $(V, K, \circ)$ is an associative algebra;
- $(V, K, \{\cdot, \cdot\})$ is a Lie algebra, namely $\{\cdot, \cdot\}$
  - is antisymmetric : $\{u, v\} = -\{v, u\}$ for all u, v in V;
  - fulfills the Jacobi identity:

$$\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0, \quad \forall\, a, b, c \in V \tag{1.1}$$

- $\{\cdot, \cdot\}$ is a derivation with respect to $\circ$ in both arguments, namely for any fixed $u \in V$ the map $\{u, \cdot\} : V \to V$ fulfills

$$\{u, a \circ b\} = \{u, a\} \circ b + a \circ \{u, b\} \tag{1.2}$$

for any $a, b \in V$, and similarly for $\{\cdot, u\}$.

The map $\{\cdot,\cdot\}$ is called *Poisson bracket.*

For our purposes the associative algebra $(V, K, \circ)$ shall be the space of smooth functions with real coefficients over a smooth manifold M into the real numbers, denoted by $C^\infty(M) \coloneqq \{f : M \to \mathbb{R}, f \text{ smooth}\}$, with the map $\circ$ being the usual (associative) functions product (not composition). Recall also that the space of vector fields on a smooth manifold is a Lie algebra under the Lie bracket (see Remark A.1).

Definition 1.3. A *Poisson manifold* is a smooth manifold M with a Poisson bracket $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ making $(C^\infty(M), \{\cdot,\cdot\})$ a Poisson algebra.

Example 1.4 (Poisson structure of symplectic manifold)**.** [LM87, p. 89] A symplectic manifold (M, ω) is defined by a closed nondegenerate 2-form $\omega \in \Omega^2(M)$, called *symplectic form.* Nondegeneracy implies that the flat isomorphism $\flat_\omega$ admits an inverse sharp isomorphism $\sharp_\omega$ (see equations (A.12), (A.14)), so ω associates every vector field to precisely one 1-form.

Hamiltonian vector fields are symplectic gradients, namely images of *exact* 1-forms via $\sharp_\omega$. Given $f \in C^\infty(M)$ its Hamiltonian vector field X_f is hence defined by[1][2]

$$X_f = -\sharp_\omega(\mathrm{d}f) \tag{1.3}$$

and the space of Hamiltonian vector fields is denoted by $\tau_H(M) \in \tau(M)$. The previous definition is equivalent to

$$\flat_\omega(X_f) \equiv \iota_{X_f}\omega \equiv \omega(X_f, \cdot) = -df \tag{1.4}$$

This provides a well defined *Hamiltonian map* $C^\infty(M) \to \tau_H(M)$, $f \mapsto X_f$.

Proposition 1.5. *A function $f \in C^\infty(M)$ is constant along the integral curves of its Hamiltonian vector field. Furthermore, given two functions $f, g \in C^\infty(M)$, f is constant along the integral curves of X_g if and only if g is constant along the integral curves of X_f.*

Proof.

$$\mathscr{L}_{X_f} f = X_f f = \mathrm{d}f(X_f) = \omega(X_f, X_f) = 0 \tag{1.5}$$

which proves the first statement, and

$$0 = \omega(X_f, X_g) + \omega(X_g, X_f) = \mathscr{L}_{X_f} g + \mathscr{L}_{X_g} f \tag{1.6}$$

which proves the second. □

We now show that $\tau_H(M)$ is Lie subalgebra of $\tau(M)$, and that $C^\infty(M)$ can be endowed with a bilinear map making it a Poisson algebra, and such that the Hamiltonian map is a Lie algebra homomorphism.

A vector field X is called *symplectic* if it leaves the symplectic form invariant, in the sense that the Lie derivative of ω along X vanishes identically, $\mathscr{L}_X\omega = 0$ (see eq. (A.42) and the discussion around it); the space of symplectic vector fields is denote by $\tau_S(M)$.

[1] We include a minus sign in the definition of Hamiltonian vector field on a symplectic manifold so that $f \mapsto X_f$ is a Lie algebra homomorphism, and not an anti-homomorphism. If τ is the tautological 1-form on a cotangent bundle T^*M with canonical coordinates (q, p), namely $\tau = p_i \mathrm{d}q^i$, this amounts to fixing the orientation of the canonical symplectic form by $\omega = +\mathrm{d}\tau$. This convention is adopted e.g. by [LM87], [Bry18], [DZ05], [Vai94]. Other classical references like [AMM78], [Arn89], [Lee12] adopt the opposite convention.

[2] A Lie algebra homomorphism is a linear map between Lie algebras compatible with the algebra structure, namely $\phi : (V, \{\cdot,\cdot\}_V) \to (W, \{\cdot,\cdot\}_W)$ such that $\phi(\{v_1, v_2\}_V) = \{\phi(v_1), \phi(v_2)\}_W$, $\forall v_1, v_2 \in V$.

Cartan magic formula (A.44) and the closedness of ω imply that a vector field X is symplectic if and only if its associated 1-form $\iota_X\omega$ is closed, since

$$\mathscr{L}_X\omega = \underbrace{\iota_x \,\mathrm{d}\omega}_{\equiv 0} + \mathrm{d}\,\iota_x\omega \tag{1.7}$$

This means that any Hamiltonian vector field is symplectic (being its associated 1-form exact, thus closed), and we have the following diagram:

$$\begin{array}{ccc} X \text{ Hamiltonian} & \Longleftrightarrow & \iota_X\omega \text{ exact} \\ & \Downarrow \Uparrow \text{locally} & \\ X \text{ symplectic} & \Longleftrightarrow & \iota_X\omega \text{ closed} \end{array} \tag{1.8}$$

The Lie bracket of symplectic vector fields is a Hamiltonian vector field: if $\mathscr{L}_X\omega = 0 = \mathscr{L}_Y\omega$,

$$\begin{aligned} \iota_{[X,Y]}\omega &= \omega\left(\mathscr{L}_X Y\right) = \mathscr{L}_X\left(\omega(Y)\right) - \left(\mathscr{L}_X\omega\right)(Y) \\ &= \iota_X\,\mathrm{d}\,\iota_Y\omega + \mathrm{d}\,\iota_X\,\iota_Y\omega = -\mathrm{d}\left(\omega(X,Y)\right) \end{aligned}$$

which indeed means that $\omega(X,Y)$ is the Hamiltonian of $[X,Y]$, or

$$X_{\omega(X,Y)} = [X,Y], \quad \forall X,Y \in \tau_S(M) \tag{1.9}$$

Define now the map

$$\begin{aligned} \{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) &\to C^\infty(M) \\ (f,g) &\longmapsto \{f,g\} \coloneqq \omega(X_f, X_g) \end{aligned} \tag{1.10}$$

Proposition 1.6. *Given a symplectic manifold (M,ω), the space $C^\infty(M)$ is a Poisson algebra under eq. (1.10), the space of Hamiltonian vector fields $\tau_H(M)$ is a Lie subalgebra of $\tau(M)$ under the Lie bracket, and the Hamiltonian map $C^\infty(M) \to \tau_H(M)$ given by eq. (1.3) is a Lie algebra homomorphism. In particular, any symplectic manifold is a Poisson manifold.*

Proof. Bilinearity and antisymmetry clearly follow from the properties of ω. The derivation property follows from the derivation nature of vector fields, eq. (A.2): note that

$$\{f,g\} = -\omega(X_g, X_f) = -\iota_{X_g}\omega(X_f) = +\mathrm{d}g(X_f) = X_f\, g$$

so that $\{f,\cdot\} = X_f$ is effectively a derivation acting on $C^\infty(M)$. Explicitly,

$$\{f,gh\} = X_f(gh) = (X_f g)h + g(X_f h) = \{f,g\}h + g\{f,g\}$$

and similarly in the second argument. Finally, since $\tau_H(M) \subset \tau_S(M)$, eq. (1.9) implies that the commutator of Hamiltonian vector fields is Hamiltonian, so $\tau_H(M)$ is indeed a Lie subalgebra:

$$X_{\omega(X_f,X_g)} = X_{\{f,g\}} = [X_f, X_g], \quad \forall f,g \in C^\infty(M) \tag{1.11}$$

For the Jacobi property of eq. (1.10), consider $f,g,h \in C^\infty(M)$. On one side, from the definition of commutator and using $\{f,\cdot\} = X_f$

$$[X_f, X_g]h = X_f X_g h - X_g X_f h = \{f,\{g,h\}\} - \{g,\{f,h\}\}$$

On the other, from eq. (1.11)

$$[X_f, X_g]h = X_{\{f,g\}}h = \{\{f,g\},h\}$$

Combining these two expressions gives indeed eq. (1.1). Thus $C^\infty(M)$ is a Poisson algebra, and eq. (1.11) says precisely that the Hamiltonian map is a Lie algebra homomorphism. ☐

1.1.1 The Poisson bivector

Consider a bivector $\pi \in \nu^2(M)$ on a manifold M, namely an antisymmetric $(2,0)$ tensor field, as defined in (A.8). Eq. (A.11) shows that π defines an antisymmetric bilinear map $C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, $(f,g) \mapsto \pi(\mathrm{d}f, \mathrm{d}g)$, which is a $C^\infty(M)$-derivation in both variables. Conversely, it can be shown [DZ05, p. 6], [LM87, p. 109] that a map with such properties, denoted by $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, defines a unique bivector π on M such that $\{f,g\} = \pi(\mathrm{d}f, \mathrm{d}g)$ for all $f,g \in C^\infty(M)$. In particular, for any pair of smooth functions f,g, $\{f,g\}$ is fully determined by a finite number of *elementary Poisson brackets* $\{x^i, x^j\}$, where x^i denotes the i-th coordinate function in a local chart. Indeed

$$\{x^i, x^j\} = \pi(\mathrm{d}x^i, \mathrm{d}x^j) = \pi^{ij} \tag{1.12}$$

$$\{f,g\} = \pi(\mathrm{d}f, \mathrm{d}g) = \pi^{ij}\partial_i f \partial_j g = \{x^i, x^j\}\partial_i f \partial_j g, \quad \forall f,g \in C^\infty(M) \tag{1.13}$$

If the map $\{\cdot,\cdot\}$ is a Poisson bracket it makes sense to ask how the Jacobi condition (1.1) looks like for its unique associated bivector. Given a local chart on M eq. (1.12) holds, and

$$\{x^i, \{x^j, x^h\}\} = \pi(\mathrm{d}x^i, \mathrm{d}\pi(\mathrm{d}x^j, \mathrm{d}x^h)) = \pi(\mathrm{d}x^i, \mathrm{d}\pi^{jh}) = \pi^{ik}\,\partial_k \pi^{jh}$$

Thus the Jacobi identity for the bivector associated to a Poisson bracket reads [DZ05, p. 8], [Vai94, p. 4]

$$\sum_{\text{cyclic } i,j,h} \pi^{ik}\,\partial_k \pi^{jh} = 0 \tag{1.14}$$

Note that this is nontrivial only in dimension greater than 2. This condition can be expressed in a coordinate-free way as $[\pi,\pi]_S = 0$, where $[\cdot,\cdot]_S$ is the so-called *Schouten-Nijenhuis bracket*; the interested reader is referred to [Vai94, p. 6] and [DZ05, p. 27] for an introduction, and to [BV88] for a detailed study.

Definition 1.7. Given a manifold M, a *Poisson bivector* is a bivector fulfilling eq. (1.14).

A Poisson bracket on a manifold M is thus equivalent to a Poisson bivector, and in the following we shall speak of a Poisson manifold (M,π).

Example 1.8 (Poisson bivector of symplectic manifold)**.** Recall from example (1.4) that a symplectic form ω on a manifold M defines a Poisson bracket $\{f,g\} = \omega(X_f, X_g)$. Per definition, the Poisson bivector associated with this Poisson bracket is $\pi \in \nu^2(M)$ such that

$$\pi(\mathrm{d}f, \mathrm{d}g) = \omega(X_f, X_g), \quad \forall f,g \in C^\infty(M) \tag{1.15}$$

As discussed around eq. (A.17), a symplectic form can map itself to a $(2,0)$ tensor field $\hat\omega$ via its musical isomorphism extended to fields of arbitrary rank, and

$$\omega(X_f, X_g) = \hat\omega\,(\flat_\omega(X_f), \flat_\omega(X_g)) = (-1)^2\,\hat\omega(\mathrm{d}f, \mathrm{d}g) \tag{1.16}$$

Thus, given eq. (A.18), a symplectic form $\omega \in \Omega^2(M)$ defines a nondegenerate Poisson bivector π equal to *minus* its inverse

$$\pi = \hat\omega = -\omega^{-1} \in \nu^2(M) \tag{1.17}$$

$$\pi(\alpha, \beta) = \omega(\alpha^{\sharp_\omega}, \beta^{\sharp_\omega}) \tag{1.18}$$

In particular, the musical isomorphisms of the nondegenerate Poisson bivector are equal to *minus* the musical isomorphisms of the symplectic form [Vai94, p. 6]: $\flat_\pi = -\flat_\omega$, and $\sharp_\pi = -\sharp_\omega$.

1.1.2 Poisson morphisms

It is natural to study the morphisms of the category of Poisson manifolds.

Definition 1.9. Let $F : (M, \pi_M) \to (N, \pi_N)$ be a smooth map between Poisson manifolds. If π_M and π_N are F-related, F is called *Poisson morphism.* As from eq. (A.33) this means that

$$\pi_M(f \circ F, g \circ F) = \pi_N(f, g) \circ F, \quad \forall f, g \in C^\infty(N) \tag{1.19}$$

Proposition 1.10. *Let $F : (M, \pi_M) \to (N, \pi_N)$ be a smooth function between Poisson manifolds. Then F is a Poisson morphism if and only if the pull-back map $F^* : (C^\infty(N), \{\cdot, \cdot\}_N) \to (C^\infty(M), \{\cdot, \cdot\}_M)$ is a Lie algebra homomorphism.*

Proof. This is just about applying definitions:

$$\begin{aligned}
\pi_M(f \circ F, g \circ F) = \pi_N(f, g) \circ F \quad &\text{iff} \\
\{f \circ F, g \circ F\}_M = \{f, g\}_N \circ F \quad &\text{iff} \\
\{F^* f, F^* g\}_M = F^*\{f, g\}_N &
\end{aligned}$$

and we are done. □

Example 1.11. Let $M = \mathbb{R}^2$ and $\pi = \sqrt{x^2 + y^2}\, \partial_x \wedge \partial_y$, i.e.

$$\pi^{ij} = \begin{pmatrix} 0 & \sqrt{x^2 + y^2} \\ -\sqrt{x^2 + y^2} & 0 \end{pmatrix}$$

Since $\det \pi = x^2 + y^2$ the Poisson structure is degenerate only at the origin. Consider a rotation $F(x, y) = (x \cos\theta + y \sin\theta, -x \sin\theta + y \cos\theta)$ with $\theta \in [0, 2\pi)$. Checking whether F is a Poisson (diffeo)morphism boils down to computing π in the rotated coordinates $(\tilde{x}, \tilde{y}) = F(x, y)$ (see eq. (A.33)). Let c, s denote resp. $\cos\theta, \sin\theta$. Then

$$\partial_x = c\tilde{\partial}_x - s\tilde{\partial}_y, \quad \partial_y = s\tilde{\partial}_x + c\tilde{\partial}_y$$

$$\begin{aligned}
\sqrt{x^2 + y^2}\, \partial_x \wedge \partial_y &= \sqrt{\tilde{x}^2 + \tilde{y}^2}(c\tilde{\partial}_x - s\tilde{\partial}_y) \wedge (s\tilde{\partial}_x + c\tilde{\partial}_y) \\
&= \sqrt{\tilde{x}^2 + \tilde{y}^2}(c^2(\tilde{\partial}_x \wedge \tilde{\partial}_y) - s^2(\tilde{\partial}_y \wedge \tilde{\partial}_x)) \\
&= \sqrt{\tilde{x}^2 + \tilde{y}^2}\, \tilde{\partial}_x \wedge \tilde{\partial}_y \quad \square
\end{aligned}$$

We will expand this example in example (1.82) discussing Poisson Lie groups action.

A Poisson morphism needs not be a diffeomorphism:

Example 1.12. Let $M = \mathbb{R}^4$ be the Poisson manifold with coordinates (q^1, p^1, q^2, p^2) and Poisson bivector $\pi_M = \partial_{q^1} \wedge \partial_{p^1} + \partial_{q^2} \wedge \partial_{p^2}$, namely

$$\pi_M^{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Let $N = \mathbb{R}^2$ be the Poisson manifold with coordinates (x, y) and Poisson bivector $\pi_N = \partial_x \wedge \partial_y$, namely

$$\pi_N^{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The map

$$\begin{aligned} F : &\mathbb{R}^4 \to \mathbb{R}^2 \\ &(q^1, p^1, q^2, p^2) \longmapsto (x, y) = (q^1, p^1) \end{aligned}$$

is a Poisson morphism. We need to check condition (A.33), $J\pi_M J^T = \pi_N \circ F$, with J Jacobian of F (since the coefficients are constant no composition is effectively needed):

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \square$$

1.1.3 Poisson submanifolds

Let (M, π) be a Poisson manifold. Given an (immersed or embedded) submanifold $\iota : S \to M$ we can ask whether S can somehow inherit a Poisson structure from M, ι being the inclusion.

Consider a function $\tilde{f} \in C^\infty(S)$ and lift it to a function $f \in C^\infty(M)$ such that $\iota^* f = \tilde{f}$. This operation is not unique: let for example $S = \{xyz = 1\} \subset \mathring{\mathbb{R}}^3_+$ $(x > 0, y > 0, z > 0)$ be the 2-dimensional connected submanifold of $\mathbb{R}^3$ parametrized by $\iota(u, v) = (x, y, z) = (u, v, \frac{1}{uv})$ with $(u, v) \in \mathring{\mathbb{R}}^2_+$. The function $\tilde{f}(u, v) = u + v + \frac{1}{uv}$ can be lifted to $g(x, y, z) = x + y + \frac{1}{xy}$ and to $h(x, y, z) = x + y + z$; clearly $g \neq h$ in general, but $g|_S = h|_S = (x + y + z)\Big|_{z=(xy)^{-1}}$, or equivalently $\iota^* g = \iota^* h = \tilde{f} = u + v + \frac{1}{uv}$.

Consider now two functions $\tilde{f}, \tilde{g} \in C^\infty(S)$ and lift them to some $f, g \in C^\infty(M)$ such that $\iota^* f = \tilde{f}$ and $\iota^* g = \tilde{g}$. Define

$$\{\tilde{f}, \tilde{g}\}_S \coloneqq \iota^* \{f, g\}_M \tag{1.20}$$

The obvious questions is: *when is this well defined?* Namely, when does the right hand side *not* depend on the lifts used?

Let f_1, f_2 be two lifts of f, and g_1, g_2 be two lifts of g, namely $\iota^* f_1 = \iota^* f_2 = \tilde{f}$, and $f_1\Big|_S = f_2\Big|_S$; and similarly for g. Then

$$\begin{aligned} (1.20) \text{ well defined iff } &\iota^*\{f_1, g_1\}_M = \iota^*\{f_2, g_2\}_M \\ \text{iff } &\{f_1, g_1\}_M\Big|_S = \{f_2, g_2\}_M\Big|_S \end{aligned} \tag{$\star$}$$

To exploit the fact that different lifts agree on S the restriction of a bracket should be the same as the bracket of restrictions, namely if

$$\{f, g\}_M\Big|_S = \{f\big|_S, g\big|_S\}_M \quad \forall f, g \in C^\infty(M) \tag{1.21}$$

holds, then $(\star)$ is true and hence (1.20) is well defined. If this is the case, the fact that $\{\cdot, \cdot\}_S$ fulfills the properties defining a Poisson bracket follows from the corresponding properties fulfilled by $\{\cdot, \cdot\}_M$: bilinearity and antisymmetry are clear; for the derivation property note that if $f, g, h \in C^\infty(M)$ lift $\tilde{f}, \tilde{g}, \tilde{h} \in C^\infty(S)$ then

$$\begin{aligned} \{\tilde{f}, \tilde{g}\tilde{h}\}_S &= \iota^*\{f, gh\}_M \\ &= \iota^*\left(\{f, g\}_M h + g\{f, h\}_M\right) = \{\tilde{f}, \tilde{g}\}_S \tilde{h} + \tilde{g}\{\tilde{f}, \tilde{h}\}_S \end{aligned}$$

and similarly for the Jacobi property. $\square$

Definition 1.13. A submanifold $\iota : S \to M$ of a Poisson manifold M is called a *Poisson submanifold* if condition (1.21) holds.

Proposition 1.14. *If $\iota : S \to M$ is a Poisson submanifold of a Poisson manifold M, then eq. (1.20) defines a Poisson bracket on S, such that the inclusion is a Poisson morphism.*

Proof. We just showed that eq. (1.20) is well defined if eq. (1.21) holds, and that in such case it is a Poisson bracket. Moreover, eq. (1.20) says precisely that the inclusion ι is a Poisson morphism. $\square$

Remark 1.15. There are several alternative ways to define Poisson submanifolds; we will mention some of them in proposition (1.31).

1.1.4 Characteristic space

Notation 1.16. Sometimes for clarity a subscript $_\pi$ or $_\omega$ is added to the musical homomorphisms of symplectic forms and Poisson bivectors. When no subscript is present we always refer to π morphisms.

Let (M, π) be an n-dimensional Poisson manifold. Since π is in general degenerate, a flat morphism is *not* defined. In other words, every 1-form $\alpha \in \Omega(M)$ is mapped to a vector field $\alpha^{\sharp_\pi} \in \mathrm{Im}\,(\sharp_\pi) \subset \tau(M)$, but not every vector field belongs to the image of $\sharp_\pi$.

For every $p \in M$ the sharp homomorphism of a Poisson bivector $\sharp : \Omega(M) \to \tau(M)$ induces a homomorphism, again called sharp morphism, between the cotangent and the tangent spaces to M at p

$$\begin{aligned} \sharp_p : T_p^* M &\to T_p M \\ \alpha &\longmapsto \alpha^{\sharp_p} \end{aligned} \tag{1.22}$$

such that $\pi_p(\alpha, \beta) = \alpha^{\sharp_p}(\beta)$ for all the covectors $\alpha, \beta \in T_p^* M$. Clearly

$$\phi^\sharp(p) = (\phi_p)^{\sharp_p} \in T_p M \tag{1.23}$$

where ϕ is a 1-form and ϕ_p its image covector at p.

Differently from the symplectic case, this homomorphism is in general not invertible, and it can have different ranks at different points on the manifold (note that, since $\mathrm{dim} T_p^* M = \mathrm{dim} T_p M$, the homomorphism $\sharp_p$ is injective iff it is surjective iff it is bijective iff its rank is n).

Definition 1.17. Given a Poisson manifold (M, π), its *rank at* $p \in M$, denoted by $\mathrm{rk}_\pi(p)$, is the rank of the homomorphism $\sharp_p$. The Poisson structure is *regular* if its rank is constant for every $p \in M$, and *nondegenerate at* p if its rank at $p \in M$ is maximal. If π is everywhere nondegenerate (M, π) is a *nondegenerate Poisson manifold.*

Remark 1.18. [LM87, p. 112] In coordinates, $(\alpha^{\sharp_p})^i = \pi_p(\alpha, \mathrm{d}x_p^i) = \pi_p^{ji} \alpha_j, \forall \alpha \in T_p^* M$, i.e. the matrix representing the sharp homomorphism is the transpose of the matrix representing π. Since the determinant of an odd antisymmetric matrix vanishes identically, the rank of a Poisson structure is an even integer. In particular, if the Poisson manifold is odd-dimensional, the Poisson structure is degenerate at every point.

The image of a cotangent space $T_p^* M$ via $\sharp_p$ as a subset of the tangent space $T_p M$ deserves a definition.

Definition 1.19. Let (M, π) be a Poisson manifold. The *characteristic space at p* is the image of the sharp morphism of π at p, eq. (1.22).

$$C_p := \operatorname{Im}(\sharp_p) \equiv \sharp_p(T_p^*M) = \{u \in T_pM : \exists \alpha \in T_p^*M : u = \alpha^{\sharp_p}\} \subset T_pM \tag{1.24}$$

If a vector belongs to a characteristic space we shall say that it is a *characteristic vector*. We will show later with eq. (1.33) that characteristic vectors are precisely images of Hamiltonian vector fields.

1.2 Hamiltonian vector fields

Let (M, π) be a Poisson manifold. In analogy to the symplectic case, gradients with respect to π close a Lie subalgebra of the space of vector fields on M, related to $C^\infty(M)$ via a Lie algebra homomorphism.

Definition 1.20. Given a Poisson manifold (M, π), *Hamiltonian vector fields* are Poisson gradients, namely images of exact 1-forms via $\sharp_\pi$. The space of Hamiltonian vector fields is denoted by $\tau_H(M)$. Thus, any $f \in C^\infty(M)$ defines the Hamiltonian vector field

$$X_f = (\mathrm{d}f)^{\sharp_\pi} = \pi(\mathrm{d}f) = \{f, \cdot\} \in \tau_H(M) \subset \operatorname{Im}(\sharp_\pi) \subset \tau(M) \tag{1.25}$$

Remark 1.21. The last equality holds because, per definition, the Poisson bracket is a $C^\infty(M)$-derivation in both its arguments, so fixing one argument one is left with a single $C^\infty(M)$-derivation, which is precisely a vector field. Indeed for a fixed $f \in C^\infty(M)$

$$\{f, g\} = \pi(\mathrm{d}f, \mathrm{d}g) = (\mathrm{d}f)^{\sharp_\pi}(\mathrm{d}g) = (\mathrm{d}f)^{\sharp_\pi}(g), \quad \forall g \in C^\infty(M)$$

Proposition 1.22. [LM87, p. 109] *The space of Hamiltonian vector fields $\tau_H(M)$ of a Poisson manifold (M, π) is a Lie subalgebra of $\tau(M)$ under the Lie bracket, and $f \mapsto X_f$ is a Lie algebra homomorphism from the Poisson algebra $C^\infty(M)$ into $\tau_H(M)$.*

Proof. This time we do not have to prove the Jacobi identity, bur rather to use it (compare with Proposition (1.6)):

$$\begin{aligned}[X_f, X_g]h = X_f X_g h - X_g X_f h &= \{f, \{g, h\}\} + \{g, \{h, f\}\} \\ &= \{\{f, g\}, h\} = X_{\{f,g\}}h, \quad \forall f, g, h \in C^\infty(M)\end{aligned}$$

which shows both that the commutator of Hamiltonian vector fields is Hamiltonian, and that the Hamiltonian vector field of $\{f, g\}$ is indeed $[X_f, X_g]$. □

Remark 1.23. Just like in the symplectic case, antisymmetry grants that smooth functions be constant along integral curves of their Hamiltonian fields: $\mathscr{L}_{X_f} f = X_f f = \{f, f\} = 0$. This is a big difference from the Riemannian symmetric world, where potentials *strictly increase* along the integral curves of their gradients, being Riemannian metrics positive definite.

We can now show the converse of example (1.8):

Proposition 1.24. *A nondegenerate Poisson manifold is a symplectic manifold.*

Proof. Let (M,π) be a nondegenerate Poisson manifold. The nondegeneracy of π allows to define the nondegenerate 2-form $\omega := -\pi^{-1}$, or

$$\omega(X,Y) = \pi(X^{\flat_\pi}, Y^{\flat_\pi}), \quad \forall X, Y \in \tau(M)$$

since $\flat_\pi = (\sharp_\pi)^{-1}$, for Hamiltonian vector fields this means that

$$\omega(X_f, X_g) = \pi(\mathrm{d}f, \mathrm{d}g) = \{f, g\}$$

We have to show that ω is closed. For two arbitrary vector fields $X, Y \in \tau(M)$

$$\begin{aligned} \iota_{[X,Y]}\omega &= \omega(\mathscr{L}_X Y) = \mathscr{L}_X\left(\omega(Y)\right) - \left(\mathscr{L}_X\omega\right)(Y) \\ &= \mathscr{L}_X\left(\iota_Y\omega\right) - \left(\iota_X\,\mathrm{d}\omega\right)(Y) - \left(\mathrm{d}\,\iota_X\omega\right)(Y) \\ &= \left(\mathrm{d}\,\iota_Y\omega\right)(X) - \left(\mathrm{d}\,\iota_X\omega\right)(Y) - \mathrm{d}\left(\omega(X,Y)\right) - \left(\mathrm{d}\omega\right)(X,Y) \end{aligned} \tag{$*$}$$

The Hamiltonian vector field of any $f \in C^\infty(M)$ is $X_f = +(\mathrm{d}f)^{\sharp_\pi} = -(\mathrm{d}f)^{\sharp_\omega}$, so any Hamiltonian vector field fulfills $\iota_{X_f}\omega = -\mathrm{d}f$. In particular this is true for the commutator of two Hamiltonian vector fields, that is Hamiltonian since the Hamiltonian map is a Lie algebra homomorphism: $X_{\{f,g\}} = [X_f, X_g]$ for all $f, g \in C^\infty(M)$, so

$$\iota_{[X_f,X_g]}\omega = -\mathrm{d}\{f,g\} = -\mathrm{d}\left(\omega(X_f, X_g)\right)$$

Comparing this with eq. ($*$) above gives

$$-\mathrm{d}\left(\omega(X_f,X_g)\right) = \iota_{[X_f,X_f]}\omega = \left(\mathrm{d}\,\iota_{X_g}\omega\right)(X_f) - \left(\mathrm{d}\,\iota_{X_f}\omega\right)(X_g) - \mathrm{d}\left(\omega(X_f,X_g)\right) - \left(\mathrm{d}\omega\right)(X_f,X_g)$$

The first two terms of the right hand side both vanish since $\iota_{X_g}\omega$ and $\iota_{X_f}\omega$ are exact, hence closed, 1-forms; the third term is exactly the left hand side, so that the last term must vanish for all $f, g \in C^\infty(M)$ for the equality to hold, which is possible only if $\mathrm{d}\omega = 0$. □

Remark 1.25. Given the nondegeneracy and the antisymmetry, the last ingredient to make ω a symplectic form - closedness - was just shown to be based on the fact that the Hamiltonian map is a Lie algebra homomorphism, which in turn depends on the Jacobi property of the Poisson bracket.

The degeneracy of π allows the existence of degenerate functions in $C^\infty(M)$, in the following sense:

Definition 1.26. Let (M,π) be a Poisson manifold. A function $f \in C^\infty(M)$ is called a *Casimir* if its Hamiltonian vector field is zero.

Remark 1.27. The following are equivalent conditions defining a Casimir function f:

- $X_f = 0$
- $\mathrm{d}f \in \ker\sharp$
- $\{f, \cdot\} = 0$
- $X_g f = 0 = \mathscr{L}_{X_g} f, \quad \forall g \in C^\infty(M)$

Remark 1.28. A constant function is a Casimir function.

