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Character varieties and Lagrangian submanifolds

Charaktervarietäten und Lagrangesche Untermannigfaltigkeiten

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Abstract

Let Γ be a finitely generated group, and let G be a reductive affine algebraic group over $\mathbb{C}$. The set of group homomorphisms $\mathrm{Hom}(\Gamma, G)$ together with an affine algebraic structure coming from G is called the *representation variety* of Γ in G. G acts on $\mathrm{Hom}(\Gamma, G)$ via conjugation. The categorical quotient $X_G(\Gamma) = \mathrm{Hom}(\Gamma, G)/\!\!/G$ is called *character variety.* For the fundamental group $\pi_1(S)$ of a surface S, the character variety of *good representations*, i.e. irreducible representations with closed orbits under the G-action, is a complex manifold $X_G^g(\pi_1(S))$ with tangent spaces $H^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho})$. Using a non-degenerate bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, William Goldman [Gol84] constructed a symplectic form ω^B on $X_G^g(\pi_1(S))$. We will use a mostly algebro-geometric approach in order to understand the construction of this symplectic form.

Afterwards, we will turn to the Lagrangian submanifold theorem as it has been proven by Adam Sikora in [Sik09]: Consider a compact connected 3-manifold M with boundary $\partial M = S$. Then the embedding $S \hookrightarrow M$ induces a map $r^* : X_G(\pi_1(M)) \to X_G(\pi_1(S))$ on the character varieties. The non-singular part of the image $Y_G(M) = [r^* X_G(\pi_1(M)) \cap X_G^g(\pi_1(S))]^{ns}$ is an isotropic submanifold of $X_G^g(\pi_1(S))$. We will see under which circumstances this isotropic submanifold is Lagrangian. Another Lagrangian submanifold is the fixed point set $\mathcal{L}_G$ of a certain anti-symplectic involution on $X_G^g(\pi_1(S))$. We will demonstrate the construction of a 3-manifold M by Laura Schaposnik and David Baraglia in [BS14], for which we have $Y_G(M) \subseteq \mathcal{L}_G$. Finally, we will briefly review the generalization of the symplectic form ω^B to a compact connected Kähler manifold as proven by Yael Karshon in [Kar92]. We will conclude by discussing to which extent the Lagrangian submanifold theorem can be generalized in this case.

Zusammenfassung

Sei Γ eine endlich erzeugte Gruppe, und sei G eine reduktive affine algebraische Gruppe über $\mathbb{C}$. Die Menge der Gruppenhomomorphismen $\mathrm{Hom}(\Gamma, G)$ zusammen mit einer von G kommenden affinen algebraischen Struktur heißt *Darstellungsvarietät* von Γ in G. Der kategorielle Quotient $X_G(\Gamma) = \mathrm{Hom}(\Gamma, G)/\!\!/G$ heißt *Charaktervarietät.* Für die Fundamentalgruppe $\pi_1(S)$ einer Fläche S ist die Charaktervarietät der *guten Darstellungen*, d.h. der irreduziblen Darstellungen mit unter der G-Wirkung abgeschlossenen Orbits, eine komplexe Mannigfaltigkeit $X_G^g(\pi_1(S))$ mit Tangentialräumen $H^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho})$. Mittels einer nicht-degenerierten Bilinearform $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ konstruierte William Goldman [Gol84] eine symplektische Form ω^B auf $X_G^g(\pi_1(S))$. Wir werden einen vorwiegend algebro-geometrischen Weg beschreiten, um die Konstruktion dieser symplektischen Form zu verstehen.

Daraufhin wenden wir uns dem Satz über lagrangesche Untermannigfaltigkeiten zu, wie er von Adam Sikora in [Sik09] bewiesen wurde: Man betrachte eine kompakte zusammenhängende 3-Mannigfaltigkeit M mit Rand $\partial M = S$. Die Inklusion $S \hookrightarrow M$ induziert eine Abbildung $r^* : X_G(\pi_1(M)) \to X_G(\pi_1(S))$ auf Charaktervarietäten. Der nicht-singuläre Teil des Bildes $Y_G(M) = [r^* X_G(\pi_1(M)) \cap X_G^g(\pi_1(S))]^{ns}$ ist eine isotrope Untermannigfaltigkeit von $X_G^g(\pi_1(S))$. Wir werden sehen unter welchen Umständen diese isotrope Untermannigfaltigkeit lagrangesch ist. Eine weitere lagrangesche Untermannigfaltigkeit ist die Fixpunktmenge $\mathcal{L}_G$ einer gewissen anti-symplektischen Involution auf $X_G^g(\pi_1(S))$. Wir demonstrieren die Konstruktion einer 3-Mannigfaltigkeit M nach Laura Schaposnik und David Baraglia [BS14], für die $Y_G(M) \subseteq \mathcal{L}_G$ gilt. Zum Schluss werden wir einen kurzen Überblick über die Verallgemeinerung der symplektischen Form ω^B für kompakte zusammenhängende Kähler-Mannigfaltigkeiten geben, wie sie von Yael Karshon in [Kar92] bewiesen wurde. Wir schließen mit einer Diskussion darüber, inwieweit der Satz über lagrangesche Untermannigfaltigkeiten auf diesen Fall verallgemeinert werden kann.

Hiermit erkläre ich, Lukas Daniel Sauer, geb. am 23.09.1994 in München, dass ich die vorliegende Masterarbeit selbst verfasst habe und keine anderen als die angegeben Quellen und Hilfsmittel verwendet habe.

Dossenheim, den 16. April 2020

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Index of notations

$\operatorname{ad} P$	the adjoint bundle associated to the principal bundle P, see p. 84
B	a (non-degenerate) $\mathbb{C}$-bilinear pairing, see p. 47, 48
$\mathbb{C}[\varepsilon]$	the $\mathbb{C}$-algebra of dual numbers $\mathbb{C}[X]/(X^2)$, see p. 28, 75
dim	the Krull dimension of a ring or topological space, see p. 31
$\dim_k$	the dimension of a k-vector space for $k = \mathbb{R}$ or $\mathbb{C}$ or the dimension of a smooth or complex manifold, see p. 31
eval^γ	the evaluation map $\mathrm{Hom}(\Gamma, G) \to G, \rho \mapsto \rho(\gamma)$, see p. 13
G	an affine algebraic group, in the major part of the text assumed to be reductive, see p. 22, 78
Γ	a finitely generated group, see p. 13
$G.\rho$	the orbit of ρ in $\mathrm{Hom}(\Gamma, G)$ with respect to the G-action, also denoted by O_ρ, see p. 22
G_ρ	the stabilizer subgroup of ρ in G, see p. 22
$\mathrm{Hom}(\Gamma, G)$	the representation variety, i.e. the set of group homomorphisms from Γ to G equipped with the structure of an affine algebraic set, see p. 13, 14
$\mathcal{H}om(\Gamma, G)$	the scheme $\mathrm{spec}(R(\Gamma, G))$, see p. 17
$\mathrm{Hom}^g(\Gamma, G)$	the set of good representations, see p. 23
$\mathrm{Hom}^i(\Gamma, G)$	the set of irreducible representations, see p. 23
ω^B	Goldman's symplectic form on the character variety, see p. 50
ω^B_ρ	Goldman's symplectic form on singular cohomology, see p. 49
$\tilde{\omega}^B$	a (non-degenerate) form on group cohomology, see p. 47
O_ρ	the orbit of ρ in $\mathrm{Hom}(\Gamma, G)$, also denoted by $G.\rho$, see p. 22
$\mathcal{O}_X$	the structure sheaf of a scheme, see p. 73
$\mathcal{O}_{X,x}$	the stalk of $\mathcal{O}_X$ at the point $x \in X$, see p. 73
$R(\Gamma, G)$	the universal representation algebra, see p. 16
$V^\vee$	the dual vector space to V, see p. 49
w	a word map, see p. 13
$X_G(\Gamma)$	the character variety $\mathrm{Hom}(\Gamma, G)/\!\!/G$, see p. 40
$\mathcal{X}_G(\Gamma)$	the scheme $\mathcal{H}om(\Gamma, G)/\!\!/G$, see p. 40
$X^g_G(\Gamma)$	the good character variety $\mathrm{Hom}^i(\Gamma, G)/\!\!/G$, see p. 41
$X^i_G(\Gamma)$	the irreducible character variety $\mathrm{Hom}^i(\Gamma, G)/\!\!/G$, see p. 41
$X/\!\!/G$	the categorical quotient, see p. 38

Introduction

At first sight, the science of mathematics may seem to be striving for two separate goals at once. On the one hand, it pursues the solution of concrete problems; on the other hand, it yearns to state the existing theory in the most general way possible. However, if all existing tools have failed to solve a concrete problem, generalization and abstraction are often the best way to find a new approach. Abstraction offers a deeper understanding of the nature of the problem, and sometimes, a cross-connection to another theory can be discovered along the way, enabling a completely new perspective.

The theory of moduli spaces is such a success story. Instead of looking at one specific geometric structure, we decide to look at the space of all such geometric structures, up to an equivalence relation. This *moduli space* bears a geometry on its own, and the study of its geometry allows drawing new conclusions on the nature of those specific geometric structures we were studying in the first place. This idea is not only relevant for geometry, but has since been in use by number theory and mathematical physics alike.

The moduli space that I am studying within this thesis is the *character variety*, the space of representations of a finitely generated group modulo conjugation. It is closely related to the moduli space of semistable Higgs bundles and to the moduli space of vector bundles with integrable connections [Sim94]. However, for this thesis, we will confine ourselves to studying the character variety *per se*. Below, I will briefly introduce the topics covered in the different sections of the thesis:

Let Γ be a finitely generated group, and let G be a reductive affine algebraic group. In Section 1, we will see that the set of homomorphisms $\mathrm{Hom}(\Gamma, G)$ carries the structure of an affine algebraic set induced by the structure of G. Together with this structure, it is called the *representation variety* of Γ in G. We will construct a scheme $\mathcal{H}om(\Gamma, G)$ with the same underlying topological space, but with a bigger structure sheaf. The representations $\rho \in \mathrm{Hom}(\Gamma, G)$ are the closed points of $\mathcal{H}om(\Gamma, G)$. Their Zariski tangent spaces $T_\rho\mathcal{H}om(\Gamma, G)$ are isomorphic to the 1-cocycles $Z^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho})$ of group cohomology.

G acts on $\mathrm{Hom}(\Gamma, G)$ via conjugation. In Section 2, we will consider the categorical quotient $X_G(\Gamma) = \mathrm{Hom}(\Gamma, G)/\!\!/G$; it is called the *character variety* of Γ in G. Let $\Gamma = \pi_1(S)$ be a *surface group*, i.e. the fundamental group of a closed compact connected surface S. The subspace of good representations, i.e. irreducible representations with closed orbits, turns out to be a complex manifold $X_G^g(\pi_1(S))$ whose tangent space at ρ is the first cohomology group $H^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho})$.

Let $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be a non-degenerate, $\mathbb{C}$-bilinear form, which is invariant under the adjoint action. In Section 3, we will construct Goldman's symplectic form ω^B on the manifold $X_G^g(\pi_1(S))$. It is induced by B and by the cup product $\cup$ of group cohomology. With respect to this symplectic structure, we will prove the Lagrangian submanifold theorem: For a compact connected 3-manifold M with boundary $\partial M = S$, the inclusion $S \hookrightarrow M$ induces a regular map $r^* : X_G(M) \to X_G(S)$ of the character varieties. The non-singular image in the good representations $Y_G(M) = [r^* X_G(\pi_1(M)) \cap X_G^g(\pi_1(S))]^{ns}$ is a disjoint union of isotropic submanifolds of $X_G^g(\pi_1(S))$, and a component of $Y_G(M)$ is Lagrangian if its inverse image contains a reduced representation.

In Section 4, we will study another Lagrangian submanifold of $X_G^g(S)$. An anti-holomorphic involution $f : S \to S$ induces an anti-symplectic involution $\hat{f} : X_G^g(S) \to X_G^g(S)$. The fixed point set of $\hat{f}$ is denoted by $\mathcal{L}_G$ and it turns out to be a Lagrangian submanifold of $X_G^g(S)$. The two submanifolds $\mathcal{L}_G$ and $Y_G(M)$ are not unrelated. We will construct a 3-manifold M with $\partial M = S$ for which the inclusion $Y_G(M) \subseteq \mathcal{L}_G$ holds.

In Section 5, we will replace the surface S by an arbitrary compact connected Kähler manifold K. We will review a generalization of the symplectic form on the character variety $X_G(K)$. Afterwards, we will briefly discuss how the Lagrangian submanifold theorem for surfaces can be generalized to the Kähler case.

I have included an appendix covering some of the theory that is needed throughout the thesis: the interplay between algebraic geometry and complex geometry, the theory of principal bundles and flat connections, and a number of isomorphism theorems for singular cohomology, group cohomology and de Rham cohomology.

Wherever possible, I will choose the tools from algebraic geometry. While this is certainly also a matter of taste, I do believe that it has one advantage: The Zariski tangent space can be calculated at all closed points of an affine algebraic set, whereas the tangent space of a subset of a manifold can only be calculated at points that are *a priori* known to be smooth. The question of which points are non-singular can thus be answered by studying the Zariski tangent space. This choice comes with a drawback, as the algebro-geometric proofs can be quite technical, and therefore might be less intuitive. At two points in the text, I will make up for this by describing an alternative proof using tools from differential geometry rather than algebraic geometry. I believe that this also illustrates how colorful this theory is: there may be several ways to prove the same statement using tools from different areas of mathematics.

I WOULD LIKE TO THANK my advisors, Anna Wienhard and Peter Albers, for the suggestion of such a rich topic, for offering multifaceted insights into the garden of geometry, and for allowing me the freedom to find my own way through the thicket that leads to the garden. Nonetheless, you always found time for a detailed explanation whenever I needed it.

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1. Representation varieties

1.1. Embedding the set of homomorphisms

Let Γ be a finitely generated group and let G be any group. Consider the set $\mathrm{Hom}(\Gamma, G)$ of group homomorphisms from Γ to G.

Pick a presentation $\Gamma = \langle \gamma_i | w_\lambda \rangle$. We choose the index set $i \in \{1, \dots, N\}$ to be finite, however this cannot necessarily be ensured for the index set $\lambda \in \Lambda$. A relation w_λ is a *word map*: this means that it is a composition of the group operation and group inversion. The evaluation on generators,

$$\mathrm{Hom}(\Gamma, G) \to G^N, \quad \rho \mapsto (r_i)_i = (\rho(\gamma_i))_i,$$

is an injective map. Via this map, we consider $\mathrm{Hom}(\Gamma, G)$ a subset of $G^N = G \times \ldots \times G$. The relations w_λ can be considered maps $w_\lambda : G^N \to G$ simply by plugging in elements of G^N instead of the γ_i. Therefore $\mathrm{Hom}(\Gamma, G)$ can be identified with $\bigcap_\lambda w_\lambda^{-1}(e) \subseteq G^N$.

The evaluation of a homomorphism is actually a restriction of a map from G^N to G. Indeed, let $\gamma \in \Gamma$ be an arbitrary element. It is a word in the generators, i.e. $\gamma = w^\gamma(\gamma_1, \dots, \gamma_N)$. The evaluation of representations at γ

$$\mathrm{eval}^\gamma : \mathrm{Hom}(\Gamma, G) \to G, \quad \rho \mapsto \rho(\gamma)$$

is the restriction of the word map

$$w^\gamma : G^N \to G, \quad (g_i)_i \mapsto w^\gamma((g_i)_i).$$

Again, w^γ can be considered a map on G^N simply by plugging in elements of G instead of elements of the free group.

We can equip $\mathrm{Hom}(\Gamma, G)$ with a G-left action. G acts via conjugation, $(g.\rho)(\gamma) = g\rho(\gamma)g^{-1}$. G also acts on G^N by conjugating component-wise. On $\mathrm{Hom}(\Gamma, G) \subseteq G^N$, the conjugation of representations and the conjugation on G^N coincide. Sometimes, it will be easier to analyze the G-operation on G^N even though we are interested in the action on $\mathrm{Hom}(\Gamma, G)$.

Lemma 1.1. *Let G be any group, and let G act on G and G^N via conjugation. A word map $w : G^N \to G$ is G-equivariant.*

Proof. This can easily be seen by induction over the word length. A word $w : G^N \to G$ of length 0 simply maps everything to $e \in G$ and is therefore trivially G-equivariant.

Let w be a word of length k. We can either write

$$w(g_1, \dots, g_N) = v(g_1, \dots, g_N)g_i \tag{1}$$

or

$$w(g_1, \dots, g_N) = v(g_1, \dots, g_N)g_i^{-1}, \tag{2}$$

for some i and a word $v : G^N \to G$ of length $k - 1$. By the induction hypothesis, v is G-equivariant. In the case of Eq. (1), we have

$$\begin{aligned} w(hg_1h^{-1}, \dots, hg_Nh^{-1}) &= v(hg_1h^{-1}, \dots, hg_Nh^{-1})hg_ih^{-1} \\ &= hv(g_1, \dots, g_N)h^{-1}hg_ih^{-1} \\ &= hv(g_1, \dots, g_N)g_ih^{-1} = hw(g_1, \dots, g_N)h^{-1}, \end{aligned}$$

which proves G-equivariance. The case of Eq. (2) is proven accordingly. $\square$

Denoting the action on G^N and G by σ_{G^N} and σ_G, respectively. The lemma shows that the diagram

$$\begin{array}{ccc} G \times G^N & \xrightarrow{\sigma_{G^N}} & G^N \\ {\scriptstyle (\mathrm{id}, w)}\downarrow & & \downarrow{\scriptstyle w} \\ G \times G & \xrightarrow{\sigma_G} & G \end{array} \tag{3}$$

commutes, a fact that will come in handy later on.

1.2. The algebraic structure of representation varieties

From now on, let G be an affine complex algebraic group. We will now construct an affine algebraic structure on $\mathrm{Hom}(\Gamma, G)$. $\mathrm{Hom}(\Gamma, G)$ equipped with this structure is called *representation variety* of Γ in G. Despite this name, there are examples where the representation variety is not irreducible.