Example 1.29. Consider the Poisson bivector[3] in $M = \mathbb{R}^3$ with coordinates $x = (x^1, x^2, x^3)$

$$\pi^{ij} = \begin{bmatrix} 0 & x_1 & x_2 \\ -x_1 & 0 & x_3 \\ -x_2 & -x_3 & 0 \end{bmatrix}$$

The image via the sharp morphism of π of a 1-form α is given by eq. (A.15): $(\alpha^\sharp)^i = \pi(\alpha, \mathrm{d}x^i) = \pi^{ji}\,\alpha_j$, or $\alpha^\sharp = \pi^T\alpha$ in matrix notation.

Consider the function $f(x) = x^1x^2$. Its differential and Hamiltonian vector fields are

$$\mathrm{d}f = x^2\,\mathrm{d}x^1 + x^1\,\mathrm{d}x^2$$
$$X_f = (\mathrm{d}f)^\sharp = -(x^1)^2\,\partial_1 + x^1x^2\,\partial_2 + \left((x^2)^2 + x^1x^3\right)\,\partial_3$$

Example 1.30. Consider the Poisson bivector in $M = \mathbb{R}^3$ with coordinates $x = (x^1, x^2, x^3)$

$$\pi^{ij} = \begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{bmatrix}$$

To build a Casimir we look for the primitive of an exact 1-form in the kernel of the sharp morphism. Such a form is for example $\alpha = x^1\,\mathrm{d}x^1 + x^2\,\mathrm{d}x^2 + x^3\,\mathrm{d}x^3$: $\alpha^\sharp \equiv 0$; it is closed since $\partial_j\,\alpha^i = 0$ for all $i \neq j$, hence exact being $\mathbb{R}^3$ simply connected. A primitive solves the PDE $\partial_i f = x^i$, so $f = \frac{(x^1)^2+(x^2)^2+(x^3)^2}{2} + \text{const}$ is a Casimir for every value of const $\in \mathbb{R}$.

Proposition 1.31 (Poisson submanifolds revisited)**.** *Let (M, π) be a Poisson manifold and $\iota : S \to M$ a submanifold. Condition (1.21) defining Poisson submanifolds is equivalent to the following conditions*

- *π is tangent to S;*
- *every Hamiltonian vector field is tangent to S.*

Proof. The proof can be found in [Mei17, p. 19] □

Example 1.32 (Submersion's level sets)**.** Let S be the level set of a submersion $F : M^n \to \mathbb{R}^{n-d}$, where M^n is an n-dimensional manifold. S is a d-dimensional submanifold of M; if all the components of F are Casimir functions, then S is a Poisson submanifold. Indeed $T_pS = \ker \mathrm{d}_pF$, and the fact that $0 = \{f, F^i\}_M = X_fF^i = \mathrm{d}F^i(X_f)$ for all $f \in C^\infty(M), i = 1, \dots, d$ means that all Hamiltonian vector fields are tangent to S.

A weaker condition is actually enough: the Lie bracket shall vanish only when restricted to S. Furthermore in practice it suffices to check this for a finite number of fundamental brackets: if

$$\{x^i, F^j\}_M\Big|_S = 0, \quad \forall i = 1, \dots, \dim M, \ \forall j = 1, \dots, \dim S \tag{1.26}$$

then

$$0 = X_i(p)\left(F^j(p)\right) = \mathrm{d}_pF^j\left(X_i(p)\right), \quad \forall p \in S$$

where X_i is the Hamiltonian vector field of the coordinate function x^i, and this is enough for S to be a Poisson submanifold.

[3] For routine operations like checking whether a bivector is Poisson, evaluating Poisson brackets or sharp morphisms, etc. we employ the Python module `PoissonGeometry`, a computational toolkit for (local) Poisson-Nijenhuis calculus on manifolds provided in [ERS19].

1.3 Poisson vector fields

Notation 1.33. In the present section no symplectic form is considered, and $\sharp$ is always the sharp morphism $\sharp_\pi$ of a Poisson bivector.

Poisson vector fields play the role of symplectic vector fields for symplectic manifolds and Killing vector fields for Riemannian manifolds, namely their flow preserves the fundamental tensor field of the theory.

Definition 1.34. Let (M, π) be a Poisson manifold. A *Poisson vector field* is a vector field X such that the Lie derivative along it of the Poisson bivector is zero:

$$\mathscr{L}_X \pi = 0 \tag{1.27}$$

The space of Poisson vector fields is denoted by $\tau_P(M) \subset \tau(M)$.

Remark 1.35. [DZ05, p. 11], [LM87, p. 121] Equivalently, a vector field is Poisson if its local flow $\Theta_t(p)$ is a Poisson diffeomorphism whenever defined.

Definition 1.36. Let (M, π) be a Poisson manifold. A *locally Hamiltonian vector field* is a Poisson vector field X that belongs to the image of $\sharp$:

$$\exists \alpha \in \Omega(M) : X = \alpha^\sharp, \quad \text{and} \quad \mathscr{L}_X \pi = 0 \tag{1.28}$$

Definition 1.37. Let (M, π) be a Poisson manifold and $\alpha \in \Omega(M)$ a 1-form. We say that α is *C-closed* if it is closed on Hamiltonian vector fields, namely

$$\mathrm{d}\alpha \left((\mathrm{d}g)^\sharp, (\mathrm{d}h)^\sharp \right) = 0, \quad \forall g, h \in C^\infty(M) \tag{1.29}$$

Clearly every closed 1-form is C-closed. It will be shown in the following that if π is nondegenerate the converse holds too.

Proposition 1.38 (Properties of Poisson vector fields)**.** [LM87, p. 121], [Vai94, p. 107] *Let (M, π) be a Poisson manifold and $X \in \tau(M)$ a vector field. The following properties are equivalent:*

(i) X is a Poisson vector field;

(ii) X is a derivation of the Poisson algebra $C^\infty(M)$ with respect to the Poisson bracket, namely

$$X\{f, g\} = \{Xf, g\} + \{f, Xg\}, \quad \forall f, g \in C^\infty(M); \tag{1.30}$$

(iii) X fulfills

$$[X, X_f] = X_{Xf} \tag{1.31}$$

Remark 1.39. Condition (iii) implies that the commutator of a Poisson vector field and a Hamiltonian vector field is Hamiltonian.

Proof. This is a straightforward computation:

$$\begin{aligned} X\{f, g\} = \mathscr{L}_X (\pi(\mathrm{d}f, \mathrm{d}g)) &= (\mathscr{L}_X \pi)(\mathrm{d}f, \mathrm{d}g) + \pi(\mathscr{L}_X \mathrm{d}f, \mathrm{d}g) + \pi(\mathrm{d}f, \mathscr{L}_X \mathrm{d}g) \\ &= (\mathscr{L}_X \pi)(\mathrm{d}f, \mathrm{d}g) + \pi(\mathrm{d}\mathscr{L}_X f, \mathrm{d}g) + \pi(\mathrm{d}f, \mathrm{d}\mathscr{L}_X g) \\ &= (\mathscr{L}_X \pi)(\mathrm{d}f, \mathrm{d}g) + \{Xf, g\} + \{f, Xg\} \end{aligned}$$

so that condition (ii) holds for all $f, g \in C^\infty(M)$ iff X is a Poisson vector field. In the second line eq. (A.45) was used. For (iii):

$$\begin{aligned}[X, X_f]g &= X\, X_f g - X_f\, Xg = X\{f, g\} - \{f, Xg\} \\ &= (\mathscr{L}_X \pi)(\mathrm{d}f, \mathrm{d}g) + \{Xf, g\} + \{f, Xg\} - \{f, Xg\} \\ &= (\mathscr{L}_X \pi)(\mathrm{d}f, \mathrm{d}g) + X_{Xf} g\end{aligned}$$

so that the last condition is fulfilled iff X is a Poisson vector field. □

Proposition 1.40. *Let (M, π) be a Poisson manifold. Then $\tau_H(M) \subset \tau_P(M)$.*

Remark 1.41. In the symplectic case the analogue result follows immediately from Cartan magic formula and relies on the closedness of the symplectic form, see eq. (1.7). Lacking a "Cartan magic formula for the Lie derivative of bivectors" it is not immediately apparent that $\mathscr{L}_{X_f}\pi$ shall be zero for a Hamiltonian vector field X_f. The conclusion follows indeed from condition (ii) and from the Jacobi identity, thus stressing the analogy between the Jacobi property of a bivector and the closedness of a 2-form mentioned in Remark (1.25).

Proof. A Hamiltonian vector field X_h identically fulfills condition (ii). Indeed for any $f, g, h \in C^\infty(M)$ we have to check whether

$$X_h\{f, g\} = \{h, \{f, g\}\} \underbrace{=}_{?} \{\{h, f\}, g\} + \{f, \{h, g\}\}$$

which is true because of the Jacobi identity. □

1.3.1 Poisson vector fields and closed forms

Notation 1.42. In this section Greek letters $\alpha, \beta, \cdots$ may represent 1-forms or covectors, depending on the context.

Diagram (1.8) shows that the relation between Hamiltonian and symplectic vector fields is precisely the same existing between exact and closed 1-forms. In particular, if the first de Rham cohomology group of a manifold is trivial, these four concepts are equivalent. The degeneracy of a Poisson bivector adds some complexity to these relations, as it is already apparent from the definition of locally Hamiltonian vector fields; in this section we shall investigate more carefully these relations.

Proposition 1.43. *Let (M, π) be a Poisson manifold and $\alpha \in \Omega(M)$ a 1-form. Then $\alpha^\sharp$ is a Locally Hamiltonian vector field if and only if α is C-closed.*

Proof sketch. This is just some gymnastics. Compute $\alpha^\sharp\{g, h\}$, $\{\alpha^\sharp g, h\}$ and $\{g, \alpha^\sharp h\}$ for some $g, h \in C^\infty(M)$ and use condition (ii) in Prop. (1.38) and eq. (A.6). See [LM87, p. 122] for details. □

Every covector is the image of an exact 1-form Given a 1-form $\alpha \in \Omega(M)$, being exact is a rather strong condition: there must exist some $f \in C^\infty(M)$ such that $\mathrm{d}f = \alpha$, so that $\alpha_p = (\mathrm{d}f)_p \in T_p^*M$ **for all** $p \in M$. On the other hand, for any $\alpha \in \Omega(M)$, **fixed** a $p \in M$, there always exists a $f^p \in C^\infty(M)$ such that $\alpha(p) = (\mathrm{d}f^p)\,(p) \in T_p^*M$; in general this holds only at p, i.e. $\alpha(q)$ and $(\mathrm{d}f^p)\,(q)$ are different covectors in T_q^*M.

For given $\alpha \in \Omega(M)$ and $p \in M$ this function is simply given by $f^p(x) = \alpha_i(p)\, x^i$, so that $\mathrm{d}f^p(x)$ is actually the constant covector $\alpha(p)$, for given coordinates x around p.

Example 1.44. In $\mathbb{R}^2$ a form is closed iff it is exact. $\alpha = \frac{y^2}{2}\,\mathrm{d}x + xy\,\mathrm{d}y$ is exact with primitive $f(x,y) = \frac{xy^2}{2} + \text{const}$, so $\mathrm{d}f = \alpha$ everywhere, while the function $f^{(x_0,y_0)}(x,y) = \frac{y_0^2}{2}\,x + x_0 y_0\,y$ fulfills $\mathrm{d}f^{(x_0,y_0)}(x_0,y_0) = \alpha(x_0,y_0)$

Every covector $\alpha \in T_p^*M$ can hence be written as

$$\alpha = (\mathrm{d}f)_p \tag{1.32}$$

for some $f \in C^\infty(M)$. Again, this holds in general only at a specific $p \in M$.

Proposition 1.45. *Characteristic vectors are precisely images of Hamiltonian vector fields, so the characteristic space at any point $p \in M$ is*

$$C_p = \{u \in T_pM : \exists f \in C^\infty(M) : u = X_f(p)\} \tag{1.33}$$

Proof. One direction is obvious: if $u = X_f(p) \in T_pM$ for some Hamiltonian vector field $X_f = (\mathrm{d}f)^\sharp$, then

$$u = (\mathrm{d}f)^\sharp(p) = (\mathrm{d}f_p)^{\sharp_p} \in \operatorname{Im}(\sharp_p)$$

Conversely, if $u = \alpha^{\sharp_p}$ for some covector $\alpha \in T_p^*M$, there exists a function $f \in C^\infty(M)$ such that $\alpha = (\mathrm{d}f)_p$ as in eq. (1.32), so

$$u = (\mathrm{d}f_p)^{\sharp_p} = (\mathrm{d}f)^\sharp(p)$$

i.e. u is the image of the Hamiltonian vector field $(\mathrm{d}f)^\sharp$ at p. The characteristic space is precisely the space of characteristic vectors, so we are done. □

The degeneracy of π is the failure of some vectors to be the image of a Hamiltonian vector field. Indeed if π is nondegenerate its sharp morphism is an isomorphism, the characteristic space is the whole tangent space, Poisson vector fields and locally Hamiltonian vector fields are the same thing, and every vector is the image of some Hamiltonian vector field. In particular we have

Proposition 1.46. *A 1-form on a nondegenerate Poisson manifold is closed if and only if it is C-closed.*

Proof. Let (M,π) be a Poisson manifold. That closure implies C-closure is obvious. Conversely, let α be a C-closed 1-form. If π is nondegenerate every vector $u \in T_pM$ can be written as $u = \left(\mathrm{d}f^\sharp\right)_p$ for some function f, so for all $u, v \in T_pM$ there exists $f, g \in C^\infty(M)$ such that

$$(\mathrm{d}\alpha)_p\,(u,v) = (\mathrm{d}\alpha)_p\left(\left(\mathrm{d}f^\sharp\right)_p, \left(\mathrm{d}g^\sharp\right)_p\right) = \mathrm{d}\alpha(\mathrm{d}f^\sharp, \mathrm{d}g^\sharp)(p) = 0$$

since α is C-closed. □

Remark 1.47. The fact that every vector of a nondegenerate Poisson manifold is the image of a Hamiltonian vector field provides another way to show that the manifold is indeed symplectic, namely that π^{-1} is closed. See [LM87, p. 112].

We have thus the following diagram to recap the relations between Hamiltonian, locally Hamiltonian and Poisson vector fields on one side, and exact, closed and C-closed 1-forms on the other, on a Poisson manifold.

$$
\begin{tikzcd}
\text{Ham. vf.} \arrow[r, Leftrightarrow, "\sharp"] \arrow[dd, Rightarrow] & \text{exact 1-form} \arrow[d, Rightarrow] \\
 & \text{closed 1-form} \arrow[d, Rightarrow, shift right] \\
\text{Loc. Ham. vf.} \arrow[r, Leftrightarrow, "\sharp"] \arrow[d, Rightarrow, shift right] & C\text{-closed 1-form} \arrow[u, Rightarrow, shift right, "*"'] \\
\text{Poisson vf.} \arrow[u, Rightarrow, shift right, "*"'] &
\end{tikzcd}
\tag{1.34}
$$

where the arrows marked with an asterisk hold if the Poisson manifold is nondegenerate.

Example 1.48. Let $M = \mathbb{R}^2$ with coordinates (x, y) be the nondegenerate Poisson manifold with $\pi = \partial_x \wedge \partial_y$. All cohomology groups are trivial, and playing with indices one can check that Poisson vector fields are Hamiltonian. In particular, we show that if X is a Poisson vector field then $\alpha = X^\flat$ is closed.

Start setting $\mathscr{L}_X\pi = 0$:

$$\mathscr{L}_X\left(\pi(\mathrm{d}x^i, \mathrm{d}x^j)\right) = (\mathscr{L}_X\pi)^{ij} + \pi\left(\mathscr{L}_X\mathrm{d}x^i, \mathrm{d}x^j\right) + \pi\left(\mathrm{d}x^i, \mathscr{L}_X\mathrm{d}x^j\right) \tag{$\star$}$$

The Lie derivatives of basis 1-forms follow from Cartan's magic formula

$$\mathscr{L}_X\mathrm{d}x^i = \mathrm{d}X^i = \partial_h X^i\,\mathrm{d}x^h$$

Solve ($\star$) for the first term of the right hand side:

$$0 = (\mathscr{L}_X\pi)^{ij} = X(\pi^{ij}) - \pi^{hk}\,\partial_h X^i\,\delta^j_k - \pi^{hk}\,\delta^i_h\,\partial_k X^j$$

the first term of the right hand side is identically zero because π has constant coefficients, and we are left with

$$\pi^{hj}\,\partial_h X^i + \pi^{ik}\,\partial_k X^j = 0 \tag{$\star\star$}$$

We now show that $\alpha = \alpha_i\,\mathrm{d}x^i = (\pi_{hi}X^h)\,\mathrm{d}x^i$ is closed, where π_{ij} is the matrix representing the $(0, 2)$ tensor field π^{-1}.

$$\mathrm{d}\alpha = \mathrm{d}(\pi_{hi}X^h) \wedge \mathrm{d}x^i = \underbrace{\pi_{hi}\,(\partial_k X^h)}_{=:T_{ik}}\,\mathrm{d}x^k \wedge \mathrm{d}x^i = (T_{ik} - T_{ki})\,\mathrm{d}x^k \otimes \mathrm{d}x^i$$

where the last equality is obtained renaming dummy indices. So α is closed if $T_{ik} = T_{ki}$:

$$\pi_{hi}\,\partial_k X^h = \pi_{hk}\,\partial_i X^h \tag{$\star\star\star$}$$

We just have to show that ($\star\star$) and ($\star\star\star$) are equivalent. Contract the former with $\pi_{ju}\pi_{iv}$:

$$\begin{aligned}
\delta^h_u \pi_{iv}\,\partial_h X^i &= \delta^k_v \pi_{ju}\partial_k X^j \\
\pi_{iv}\,\partial_u X^i &= \pi_{ju}\,\partial_v X^j
\end{aligned}$$

which is precisely ($\star\star\star$), renaming dummy indices. $\square$

1.4 Symplectic foliation

In this section we show that the motion along integral curves of Hamiltonian vector fields on a Poisson manifold M is confined to immersed submanifolds[4] carrying a symplectic structure whose Poisson bracket coincides with that of M. To do this we need the concepts of *generalized distribution* and *generalized foliation*. This section is based on [LM87, pp. 130, 382], [Vai94, p. 19], [DZ05, p. 16].

Quotient away the degeneracy of π Let's start by recognizing the existence of a symplectic structure on characteristic spaces. Let (M, π) be a Poisson manifold: for every $p \in M$ π induces two *nondegenerate* antisymmetric bilinear forms.[5] The first is

$$\begin{aligned} \dot{\pi}_p : {}^{T_p^*M}/_{\ker \sharp_p} \times {}^{T_p^*M}/_{\ker \sharp_p} &\to \mathbb{R} \\ (\dot{\alpha}, \dot{\beta}) &\longmapsto \pi_p(\alpha, \beta) \text{ for any } \alpha \in \dot{\alpha}, \beta \in \dot{\beta} \end{aligned} \tag{1.35}$$

where $\dot{\alpha}$ is an equivalence class for the equivalence relation on T_p^*M $\alpha_1 \sim \alpha_2 \iff \sharp_p(\alpha_1) = \sharp_p(\alpha_2)$, and similarly for β. This is indeed well defined: two covectors in the same class differ for an element in the kernel of $\sharp_p$, so

$$\begin{aligned} \pi_p(\alpha + \ker \sharp_p, \beta + \ker \sharp_p) &= (\alpha + \ker \sharp_p)^{\sharp_p}(\beta + \ker \sharp_p) \\ &= -(\beta + \ker \sharp_p)^{\sharp_p}(\alpha) = \pi_p(\alpha, \beta) \end{aligned}$$

for all $\alpha, \beta \in T_p^*M$. Furthermore, as claimed, the map $\dot{\pi}_p$ is nondegenerate. Its only degenerate element is $\ker \sharp_p = \dot{0}$, which is precisely the zero element of ${}^{T_p^*M}/_{\ker \sharp_p}$:

$$\dot{\pi}_p(\dot{\alpha}, \cdot) \equiv 0 \iff \pi_p(\alpha, \cdot) = 0 \, \forall \alpha \in \dot{\alpha} \iff \dot{\alpha} = \dot{0} \quad \square$$

Similarly, π induces a nondegenerate antisymmetric bilinear form on characteristic spaces:

$$\begin{aligned} \dot{\omega}_p : C_p \times C_p &\to \mathbb{R} \\ (u, v) &\longmapsto \pi_p(\alpha, \beta) \text{ for any } \alpha, \beta \in T_p^*M \text{ such that } \alpha^{\sharp_p} = u, \beta^{\sharp_p} = v \end{aligned} \tag{1.36}$$

Again this is well defined: if $(\alpha')^{\sharp_p} = \alpha^{\sharp_p}$ then $\alpha' - \alpha \in \ker \sharp_p$, and similarly for β, so that $\pi_p(\alpha', \beta') = \pi_p(\alpha, \beta)$. It is also nondegenerate: $\dot{\omega}_p(u, \cdot) \equiv 0 \iff \pi_p(\alpha, \cdot) = 0 \, \forall \alpha \in \sharp_p^{-1}(u) \iff \sharp_p^{-1}(u) = \ker \sharp_p \iff u = 0 \quad \square$

In the next section we will see that the tangent spaces to the symplectic submanifolds mentioned in the introduction are precisely the characteristic spaces endowed with eq. (1.36).

1.4.1 Generalized distributions

Definition 1.49. A *generalized distribution* D on M is a subset of the tangent bundle $D \subset TM$ such that, for every point $p \in M$, the fiber $D_p = D \cap TM$ is a vector subspace of T_pM.

The dimension of D_p, which in general depends on p, is called *rank* of the distribution D at p and is denoted by rk_p. If the rank $\mathrm{rk}_p = k$ is constant everywhere the distribution is said to be *regular of rank k*, or a *k-distribution*, as opposed to *generalized*. When we speak of *distribution* we always refer to a generalized distribution.

[4] An immersed submanifold S of M is the image of an injective immersion $F : S \to M$ endowed with a manifold structure induced by that of M. See [AT11, p. 91] or [Lee12, p. 108].

[5] See [LM87, p. 8] for a deeper treatment of the results in this section.

Definition 1.50. The space of *sections* τ_D of the distribution D is the subset of $\tau(M)$ such that $X_p \in D_p$ for every $p \in M$ where $X \in \tau_D$ is defined.

Definition 1.51. A distribution D is *spanned* by a collection $A \subset \tau(M)$ of vector fields if

$$D_p = \mathrm{Span}\,(X_i(p))\,, \quad \forall X_i \in A \text{ defined at } p \tag{1.37}$$

In this case we write $D = \mathrm{Span}\,(A)$.

A generalized distribution provides a rk_p-dimensional vector subspace of T_pM for every point on a manifold. The question is: *is it possible to find, for every point p on the manifold, a rk_p-dimensional submanifold whose tangent space at p is precisely the given vector subspace?* This is an "integration" problem; to get a feeling of it consider initially distributions of constant rank.

Example 1.52 (Regular distributions and Frobenius theorem)**.** Given a regular distribution, we look for an *immersed* submanifold S of M, everywhere tangent to the fibers of the distribution. In dimension one this is a familiar problem. A natural way to obtain a one dimensional regular distribution is to choose a never vanishing vector field $X \in \tau(M)$: then the fiber $D_p \coloneqq \mathrm{Span}\,(X_p)$ is a one dimensional vector subspace of T_pM, and the union of these fibers is the distribution spanned by X. We need an immersed submanifold $\gamma : \mathbb{R} \supset I \to M$ such that

$$T_{\gamma(t)}\mathrm{Im}\,(\gamma) = \mathrm{Span}\,(\dot\gamma(t)) = \mathrm{Span}\left(X_{\gamma(t)}\right), \quad \forall t \in I \tag{1.38}$$

Each integral curve γ of X is an injective immersion ([Lee12, p. 271]), so the image of every integral curve is an immersed 1-dimensional submanifold clearly fulfilling the condition of eq. (1.38). Finding submanifolds tangent to the fibers of a one dimensional distribution generated by a never vanishing vector field amounts to finding the integral curves of the vector field, hence to solving an ODE system. This is always possible, and we say that we *integrate* the distribution.

We can try to do the same in dimension two. Let X, Y be two independent vector fields so that $D_p = \mathrm{Span}\,(X_p, Y_p)$ is a 2-dimensional subspace of T_pM for every p. We need an immersed submanifold $\psi : \mathbb{R}^2 \supset A \to M$ such that

$$T_{\psi(x)}\mathrm{Im}\,(\psi) = \mathrm{Span}\,(\partial_1\psi(x), \partial_2\psi(x)) = \mathrm{Span}\left(X_{\psi(x)}, Y_{\psi(x)}\right), \quad \forall\, x \in A \tag{1.39}$$

This is a PDE system; *Frobenius theorem*[6] provides a necessary and sufficient condition that X and Y must fulfill for solutions to exist. Rather than discussing it we focus directly on the generalized case, for which we need a few definitions more.

Definition 1.53. A generalized distribution D on a manifold M is *smooth* if, for any point $p \in M$ and for any vector u in the fiber D_p, there exists a section $X \in \tau_D$ defined in an open neighborhood of p *realizing it*, namely such that $X_p = u$.

Remark 1.54. [DZ05, p. 17] If a distribution D is smooth then it is spanned by its sections, namely $D = \mathrm{Span}\,(\tau_D)$ in the sense of definition (1.51). Indeed, for any basis of D_p, every basis vector is realized by a section.

Definition 1.55. A distribution D is *involutive* if τ_D is closed under the Lie bracket, namely $[X, Y] \in \tau_D$ for every pair $X, Y \in \tau_D$, whenever defined.

The following definition gives the higher dimensional analogue of images of vector fields integral curves:

[6] Lee12, p. 496.

Definition 1.56. Consider a manifold M and a distribution D on M. An immersed submanifold $\iota : S \to M$ is an *integral manifold of D* if $T_pS \subset D_p$ for all $p \in S$. Furthermore, the integral manifold S is said

- *of maximal dimension* if $T_pS = D_p$ for all $p \in S$;
- *maximal* if S is not contained in any other integral manifold.

We are almost done with definitions: just two more, providing the property that we would like a distribution to fulfill, and a necessary and sufficient condition for such property to hold.

Definition 1.57. Let D be a distribution on a manifold M. D is *integrable* if for all $p \in M$ there exists an integral manifold S_p of D maximal and of maximal dimension containing p.

The crucial bit in the regular case (Frobenius theorem) is the concept of involutivity: *a regular distribution is integrable if and only if it is involutive.* In the generalized case we need something more:

Definition 1.58. A distribution D on a manifold M is *invariant* under a subset of vector fields $A \subset \tau(M)$ if the push-forward of a fiber is equal to the fiber along the flow, namely for all $X \in A$ (see eq. (A.21) and the discussion around it)

$$\left(\mathrm{d}_p\Theta_t^X\right)(D_p) = D_{\Theta_t^X(p)} \tag{1.40}$$

whenever X and its flow Θ^X are defined.

We are finally in position to give

Theorem 1.59 (Stefan–Sussmann)**.** *Let D be a smooth generalized distribution on a smooth manifold M. Then D is integrable if and only if it is invariant under its sections.*

Proof. The proof can be found in different flavors in [LM87, pp. 389, 390], [Vai94, p. 20], [DZ05, p. 17].

First for the integrable $\Rightarrow$ invariant part. The idea is that orbits of sections are contained in integral manifolds. Consider $p \in M$ and the integral manifold S_p containing it, so that $T_qS_p = D_q$ for all $q \in S_p$. Let $X \in \tau_D$ be a section, so that $X_q \in D_q = T_qS_p$ for all $q \in S_p$ where X is defined. This means that $X|_{S_p}$ is tangent to S_p, and since S_p is maximal it is invariant for the flow: $\Theta_t^X(p) \in S_p$ whenever defined. Since the flow is a diffeomorphism whenever defined, the push-forward of a tangent space is still a tangent space: $\left(\mathrm{d}_p\Theta_t^X\right)(T_pS_p) = T_{\Theta_t^X(p)}S_p$. Being D integrable $T_pS_p = D_p$, so $D_{\Theta_t^X(p)} = T_{\Theta_t^X(p)}S_p = \left(\mathrm{d}_p\Theta_t^X\right)(D^p)$, and we are done.

Conversely, let eq. (1.40) hold for every section $X \in \tau_D$, whenever defined. Since the distribution is smooth it is spanned by its sections: if the rank of the distribution at $p \in M$ is an integer k there exist k sections $X_1, \cdots X_k$ defined in a neighborhood of p such that $\{X_1(p), \cdots, X_k(p)\}$ is a basis for D_p. The idea is to build the integral manifold through p as the image of the composition of the flows of these sections. Define $\chi^p(t_1, \cdots t_k) := \Theta_{t_k}^{(k)} \circ \cdots \circ \Theta_{t_1}^{(1)}(p)$, where $\Theta_{t_i}^{(i)}$ is the flow of the section X_i. The map χ is defined in an open neighborhood of the origin in $\mathbb{R}^k$, and this neighborhood can be restricted to make it injective. Its differential is injective:

$$\begin{aligned}
\mathrm{d}_{T=0}\chi^p(T) : T_0\mathbb{R}^k &\to T_pM \\
\left.\frac{\partial}{\partial_{t_i}}\right|_{T=0} &\longmapsto X_i(p), \quad i = 1, \cdots, k
\end{aligned}$$

where $T = (t_1, \cdots, t_k)$, so χ is an immersion and $S_p := \mathrm{Im}\,(\chi^p)$ is a k-dimensional immersed submanifold of M through p. The tangent spaces to S_p are

$$T_{\chi^p(T)} S_p = (\mathrm{d}_p \chi_T)\,(T_p S_p)$$

and

$$T_p S_p = (\mathrm{d}_{T=0} \chi^p)\,(\mathrm{Span}\,(\partial_{t_1}|_{T=0}, \cdots, \partial_{t_k}|_{T=0})) = \mathrm{Span}\,(X_1(p), \cdots, X_k(p)) = D_p$$

Finally, since eq. (1.40) holds for every section,

$$(\mathrm{d}_p \chi_T)\,(D_p) = D_{\chi^p(T)}$$

Combining the last three expressions we get

$$T_{\chi^p(T)} S_p = D_{\chi^p(T)}$$

whenever defined, and we are done. □

Generalized foliations As a consequence of the previous theorem the integral manifolds of an integrable smooth distribution D on a manifold M are disjoint connected immersed submanifolds of M whose union is M: they form what is called a *generalized foliation*, and each of them is called a *leaf*[7]. The second part of the proof above hints that the leaf through a point $p \in M$ is the orbit of p under the action of the group of local diffeomorphisms obtained composing the flows of the sections of the distribution; this is made rigorous in [LM87, p. 389].

1.4.2 The characteristic distribution of a Poisson manifold

The next step is to show that the characteristic spaces of a Poisson manifold form a smooth integrable distribution spanned by Hamiltonian vector fields[8].

Definition 1.60. The *characteristic distribution* C of a Poisson manifold (M, π) is the disjoint union of the characteristic spaces:

$$C = \sqcup_{p \in M} C_p \tag{1.41}$$

Note that C is indeed a generalized distribution: its fibers are the characteristic spaces, vector subspaces of tangent spaces whose dimension $\mathrm{dim} C_p = \mathrm{rk}_p$ depends on p.

Each characteristic space is spanned by images of Hamiltonian vector fields: this means that, if the rank at $p \in M$ is k, any choice of k independent Hamiltonian vector fields defined in an open neighborhood of p is a local basis for the characteristic space C_p. This means that

Proposition 1.61. *The characteristic distribution C of a Poisson manifold (M, π) is a smooth generalized distribution spanned by Hamiltonian vector fields.*

Proof. Equation (1.33) shows that a vector belongs to a characteristic space if and only if it is the image of a Hamiltonian vector field. This means that the characteristic distribution is smooth, and a vector field is a section if and only if it is Hamiltonian. □

We are ready to enunciate the main result of this section:

[7] LM87, p. 385.
[8] OR04, p. 130.

Theorem 1.62 (Symplectic Foliation)**.** *Let (M, π) be a Poisson manifold and C the associated characteristic distribution. C is integrable, and its leaves S are nondegenerate Poisson submanifolds of (M, π), namely symplectic manifolds with the unique symplectic structure making the inclusion $\iota : S \to M$ a Poisson morphism.*

Proof sketch. If the rank of π is constant involutivity is enough to have integrability (Frobenius theorem). Recall from Proposition (1.22) that the Hamiltonian map is a Lie algebra homomorphism: $X_{\{f,g\}} = [X_f, X_g]$ for all $f, g \in C^\infty(M)$. This means that the characteristic distribution is involutive, hence integrable.

If the rank of π varies over M we need Stefan-Sussmann theorem, i.e. we have to check that C is invariant under Hamiltonian vector fields. Loosely speaking, since Hamiltonian vector fields are Poisson they preserve the Poisson structure (in the sense that $\mathscr{L}_{X_f}\pi = 0$), so they also preserve the characteristic distribution. See [OR04, p. 131] for a rigorous proof of this fact and of this theorem in general.

Since the rank of a Poisson bivector is even the integral manifolds of the characteristic distribution are even dimensional (they have the same dimension of their tangent spaces, the characteristic spaces). Furthermore, since all Hamiltonian vector fields are tangent to integral manifolds, these are indeed Poisson submanifolds (proposition (1.31)) with Poisson bracket given by eq. (1.20).

The last bit is nondegeneracy. The Poisson bivector of an integral manifold is the Poisson bivector of M restricted to the integral manifold, so for all $p \in M$ the characteristic space of S_p at a point $q \in S_p$ is the whole tangent space to S_p at q: $C_q(S_p) = C_q(M) = T_q S_q = T_q S_p$, for all $q \in S_p$. Hence the Poisson structure of S_p is nondegenerate, and it induces on the leaf a symplectic form that at every point coincides with the symplectic form on the corresponding characteristic space given by eq. (1.36). □

Remark 1.63. As mentioned at the beginning of this section the Symplectic Foliation theorem implies that two points p, q of a Poisson manifold M belong to the same symplectic leaf if and only if they can be connected by a piecewise-smooth curve consisting of integral curves of Hamiltonian vector fields[9].

Example 1.64. Consider the degenerate rank-2 Poisson manifold $\pi = \partial_x \wedge \partial_y - \partial_x \wedge \partial_z + \partial_y \wedge \partial_z$ on $M = \mathbb{R}^3$, namely

$$\pi^{ij} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Two independent Hamiltonian vector fields span the characteristic space and generate the symplectic leaf at every point. Let X be the Hamiltonian vector field generated by $f(x, y, z) = x$ and Y the Hamiltonian vector field generated by y:

$$X = (\mathrm{d}x)^\sharp = \pi^T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \partial_y - \partial_z, \quad Y = (\mathrm{d}y)^\sharp = \pi^T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -\partial_x + \partial_z$$

Characteristic spaces and induced symplectic form The characteristic space at every point is the 2-dimensional vector subspace of $T_p\mathbb{R}^3 \cong \mathbb{R}^3$ spanned by X and Y: $C_p = \mathrm{Span}\,(X, Y) : x + y + z = 0$. As a vector subspace of $\mathbb{R}^3$ C_p inherits the nondegenerate Poisson

[9] DZ05, p. 20.

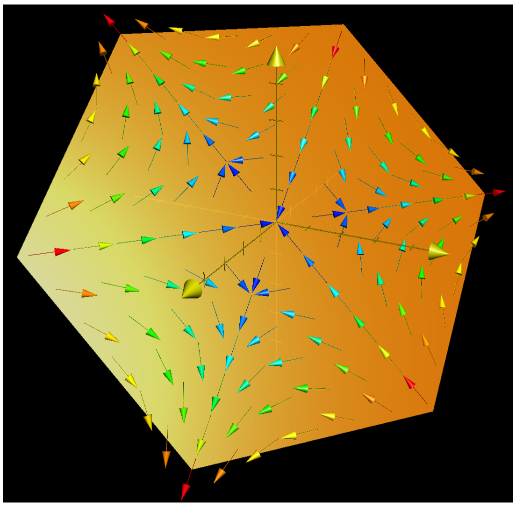

Figure 1.1: Symplectic leaf through the origin $x+y+z=0$, with quiver plot of the tangent Hamiltonian vector field (1.42). The colors of the arrows represent the norm of the corresponding vector, cold colors meaning smaller norm (the origin is a fixpoint).

structure eq. (1.36). The kernel of $\sharp_p$ is the 1-dimensional vector subspace of $T_p^*\mathbb{R}^3 \cong \mathbb{R}^3$ (with euclidean metric):

$$\pi^T\alpha = 0 \Rightarrow \ker \sharp_p = \mathrm{Span}\left((1,1,1)\right) : x = y = z$$

Take for example $\alpha = \mathrm{d}x + 2\mathrm{d}y = (1,2,0)$ and $\alpha' = \mathrm{d}y - \mathrm{d}z = (0,1,-1)$. Their difference $\alpha - \alpha' = (1,1,1)$ is a kernel element, and one immediately checks that

$$\pi^T \cdot \alpha = \pi^T \cdot \alpha' = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

Thus $u := \alpha^\sharp = (\alpha')^\sharp = -2\,\partial_x + \partial_y + \partial_z = (-2,1,1)$ is a characteristic vector; indeed $-2+1+1 = 0$. Its components (t,s) in the X, Y basis are given by

$$\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = tX + sY = t\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

namely $u = X + 2Y$. Choose another covector - say, $\beta = \mathrm{d}x + \mathrm{d}z$: with the same procedure, $v := \beta^\sharp = -Y = (1,0,-1)$ is another characteristic vector; indeed $1 - 1 = 0$. The induced symplectic form (1.36) $\dot{\omega}$ acting on u, v is

$$\dot{\omega}(u,v) = \pi(\alpha,\beta) = \pi(\alpha',\beta) = u \cdot \beta = (-2,1,1)\cdot(1,0,1) = -1 \tag{$\star$}$$

with the euclidean metric.

Symplectic leaves The flows of X and Y are respectively

$$\Theta_t(x,y,z) = (x, y+t, z-t), \quad \theta_s(x,y,z) = (x-s, y, z+s)$$

The symplectic leaf through $p = (a,b,c)$ is parametrized by

$$(x,y,z) = \psi_{(a,b,c)}(t,s) = \Theta_t \circ \theta_s(a,b,c) = (a-s,\, b+t,\, c+s-t)$$

The symplectic leaf is the image of $\psi_{(a,b,c)}$, the (affine) plane $S : z = a + b + c - (x + y)$, that can be written as the zero level set of $F(x, y, z) = x + y + z - (a + b + c)$, which is a smooth submersion. This plane is indeed a Poisson submanifold since F is a Casimir, as discussed in the example (1.32):

$$\begin{aligned}\{x^i, F\} &= \pi(\mathrm{d}x^i, \mathrm{d}F) = \pi^{ij}\,\partial_j F = (\pi \cdot \mathrm{d}F)^i \\ &= \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0\end{aligned}$$

The induced nondegenerate Poisson structure on the plane is such that $\psi_{(a,b,c)}$ is a Poisson morphism, namely we look for the 2×2 Poisson bivector $\psi_{(a,b,c)}$-related to π:

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = J \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} J^T$$

where J is the Jacobian of $\psi_{(a,b,c)}$ and A is to be determined. Replacing

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$$

in the above expression leads to $A = 1$, so the nondegenerate Poisson bivector induced on the plane is $\pi_S = \partial_t \wedge \partial_s$, corresponding to the canonical symplectic form $\omega_S = \mathrm{d}t \wedge \mathrm{d}s$ (since in general $\omega_S = -\pi_S^{-1}$ field-wise, and in this canonical case with the symplectic unit matrix $\pi_S^{-1} = -\pi_S$ matrix-wise, π_S and ω_S are represented by the same matrix).

We can now check that $\omega_S(u, v)$ agrees with ($\star$): in the $\{X, Y\}$ basis $u = X + 2Y$ and $v = -Y$, so

$$\omega_S(u, v) = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -1 \quad \square$$

Finally, consider for example the function $f(x, y, z) = xyz$. Its Hamiltonian vector field is

$$X_f = x(y - z)\,\partial_x + y(z - x)\,\partial_y + z(x - y)\,\partial_z \tag{1.42}$$

It is indeed tangent to symplectic leaves, since its components add up to zero. Fig. (1.1) shows the symplectic leaf through the origin $x + y + z = 0$ and a quiver plot of this Hamiltonian vector field restricted to the leaf.

1.5 Poisson reduction

The concept of Poisson submanifold is the answer to the question "when does a regular *injection into* a Poisson manifold define a new Poisson manifold?" The idea of *Poisson reduction* is to study the spaces arising as *quotients*, namely images of some *surjective* maps defined *on* Poisson manifolds.

If X is a topological space, Y a set and $f : X \to Y$ a surjective map, Y can be endowed with a topology, called *quotient* topology, such that

$$U \subset Y \text{ is open} \iff f^{-1}(U) \subset X \text{ is open} \tag{$\star$}$$

In particular, $\Rightarrow$ is equivalent to f being continuous. A surjective map between topological spaces fulfilling condition ($\star$) is called *quotient map*.

A quotient map $f : X \to Y$ induces on its domain X the equivalence relation $x_1 \sim x_2 \iff f(x_1) = f(x_2)$. The canonical projection $\pi : X \to {}^X\!/_\sim$ is surjective, hence a quotient map endowing the quotient space ${}^X\!/_\sim$ with the quotient topology. The quotient space is homeomorphic to Y, the identification being the obvious one between level sets of f in X and their images in Y[10].

Let $f : M \to N$ be a surjective smooth map between smooth manifolds. When is f a quotient map? Smoothness implies continuity, so what f needs is the $\Leftarrow$ property in condition ($\star$). The next result, proved in [Lee12, p. 89], provides a sufficient condition for this:

Proposition 1.65. *A smooth submersion between smooth manifolds is open.*

Openness implies the $\Leftarrow$ property in condition ($\star$), so a smooth surjective submersion between smooth manifolds is a quotient map.

Conversely, let $f : M \to X$ be a quotient map between a smooth manifold and a topological space. An important question is whether X, the quotient of M (namely its image via a quotient map), is a smooth manifolds itself, or at least preserves some smoothness property. One is in particular interested in quotient maps arising from smooth Lie groups actions on manifolds.

1.5.1 Lie groups actions

Notation 1.66. Let G be a Lie group. We will denote by $\mathrm{Lie}(G) \in \tau(G)$ its Lie algebra, namely the set of left-invariant vector fields; and by $\mathfrak{g} = T_eG$ the tangent space to G at the identity element e, isomorphic to $\mathrm{Lie}(G)$. The exponential map is

$$\begin{aligned} \exp : \mathrm{Lie}(G) &\to G \\ X &\longmapsto \exp X = \gamma(1) \end{aligned} \tag{1.43}$$

where γ is the integral curve of X through e.

Definition 1.67. Let G be a Lie group and M a smooth manifold. A *smooth right action* of G on M is a smooth map

$$\begin{aligned} \psi : M \times G &\to M \\ (p, g) &\longmapsto \psi(p, g) \equiv p \cdot g \end{aligned} \tag{1.44}$$

such that $p \cdot \mathrm{e} = p$ and $p \cdot (gh) = (p \cdot g) \cdot h$ for all p in M, g, h in G. Furthermore:

- $\psi^p : G \to M$ is the *orbit map* $\psi^p(g) = p \cdot g$
- $\psi_g : M \to M$ is the diffeomorphism $\psi_g(p) = p \cdot g$ with inverse ψ_{-g}

All actions considered in the following are smooth, even when the adjective is omitted.

Let G be a Lie group acting from the right on a smooth manifold M. A Lie algebra vector field $X \in \mathrm{Lie}(G)$ defines a vector field $\hat{X} \in \tau(M)$, called *infinitesimal generator*, whose flow is the action of the group elements along the integral curve of X through e. Let g_t be such a group

[10] These facts can be found in any book about General Topology, e.g. [Eng89, p. 90]. See also the first chapters of [Kos80] for a concise treatment and the notes [Hat05] for a good combination of motivation, intuitive explanations, and rigorous details.

element, at parameter distance t along the integral curve of X through e, or $g_t = \exp tX$, for some t in the vicinity of zero in $\mathbb{R}$. This defines the flow on M

$$\Theta(t,p) = p \cdot g_t = p \cdot \exp(tX) \tag{1.45}$$

realized by the vector field on M

$$\hat{X}(p) = \frac{\mathrm{d}}{\mathrm{d}t}\, p \cdot \exp(tX)\Big|_{t=0} \tag{1.46}$$

Proposition 1.68. *Let G be a Lie group acting from the right on a smooth manifold M. The map*

$$\begin{aligned} \hat{}: \mathrm{Lie}(G) &\to \tau(M) \\ X &\longmapsto \hat{X} \end{aligned} \tag{1.47}$$

defined by eq. (1.46) is a Lie algebra homomorphism with respect to the Lie bracket, namely

$$\widehat{[X,Y]_G} = [\hat{X},\hat{Y}]_M \tag{1.48}$$

Proof sketch. The proof, that can be found in [Lee12, p. 526], relies on three ingredients:

1. Right group action and left translation L_g give $\psi^p \circ L_g = \psi^{p \cdot g}$, with $p \in M, g \in G$;
2. Left invariance of a Lie algebra element X gives $(\mathrm{d}_e L_g)(X_e) = X_g$;
3. the infinitesimal generator can be written as $\hat{X}(p) = (\mathrm{d}_e \psi^p)(X_e) \in T_p M$.

With these three, it is shown that $\hat{X}$ and X are ψ^p-related vector fields, and the conclusion follows from the naturality of the Lie bracket [Lee12, p. 188]. □

Remark 1.69. Right group action was chosen to have a Lie algebra homomorphism; left action gives a Lie algebra anti-homomorphism [Lee12, p. 529].

Definition 1.70. Let G be a Lie group acting from the right on a smooth manifold M.

- The *orbit* of $p \in M$ is the subset of M reachable from p via the action of G

$$p \cdot G = \mathrm{Im}\,(\psi^p) = \{p \cdot g : g \in G\} \subset M$$

- The *isotropy group* or *stabilizer* of $p \in M$ is the subset of group elements fixing p:

$$G_p = \{g \in G : p \cdot g = p\}$$

  and it clearly is a subgroup of G.

The stabilizers of points in the same orbit are related by a simple expression. Take $p \in M$ and $g \in G$: since G_p fixes p, we have $p \cdot g = p \cdot g g^{-1} G_p g$, so the stabilizer of $p \cdot g$ is

$$G_{p \cdot g} = g^{-1} G_p g \tag{1.49}$$

1.5.2 Quotient spaces by group action

These definitions allow to formalize the idea of a quotient map arising from a smooth Lie group action mentioned in the introduction[11]. Define an equivalence relation on M by $p \sim q \iff p \cdot G = q \cdot G$, with $p, q \in M$. Equivalently, $p \sim q \iff \exists g \in G : q = p \cdot g$. The equivalence classes for this relation are precisely the orbits of G in M. The set of orbits ${}^{M}/_{G}$ with the quotient topology induced by the canonical projection $\pi : M \to {}^{M}/_{G}$ is called *quotient space* or *orbit space*. The canonical projection is per definition a quotient map; [Kos80, p. 38] or [Lee12, p. 541] show that it is also open, if the action of G is continuous.

The properties of the quotient space crucially depend on the properties of the group action:

Definition 1.71. Let G be a Lie group acting from the right on a smooth manifold M. The action of G is called

- *proper* if the map $M \times G \to M \times M$, $(p, g) \mapsto (p \cdot g, p)$ is proper[12];
- *free* if all isotropy groups are trivial: $G_p = \{e\}$ for all $p \in M$, meaning that every point is fixed only by the identity.
- *simple* if the orbit space ${}^{M}/_{G}$ has a smooth manifold structure such that the canonical projection $\pi : M \to {}^{M}/_{G}$ is a submersion.

A crucial result proved in [Lee12, p. 544] is that a smooth free proper action is simple:

Theorem 1.72 (Quotient Manifold Theorem)**.** *If a Lie group G acts smoothly, freely and properly on a smooth manifold M then the orbit space ${}^{M}/_{G}$ is a smooth manifold of dimension $dimM - dimG$ with unique smooth structure such that the canonical projection $\pi : M \to {}^{M}/_{G}$ is a smooth submersion.*

The group action that we will use to derive the Poisson structure of a simplex is proper but not free, as discussed in section (2.2). In this case the quotient space is not a manifold itself, but only a "collection of manifolds fitting together nicely"; to make this rigorous we need some definitions.

Stratified spaces

Definition 1.73 (Stratified space)**.** Let X be a topological space, and $\mathcal{S} = \{S_i\}_{i \in I}$ a locally finite partition of X such that

- the pieces of $\mathcal{S}$ are locally closed[13] smooth manifolds $S_i \subset X$, called *strata*;
- the strata fulfill the *frontier condition*: if a stratum meets the closure of another, the first stratum is contained in the closure of second. $S_i \cap \bar{S}_j \neq \emptyset \Rightarrow S_i \subset \bar{S}_j$.

The pair $(X, \mathcal{S})$ is called *stratified space*[14], or *stratification* of X.

[11] [Lee12, p. 540], [Kos80, p. 36]

[12] A map between topological spaces is *proper* if the preimage of compact sets is compact, [Lee12, p. 610]. All the actions we will encounter are proper, and every continuous action by a compact Lie group on a manifold is proper, [Lee12, p. 544]. Properness is a very powerful assumption for a group action because it guarantees that some of the technically important properties of compact group actions are still valid, see [OR04, p. 59].

[13] Local closure is what makes an immersed submanifold of a manifold an embedded submanifold, [OR04, p. 31], [Lee12, p. 85]

[14] This is actually what is called a *decomposed space*, [OR04, p. 31]. The definition of stratified space requires some more technicalities, but the decomposition of a space defines a stratification, and vice versa.

Example 1.74. It will be shown in the next chapter that the standard simplex defined in section (2.1) is a stratified space with the faces as strata. Intuitively, the simplex is not a smooth manifold (it has corners, edges and so on), but the faces, that do not include the boundaries, are submanifolds of the hosting Euclidean space. Faces also fulfill the frontier condition: the intersection of the closure of a face with a smaller face can be non-empty, and in such case the smaller face is contained in the closure of the bigger face (think of the 2-face and an edge of a 2-simplex).

Remark 1.75. Analogously to what is done with smooth manifolds, *stratified charts* and *charts compatibility* can be defined for stratified spaces. Loosely speaking, the intersection of each stratum with a small enough open subset of the stratified space has to be diffeomorphic to a submanifold of an Euclidean space; clearly the dimension of this submanifold is not the same for different strata. This leads to the definition of a *smooth stratified space*; similarly, the algebra of smooth functions $C^\infty(X)$ is defined for a smooth stratified space $(X, \mathcal{S})$. See [OR04, p. 32] for details.

Type submanifolds The next ingredient we need is the characterization of the strata of a stratified space arising as a quotient space. This section is based on [OR04, p. 75].

Definition 1.76. Let G be a Lie group acting from the right on a manifold M, and H a subgroup of G.

- The *conjugacy class* of H is the set of subgroups of G defined by

$$(H) := \{L \subset G : L = g^{-1}Hg, g \in G\} \tag{1.50}$$

- the *H-isotropy type submanifold* M_H is set of points of M whose stabilizer is H:

$$M_H := \{p \in M : G_p = H\} \subset M \tag{1.51}$$

- the *H-orbit type submanifold* $M_{(H)}$ is the set of points of M whose stabilizer belongs to (H):

$$M_{(H)} = \{p \in M : G_p \in (H)\} \subset M \tag{1.52}$$

Proposition 1.77. *Let G be a Lie group acting from the right on a manifold M, and H a subgroup of G. The H-orbit type submanifold is the orbit of the H-isotropy type submanifold:*

$$M_{(H)} = M_H \cdot G \tag{1.53}$$

Proof. Start with $M_{(H)} \subset M_H \cdot G$. Consider $p \in M_{(H)}$: per definition, $G_p = g^{-1}Hg$ for some $g \in G$. Recalling eq. (1.49), $G_{p\cdot g^{-1}} = gG_pg^{-1} = gg^{-1}Hgg^{-1} = H$, so $p \cdot g^{-1} \in M_H$, and since $p = (p \cdot g^{-1}) \cdot g$ we conclude that $p \in M_H \cdot G$.

Conversely, since H is its own conjugacy orbit via the identity element, $H \in (H)$ and $M_H \subset M_{(H)}$. Furthermore, $M_{(H)}$ is G-invariant: if $p \in M_{(H)}$ then $G_p = g^{-1}Hg$ for some $g \in G$. For any $h \in G$, $G_{p\cdot h} = h^{-1}G_ph = h^{-1}g^{-1}Hgh = (gh)^{-1}H(gh)$. Since $gh \in G$, $p \cdot h \in M_{(H)}$, so $M_{(H)} \cdot G = M_{(H)}$.

These two facts mean that if $p \in M_H \cdot G$ then $p \in M_{(H)} \cdot G = M_{(H)}$, i.e. $M_H \cdot G \subset M_{(H)}$ $\square$

With these definitions at hand we can finally state the Stratification Theorem, discussed in detail in [OR04, p. 84]:

Theorem 1.78 (Stratification Theorem)**.** *If a Lie group G acts smoothly and properly on a smooth manifold M then the orbit space ${}^{M}/_{G}$ is a smooth stratified space whose strata are the connected components of ${}^{M_{(H)}}/_{G}$, the reduced orbit type submanifolds, as H ranges over all possible subgroups of G, namely*

$$ {}^{M}/_{G} = \bigsqcup_{(H)} \textit{conn. comp. of } {}^{M_{(H)}}/_{G} \tag{1.54}$$

is a smooth stratification.

Remark 1.79. It is interesting to see intuitively how adding freeness to the hypothesis of the previous theorem turns a stratified quotient space into a smooth manifold, as expected from the Quotient Manifold Theorem. Indeed if the action is free all stabilizers are trivial, so that $M_H = M$ for $H = \{e\}$, and $M_H = \emptyset$ for all other subgroups $H \subset G$. Thus the only nontrivial orbit type submanifold is $M_{(\{e\})} = M_{\{e\}} \cdot G = M \cdot G = M$, and ${}^{M}/_{G}$ is a stratified space with only one strata ${}^{M_{(\{e\})}}/_{G}$, namely itself.

Orbit space smooth functions Let ψ be a smooth proper action of a Lie group G on a manifold M. A function $f \in C^\infty(M)$ is G-*invariant* if it is constant on the orbits of G in M; the space of G-invariant functions is denoted by $C^\infty(M)^G$.

The algebra of smooth functions of the smooth stratified orbit space, whose existence was discussed in remark (1.75), can be identified with the smooth G-invariant functions via the pullback of the canonical projection $\pi : M \to {}^{M}/_{G}$, as in [ORF09, p. 1271]:

$$C^\infty({}^{M}/_{G}) = \{f \in C^0({}^{M}/_{G}) : \pi^* f \in C^\infty(M)^G\} \cong C^\infty(M)^G \tag{1.55}$$

Indeed the pull-back of $f \in C^0({}^{M}/_{G})$ is G-invariant: $\pi^* f \circ \psi_g(p) = f \circ \pi \circ \psi_g(p) = f \circ \pi(p) = \pi^* f(p)$ for all $p \in M, g \in G$. Conversely, a G-invariant function $g \in C^\infty(M)^G$ defines a unique function $\tilde{g} \in C^0({}^{M}/_{G})$ by $\tilde{g}(p \cdot G) = g(q)$ for any $q \in p \cdot G$; clearly $\pi^* \tilde{g} = g$.

1.5.3 Poisson actions

The Quotient Manifold Theorem and the Stratification Theorem describe the smooth structure of quotients spaces. Another important question is whether quotient spaces of Poisson manifolds inherit a Poisson structure. Unsurprisingly this is true if the group action has an additional property.

Definition 1.80. Let ψ be the action of a Lie group G on a Poisson manifold (M, π). The action is called *Poisson* or *canonical* if G acts by Poisson morphisms, i.e. for all $g \in G$ and for all $f, g \in C^\infty(M)$

$$\psi_g^*\{f, g\} = \{\psi_g^* f, \psi_g^* g\} \tag{1.56}$$

Remark 1.81. [LM87, p. 187] If the action ψ of a Lie group G on a Poisson manifold M is Poisson then the vector field $\hat{X} \in \tau(M)$ associated to any vector field $X \in \mathrm{Lie}(G)$ as in eq. (1.46) is a Poisson vector field, namely $\mathscr{L}_{\hat{X}} \pi = 0$. Conversely, if $\hat{X}$ is Poisson for every $X \in \mathrm{Lie}(G)$ and G is connected, then the action of G on M is Poisson.

Example 1.82. The transformation discussed in example (1.11) is the action of the matrix Lie group $SO(2)$ on $\mathbb{R}^2$. The infinitesimal generator is

$$\hat{X}(x,y) = \left.\frac{\mathrm{d}}{\mathrm{d}\theta}\right|_{\theta=0}(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta) = y\,\partial_x - x\,\partial_y$$

We verified already that the action is Poisson, so $\hat{X}$ should be a Poisson vector field. It is indeed even Hamiltonian, of Hamiltonian function $f = -\sqrt{x^2+y^2}$.

Quotient Poisson structure If the action of a Lie group G on a Poisson manifold M is canonical the space of G-invariant functions $C^\infty(M)^G$ is a Poisson subalgebra of $C^\infty(M)$. Consider indeed $f, g \in C^\infty(M)^G$: their bracket fulfills

$$\psi_g^*\{f,g\} = \{\psi_g^* f, \psi_g^* g\} = \{f,g\} \tag{$\star$}$$

so that $\{f,g\} \in C^\infty(M)^G$.

Since eq. (1.55) says that $C^\infty(M)^G \cong C^\infty({}^M/_G)$ via $\pi^*\hat{f} = f$ for a smooth, proper and canonical action, $(\star)$ means that the algebra of smooth functions of the quotient space inherits a natural Poisson algebra structure, defined by

$$\{\hat{f},\hat{g}\}_{M/G} \coloneqq \widehat{\{f,g\}_M}, \qquad \text{or} \qquad \pi^*\{\hat{f},\hat{g}\}_{M/G} = \{\pi^*\hat{f}, \pi^*\hat{g}\}_M \tag{1.57}$$

From the second expression it is apparent that the canonical projection is a Poisson morphism for these structures. Note that the above expression defines a Poisson bracket on $C^\infty({}^M/_G)$ whether the quotient space is a manifold or not. This idea is made precise in the following theorem, which says actually much more than what enunciated here; see [OR04, p. 364].

Theorem 1.83 (Regular Poisson Reduction)**.** *Let $(M, \{\cdot,\cdot\}_M)$ be a Poisson manifold, and let G be a Lie group acting smoothly, properly, freely and canonically on it via the map $\psi : M \times G \to M$. $\pi : M \to {}^M/_G$ denotes the canonical projection. Then*

- *the orbit space ${}^M/_G$ is a Poisson manifold with the Poisson bracket $\{\cdot,\cdot\}_{M/G}$ uniquely characterized by*

$$\{f,g\}_{M/G}(\pi(p)) \coloneqq \{\pi^* f, \pi^* g\}_M(p) \tag{1.58}$$

  for all $p \in M$, $f, g \in C^\infty({}^M/_G)$.

- *The above Poisson structure is the only one for which the canonical projection is a Poisson morphism.*

Proof. The Quotient Manifold Theorem assures that the quotient space is a smooth manifold. For the Poisson structure we need to check that eq. (1.58) is well defined, namely that it agrees on points in the same orbit. Let $p = \psi_g(q)$ for some $q \in M, g \in G$; then

$$\begin{aligned}
\{f,g\}_{M/G}(\pi(p)) &= \{\pi^* f, \pi^* g\}_M(p) \\
&= \{\pi^* f, \pi^* g\}_M(\psi_g(q)) \\
&= \{f \circ \pi \circ \psi_g, g \circ \pi \circ \psi_g\}_M(q) \\
&= \{f \circ \pi, g \circ \pi\}_M(q) \\
&= \{f,g\}_{M/G}(\pi(q))
\end{aligned}$$

where in the third line we used the fact that the action is Poisson, and in the fourth that $\pi \circ \psi_g = \pi$ per definition of projection.

Similarly to what happened with eq. (1.21), the properties defining a Poisson bracket are inherited from the Poisson character of the bracket $\{\cdot,\cdot\}_M$. Bilinearity and antisymmetry are clear; for Leibniz

$$\begin{aligned}\{f, gh\}_{M/G}(\pi(p)) &= \{f\circ\pi, (gh)\circ\pi\}_M(p)\\ &= \{f\circ\pi, g\circ\pi\}_M(p)\,(h\circ\pi)(p) + (g\circ\pi)(p)\{f\circ\pi, h\circ\pi\}_M(p)\\ &= \left(\{f,g\}_{M/G}h + g\{f,h\}_{M/G}\right)(\pi(p))\end{aligned}$$

and similarly for Jacobi.

Let $(\cdot,\cdot)_{M/G}$ be another Poisson bracket on the quotient space fulfilling eq. (1.58): since π is surjective, $(\cdot,\cdot)_{M/G}$ and $\{\cdot,\cdot\}_{M/G}$ agree on every point of the quotient space, namely they are the same.