Proposition 1.2. *Via the above inclusion,* $\mathrm{Hom}(\Gamma, G)$ *possesses the structure of an affine algebraic set. Up to isomorphism, this structure is independent of the choice of generators.*

It is immediately clear that the G-action $g.\rho(\cdot) = g\rho(\cdot)g^{-1}$ and the evaluation map $\mathrm{eval}^\gamma : \mathrm{Hom}(\Gamma, G) \to G, \rho \mapsto \rho(\gamma)$ are regular maps, as they descend from G^N.

Proof of Proposition 1.2. Some ideas in this proof are taken from [Wei64]. G^N is an affine algebraic set because G is. In other words, G^N is the zero locus $Z(S) \subset \mathbb{C}^n$ of a set $S \subset \mathbb{C}[T_1, \ldots, T_n]$ of polynomial functions. We now want to show that this already gives us an affine algebraic structure on $\mathrm{Hom}(\Gamma, G)$. Each word map w_λ can be regarded as a map $G^N \to G$ simply by plugging in elements of G instead of the γ_i. As mentioned in Section 1.1, we have $\mathrm{Hom}(\Gamma, G) = \bigcap_\lambda w_\lambda^{-1}(e)$ as a subset of G^N. Each $w_\lambda : G^N \to G$ consists of multiplications and inversions, which are regular maps because G is an algebraic group. So, the w_λ give us a set of polynomials S' such that

$$\bigcap_\lambda w_\lambda^{-1}(e) = Z(S) \cap Z(S') = Z(S \cup S') \subset \mathbb{C}^n,$$

so $\mathrm{Hom}(\Gamma, G)$ is indeed a zero locus of polynomials, i.e. an affine algebraic set. We see here that Λ does not have to be finite: An infinite intersection of Zariski-closed sets is still closed, so an infinite intersection will again be an affine algebraic set.

It remains to be shown that this structure is independent of the choice of generators of Γ. Let γ_j with $j \in \{1, \ldots, N'\}$ be another finite set of (generating) elements. $\{\gamma_i, \gamma_j\}$ is also a set of generators, so that $\Gamma = \langle \gamma_i, \gamma_j | w_{\lambda'} \rangle$ for a new set of relations $\{w_{\lambda'}\}$. As γ_i are already generators, we have word maps such that $\gamma_j = w^{\gamma_j}(\gamma_i)$. We can consider these maps as regular functions $G^N \to G$, so that we get a regular function

$$G^N \to G^{N+N'}, (g_i) \mapsto (g_i, w^{\gamma_j}(g_i)).$$

By construction, it can be restricted to a map $\bigcap_\lambda w_\lambda^{-1}(e) \to \bigcap_{\lambda'} w_{\lambda'}^{-1}(e)$ and this map is bijective, as both sets represent $\mathrm{Hom}(\Gamma, G)$. The inverse function is the restriction of the natural projection map $G^{N+N'} \to G^N, (g_i, g_j) \mapsto (g_i)$. Thus, the inverse function is also regular. Therefore, the two affine structures can be identified by regular isomorphisms. This shows the affine algebraic structure of $\mathrm{Hom}(\Gamma, G)$ is independent of the set of generators up to isomorphism. □

Lemma 1.3. *1. Let $r : \Gamma \to \Delta$ be group homomorphism of two finitely generated groups. Then the induced map $r^* : \mathrm{Hom}(\Delta, G) \to \mathrm{Hom}(\Gamma, G), \rho \mapsto \rho \circ r$ is a regular map. Obviously, this map commutes with the G-operation.*

2. Let $s : G \to H$ be a homomorphism of two algebraic groups (i.e. a regular group homomorphism). Then the induced map $s_* : \mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Gamma, H), \rho \mapsto s \circ \rho$ is regular.

Proof. 1. Choose finite generating sets γ_i and δ_j of Γ and Δ, respectively. Consider embeddings $\mathrm{Hom}(\Gamma, G) \subset G^N$ and $\mathrm{Hom}(\Delta, G) \subset G^{N'}$ as explained above. Under these identifications, the induced map $\bar{r}$ maps $(\rho(\delta_j))_j \in G^{N'}$ to $(\rho(r(\gamma_i)))_i \in G^N$. However, $r(\gamma_i)$ is a word map in the δ_j. Therefore, as ρ is a homomorphism, $\rho(r(\gamma_i))$ is also a word map in the $\rho(\delta_j)$. Any word map is regular as a composition of multiplications and inversions, which proves that r is regular.

2. The morphism $s^N : G^N \to H^N$ is regular as a product of regular maps. The induced map s_* is the restriction of s^N to $\mathrm{Hom}(\Gamma, G)$. This proves regularity. □

Example 1.4. *The fundamental group $\pi_1(M)$ of any compact finite-dimensional manifold M is finitely presented.* $\mathrm{Hom}(\pi_1(M), G)$ *is the representation variety which we will mostly study in this thesis. Surface groups $\pi_1(S)$ for closed compact connected surfaces S have a particularly simple structure, and many statements are only known in this case. In particular, we will only prove the Lagrangian submanifold Theorem 3.12 for character varieties of surfaces. Possible generalizations will be discussed in Section 5.*

The finitely generated group Γ becomes a topological group when equipped with the discrete topology. We have the following statement on the topology of $\mathrm{Hom}(\Gamma, G)$.

Remark 1.5. *Let G be an affine algebraic group G. Equip G^N with the complex analytic topology. The subspace topology on* $\mathrm{Hom}(\Gamma, G) \subseteq G^N$ *via the above inclusion coincides with the compact-open topology.*

Proof. For compact sets $K \subseteq \Gamma$ and open sets $U \subseteq G$, the sets

$$V(K, U) = \{\rho \in \mathrm{Hom}(\Gamma, G), \rho(K) \subseteq U\}$$

are a subbase of the compact-open topology. Note that $V(K, U) = \bigcap_{\gamma \in K} (\mathrm{eval}^\gamma)^{-1}(U)$. We begin by showing that these sets are open in the subspace topology of $\mathrm{Hom}(\Gamma, G) \subseteq G^N$ as well.

Note that Γ is discrete, so a compact set K is in fact a finite subset of Γ, i.e. a finite set of words $w_\nu(\gamma_i)$. As before, a word can be considered a continuous map $w_\nu : G^N \to G, (g_i) \mapsto w_\nu(g_i)$, and $w_\nu^{-1}(U)$ is an open subset of G^N. The evaluation is the restriction of w_ν, so we have

$$V(K, U) = \bigcap_{\gamma \in K} (\mathrm{eval}^\gamma)^{-1}(U) = \mathrm{Hom}(\Gamma, G) \cap \bigcap_\nu w_\nu^{-1}(U).$$

This is an open set in the subset topology because there are only finitely many v.

Now, consider an open set in the subspace topology, i.e. $\mathrm{Hom}(\Gamma, G) \cap \prod_i U_i$ for U_i open in G. It remains to be shown that this is an open set in the compact-open topology. Clearly, we have

$$\mathrm{Hom}(\Gamma, G) \cap \prod_i U_i = \bigcap_i V(\{\gamma_i\}, U_i),$$

which is an open set in the compact-open topology. Thus, the two topologies are identical. □

Note that the last argument does not work when equipping G^N with the Zariski topology: The Zariski topology of the product is finer than the product topology, so that there could be open sets in G^N which are not of the form $\prod_i U_i$. At least, the argument shows that the compact-open topology on $\mathrm{Hom}(\Gamma, G)$ with respect to the Zariski topology on G is coarser than the Zariski topology on $\mathrm{Hom}(\Gamma, G)$.

1.3. The universal representation algebra

Any affine algebraic set $X = \mathrm{specm}(\mathcal{O}_X(X))$ defines a scheme $X^{\mathrm{sch}} = \mathrm{spec}(\mathcal{O}_X(X))$. The underlying topological space of X consists of the closed points of X^{sch}, see Theorem A.3 for details. Since all the information on X can be retrieved from X^{sch}, we will use the symbol X for both the affine algebraic set and the scheme whenever the context is clear. In particular, $\mathrm{Hom}(\Gamma, G)$ and G define schemes $\mathrm{Hom}(\Gamma, G)^{\mathrm{sch}} = \mathrm{spec}(\mathcal{O}_{\mathrm{Hom}}(\mathrm{Hom}))$ and $G^{\mathrm{sch}} = \mathrm{spec}(\mathcal{O}_G(G))$. G^{sch} is a group scheme over $\mathbb{C}$. In this section, we will define another affine scheme over $\mathbb{C}$ with the same underlying topological space as $\mathrm{Hom}(\Gamma, G)^{\mathrm{sch}}$. This scheme, denoted by $\mathcal{H}om(\Gamma, G)$, has a "bigger" structure sheaf and turns out to be more useful when calculating tangent spaces, see Section 1.5.

Consider the points functor

$$G(-) : \mathbf{Alg}_{\mathbb{C}} \to \mathbf{Grp}, \quad A \mapsto G(A) = \mathrm{Hom}_{\mathbf{Alg}_{\mathbb{C}}}(\mathcal{O}_G(G), A),$$

it maps any $\mathbb{C}$-algebra A to the set of $\mathbb{C}$-algebra homomorphisms from $\mathcal{O}_G$ to A. $G(A)$ naturally carries the structure of a group and, in particular, we obtain $G(\mathbb{C}) = G$.

Definition 1.6. A commutative $\mathbb{C}$-algebra $R(\Gamma, G)$ together with a group homomorphism $\rho_U : \Gamma \to G(R(\Gamma, G))$ is called *universal representation algebra of* Γ *into* G, if it fulfills the following universal property: For every commutative $\mathbb{C}$-algebra A and every representation $\rho : \Gamma \to G(A)$, there is a unique $\mathbb{C}$-algebra homomorphism $f : R(\Gamma, G) \to A$ such that the diagram

$$\begin{tikzcd} & G(R(\Gamma, G)) \arrow[d, "G(f)"] \\ \Gamma \arrow[ur, "\rho_U"] \arrow[r, "\rho"] & G(A) \end{tikzcd}$$

commutes. Here, $G(f)$ is the representation induced by f. The homomorphism ρ_U is also called *universal representation.*

Proposition 1.7. *For every finitely generated group* Γ *and every affine complex algebraic group* G*, the following holds:*

1. *The universal representation algebra exists and is unique up to unique isomorphism of $\mathbb{C}$-algebras. ρ_U is unique up to composition with the respective induced group isomorphism.*

2. *We will write $\mathcal{H}om(\Gamma, G) = \operatorname{spec}(R(\Gamma, G))$. This scheme's reduction is given by $\mathcal{H}om(\Gamma, G)_{\mathrm{red}} = \operatorname{Hom}(\Gamma, G)^{\mathrm{sch}}$.*

3. *The underlying topological spaces of $\mathcal{H}om(\Gamma, G)$ and $\operatorname{Hom}(\Gamma, G)^{\mathrm{sch}}$ coincide. In particular, the set of closed points of $\mathcal{H}om(\Gamma, G)$ is exactly $\operatorname{Hom}(\Gamma, G)$.*

Proof. 1. The following explicit construction of $R(\Gamma, G)$ has been performed in [Sik09, Lem. 26].

G is an algebraic group, so $G = \operatorname{specm}(H)$, where $H = \mathcal{O}_G(G)$ is the $\mathbb{C}$-Hopf algebra of global sections.

Step 1.i: Construction for a free group. We will first construct $R(F, G)$ for the finitely generated free group $F = \langle \gamma_i \rangle$, $i \in \{1, \dots, N\}$. Define

$$R(F, G) = \bigotimes_{i=1}^{N} H.$$

We need a group homomorphism

$$\rho_U^F : F \to G(R(F, G)) = \operatorname{Hom}_{\mathbb{C}}\left(H, \bigotimes_i H\right).$$

We set

$$\rho_U^F(\gamma_i) = \iota_i : H \to \bigotimes_i H, \quad x \mapsto 1 \otimes \ldots \otimes x \otimes \ldots \otimes 1,$$

which embeds H exactly at the i-th copy in $R(F, G)$. We can extend this definition to a group homomorphism by setting

$$x \cdot y \in \Gamma \mapsto \rho_U^F(x) \cdot \rho_U^F(y) \in G(R(\Gamma, G))$$

and

$$x^{-1} \mapsto \rho_U^F(x)^{-1} \in G(R(\Gamma, G)),$$

where the group structure on the right-hand side is induced by the Hopf algebra structure of $H = \mathcal{O}_G(G)$. Explicitly, this would mean

$$\rho_U^F(x) \cdot \rho_U^F(y) = m \circ (\rho_U^F(x) \otimes \rho_U^F(y)) \circ \mu^\sharp, \quad \rho_U^F(x)^{-1} = \rho_U^F(x) \circ \iota^\sharp,$$

where $\mu^\sharp : H \to H \otimes H$ is the coproduct, $\iota^\sharp : H \to H$ is the antipode, and $m : H \times H \to H$ is the multiplication of the Hopf algebra, but we will never use this explicit definition.

We have a well-defined group homomorphism $\rho_U^F : F \to G$ and want to check whether it fulfills the universal property. For every $\mathbb{C}$-algebra A together with a representation $\rho : \Gamma \to G(A) = \operatorname{Hom}_{\mathbb{C}}(H, A)$, we have ring homomorphisms

$$\rho(\gamma_i) : H \to A.$$

and thus by definition of the tensor product, we canonically obtain a unique induced ring homomorphism

$$f^F_\rho : \bigotimes_i H \xrightarrow{\rho(\gamma_1)\otimes\ldots\otimes\rho(\gamma_N)} A.$$

By definition, the diagram

$$\begin{tikzcd} & G(R(F,G)) \arrow[d, "G(f^F_\rho)"] \\ F \arrow[ur, "\rho^F_U"] \arrow[r, "\rho"] & G(A) \end{tikzcd}$$

commutes. By construction, f^F_ρ is unique with this property. So, any other algebra $\tilde{R}$ fulfilling the universal property is uniquely isomorphic to $R(F,G)$.

Step 1.ii: Construction for any finitely generated group. Now, consider an arbitrary finitely generated group $\Gamma = \langle \gamma_i | w_\lambda \rangle = F/\langle w_\lambda(\gamma_i)_i, \lambda \in \Lambda \rangle$. Let $\pi_\Gamma : F \longrightarrow \Gamma$ be the canonical projection. Then the ρ^F_U above is possibly no longer well-defined, because

$$w_\lambda(\gamma_1, \ldots, \gamma_N) = e \text{ in } \Gamma,$$

but we could possibly have

$$\rho_U(w_\lambda(\gamma_1, \ldots, \gamma_N)) = w_\lambda(\rho_U(\gamma_1), \ldots, \rho_U(\gamma_N)) \neq e_{G(R(F,G))} \text{ in } G(R(F,G)).$$

We correct this by constructing the following ideal

$$I_\Gamma = \left\langle w_\lambda(\rho_U(\gamma_i)(x) - e_{G(R(F,G))}(x), \text{ for } x \in H, \lambda \in \Lambda \right\rangle \subseteq R(F,G).$$

We define

$$R(\Gamma,G) = R(F,G)/I_\Gamma,$$

and set $\pi_{R(\Gamma,G)} : R(F,G) \longrightarrow R(\Gamma,G)$ to be the canonical projection. Now we define ρ_U to be the canonical group homomorphism that makes

$$\begin{tikzcd} F \arrow[r, "\rho^F_U"] \arrow[d, "\pi_\Gamma"'] & G(R(F,G)) \arrow[d, "G(\pi_{R(\Gamma,G)})"] \\ \Gamma \arrow[r, "\rho_U"] & (R(\Gamma,G)) \end{tikzcd}$$

commute. Furthermore, the fact that $\rho : \Gamma \longrightarrow G(A)$ is a group homomorphism implies that $w_\lambda(\rho(\gamma_i)) = e_{G(A)}$. In particular, the ring homorphism $f^F_{f\circ\pi_\Gamma}$ factorizes over I_Γ, which means that we have a commutative diagram

$$\begin{tikzcd} R(F,G) \arrow[r, "f^F_{\rho\circ\pi_\Gamma}"] \arrow[d, "\pi_{R(\Gamma,G)}"'] & A \arrow[d, equal] \\ R(\Gamma,G) \arrow[r, "f_\rho"] & A. \end{tikzcd}$$

This shows that $R(\Gamma,G)$, ρ_U and f_ρ are well-defined, and it can be easily seen that they fulfill the universal property. This proves point 1 of Proposition 1.7. ◻

Before we continue with parts 2 and 3 of the proof, remark the following:

Remark 1.8. *Let $w^\sharp : H \to \bigotimes_i H$ be the $\mathbb{C}$-algebra homomorphism dual to a word map $w : G^N \to G$. Then $w(\rho_U(\gamma_i)_i) \equiv w^\sharp$. In particular, the ideal I_Γ is given by*

$$I_\Gamma = \left\langle w_\lambda^\sharp(x) - e_{G(R(F,G))}(x), \text{ for } x \in \mathcal{O}_G(G), \lambda \in \Lambda \right\rangle \subseteq R(F,G).$$

Proof. We defined $\rho_U(\gamma_i) : H \to \bigotimes_i H$ to be $a \mapsto 1 \otimes \ldots \otimes a \otimes \ldots \otimes 1$, so

$$\rho_U(\gamma_1) \otimes \ldots \otimes \rho_U(\gamma_N) : \bigotimes_i H \to \bigotimes_i H$$

is simply the identity. Note that $w(\rho_U(\gamma_i)_i)$ is simply the evaluation of the word map $w : G(\bigotimes_i H)^N \to G(\bigotimes_i H)$ at the element $\rho_U(\gamma_i)_i$. According to the universal property of the tensor product, we have

$$G(\bigotimes_i H)^N = \prod_i \mathrm{Hom}_{\mathbb{C}}(H, \bigotimes_i H) = \mathrm{Hom}_{\mathbb{C}}(\bigotimes_i H, \bigotimes_i H) = G^N(\bigotimes_i H).$$

Therefore, the word map $w : G(\bigotimes_i H)^N \to G(\bigotimes_i H)$ is nothing but the induced morphism $G(w) : G^N(\bigotimes_i H) \to G(\bigotimes_i H)$, i.e. $\phi \mapsto \phi \circ w_\lambda^\sharp$. In particular,

$$w(\rho_U(\gamma_i)_i) = G(w)(\rho_U(\gamma_1) \otimes \ldots \otimes \rho_U(\gamma_N)) = \mathrm{id} \circ w^\sharp = w^\sharp.$$

□

Proof of Proposition 1.7.2 and .3. Using the Remark 1.8, we can now complete the proof of the theorem.