Finally, having established that $\{\cdot,\cdot\}_{M/G}$ is a Poisson structure, eq. (1.58) says that π is a Poisson morphism; if there was another Poisson structure making π a Poisson morphism, this structure would fulfill eq. (1.58) as well; again, since π is surjective, it would coincide with $\{\cdot,\cdot\}_{M/G}$. $\square$

The last step is the analogue of the Stratification Theorem: what happens to the quotient of a Poisson manifold via a smooth, proper, canonical, *non* free action? The idea is the following:

- Even if a smooth stratified space is not a manifold, eq. (1.57) defines a Poisson algebra on it;
- the strata inherit a Poisson structure such that the inclusion is a Poisson morphism.

Let's make this precise:

Definition 1.84 (Poisson Stratified space)**.** Let X be a topological space. A *Poisson stratification* of X is a smooth stratification $(X,\mathcal{S})$ of X together with a Poisson algebra $(C^\infty(X), \{\cdot,\cdot\}_X)$, where $C^\infty(X)\subset C^0(X)$ is the space of smooth functions associated with S introduced in Remark (1.75), such that each stratum S_i is a Poisson manifold and the inclusion $\iota: S_i\to X$ is a Poisson morphism. $(X,\mathcal{S},\{\cdot,\cdot\}_X)$ is called *Poisson stratified space.*

Theorem 1.85 (Singular Poisson Reduction)**.** *Let G be a Lie group acting smoothly, properly and canonically on a Poisson manifold M. Then $({}^M/_G,\ \mathcal{S},\ \{\cdot,\cdot\}_{M/G})$ is a Poisson stratified space with the unique Poisson bracket such that the canonical projection is a Poisson morphism, whose strata are the connected components of ${}^{M_{(H)}}/_G$, the reduced orbit type submanifolds.*

Proof sketch. The Stratification Theorem assures that the quotient space is a smooth stratified space; its Poisson algebra is given by eq. (1.57). To see that the inclusions of the strata into the quotient space are Poisson morphisms we refer to [ORF09, p. 1273]. $\square$

In the next chapter we present an example of the results of the previous sections: via a double reduction, the Regular and Singular Poisson Reduction Theorems show that the standard simplex is a Poisson stratified space.

Chapter 2

A stratified Poisson structure for the standard simplex

The goal of this chapter is to describe a stratified Poisson structure for the simplex, the space hosting the replicator dynamical system studied in the next chapter. This Poisson structure is relevant because, as will be discussed in the next chapter, the replicator vector field for zero sum games is under some circumstances Hamiltonian with respect to it.

2.1 The standard simplex

The *standard simplex* is

$$\Delta^n = \{x \in \mathbb{R}^{n+1} : x^i \geq 0, \sum_i x^i = 1\} \subset \mathbb{R}^{n+1} \tag{2.1}$$

We refer to [Nak03, Section 3.2] and [Hat02, pp. 9, 103] for a comprehensive treatment of the role simplices play in Homology theory, but in the present work we are only interested in their defining property: a point on a simplex has non negative components that add up to 1, so it is suitable to represent a discrete probability distribution.

The unit vectors along the coordinate axes are called *vertices* of the simplex. A 0-simplex is the singleton $\{1\} \subset \mathbb{R}$, which is a vertex. A 1-simplex in $\mathbb{R}^2$ is the segment $x + y = 1$ with x

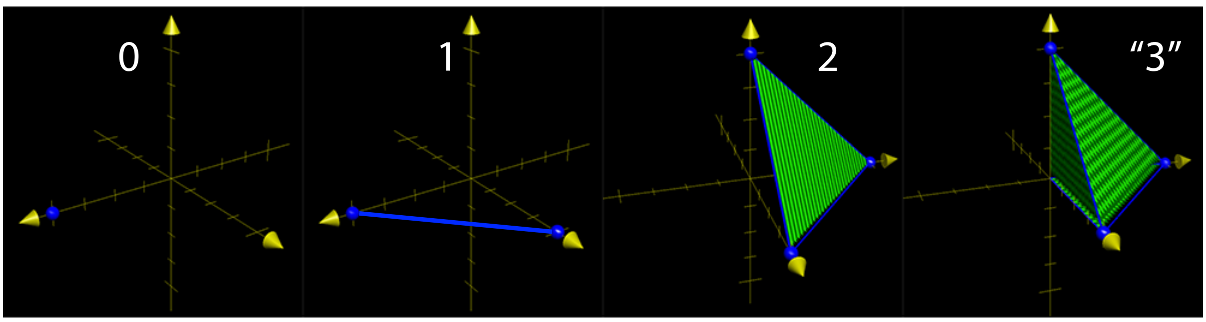

Figure 2.1: In three dimensions the 0, 1 and 2 dimensional simplices and the parameters space of the 3-simplex can be fully visualized. See Remark (3.28).

and $y \geq 0$. It is the disjoint union of the two vertices $(1,0)$ and $(0,1)$ and the open segment $x+y=1$ with x and y *strictly* positive; this open segment is called *1-face.* A 2-simplex is the triangle in $x+y+z=1$ in $\mathbb{R}^3$ with all coordinates non negative; it is the disjoint union of three vertices, three 1-faces and one 2-face, that is the interior of the triangle. The pattern is clear; see fig. (2.1).

Define $I = \{0,\dots,n\}$. For a point $x = (x^0,\dots,x^n) \in \Delta^n$ the indices corresponding to the nonzero components form the *support* of x:

$$\operatorname{supp}(x) = \{i \in I : x^i > 0\} \tag{2.2}$$

The complement of the support in I is $\operatorname{supp}^*(x) = \{i \in I : x^i = 0\}$

A *d-face* of an n-simplex $\Delta^n \subset \mathbb{R}^{n+1}$ is the collection of all points $x = (x^0,\dots,x^n)$ that

1. belong to the simplex, $\sum_{i \in I} x_i = 1$;
2. have $d+1$ strictly positive components and $n-d$ zero components.

A d-face is then defined by a subset $J \subset I$ with $d+1$ elements: $\mathring{\Delta}^J := \{x \in \Delta^n : \operatorname{supp}(x) = J\}$. J also defines the *face with boundaries*[1] $\Delta^J := \{x \in \Delta^n : \operatorname{supp}(x) \subset J\}$.

The number of d-faces of a n-simplex is[2] $\binom{n+1}{d+1}$. Thus Δ^n has only one n-face, called *interior*, consisting of points with all $n+1$ components positive, i.e. $\mathring{\Delta}^n = \{x \in \Delta^n : x^i > 0 \text{ for all } i\}$. On the other extreme, an n-simplex has $n+1$ 0-faces, which are precisely its vertices. All points that do not belong to the interior of Δ^n form its *boundary*.

Example 2.1. Consider the standard 2-simplex, so $I = \{0,1,2\}$, and choose $J = \{0,1\}$. Δ^J is the set of points such that $x^i > 0 \Rightarrow i \in J$, i.e. $i \notin J \Rightarrow x^i = 0$. Thus, $\Delta^J = \{x \in \Delta^2 : x^2 = 0\}$, which is the segment $x^0 + x^1 = 1$, extremes *included*.

For the set $\mathring{\Delta}^J$ the implication holds in both directions: $x^i = 0 \iff i \notin J$, so $\mathring{\Delta}^J = \{x \in \Delta^2 : x^0 > 0, x^1 > 0, x^2 = 0\}$, which is, as expected, the 1-face $x^0 + x^1 = 1$ (extremes *excluded*).

A d-face $\mathring{\Delta}^J$ of Δ^n is a d-dimensional submanifold of $\mathbb{R}^{n+1}$ (recall that J contains $d+1$ elements). Consider

$$\begin{aligned} S^d : M \subset \mathbb{R}^{n+1} &\to \mathbb{R}^{n+1-d} \\ x = (x^0, \cdots, x^n) &\longmapsto \left(\sum_{i=0}^{n} x^i, x^j\right) \quad \text{for all } j \notin J \end{aligned} \tag{2.3}$$

where M is the $(n+1)$-dimensional manifold $\mathbb{R}^{n-d} \times \left\{x : x \in \operatorname{int}\left(\mathbb{R}_+^{d+1}\right)\right\}$. Reordering the coordinates without loss of generality the Jacobian of S^d is the $(n+1-d) \times (n+1)$ block matrix

$$\left(\begin{array}{c|c} 1 \cdots 1 & 1 \cdots 1 \\ \hline \mathbb{1} & \mathbf{0} \end{array}\right) \tag{2.4}$$

where $\mathbb{1}$ is the identity matrix. Its rank is maximal everywhere, so S^d is a smooth submersion. The d-face is the level set $\left(S^d\right)^{-1}(1,0,\cdots,0)$ with $n-d$ zeros, hence a d-dimensional submanifold of $\mathbb{R}^{n+1}$.

[1] AL84, p. 234.
[2] Nak03.

2.2 Simplex singular Poisson reduction

Notation In this section

- $i = 0, 1, \ldots, n$
- $\mu = 1, \ldots, n$
- $\lambda = a + ib = \rho e^{i\alpha} \in G = \mathbb{C}^* = \mathbb{C} - \{0\}$
- $z = (x_i + iy_i) = (r_i e^{i\theta_i}) \in M = \mathbb{C}^{n+1} - \{0\}$
- $T = (e^{i\phi_\mu}) \in \mathbb{T}^n$
- $[z] \in \mathbb{C}P(n)$ has coordinates (R_μ, Θ_μ)

The idea is now to identify the simplex as the Poisson stratified space arising from two Poisson actions on a Poisson manifold. This procedure is discussed in [ORF09, p. 1276] and [AD14, p. 17].

Consider the real $2(n+1)$-dimensional manifold $M = \mathbb{C}^{n+1} - \{0\}$ with quadratic Poisson structure $\{z_i, z_j\} = a_{ij}\, z_i z_j$, $i, j = 0, \cdots n$ (without summation over repeated indices), with $A = (a_{ij})$ some fixed antisymmetric $(n+1)$-square matrix. As explained in [Mei17, p. 8], this is indeed a Poisson structure. Holomorphic, real and polar coordinates are related by the transition functions $z_i = x_i + i\, y_i = r_i\, e^{i\,\theta_i}$, $i = 0, \cdots n$. By direct computation the bracket in these coordinates is

$$
\begin{aligned}
\{w_i, w_j\} &= a_{ij}\, w_i w_j, \quad w_i = x_i, y_i \\
\{r_i, r_j\} &= a_{ij}\, r_i r_j \\
\{r_i, \theta_j\} &= 0 = \{\theta_i, \theta_j\}
\end{aligned}
$$

Consider now the Lie group $G = \mathbb{C}^* \equiv \mathbb{C} - \{0\}$; a group element is denoted by $\lambda = a + ib = \rho e^{i\alpha} \in \mathbb{C}^*$. This group can act on M by element-wise complex multiplication ψ: in polar coordinates

$$\psi_\lambda(z) = \rho e^{i\alpha} \cdot (r_0, \ldots r_n, \theta_0, \ldots, \theta_n) = (\rho\, r_0, \ldots \rho\, r_n, \alpha + \theta_0, \ldots, \alpha + \theta_n) \tag{2.5}$$

The action is free $\lambda \cdot z = z$ iff $\rho\, r_i = r_i$ with $\rho \neq 0$, and $\alpha + \theta_i = \theta_i + 2\pi n$, for all i and any integer n. Since at least one $r_i \neq 0$ this means $\rho = 1$ and $\alpha = 2\pi n$, i.e. λ is the identity element of $\mathbb{C}^*$.

The action is Poisson We must check that $\psi_\lambda(z)$ is a Poisson morphism. In local coordinates we have to check eq. (A.33), namely that $J\pi J^T = \pi \circ \psi$, where J is the Jacobian of ψ. In polar coordinates every derivative with respect to r_i gives a ρ term and every derivative with respect to θ_1 gives a 1, so the Jacobian is the $2(n+1)$ square matrix

$$J = \left(\begin{array}{c|c} D_\rho & \mathbf{0} \\ \mathbf{0} & \mathbb{1} \end{array}\right)$$

where $D_\rho = \rho\mathbb{1}$ is the $(n+1)$-square matrix with ρ on the diagonal. The matrix of the Poisson bivector is given by elementary Poisson brackets, and it is nonzero only for $\{r_i, r_j\}$ terms:

$$\pi = \left(\begin{array}{c|c} D_r A D_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{array}\right)$$

where $(D_r)_{ij} = \delta_{ij} r_i$. Indeed $(D_r A D_r)_{ij} = \sum_{h,k} \delta_{ih} r_i A_{hk} \delta_{kj} r_i = A_{ij} r_i r_j = \{r_i, r_j\}$, so

$$\begin{aligned} J\pi J^T &= \left(\begin{array}{c|c} D_\rho & \mathbf{0} \\ \hline \mathbf{0} & \mathbb{1} \end{array}\right) \left(\begin{array}{c|c} D_r A D_r & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) \left(\begin{array}{c|c} D_\rho & \mathbf{0} \\ \hline \mathbf{0} & \mathbb{1} \end{array}\right) \\ &= \left(\begin{array}{c|c} D_{\rho r} A D_{\rho r} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) = \pi \circ \psi \qquad \square \end{aligned}$$

By the Regular Poisson Reduction Theorem, ${}^{M}/_{G} = {}^{\mathbb{C}^{n+1}}/_{\mathbb{C}^*}$ is a Poisson manifold of real dimension $2(n+1-1) = 2n$, with Poisson structure such that the canonical projection is a Poisson morphism. This quotient space is called *complex projective space* and is denoted by $\mathbb{C}P(n)$. An orbit $[z] = \mathbb{C}^* \cdot z$ is a line through the origin in complex space, which is a plane in real space.

Example 2.2. Consider $M = \mathbb{C}^2 - \{0\}$. The matrix A and the Poisson bivector in real coordinates are

$$A = \begin{bmatrix} 0 & a_{01} \\ -a_{01} & 0 \end{bmatrix}, \quad \pi = \begin{bmatrix} 0 & a_{01}x_0x_1 & 0 & a_{01}x_0y_1 \\ -a_{01}x_0x_1 & 0 & -a_{01}x_1y_0 & 0 \\ 0 & a_{01}x_1y_0 & 0 & a_{01}y_0y_1 \\ -a_{01}x_0y_1 & 0 & -a_{01}y_0y_1 & 0 \end{bmatrix}$$

The orbit of the point $(\hat r_0, \hat r_1, \hat\theta_0, \hat\theta_1)$ with $\hat r_0 \neq 0$ under the action $(\rho\,\hat r_0, \rho\,\hat r_1, \alpha + \hat\theta_0, \alpha + \hat\theta_1)$ for $\rho \neq 0$ is the set of points $(r_0, r_1, \theta_0, \theta_1)$ in $\mathbb{R}^4 - \{0\}$ such that

$$\begin{cases} r_1 = \frac{\hat r_1 \, r_0}{\hat r_0} \\ \theta_0 = \theta_1 + \hat\theta_0 - \hat\theta_1 \end{cases}$$

which is a plane.

If a point $z \in M$ has, say, $r_0 \neq 0$, then all the points in its orbit $[z]$ have the same nonzero component. This can be used to build well defined coordinates for $[z] \in \mathbb{C}P(n)$ as $(R_\mu = \frac{r_\mu}{r_0}, \Theta_\mu = \theta_i - \theta_0)$, and a direct computation shows that all brackets involving Θ_μ vanish, while

$$\{R_\mu, R_\nu\} = (a_{\mu\nu} - a_{\mu 0} - a_{0\nu})\, R_\mu R_\nu$$

A further reduction is required to identify the quotient space with the simplex. Let $T \in \mathbb{T}^n$ be an element of the n-torus $\mathbb{T}^n$: this group can act on $\mathbb{C}P(n)$ by

$$\begin{aligned} \psi_T([z]) = T \cdot [z] &= (e^{i\phi_1}, \ldots, e^{i\phi_n}) \cdot [(z_0, z_1, \ldots, z_n)] \\ &= [(z_0, e^{i\phi_1} z_1, \ldots, e^{i\phi_n} z_n)] \end{aligned} \tag{2.6}$$

This is well defined: consider $z \in M$ and $\lambda \cdot z \in [z]$ for some $\lambda \in G$. Evaluating $T \cdot [z]$ with z gives $[(z_0, e^{i\phi_\mu} z_\mu)]$, while using $\lambda \cdot z$ we get

$$[(\lambda\, z_0,\, \lambda\, e^{i\phi_\mu} z_\mu)] = [\lambda(z_0, e^{i\phi_\mu} z_\mu)] = [(z_0, e^{i\phi_\mu} z_\mu)]$$

One can check again that this action is Poisson, *but it is not free.* For example, if $z_0 = 0$ and $\phi_\mu = \phi$ for all μ, the action is a global rescaling which fixes the class: $[(0, z_\mu)] \mapsto [(0, e^{i\phi} z_\mu)] = [e^{i\phi}(0, z_\mu)] = [(0, z_\mu)]$. We will investigate in detail the orbit type manifolds for this action in the next section.

By the Singular Poisson Reduction Theorem the quotient space ${}^{\mathbb{C}P(n)}/_{\mathbb{T}^n}$ is a Poisson stratified space. The action of the torus quotients the angles away: for $z, w \in M$, $z \sim_{C^*} w$ if the radial components of w are proportional to the radial components of z by the same proportionality constant, *and* the angular components of w differ by the angular components of z by a constant angle; $[z] \sim_{\mathbb{T}^n} [w]$ if the radial components of any representative element of w are proportional to the radial components of any representative element of $[z]$, regardless of angular components. In other words a point in ${}^{\mathbb{C}P(n)}/_{\mathbb{T}^n}$ has n degrees of freedom:

$$\begin{aligned} \pi : \mathbb{C}P(n) &\to {}^{\mathbb{C}P(n)}/_{\mathbb{T}^n} \\ [(r_i, \theta_i)] &\longmapsto [(r_i)] \end{aligned} \tag{2.7}$$

or in the complex projective space coordinates (Θ_μ, R_μ)

$$\begin{aligned} \pi : \mathbb{C}P(n) &\to {}^{\mathbb{C}P(n)}/_{\mathbb{T}^n} \\ (R_\mu, \Theta_\mu) &\longmapsto (R_\mu) \end{aligned} \tag{2.8}$$

Consider now the map

$$\begin{aligned} \xi : \mathbb{C}P(n) &\to \Delta^n \subset \mathbb{R}^{n+1} \\ [z] &\longmapsto \left(\frac{r_0^2}{r_0^2 + \cdots + r_n^2}, \cdots, \frac{r_n^2}{r_0^2 + \cdots + r_n^2} \right) \end{aligned} \tag{2.9}$$

This is well defined for a representative of $[z]$, and is onto the standard simplex, so it is a quotient map defining the quotient topology on $\Delta^n \cong {}^{\mathbb{C}P(n)}/_{\sim_\xi}$. In $\mathbb{C}P(n)$ coordinates

$$\begin{aligned} \xi : \mathbb{C}P(n) &\to \Delta^n \subset \mathbb{R}^{n+1} \\ [z] &\longmapsto \left(\frac{1}{1 + R_1^2 + \cdots + R_n^2}, \frac{R_1^2}{1 + R_1^2 + \cdots + R_n^2}, \cdots, \frac{R_n^2}{1 + R_1^2 + \cdots + R_n^2} \right) \end{aligned} \tag{2.10}$$

showing that $[z] \sim_\xi [w] \iff [z] \sim_{\mathbb{T}^n} [w]$, i.e. ${}^{\mathbb{C}P(n)}/_{\sim_\xi} \cong {}^{\mathbb{C}P(n)}/_{\mathbb{T}^n}$, and finally

$$\Delta^n \cong {}^{\mathbb{C}P(n)}/_{\mathbb{T}^n} \tag{2.11}$$

This means that the standard simplex is a Poisson stratified space with unique Poisson structure such that eq. (2.7) is a Poisson morphism.

2.2.1 The Poisson bracket on Δ^n

Let $x = (x^0, \ldots, x^n)$ be the coordinates in $\mathbb{R}^{n+1} \supset \Delta^n$. The Singular Poisson Reduction theorem says that the pullback of the elementary Poisson bracket of the simplex is the bracket of the pulled back coordinates functions:

$$\{x^i, x^j\}_{\Delta^n}(\xi([z])) = \{\xi^i, \xi^j\}_{C^{n+1}-\{0\}}(z), \quad i, j = 0, \ldots, n$$

Since in general $\{f, g\} = \sum \{x^i, x^j\}\, \partial_i f\, \partial_j g$ we have to compute $\sum_{h,k} \{r_h, r_k\}\, \partial_h \xi^i\, \partial_k \xi^j$ with[3]

$$\xi^i = \frac{r_i^2}{\sum_{j=0}^n r_j^2}$$

[3]Note that the position of indices for coordinates has no particular meaning, and may change according to the mood.

This is some gymnastics with indices; $(\sum)$ is a shorthand for $\sum_{j=0}^{n} r_j^2$ and P_{hk} a shorthand for $\{r_h, r_k\} = a_{hk} r_h r_k$.

$$\partial_h \xi^i = \frac{2r_i \delta_{ih}(\sum) - 2r_i^2 r_h}{(\sum)^2}$$

$$\sum_{h,k} \{r_h, r_k\}\, \partial_h \xi^i\, \partial_k \xi^j = \sum_{h,k} P_{hk} \frac{2r_i \delta_{ih}(\sum) - 2r_i^2 r_h}{(\sum)^2} \frac{2r_j \delta_{jk}(\sum) - 2r_j^2 r_k}{(\sum)^2}$$
$$= \frac{4r_i r_j P_{ij}}{(\sum)^2} - \sum_k \frac{4r_i r_j^2 P_{ik} r_k}{(\sum)^3} - \sum_k \frac{4r_i^2 r_j P_{kj} r_k}{(\sum)^3}$$

where in the second line the fourth term of the product vanishes because it contains the contraction of the antisymmetric quantity P_{hk} with the symmetric term $r_h r_k$. Replace $\xi^i = r_i^2/(\sum)$ and P_{hk}, factor and rename $a_{ij} \to 4a_{ij}$ to absorb the factor of 4 to get the result:

$$\boxed{\{x^i, x^j\}_{\Delta^n} = x^i x^j \left(a_{ij} - \sum_{k=0}^{n} (a_{ik} + a_{kj}) x^k \right), \quad i, j = 0, \dots, n} \tag{2.12}$$

An explicit check of the Jacobi condition (1.14) shows that eq. (2.12) actually defines a Poisson structure for the whole $\mathbb{R}^{n+1}$. The next steps is to identify the strata, namely the Poisson manifolds that inherit this Poisson structure.

2.2.2 Poisson strata of Δ^n

According to the Singular Poisson Reduction Theorem the Poisson manifolds stratifying the simplex are the connected components of the reduced orbit type submanifolds ${}^{\mathbb{C}P(n)_{(H)}}/_{\mathbb{T}^n}$. To understand their nature we start investigating the isotropy type submanifolds $\mathbb{C}P(n)_H$ as H ranges over all possible subsets of $\mathbb{T}^n$.

It is important to understand the logic behind the definitions (1.70) and (1.76) of isotropy subgroup and isotropy type submanifold. Let G be a Lie group acting on a manifold M: given $H \subset G$, M_H contains *all and only* the points for which H contains *all and only* the group elements fixing the point. This provides three necessary and sufficient conditions to say that a subset of M is the isotropy type submanifold of some $H \subset G$ ($p \in M_H \Rightarrow G_p = H$ gives the first two, and $p \in M_H \Leftarrow G_p = H$ gives the third):

1. $\forall p \in M_H, T \in H \Rightarrow T \cdot p = p$
2. $\forall p \in M_H, T \cdot p = p \Rightarrow T \in H$
3. $p \notin M_H \Rightarrow G_p \neq H$

In the following $M = \mathbb{C}P(n)$ and $G = \mathbb{T}^n$. The identity element of $\mathbb{T}^n$ is id $= (e^{i0}, \dots, e^{i0})$. Clearly if H is not a subgroup of G, $M_H = \emptyset$ (if $p \in M_H, \mathrm{id} \notin G_p$, which is absurd). So we are interested in the subgroups of $\mathbb{T}^n$, $H = \{\mathrm{id}\} \cup (\dots)$. Keep in mind that $\mathbb{T}^n \ni T = (e^{i\phi_1}, \dots, e^{i\phi_n})$ has n entries, with each $\phi_i \in [0, 2\pi)$, while $\mathbb{C}P(n) \ni [z] = [z_0, z_1, \dots, z_n]$ has $n+1$.

$H_0 = \{\mathbf{id}\}$ We begin with the trivial subgroup and claim that its isotropy type manifold is

$$M_0 = \{[z] : z_i \neq 0 \text{ for all } i = 0, \dots, n\} \tag{2.13}$$

Condition (1) is trivial (id fixes everything).

For condition (2), consider a generic $T = (e^{i\phi_1}, \dots, e^{i\phi_n})$:

$$T \cdot [z] = [z] \Rightarrow [z_0, e^{i\phi_\mu} z_\mu] = [z_0, z_\mu]$$

where z_μ means $z_1, \dots, z_n$. The above equality holds only if T is a global rescaling; since z_0 does not change, this is possible only if $1 = e^{i\phi_1} = \dots = e^{i\phi_n}$, namely $T = \text{id}$. $\square$

For condition (3), consider $[z] \notin M_0$; then $z_k = 0$ for at least one value of k, and $G_{[z]}$ is strictly bigger than H_0: for example

$$[z_0, z_1, \cdots, z_{k-1}, \underbrace{0}_{k}, z_{k+1} \cdots, z_n] \text{ is fixed by } T = (1, \dots, 1, \underbrace{e^{i\phi}}_{k}, 1, \dots, 1)$$

for all values of ϕ. $\square$

$H_1^{(0)} = \{\mathbf{id}\} \cup \{(e^{i\phi}, \dots, e^{i\phi}) : \phi \in [0, 2\pi)\}$ Consider now the subgroups containing elements with all the components equal. We claim that the isotropy type manifold is

$$M_1^{(0)} = \{\, [0, z_1, \dots, z_n] : z_\mu \neq 0 \text{ for all } \mu = 1, \dots n\} \tag{2.14}$$

1. $(e^{i\phi}, \dots, e^{i\phi}) \cdot [0, z_1, \dots, z_n] = [0, e^{i\phi} z_1, \dots, e^{i\phi} z_n] = [0, z_1, \dots, z_n]$, being a global rescaling.
2. $(e^{i\phi_1}, \dots, e^{i\phi_n}) \cdot [z] = [0, e^{i\phi_1} z_1, \dots, e^{i\phi_n} z_n] = [z]$ iff $e^{i\phi_1} = \dots = e^{i\phi_n}$, i.e. $T \in H_1^{(0)}$. $\square$
3. Consider $[z] \notin M_1^{(0)}$. If it has less zero components, i.e. if $z_0 \neq 0$, then $H_1^{(0)}$ cannot fix it and $G_{[z]}$ is strictly smaller than $H_1^{(0)}$; if there are more zero components, a less restrictive group element can fix it, and $G_{[z]}$ is strictly bigger than $H_1^{(0)}$.

With the same logic one can prove all the other cases:

$$\begin{aligned} H_1^{(k)} &= \{\text{id}\} \cup \{(1, \dots, 1, \underbrace{e^{i\phi}}_{k}, 1, \dots, 1) : \phi \in [0, 2\pi)\} \\ M_1^{(k)} &= \{\text{precisely 1 component} = 0, \text{ the } k\text{-th}\} \\ &= \{[z_0, z_1, \dots, \underbrace{0}_{k}, \dots, z_n] : z_i \neq 0 \text{ for } i \neq k\} \end{aligned} \tag{2.15}$$

$$\begin{aligned} H_2^{(0,k)} &= \{\text{id}\} \cup \{(e^{i\phi}, \dots, e^{i\phi}, \underbrace{e^{i\psi}}_{k}, e^{i\phi}, \dots, e^{i\phi}) : \phi, \psi \in [0, 2\pi)\} \\ M_2^{(0,k)} &= \{\text{precisely 2 components} = 0, \text{ the } \mathit{0}\text{-th and the } k\text{-th}\} \\ &= \{\, [0, z_1, \dots, z_{k-1}, 0, z_{k+1}, \dots, z_k]\} \end{aligned} \tag{2.16}$$

$$\begin{aligned} H_2^{(h,k)} &= \{\text{id}\} \cup \{(1, \dots, 1, \underbrace{e^{i\phi}}_{h}, 1, \dots, 1, \underbrace{e^{i\psi}}_{k}, 1, \dots, 1) : \phi, \psi \in [0, 2\pi)\} \\ M_2^{(h,k)} &= \{\text{precisely 2 components} = 0, \text{ the } h\text{-th and the } k\text{-th}\} \end{aligned} \tag{2.17}$$

And in general

$$H_m^{\overbrace{(i_1, \dots, i_m)}^{\text{all } m \neq 0}} = \{\text{id}\} \cup \{i_1, \dots, i_m \text{ free} : e^{i\phi_{i_1}}, \dots, e^{i\phi_{i_m}};\ \text{the others } = 1\}$$

$$M_m^{\overbrace{(i_1, \dots, i_m)}^{\text{all } m \neq 0}} = \{[z] \text{ with precisely } m \text{ components } = 0 : \text{the } i_1\text{-th}, \dots, \text{the } i_m\text{-th}\}$$

$$H_m^{\overbrace{(0,i_1,\dots,i_{m-1})}^{m,\text{ including }0}} = \{\mathrm{id}\} \cup \{i_1,\dots,i_{m-1} \text{ free} : e^{i\phi_{i_1}},\dots,e^{i\phi_{i_{m-1}}};\ \text{the others free but equal to each other}\ \}$$

$$M_m^{\overbrace{(0,i_1,\dots,i_{m-1})}^{m,\text{ including }0}} = \{[z] \text{ with precisely } m \text{ components } = 0 : \text{the 0-th, the } i_1\text{-th},\dots,\text{the } i_{m-1}\text{-th}\}$$

At top level these two types coincide:

$$\begin{aligned}
H_n^{\overbrace{(1,2,\dots,n)}^{\text{all but }0}} &= 1,2,\dots,n \text{ free, others} = 1\text{; but there is no other}\\
&= \text{all free } = \mathbb{T}^n\\
H_n^{(0,1,\dots,k-1,k+1,,\dots,n)} &= 1,\dots,k-1,k+1,\dots,n \text{ free; } k\text{-th free and "equal to each other", but it is alone}\\
&= \text{all free } = \mathbb{T}^n
\end{aligned}$$

So $H_n = \mathbb{T}^n$; correspondingly $M_n^{(\dots)}$ contains precisely n nonzero components out of $n+1$, determined by $(\dots)$. In this case Condition (1) holds because any group element acts on a class with a single nonzero component as a global rescaling; Condition (2) is trivial because $\mathbb{T}^n$ is the whole group; for Condition (3), if $[z] \neq M_n^{(\dots)}$ because it has less zeros, $G_{[z]}$ is strictly smaller; if $[z]$ had more zeros - namely, all components - it would not belong to the manifold, since we started from $\mathbb{C}^{n+1} - \{0\}$.

This completes the description of the isotropy type submanifolds M_H; the step to the reduced orbit types is now short. $M_m^{(\dots)}$ are the classes in $\mathbb{C}P(n)$ with m zero components out of $n+1$. Recall proposition (1.77): the orbit type $M_{(m)}^{(\dots)} = \mathbb{T}^n \cdot M_m^{(\dots)}$ is just the orbit in $\mathbb{C}P(n)$ of these points, quotienting the angles away; and the strata $M_{(m)}^{(\dots)}/_{\mathbb{T}^n}$ are the points of the n-simplex Δ^n with precisely m components $= 0$, namely the $(n-m)$-faces. In one extreme case $m=0$, $H_0 = \{\mathrm{id}\}$ and $M_{(0)}/_{\mathbb{T}^n}$ is the set of simplex points with all nonzero components, the n-face, i.e. the interior; in the other $m=n$, $H_n = \mathbb{T}^n$ and $M_{(n)}^{(\dots)}/_{\mathbb{T}^n}$ has only one nonzero component; it is indeed a 0-face, i.e. a vertex, which one depending on the indices in $(\dots)$.

In conclusion *the faces of the simplex Δ^n are Poisson manifolds that inherit the Poisson structure* (2.12). They actually are Poisson submanifolds of $\mathbb{R}^{n+1}$: we just have to check condition (1.26) for all the components of the submersion (2.3) defining the faces, restricting the bracket to the considered face. Recall that

$$\{x^i, x^j\}_{\Delta^n} = x^i x^j \left(a_{ij} - \sum_{k=0}^{n} (a_{ik} + a_{kj}) x^k \right), \quad i,j = 0,\dots,n$$

so for all i, $\{x^i, x^j\}_{\Delta^n} = 0$ whenever $x^j = 0$; and it can be seen that for all i, $\{x^i, \sum_{j=0}^n x^j\}_{\Delta^n} = 0$ anywhere on the simplex, namely whenever $\sum_{j=0}^n x^j = 1$. This shows that the faces are Poisson submanifolds of $\mathbb{R}^{n+1}$; in particular all Hamiltonian vector fields with respect to this structure are tangent to the faces.

Example 2.3. The Poisson structure defined by

$$A = \begin{bmatrix} 0 & a_{01} & a_{02} \\ -a_{01} & 0 & a_{12} \\ -a_{02} & -a_{12} & 0 \end{bmatrix}$$

on the 2-simplex is

$$\begin{aligned}
\{x^0, x^1\} &= -x_1 x_2 \left(a_{01} - a_{02} + a_{12}\right)\left(x_1 + x_2 - 1\right) \\
\{x^0, x^2\} &= x_1 x_2 \left(a_{01} - a_{02} + a_{12}\right)\left(x_1 + x_2 - 1\right) \\
\{x^1, x^2\} &= -x_1 x_2 \left(a_{01} - a_{02} + a_{12}\right)\left(x_1 + x_2 - 1\right)
\end{aligned}$$

where x^0 was replaced by $1 - x^1 - x^2$. It is clear that $\{x^i, x^0 + x^1 + x^2\} \equiv 0$.

2.2.3 Constant Poisson structure

As a last result for this section we present, following [AD14, p. 11], a canonical coordinates transformation putting the Poisson structure of the simplex into a constant form. In this section Latin indices $i, j, \ldots$ run over $1, \ldots, n$ and Greek indices $\mu, \nu, \ldots$ run over $0, \ldots, n$.

Proposition 2.4. *Consider the map*

$$\begin{aligned}
\phi : \mathbb{R}^n &\to \mathring{\Delta}^n \subset \mathbb{R}^{n+1} \\
(y^1, \ldots, y^n) &\longmapsto (x^0, x^1, \ldots, x^n) \\
x^0 = \frac{1}{1 + \sum_j e^{y^j}}, &\quad x^i = \frac{e^{y^i}}{1 + \sum_j e^{y^j}}
\end{aligned} \tag{2.18}$$

Let A be an $(n+1)$ square antisymmetric matrix with fixed coefficients, E the $n \times (n+1)$ matrix

$$E = \left(\begin{array}{c|c} 1 & \\ \vdots & -\mathbb{1} \\ 1 & \end{array}\right) \tag{2.19}$$

and define the $n \times n$ matrix $B = EAE^T$. Then $\phi : (\mathbb{R}^n, B) \to (\mathring{\Delta}^n, \{\cdot,\cdot\}_{\Delta^n})$ is a Poisson diffeomorphism, where $\{\cdot,\cdot\}_{\Delta^n}$ is defined by eq. (2.12) using the matrix A.

Proof. The point $\phi(y)$ belongs indeed to $\mathring{\Delta}^n$ for all $y \in \mathbb{R}^n$, since $\sum_\mu x^\mu = 1$ and $x^\mu > 0$ for all μ. ϕ is a diffeomorphism whose inverse is $y^i = \ln \frac{x^i}{x^0}$, well defined since all $x^\mu > 0$. Note that the origin of $\mathbb{R}^n$ corresponds to the center of Δ^n: $0 \mapsto (1/(n+1), \ldots, 1/(n+1))$.

The following is just indices juggling. The matrix E can be written as

$$E_{i\mu} = \begin{cases} E_{i0} = 1 \\ E_{ij} = -\delta_{ij} \end{cases} \tag{2.20}$$

so that B is the antisymmetric constant coefficients matrix $B_{ij} = E_{i\mu} E_{j\nu} A_{\mu\nu}$, summing over repeated indices, which defines a Poisson structure for $\mathbb{R}^n$. We have to check that $J_\phi B J_\phi^T = \pi \circ \phi$.

We claim, and show later, that the matrix form of π is

$$\pi(x) = T_x D_x A D_x T_x^T \tag{2.21}$$

with $D_x = \mathrm{diag}(x^0, \ldots, x^n)$, i.e. $(D_x)_{\mu\nu} = \delta_{\mu\nu} x_\mu$, and $(T_x)_{\mu\nu} = x_\mu - \delta_{\mu\nu}$.

The Jacobian of ϕ is

$$J_{\mu i} = \partial_i \phi^\mu = \delta_{\mu i} x_i - x_\mu x_i$$

with no summation over repeated indices. This is particularly simple because the terms containing y appear only as $e^{y_i}/(1+\sum_j e^{y^j})$ and can be replaced by x^i. This takes care of the composition part $\pi \circ \phi$, since we are already looking at the Jacobian in x coordinates.

Since $JBJ^T = J\,EAE^T\,J^T \overbrace{=}^{?} \pi = T_x D_x A D_x T_x^T$ the problem reduces to checking whether

$$JE = T_x D_x \tag{2.22}$$

Split the left hand side according to eq. (2.20): $(JE)_{\mu 0} = \sum_i J_{\mu i} \underbrace{E_{i0}}_{1} = \sum_i J_{\mu i}$. Split this as well and massage to obtain

$$(JE)_{00} = -x_0(1-x_0), \quad (JE)_{j0} = x_j x_0 \tag{2.23}$$

The other piece of JE is

$$(JE)_{\mu h} = J_{\mu i} E_{ih} = -J_{\mu h} = -\delta_{\mu h} x_h + x_\mu x_h \tag{2.24}$$

Similarly, for the right hand side $T_x D_x$:

$$(TD)_{\mu\nu} = \sum_\alpha (x_\mu - \delta_{\mu\alpha})\delta_{\alpha\nu} x_\alpha = x_\mu x_\nu - \delta_{\mu\nu} x_\nu$$

Splitting this into $(TD)_{00}$, $(TD)_{j0}$ and $(TD)_{\mu h}$ gives precisely the corresponding JE expressions.

Finally for eq. (2.21), using the expression just derived for TD:

$$\begin{aligned}
&\left(TDA(TD)^T\right)_{\mu\nu} = (TD)_{\mu\alpha} A_{\alpha\beta} (TD)_{\nu\beta} \\
&= (x_\mu x_\alpha - \delta_{\mu\alpha} x_\mu) A_{\alpha\beta} (x_\nu x_\beta - \delta_{\nu\beta} x_\nu) \\
&= x_\mu x_\alpha A_{\alpha\beta} x_\nu x_\beta - x_\mu x_\alpha A_{\alpha\beta} \delta_{\nu\beta} x_\nu - \delta_{\mu\alpha} x_\mu A_{\alpha\beta} x_\nu x_\beta + \delta_{\mu\alpha} x_\mu A_{\alpha\beta} \delta_{\nu\beta} x_\nu
\end{aligned}$$

with summation over α and β. The first term vanishes as it contains $A_{\alpha\beta} x_\alpha x_\beta$; the remaining three give $x_\mu x_\nu \left(A_{\mu\nu} - \sum_\alpha (A_{\mu\alpha + A_{\alpha\nu}}) x^\alpha\right) = \pi_{\mu\nu}$. □

A constant coefficient Poisson structure simplifies matters greatly. Its explicit expression, using eq. (2.20), is

$$B_{ij} = E_{i\mu} E_{j\nu} A_{\mu\nu} = A_{ij} - (A_{i0} + A_{0j}) \tag{2.25}$$

Recall that A is an $n+1$ antisymmetric square matrix, and B is an n antisymmetric square matrix: B can be nondegenerate only if n is even, i.e. if $\det A = 0$.

2.3 Simplex symplectic foliation

The symplectic leaves of the Poisson manifold $(\mathring{\Delta}^n, \{\cdot,\cdot\}_{\Delta^n})$ are the images of the symplectic leaves of $(\mathbb{R}^n, B)$ via the diffeomorphism ϕ defined in prop. (2.4), which are very easy to compute.

Since B has constant coefficients the characteristic distribution of $(\mathbb{R}^n, B)$ is regular, namely it has constant rank: the (even) number of independent Hamiltonian vector fields spanning the distribution is equal to the number of linear independent rows of B, say $2k$. These Hamiltonian vector fields are generated by the coordinates functions corresponding to these rows: say the j-th is one of them, then

$$Y_j \coloneqq (\mathrm{d}y^j)^\sharp \Rightarrow Y_j^i = B(\mathrm{d}y^j, \mathrm{d}y^i) = B^{ji}$$

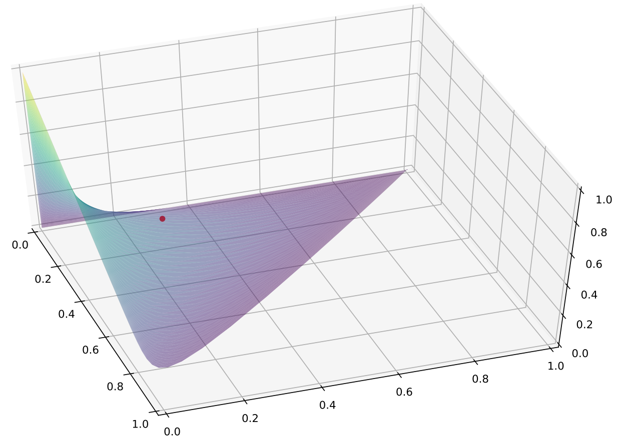

Figure 2.2: Symplectic leaf through the center of the 3-simplex

i.e. the j-th row of B provides the constant components of the Hamiltonian vector field Y_j. Thus, the symplectic leaf of $(\mathbb{R}^n, B)$ through p is the even dimensional affine subspace through p whose linear part is generated by the linear independent rows of B.

Example 2.5. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 2 & -1 \\ 0 & -2 & 0 & -2 \\ -1 & 1 & 2 & 0 \end{bmatrix}$$

The Poisson structure it induces on the 3-simplex is

$$\pi = \begin{bmatrix} 0 & -x_1x_2\left(3x_1+3x_2+2x_3-3\right) & x_1x_3\left(x_1+x_3-1\right) \\ x_1x_2\left(3x_1+3x_2+2x_3-3\right) & 0 & x_2x_3\left(4x_1+3x_2+3x_3-3\right) \\ -x_1x_3\left(x_1+x_3-1\right) & -x_2x_3\left(4x_1+3x_2+3x_3-3\right) & 0 \end{bmatrix}$$

corresponding to the constant rank-2 Poisson structure

$$B = \begin{bmatrix} 0 & 3 & -1 \\ -3 & 0 & -3 \\ 1 & 3 & 0 \end{bmatrix}$$

Choose the first and the third rows to span the symplectic leaves: $y^1 = b, y^2 = 3a + 3b, y^3 = -a$, i.e. $y^2 = 3y^1 - 3y^3$. Thus y^1 and y^3 parametrize the symplectic leaf in $\mathring{\Delta}^3$ through $\phi(0,0,0) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \in \mathring{\Delta}^3$ as

$$x^1 = \frac{e^{y^1}}{1+e^{y^1}+e^{3y^1-3y^3}+e^{y^3}}, x^2 = \frac{e^{3y^1-3y^3}}{1+e^{y^1}+e^{3y^1-3y^3}+e^{y^3}}, x^3 = \frac{e^{y^3}}{1+e^{y^1}+e^{3y^1-3y^3}+e^{y^3}}$$

This leaf is plotted in Fig. (2.2).

In the next chapter, after discussing some general features of Evolutionary Game Theory, we will use the Poisson structure $\{\cdot,\cdot\}_\Delta$ to recognize the Hamiltonian nature of the zero-sum replicator vector field.

Chapter 3

Evolutionary Game Theory

3.1 Introduction

Remark 3.1. See [Sig17] for a beautiful discussion about the ecology, the genetics and the sociobiology underlying the ideas outlined in this section, and [HS98] for their mathematical formalization.

The concept of *population* is very broad: in general terms, it is a large set of *interacting individuals*, where an individual is an elementary unit whose nature depends on the system under consideration. Given a way to measure the well-being, the fitness, the success of an individual within a population, the *interaction* is the set of processes through which individuals influence each other's well-being. Individuals may gather in *species*, groups sharing some common traits and such that the interaction between individuals of the *same* species is quantitatively and/or qualitatively different from the interaction between individuals of *different* species.

Ecosystems provide an archetypal example: in nature, thousands of different species interact in extremely complex patterns, the well-being being measured in terms of Darwinian fitness, i.e. reproductive success. Many other more abstract situations may be considered, as economical systems with richness as success measure; political or cultural systems, with the success of ideals, ideas or concepts measured in terms of their diffusion; and so on.

Consider for definiteness an ecological system with some interacting species. The system is sustained by the flow of an incoming resource, for example the light of the Sun allowing the growth of the plants some animals feed on; the reproductive success of a species depends on the amount of resource it can effectively exploit. Lotka[1] and Volterra[2] proposed two basic types of interaction for this situation.

Species depending on the same resource are said to interact via a P (parallel) process: one may effectively increase (for example providing shelter from predators, or decomposing the resource as fungi do) or decrease (for example wasting) the amount of resource the other can exploit. The interaction needs not be symmetric: two species may damage each other (*competition*), benefit from each other (*mutualism*), or one can exploit the other (*parasitism*). On the other hand if a species can exploit a resource only after another species consumed it, as in a predator-prey relationship, we speak of an S (serial) process.

[1] Lot20; Lot26.
[2] Vol26; Vol27.

The interaction between individuals may be *local*, i.e. the success of an individual may depend on the outcome of an encounter with another individual. Think for example of two animals of the same species engaging in a fight to establish who is the alpha male, or of an animal feeding on another. It can also happen that the success of a individual depends on the current state of the population as a whole, in which case we speak of *global* interaction. For example, the reproductive success of an individual in terms of number of *grandchildren* depends on the sex of its offspring: If there are many males in a population, it is convenient to have female offspring. As we will discuss, global interaction processes may lead to non-linear interaction functions.

The spatial structure of a system imposes some additional modeling constraints. If the individuals, moving in their environment, have a range of motion which is much smaller than the domain spatial scale, the spatial structure of the environment affects crucially the interaction between individuals and hence the development of the population. On the other hand, if individuals can cover the whole spatial domain, the structure of the domain and the spatial distribution of individuals are irrelevant, and we speak of a *well mixed system*.

3.2 Lotka-Volterra dynamical system

One of the main goals of *mathematical ecology* is to investigate the dynamics of *population densities*, i.e. the appropriately normalized numbers of individuals in the species composing a population, upon some interaction modeling choices. The archetypal model is the Lotka-Volterra dynamical system $\dot{n} = X_{\mathrm{LV}}(n)$, which is a deterministic development equation for species interacting via S and P process in a well-mixed system.

Consider a single species, and let $n \geq 0$ be the number of individuals in it. The first assumption to model the evolution $n(t)$ of this species density is *nothing from nothing*: if the species initially counts no individuals, its density n will be zero forever. This means that $n = 0$ shall be a fixpoint of the vector field, i.e. $\dot{n} \propto n$. Furthermore, one assumes that the resource available in the environment can support at most a finite number of individuals, so X_{LV} must have a positive fixpoint $n^* > 0$, called *carrying capacity*. Finally assume that the species has an *intrinsic growth rate* $\mu > 0$ describing its reproductive efficiency, i.e. μ is the difference between the number of individuals that are born and that die per unit time. The simplest equation modeling these features is[3]

$$\dot{n} = \mu n \left(1 - \frac{n}{n^*}\right) \tag{3.1}$$

To express the carrying capacity in term of environmental quantities define the *individual consumption rate* a as the amount of resource consumed by an individual of the species per unit time, and the *resource input rate* q as the amount of resource that flows in the system per unit time. Clearly $n^* = q/a$: indeed an/q is the fraction of the available resource that n individuals consume, and per definition of carrying capacity $an^*/q = 1$. The evolution equation for a single species density in a limited environment becomes

$$\dot{n} = \mu n \left(1 - \frac{a}{q} n\right) \tag{3.2}$$

Consider now N species interacting via P processes, i.e. sharing a common resource, and let $n_i \geq 0$ be the number of individuals in the i-th species, $i = 1, \ldots, N$. The intrinsic growth rate of the i-th species is μ_i, and its individual consumption rate a_{ii} (the reason for the double index

[3] Lot26.

will be clear in a second). We need to add interaction the the development equation for the i-th species density

$$\dot{n}_i = \mu_i n_i \left(1 - \frac{a_{ii} n_i}{q} - (*)\right) \tag{3.3}$$

where $(*)$ is the effect all other species have on the resource consumption of species i.

Define a_{ij} as the impact an individual of the species j has on the resource consumption of the whole species i; it has the same units of a_{ii}, namely it is an effective resource consumption rate. Because of the choice of sign in the previous equation $a_{ij} < 0$ means that the species j is beneficial for the species i, while $a_{ij} > 0$ means that the species j damages the species i. With this definition we have that

$$(*) = \frac{\sum_{j\neq i} a_{ij} n_j}{q} \tag{3.4}$$

Note that we assumed that μ_i and a_{ij} *do not depend on* n_i. This is a strong assumption: in general, both the intrinsic growth rate and the interaction with other species may depend in a complex way on the number of individuals in a given species. Replacing $(*)$ in the development equation

$$\dot{n}_i = \mu_i n_i \left(1 - \frac{a_{ii} n_i}{q} - \frac{\sum_{j\neq i} a_{ij} n_j}{q}\right) \tag{3.5}$$

shows that the sign of $\frac{\sum_{j\neq i} a_{ij} n_j}{q}$ determines whether the i-th species is globally damaged or benefits from the interaction. Indeed $\dot{n}_i(n_i^{**}) = 0$ for $n_i^{**} \neq 0$ fulfilling

$$a_{ii} n_i^{**} = q - \sum_{j\neq i} a_{ij} n_j$$

so that if $\sum_{j\neq i} a_{ij} n_j < 0$ then n_i^{**} is bigger than $n_i^* = \frac{q}{a_{ii}}$, its carrying capacity *in absence of interaction*, and the i-th species globally benefits from the interaction. Analogously, $1 - \frac{\sum_{j\neq i} a_{ij} n_j}{q}$ is the fraction of resource the i-th species has available to grow (its budget), that can be more or less than the natural 1, while $\frac{a_{ii} n_i}{q}$ is the resource the i-th species is consuming at the moment. The species can grow if $\dot{n}_i > 0$, namely if what it is consuming at the moment is less than the available budget.

Let's now add interactions of type S between these species. Rewrite the evolution equation as

$$\dot{n}_i = n_i \left(\mu_i - \frac{\mu_i \sum_j a_{ij} n_j}{q}\right) = n_i\, \mu_i^{\text{eff}_P}(n) \tag{3.6}$$

where all interaction is enclosed in the *effective growth rate* $\mu_i^{\text{eff}_P}(n)$. Recall that P-interaction coefficients a_{ii} have dimension of resource over time, as q does, while the growth rates (both intrinsic and effective) have dimension of inverse time. We model S-interaction coefficients b_{ij} with dimension of number of individuals over time, i.e. inverse time. What defines a predator-prey relationship between two species is that the intrinsic growth rate of the prey shall be positive (if alone, it lives); the intrinsic growth rate of the predator shall be negative (if alone, it starves to death); the more preys there are, the more predator grows: $\partial_{n_{\text{prey}}} \mu_{\text{predator}}^{\text{eff}_S} > 0$; the more predator there are, the more preys die: $\partial_{n_{\text{predator}}} \mu_{\text{prey}}^{\text{eff}_S} < 0$. With this convention we add the S-interaction to the development equation as

$$\dot{n}_i = n_i \left(\mu_i - \frac{\mu_i \sum_j a_{ij} n_j}{q} - \sum_j b_{ij} n_j\right) \tag{3.7}$$

so that if $b_{ij} < 0$ i eats j, while if $b_{ij} > 0$ j eats i.

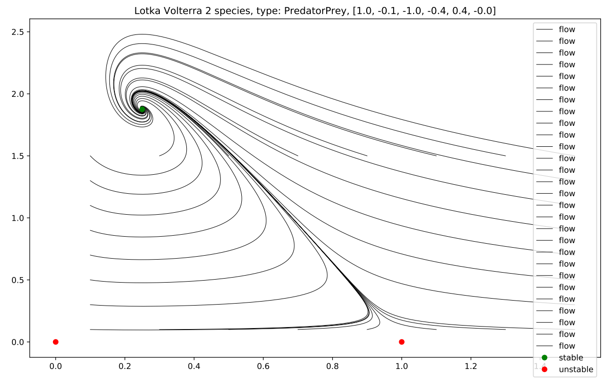

Figure 3.1: Lotka Volterra predator prey system with interior asymptotically stable fixpoint

Example 3.2. We want to model a 2-species pure predator-prey system:

$$\begin{aligned}\dot{n}_1 &= n_1\left(\mu_1 - \mu_1\frac{a_{11}n_1}{q} - \mu_1\frac{a_{12}n_2}{q} - b_{11}n_1 - b_{12}n_2\right)\\ \dot{n}_2 &= n_2\left(\mu_2 - \mu_2\frac{a_{21}n_1}{q} - \mu_2\frac{a_{22}n_2}{q} - b_{21}n_1 - b_{22}n_2\right)\end{aligned}$$

Let species 1 be the predator and species 2 the prey. Then $\mu_1 < 0$ since the predator alone starves to death; $a_{11} = 0$ since the predator does not interact directly with the environment; $a_{12} = 0$ since predator and prey interact only via predation; $b_{11} = 0$ excluding cannibalism; $b_{12} < 0$ since the predator eats the prey. For the prey $\mu_2 > 0$ since alone it grows; $a_{21} = 0$ for the same reason as a_{12}; $a_{22} > 0$ is the predator consumption of the environment resource; $b_{21} > 0$ since the prey is eaten; $b_{22} = 0$ for the same reason as b_{11}. Thus we are left the classical Lotka-Volterra predator-prey equation:

$$\begin{aligned}\dot{n}_1 &= n_1\left(-|\mu_1| + |b_{12}|\, n_2\right)\\ \dot{n}_2 &= n_2\left(\mu_2 - \mu_2\frac{a_{22}n_2}{q} - b_{21}n_1\right)\end{aligned}$$

where all coefficients in $|\cdot|$ are negative and all the others are positive. It is well known that this system can admit an asymptotically stable interior fixpoint; we don't go into details here, and refer to [HS98, p. 17] for a detailed study. A typical phase diagram is shown in fig. (3.1).

For the modeling logic it was convenient to work with dimensionful quantities and to distinguish between S and P coefficients. In the following we won't care about the difference between these interactions, so it is convenient to collect everything in a single interaction matrix. We also change sign convention and notations for compatibility with later results, so that the Lotka-Volterra dynamical system for the evolution of the population densities of N interacting species is written as

$$\boxed{\dot{y}_\mu = y_\mu\left(r_\mu + \sum_\nu a_{\mu\nu}\, y_\nu\right), \quad \mu,\nu = 1,\ldots,N} \tag{3.8}$$

3.3 Population games

So far we considered a population divided in N species, and we looked at the evolution of the distribution of species numbers (or densities) $n(t)$, the point in $\mathbb{R}^N_+$ saying how many individuals each species counts. This point moves according to some dynamics, for example the Lotka-Volterra system, driven by the interaction within and between species. A species is successful if the way it interacts (within and between) enhances its fitness, so that it counts many individuals. What about stability? Usually ecosystems are way too complex to expect that the numbers of individuals in the various species settle down to some constant value. Roughly speaking, we define the equilibrium of a population dynamical system to be a state stable against invasion: once the ecosystem settles down to it, a perturbation in the number of individuals of a single species can not disrupt the status quo. Note that this is an implicit definition!

These ideas can be rephrased in the language of game theory. Instead of gathering the individuals of a population in species according to some common features, assume the existence of a finite set of behaviors the individuals can employ, called *pure strategies*. The interaction happens now through what is called a *game*, and the fitness or well-being of an individual, called *payoff*, depends on the pure strategy he decides to employ. As before, the interaction can be *local*, meaning that the payoff of an individual employing a certain strategy depends on the outcome of a pairwise encounter with another individual; or *global* if no actual pairwise encounter occurs. We stress that the analogy (to be made precise) is between species and *pure* strategies.

Consider $n + 1$ pure strategies; the analogue of a distribution of species numbers is called *strategy* $x \in \mathbb{R}^{n+1}$. For an individual, it is the discrete probability distribution of pure strategies usage: an individual employing the strategy $x = (x_0, \dots, x_n)$ uses the i-th pure strategy with probability x_i. Crucially this means that $x_i \geq 0$ and $\sum_{x_i} = 1$, namely x is a point on the n-simplex: $x \in \Delta^n \subset \mathbb{R}^{n+1}$. An *average population strategy* is equivalently the strategy employed on average by a random individual in the population, and the distribution of pure strategies in the population.

What does it mean now for a *strategy* to be stable? In analogy to the implicit definition of population equilibrium given above, we say that a strategy is *evolutionarily stable* if, whenever all the individuals of the population employ it, a small minority of individuals that start using a different strategy can not invade, i.e. the number of individuals using the different strategy stays small.

When we say that "the number of individuals *stays* small" we are implicitly assuming some underlying dynamics of the average population strategy x, i.e. the analogue of the Lotka-Volterra system for the quantity $x(t) \in \Delta^n$. But what is t? With the duality between population and game setting in mind, we can speak of the strategy played during the next round of a game, the number of individuals born in the next generation, the opinion of a crowd during the next election, the number of fingers humans will have in 2000 years. The considered timescale clearly matters when trying to model the evolution of a population strategy, or of species numbers. On short timescales processes like imitation and learning have to be taken into account; see [HS98, pp. 86, 101] for an introduction to Imitation Dynamics and Adaptive Dynamics. On long timescales mutations may occur, new strategies or new species may arise, and one enters the world of Population Genetics and Game Dynamics, for which we direct the reader to [HS98, p. 233].

We focus here on the timescale in the middle, namely on *inheritance* as the mechanism driving strategies evolution. This assumption is know as *like begets like*, and leads to the *replicator* *equation* describing the evolution of $x(t) \in \Delta$, the average population strategy. Remarkably, this system for $n+1$ strategies, whose dynamics takes place on the n-dimensional simplex, is equivalent to the Lotka-Volterra system for n species in $\mathbb{R}^n_+$, making precise the intuitive analogy between species and strategies outlined above: the diffeomorphism between the interior of $\Delta^n \ni x$ and $\mathbb{R}^n_+ \ni y$

$$y_i = \frac{x^i}{x_0}, \quad i = 1, \ldots, n \tag{3.9}$$

maps the orbits of the replicator system on those of a Lotka-Volterra system (with the conventions of eq. (3.8); see [HS98, p. 77]). To further investigate these ideas (and much more) we refer to the milestone by John Maynard Smith. *Evolution and the Theory of Games*. 1St Edition. 1982.

In the rest of this work we focus on the replicator dynamical system and in particular on the subclass of "zero-sum games", that admit (in some circumstances) an Hamiltonian formulation with respect to the Poisson structure of the simplex derived in the previous chapter.

3.3.1 Hawks and Doves

Before moving to definitions we propose a classical example to gain some intuition about the game theoretical framework. In nature animals belonging to the same species often engage in non lethal fights to establish dominance, with the ultimate Darwinian goal of reproduction. To maximize its payoff an individual should carefully decide whether to escalate a fight or not: retiring too often means appearing weak, thus loosing appeal; fighting too often may result in serious injuries.

Consider two pure strategies: a *dove* strategist shows off and provokes the opponent, but quits if the opponent actually escalates the fight; a *hawk* strategist fights until his or his opponent defeat, no matter what. Assume that avoiding a fight has no consequences, winning a fight increases the winner's fitness by a gain G, and loosing a fight decreases the loser's fitness by a cost C.

For any individual in this population a fight can have four possible outcomes: he can play hawk or dove, and his opponent as well. If a dove meets a dove both will show off and try to scare the opponent, but one of them will eventually quit without fighting, so that the payoff to play dove against dove is on average $G/2$. A dove meeting a hawk always quits, so its payoff is 0; a hawk meeting a dove always wins G; and an hawk meeting an hawk gains G half of the times and looses C half of the times, so that the average payoff is $(G-C)/2$. This is encoded in the *payoff matrix*

	meeting a dove	meeting a hawk
a dove gets	$G/2$	0
a hawk gets	G	$\frac{G-C}{2}$

(3.10)

A *strategy* for this game is a point $x = (x_0, x_1) \in \Delta^1$: a player using x plays *dove* with a probability $x_0 = 1 - x_1$, and *hawk* with probability x_1.

The payoff g for a player using the strategy $x \in \Delta^1$ against a player using the strategy $y \in \Delta^1$ is the sum of the products of the probabilities of a particular outcome, times the payoff of the corresponding outcome:

$$
\begin{aligned}
g(x,y) &= x_0y_0\,G/2 + x_0y_1\,0 + x_1y_0\,G + x_1y_1\,\frac{G-C}{2} \\
&= (1-x_1)(1-y_1)\,G/2 + x_1(1-y_1)\,G + x_1y_1\,\frac{G-C}{2}
\end{aligned}
$$

For a fixed opponent's strategy y this is just a linear function in x, in particular a bundle of straight lines. It's worth studying its property in general, before going back to this example.