2. Earlier, we defined the subvariety $\mathrm{Hom}(\Gamma, G) = \bigcap_\lambda w_\lambda^{-1}(e) \subseteq G^N$. A subvariety always carries the reduced structure, so

$$\mathcal{O}_{\mathrm{Hom}}(\mathrm{Hom}) = \mathcal{O}_{G^N}(G^N)/\sqrt{J} = \left(\bigotimes_i H\right)/\sqrt{J}.$$

The ideal J is generated by all the $w_\lambda^\sharp(x) - e_{G^N}(x)$. $\sqrt{J}$ is its radical ideal. In particular, Remark 1.8 shows that $J = I_\Gamma$, which means that

$$\mathcal{O}_{\mathrm{Hom}}(\mathrm{Hom}) = R(F,G)/\sqrt{I_\Gamma} = R(\Gamma, G)/\sqrt{0},$$

so that $\mathrm{Hom}(\Gamma, G)$ is indeed the reduction of $\mathcal{H}om(\Gamma, G)$.

3. This is a general property of the reduction of a scheme [see Sta19, Section 01IZ]. □

We have two closed immersions

$$\mathrm{Hom}(\Gamma, G) \hookrightarrow \mathcal{H}om(\Gamma, G) \hookrightarrow G^N.$$

The G-action on G^N descends to a G-action on $\mathrm{Hom}(\Gamma, G)$. This is not yet sufficent to prove that the action descends to $\mathcal{H}om(\Gamma, G)$ as well, even though the two schemes coincide topologically. However, we can show this by proving the invariance of the ideal I_Γ.

Proposition 1.9. *The G-action on G^N descends to a G-action on the closed subscheme $\mathcal{H}om(\Gamma, G)$. This means that we have a commutative diagram*

$$\begin{tikzcd} G \times G^N \arrow[r, "\sigma"] & G^N \\ G \times \mathcal{H}om(\Gamma, G) \arrow[u, "(\mathrm{id},\iota)"] \arrow[r, "\sigma|"] & \mathcal{H}om(\Gamma, G). \arrow[u, "\iota"'] \end{tikzcd}$$

Proof. As before, denote the Hopf algebra of global sections by $H = \mathcal{O}_G(G)$. The diagram above is dual to the diagram

$$\begin{tikzcd} \bigotimes_i H \arrow[r, "\sigma^\sharp"] \arrow[d, "\iota^\sharp"'] & H \otimes \bigotimes_i H \arrow[d, "\mathrm{id} \otimes \iota^\sharp"] \\ \bigotimes_i H/I_\Gamma \arrow[r, dashed, "\bar{\sigma}^\sharp"] & H \otimes \bigotimes_i H/I_\Gamma, \end{tikzcd}$$

where $\sigma^\sharp$ is the dual action, $\iota^\sharp$ is the natural projection, and I_Γ is the ideal from Proposition 1.7. We need to show that the dashed arrow is well-defined, i.e. that $\sigma^\sharp(I_\Gamma) \subseteq H \otimes I_\Gamma$. Taking into account Remark 1.8, it is sufficient to show that

$$\sigma^\sharp(w_\lambda^\sharp(x) - e_{G(\bigotimes_i H)}(x)) \in H \otimes I_\Gamma$$

for any $x \in H$ and any $\lambda \in \Lambda$. From Eq. (3) after Lemma 1.1, we obtain $\sigma^\sharp \circ w_\lambda^\sharp = (\mathrm{id} \otimes w_\lambda^\sharp) \circ \sigma_G^\sharp$, where σ_G is the conjugation action on G.

Note that $e_{G(\bigotimes_i H)}$ is given by $s^\sharp \circ e^\sharp$, where $s^\sharp : \mathbb{C} \to \bigotimes_i H$ is the $\mathbb{C}$-algebra structure morphism (dual to $s : G^N \to \mathrm{spec}(\mathbb{C})$) and $e^\sharp : H \to \mathbb{C}$ is the counit (dual to $e : \mathrm{spec}(\mathbb{C}) \to G$). The following diagram commutes:

$$\begin{tikzcd} G \times G^N \arrow[r, "\sigma"] \arrow[d, "(\mathrm{id},s)"'] & G^N \arrow[d, "s"] \\ G \times \mathrm{spec}(\mathbb{C}) \arrow[r] \arrow[d, "(\mathrm{id},e)"'] & \mathrm{spec}(\mathbb{C}) \arrow[d, "e"] \\ G \times G \arrow[r, "\sigma_G"] & G. \end{tikzcd}$$

This shows that $\sigma^\sharp \circ e_{G(\bigotimes_i H)} = \sigma^\sharp s^\sharp e^\sharp = (\mathrm{id} \otimes s^\sharp e^\sharp) \circ \sigma_G^\sharp = (\mathrm{id} \otimes e_{G(\bigotimes_i H)}) \circ \sigma_G^\sharp$. In summary, we see that

$$\begin{aligned} \sigma^\sharp(w_\lambda^\sharp(x) - e_{G(\bigotimes_i H)}(x)) &= (\mathrm{id} \otimes w_\lambda^\sharp) \circ \sigma_G^\sharp(x) - (\mathrm{id} \otimes e_{G(\bigotimes_i H)}) \circ \sigma_G^\sharp(x) \\ &= (\mathrm{id} \otimes (w_\lambda^\sharp - e_{G(\bigotimes_i H)})) \circ \sigma_G^\sharp(x), \end{aligned}$$

which is an element of $H \otimes I_\Gamma$. This remained to be shown. $\square$

Lemma 1.10. *Let Γ and Δ be two finitely generated groups, and let $r : \Gamma \to \Delta$ be a group homomorphism. There is an induced $\mathbb{C}$-algebra morphism $(r^*)^\sharp : R(\Gamma, G) \to R(\Delta, G)$ and thus a morphism of $\mathbb{C}$-schemes $r^* : \mathcal{H}om(\Delta, G) \to \mathcal{H}om(\Gamma, G)$.*

Proof. This follows almost immediately from the universal property. Let $\rho_U : \Delta \to G(R(\Delta, G))$ be the universal representation of Δ in G. Consider the composition $\rho_U \circ r : \Gamma \to G(R(\Delta, G))$. In accordance with the universal property, it defines a unique $\mathbb{C}$-algebra morphism $(r^*)^\sharp$ as required. $\square$

Let A be a $\mathbb{C}$-algebra. The evaluation map $\mathrm{eval}^\gamma : \mathrm{Hom}(\Gamma, G) \to G, \rho \mapsto \rho(\gamma)$ is a regular map, so that it induces a map $\mathrm{eval}^\gamma_{(A)} : \mathrm{Hom}(\Gamma, G)(A) \to G(A)$ on A-valued points. Via the closed immersion of schemes $\iota : \mathrm{Hom}(\Gamma, G) \hookrightarrow \mathcal{H}om(\Gamma, G)$, we obtain an embedding $\iota_{(A)} : \mathrm{Hom}(\Gamma, G)(A) \hookrightarrow \mathcal{H}om(\Gamma, G)(A)$.

Lemma 1.11. *Let ϕ be an element of $\mathrm{Hom}(\Gamma, G)(A)$, and consider it as an element of $\mathcal{H}om(\Gamma, G)(A)$ via ι_*. We have*

$$\mathrm{eval}^\gamma_{(A)}(\phi) = G(\phi)(\rho_U(\gamma))$$

for any element $\gamma \in \Gamma$.

Proof. $\phi \in \mathrm{Hom}(\Gamma, G)(A)$ is a homomorphism $\mathcal{O}_{\mathrm{Hom}}(\mathrm{Hom}) \to A$ of $\mathbb{C}$-algebras. $\mathrm{eval}^\gamma : \mathrm{Hom}(\Gamma, G) \to G$ corresponds to a homomorphism $(\mathrm{eval}^\gamma)^\sharp : \mathcal{O}_G(G) \to \mathcal{O}_{\mathrm{Hom}}(\mathrm{Hom})$ of $\mathbb{C}$-algebras. By definition, we simply have $\mathrm{eval}^\gamma_{(A)}(\phi) = \phi \circ \mathrm{eval}^\sharp_\gamma$. The element $\rho_U(\gamma) \in G(R(\Gamma, G))$ is a homomorphism $\mathcal{O}_G(G) \to R(\Gamma, G)$ of $\mathbb{C}$-algebras, and we have

$$G(\phi)(\rho_U(\gamma)) = \phi \circ \iota^\sharp \circ \rho_U(\gamma).$$

Therefore, it suffices to show that $\iota^\sharp \circ \rho_U(\gamma) = \mathrm{eval}^\gamma$. This can be seen easily, as both $\rho_U(\gamma)$ and eval^γ are induced by the word map $w^\gamma : G^N \to G$. This means that the diagram

$$\begin{tikzcd}
 & \mathcal{O}_{G^N}(G^N) \arrow[d] \\
\mathcal{O}_G(G) \arrow[ur, "(w^\gamma)^\sharp"] \arrow[r, "\rho_U(\gamma)"] \arrow[dr, "(\mathrm{eval}^\gamma)^\sharp"'] & R(\Gamma, G) \arrow[d, "\iota^\sharp"] \\
 & \mathcal{O}_{\mathrm{Hom}}(\mathrm{Hom})
\end{tikzcd}$$

commutes, which ends our proof. $\square$

1.4. Subsets of the representation variety

Due to their more convenient differential or algebraic properties, it is convenient to consider a couple of subsets of the representation variety $\mathrm{Hom}(\Gamma, G)$. Let G be an affine algebraic group. A subgroup $B \subseteq G$ is called *Borel subgroup* if it is a maximal connected solvable subgroup. A *parabolic subgroup* $P \subseteq G$ is a subgroup that contains a Borel subgroup. A subgroup is parabolic if and only if the homogenous space G/P is a complete variety [see Bor91, Cor. 11.2].

Definition 1.12. Let G be an affine algebraic group, and let H be a subgroup.

1. H is called *irreducible* if it is not contained in any proper parabolic subgroup of G.

2. H is called *completely reducible* if for every parabolic subgroup $P \subset G$ that contains H, there is a Levi subgroup of P containing H as well.

3. We call a representation $\rho \in \mathrm{Hom}(\Gamma, G)$ *irreducible* (or *completely reducible*) if $\rho(\Gamma) \subset G$ is irreducible or completely reducible, respectively.

4. Let $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation. We call ρ Ad-*irreducible* or Ad-*completely reducible* if the representation $\mathrm{Ad} \circ \rho : \Gamma \to \mathrm{GL}(\mathfrak{g})$ is irreducible or completely reducible, respectively.[1]

Obviously, every irreducible subgroup is completely reducible and thus, irreducible representations are completely reducible.

Example 1.13. *This example shows that the above definitions coincide with the classical notions of irreducible and completely reducible representations. Let V be a $\mathbb{C}$-vector space and let $G = \mathrm{GL}(V)$ be the general linear group. Consider a homomorphism $\rho : \Gamma \to \mathrm{GL}(V)$, i.e. a representation of Γ in V. Then the following holds:*

1. *The subgroup $\rho(\Gamma)$ is an irreducible subgroup of $\mathrm{GL}(V)$ if and only if the representation in V is irreducible, i.e. if the only Γ-invariant subspaces of V are 0 and V.*

2. *The subgroup $\rho(\Gamma)$ is a completely reducible subgroup of $\mathrm{GL}(V)$ if and only if the representation in V is completely reducible (or* semi-simple*), i.e. if V is the direct sum of irreducible Γ-invariant subspaces.*

These results can be found in [Ser05, Sec. 1.3, Sec. 3.1].

By $G.\rho \subset \mathrm{Hom}(\Gamma, G)$, we will denote the orbit of ρ under the G-action. Whenever the group G is obvious, we will write $O_\rho = G.\rho$. By $G_\rho \subset G$, we will denote the stabilizer of ρ. By $C(G) \subset G$, we will denote the center of G. This said, we define:

Definition 1.14. A representation $\rho \in \mathrm{Hom}(\Gamma, G)$ is *good*, if O_ρ is closed in G and the stabilizer of ρ coincides with the center, $G_\rho = C(G)$.[2]

Let G be an affine algebraic group. The unity component of the intersection of all Borel subgroups is a group called the *radical* of G,

$$RG = \left(\bigcap_{B \subseteq G \text{ Borel}} B \right)^0 .$$

Its subgroup of unipotent elements is denoted by RG^u. G is called *reductive*, if the unipotent radical RG^u is trivial. We have the following topological properties and inclusions.

Proposition 1.15. *Let G be a reductive group. Let Γ be any finitely generated group. We have:*

1. *The set of irreducible representations $\mathrm{Hom}^i(\Gamma, G)$ is a Zariski-open subset of $\mathrm{Hom}(\Gamma, G)$.*

[1] In [BS14], Ad-completely reducible representations are called reductive.
[2] In [BS14], this is called a simple point.

2. *The set of* Ad*-irreducible representations is a Zariski-open subset of* $\mathrm{Hom}(\Gamma, G)$.

3. *The set of good representations is an open subset of* $\mathrm{Hom}^i(\Gamma, G)$ *in the complex topology.*

4. *Every* Ad*-irreducible representation is good.*

We will denote the subset of good representations and the subset of irreducible representations by

$$\mathrm{Hom}^g(\Gamma, G) \text{ and } \mathrm{Hom}^i(\Gamma, G),$$

respectively. Before proving this proposition, I would like to remark the following. The proof of point 3 uses a principal orbit type theorem for Lie groups, so that it only shows that $\mathrm{Hom}^g(\Gamma, G)$ is open in the complex topology. For the proof of Theorem 3.12, it would be highly useful to know that $\mathrm{Hom}^g(\Gamma, G)$ is Zariski-open as well. I do not have a proof for this. However, there is a principal orbit type theorem for reductive affine algebraic group claiming that there is a non-empty Zariski-open subset of principal orbits [see Wal18, Thm. 12]. In view of this theorem, I consider it highly plausible that $\mathrm{Hom}^g(\Gamma, G)$ should contain a non-empty, G-invariant subset that is Zariski-open in $\mathrm{Hom}(\Gamma, G)$. An in-depth treatment exceeding the scope of this thesis would be necessary in order to prove this. I will denote such a subset by $\mathrm{Hom}^{gg}(\Gamma, G)$. Perhaps, some algebraic statements on $\mathrm{Hom}^g(\Gamma, G)$ (such as the construction of a categorical quotient) should be performed on $\mathrm{Hom}^{gg}(\Gamma, G)$ instead.

Proof of Proposition 1.15. 1. The following proof is based on the proof of [Sik09, Prop. 27]. A representation $\rho : \Gamma \to G$ is called irreducible if its image is not contained in any proper parabolic subgroup $P \subsetneq G$. Therefore, $\mathrm{Hom}^i(\Gamma, G) \subset \mathrm{Hom}(\Gamma, G)$ is the complement of

$$\bigcup_{P \subsetneq G \text{ par.}} \mathrm{Hom}(\Gamma, P) \subseteq \mathrm{Hom}(\Gamma, G),$$

and it is sufficient to show that this set is closed.

Fix any Borel subgroup $B \subseteq G$. Every parabolic subgroup is conjugate to precisely one "standard subgroup", i.e. a parabolic subgroup that contains B [see Bor91, Cor. 11.17]. It can be shown that there are only finitely many standard subgroups[3]. We define

$$X_P = \bigcup_{g \in G} \mathrm{Hom}(\Gamma, gPg^{-1}).$$

It follows that

$$\bigcup_{P \text{ par.}} \mathrm{Hom}(\Gamma, P) = \bigcup_{P \text{ stand. par.}} X_P,$$

and the union on the right-hand side is finite. As the finite union of closed sets is closed, it suffices to show that X_P is closed for every standard parabolic subgroup P. According to the definition of parabolic subgroups, $P \subseteq G$ is a closed subgroup and G/P is complete, so

$$^G P = \bigcup_{g \in G} gPg^{-1}$$

[3] According to [Bor91, Prop. 14.18], all standard parabolic subgroups $B \subseteq P_I \subsetneq G$ can be indexed by subsets $I \subseteq \Delta$, where Δ is the set of vertices of some Dynkin diagram related to B. In particular, Δ is finite, so that there are only finitely many standard parabolic subgroups P_I.

is closed by [Bor91, Lem. 11.9]. This implies that $({}^G P)^N$ is closed in G^N.[4] With the inclusion $\mathrm{Hom}(\Gamma, G) \subseteq G^N$, we have

$$X_P = ({}^G P)^N \cap \mathrm{Hom}(\Gamma, G) \subseteq G^N.$$

$\mathrm{Hom}(\Gamma, G)$ is the intersection of some kernels, so it is closed. Therefore, X_P is the intersection of two closed sets, and thus it is closed.

2. This proof follows [Sik09, Cor. 28]. The map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a morphism of algebraic groups [see Mil17b, Sec. 10.20], and the induced map $\mathrm{Ad}_* : \mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Gamma, \mathrm{GL}(\mathfrak{g}))$ is a regular map by Lemma 1.3. By definition, the set of Ad-irreducible representations is given by

$$\mathrm{Ad}_*^{-1}(\mathrm{Hom}^i(\Gamma, \mathrm{GL}(\mathfrak{g}))).$$

According to bullet point 1, it is the inverse image of a Zariski-open set, so it is Zariski-open itself.

We omit the proofs of bullet points 3 and 4; they can be found in Prop. 33 and Cor. 13 of [Sik09], respectively. □

In the reductive case, we have the following equivalent characterization of closed orbits.