3.3.2 Slope, focus, Nash

The above setup with two generic pure strategies E_0, E_1 and generic payoff coefficients gives the matrix

	meeting E_0	meeting E_1
E_0 gets	g_{00}	g_{01}
E_1 gets	g_{10}	g_{11}

(3.11)

A strategy is $x = (x_0, x_1) \in \Delta^1$; x_i is the probability to play E_i, $i = 0, 1$. The payoff of an x-strategist against a y-strategist is

$$
g(x,y) = g_{00}x_0y_0 + \cdots + g_{11}x_1y_1 = \sum_{ij} g_{ij}x_iy_j \tag{3.12}
$$

Fix y and replace $x_1 = 1 - x_0$ to get a function of x_0 as $\text{const}_1 x_0 + \text{const}_2$:

$$
\begin{aligned}
g(x,y) &= g_{0j}x^0y^j + g_{1j}(1-x^0)y^j \\
&= x^0\left(g_{0j}y^j - g_{1j}y^j\right) + g_{1j}y^j
\end{aligned} \tag{3.13}
$$

summation over repeated indices understood. The term between brackets is the angular coefficients of the lines bundle. If it is zero, $g(x,y) = g_{1j}y^j$ does not depend on x. This leads to a very important definition, to be made precise in the following: the **slope** strategy is the one that flattens the opponent's payoff. If *you* play slope, *my* (average) payoff does not change, no matter what I do.

Note that the payoff function is not symmetric: $g(x, x_{\text{slope}})$ per definition does not depend on x; but this does not say anything about $g(x_{\text{slope}}, x)$. In other words if I play slope your payoff is flattened, but my payoff changes as you change your strategy.

The slope point solves $\sum_j g_{0j}y^j - g_{1j}y^j = 0$, namely ($s$ stands for *slope*)

$$
x_s^0 = \frac{g_{11} - g_{01}}{g_{11} + g_{00} - g_{01} - g_{10}}, \quad x_s^1 = 1 - x_s^0 \tag{3.14}
$$

Note that, depending on the coefficients of g, the slope point may not exist; and if it exists it may not belong to the simplex, or it may belong to its boundary (i.e. be a vertex), or it may belong to its interior.

The concept mirroring a slope strategy is that of *focus* strategy: the one fixing the payoff of the player using it, no matter what the opponent does. A bundle of lines has indeed in general a focus point where all lines converge. To find it set $g(x, y_a) = g(x, y_b)$ for any choice of $y_a, y_b \in \Delta^1$ and solve for x; a smart choice is $g(x, x_s) = g(x, (1,0))$. The result is called **focus** strategy. The actual expression of the focus point is not enlightening nor useful in the following, just a

```python
# Tune payoff matrices to represent all the possible cases

payoff_matrix_1 = np.array( [[2, 1],
                             [1, 0]] ).reshape(4)

payoff_matrix_2 = np.array( [[0, 0],
                             [1, 1]] ).reshape(4)

payoff_matrix_3 = np.array( [[1, 1],
                             [1, 1]] ).reshape(4)

payoff_matrix_4 = np.array( [[2, 1],
                             [0, 0]] ).reshape(4)

payoff_matrix_5 = np.array( [[1, 0.2],
                             [0, 1]] ).reshape(4)

payoff_matrix_6 = np.array( [[0, 0],
                             [2, 1]] ).reshape(4)

payoff_matrix_7 = np.array( [[0, 1.2],
                             [1, 0]] ).reshape(4)
```

Figure 3.2: Payoff matrices corresponding to the seven cases discussed in the text

fraction similar to x_s^0, but keep in mind its meaning: If *I* play focus, *my* (average) payoff does not change, no matter what you do.

A very important notion, that will turn out to be strictly related to that of slope strategy, is that of **Nash** strategy, defined as a strategy that is *a best reply to itself*: no strategy can do better than x_N against x_N, but there may be strategies doing just as well, i.e. $g(x_N, x_N) \geq g(x, x_N)$ for all possible strategies. The expression of the slope strategy eq. (3.14) allows to fully catalog the Nash strategies for this class of games.

Remark 3.3. The notion of *set of best replies* will be made precise in the following.

Let $\beta(y)$ be the set of best replies to y. If the angular coefficient of $g(x, y)$ is always positive, for any fixed y the best thing to do is to have x^0 as big as possible, so the best reply to any y is $\beta(y) = \{(1, 0)\}$. Since a Nash strategy has to belong to the set of its own best replies, the only possible Nash strategy is $(1, 0)$ itself.

Similarly if the angular coefficient of $g(x, y)$ is always negative the best thing to do against any y is to have x^0 as small as possible, so $\beta(y) = \{(0, 1)\}$ for all y, and the only Nash strategy is $(0, 1)$.

If the angular coefficient is identically zero the situation is rather uninteresting; but if the angular coefficient is zero because the slope strategy exists and $y = x_s$, then the set of best replies to it is the whole Δ^1: $g(x, x_s)$ is constant, in particular maximal for all choices of x. In this case the slope strategy is a Nash strategy, since it belongs to its set of best replies.

A careful analysis of the coefficients of g appearing in eq. (3.14) reveals 7 possible scenarios. Denote by D the denominator of x_s^0 and $A := g_{01} - g_{11}$:

- $D = 0$; no slope point exists
  - **(1)** $A > 0$, positive angular coefficient, $(1, 0)$ Nash;
  - **(2)** $A < 0$, negative angular coefficient, $(0, 1)$ Nash;
  - **(3)** $A = 0$, identically zero angular coefficient, every point is slope and Nash.
- $D > 0$

    - **(4)** $A \geq 0$, positive angular coefficient, $(1, 0)$ Nash;
    - **(5)** $A < 0$, $(0, 1)$ and $(1, 0)$ Nash, x_s Nash if it belongs to $[0, 1]$.
- $D < 0$
    - **(6)** $A \leq 0$, negative angular coefficient, $(0, 1)$ Nash;
    - **(7)** $A > 0$, only x_s is Nash

Fig. (3.2) shows seven payoff matrices realizing these cases, and figure (3.3) the corresponding payoff plots. The different plotted lines correspond to a different fixed value of the second player's strategy (q in the plots is y^0 in the notation employed here, and p is x^0). Consider for example the first case: The angular coefficient is positive for all values of y and the best strategy for the first player is always $(1, 0)$, which is the unique Nash strategy. When existing, slope and focus strategies are plotted both as employed by the first and the second player.

3.3.3 Evolutionary stability

Notation 3.4. The letters p and q appearing in the plots of this section correspond to the 0-th component of a point $x = (x^0, x^1) \in \Delta^1$.

Let's take a closer look at these seven cases. Case 3 is degenerate: no matter what both players do, the average payoff is just always the same. The cases 1, 2, 4 and 6 are not very interesting either: the Nash point is always a vertex of the simplex. As discussed above, in these cases not only it is a best reply to itself, but it is the *unique* one: $\beta(x_N)$ contains only x_N. In this case we speak of *strict* Nash equilibrium.

The comparison between the 5th and 7th cases allows to gain some more intuition about the focus strategy and the slope strategy. The former is known as *maximin*: look at the dotted vertical red line. If player one uses the focus strategy he fixes his payoff, no matter what the second player does, as the definition of the focus strategy requires. This means that player one gets *the best of the worst case scenario*: if he played anything different, the second player could adopt a strategy reducing the payoff of the first player. Whether the second player is interested in doing this or not is a different story.

More importantly, as discussed above, when the slope strategy exists it is Nash. Look at the horizontal green lines: if player two plays the slope strategy, the payoff of player one is flattened. The crucial question is: *does player one has any interest in playing the Nash strategy himself?* The answer is: it depends.

In a population game setting as the one we are considering the second entry of the payoff function $g(x, y)$ can be thought as the average population strategy: an x strategist locally interacting meets random individuals playing y on average; an x strategist globally interacting has a payoff that depends on the population state as a whole. The interpretation holds in both scenarios. In this framework we will see that the implicit definition of evolutionarily stable strategy we gave - if the whole population adopts it, a mutant minority cannot invade - is equivalent to

1. the strategy is Nash, and
2. given an alternative best reply to it, this alternative strategy performs *against itself* worst than how the Nash strategy performs against it.

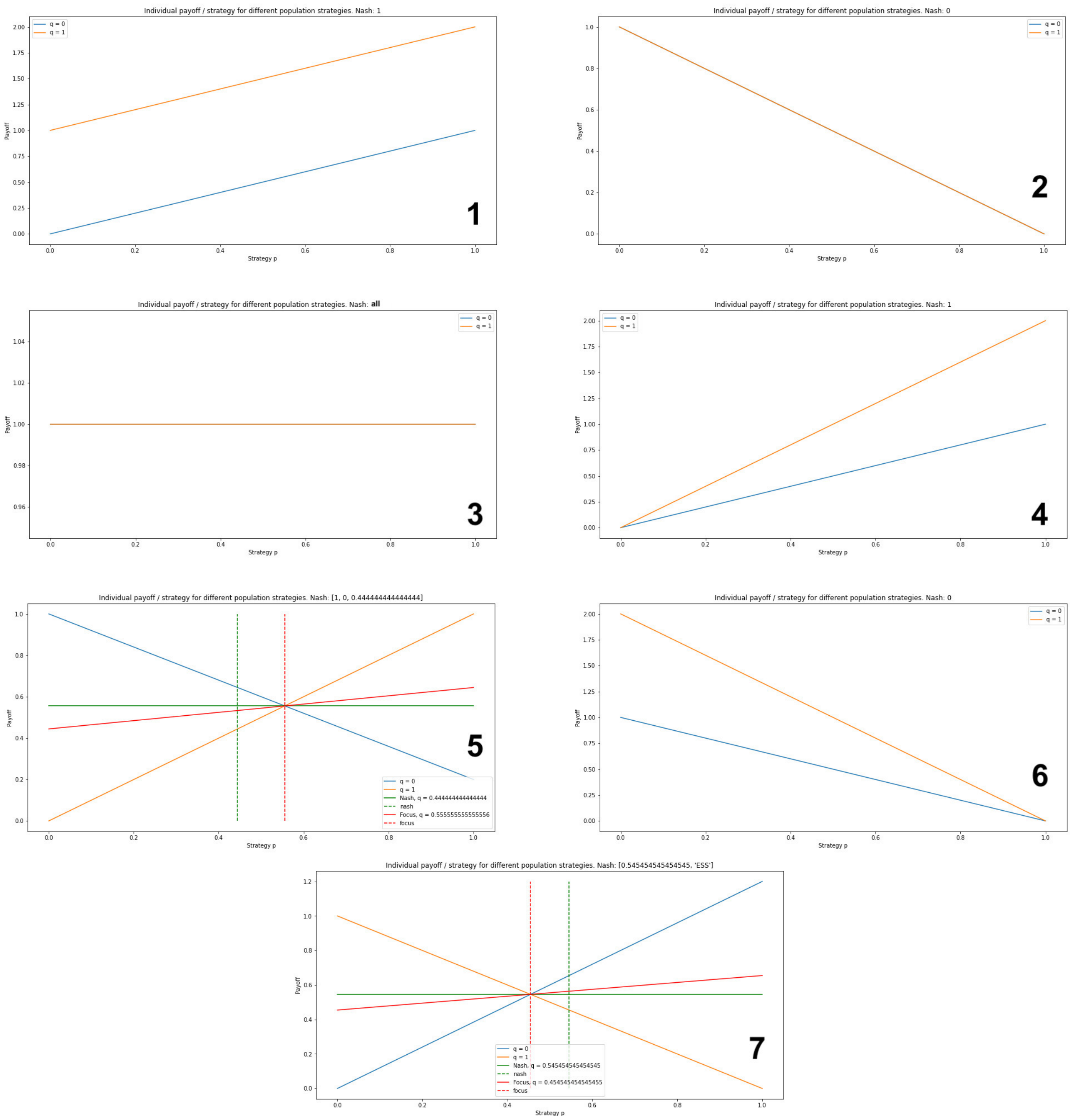

Figure 3.3: Payoff of the first player for fixed strategy of the second player in the seven cases discussed in the text. The last case, to which the Hawks and Doves game belongs, is the only one with an interior evolutionarily stable strategy.

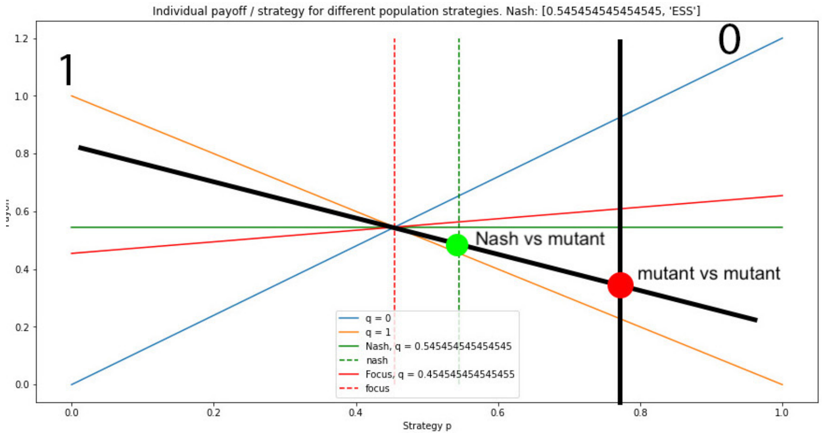

Figure 3.4: The Nash strategy is evolutionarily stable. Mutants not adopting it do worst against themselves than natives adopting it do against mutants.

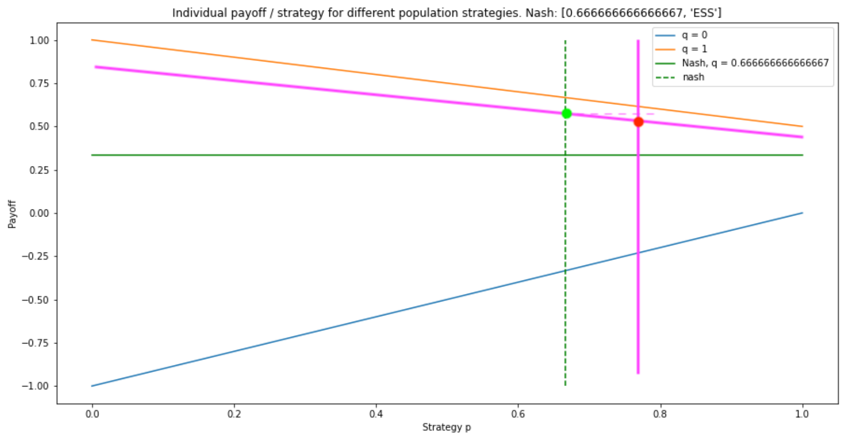

Figure 3.5: Payoff and evolutionarily stable strategy for the hawks and doves games with $G = 1$, $C = 3$.

With this definition in mind it is clear that the Nash strategy of case 7 is evolutionarily stable, while the interior Nash strategy of case 5 is not. Consider indeed figure (3.4). Imagine that player 1 does *not* adopt the Nash strategy, but rather some other strategy closer to 1 (black vertical line). When he meets some other individual adopting the same non-Nash strategy, his payoff will be given by the black diagonal line, so he would have better sticked to the Nash strategy. Convince yourself that the situation in case 5 is reversed.

Note that this interpretation makes sense strictly in a population game setting: the notion of Nash strategy is well defined also in a 1 versus 1 game between two players without any concept of population; but the notion of *stability* defined by condition (2) above strictly depends on the interaction between individuals. What happens indeed is that *mutants check their own growth* via a self-interaction process: if a Nash strategy is stable, mutants not adopting it do not underperform against natives; but whenever a mutant meets another mutant he realizes his payoff would be higher if he had used the stable Nash strategy.

Back to hawks and doves The hawks and doves payoff matrix introduced at the beginning of this section belongs to the 7th group if $G < C$, i.e. if the gain of winning a fight is smaller than the cost of losing it. From eq. (3.14) the slope strategy is

$$x_s = \left(1 - \frac{G}{C}, \frac{G}{C}\right) \tag{3.15}$$

Since this is an interior point of Δ^1 for $G < C, G > 0, C > 0$, it is a Nash strategy. Furthermore it is stable in the sense of eq. (2): apply the same graphical logic to fig. (3.5). Recall that the component plotted is the 0-th one, corresponding to the dove strategy. Escalating less often, i.e. playing more dove, results in a vertical line on the right hand side of the green vertical line (vertical pink line), and correspondingly a payoff function tilted toward the orange line (diagonal pink line), so a Nash strategist does against this "hyperdove" better than what the "hyperdove" does against its own type - it is better to fight more often. The same reasoning applies for a mutant fighting too much, corresponding to a vertical line on the left hand side of the green vertical line, and a payoff tilted toward the blue diagonal line. The Nash strategy is the sweet spot corresponding to the optimal frequency of engaged fights, which in this case is precisely the ratio G/C between the gain of a won fight and the cost of a lost one.

3.4 Normal form games

We now need some definitions to formalize the ideas of the previous section; we follow [HS98, p. 57].

Definition 3.5. An N-*normal form game* (Δ^N, g) is the collection of

- a set of $N+1$ *pure strategies* $\{R_0, \dots, R_N\}$;
- a *game space* $\Delta^N \in \mathbb{R}^{N+1}$
- a *population* of interacting individuals;
- a *payoff function*

$$\begin{aligned} g : \Delta^N \times \Delta^N &\to \mathbb{R} \\ p, q &\longmapsto g(p,q) \end{aligned} \tag{3.16}$$

A point in game space is called a *strategy*, and $g(p,q)$ is the *payoff of the strategy* p *against the strategy* q.

It is hard, and useless, to give a precise definition of *pure strategy*. We just think of it as an entity in some abstract strategy space, that can correspond to a behavior, a physical trait, a belief, ..., of an individual in the population. This definition is very general, allowing for different interpretations depending on the circumstances.

A strategy can be thought of as the discrete probability distribution of pure strategies usage for a single individual (an individual employing the strategy $p = (p_0, \dots, p_N)$ uses the i-th pure strategy with probability p_i), or as the distribution of pure strategies in the population (in a population with average strategy $p = (p_0, \dots, p_N)$ a fraction p_i of individuals uses on average the i-th pure strategy).

The $N+1$ vertices $\{e_0, \dots, e_N\}$ of the game space clearly correspond to the pure strategies, meaning that an individual playing the strategy e_i uses the pure strategy R_i 100% of the times, or that in population with average strategy e_i 100% of the individuals use on average the pure strategy R_i. For this reason we shall often refer to $\{e_i\}$ as the pure strategies.

3.4.1 The payoff function

Geometrically, as a point in $\mathbb{R}^{N+1}$, a strategy is a linear combination of pure strategies:

$$p = p^i\, e_i \in \Delta^N \tag{3.17}$$

summation over repeated indices understood. With the double interpretation of strategy in mind, $g(p,q)$ can be the payoff of a p-strategist against a q-strategist (local interaction), or the payoff of a p-strategist in a population with average strategy q (global interaction). In both cases it is reasonable to assume that g is linear in the first argument:

$$\begin{aligned} g(p,q) = g(p^i e_i, q) &= \sum_i (\text{prob. to use } i\text{-th pure strategy}) \cdot (\text{payoff of } i\text{-th pure strategy vs } q) \\ &= p^i\, g(e_i, q) =: p^i g_i(q) \end{aligned} \tag{3.18}$$

On the other hand the function just defined

$$\begin{aligned} g_i : \Delta^n &\to \mathbb{R} \\ q &\longmapsto g_i(q) \end{aligned} \tag{3.19}$$

representing the payoff of i-th pure strategy against the strategy q needs *not* be linear in the case of global interaction, as shown in the following example:

Example 3.6 (Sex ratio)**.** Consider a population of individuals with two pure strategies defined by *having male offspring* and *having female offspring*. The success of a strategy depends on the state of the population as a whole (i.e. the ratio of males and females), not on the outcome of random pairwise encounters between individuals. This is a 1-normal form game with global interaction and payoff $g(p,q)$ equal to the success of the strategy p in a population with a sex ratio q, measured in number of descendants. A simple argument[4] leads to

$$g(p,q) = \frac{p^0}{q^0} + \frac{p^1}{q^1} \quad \text{i.e.} \quad g_i(q) = \frac{1}{q^i} \tag{3.20}$$

which is readily interpreted: the more males there currently are in a population, the less convenient it is to have male offspring, and the same for females.

Thus we consider a payoff function linear in the first argument and possibly nonlinear in the second. If the payoff function happens to be linear also in the second argument we define a *payoff matrix*; this is the case of games based on pairwise random encounter, like Hawks and Doves, in which it makes sense to speak of an "opponent's strategy":

$$\begin{aligned} g_i(q) &= \sum_j (\text{payoff of } i\text{-th pure strategy vs } j\text{-th pure strategy}) \cdot (\text{prob. opponent uses } j\text{-th pure strategy}) \\ &= g_i(e_j)\, q^j =: g_{ij}\, q^j \end{aligned} \tag{3.21}$$

Definition 3.7. The *payoff matrix* of an N-normal form game with bilinear payoff function g and pure strategies $\{e_0, \dots, e_N\}$ is the square $(N+1)$-matrix

$$g_{ij} \coloneqq g(e_i, e_j), \quad i,j = 0, \dots, N \tag{3.22}$$

Example 3.8. See equations (3.10), (3.11).

[4] HS98, p. 60.

Remark 3.9. Whenever we speak of a *bilinear* normal form game we mean that not only the payoff function is linear in the first argument (which is always true), but also in the second. Similarly, *nonlinear* means linear in the first argument and nonlinear in the second argument. **In the following we shall consider only bilinear normal form games**, since our final goal it to study zero-sum games, that are indeed bilinear; but we stress that most of the given results hold also in the nonlinear case, taking care of adding a notion of *locality* to some definitions. We mention only this major difference between the nonlinear and the bilinear cases: in the latter if an evolutionarily stable strategy exists *in the interior* of the game space, than no other Nash strategy can exist; on the other hand, in the nonlinear case more evolutionarily stable strategies can coexist in the interior of the game space. This is a consequence of the definition of *strength* of a strategy, which is equivalent to that of evolutionarily stable strategy *locally* in the nonlinear case and *globally* in the bilinear case. The boundary can host multiple evolutionarily stable strategies both in the bilinear and nonlinear case. See [HS98, pp. 63,65].

3.4.2 Nash strategies and Evolutionarily stable strategies

The concept of *slope strategy* readily generalizes to the current framework. Consider an N-normal form game with payoff function g, and replace $p^0 = 1 - \sum_{\mu=1}^{n} p^\mu$ in $g(p,q)$:

$$\begin{aligned} g(p,q) &= p^i g_i(q) = p^0 g_0(q) + p^\mu g_\mu(q) \\ &= g_0(q) + \sum_{\mu=1}^{N} p^\mu \big(g_\mu(q) - g_0(q) \big) \end{aligned} \tag{3.23}$$

If the term between brackets of the right hand side vanishes $g(p,q)$ does not depend on p, making q a slope strategy:

Definition 3.10. Consider an N-normal form game with payoff function g. A *slope strategy* is a strategy $q \in \Delta^N$ such that

$$g_0(q) = g_1(q) = \cdots = g_N(q) \quad \text{so that} \quad g(p,q) = g_0(q)\, \forall p \in \Delta^N \tag{3.24}$$

Remark 3.11. A slope strategy for a normal form game may not exist, may exist on the boundary of the simplex, or may exist in the interior of the simplex.

A generic strategy may fulfill the slope condition only partially:

Definition 3.12. Consider an N-normal form game with payoff function g. A strategy p is said to have *slope support* if all of its nonzero components fulfill the slope condition (3.24), namely if

$$g_i(p) = g_j(p) \quad \forall i,j \in \operatorname{supp}(p) \tag{3.25}$$

Note that if p has slope support nothing is said about $g_i(p)$, that may have any value, positive, zero or negative, for $i \notin \operatorname{supp}(p)$.

Definition 3.13. Consider an N-normal form game with payoff function g and a strategy $q \in \Delta^N$. The *set of best replies to* q, denoted by $\beta(q)$, is the maximal level set of $g(\cdot,q)$: a strategy p belongs to $\beta(q)$ if and only if $g(p,q)$ assumes its maximal possible value for q fixed, i.e.

$$\beta(q) = \left\{ p \in \Delta^N : g(p,q) = \max_{p' \in \Delta^N} g(p',q) \right\} \tag{3.26}$$

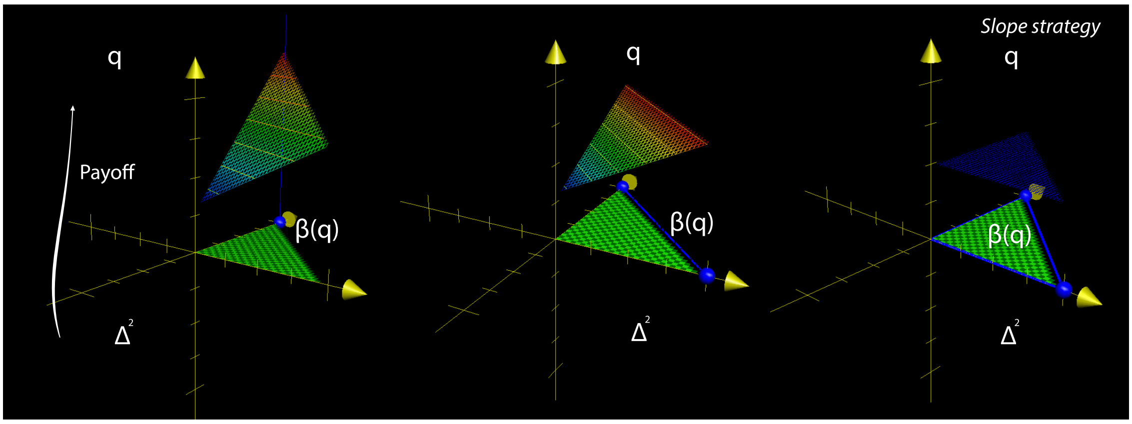

Figure 3.6: The set of best replies is a nonempty union of faces of the game space.

Proposition 3.14. *Consider an N-normal form game. The set of best replies to any point in the game space is a nonempty union of faces of Δ^N, containing a vertex at least and the whole simplex at most.*

Proof sketch. The idea is the same we adopted in deriving the slope strategy eq. (3.13): since the payoff function is linear in the first argument, once the second argument is fixed the payoff function is just a linear function from Δ^N to $\mathbb{R}$, whose graph is a "tilted N-simplex", no matter what the dependency on the second argument is. Recall that a 1-simplex is just a closed segment, so for a 1-normal game all the colored horizontal and diagonal lines - "tilted 1-simplices" - of Fig. (3.3) are precisely graphs of $g(p, q)$ for different fixed values of q. In that case we saw, right after Remark (3.3), that the set of best replies to a generic point was a vertex, and the set of best replies to the slope strategy - when it exists - was the whole Δ^1.

The situation for Δ^2 is similar and is better understood graphically. From eq. (3.23), for a fixed value of q

$$g(p, q) = a + b\,p^1 + c\,p^2$$

where a, b, c are constant numbers determined via $g_i(q)$. Follow on Fig. (3.6): if q is such that $b \neq c \neq 0$ the graph of the 2-simplex is maximally tilted, and $\beta(q)$ is one of the three vertices; if q is such that $a = b \neq 0$, a whole edge of the graph of the 2-simplex has the same height; this may be less than the height of the vertex not belonging to the edge, in which case $\beta(q)$ is that vertex; or more than the height of the vertex not belonging to the edge, in which case $\beta(q)$ is the corresponding whole edge of Δ^2. Finally, if $b = c = 0$, namely if q is the slope strategy, the whole graph has the same height, and $\beta(q)$ is the whole Δ^2. In higher dimension the logic is the same. □

Definition 3.15. Consider an N-normal form game with payoff function g. A *Nash strategy* is a strategy p that belongs to its set of best replies: $p \in \beta(p)$, i.e.

$$g(p, p) \geq g(q, p) \quad \forall q \in \Delta^N \tag{3.27}$$

A *strict Nash strategy* is a strategy p that is its *unique* best reply: $\beta(p) = \{p\}$, i.e.

$$g(p, p) > g(q, p) \quad \forall q \in \Delta^N, q \neq p \tag{3.28}$$

An *alternative best reply* to a Nash strategy p is a strategy q that belongs to $\beta(p)$ and is different from p, i.e. such that $g(p, p) = g(q, p)$ with $q \neq p$. In this case we write $q \in \hat{\beta}(p)$.

Remark 3.16. Strict Nash strategies have no alternative best replies and can only be vertices of the game space.

Note that if p is a Nash strategy then $p \in \beta(p)$ implies that $\beta(p)$ contains the **closure** of the face p belongs to. This observation shows that slope strategies and Nash strategies are strictly related:

Proposition 3.17 (Slope strategies and Nash strategies)**.** *Consider an N-normal form game with payoff function g. Then*

- *If a slope strategy exists then it is a Nash strategy;*
- *A Nash strategy has slope support;*
- *An interior strategy is Nash if and only if it is slope.*

Proof. For the first point, per definition $g(p, p_{\text{slope}})$ is constant for all $p \in \Delta^N$ and the set of best replies to the slope strategy is the whole simplex, to which the slope strategy belongs.

For the second point, if a point is *strictly* Nash there is nothing to prove. Consider then a Nash strategy which is not a vertex: $p \in \beta(p)$ implies that the closure of the face x belongs to is a subset of $\beta(p)$. Then all the points in the closure of the face (except p itself) are alternative best replies; in particular the vertices are alternative best replies:

$$g(p, p) = g(e_i, p) = g_i(p) \quad \forall i \in \text{supp}(x)$$

which implies that p has slope support.

The last point is an immediate consequence of the first two: an interior Nash strategy has slope support, and its support is full, so it is a slope point; conversely a slope strategy is always Nash. □

The crucial property making a Nash strategy *stable* is the one introduced in section (3.3.3).

Definition 3.18. Consider an N-normal form game with payoff function g. A Nash strategy $p \in \Delta^N$ is called *stable* if

$$g(p, q) > g(q, q) \quad \forall q \in \hat{\beta}(p) \tag{3.29}$$

namely if its payoff against any of its alternative best replies is higher than the payoff the alternative best reply has *against itself.*

Remark 3.19. A *strict* Nash strategy p is always trivially stable since $\hat{\beta}(p) = \emptyset$.