Proposition 1.16. *Let G be a reductive group, and let $\rho \in \mathrm{Hom}(\Gamma, G)$ be a representation. $O_\rho \subseteq \mathrm{Hom}(\Gamma, G)$ is closed if and only if ρ is completely reducible.*

We omit the proof. It can be found in Thm. 30 of [Sik09]. As a consequence of this theorem, we have several equivalent characterizations of a good representation.

Proposition 1.17. *Let G be a reductive group, and $\rho \in \mathrm{Hom}(\Gamma, G)$ be any representation. Then the following three statements are equivalent:*

1. *ρ is a good representation, i.e. $O_\rho \subseteq G$ is closed and $G_\rho = C(G)$.*
2. *ρ is completely reducible and $G_\rho = C(G)$.*
3. *ρ is irreducible and $G_\rho = C(G)$.*

The theorem is a direct consequence of [Sik09, Cor. 17] and Proposition 1.16. The proof will be omitted.

For all good ρ, we have $G_\rho = C(G)$, so the G-action induces a $G/C(G)$-action on $\mathrm{Hom}^g(\Gamma, G)$. The group $G/C(G)$ is itself a smooth affine algebraic group by [Mil17b, Cor. 5.26, Prop. 5.29, Thm. 7.18]. Obviously, the orbits of the actions are identical, i.e. $G.\rho = G/C(G).\rho$ for all good representations ρ. The subset of good representations has the following property which will be of use later on, in the proof of Proposition 2.7.

Proposition 1.18. *The $G/C(G)$-action on $\mathrm{Hom}^g(\Gamma, G)$ is proper in both the complex and the Zariski topology.*

[4] It is well known that the Zariski topology on the fiber product of schemes $X \times_S Y$ is not the product topology. We will show that the product of two closed subschemes is nevertheless a closed subscheme in the product. Let $U \to X$ and $V \to Y$ be closed immersions. As closed immersions are stable under base change and under composition, we know that $U \times_S V \to U \times_S Y \to X \times_S Y$ is a closed immersion as well.

Proof. For the action of an affine algebraic group on an affine algebraic set, a point is called stable if its orbit is closed and its stabilizer is finite.[5] Let ρ be a good representation. We will see that ρ is a stable point under the $G/C(G)$-action.

By definition, all orbits $G.\rho$ of good representations are closed, and the stabilizer of the action of $\tilde{G} = G/C(G)$ is trivial, so all good representations are stable points of the $\tilde{G}$ action. According to [JM87, Prop.1.1, Lem.1.2], this implies that the G-action is proper on the set of stable points in both topologies. In particular, the G-action is proper on $\mathrm{Hom}^g(\Gamma, G)$. □

1.5. Tangent spaces of the representation variety

We saw that $\mathrm{Hom}(\Gamma, G)$ inherits an affine algebraic structure from G^N. Considering G^N a complex manifold, we can calculate the tangent space $T_\rho \mathrm{Hom}(\Gamma, G)$ at a smooth point using an argument by Goldman.

More generally, we can calculate the Zariski tangent space $T_\rho \mathcal{H}om(\Gamma, G)$ of the scheme $\mathcal{H}om(\Gamma, G)$. We will see that the Zariski tangent spaces of $\mathcal{H}om(\Gamma, G)$ and $\mathrm{Hom}(\Gamma, G)$ are isomorphic on the non-singular points of $\mathcal{H}om(\Gamma, G)$. As explained in Proposition A.7, the theory of GAGA ("Géométrie algébrique et géométrie analytique") shows that the Zariski tangent space and the complex tangent space are isomorphic for $\mathrm{Hom}(\Gamma, G)$. Nevertheless, we will demonstrate both techniques in order to show how the different approaches of differential and algebraic geometry come into effect.

We will see that $T_\rho \mathcal{H}om(\Gamma, G)$ is isomorphic to the 1-cocycles of group cohomology. For now, the following definition of 1-cocycles is sufficient for our work: Let $\mathfrak{g}$ be the Lie algebra of G, and let $\rho : \Gamma \to G$ be a representation. We can turn $\mathfrak{g}$ into a Γ−module by setting $\gamma.x = \mathrm{Ad}_{\rho(\gamma)} x$ for any $\gamma \in \Gamma$ and $x \in \mathfrak{g}$. We denote this Γ-module by $\mathfrak{g}_{\mathrm{Ad}\,\rho}$. We define the group of *1-cocycles of group cohomology* by

$$Z^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho}) = \{u : \Gamma \to \mathfrak{g}_{\mathrm{Ad}\,\rho}, u(\gamma\delta) = u(\gamma) + \mathrm{Ad}_{\rho(\gamma)} u(\delta) \text{ for all } \gamma, \delta \in \Gamma\}.$$

The subgroup of *1-coboundaries of group cohomology* is defined to be

$$B^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho}) = \{u : \Gamma \to \mathfrak{g}_{\mathrm{Ad}\,\rho}, u(\gamma) = x - \mathrm{Ad}_{\rho(\gamma)} x \text{ for some } x \in \mathfrak{g}\}.$$

The quotient

$$H^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho}) = Z^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho}) / B^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho})$$

is called the *first cohomology group*. Note that $\mathfrak{g}$ carries the structure of $\mathbb{C}$-vector space, so $Z^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho})$, $B^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho})$ and $H^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho})$ are $\mathbb{C}$-vector spaces as well.

1.5.1. A differential calculation

Let G be a real or complex Lie group. As we saw before, we can consider $\mathrm{Hom}(\Gamma, G)$ a subset of G^N. In the following calculations, we equip G^N with its real or complex topology, and $\mathrm{Hom}(\Gamma, G)$ with the corresponding subset topology. We will calculate the tangent space at a smooth point ρ of $\mathrm{Hom}(\Gamma, G)$.

[5] This definition does not precisely match the definition used within geometric invariant theory, but is the definition provided by [JM87, 1.def.].

Definition 1.19. Let M be finite-dimensional complex manifold, and let $X \subseteq M$ be a subset equipped with the subspace topology. We call $p \in X$ a *smooth point* of X, if it has an open neighborhood $U \subseteq X$ such that U is a submanifold of M.

With this definition, we can calculate the tangent space at smooth points $\rho \in \operatorname{Hom}(\Gamma, G)$.

Theorem 1.20. *For any real or complex Lie group G and any smooth point $\rho \in \operatorname{Hom}(\Gamma, G)$ in the aforementioned differential sense, we have an isomorphism*

$$T_\rho \operatorname{Hom}(\Gamma, G) \cong Z^1(\Gamma, \mathfrak{g}_{\operatorname{Ad}\rho}),$$

between the tangent space and the 1*-cocycles of group cohomology.*

Proof. The proof is due to [Gol84, Sec. 1.2]. Let $X \in T_\rho \operatorname{Hom}(\Gamma, G)$ be a tangent vector. We will construct an element $u \in Z^1(\Gamma, \mathfrak{g}_{\operatorname{Ad}\rho})$ that corresponds to it.

Under the inclusion $\operatorname{Hom}(\Gamma, G) \subseteq G^N$, X corresponds to an element of $\mathfrak{g}^N$.[6] Choose a smooth neighborhood U of ρ as above, and choose a curve ρ_t in U such that $\rho_0 = \rho$ and the derivative is $\frac{d}{dt}\rho_t|_{t=0} = X$. Evaluation at an element $\gamma \in \Gamma$ is a smooth map, as it is merely the word map $w^\gamma : G^N \to G$, where w^γ is the word representing γ. Thus, $\rho_t(\gamma)$ is a smooth path in G for any given $\gamma \in \Gamma$. According to the inverse function theorem, the Lie exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism from a small neighborhood of $0 \in \mathfrak{g}$ to a small neighborhood of $e \in G$. Furthermore, right translation is a diffeomorphism on G, so we obtain a map $\Phi : V \times \Gamma \to \mathfrak{g}$ with

$$\rho_t(\gamma) = \exp(\Phi(t, \gamma))\rho(\gamma),$$

where $V \subseteq \mathbb{R}$ is some small neighborhood of 0. Φ is smooth in t because $\rho_t(\gamma)$ is. Therefore, the Taylor expansion of Φ in t is well defined near 0. Using $\Phi(0, \gamma) = 0$, we obtain

$$\Phi(t, \gamma) = t \cdot u(\gamma) + t^2 \cdot h(\gamma, t),$$

where $\lim_{t \to 0} h(\gamma, t) \to 0$. The homomorphism property $\rho_t(\gamma\delta) = \rho_t(\gamma)\rho_t(\delta)$ implies

$$\exp(\Phi(t, \gamma\delta))\rho(\gamma\delta) = \exp(\Phi(t, \gamma))\rho(\gamma)\exp(\Phi(t, \delta))\rho(\delta),$$

which can be simplified to be

$$\begin{aligned}
\exp(\Phi(t, \gamma\delta)) &= \exp(\Phi(t, \gamma)) \cdot \rho(\gamma)\exp(\Phi(t, \delta))\rho(\gamma)^{-1} \\
&= \exp(\Phi(t, \gamma)) \cdot \exp(\operatorname{Ad}_{\rho(\gamma)} \Phi(t, \delta)) \\
&= \exp(\Phi(t, \gamma) + \operatorname{Ad}_{\rho(\gamma)} \Phi(t, \delta) + \frac{1}{2}[\Phi(t, \gamma), \operatorname{Ad}_{\rho(\gamma)} \Phi(t, \delta)] + \dots) \\
&= \exp(t \cdot u(\gamma) + t \cdot \operatorname{Ad}_{\rho(\gamma)} u(\delta) + O(t^2)),
\end{aligned}$$

where we used the Baker-Campbell-Hausdorff formula for combining the two exponential functions. This formula holds for sufficiently small t. As exp is a local diffeomorphism, we have $tu(\gamma\delta) = tu(\gamma) + t \operatorname{Ad}_{\rho(\gamma)} u(\delta) + O(t^2)$, and deriving by t at $t = 0$ implies

$$u(\gamma\delta) = u(\gamma) + \operatorname{Ad}_{\rho(\gamma)} u(\delta),$$

[6] We identify all tangent spaces $T_g G$ with $\mathfrak{g}$ via left translation.

which is the property of being a 1-cocycle. We will denote this u by u^X as it depends on X.

On the other hand, consider an arbitrary 1-cocycle u. Unfortunately, the function $\tilde{\rho}_t : \Gamma \to G$ defined by $\tilde{\rho}_t(\gamma) = \exp(tu(\gamma))\rho(\gamma)$ for all $\gamma \in \Gamma$ is not a group homomorphism (as the higher terms of the Baker-Campbell-Hausdorff series are missing when comparing $\tilde{\rho}_t(\gamma\delta)$ to $\tilde{\rho}_t(\gamma)\tilde{\rho}_t(\delta)$). However, it is "topologically very close" to being a group homomorphism, so we will only need to alter our definition a little bit. More precisely, consider the group homomorphism ρ_t^u defined by

$$\rho_t^u(\gamma_i) = \exp(tu(\gamma_i))\rho(\gamma_i)$$

on all the generators of Γ. Then $\rho_0^u = \rho$ and $\frac{d}{dt}\rho_t^u|_{t=0}$ is an element of the tangent space $T_\rho \operatorname{Hom}(\Gamma, G)$.

It remains to be shown that

$$T_\rho \operatorname{Hom}(\Gamma, G) \to Z^1(\Gamma, \mathfrak{g}_{\operatorname{Ad}\rho}), \quad X \mapsto u^X(\cdot) = \frac{d}{dt}\rho_t^X(\cdot)\rho(\cdot)^{-1}|_{t=0},$$

where ρ_t^X is the curve in $\operatorname{Hom}(\Gamma, G)$ with tangent vector X at ρ, and

$$Z^1(\Gamma, \mathfrak{g}_{\operatorname{Ad}\rho}) \to T_\rho \operatorname{Hom}(\Gamma, G), \quad u \mapsto X^u = \frac{d}{dt}\rho_t^u(\cdot)\rho(\cdot)^{-1}|_{t=0}$$

are inverse to one another.

Represent X as an element of $\mathfrak{g}^N$ via $X = (\frac{d}{dt}\rho_t^X(\gamma_i)|_{t=0})_i$. Then,

$$\begin{aligned}
\frac{d}{dt}\rho_t^{(u^X)}(\gamma_i)|_{t=0} = dR_{\rho(\gamma_i)}(d\exp)_0(u^X(\gamma_i)) &= dR_{\rho(\gamma_i)}(d\exp)_0(\frac{d}{ds}\rho_s^X(\gamma_i)\rho(\gamma_i)^{-1}|_{s=0}) \\
&= dR_{\rho(\gamma_i)}(d\exp)_0 dR_{\rho(\gamma_i)^{-1}}(\frac{d}{ds}\rho_s^X(\gamma_i)|_{s=0}) \\
&= \frac{d}{ds}\rho_s^X(\gamma_i)|_{s=0},
\end{aligned}$$

where we took benefit of the fact that $(d\exp)_0$ is nothing but the identity map. In particular, $X^{(u^X)} = X$. The other direction, i.e. $u^{(X^u)} = u$ for any $u \in Z^1(\Gamma, \mathfrak{g}_{\operatorname{Ad}\rho})$, follows similarly. (In order to check this equality, please note that it is sufficient to check the equality on generators due to the cocycle property.) □

André Weil chooses a different approach in [Wei64, Sec. 3]. We saw before that $\operatorname{Hom}(\Gamma, G) = \bigcap_\lambda w_\lambda^{-1}(e)$. Weil calculates equations that characterize the kernels $\ker(dw_\lambda)_\rho$, and shows that elements solving these equations for all λ correspond to 1-cocycles. Thus, if we have

$$T_\rho \operatorname{Hom}(\Gamma, G) = \bigcap_\lambda \ker(dw_\lambda)_\rho, \tag{4}$$

we are done. This approach may face two difficulties: Firstly, the intersection is infinite in case Γ is not finitely generated. Secondly, we may need extra requirements for the point ρ, e.g. a transverse intersection, in order to ensure that Eq. (4) is correct.

Both Goldman's and Weil's approach only enable us to calculate the tangent space when assuming that the point is already smooth. In particular, we need a more general approach if we want to find out which points are smooth. Accordingly, in Section 1.5.2 below, we will avail ourselves of the machinery of algebraic geometry in order to determine the Zariski tangent spaces at all closed points ρ.

1.5.2. An algebraic calculation

Every representation $\rho \in \operatorname{Hom}(\Gamma, G)$ can be considered a closed point of the scheme $\mathcal{H}om(\Gamma, G)$. This means that ρ corresponds to a maximal ideal $\mathfrak{m}_\rho \subset R(\Gamma, G)$ of the universal representation algebra. This ideal can be calculated using the universal property: ρ defines a unique projection $r_\rho : R(\Gamma, G) \to \mathbb{C}$, so $\ker r_\rho$ is a maximal ideal. We identify $\rho \in \operatorname{Hom}(\Gamma, G)$ with $\mathfrak{m}_\rho \in \mathcal{H}om(\Gamma, G)$. Thusly, it makes sense to consider the Zariski tangent spaces $T_\rho\mathcal{H}om(\Gamma, G) := T_{\mathfrak{m}_\rho}\mathcal{H}om(\Gamma, G)$.

The adjoint representation Ad is a representation of G in its Lie algebra $\mathfrak{g}$. Choosing an element $\rho \in \operatorname{Hom}(\Gamma, G)$, we obtain a Γ-operation on the Lie algebra. We will designate this Γ-module by $\mathfrak{g}_{\operatorname{Ad}\rho}$ as explained in the beginning of Section 1.5.

Theorem 1.21. *Let G_ρ be the centralizer of $\rho(\Gamma)$ in G. We then have an G_ρ-equivariant isomorphism of $\mathbb{C}$-vector spaces*

$$T_\rho\mathcal{H}om(\Gamma, G) \cong Z^1(\Gamma, \mathfrak{g}_{\operatorname{Ad}\rho})$$

between the Zariski tangent space and the 1-cocycles of Γ in $\mathfrak{g}_{\operatorname{Ad}\rho}$ in the sense of group cohomology.

For the proof, we will need the following lemma. Consider the $\mathbb{C}$-algebra of *dual numbers* $\mathbb{C}[\varepsilon] = \mathbb{C}[X]/(X^2)$ with the element $\varepsilon^2 = 0$. The set of $\mathbb{C}[\varepsilon]$-valued points is a group because G is an affine algebraic group. The embedding $\iota : \mathbb{C} \hookrightarrow \mathbb{C}[\varepsilon]$ induces an embedding $G = G(\mathbb{C}) \hookrightarrow G(\mathbb{C}[\varepsilon])$. On the other hand, the projection $\pi : \mathbb{C}[\varepsilon] \to \mathbb{C}, \varepsilon \mapsto 0$ induces a group homomorphism $G(\pi) : G(\mathbb{C}[\varepsilon]) \to G$. Note that

$$\pi \circ \iota = \operatorname{id}_{\mathbb{C}} \tag{5}$$

and therefore also

$$G(\pi) \circ G(\iota) = \operatorname{id}_{G(\mathbb{C})} .$$

Lemma 1.22. *There is a well-defined map $v : G(\mathbb{C}[\varepsilon]) \to \mathfrak{g}$ that fulfills*

$$v(g_1 g_2) = v(g_1) + \operatorname{Ad}_{G(\pi)(g_1)} v(g_2). \tag{6}$$

If g is already an element of G (embedded into $G(\mathbb{C}[\varepsilon])$ as explained above), then $\operatorname{Ad}_{G(\pi)(g)} = \operatorname{Ad}_g$.