We are finally in the position to relate the game-theoretical notion of Nash equilibrium to the evolutionary notion of equilibrium hinted in section (3.3):

Definition 3.20. Consider an N-normal form game with payoff function g. A strategy is called *evolutionarily stable* (*ES*) if, whenever most of the population is using it, a minority starting to use a different strategy cannot invade, i.e. the payoff of using the ES strategy is higher than the payoff of using any other strategy if most of the population uses the ES strategy. Formally, $p \in \Delta^N$ is an ES strategy if

$$g\left(p, \epsilon q + (1-\epsilon)p\right) > g\left(q, \epsilon q + (1-\epsilon)p\right), \quad \forall q \neq p, q \in \Delta^N \tag{3.30}$$

for all positive ϵ smaller than some positive *invasion threshold.*

Remark 3.21. The inequality needs *not* to hold for $\epsilon = 0$: if this was the case p would be a *strict* Nash strategy, which is not required for a strategy to be ES.

Theorem 3.22. *Consider a N-normal form game. A strategy p is ES if and only if it is a stable Nash strategy.*

Proof. Exploiting the bilinearity of the payoff function the expression defining an ES strategy can be rewritten as

$$\epsilon \left[g(p,q) - g(q,q)\right] + (1-\epsilon)\left[g(p,p) - g(q,p)\right] > 0 \quad \forall q \neq p, 0 < \epsilon < k \tag{3.31}$$

If the $\mathcal{O}(1)$ term is negative the inequality cannot hold, so the $\mathcal{O}(1)$ term must be non-negative, which is possible if and only if p is a Nash strategy.

If the $\mathcal{O}(1)$ term is positive we don't care about the $\mathcal{O}(\epsilon)$ term; but the $\mathcal{O}(1)$ term is positive if and only if p is a *strict* Nash strategy, hence stable.

If the $\mathcal{O}(1)$ term is zero, i.e. if p is a non-strict Nash strategy and q is an alternative best reply, then the $\mathcal{O}(\epsilon)$ term must be positive, which is true if and only if p is stable. □

This is conceptually a remarkable result, showing that the mechanism providing evolutionary stability to a strategy is the self interaction process - namely the stability property of a Nash equilibrium - discussed in section 3.3.3. Both the definition of ES strategy and that of stable Nash strategy are though quite unpractical to check; luckily there exists a third equivalent condition of much easier applicability.

Definition 3.23. Consider an N-normal form game with payoff function g. A strategy p is called *locally strong* if

$$g(p,q) > g(q,q) \quad \forall q \neq p \quad \text{locally} \tag{3.32}$$

i.e. for all q in some neighborhood of p in Δ^N.

Proposition 3.24. *Consider an N-normal form game with payoff function g. A strategy is ES if and only if it is strong.*

Proof. See [HS98, p. 63]. □

Thus the concepts of stable Nash strategy, evolutionarily stable strategy and locally strong strategy are equivalent. An immediate but powerful corollary of this fact is that interior ES strategies are unique:

Proposition 3.25. *Consider an N-normal form game with payoff function g. If p is an* ***interior*** *ES strategy than no other Nash strategy (and in particular no other ES strategy) can exist.*

Proof. An interior Nash strategy p is slope, so $\beta(p) = \Delta^N$ and the stability property of the Nash point (3.29) becomes

$$g(p,q) > g(q,q) \quad \forall q \neq p \tag{3.33}$$

meaning that the strength condition (3.32) holds *globally*, so that no other Nash point except p can exist. □

Example 3.26 (Hawks and Doves revisited)**.** Let's have a look at the Hawks and Doves game under this new light. The payoff matrix is

$$\begin{pmatrix} G/2 & 0 \\ G & \frac{G-C}{2} \end{pmatrix} \tag{3.34}$$

The first thing to do is to check the existence of a slope strategy with eq. (3.24): a simple calculation shows that

$$p_s = \left(1 - \frac{G}{C}, \frac{G}{C}\right) \tag{3.35}$$

as in eq. (3.15). For $G < C$ this is an interior slope, hence Nash, strategy. From graphical inspection in the previous section we concluded that it is evolutionarily stable; indeed all we have to check is whether the strength condition (3.32) holds around p_s. This is equivalent to

$$\frac{(Cp^1 - G)^2}{2C} > 0 \tag{3.36}$$

which clearly holds, actually globally, as expected from the last proposition, confirming the evolutionarily stability of the slope point and its uniqueness.

3.5 Replicator dynamics

In the previous section everything was static: if a population average strategy is evolutionarily stable, or equivalently a stable Nash strategy, then no mutants can invade. But how does a population reach an ES strategy? To answer this question we have to model the evolution of the average population strategy driven by the interaction between the individuals of the population, that is a dynamical system, i.e. a vector field, on the simplex.

Consider an n-normal form game with payoff function f; in this section we will speak equivalently of "payoff" or "fitness". The first basic assumption we make to model the evolution of the average population strategy $x(t) \in \Delta^n$ is *nothing from nothing*[5]: $\dot{x}^i \propto x^i$, thus preventing the appearance of strategies or traits that are initially absent. Secondly we assume that the success of a strategy, i.e. its effective growth rate $\dot{x}^i/x^i$, is measured as the difference between the payoff or fitness of the strategy i in the current population state, namely $f_i(x)$, and *the average population fitness*. This quantity is the answer to the question "how does it feel to be an average member of the population nowadays?" In other words, it is the fitness of an individual employing the current average population strategy (giving $f(x, \cdot)$), interacting (locally or globally) within the population (giving $f(\cdot, x)$):

$$\bar{f}(x) \coloneqq f(x, x) = \sum_j x^j f_j(x) \tag{3.37}$$

With no further hesitation we present the dynamical system for the evolution of the average population strategy $x \in \Delta^n \in \mathbb{R}^{n+1}$ for an n-normal form game with payoff function f based on these assumptions, known as *replicator system*:

$$\dot{x}^i = x^i \left(f_i(x) - \bar{f}(x)\right), \quad i = 0, \dots, n \tag{3.38}$$

[5] Equations aiming at modeling *mutations* shall first of all remove this constraint; see for example [HS98, p. 265] for an introduction to mutation and recombination dynamics.

3.5.1 The replicator vector field

The replicator vector field for an n-normal form game is $X^i(x)\,\partial_i$ in $\mathbb{R}^{n+1}$ with

$$X^i(x) = x^i\left(f_i(x) - \bar{f}(x)\right), \quad i = 0, \dots, n \tag{3.39}$$

For the replicator dynamical system to be well defined this vector field shall be tangent to Δ^n. Recall the submersion (2.3) defining the faces of the simplex as level sets: the tangent space to a face at a point is the kernel of the differential of this submersion at the point, so $X \in \tau(\mathbb{R}^{n+1})$ is tangent to a d-face iff $\mathrm{d}_x S^d(X_x) = 0$ for all the points of the face. Recalling the structure of the Jacobian of the submersion eq. (2.4) we have (up to a permutation of the coordinates)

$$d_x S(X_x) = \left(\begin{array}{c|c} 1 \cdots 1 & 1 \cdots 1 \\ \hline \mathbb{1} & \mathbf{0} \end{array}\right) \begin{pmatrix} X^0 \\ \vdots \\ X^n \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^n X^i \\ X^0 \\ \vdots \\ X^{n-d-1} \end{pmatrix}$$

All the components of the replicator vector field fulfill $X^i(x) \propto x^i$, so $X^i(x)\Big|_{\text{face}} = 0$ since all the points on the considered d-face have per definition the considered $n-d$ components equal to zero. It remains to be shown that the sum of all the components of the replicator vector field is zero on any face, i.e. on the whole simplex:

$$\sum_{i=0}^n X^i = \sum_{i=0}^n x^i f_i - \bar{f}\sum_{i=0}^n x^i = \bar{f}\left(1 - \sum_{i=0}^n x^i\right) \tag{3.40}$$

per definition of $\bar{f}$, so clearly

$$\left(\sum_{i=0}^n X^i(x)\right)\Bigg|_{\Delta^n} \equiv 0 \tag{3.41}$$

Similarly it can be shown that the same result holds summing only over the components corresponding to the support of a face and restricting to the face:

$$\left(\sum_{i\in\text{supp(face)}}^n X^i(x)\right)\Bigg|_{\text{face}} \equiv 0 \tag{3.42}$$

where the support of a face is obviously the support of any point in the face.

This proves that the replicator vector field is tangent to all the faces of the simplex, that are Poisson submanifolds of $\mathbb{R}^{n+1}$ endowed with the Poisson structure $\{\cdot,\cdot\}_\Delta$ derived in the previous chapter; it must then be possible to write it in terms of the actual degrees of freedom of any considered face.

Notation 3.27. In the following Greek indices run over $0, \dots, n$ and Latin indices over $1, \dots, n$.

Consider for example the interior $\mathring{\Delta}^n$ with coordinates $(u^1, \dots, u^n)$. As a submanifold of $\mathbb{R}^{n+1} \ni x = (x^0, \dots, x^n)$ it is parametrized by

$$\begin{cases} x^0 = 1 - \sum_j u^j \\ x^i = u^i \end{cases} \tag{3.43}$$

so that $\partial_{u^i} = \partial_{x^i} - \partial_{x^0}$, with u belonging to the (open) *parameters space*, the open subset of $\mathbb{R}^n$ such that

$$u^i > 0, \quad \sum_i u^i < 1 \tag{3.44}$$

Remark 3.28. The closure of this space is just the projection of $\Delta^n \subset \mathbb{R}^{n+1}$ in $\mathbb{R}^n$; this is what is actually plotted in Fig (2.1) for the "3"-simplex, in case you wandered about the quotes.

The idea is to replace $x^0 = 1 - \sum_i x^i$ and note that the ∂_{x^0} term provides one term for each ∂_{x^i} term that allows X to be factored as

$$\begin{aligned} X &= X^\mu \partial_\mu = x^\mu \left(f_\mu(x) - \bar{f}(x)\right) \\ &= x^i \left(f_i(x) - f_0(x) - \sum_k x^k \left(f_k(x) - f_0(x)\right)\right) \underbrace{\left(\partial_{x^i} - \partial_{x^0}\right)}_{\partial_{u^i}} \end{aligned} \tag{3.45}$$

For many practical applications, analytical and numerical, this reduced equation is much more convenient to use. It contains all the information of the full system but operates in parameters space, hence effectively with one dimension less. It is for example computationally less expensive to solve the replicator flow ODE in parameters space and to map it on the simplex with eq. (3.43); and the reduced equation clearly shows that a slope strategy is a fixpoint for the replicator dynamical system. We will come back to this issue in a second. On the other hand, when the explicit bilinear nature of the interaction is taken into account it is convenient to work with the full system, because the reduced system contains the mixed terms $f_i(x) = f_{i\mu}x^\mu$, cumbersome to work with.

3.5.2 Evolutionary stability and dynamical stability

How do the static equilibrium notions described in the previous sections relate with the notions of equilibrium of the replicator dynamical system?

Proposition 3.29. *Consider an n-normal form game with payoff function f and replicator dynamical system $\dot{x} = X(x)$. A strategy is a fixpoint for the replicator dynamical system if and only if it has slope support.*

Proof. That a slope strategy is a fixpoint is clear from the reduced system above. More generally let x have slope support. If $i \notin \mathrm{supp}(x)$, $x^i = 0 \Rightarrow X^i(x) = 0$. If $i \in \mathrm{supp}(x)$,

$$f_i(x) - \bar{f}(x) = f_i(x) - \sum_j x^j f_j(x) = f_i(x) - \sum_{j \in \mathrm{supp}(x)} x^j \underbrace{f_j(x)}_{=\,\mathrm{const}=f_i(x)} = f_i(x) - f_i(x) \underbrace{\sum_{j \in \mathrm{supp}(x)} x^j}_{1} = 0$$

Conversely, if x is a fixpoint, either $x^i = 0$, or $f_i(x) = \bar{f}(x)$, meaning that $f_i(x) = f_j(x)$ for all $i, j \in \mathrm{supp}(x)$, i.e. x has slope support. □

We can summarize these results and those of Prop. (3.17) as follows:

Proposition 3.30. *Consider a normal form game with replicator dynamics. The following relations between strategies hold (bd. stands for boundary):*

- *Slope* $\Rightarrow$ *Nash*
- *Nash* $\Rightarrow$ *has slope support*
- *Replicator fixpoint* $\iff$ *has slope support*
- *Slope* $\Rightarrow$ *Replicator fixpoint*

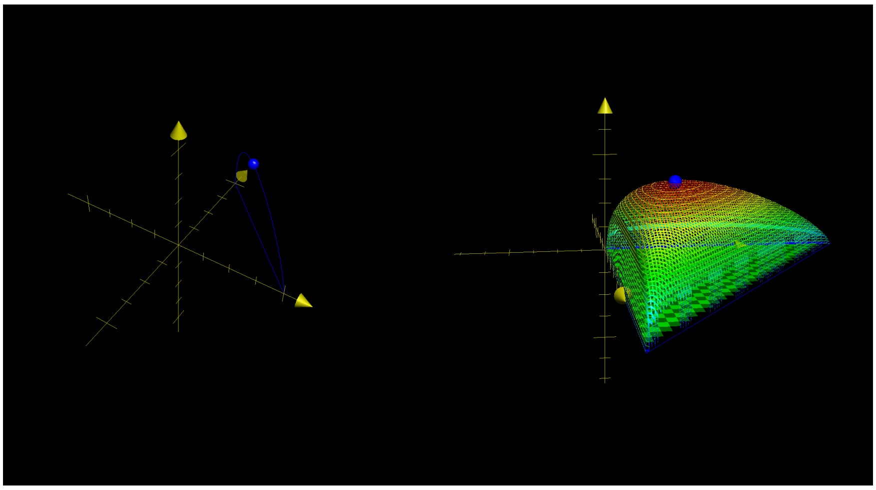

Figure 3.7: Lyapunov function for the 1-simplex and the 2-simplex

- *Interior Nash* $\iff$ *interior slope* $\iff$ *interior Replicator fixpoint*
- *Bd. slope* $\Rightarrow$ *bd. Nash* $\Rightarrow$ *bd. with slope support* $\iff$ *bd. Replicator fixpoint*

In particular Nash strategies are fixpoints for the replicator dynamical system. Note that Nash $\not\Rightarrow$ slope, and replicator fixpoint $\not\Rightarrow$ slope, because nothing can be said about $f_i(x)$ for $i \notin \operatorname{supp}(x)$.

The next remarkable result relates the *stability* properties of dynamical and evolutionary equilibria, defining *en passant* a function that will be of crucial importance in the following. The proof relies on Lyapunov's theorem, that can be found in any classical text of mechanics, as [Arn89], or in an agile form in [HS98, p. 19]. We take the chance to recall the definitions of ω and α limit of integral curves of a vector field.

Definition 3.31. Consider the autonomous dynamical system $\dot{x} = X(x)$ on a smooth manifold M. The ω*-limit* of a point p, or equivalently of the integral curve of X through p, is

$$\omega(p) = \{q \in M : q \text{ is accumulation point for the sequence } x_k = x(t_k) \text{ for some sequence } t_k \to \infty\}$$

where $x(t)$ is the integral curve through p. This means that for any point $q \in \omega(p)$ it is possible to find a sequence $(t_k)_{k\in\mathbb{N}}, t_k \to \infty$ such that the sequence $x_k = x(t_k)$ in M has q as accumulation point; in particular every neighborhood of q is visited infinitely many times by x_k. α-limits are defined the same way, with $t_k \to -\infty$.

Proposition 3.32. *Consider an n-normal form game with payoff function f and replicator dynamical system $\dot{x} = X(x)$. An evolutionarily stable strategy is an **asymptotically** stable fixpoint for the replicator dynamical system.*

Proof. For a fixed point $\hat{x} \in \Delta^n$ define the function

$$P_{\hat{x}}(x) = \prod_i x_i^{\hat{x}_i} \tag{3.46}$$

The support of $\hat{x}$ determines the domain where $P_{\hat{x}}$ is strictly positive:

- if $\hat{x}_i = 0$, x_i does not contribute, so $x_i = 0$ is allowed and gives $P_{\hat{x}}(x) = 0$;
- if $\hat{x}_i > 0$, x_i must be > 0. Thus the domain $D \in \Delta^n$ where $P_{\hat{x}}$ is positive is

$$D = \{x \in \Delta^n : \operatorname{supp}(\hat{x}) \subset \operatorname{supp}(x)\} \tag{3.47}$$

namely $\hat{x}_i > 0 \Rightarrow x_i > 0$.

It can be shown that $P_{\hat{x}}$ has on Δ^n a unique maximum at $\hat{x}$ (see Fig. (3.7) and [HS98, p. 71]). On the domain D consider

$$L_{\hat{x}}(x) = \ln P_{\hat{x}}(x) = \sum_i \hat{x}_i \ln x_i \tag{3.48}$$

The action of the replicator vector field on this function is

$$XL_{\hat{x}}(x) = \sum_j X^j \, \partial_j L_{\hat{x}} = \sum_j X^j \frac{\hat{x}_j}{x_j} = \sum_j \hat{x}_j \left(f_j(x) - \bar{f}(x)\right)$$

$$\boxed{XL_{\hat{x}}\,(x) = f(\hat{x}, x) - f(x, x)} \tag{3.49}$$

Recall eq. (3.32): This quantity is what has to be positive in a neighborhood of a point for it to be strong, i.e. an ES strategy. This means that a strategy $\hat{x}$ is evolutionarily stable if and only if $P_{\hat{x}}$ is a strict local Lyapunov function. Since we are assuming $\hat{x}$ to be an ES strategy $P_{\hat{x}}$ is such Lyapunov function, proving the asymptotic stability of $\hat{x}$. □

3.6 Zero-sum replicator systems

In the following we focus on *zero-sum* normal games, i.e. bilinear normal games with an antisymmetric payoff matrix, where the gain of a player is exactly the loss of the other. This is a strong constraint, leading to structurally unstable and fine tuned systems, that deserves a comment. Zero-sum games were extensively studied in the early years of game theory[6], since most of the so-called *parlour* games[7] belong to this category, and proved very useful in the theoretical development of the theory. In the present context we shall show in a moment that they represent the bridge between the Poisson geometry studied in the first chapters and the replicator dynamical system. In addition to its interest as a special case of evolutionary game models this class of systems has a separate pattern of applications all its own: Nagylaki[8] has introduced the antisymmetric *discrete* replicator equation as a model for gene conversion; see also [AL84]. But most real-life games are not zero-sum: Hamiltonian systems, being structurally unstable, are appropriate only when their conservative properties are essential features in the considered theory. It is nevertheless interesting, and may open the road to further unexpected directions of research, to recognize the Hamiltonian nature of this relatively simple system.

The first striking peculiarity of a zero-sum normal game is that the average population fitness vanishes identically: if $f_i(x) = \sum_i A_{ij} x^j$ with $A_{ij} + A_{ji} = 0$, then $\bar{f}(x) = f(x, x) = \sum_i x^i f_i(x) = \sum_{ij} A_{ij} x^i x^j = 0$. This reduces the replicator system to

$$\dot{x}^i = x^i A_i \tag{3.50}$$

with $A_i = \sum_j A_{ij} x^j$, so that an interior fixpoint $\hat{x}$ solves the linear system $\sum_j A_{ij} \hat{x}^j = 0$ for all values of i.

[6] Sig17, p. 186.
[7] Gar00.
[8] Nag83.

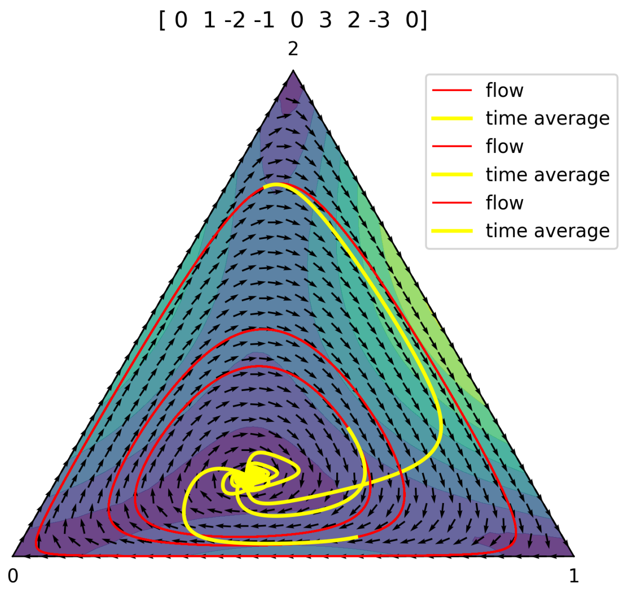

Figure 3.8: Typical trajectories for a rock-paper-scissor game. For every trajectory $x(t)$ (red) its time average $y_i(t) = \frac{\int_0^t x_i(\tau)\mathrm{d}\tau}{t}$ (yellow) is also plotted.

Rock Paper Scissor We start with a very simple example. Consider the zero-sum game defined by the payoff matrix

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \tag{3.51}$$

corresponding to three strategies beating each others cyclically. The slope system $f_i(x) = \text{const}$ is $x_0 = x_1 = x_2, x_0 + x_1 + x_2 = 1$, so that the center of the game space $\hat{x} = (1/3, 1/3, 1/3)$ is an internal Nash equilibrium. It is not evolutionarily stable since it is not strong: $f(\hat{x}, x) = 0$ identically for any x, not fulfilling eq. (3.32).

The replicator system $\dot{x} = X(x)$ reads

$$\begin{cases} \dot{x}^0 = x^0(x^1 - x^2) \\ \dot{x}^1 = x^1(-x^0 + x^2) \\ \dot{x}^2 = x^2(x^0 - x^1) \end{cases} \tag{3.52}$$

Endow the 2-simplex with the Poisson structure (2.12), recalling that $x^0 + x^1 + x^2 = 1$:

$$\begin{aligned} \pi^{01} &= x^0 x^1 \left(A_{01} - A_{01}x^0 - A_{01}x^1 - (A_{02} + A_{21}x^2)\right) \\ &= x^0 x^1 (1 - x^0 - x^1 + 2x^2) = 3\, x^0 x^1 x^2 \end{aligned}$$

and similarly for the remaining two components, so that

$$\begin{cases} \pi^{01} = 3\, x^0 x^1 x^2 \\ \pi^{02} = -3\, x^0 x^1 x^2 \\ \pi^{12} = 3\, x^0 x^1 x^2 \end{cases} \tag{3.53}$$

The replicator vector field X is Hamiltonian with respect to this Poisson structure. It belongs indeed to the image of the sharp morphism of π: $\alpha^\sharp = X$ for some 1-form α, that is $X^i = \pi^{ji}\alpha_j$ or in matrix notation $X = -\pi \cdot \alpha$. This is readily solved by

$$\alpha = \frac{1}{3}\left(\frac{\mathrm{d}x^0}{x^0} + \frac{\mathrm{d}x^1}{x^1} + \frac{\mathrm{d}x^2}{x^2}\right) \tag{3.54}$$

This 1-form is exact and its primitive, i.e. the Hamiltonian function of the replicator vector field, is $H(x) = \frac{1}{3}\ln(x^0) + \frac{1}{3}\ln(x^1) + \frac{1}{3}\ln(x^2) + \mathrm{const}$; note that this is precisely $L_{\hat{x}}(x)$ as defined in eq. (3.48). Some typical trajectories for different values of the coefficients but same qualitative behavior are plotted in fig. (3.8).

The properties of this example are actually very general:

Proposition 3.33. *For a normal form zero-sum game a strategy is evolutionarily stable if and only if it is a vertex e_j such that $f_j(x) > 0$ in a neighborhood of e_j.*

Proof sketch. An ES strategy $\hat{x}$ of a zero-sum game fulfills

$$f(\hat{x}, x) > 0 \tag{3.55}$$

in a neighborhood of $\hat{x}$. If $\hat{x}$ is an equilibrium, but not pure, one can always find a "pathological" point x in its neighborhood and still on the simplex (take x with the same support of $\hat{x}$, and with nonzero components slightly perturbed so that x is indeed on the simplex) so that $f(\hat{x}, x) = 0$. A vertex, on the other hand, cannot be perturbed this way. The converse is obvious. □

Thus ES strategies are not very interesting for zero-sum games: if they exist, they are vertices. Still two mutually exclusive classes of fixpoints on which the dynamics depends can be identified[9]: one is that of interior fixpoints, discussed in the following and leading to Hamiltonian systems; the other consists of boundary fixpoints with some sign-definiteness property, and corresponds to a dynamics in which some strategies are *eliminated by competition* and all interior trajectories converge to a face. The second scenario, related to the concept of *survival of the fittest*, is not examined in detail in this work but deserves anyway a comment.

As discussed above ES strategies are asymptotically stable. The converse is trickier: It holds for *symmetric* games[10] and for *nondegenerate* games[11], where a *degenerate* game is built defining some *types* in game space, and using them as a degenerate set of pure strategies generating a higher dimensional, degenerate system. Thus a nondegenerate zero-sum game without ES strategies does not have asymptotically stable fixpoints either, and if boundaries fixpoints exist (second scenario) then trajectories converge to a face, eliminating by competition some strategies.

Which face is reached depends on the existing boundary fixpoints, as described by Akin and Losert, and this determines the number of eliminated strategies. Once the trajectory settles on (that means, arrives close to) the face the dynamics (not converging to a fixpoint) shall be investigated; for example the analysis of the system with payoff matrix

$$\begin{bmatrix} 0 & -1.5 & 1.3 & -2.5 \\ 1.5 & 0 & -2.0 & 2.0 \\ -1.3 & 2.0 & 0 & -1.0 \\ 2.5 & -2.0 & 1.0 & 0 \end{bmatrix}$$

[9] AL84, p. 239.
[10] HS98, p. 82.
[11] HS98, p. 73.

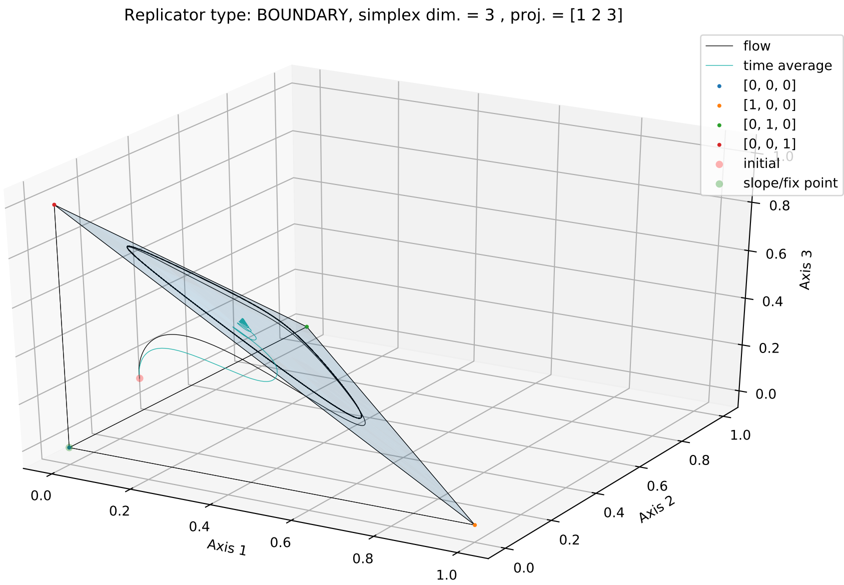

Figure 3.9

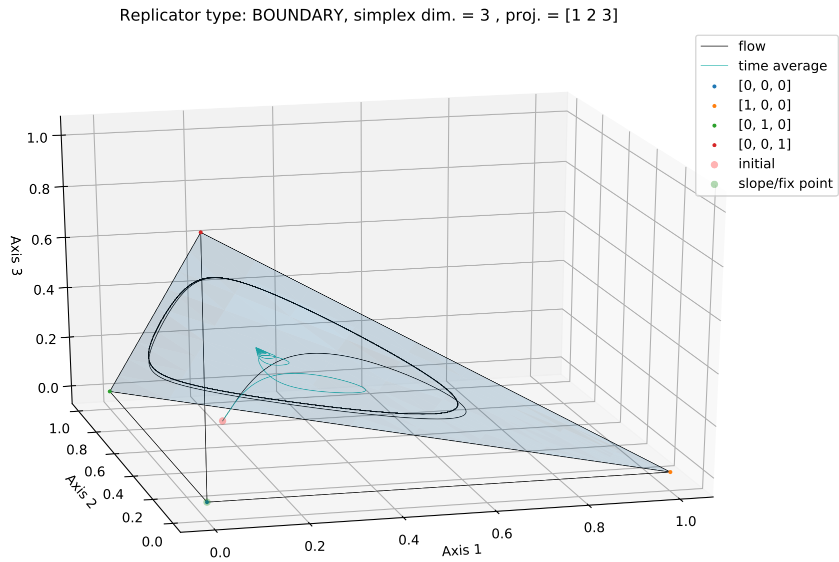

Figure 3.10: Boundary dynamics from two angles of view of a nondegenerate zero-sum replicator with boundary fixpoints. No interior fixpoints exist, no ES strategies exist, no asymptotically stable strategies exist. One strategy out of four is eliminated by competition, the remaining three strategies exhibit an oscillatory behavior that resembles the interior dynamics of a rock paper scissor game, namely of a zero-sum game admitting an interior equilibrium. The black line is the trajectory and the azure is it's time average, converging to the boundary fixpoint. A fixpoint, not shown, exists also in the $x^3 = 0$ face, containing the α-limit of interior trajectories.

reveals the existence of two boundary fixpoints, one determining the face $x^0 = 0$ containing the ω-limit of interior orbits, the other determining the face $x^3 = 0$ containing the α-limit of interior orbits. Figures (3.9) and (3.10) show a trajectory orbiting the boundary fixpoint in the ω-limit face.

Both the degeneracy of games and the boundary competition dynamics may be worth of examination in a future work.