Proof of the lemma. The Zariski tangent space gives us an abstract notion of the Lie algebra

$$\mathfrak{g} \cong T_e G = \ker\left(G(\pi) : G(\mathbb{C}[\varepsilon]) \to G(\mathbb{C})\right),$$

as explained in Definition A.5.1 and Section A.5, where $\pi : \mathbb{C}[\varepsilon] \to \mathbb{C}$ is defined by $\varepsilon \mapsto 0$. Now, consider an element $g \in G(\mathbb{C}[\varepsilon])$. Technically, $G(\pi)(g)$ is an element of $G(\mathbb{C})$. Nevertheless, it can be considered an element of $G(\mathbb{C}[\varepsilon])$ via the embedding $G(\iota) : G(\mathbb{C}) \to G(\mathbb{C}[\varepsilon])$. Therefore,

$$v(g) = g \cdot (G(\pi)(g))^{-1}$$

is a well-defined element of $G(\mathbb{C}[\varepsilon])$. We want to show that it is already in $\ker G(\pi) \cong \mathfrak{g}$. The fact that $G(\pi) \circ G(\iota) = \operatorname{id}_{G(\mathbb{C})}$ implies

$$G(\pi)(g \cdot G(\pi)(g)^{-1}) = G(\pi)(g) G(\pi)(g)^{-1} = e,$$

which shows that $v(g)$ is an element of the tangent space.

It remains to be shown that Eq. (6) holds. This follows by direct calculation:

$$\begin{aligned}
v(g_1g_2) &= g_1g_2(G(\pi)(g_1g_2))^{-1} \\
&= g_1g_2(\pi_*g_2)^{-1}(G(\pi)g_1)^{-1} \\
&= g_1(\pi_*g_1)^{-1} \cdot (G(\pi)g_1)g_2(G(\pi)g_2)^{-1}(G(\pi)g_1)^{-1} \\
&= v(g_1) \cdot \mathrm{Ad}_{G(\pi)g_1}(v(g_2)),
\end{aligned}$$

when using the characterization of the adjoint representation as described in Appendix A.5, Eq. (32). In any additive characterization of $\mathfrak{g}$, $\cdot$ becomes $+$, so we obtain precisely Eq. (6). □

Proof of Theorem 1.21. The proof follows [Sik09, Thm. 35]. Let $\mathfrak{m}_\rho$ be a closed point of $\mathcal{H}om(\Gamma, G)$. As we are dealing with an affine scheme, its Zariski tangent space at $\mathfrak{m}_\rho$ can be calculated as the space

$$T_\rho\mathcal{H}om(\Gamma, G) = \left\{ f : R(\Gamma, G) \to \mathbb{C}[\varepsilon], \quad \pi \circ f = r_\rho \right\},$$

where f is a homomorphism of $\mathbb{C}$-algebras, $r_\rho : R(\Gamma, G) \to R(\Gamma, G)/\mathfrak{m}_\rho \cong \mathbb{C}$ is the natural projection and $\pi : \mathbb{C}[\varepsilon] \to \mathbb{C}$ is defined by $\varepsilon \to 0$, see Appendix A.2 for further details. From such a f, we now need to construct a map $\Gamma \to \mathfrak{g}$ that fulfills the cocycle condition. Using the universal representation ρ_U, we have a group homomorphism

$$\psi_f : \Gamma \xrightarrow{\rho_U} G(R(\Gamma, G)) \xrightarrow{G(f)} G(\mathbb{C}[\varepsilon]).$$

The map $u_f = v \circ \psi_f : \Gamma \to \mathfrak{g}$ fulfills the cocycle condition, because ψ_f is a group homomorphism and because of Eq. (6). Therefore, we have a map

$$\Psi_\rho : T_\rho\mathcal{H}om(\Gamma, G) \to Z^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho}), \quad f \mapsto u_f.$$

Conversely, consider an element $u \in Z^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho})$. We will see that it defines an element of $T_\rho\mathcal{H}om(\Gamma, G)$. u defines a group homomorphism

$$\phi_u : \Gamma \to G(\mathbb{C}[\varepsilon]), \quad \gamma \mapsto u(\gamma) \cdot \rho(\gamma).$$

Indeed, we have

$$\begin{aligned}
\phi_u(\gamma\delta) &= u(\gamma\delta) \cdot \rho(\gamma\delta) \\
&= u(\gamma) \cdot \mathrm{Ad}_{\rho(\gamma)}\, u(\delta) \cdot \rho(\gamma\delta) \\
&= u(\gamma)\rho(\gamma)u(\delta)\rho(\gamma)^{-1}\rho(\gamma)\rho(\delta) \\
&= \phi_u(\gamma)\phi_u(\delta),
\end{aligned}$$

which shows that ϕ_u is a group homomorphism. According to the universal property, a unique homomorphism $f_u : R(\Gamma, G) \to \mathbb{C}[\varepsilon]$ exists, such that $\phi_u = G(f_u) \circ \rho_U$. We have $\mathfrak{g} \cong \ker G(\pi)$. In particular, we have $u(\gamma) \in \ker G(\pi)$. The fact that $\rho(\gamma) \in G$ shows that $G(\pi)(\rho(\gamma)) = \rho(\gamma)$, so

$$G(\pi)(\phi_u(\gamma)) = G(\pi)(u(\gamma))G(\pi)(\rho(\gamma)) = e \cdot \rho(\gamma) = \rho(\gamma) \in G.$$

This shows that $G(\pi) \circ \phi_u = \rho$. The universal property implies that $\pi \circ f_u = r_\rho$. This shows that f_u is an element of $T_\rho\mathcal{H}om(\Gamma, G)$. This means that we have a map

$$\Phi_\rho : Z^1(\Gamma, \mathfrak{g}_{\operatorname{Ad}\rho}) \to T_\rho\mathcal{H}om(\Gamma, G), \quad u \mapsto f_u.$$

It remains to be shown that Ψ_ρ and Φ_ρ are inverse to one another.

We will prove that $\Phi_\rho(\Psi_\rho(f)) = \Phi_\rho(u_f) = f$. $\Phi_\rho(u_f)$ is defined to be the unique $\mathbb{C}$-algebra homomorphism $f_{u_f} : R(\Gamma, G) \to \mathbb{C}[\varepsilon]$ such that

$$G(f_{u_f}) \circ \rho_U(\gamma) \overset{!}{=} \phi_{u_f}(\gamma) = u_f(\gamma)\rho(\gamma).$$

However,

$$u_f(\gamma) = v \circ \psi_f(\gamma) = (G(f) \circ \rho_u)(\gamma) \cdot (G(\pi) \circ G(f) \circ \rho_u)^{-1} = (G(f) \circ \rho_U)(\gamma) \cdot \rho(\gamma)^{-1},$$

so by uniqueness of f_{u_f}, we obtain $f = f_{u_f}$. Similarly, it follows that $\Psi_\rho(\Phi_\rho(u)) = u$. The G_ρ-equivariance of Φ_ρ follows from the definitions. This ends our proof. □

This isomorphism is natural in the following sense: Let $r : \Gamma \to \Delta$ be a group homomorphism between two discrete groups, then we have an induced morphism $r^* : \mathcal{H}om(\Delta, G) \to \mathcal{H}om(\Gamma, G)$ on the representation varieties according to Lemma 1.10. Furthermore, we have an induced map $r^* : Z^1(\Delta, \mathfrak{g}_{\operatorname{Ad}\rho}) \to Z^1(\Gamma, \mathfrak{g}_{\operatorname{Ad}\rho r})$ on the 1-cocycles, which is defined by $u \mapsto u \circ r$.

Lemma 1.23. *With the above maps, the isomorphism Ψ_ρ from Theorem 1.21 makes the diagram*

$$\begin{array}{ccc} T_\rho\mathcal{H}om(\Delta, G) & \xrightarrow{d\bar{r}} & T_{\rho r}\mathcal{H}om(\Gamma, G) \\ \downarrow{\scriptstyle \Psi_\rho} & & \downarrow{\scriptstyle \Psi_{\rho r}} \\ Z^1(\Delta, \mathfrak{g}_{\operatorname{Ad}\rho}) & \xrightarrow{r^*} & Z^1(\Gamma, \mathfrak{g}_{\operatorname{Ad}\rho r}) \end{array}$$

commute.

Proof. We choose the same characterization of $T_\rho\mathcal{H}om(\Delta, G)$ as in Theorem 1.21, i.e. tangent vectors $\mathbb{C}$-algebra homomorphisms $f : R(\Delta, G) \to \mathbb{C}[\varepsilon]$ with $\pi f = r_\rho$. The differential is simply given by $d\bar{r}(f) = f \circ \bar{r}^\sharp$, where $\bar{r}^\sharp : R(\Gamma, G) \to R(\Delta, G)$ is the homomorphism from Lemma 1.10. We have a commutative diagram

$$\begin{array}{ccccc} \Gamma & \xrightarrow{\rho_U} & G(R(\Gamma, G)) & \xrightarrow{G(d\bar{r}(f))} & G(\mathbb{C}[\varepsilon]) \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle G(\bar{r}^\sharp)} & & \| \\ \Delta & \xrightarrow{\rho_U} & G(R(\Delta, G)) & \xrightarrow{G(f)} & G(\mathbb{C}[\varepsilon]), \end{array}$$

where the upper and the lower lines are the definitions of $\psi_{d\bar{r}(f)}$ and ψ_f, respectively, as in the proof of Theorem 1.21. The commutativity shows that $\psi_f \circ r = \psi_{d\bar{r}(f)}$. This proves that

$$r^* \circ \Psi_\rho(f) = r^*(v \circ \psi_f) = v \circ \psi_f \circ r = v \circ \psi_{dr^*(f)} = \Psi_{\rho r} \circ dr^*(f),$$

which was to be shown. □

1.6. Reduced and non-singular points of the representation variety

In algebraic geometry, a point p of an affine algebraic set X is called *non-singular* if it is contained in only one irreducible component $C \subseteq X$, and $\dim_{\mathbb{C}} T_pX = \dim C$. Note that this definition is equivalent to the stalk $\mathcal{O}_{X,p}$ being a regular local ring, see Proposition A.9.

With this said, we can relate the non-singular points of the representation variety to the non-singular points of $\mathcal{H}om(\Gamma, G)$. First, we will define the notion of a reduced representation.

Definition 1.24. We call a representation $\rho \in \mathrm{Hom}(\Gamma, G)$ *reduced*, if it is a reduced point of $\mathcal{H}om(\Gamma, G)$, i.e. $\mathcal{O}_{\mathcal{H}om(\Gamma,G),\rho}$ – the stalk at the point – has no non-trivial nilpotent elements.

Lemma 1.25. *Consider a closed point $\rho \in \mathrm{Hom}(\Gamma, G)$. ρ is non-singular in $\mathcal{H}om(\Gamma, G)$ if and only if ρ is non-singular in* $\mathrm{Hom}(\Gamma, G)$ *and ρ is a reduced representation.*

Proof. Let $R = R(\Gamma, G)$, and let $\mathfrak{m}$ be the maximal ideal corresponding to ρ. Then, we have canonical isomorphisms $\mathcal{O}_{\mathcal{H}om(\Gamma,G),\rho} \cong R_{\mathfrak{m}}$ and $\mathcal{O}_{\mathrm{Hom}(\Gamma,G),\rho} \cong (R/\sqrt{0})_{\mathfrak{m}}$, where $R_{\mathfrak{m}}$ is the localization $(R\backslash\mathfrak{m})^{-1}R$. Localization is an exact functor, so the short exact sequence

$$0 \to \sqrt{0} \to R \to R/\sqrt{0} \to 0$$

is translated into a short exact sequence

$$0 \to (R\backslash\mathfrak{m})^{-1}\sqrt{0} \to R_{\mathfrak{m}} \to (R/\sqrt{0})_{\mathfrak{m}} \to 0.$$

According to [AM69, Cor 3.12], the nilradical of the localization is the localization of the nilradical, i.e. $(R\backslash\mathfrak{m})^{-1}\sqrt{0} = \sqrt{R_{\mathfrak{m}} \cdot 0}$. Therefore, the above exact sequence gives us a canonical isomorphism

$$R_{\mathfrak{m}}/\sqrt{R_{\mathfrak{m}} \cdot 0} \cong (R/\sqrt{0})_{\mathfrak{m}}. \tag{7}$$

In other words, the reduction of the stalk is the stalk of the reduction.

Now suppose ρ is non-singular in $\mathcal{H}om(\Gamma, G)$. This means that $R_{\mathfrak{m}}$ is regular. Every regular local ring is a domain [Sta19, Lemma 00NP]. Hence, every regular local ring is reduced. This means that $\sqrt{R_{\mathfrak{m}} \cdot 0} = 0$ and the Eq. (7) then gives us an isomorphism $R_{\mathfrak{m}} \cong (R/\sqrt{0})_{\mathfrak{m}}$. In particular, $(R/\sqrt{0})_{\mathfrak{m}}$ is regular as well and therefore ρ is non-singular in $\mathrm{Hom}(\Gamma, G)$.

On other hand, suppose ρ is non-singular in $\mathrm{Hom}(\Gamma, G)$ and reduced in $\mathcal{H}om(\Gamma, G)$, i.e. $\sqrt{R_{\mathfrak{m}} \cdot 0} = 0$. Then we obtain the same isomorphism as before, and regularity follows in the same way. □

For surface groups, we will now see that irreducible representations are non-singular. First, we will need to calculate the dimension of $Z^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho})$. Note that the stabilizer subgroup G_ρ of the G-action on $\mathrm{Hom}(\Gamma, G)$ at ρ equals the centralizer subgroup of the set $\rho(\Gamma)$, i.e. $G_\rho = C_G(\rho(\Gamma)) \subseteq G$.

We use two different notions of dimension: By $\dim_k$ for $k = \mathbb{R}$ or $k = \mathbb{C}$, we denote the dimension of a k-vector space, and more generally the dimension of smooth or complex manifold. By dim, we denote the Krull dimension of a ring, i.e. the maximal length of chains of prime ideals, or the Krull dimension of a topological space, i.e. the maximal length of chains of irreducible closed sets.

Lemma 1.26. *Let $\pi_1(S)$ be the fundamental group of a surface S of genus g. The dimension of the 1-cocycles is given by*

$$\dim_k Z^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho}) = (2g-1)\dim_k G + \dim_k G_\rho,$$

for $k = \mathbb{R}$ or $\mathbb{C}$.

Proof. To begin with, let $\Gamma = \langle \gamma_i | w_\lambda \rangle$ be an arbitrary finitely generated group and consider $w_\lambda : G^N \to G$ as a map as before. In [Gol84, Sec. 3.6], Goldman argues that the 1-cocycles can be identified with the set

$$Z^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho}) = \left\{ (u_1, \dots, u_N) \in \mathfrak{g}^N \,\middle|\, dw_\lambda(u_1, \dots, u_N) = 0 \text{ for all } \lambda \right\}.$$

In the case of a surface group $\Gamma = \pi_1(S)$, we have a set of generators $\gamma_1, \delta_1, \dots, \gamma_g, \delta_g$ and a single relation $w(\gamma_1, \dots, \delta_g) = \prod_i [\gamma_i, \delta_i]$. Hence, the dimension of the kernel of dw remains to be calculated in order to calculate the dimension of $Z^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho})$. In [Gol84, Prop. 3.7], Goldman calculates the $\mathrm{rank}_k(dw)_r$ at a point $r = (r_1, s_1 \dots, r_g, s_g) \in G^{2g}$. It is given by

$$\mathrm{rank}_k(dw)_r = \dim_k G - \dim_k C_G(\{r_1, s_1, \dots, r_g, s_g\}),$$

where $C_G(\{r_1, s_1, \dots, r_g, s_g\})$ is the centralizer of the set. If the point $r \in G^{2g}$ represents an element $\rho \in \mathrm{Hom}(\pi_1(S), G) \subseteq G^{2g}$, then we have

$$C_G(\{r_1, s_1, \dots, r_g, s_g\}) = C_G(\rho(\pi_1(S))).$$

It follows that

$$\begin{aligned} \dim_k \ker(dw_\rho) &= \dim_k G^{2g} - \mathrm{rank}_k(dw)_\rho \\ &= (2g-1)\dim_k G + \dim_k C_G(\rho(\pi_1(S))). \end{aligned}$$

Finally, note that $C_G(\rho(\pi_1(S))) = G_\rho$ is the stabilizer of the action on the representation variety. Hence, our statement follows. ◻

Let ρ be any representation. The stabilizer G_ρ always contains the center $C(G)$, so we expect the dimension of $T_\rho\mathcal{H}om(\pi_1(S), G)$ to be minimal for $\dim_{\mathbb{C}} C(G) = \dim_{\mathbb{C}} G_\rho$. In the next proposition, we will see that irreducible representations have minimal tangent space dimension. Thus, they are non-singular.

Proposition 1.27. *Let G be a reductive affine algebraic group and let $\pi_1(S)$ be a surface group. All elements $\rho \in \mathrm{Hom}^i(\pi_1(S), G)$ are non-singular in $\mathcal{H}om(\pi_1(S), G)$ (and thus also in $\mathrm{Hom}(\pi_1(S), G)$).*

Possibly this proposition only holds with a certain restriction: ρ is definitely non-singular when it is contained in a closed irreducible component $C \subseteq \mathrm{Hom}(\Gamma, G)$ that is "irredundant". I will explain what this means within the proof.