3.6.1 Hamiltonian zero-sum replicator systems

We can finally describe the Hamiltonian nature of zero-sum replicator systems in the first scenario, namely if interior fixpoints exist:

Theorem 3.34. *Consider an n-normal form zero-sum game with payoff function f and replicator dynamical system $\dot{x} = X_{rep}(x)$. If the system admits an interior fixpoint $\hat{x}$, then the replicator vector field is Hamiltonian with respect to (minus) the Poisson structure $\{\cdot,\cdot\}_{\Delta^n}$ (2.12). Its Hamiltonian function, defined in the interior, is $H = -L_{\hat{x}}$, where $L_{\hat{x}}$ is the function defined in eq. (3.48).*

Proof. We check explicitly that $(\mathrm{d}H)^\sharp = X_{\mathrm{rep}}$. Note that 3 minus signs contribute to an overall minus sign outside the brackets: to obtain a *convex* Hamiltonian function we choose $H = -L_{\hat{x}} = -\sum_i \hat{x}_i \ln x_i$; we then have to change the sign of the Poisson structure, so that $\pi^{ij} = -x^i x^j \left(A_{ij} - \sum_h \left(A_{ih} + A_{hj}\right) x^h\right)$; and with the adopted convention $(\mathrm{d}H)^{\sharp^i} = \sum_j \pi^{ji}\partial_j H = -\sum_j \pi^{ij}\partial_j H$. Thus

$$
\begin{aligned}
(\mathrm{d}H)^{\sharp^i} = -\sum_j \pi^{ij}\partial_j H = -\sum_j x^i x^j \frac{\hat{x}^j}{x^j}\left(A_{ij} - \sum_h A_{ih}x^h - \sum_h A_{hj}x^h\right) \\
= -x^i\left(\sum_j A_{ij}\hat{x}^j - \sum_h A_{ih}x^h \sum_j \hat{x}^j - \sum_{j,h} x^h A_{hj}\hat{x}^j\right) \\
= -x^i\left(f_i(\hat{x}) - f_i(x) - \sum_h x^h f_h(\hat{x})\right)
\end{aligned}
$$

The first and third terms vanish because for an internal fixpoint $f_i(\hat{x}) = 0$ for all i, so

$$
(\mathrm{d}H)^{\sharp^i}(x) = x^i f_i(x) = X^i_{\mathrm{rep}}(x)
$$

□

The trajectories of this Hamiltonian vector field are described in the next theorem:

Theorem 3.35. *Consider an n-normal form zero-sum game with at least one interior fixpoint, and denote by E_0 the set of all interior fixpoints. For any interior non-equilibrium point $p \in \mathring{\Delta}^n - \{E_0\}$* ***the closure*** *of the orbit through p is a compact invariant set containing $\omega(p)$ and $\alpha(p)$ and contained in $\mathring{\Delta}^n - \{E_0\}$. In particular, the closure of the orbit contains no equilibria.*

Proof. See [AL84, p. 239]. □

This means that all the interior equilibria are stable, but not asymptotically stable. A trajectory starting close to an equilibrium stays in some neighborhood of it, without approaching it and without diverging to the boundary; this fills the interior of the simplex with invariant manifolds containing no equilibria. The orbits can be periodic, as in the case of the Rock-Paper-Scissor game; or they can keep orbiting in a bounded region, never intersecting themselves.

Example 3.36. Consider the zero-sum game on the 4-simplex given by the matrix

$$\begin{bmatrix} 0 & 1.0 & -1.0 & 0.5 & 0.3 \\ -1.0 & 0 & -1.2 & 1.4 & 1.4 \\ 1.0 & 1.2 & 0 & 1.2 & -1.8 \\ -0.5 & -1.4 & -1.2 & 0 & 1.5 \\ -0.3 & -1.4 & 1.8 & -1.5 & 0 \end{bmatrix}$$

An interior fixpoint exists, of coordinates $(x^1 : 0.0362, x^2 : 0.1843, x^3 : 0.1148, x^4 : 0.3021)$ and x^0 so that the point belongs to the 4-simplex, hence the replicator system is Hamiltonian. The orbits plotted in Fig. (3.13) display the non-periodic bounded behavior described above. The system has an interior fixpoint (stable but not asymptotically) and no ES strategies.

The Hamiltonian nature of the system opens the door to further theoretical investigation. Future directions of research may include the study of the symplectic leaves and of quasi-periodic tori employing the constant Poisson structure (2.25) and the convexity of the Hamiltonian function.

3.7 Conclusions

The first steps of this work have been to build a Poisson structure for the simplex and to recognize the nature of Poisson manifolds of its faces. The simplex was considered in the first place as a space of discrete probabilities distributions hosting normal form games. The geometrical nature of the simplex becomes more relevant with the introduction of the replicator dynamical system, modeling the evolution of the average population strategy; and crucial with the realization that the replicator vector field is Hamiltonian under the simplex Poisson structure.

The focus has been mainly geometrical, beyond some more phenomenological sections at the beginning of the last chapter. This may sound striking, and the relatively small dynamical investigation performed may appear not to justify the amount of developed theoretical machinery. Still, this is mainly a limitation of time and space: This work is more of a starting point than an arrival, inasmuch the open questions are still many.

First of all degenerate normal form games shall be considered. As mentioned in the text they are built defining some *types* in game space, and using them as a degenerate set of pure strategies generating a higher dimensional, degenerate replicator system. The main result in this context is that the preimage of an evolutionarily stable strategy in game space is an attractor set in the degenerate space; see for example [HS98, p. 69].

Akin and Losert [AL84] describe in detail the zero-sum replicator dynamics both in the continuous and in the discrete case. The continuous case with interior fixpoints was described in Theorem (3.35). From this point one can write the Hamiltonian function in the coordinates of the constant Poisson structure B and exploit its convexity[12] properties to describe the orbits.

[12] HWZ98.

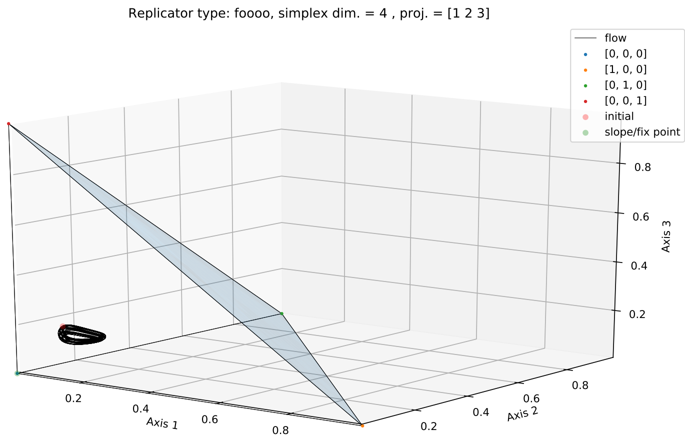

Figure 3.11

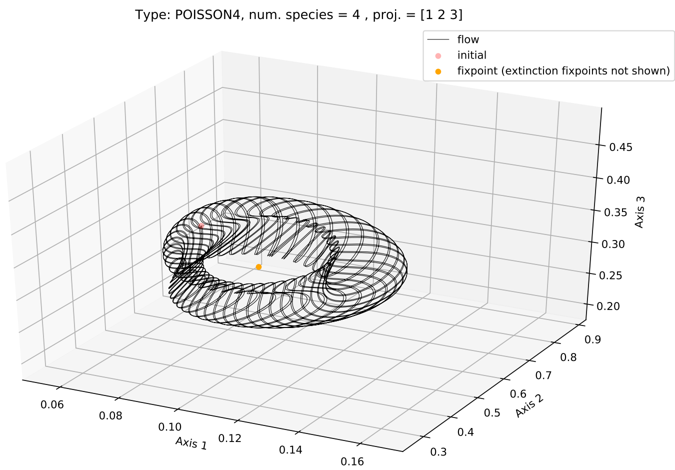

Figure 3.12

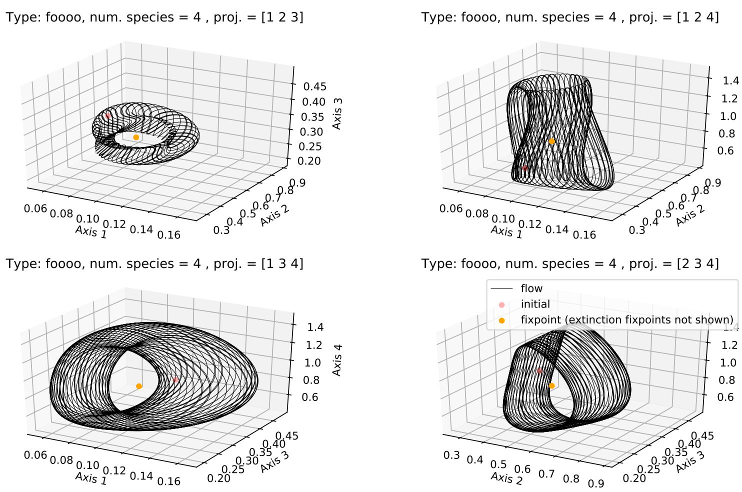

Figure 3.13: Top: projection on the 3 simplex of an orbit of the Hamiltonian replicator system of example (3.36). Middle and bottom: projection of an orbit of the Lotka-Volterra system equivalent to the same replicator system. The orbits seem to suggest a quasi-periodic tori behavior.

The continuous case with boundary fixpoints seems to rephrase the idea of *survival of the fittest*: as we mentioned if boundary fixpoints with some sign-definiteness property exist then trajectories converge to faces (not necessarily vertices) of the simplex, corresponding to *some* strategies surviving - not necessarily only one.

For a meaningful comparison with the survival of the fittest statement one should look carefully at the hypothesis upon which it is formulated; In particular a discrete replicator equation may lead to completely different dynamics than that of the continuous model, as discussed by Akin and Losert. The discrete replicator model, more suitable for many real life applications, may furthermore lead to interesting chaotic dynamics. [PMC18]

The equivalence between a replicator system and a Lotka-Volterra system may also be worth investigating more carefully. [DFO98].

The replicator system is the first building block of *bimatrix games*, involving interaction between two species, and *polimatrix games*, involving interaction within and between many species; the Poisson-Hamiltonian structure exists also in this more general case. [Hof96][AD14]

Finally not only quantitative but also qualitatively different interaction can be taken into account, leading to imitation, best-response, adaptive, mutator dynamics. [HS98][Aki79] [GP04]

I hope to tackle some of these issues in a future work.

Appendix A

Differential geometry

Here is a collection of some definitions and useful results, and an overview of the adopted conventions (track the minus signs) and notations. The standard reference is [Lee12] (or [AT11], in Italian).

Local coordinates $(x^1, \dots, x^n)$ on a smooth manifold M induce the standard coordinates local bases $\{\partial_i\}_{i \in I}$ for vector fields and $\{\mathrm{d}x^i\}_{i \in I}$ for 1-forms, with $I = \{1, \dots, \dim M\}$. Einstein summation convention over repeated indices is understood if not otherwise specified, so that $X = X^i \partial_i \in \tau(M)$ is a vector field and $\alpha = \alpha_i \mathrm{d}x^i \in \Omega^1(M)$ is a 1-form, with components $X^i, \alpha_i \in C^\infty(M)$. For the vector given by a vector field X at a point $p \in M$ we write $X(p) \equiv X_p \in T_pM$; similarly for a covector $\alpha_p \in T_p^*M$.

The exterior derivative of a form $\omega \in \Omega^k(M)$ is denoted by $\mathrm{d}\omega \in \Omega^{k+1}(M)$. With slight abuse of notation the action of a vector field on a function in $C^\infty(M)$ (as a derivation) and on a 1-form (as its dual object) is denoted the same way; see [Lee12, p. 180] for the canonical isomorphism between vector fields and $C^\infty(M)$ derivations. So for $f \in C^\infty(M)$ we have

$$Xf = X(\mathrm{d}f) = \mathrm{d}f(X) = X^i \partial_i f = X^i (\mathrm{d}f)_i \in C^\infty(M) \tag{A.1}$$

Derivations fulfill the Leibniz rule

$$X(fg) = (Xf)\, g + f\,(Xg), \quad \forall f, g \in C^\infty(M),\ X \in \tau(M) \tag{A.2}$$

The commutator (or Lie bracket) of vector fields is the map $[\cdot,\cdot] : \tau(M) \times \tau(M) \to \tau(M)$ such that for $X, Y \in \tau(M)$

$$\forall f \in C^\infty(M) \quad [X,Y]f = X(Yf) - Y(Xf) \tag{A.3}$$

Remark A.1. Since the Lie bracket is bilinear, antisymmetric and fulfills the Jacobi identity [Lee12, p. 188], the space of vector fields $\tau(M)$ on a smooth manifold is a Lie algebra under the Lie bracket.

The interior product of a form with a vector field $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ is

$$\iota_X \omega(Y_1, \dots, Y_{k-1}) = \omega(X, Y_1, \dots, Y_{k-1}) \tag{A.4}$$

Forms and multivectors A k-form $\omega \in \Omega^k(M)$, with $\dim M = n \geq k$, is written as

$$\begin{aligned}\omega &= \sum_{*} \omega_{i_1\dots i_k}\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \\ &= \frac{\omega_{i_1\dots i_k}}{k!}\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \\ &= \omega_{i_1\dots i_k}\, \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k}\end{aligned} \tag{A.5}$$

where $\sum_*$ means $\sum_{1\leq i_1<i_2<\cdots<i_k\leq n}$, and is a sum over $\binom{n}{k}$ terms. In the second and third line the summation over all the possible values $i_1 = 1,\dots,n;\dots; i_k = 1,\dots,n$ is understood. The coefficients $\omega_{i_1\dots i_k}$ in the first line are precisely the $\binom{n}{k}$ degrees of freedom of the form, from which the n^2 coefficients $\omega_{i_1\dots i_k}$ in the second and third line are defined by antisymmetrization.

If $\alpha \in \Omega^1(M)$ and $X, Y \in \tau(M)$

$$\mathrm{d}\alpha(X,Y) = X\left(\alpha(Y)\right) - Y\left(\alpha(X)\right) - \alpha\left([X,Y]\right) \in C^\infty(M) \tag{A.6}$$

This formula extends easily to the differential of a $k-1$ form acting on k vector fields [Lee12, p. 370], using Cartan magic formula (A.44) and knowing how to compute the Lie derivative of any form by standard techniques (see for example eq. (A.43)), for example for a 2-form

$$\mathrm{d}\omega(X,Y,Z) = \sum_{\text{cyclic } X,Y,Z} X\left(\omega(Y,Z)\right) - \omega\left([X,Y],Z\right) \tag{A.7}$$

Antisymmetric $(k,0)$ tensor fields, called **k-vector fields**, are the natural contravariant version of k-forms; their space is denoted by $\nu^k(M)$. Let $\bigwedge^k(T_pM) \subset T_pM \otimes \cdots \otimes T_pM$ be the *antisymmetric* tensor product of k copies of the tangent space to M at p. An element in this space is called *k-vector*, and the disjoint union of these vector spaces $\bigsqcup_{p\in M} \bigwedge^k(T_pM)$ is called *k-tangent bundle*. A k-vector field π is a section of this bundle:

$$\begin{aligned}\pi : M &\to \bigsqcup_{p\in M} \bigwedge^k(T_pM) \\ p &\longmapsto \pi_p \in \bigwedge^k(T_pM)\end{aligned} \tag{A.8}$$

$$\pi_p : \underbrace{T_p^*M \times \cdots \times T_p^*M}_{k} \to \mathbb{R} \text{ multilinear antisymmetric} \tag{A.9}$$

Just like for rank-1 vector fields, k-vector fields are derivations in all entries when acting on smooth functions, and of course act as normal tensor fields on 1-forms; for example for a bivector π, in analogy to eqs. (A.1), (A.2), we have for $f, g, h \in C^\infty(M)$ [DZ05, p. 6]

$$\pi(f,g) = \pi(\mathrm{d}f, \mathrm{d}g) = \pi^{ij}\, \partial_i f\, \partial_j g \in C^\infty(M) \tag{A.10}$$

$$\pi(f, gh) = \pi(f,g)\, h + g\, \pi(f,h) \tag{A.11}$$

and similarly in the first argument.

Musical homomorphisms [Lee12, p. 342] Any $(0,2)$ tensor field ω on M defines a homomorphism called *flat*[1], $\flat_\omega : \tau(M) \to \Omega(M)$, defined by

$$\forall X \in \tau(M), \quad \flat_\omega X(\cdot) = \omega(X, \cdot) \in \Omega(M) \tag{A.12}$$

Clearly $\flat_\omega(X) = \iota_X(\omega)$. If there is no risk of ambiguity about the $(0,2)$ tensor field being used we shall write $\flat_\omega(X)$ as $X^\flat$. In coordinates,

$$X_i^\flat = X^\flat(\partial_i) = \omega(X, \partial_i) = \omega_{ji} X^j \tag{A.13}$$

Similarly, a $(2,0)$ tensor field π defines the *sharp* morphism $\sharp_\pi : \Omega(M) \to \tau(M)$ defined by

$$\forall \alpha \in \Omega(M), \quad \sharp_\pi \alpha(\cdot) = \pi(\alpha, \cdot) \in \tau(M) \tag{A.14}$$

Again we write $\sharp_\pi(\alpha) = \alpha^\sharp$, and

$$(\alpha^\sharp)^i = \alpha^\sharp(\mathrm{d}x^i) = \pi(\alpha, \mathrm{d}x^i) = \pi^{ji} \alpha_j \tag{A.15}$$

If a rank-2 tensor field is nondegenerate its natural musical homomorphism is invertible, hence an isomorphism. So if ω (0,2) is nondegenerate, ω^{-1} is the (2,0) tensor field whose sharp morphism is the inverse of ω's flat morphism, and $(\omega^{-1})^{ij} = (\omega_{ij})^{-1}$, matrix-wise. For brevity one usually writes $\omega^{ij} := (\omega^{-1})^{ij}$.

We now focus on **antisymmetric** rank-2 fields, namely 2-forms and bivectors. For example, if π is a bivector and α, β 1-forms one has

$$\pi(\alpha, \beta) = \alpha^\sharp(\beta) = -\pi(\beta, \alpha) = -\beta^\sharp(\alpha) = \pi^{ij} \alpha_i \beta_j \in C^\infty(M) \tag{A.16}$$

By multilinearity, musical homomorphisms can be extended to tensor fields of arbitrary rank; in particular, a *nondegenerate* 2-form ω can map itself to a $(2,0)$ field $\hat{\omega}$:

$$\hat{\omega}^{hk} = \omega^{ih} \omega^{jk} \omega_{ij} = -\omega^{hk}$$

This is equivalent to

$$\omega(X, Y) = \hat{\omega}(X^\flat, Y^\flat), \quad \forall X, Y \in \tau(M) \tag{A.17}$$

and shows that a nondegenerate 2-form ω maps itself via its extended musical isomorphism to *minus* its inverse:

$$\boxed{\hat{\omega} = -\omega^{-1} \quad (2,0)} \tag{A.18}$$

Analogue formulas hold for nondegenerate bivectors.

Differential and codifferential [Lee12, pp. 55, 181, 284]. Given a homomorphism between vector spaces $\phi : V \to W$, dual vectors are naturally "pulled-back":

$$\begin{aligned} \phi^* : W^* &\to V^* \\ \xi &\longmapsto \phi^* \xi := \xi \circ \phi \end{aligned} \tag{A.19}$$

namely $\phi^* \xi$ is the dual vector mapping $v \in V$ to $\xi(\phi(v))$; ϕ^* is called the *dual* of ϕ.

[1] The nomenclature comes from musical theory: *flat* $\flat$ "lowers the indices" of tensors, i.e. produces 1-forms from vector fields, while *sharp* $\sharp$ "raises indices", i.e. produces vector fields from 1-forms.

Given a smooth function between manifolds $F: M \to N$ one can build from it a homomorphism between vector spaces, and consider its dual. First define the pull-back of smooth functions $F^*: C^\infty(N) \to C^\infty(M)$ such that the following diagram commutes for all $f \in C^\infty(N)$:

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[dr, "F^* f"'] & N \arrow[d, "f"] \\ & \mathbb{R} \end{tikzcd} \quad \text{namely} \quad \begin{aligned} F^* : C^\infty(N) &\to C^\infty(M) \\ f &\longmapsto f \circ F \end{aligned} \tag{A.20}$$

This allows to define the **differential** of F at $p \in M$ as a linear map between the tangent space to M at p and the tangent space to N at $F(p)$:

$$\begin{aligned} \mathrm{d}_p F : T_p M &\to T_{F(p)} N \\ X_p &\longmapsto \mathrm{d}_p F\left(X_p\right) := X_p \circ F^* \end{aligned} \tag{A.21}$$

A vector in $T_p M$ mapped via $\mathrm{d}_p F$ is said to be *pushed forward*, and the definition implies that a pushed vector acts as a derivation on a function in $C^\infty(N)$ as the original vector acts as a derivation on the pulled function:

$$\left(\mathrm{d}_p F X_p\right)(f) = X_p(f \circ F) \in C^\infty(M) \tag{A.22}$$

Remark A.2. [Lee12, p. 63] Chosen two local charts at $p \in M$ and $F(p) \in N$, the matrix representing the homomorphism $d_p F$ in the corresponding coordinates bases is the Jacobian matrix of F at p.

Applying eq. (A.19) to the differential (A.21) gives the **pull-back of covectors**:

$$\begin{aligned} \left(\mathrm{d}_p F\right)^* : T^*_{F(p)} N &\to T^*_p M \\ \xi_{F(p)} &\longmapsto \left(\mathrm{d}_p F\right)^*\left(\xi_{F(p)}\right) := \xi_{F(p)} \circ \mathrm{d}_p F \end{aligned} \tag{A.23}$$

Pull-back Recall that F is just a smooth function from M to N, not necessarily invertible. Interestingly, for all 1-forms in the "right" space $\alpha \in \Omega(N)$, this is always enough to define explicitly a unique 1-form in the "left" space $F^*\alpha \in \Omega(M)$, called (without surprise) the *pull-back* of α, such that

$$\begin{aligned} \forall p \in M, \quad \left(F^* \alpha\right)_p &:= \text{pull-back of the covector } \alpha_{F(p)} \\ &= \left(\mathrm{d}_p F\right)^*\left(\alpha_{F(p)}\right) \in T^*_p M \end{aligned} \tag{A.24}$$

The map $F^*: \Omega(N) \to \Omega(M)$ is called **pull-back of 1-forms**.

Remark A.3. [Lee12, p. 286] In local coordinates the components of a 1-form α pulled back along $F: M \to N$ are

$$\left(F^* \alpha\right)_i = \partial_i F^h \left(\alpha_h \circ F\right) \in C^\infty(M) \tag{A.25}$$

Note that $\alpha_h \in C^\infty(N)$ and $\partial_i F^h \in C^\infty(M)$.

Vector fields relation On the other hand, eq. (A.21) does *not* allow to define a natural "pushed-forward vector field": a definition analogue to (A.24), would be

$$\forall q \in N, \quad T_q N \ni \left(F_* X\right)_q := \text{push-forward of the vector } X_{F^{-1}(q)} \tag{A.26}$$

which is clearly not well defined, since in general F is nor injective, nor surjective.

For this reason, one needs a different concept to relate vector fields via a smooth function $F : M \to N$. So, $X \in \tau(M)$ and $Y \in \tau(N)$ are said to be ***F-related*** if

$$\mathrm{d}_p F(X_p) = Y_{F(p)} \in T_{F(p)}N, \quad \forall p \in M \tag{A.27}$$

Equivalently,

$$X(f \circ F) = Y(f) \circ F \in C^\infty(M), \quad \forall f \in C^\infty(N) \tag{A.28}$$

Note that this is an implicit definition: for given F and X there may be none, or multiple, vector fields F-related to X.

Remark A.4. If there exists a vector field $Y \in \tau(N)$ F-related to $X \in \tau(M)$ for some $F : M \to N$, then the components of X and Y in local coordinates are related by

$$Y^i \circ F = \partial_h F^i \, X^h \in C^\infty(M) \tag{A.29}$$

Given a function $F : M \to M$ a vector field X is said ***F-invariant*** if it is F-related to itself:

$$\mathrm{d}_p F(X_p) = X_{F(p)}, \quad \forall p \in M \tag{A.30}$$

The concept of F-relation can be extended to higher rank vector fields via eq. (A.28): for example, given the usual smooth, not necessarily invertible $F : M \to N$, two bivectors $\pi \in \nu^2(M), \hat{\pi} \in \nu^2(N)$ are F-related if

$$\pi(f \circ F, g \circ F) = \hat{\pi}(f, g) \circ F, \quad \forall f, g \in C^\infty(N) \tag{A.31}$$

In coordinates this means

$$\hat{\pi}^{ij} \circ F = \pi^{hk} \, \partial_h F^i \, \partial_k F^j \, \in C^\infty(M) \tag{A.32}$$

or in matrix notation, with $J^i_{\ h} = \partial_h F^i$

$$\boxed{\hat{\pi} \circ F = J \, \pi \, J^T} \tag{A.33}$$

This is relevant whenever we have to check in practice whether a given map is a Poisson map (see main text, chapter 1).

Push-forward *If $F : M \to N$ is a diffeomorphism*, eq. (A.26) provides a well defined *push-forward of vector fields $F_* : \tau(M) \to \tau(N)$*:

$$\forall q \in N, \quad (F_* X)_q := \mathrm{d}_{F^{-1}(q)} F \left(X_{F^{-1}(q)} \right) \in T_q N \tag{A.34}$$

Clearly, for $p \in M$, $(F_* X)_{F(p)} = \mathrm{d}_p F(X_p)$, so that the pushed-forward vector field $F_* X$ is the unique vector field F-related to X; then, if $F : M \to M$ is a diffeomorphism, X is F-invariant iff it is push-forward invariant, i.e. $X = F_* X \in \tau(M)$.

Flow of a vector field [Lee12, p. 205] Given a vector field $X \in \tau(M)$ its flow is the "collection of all its integral curves, letting the initial point vary on the manifold". In other words, the flow of the (complete) vector field X is the map

$$\begin{aligned} \Theta : &\mathbb{R} \times M \to M \\ &(t,p) \longmapsto \Theta(t,p) \equiv \Theta^p(t) \equiv \Theta_t(p) \end{aligned} \tag{A.35}$$

where $\Theta^p : \mathbb{R} \to M$, obtained fixing $p \in M$, is the unique maximal integral curve of X through p, while $\Theta_t : M \to M$, obtained fixing $t \in \mathbb{R}$, can be shown to be a local diffeomorphism of some U open $\ni p$ with inverse Θ_{-t}.

Both these maps are extremely useful:

- The curve $\Theta^p(t)$ allows "to study smooth functions along the flow of X", that is to relate the derivative of $f \in C^\infty(M)$ along an integral curve of X to the action of X on f:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(f \circ \Theta^p\right)(t) &= \partial_i f\left(\Theta^p(t)\right) \dot{\Theta^p}^i(t) \\ &= \mathrm{d}_{\Theta^p(t)} f\left(\dot{\Theta}^p(t)\right) = \mathrm{d}_{\Theta^p(t)} f\left(X_{\Theta^p(t)}\right) \\ &= (Xf)(\Theta^p(t)) = (\mathscr{L}_X f) \circ \Theta^p\,(t) \in \mathbb{R} \end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(f \circ \Theta^p\right) = (Xf) \circ \Theta^p : \mathbb{R} \to \mathbb{R}} \tag{A.36}$$

- The local diffeomorphism $\Theta_t(p)$ allows to measure the rate of change of vector fields (actually, arbitrary tensor fields) along the flow of X, replacing F with Θ_t in the definition (A.30) of F-invariance: if Θ is the flow of X, Y is said to be X**-invariant** if it is Θ_t-invariant, whenever $\Theta(t,p)$ is defined:

$$\mathrm{d}_p\Theta_t\left(Y_p\right) = Y_{\Theta_t(p)} \tag{A.37}$$

In particular, it can be shown that *any vector field is invariant with respect to itself*:

$$\mathrm{d}_p\Theta_t\left(X_p\right) = X_{\Theta_t(p)} \tag{A.38}$$

whenever both sides are defined, where Θ is the flow of X.

Lie derivative [Lee12, pp. 228, 321]

Let Θ be the flow of the vector field $X \in \tau(M)$. Since Θ_t is a local diffeomorphism it makes sense to consider the push-forward of a vector field $Y \in \tau(M)$ along it. Because of eq. (A.34), though, this operations involves Y "along the past flow" of X: $\left((\Theta_t)_* Y\right)(p)$ is Y evaluated at $\Theta_{-t}(p)$, and pushed forward in T_pM.

To gather information about Y along the future flow of X one has to push it forward along the *inverse* of Θ_t, that is Θ_{-t}: define $\hat{Y}_t \coloneqq (\Theta_{-t})_* Y \in \tau(M)$ so that for $p \in M$

$$\hat{Y}_t(p) = \left((\Theta_{-t})_* Y\right)(p) = \mathrm{d}_{\Theta_t(p)}\Theta_{-t}\left(Y_{\Theta_t(p)}\right) \in T_pM \tag{A.39}$$

and note that $\hat{Y}_{t=0} \equiv Y$. The *Lie derivative* of Y along X is the vector field $\mathscr{L}_X Y$ defined by

$$\left(\mathscr{L}_X Y\right)(p) = \lim_{t \to 0} \frac{\hat{Y}_t(p) - \hat{Y}_0(p)}{t - 0} = \frac{\mathrm{d}}{\mathrm{d}t}\hat{Y}_t(p)\Big|_{t=0} \in T_pM \tag{A.40}$$

for all $p \in M$. Remarkably, it turns out that $\mathscr{L}_X Y = [X, Y]$.

With the same logic one can define the Lie derivative of a 1-form α along a vector field X. Since forms are naturally pulled back one can use directly Θ_t and not its inverse: let $\hat{\alpha}_t \coloneqq \Theta_t^* \alpha \in \Omega(M)$, so that

$$\hat{\alpha}_t(p) = (\Theta_t^* \alpha)(p) = (\mathrm{d}_p \Theta_t)^* \left(\alpha_{\Theta_t(p)}\right) \in T_p^* M \tag{A.41}$$

Again $\hat{\alpha}_{t=0} \equiv \alpha$, and $\mathscr{L}_X \alpha$ is the 1-form such that, for all $p \in M$

$$(\mathscr{L}_X \alpha)(p) = \lim_{t \to 0} \frac{\hat{\alpha}_t(p) - \hat{\alpha}_0(p)}{t - 0} = \frac{\mathrm{d}}{\mathrm{d}t} \hat{\alpha}_t(p) \Big|_{t=0} \in T_p^* M \tag{A.42}$$

With similar methods it is possible to extend the Lie derivative operator to tensor fields of arbitrary rank, obtaining a map $\mathscr{L}_X : \tau_l^k(M) \to \tau_l^k(M)$. In practice it suffices to know that $\mathscr{L}_X Y = [X, Y]$ for vector fields and $\mathscr{L}_X f = Xf$ for $f \in C^\infty(M)$; the Lie derivative of an arbitrary tensor field is then obtained by a "generalized Leibniz rule". For example the Lie derivative of a 1-form α along a vector field X is computed from

$$\mathscr{L}_X (\alpha(Y)) = (\mathscr{L}_X \alpha)(Y) + \alpha (\mathscr{L}_X Y) \tag{A.43}$$

for all $Y \in \tau(M)$. Since one knows how to compute the left hand side and the second term of the right hand side, the first term of the right hand side is determined.

Finally, the following results hold:

$$\mathscr{L}_X \omega = \iota_X \mathrm{d}\omega + \mathrm{d}\iota_X \omega \tag{A.44}$$

$$\mathscr{L}_X \mathrm{d}f = \mathrm{d}\mathscr{L}_X f \tag{A.45}$$

$$\mathscr{L}_{[X,Y]} = \mathscr{L}_X \mathscr{L}_Y - \mathscr{L}_Y \mathscr{L}_X \tag{A.46}$$

Equation (A.44) is known as *Cartan magic formula*.

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It’s done!

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