Proof. The proof follows [Sik09, Prop. 37]. The stabilizer of an irreducible subgroup is a finite extension of the center, so $\dim_{\mathbb{C}} G_\rho = \dim_{\mathbb{C}} C_G(\rho(\pi_1(S))) = \dim_{\mathbb{C}} C(G)$ [see Sik09, Cor. 17]. In particular, the isomorphism from Theorem 1.21 together with Lemma 1.26 implies

$$\dim_{\mathbb{C}} T_\rho \mathcal{H}om(\pi_1(S), G) = (2g-1)\dim_{\mathbb{C}} G + \dim_{\mathbb{C}} C(G). \tag{8}$$

Let C be any irreducible component of $\mathrm{Hom}(\pi_1(S), G)$. We claim that it suffices to show that

$$\dim C \geqslant \dim_{\mathbb{C}} T_\rho \mathcal{H}om(\pi_1(S), G). \tag{9}$$

Indeed, suppose that this inequality holds. Using the irreducibility of C and the closed immersion $\mathrm{Hom}(\pi_1(S), G) \hookrightarrow \mathcal{H}om(\pi_1(S), G)$, we can see that

$$\dim C \leqslant \dim_{\mathbb{C}} T_\rho \mathrm{Hom}(\pi_1(S), G) \leqslant \dim_{\mathbb{C}} T_\rho \mathcal{H}om(\pi_1(S), G),$$

for all irreducible representations $\rho \in C$. Therefore, Eq. (9) must be an equality. In accordance with [AM69, Thm. 11.25], we have $\dim C = \dim \mathcal{O}_{\mathrm{Hom},\rho}$, and it follows that

$$\dim C = \dim \mathcal{O}_{\mathrm{Hom},\rho} \leqslant \dim \mathcal{O}_{\mathcal{H}om,\rho} \leqslant \dim_{\mathbb{C}} T_\rho \mathcal{H}om(\pi_1(S), G) = \dim C.$$

In particular, $\dim \mathcal{O}_{\mathcal{H}om,\rho} = \dim_{\mathbb{C}} T_\rho \mathcal{H}om(\pi_1(S), G)$. Therefore, $\mathcal{O}_{\mathcal{H}om,\rho}$ is regular. This means that ρ is a non-singular point of $\mathcal{H}om(\pi_1(S), G)$.

Step i: G is semi-simple. First, we will prove that Eq. (9) holds if G is semi-simple. Then the center $C(G)$ is discrete,[7] and in particular $\dim_{\mathbb{C}} C(G) = 0$. Therefore, Eq. (8) boils down to

$$\dim_{\mathbb{C}} T_\rho \mathcal{H}om(\pi_1(S), G) = (2g-1)\dim_{\mathbb{C}} G.$$

Now $\pi_1(S)$ is generated by $2g$ generators and one single relation w, so $\mathrm{Hom}(\pi_1(S), G)$ is simply the fiber $w^{-1}(e)$ of $w : G^{2g} \to G$. The word w is a regular map of irreducible varieties, so we have

$$\dim C \geqslant \dim G^{2g} - \dim G = (2g-1)\dim G$$

for any irreducible component C of $\mathrm{Hom}(\pi_1(S), G)$ according to the fiber dimension theorem [see Sha13, Thm 1.25]. The topological spaces of $\mathrm{Hom}(\pi_1(S), G)$ and $\mathcal{H}om(\pi_1(S), G)$ coincide, which means that C can be considered an irreducible component of the latter. As the Krull dimension is an entirely topological property, Eq. (9) follows for a semi-simple G.

Step ii: G is reductive. Now, we will prove that Eq. (9) holds for any reductive G. According to [Bor91, Prop. 14.2], we have a surjective map

$$\phi : C^0(G) \times DG \to G, \quad (g, h) \mapsto gh,$$

where $C^0(G) \leqslant G$ is the connected component at the identity of the center $C(G)$ and $DG \leqslant G$ is the derived subgroup (which is semi-simple). As elements of $C^0(G)$ commute with elements

[7] Consider a semi-simple Lie group G with Lie algebra $\mathfrak{g}$. The Lie algebra $\mathfrak{c}$ of the center $C(G)$ is an abelian ideal of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is semi-simple, so it does not contains any non-trivial abelian ideals. It follows that $\mathfrak{c} = 0$, so $C(G)$ is discrete.

of DG, ϕ is in fact a group homomorphism. The kernel of ϕ can be identified with $C^0(G) \cap DG$ via

$$C^0(G) \cap DG \hookrightarrow C^0(G) \times DG, \quad g \mapsto (g^{-1}, g).$$

[Bor91, Prop. 14.2] states that the kernel is finite. This implies that $C^0(G) \times DG \to G$ is a finite morphism of schemes,[8] which in turn shows that the induced morphism

$$\phi_* : \mathrm{Hom}(\pi_1(S), C^0(G)) \times \mathrm{Hom}(\pi_1(S), DG) \to \mathrm{Hom}(\pi_1(S), G)$$

is finite [using Sta19, Lem. 035D]. Let C be an irreducible component of $\mathrm{Hom}(\pi_1(S), G)$. Suppose furthermore that C is "irredundant" for the covering of $\operatorname{im} \phi_*$. By this, I mean that the image $\phi_*(Hom(\pi_1(S), C^0(G)) \times \mathrm{Hom}(\pi_1(S), DG))$ is no longer covered if C is removed from the covering. We claim that there exists an irreducible component D of $\mathrm{Hom}(\pi_1(S), C^0(G)) \times \mathrm{Hom}(\pi_1(S), DG)$ with

$$\dim C \geqslant \dim D. \tag{10}$$

Indeed, consider the decomposition of $\mathrm{Hom}(\pi_1(S), C^0(G)) \times \mathrm{Hom}(\pi_1(S), DG) = \bigcup D_i$ into closed irreducible components D_i. Then the image $\phi_*(D_i)$ is also irreducible, because ϕ_* is continuous, and it is closed because ϕ_* is finite. As a closed irreducible set, $\phi_*(D_i)$ is contained in an irreducible component of $\mathrm{Hom}(\pi_1(S), G)$. As C is irredundant, there must be a $D = D_i$ such that $\phi_*(D_i) \subseteq C$. Because ϕ_* is finite and closed immersion are finite, $\phi_*|_D : D \to \mathrm{Hom}(\pi_1(S), G)$ is finite as well. [Sta19, Lem. 035D] shows that the restriction of the image $\phi_*|_D : D \to C$ is finite as well. For finite morphisms, we have $\dim C \geqslant \dim D$ according to [Sta19, Lem. 01WJ, Lem. 0ECG].

The finiteness and surjectivity of ϕ together with the same lemmas [Sta19, Lem. 01WJ, Lem. 0ECG] also show that

$$\dim G = \dim C^0(G) + \dim DG. \tag{11}$$

We claim that $\mathrm{Hom}(\pi_1(S), C^0(G)) = (C^0(G))^{2g}$. Indeed, the relation w on a surface group is the product of commutators, and $C^0(G)$ is abelian, so every tuple $(c_1, d_1, \dots, c_g, d_g) \in C^0(G)^{2g}$ fulfills the relation. It remains to calculate $\dim DG$. DG is semi-simple. Our calculation for a semi-simple group shows that every irreducible component of $\mathrm{Hom}(\pi_1(S), DG)$ has dimension greater or equal than $(2g-1)\dim DG$. In particular, an irreducible component C of $\mathrm{Hom}(\pi_1(S), G)$ has a dimension of at least

$$\begin{aligned}
\dim C &\geqslant \dim D && \text{by Eq. (10)} \\
&\geqslant 2g \dim C^0(G) + (2g-1)\dim DG \\
&= (2g-1)\dim G + \dim C^0(G) && \text{by Eq. (11),} \\
&= (2g-1)\dim G + \dim C(G) && \text{as all components of } C(G) \text{ are conjugate,} \\
&= (2g-1)\dim_{\mathbb{C}} G + \dim_{\mathbb{C}} C(G) && \text{as } G \text{ and } C(G) \text{ are non-singular.}
\end{aligned}$$

[8] Let $\phi : G \to H$ be a morphism of group schemes over a field k. Then the kernel scheme is defined to be $\ker(\phi) = G \times_H \mathrm{spec}(k)$ [see Mil17b, Sec. 1.e]. $\ker(\phi)$ is finite (as a set) if and only if the structure morphism $\ker(\phi) \to \mathrm{spec}(k)$ is finite [see Mil17b, Prop. 11.2]. Finite morphisms are stable under base change, and $\phi : G \cong G \times_H \mathrm{spec}(k) \times_k H \to \mathrm{spec}(k) \times_k H \cong H$ is the base change of the structure morphism. Hence, $\phi : G \to H$ is finite.

Using Lemma 1.26, this proves that Eq. (9) holds for any reductive group G and any irredundant component $C \in \mathrm{Hom}(\Gamma, G)$. □

If ϕ_* is surjective, then every connected component $C \in \mathrm{Hom}(\Gamma, G)$ is irredundant. If ϕ_* is not surjective, Eq. (10) could be wrong, as the following example illustrates. Consider the finite morphism of varieties

$$\mathrm{spec}(\mathbb{C}[X, Y]) \to \mathrm{spec}(\mathbb{C}[X, Y, Z]/(XZ, YZ))$$

which is induced by the ring map $\mathbb{C}[X, Y, Z]/(XZ, YZ) \to \mathbb{C}[X, Y], Z \mapsto 0$. Geometrically speaking, it is the embedding of the two-dimensional complex plane into the space consisting of a plane and a line perpendicular to it. It is a closed immersion, which implies that it is finite. $\mathrm{spec}(\mathbb{C}[X, Y])$ consists of one irreducible component of dimension 2. The components of $\mathrm{spec}(\mathbb{C}[X, Y, Z]/(XZ, YZ))$ are the closed subvarieties

$$\mathrm{spec}(\mathbb{C}[X, Y, Z]/(Z)) \text{ and } \mathrm{spec}(\mathbb{C}[X, Y, Z]/(X, Y)),$$

i.e. a plane and a line. Their dimensions are 2 and 1, respectively. In particular,

$$\dim \mathrm{spec}(\mathbb{C}[X, Y]) > \dim \mathrm{spec}(\mathbb{C}[X, Y, Z]/(X, Y))$$

even though the morphism is finite. This shows that Eq. (10) is wrong in this case.

In order to show that Proposition 1.27 is correct in full generality, we would need to show that every connected component is irredundant, e.g. by showing that ϕ_* is surjective – or we would need to show that the dimension of all *redundant* components is equal to $\dim_{\mathbb{C}} T_\rho \mathcal{H}om(\pi_1(S), G)$ by some other argument.

1.7. Tangent spaces of orbits

Consider the orbit $O_\rho \subset \mathrm{Hom}(\Gamma, G)$ of ρ under the G-action. As G is a reduced and non-singular group scheme G, it follows that O_ρ is a non-singular affine algebraic subspace of $\mathrm{Hom}(\Gamma, G)$ [by Mil17b, Prop. 7.11]. We will now calculate the tangent space of the orbit.

Theorem 1.28. *The isomorphism in Theorem 1.21 can be restricted to an isomorphism*

$$T_\rho O_\rho \cong B^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho}),$$

between the Zariski tangent space of the orbit and the 1-coboundaries of group cohomology. These subsets are preserved by the action of G_ρ on $T_\rho \mathcal{H}om(\Gamma, G)$ and $Z^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho})$.

To prove the theorem, we will need the following lemma.

Lemma 1.29. *Let G be a reductive algebraic group, and let X be an affine algebraic set. Consider a closed point $\rho \in X$. The homogenous space G/G_ρ exists as an affine algebraic set and is isomorphic to the orbit $O_\rho \subseteq X$. Consider the projection morphism $G \to G/G_\rho$. Its differential*

$$T_e G \to T_e(G/G_\rho) \cong T_\rho O_\rho$$

is an epimorphism and $\dim_{\mathbb{C}} T_\rho O_\rho = \dim_{\mathbb{C}} \mathfrak{g} - \dim_{\mathbb{C}} T_e G_\rho$.

Proof. By [Mil17b, Prop. 7.12 and Lem. 7.17], the quotient G/G_ρ exists and is isomorphic to O_ρ. Consider the canonical projection $G \to G/G_\rho$. We will now prove that the differential $T_eG \to T_eG/G_\rho$ is an epimorphism. In Proposition 2.3, we will see that G/G_ρ is isomorphic to $G/\!\!/G_\rho$, so we can make use of Luna's slice theorem.

Let G_ρ act on G via left translation. The stabilizer subgroup of e with respect to this action is trivial. By Luna's Étale Slice Theorem A.14, we have a subvariety $V \subseteq G$ containing e such that $G_\rho \times V \to G$ is étale, its image $U \subseteq G$ is open and affine, and $V \to U/\!\!/G_\rho$ is étale as well. In particular, we have isomorphisms $T_eG \cong T_eG_\rho \oplus T_eV$ and $T_eG/\!\!/G_\rho T_eU/\!\!/G_\rho \cong T_eV$. Therefore, $T_eG \to T_eG/\!\!/G_\rho$ is an epimorphism and the dimension formular follows as well. □

Proof of Theorem 1.28. This proof follows [Sik09, Thm. 43]. The orbit $O_\rho \subseteq \operatorname{Hom}(\Gamma, G)$ is the image of the map $\sigma_\rho : G \to \operatorname{Hom}(\Gamma, G), \sigma_\rho(g) = g\rho g^{-1}$. We have an induced map on the tangent spaces

$$d\sigma_\rho : \mathfrak{g} \cong T_eG \to T_\rho \operatorname{Hom}(\Gamma, G),$$

whose image is $T_\rho O_\rho$. This is easy to see: Both G and O_ρ are non-singular, which means that σ_ρ is a smooth surjective map of manifolds, inducing a surjection on the tangent spaces.

Let us embed $\mathfrak{g} = \ker G(\pi) \subseteq G(\mathbb{C}[\varepsilon])$ and $T_\rho \operatorname{Hom}(\Gamma, G) \subseteq \operatorname{Hom}(\Gamma, G)(\mathbb{C}[\varepsilon])$ as explained in the proof of Theorem 1.21. For an element $f : \mathcal{O}_G(G) \to \mathbb{C}[\varepsilon]$ in $\mathfrak{g}$, we have $d\sigma_\rho(f) = f \circ \sigma_\rho^\sharp$. Therefore, $d\sigma_\rho$ is in fact a restriction of the map

$$(\sigma_\rho)_{\mathbb{C}[\varepsilon]} : G(\mathbb{C}[\varepsilon]) \to \operatorname{Hom}(\Gamma, G)(\mathbb{C}[\varepsilon]), \quad g \mapsto g\rho g^{-1},$$

where ρ is considered an element of $\operatorname{Hom}(\Gamma, G)(\mathbb{C}[\varepsilon])$ via the inclusion $\operatorname{Hom}(\Gamma, G)(\mathbb{C}) \hookrightarrow \operatorname{Hom}(\Gamma, G)(\mathbb{C}[\varepsilon])$.[9] For any element $\gamma \in \Gamma$, the evaluation map $w^\gamma : \operatorname{Hom}(\Gamma, G) \to G, \rho \mapsto \rho(\gamma)$ is a regular map, and it is G-equivariant as explained in Lemma 1.1. eval^γ can thus be considered a G-equivariant morphism of schemes, which is exactly the evaluation map on closed points ρ.

Let now $f \in \mathfrak{g}$, so $(\sigma_\rho)_{\mathbb{C}[\varepsilon]}(f)$ is an element of $T_\rho \operatorname{Hom}(\Gamma, G)$. We want to understand $\Psi_\rho((\sigma_\rho)_{\mathbb{C}[\varepsilon]}(f)) = \upsilon \circ \psi_{(\sigma_\rho)_{\mathbb{C}[\varepsilon]}(f)}$, so we first need to understand the group homomorphism

$$\psi_{(\sigma_\rho)_{\mathbb{C}[\varepsilon]}(f)} : \Gamma \xrightarrow{\rho_U} G(R(\Gamma, G)) \xrightarrow{G((\sigma_\rho)_{\mathbb{C}[\varepsilon]}(f))} G(\mathbb{C}[\varepsilon]).$$

Lemma 1.11 shows us that $G((\sigma_\rho)_{\mathbb{C}[\varepsilon]}(f))(\rho_U(\gamma)) = \mathrm{eval}^\gamma(\sigma_\rho)_{\mathbb{C}[\varepsilon]}(f)$, and we know that eval^γ is precisely $\rho \mapsto \rho(\gamma)$ on $\mathbb{C}$-rational points $\rho \in \operatorname{Hom}(\Gamma, G)$, so

$$\psi_{(\sigma_\rho)_{\mathbb{C}[\varepsilon]}(f)}(\gamma) = f\rho(\gamma)f^{-1}.$$

Now we have

$$\begin{aligned}
\Psi_\rho((\sigma_\rho)_{\mathbb{C}[\varepsilon]}(f))(\gamma) &= \upsilon(w\rho(\gamma)f^{-1}) \\
&= f\rho(\gamma)f^{-1}G(\pi)(f\rho(\gamma)f^{-1})^{-1} \\
&= f\rho(\gamma)f^{-1}G(\pi)(f)G(\pi)(\rho(\gamma))^{-1}G(\pi)(f)^{-1} \\
&= f\rho(\gamma)f^{-1}G(\pi)(\rho(\gamma))^{-1} && \text{as } f \in \ker G(\pi) \\
&= f\rho(\gamma)f^{-1}\rho(\gamma)^{-1} && \text{as } \pi \circ \iota = \mathrm{id}, \text{ see Eq. (5)} \\
&= f \cdot (\mathrm{Ad}_{\rho(\gamma)} f)^{-1},
\end{aligned}$$

[9] G acts on $\operatorname{Hom}(\Gamma, G)$ via conjugation, so we have a group action of schemes G on $\operatorname{Hom}(\Gamma, G)$, and therefore also on $\mathbb{C}[\varepsilon]$-valued points.

or additively written,

$$\Psi_\rho((\sigma_\rho)_{\mathbb{C}[\varepsilon]}(f))(\gamma) = f - \mathrm{Ad}_{\rho(\gamma)} f,$$

which is precisely the condition for $\Psi_\rho((\sigma_\rho)_{\mathbb{C}[\varepsilon]}(f)) \in B^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho})$. This means that the image of $T_\rho O_\rho$ under $\Psi_\rho : T_\rho \mathcal{H}om(\Gamma, G) \to Z^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho})$ lies in $B^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho})$.

Now, we need to show now that the restriction $\Psi_\rho : T_\rho O_\rho \to B^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho})$ is an isomorphism. Ψ_ρ is injective, so it is sufficient to show that it is surjective on $B^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho})$. According to Lemma 1.29, we have $\dim_{\mathbb{C}} T_\rho O_\rho = \dim_{\mathbb{C}} \mathfrak{g} - \dim_{\mathbb{C}} T_e G_\rho$.

We will now estimate $\dim_{\mathbb{C}} B^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho})$ as well. By the definition of 1-coboundaries, the map

$$\mathfrak{g} \twoheadrightarrow B^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho}), f \mapsto [\gamma \mapsto f - \mathrm{Ad}_{\rho(\gamma)} f] \tag{12}$$

is an epimorphism of $\mathbb{C}$-vector spaces. $\mathrm{Ad}_{\rho(\gamma)} : \mathfrak{g} \to \mathfrak{g}$ is the differential of the conjugation map $G \to G, g \mapsto \rho(\gamma) g \rho(\gamma)^{-1}$. For any γ, this map is the identity on G_ρ[10], so $\mathrm{Ad}_{\rho(\gamma)}$ is the identity on the Lie algebra $\mathrm{Lie}(G_\rho)$. Therefore, $\mathrm{Lie}(G_\rho) \cong T_e G_\rho$ is in the kernel of the map in Eq. (12), which implies $\dim_{\mathbb{C}} B^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \leqslant \dim_{\mathbb{C}} \mathfrak{g} - \dim_{\mathbb{C}} T_e G_\rho$. Thus, $\dim_{\mathbb{C}} B^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \leqslant \dim_{\mathbb{C}} T_\rho O_\rho$, so $\Psi_\rho : T_\rho O_\rho \to B^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho})$ is surjective and thus an isomorphism. □

[10] $h \in G_\rho$ means $h\rho(\gamma)h^{-1} = \rho(\gamma)$, i.e. $\rho(\gamma)h\rho(\gamma)^{-1} = h$ for all $\gamma \in \Gamma$.

2. Character varieties

2.1. Geometric invariant theory: The categorical quotient

Let S be a base scheme in the sense of algebraic geometry, and let X be a scheme over S. Let $\sigma : G \times_S X \to X$ be the action of a group scheme on X.

Definition 2.1. Consider an S-scheme $X /\!\!/ G$ together with an S-morphism $\pi : X \to X /\!\!/ G$ such that the diagram

$$\begin{tikzcd} G \times_S X \arrow[r, "\sigma"] \arrow[d, "p_2"] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"] & X /\!\!/ G \end{tikzcd}$$

commutes. We call $X /\!\!/ G$ a *categorical quotient* of X by G, if the following universal property holds: For any other S-morphism $\phi : X \to Z$ such that $\phi \circ p_2 = \phi \circ \sigma$, there exists a unique S-morphism $\bar{\phi} : X /\!\!/ G \to Z$ such that

$$\begin{tikzcd} G \times_S X \arrow[r, "\sigma"] \arrow[d, "p_2"] & X \arrow[d, "\pi"] \arrow[ddr, bend left, "\phi"] & \\ X \arrow[r, "\pi"] \arrow[drr, bend right, "\phi"] & X /\!\!/ G \arrow[dr, dashed, "\bar{\phi}"] & \\ & & Z \end{tikzcd}$$

is a commutative diagram as well.

For our needs, the categorical quotient always exists, as the following theorem shows.

Proposition 2.2. *Let k be a field with $\operatorname{char} k = 0$. For an affine scheme $X = \operatorname{spec}(R)$ over a k and a reductive algebraic group G over k, the categorical quotient $X /\!\!/ G$ always exists and is affine as well. If X is an affine algebraic variety, then $X /\!\!/ G$ is as well.*

A proof can be found in [MFK94, Thm. 1.1.]. We will omit the proof and most of the theory leading to it, but some of its details are necessary in order to prove some properties of the quotient. The categorical quotient is defined to be $X /\!\!/ G = \operatorname{spec}(R^G)$, where $R^G \subseteq R$ is the so-called *ring of invariants* under the G-action, and the projection $X \to X /\!\!/ G$ is induced by the inclusion $R^G \hookrightarrow R$. The ring of invariants is constructed as follows:

The group action $\sigma : G \times_k X \to X$ induces a k-algebra homomorphism $\sigma^\sharp : R \to \mathcal{O}_G(G) \otimes_k R$, the *dual action on R*. There exists a canonical homomorphism $E : R \to R$ of $\mathbb{C}$-vector spaces, which is equivariant under the dual action, idempotent, and projects to the elements that remain invariant under the dual action. This homomorphism is called the *Reynolds operator*, see [MFK94, Def. 1.5 f.] for details. Its image is again a $\mathbb{C}$-algebra, and we denote the ring of invariants by $R^G = E(R) \subseteq R$.

Proposition 2.3. *Let X and Y be two affine schemes over k. Let G be a reductive algebraic group over k acting on X and Y. The categorical quotient has the following properties:*

1. *If X is of finite type over k, then $X /\!\!/ G$ is as well. The respective implication holds for integral, irreducible and reduced X. In particular, if X is an affine algebraic set, then $X /\!\!/ G$ is as well.*

2. *A G-morphism $Y \to X$ canonically induces a morphism $Y /\!\!/ G \to X /\!\!/ G$.*

3. *If the G-morphism $Y \hookrightarrow X$ is a closed immersion, then $Y /\!\!/ G \hookrightarrow X /\!\!/ G$ is a closed immersion as well.*

4. *For an affine algebraic group G and a subgroup $H \subseteq G$, there is a canonical homorphism between the homogeneous space and the categorical quotient, $G/H \cong G /\!\!/ H$. The* homogeneous space *is an algebraic scheme G/H together with a G-action $\sigma : G \times G/H \to G/H$ and a point $o \in G/H(\mathbb{C})$ such that $\sigma_o : G \to G/H$ realizes $G(\cdot)/H(\cdot)$ as a fat subfunctor of $G/H(\cdot)$, see [Mil17b, Def. 5.20] for details.*

Proof. 1. These general properties can be found in [MFK94, Sec. 0.2, p.5]. The fact that $X /\!\!/ G$ is of finite type is Hilbert's finiteness theorem, see [Wal17, Thm. 3.11] or [MFK94, Thm. 1.11] for proofs. Hilbert's 14^{th} problem conjectured that this is true for $\operatorname{char} k = p \neq 0$ as well. However, Masayoshi Nagata famously constructed a counterexample [Nag59].

2. You can derive this as a direct consequence of the universal property. However, we will also see how to construct this morphism explicitly in the proof of 3.

3. $\iota : Y \hookrightarrow X$ is a closed immersion, so we have $X = \operatorname{spec}(R)$ and $Y = \operatorname{spec}(R/I)$ for some ideal $I \subseteq R$. The dual map corresponding to ι is the natural projection $\iota^\sharp = p : R \to R/I$ (which commutes with the dual action). Thus, we have a diagram

$$\begin{array}{ccc} R & \xrightarrow{p} & R/I \\ {\scriptstyle E_X}\downarrow & & \downarrow{\scriptstyle E_Y} \\ R & \xrightarrow{p} & R/I \end{array}$$

where E_X and E_X are the respective Reynolds operators. The commutativity $E_Y \circ p = p \circ E_X$ is a general property of the Reynolds operator [see MFK94, Sec. 1.1, p.27]. Restricting the bottom row to the images of E_X and E_Y gives us

$$\begin{array}{ccc} R & \xrightarrow{p} & R/I \\ {\scriptstyle E_X}\downarrow & & \downarrow{\scriptstyle E_Y} \\ R^G & \xrightarrow{p|_{R^G}} & (R/I)^G. \end{array}$$

The commutativity implies that the bottom line of the diagram is surjective as well. In particular, its dual morphism $\bar{\iota} : Y /\!\!/ G \to X /\!\!/ G$ is a closed immersion.

4. In [MFK94], there is a third notion of a quotient, the geometric quotient, which we will define in next paragraph. [MFK94, Prop. 0.2] shows that every geometric quotient is a categorical quotient (and hence unique up to unique isomorphism). Hence, it suffices to show that G/H is a geometric quotient.

What is a geometric quotient? Let $\sigma : G \times_S X \to X$ be a group action. An S-scheme Y together with an S-morphism $\pi : X \to Y$ is called a *geometric quotient* of X by G if the following four properties hold: (i) $\pi\sigma = \pi p_2$, (ii) π is surjective and the image of $(\sigma, p_2) : G \times_S X \to X \times_S X$ is $X \times_S X$, (iii) a subset $U \subseteq Y$ is open if and only if $\pi^{-1} \subseteq X$ is open, (iv) we have an isomorphism of sheaves $O_Y \cong \pi_* O_X^G$.

The homogeneous space G/H has a G-action $\sigma : G \times G/H \to G/H$ and a designated point $o \in G/H(\mathbb{C})$. We want to show that $\sigma_o : G \to G/H$ fulfills (i)-(iv). Restricting the group operation $\mu : G \times_{\mathbb{C}} G \to G$, we have

$$\begin{array}{ccc} H \times_{\mathbb{C}} G & \xrightarrow{\mu} & G \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \sigma_o} \\ G & \xrightarrow{\sigma_o} & G/H. \end{array}$$

For smooth and affine G, we can assume that $H = G_o$ is the isotropy group of the point $o \in G/H(\mathbb{C})$ with respect to the G-action on G/H [see Mil17b, Thm. 7.18]. In particular, $\sigma_o \circ \mu = \sigma_o \circ p_2$. This proves (i).

For (ii) note that $(\mu, p_2) : H \times_{\mathbb{C}} G \to G \times_{\mathbb{C}} G$ trivially has the image $G \times_C G$. Furthermore, $\mu_o : G \to G/H$ is faithfully flat [see Mil17b, Prop. 5.25], and therefore also surjective. The fact that μ_o is faithfully flat also proves that it fulfills (iii) using [Gro65, Cor. 2.3.12]. Finally (iv) is a consequence of flatness and of [Har77, Prop. 9.3]. A more detailed explanation of (iv) can be found in the Michel Brion's notes [Bri09].

This shows that G/H is a geometric quotient. As every geometric quotient is categorical and categorical quotients are unique, we obtain a unique isomorphism $G/H \cong G/\!\!/H$. □

Classically, we can consider the action of a group G on a set X, and denote the space of orbits by X/G. The following theorem explains to which extent the categorical quotient can be considered an orbit space.

Theorem 2.4. *If X is affine and G acts on it, then $X/\!\!/G$ is a variety parametrizing the closed orbits.*

A proof can be found in [Wal17, Thm. 3.20].

2.2. Definitions of the character variety

G acts on $\mathrm{Hom}(\Gamma, G)$ by conjugation, $g.\rho(\cdot) = g\rho(\cdot)g^{-1}$. This action can be considered the action of an affine algebraic group on an affine algebraic set, as explained after Proposition 1.2.

Definition 2.5. If it exists, the categorical quotient $X_G(\Gamma) = \mathrm{Hom}(\Gamma, G)/\!\!/G$ is called the *character variety* of Γ. We also define $\mathcal{X}_G(\Gamma) = \mathcal{H}om(\Gamma, G)/\!\!/G$.

As explained in Proposition 2.2, the quotient exists in the case where G is reductive. As the quotient is of finite type whenever X is, the character variety is also an affine algebraic set. Despite being called a variety, $X_G(\Gamma)$ is not necessarily irreducible. The name character variety is motivated by the fact that the points of $X_G(\Gamma)$ correspond to the characters of representations in some special cases such as $G = \mathrm{PSL}_2(\mathbb{C})$ [see Lou11, Lem. 3.1].

The set $\mathrm{Hom}^i(\Gamma, G)$ of irreducible representations subset of $\mathrm{Hom}(\Gamma, G)$ is preserved by the G-operation. This is clear from the definition as the conjugate of a parabolic subgroup is also parabolic.

Proposition 2.6. *We have equations* $\mathrm{Hom}^i(\Gamma, G)/\!\!/G = \mathrm{Hom}^i(\Gamma, G)/G$ *and* $\mathrm{Hom}^g(\Gamma, G)/\!\!/G = \mathrm{Hom}^g(\Gamma, G)/G$. *We will denote these quotients by* $X^i_G(\Gamma)$ *and* $X^g_G(\Gamma)$, *respectively.*

In addition, Sikora showed that $X^i_G(\pi_1(S))$ is a complex orbifold [Sik09, Prop. 49] for a surface group $\pi_1(S)$. A calculation by Goldman shows that the complex dimension of this orbifold amounts to $(2g-2)\dim G + 2\dim C(G)$, where g is the genus of our surface [Gol84, Prop. 1.2]. However, we will not need these facts here, so we omit the proofs.

Proof of Proposition 2.6. By definition, every irreducible representation is completely reducible. Hence, all orbits in $\mathrm{Hom}^i(\Gamma, G)$ are closed according to Proposition 1.16. According to Theorem 2.4, every point of $\mathrm{Hom}^i(\Gamma, G)/\!\!/G$ corresponds to a unique closed orbit. As all orbits are closed, $\mathrm{Hom}^i(\Gamma, G)/\!\!/G = \mathrm{Hom}^i(\Gamma, G)/G$ follows.

Every good representation is irreducible as we saw in Proposition 1.17. Hence, the second equation follows with the exact same argument. □

Now, consider the subset $\mathrm{Hom}^g(\Gamma, G) \subset \mathrm{Hom}^i(\Gamma, G)$ of good representations. It is open according to Proposition 1.15.

Proposition 2.7. *For every surface group* $\pi_1(S)$ *and every reductive group* G, *the set* $X^g_G(\pi_1(S)) = \mathrm{Hom}^g(\pi_1(S), G)/G$ *is a non-singular open subset of* $X^i_G(\pi_1(S))$. *In particular, it can be considered a complex manifold.*

Proof. We follow the proof in [Sik09, Cor. 50]. Due to the fact that the stabilizers of good representations are exactly the center, the orbits fulfill $G.\rho = (G/C(G)).\rho$ for any good ρ. This implies that

$$\mathrm{Hom}^g(\pi_1(S), G)/G = \mathrm{Hom}^g(\pi_1(S), G)/(G/C(G)). \tag{13}$$

As G is smooth, $G/C(G)$ is as well [see Mil17b, Cor. 5.26]. Thus – as a consequence of Proposition 1.18 and Proposition 1.27 – we have a free proper action of the complex algebraic group $G/C(G)$ on the complex manifold $\mathrm{Hom}^g(\pi_1(S), G)$, so the quotient is a complex manifold as well. By Eq. (13), this quotient is already $X^g_G(\pi_1(S))$, which ends the proof. □

According to Proposition 1.7, we have a closed immersion $\mathrm{Hom}(\pi_1(S), G) \hookrightarrow \mathcal{H}om(\pi_1(S), G)$. Proposition 2.3 shows that it induces a closed immersion $X_G(\pi_1(S)) \hookrightarrow \mathcal{X}_G(\pi_1(S))$. Summarizing the results so far, we can write the following chain of inclusions:

$$X^g_G(\pi_1(S)) \subset_{\mathbb{C}-op} X^i_G(\pi_1(S)) \subset X_G(\pi_1(S)) \subset_{cl} \mathcal{X}_G(\pi_1(S)).$$

Remark 2.8. *Consider the fundamental group* $\pi_1(S, z_0)$ *and* $\pi_1(S, z_0')$ *at two different base points* z_0 *and* $z_0' \in S$. *A path* δ *connecting* z_0 *and* z_0' *induces an isomorphism*

$$r_\delta : \pi_1(S, z_0) \xrightarrow{\sim} \pi_1(S, z_0'), \quad \gamma \mapsto \delta\gamma\delta^{-1}.$$

Another isomorphism ϕ'_δ differs from ϕ_δ by conjugation with an element in $\pi_1(S, z'_0)$. Hence, the induced isomorphism

$$(r_\delta)^* : \mathrm{Hom}(\pi_1(S, z'_0), G) \xrightarrow{\sim} \mathrm{Hom}(\pi_1(S, z_0), G)$$

is "independent of δ up to conjugation in G". Therefore, the good character variety $X^g_G(\pi_1(S))$ is independent of the base point.

2.3. Tangent spaces of character varieties

From now on, we will assume G as being reductive, so the character variety exists. Using the projections $\mathrm{Hom}(\Gamma, G) \to X_G(\Gamma)$ and $\mathcal{H}om(\Gamma, G) \to \mathcal{X}_G(\Gamma)$, we can consider $\rho \in \mathrm{Hom}(\Gamma, G)$ a point of $X_G(\Gamma)$ and $\mathcal{X}_G(\Gamma)$. We will denote the image of ρ in $X_G(\Gamma)$ and $\mathcal{X}_G(\Gamma)$ by $[\rho]$.

We have a closed immersion $\mathrm{Hom}(\Gamma, G) \hookrightarrow \mathcal{H}om(\Gamma, G)$, which induces a closed immersion of schemes

$$X_G(\Gamma) \hookrightarrow \mathcal{X}_G(\Gamma),$$

which in turn induces an embedding on the tangent spaces

$$T_{[\rho]}X_G(\Gamma) \hookrightarrow T_{[\rho]}\mathcal{X}_G(\Gamma)$$

according to [Sta19, Lem. 0B2G, Lem. 04XV].[11]

Proposition 2.9. *If ρ is reduced, this embedding is an isomorphism.*

In particular, we have an isomorphism $T_{[\rho]}X_G(\pi_1(S)) \cong T_{[\rho]}\mathcal{X}_G(\pi_1(S))$ for good representations $\rho \in \mathrm{Hom}(\pi_1(S), G)$ by Lemma 1.25 and Proposition 1.27.

Proof. The closed immersion $X_G(\Gamma) \hookrightarrow \mathcal{X}_G(\Gamma)$ induces a surjection $\mathcal{O}_{\mathcal{X},\rho} \twoheadrightarrow \mathcal{O}_{X,\rho}$ on the stalks. It is suffices to show that this surjection is an isomorphism whenever ρ is reduced. Write $R = R(\Gamma, G)$ and denote by $\mathfrak{m} \subset R$ the maximal ideal corresponding to ρ. Denote by $R^G \subseteq R$ the ring of invariants and $\mathfrak{n} = \mathfrak{m} \cap R^G$ the corresponding prime ideal. We will see that the sequence

$$0 \to (R^G \cap \sqrt{0})_\mathfrak{n} \to (R^G)_\mathfrak{n} \xrightarrow{\phi} ((R/\sqrt{0})^G)_\mathfrak{n} \to 0$$

is exact. This follows simply by restricting the exact sequence

$$0 \to \sqrt{0} \to R \to R/\sqrt{0} \to 0$$

using the Reynolds operator as in the proof of Proposition 2.3, and then applying the fact that the localization functor is exact. The arrow $(R^G)_\mathfrak{n} \to ((R/\sqrt{0})^G)_\mathfrak{n}$ is the same as $\mathcal{O}_{\mathcal{X},\rho} \twoheadrightarrow \mathcal{O}_{X,\rho}$. In order to show that this is an isomorphism in case ρ is reduced, we only need to show that $(R^G \cap \sqrt{0})_\mathfrak{n} = 0$.

[11] If you do not want to use abstract nonsense, the statement quickly follows like this: The closed immersion corresponds to a projection $R \twoheadrightarrow R/\sqrt{0}$, which induces a surjective map on stalks, which induces a surjective map on the cotangent spaces. A surjective map on the cotangent spaces induces an injective map on the tangent spaces.

The canonical map

$$(R^G)_{\mathfrak{n}} \longrightarrow R_{\mathfrak{m}}, \quad \frac{a}{b} \mapsto \frac{a}{b} \tag{14}$$

is in fact injective. Indeed, suppose

$$\frac{a}{b} \equiv \frac{a'}{b'} \quad \text{in } R_{\mathfrak{m}}$$

for some $a, a' \in R^G$ and $b, b' \in R^G \backslash \mathfrak{n}$. By the definition of the localization, this means that there exists a $c \in R\backslash\mathfrak{m}$ such that

$$c(ab' - a'b) \in R\backslash\mathfrak{m}.$$

Applying the Reynolds operator $E : R \longrightarrow R$ together with the Reynolds identity [see MFK94, Sec. 1.2], we obtain

$$E(c(ab' - a'b)) = E(c)(ab' - a'b) \in R^G\backslash\mathfrak{n}.$$

Furthermore, $E(c) \in R^G\backslash\mathfrak{n}$, which means that

$$\frac{a}{b} \equiv \frac{a'}{b'} \quad \text{in } (R^G)_{\mathfrak{n}}.$$

This shows that the canonical map Eq. (14) is indeed injective. In particular, we have an embedding $(R^G \cap \sqrt{0})_{\mathfrak{n}} \subseteq (\sqrt{0})_{\mathfrak{m}} \subseteq R_{\mathfrak{m}}$. For a reduced ρ, we have $(\sqrt{0})_{\mathfrak{m}} = 0$, which implies that ϕ is indeed an isomorphism. $\square$

Theorem 2.10. *1. If $\rho \in \mathrm{Hom}(\Gamma, G)$ is completely reducible, then the isomorphism of Theorem 1.21 combined with the natural projection $\mathcal{H}om(\Gamma, G) \longrightarrow \mathcal{X}_G(\Gamma)$ induces a natural linear map $\phi : H^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\circ\rho}) \longrightarrow T_{[\rho]}\mathcal{X}_G(\Gamma)$.*

2. If ρ is good, then ϕ is an isomorphism. Note that ρ is irreducible according to Proposition 1.17, so ϕ exists.

Proof. This proof follows [Sik09, Thm. 53].

1. Consider a completely reducible ρ. According to Proposition 1.16, complete reducibility implies that the orbit O_ρ is closed in $\mathrm{Hom}(\Gamma, G)$ (and therefore also in $\mathcal{H}om(\Gamma, G)$). Therefore, we can apply Luna's Étale Slice Theorem A.14, and obtain an étale slice $V \subseteq \mathcal{H}om(\Gamma, G)$, i.e. a smooth affine variety that has the following properties: It contains ρ and is preserved by the G_ρ-action. Furthermore, the canonical morphism

$$(G \times V)/\!\!/G_\rho \xrightarrow{\psi} \mathcal{H}om(\Gamma, G)$$

is étale, and the canonical morphism

$$V/\!\!/G_\rho \longrightarrow \mathcal{H}om(\Gamma, G)/\!\!/G = \mathcal{X}_G(\Gamma)$$

is étale at ρ. Being étale means that the differential maps

$$T_{(e,\rho)}(G \times V)/\!\!/G_\rho \longrightarrow T_\rho\mathcal{H}om(\Gamma, G) \tag{15}$$

as well as

$$T_\rho V/\!\!/G_\rho \to T_{[\rho]}\mathcal{X}_G(\Gamma) \tag{16}$$

are isomorphisms. The natural projection $\mathcal{H}om(\Gamma, G) \to \mathcal{X}_G(\Gamma)$ induces a map

$$T_\rho\mathcal{H}om(\Gamma, G) \to T_{[\rho]}\mathcal{X}_G(\Gamma)$$

on tangent spaces. All of the above maps are canonical, so this differential map can be split up as follows:

$$T_\rho\mathcal{H}om(\Gamma, G) \xrightarrow{\cong} T_{(e,\rho)}(G\times V)/\!\!/G_\rho \to T_\rho V/\!\!/G_\rho \xrightarrow{\cong} T_{[\rho]}\mathcal{X}_G(\Gamma), \tag{17}$$

where the first map is given by the inverse of Eq. (15), the third map is given by Eq. (16), and the second map is the differential

$$(dp)_{(e,\rho)} : T_{(e,\rho)}(G\times V)/\!\!/G_\rho \to T_\rho V/\!\!/S_\rho$$

of the canonical map

$$p : (G\times V)/\!\!/G_\rho \to V/\!\!/G_\rho. \tag{18}$$

The point $\rho \in V$ is represented by a morphism $\rho : \mathrm{spec}(\mathbb{C}) \to V$. Hence, we have a constant $\mathbb{C}$-morphism $G \to \mathbb{C} \xrightarrow{\rho} V$. Therefore, the morphism $G \xrightarrow{(\mathrm{id},\rho)} (G\times V)/\!\!/G_\rho \to V/\!\!/G_\rho$ is constant. In consequence, the image of T_eG in $T_{(e,\rho)}(G\times V)/\!\!/G_\rho$ is contained in the kernel of $(dp)_{(e,\rho)}$. As we show in Proposition A.16, the image of T_eG corresponds to $T_\rho O_\rho$ under the isomorphism in Eq. (15), and therefore, the map in Eq. (17) factorizes over

$$T_\rho\mathcal{H}om(\Gamma, G)\big/T_\rho O_\rho \xrightarrow{\cong} \left(T_{(e,\rho)}(G\times V)/\!\!/G_\rho\right)\big/\left(\mathrm{im}\, T_eG\right) \to T_\rho V/\!\!/G_\rho \xrightarrow{\cong} T_\rho\mathcal{X}_G(\Gamma). \tag{19}$$

Using the isomorphism from Theorem 1.20 respectively Theorem 1.28, we obtain the first statement.

2. If ρ is good, then it is completely reducible according to Proposition 1.17, so the map from 1. exists in this case. Per definition, $G_\rho = C(G)$, which acts trivially on $\mathrm{Hom}(\Gamma, G)$, and thus also on V. The p in Eq. (18) thus becomes

$$G/C(G)\times V \to V.$$

Its differential is the projection $(dp)_{(e,\rho)} : T_eG/C(G)\times T_\rho V \to T_\rho V$. In 1., we saw that the image of T_eG is contained in the kernel of $(dp)_{(e,\rho)}$. Now, the image of $T_eG \to T_eG/C(G)\times T_\rho V$ is precisely the kernel. Therefore, the map in Eq. (19) boils down to

$$T_\rho\mathcal{H}om(\Gamma, G)/T_\rho O_\rho \xrightarrow{\sim} T_{(e,\rho)}(G/C(G)\times V)/(\mathrm{im}\, T_eG) \to T_\rho V \xrightarrow{\sim} T_\rho\mathcal{X}(\Gamma),$$

which must be an isomorphism. □

The isomorphism from Theorem 2.10.2 is also natural in the following sense. Let $r : \Gamma \to \Delta$ be a group homomorphism between two discrete groups and let G be a reductive affine algebraic group. If $\rho : \Delta \to G$ is a representation, then $\rho\circ r$ is a representation of Γ in G. r induces a homomorphism $r^* : H^1(\Delta, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \to H^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho r})$. We have:

Lemma 2.11. *Let G be reductive, and let $\rho : \Delta \to G$ be a representation. If $\rho \circ r : \Gamma \to G$ is good, then ρ is good as well, and with the isomorphism ϕ from Theorem 2.10.2, the diagram*

$$\begin{array}{ccc} H^1(\Delta, \mathfrak{g}_{\operatorname{Ad}\rho}) & \xrightarrow{r^*} & H^1(\Gamma, \mathfrak{g}_{\operatorname{Ad}\rho r}) \\ \big\downarrow{\scriptstyle\phi} & & \big\downarrow{\scriptstyle\phi} \\ T_{[\rho]}\mathcal{X}_G(\Delta) & \xrightarrow{dr^*} & T_{[\rho]}\mathcal{X}_G(\Gamma) \end{array}$$

commutes.

Proof. The isomorphism ϕ comes from the isomorphism Ψ_ρ^{-1} from Theorem 1.21, so the commutativity of the diagram follows from Lemma 1.23. It remains to be shown that $\rho \circ r$ being good implies that ρ is good as well. According to Proposition 1.17, it suffices to show that $G_\rho = C(G)$ and that ρ is irreducible.

We commence with the former. From the definition of the G-action, it directly follows that $G_\rho \supseteq C(G)$. Furthermore, any element $g \in G$ that leaves ρ invariant will also leave $\rho \circ r$ invariant. Therefore, $G_\rho \subseteq G_{\rho r}$. As ρr is good, we have $G_{\rho r} = C(G)$, which shows that $G_\rho = C(G)$.

ρr is irreducible, which means that $(\rho \circ r)(\Gamma) \subseteq G$ is not contained in any proper parabolic subgroup $P \subset G$. However, we have an embedding $(\rho \circ r)(\Gamma) \subseteq \rho(\Delta)$, so that $\rho(\Delta)$ cannot be contained in any proper parabolic subgroup either, i.e. ρ is irreducible. □

3. The symplectic structure of character varieties of surface groups

3.1. Non-degenerate forms in group cohomology

Let Γ be any group. In the following section, we will call any abelian group with a Γ-action a Γ-left module[12] (short: Γ-module) and by $\otimes_\Gamma$, we mean $\otimes_{\mathbb{Z}[\Gamma]}$.

Let M be a Γ-module. Let (F_*, ∂) be a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\Gamma]$-module, then the group homology of Γ with coefficients in M is defined as the homology of the chain complex $F_* \otimes_\Gamma M$,

$$H_k(\Gamma, M) = H_k(F_* \otimes_\Gamma M).$$

Accordingly, consider the cochain complex (C^*, δ) given by $C^p = \mathrm{Hom}_\Gamma(F_p, M)$ with $(\delta u)(x) = u(\partial x)$[13], then the group cohomology is defined as the cohomology of the cochain complex

$$H^k(\Gamma, M) = H^k(C^*).$$

Now, let M and N be two Γ-modules. We have a chain map

$$\mathrm{Hom}_\Gamma(F_*, M) \otimes_\Gamma (F_* \otimes_\Gamma N) \to M \otimes_\Gamma N, u \otimes (x \otimes n) \mapsto u(x) \otimes n,$$

which induces a pairing of Γ-modules

$$\langle \cdot, \cdot \rangle_\Gamma : H^k(\Gamma, M) \times H_k(\Gamma, N) \to M \otimes_\Gamma N.$$

In fact, we need to be cautious here, as the tensor product $M \otimes_\Gamma N$ is defined for a Γ-right module M and a Γ-left module N. However, we can turn a left module M into a right module via $m.\gamma = \gamma^{-1} m$.

The pairing $\langle \cdot, \cdot \rangle_\Gamma$ has the following properties:

Proposition 3.1.

1. *Consider two Γ-modules M and N. For $u \in H^k(\Gamma, M)$ and $v \in H_k(\Gamma, N)$, we have $u \cap v = \langle u, v \rangle$ in $H_0(\Gamma, M \otimes_\Gamma N) = M \otimes_\Gamma N$.*

2. *Consider three Γ-modules M_1, M_2 and M_3. For elements $u \in H^k(\Gamma, M_1)$, $v \in H^l(\Gamma, M_2)$ and $w \in H_{k+l}(\Gamma, M_3)$, the following adjunction formula holds:*

$$\langle u \cup v, w \rangle = \langle u, v \cap w \rangle.$$

3. *Let $r : \Gamma \to \Delta$ be a group homomorphism. We can consider a Δ-module M as a Γ-module via r and denote it by M_r. r induces maps $r_* : H_k(\Gamma, M_r) \to H_k(\Delta, M)$ and $r^* : H^k(\Delta, M) \to H^k(\Gamma, M_r)$ on the (co)homology groups. For $u \in H^k(\Delta, M)$ and $v \in H^k(\Gamma, M_r)$, we have the following duality property*

$$\pi_r \langle r^* u, v \rangle_\Gamma = \langle u, r_* v \rangle_\Delta,$$

where $\pi_r : M_r \otimes_\Gamma M_r \to (M \otimes_\Delta M)_r, x \otimes y \mapsto x \otimes y$ is the canonical Γ-module homomorphism induced by $M \times M \to (M \otimes_\Delta M)$.

[12]Using the group ring, this is the same as a $\mathbb{Z}[\Gamma]$-module.

[13][Bro05, Sec. III.1] actually defines $(\delta u)(x) = (-1)^{p+1} u(\partial x)$, which makes some things more canonical. However, the definitions for cocycles, coboundaries and cohomology groups remain untouched by the choice of sign.

Proof. The first and the second properties are discussed in [Bro05, Sec. V.3].

For the third property, let F_*^Γ and F_*^Δ be projective resolutions of Γ respectively Δ, and let $\tau : F_*^\Gamma \to F_*^\Delta$ be an augmentation-preserving Γ-chain map compatible with r. Thus, the maps r_* and r^* are induced by

$$\tau \otimes \mathrm{id} : F_*^\Gamma \otimes_\Gamma M_r \to F_*^\Delta \otimes_\Delta M, x \otimes m \mapsto \tau(x) \otimes m$$

respectively

$$\mathrm{Hom}(\tau, \mathrm{id}) : \mathrm{Hom}_\Delta(F_*^\Delta, M) \to \mathrm{Hom}_\Gamma(F_*^\Gamma, M_r), u \mapsto u \circ \tau.$$

Let $u \in H^k(\Delta, M)$ and $v \in H_k(\Gamma, M_r)$ be represented by $u \in \mathrm{Hom}_\Delta(F_p^\Delta, M)$ and $x_v \otimes m_v \in F_p^\Gamma \otimes_\Gamma M$, then we have

$$\langle r^* u, v\rangle_\Gamma = u(\tau(x_v)) \otimes m_v \in M_r \otimes_\Gamma M_r$$

as well as

$$\langle u, r_* v\rangle_\Delta = u(\tau(x_v)) \otimes m_v \in M \otimes_\Delta M,$$

which shows our statement. □

Now, assume that M and N bear the structure of $\mathbb{C}$-vector spaces, such that the scalar multiplication commutes with the action of Γ. By scalar multiplication on M, the chain complexes and cochain complexes defined above naturally become complexes of $\mathbb{C}$-vector spaces with $\mathbb{C}$-linear (co)boundary operators. Therefore, the (co)homology groups are $\mathbb{C}$-vector spaces.

Consider a $\mathbb{C}$-bilinear pairing

$$B : M \times N \to \mathbb{C}$$

that is invariant under the Γ-action. If we consider $\mathbb{C}$ as a Γ-module via the trivial action, then B is in fact $\mathbb{Z}[\Gamma]$-bilinear as well. We now define the following $\mathbb{Z}$-bilinear pairing by composing the pairing defined above with B:

$$\tilde{\omega}^B : H^k(\Gamma, M) \times H_k(\Gamma, N) \xrightarrow{\langle\cdot,\cdot\rangle_\Gamma} M \otimes_\Gamma N \xrightarrow{B} \mathbb{C}\,.$$

Even though $M \otimes_\Gamma N$ is not canonically a $\mathbb{C}$-module – but rather a $\mathbb{C} \times \mathbb{C}$-module – the whole pairing is $\mathbb{C}$-bilinear. If we assume that B induces an isomorphism of $\mathbb{C}$-vector spaces

$$M \xrightarrow{\cong} \mathrm{Hom}_\mathbb{C}(N, \mathbb{C}), \quad m \mapsto B(m, \cdot), \tag{20}$$

we obtain:

Proposition 3.2. *If B is a bilinear pairing inducing $M \cong \mathrm{Hom}_\mathbb{C}(N, \mathbb{C})$ as described above, then the pairing $\tilde{\omega}^B$ is non-degenerate, i.e. it induces isomorphisms*

$$H^k(\Gamma, M) \xrightarrow{\cong} \mathrm{Hom}_\mathbb{C}(H_k(\Gamma, N), \mathbb{C}), \quad u \mapsto \tilde{\omega}^B(u, \cdot)$$

of $\mathbb{C}$-vector spaces for all k.

Proof. The groups $H^k(\Gamma, M)$ are the cohomology groups of the cochain complex $(\mathrm{Hom}_\Gamma(F_*, M), \delta)$ for some projective resolution (F_*, ∂) of $\mathbb{Z}$ as $\mathbb{Z}[\Gamma]$-module. We have

$$\begin{aligned}(\mathrm{Hom}_\Gamma(F_*, M), \delta) &= (\mathrm{Hom}_\Gamma(F_*, \mathrm{Hom}_\mathbb{C}(N, \mathbb{C})), \delta) && \text{according to the choice of } B \text{, see Eq. (20)}\\ &= (\mathrm{Hom}_\mathbb{C}(F_* \otimes_\Gamma N, \mathbb{C})), \delta) && \text{as } \otimes \text{ and Hom are adjoint}\\ &= \mathrm{Hom}_\mathbb{C}((F_* \otimes_\Gamma N, \partial), \mathbb{C}),\end{aligned}$$

so the cohomology groups of the two cochain complexes are the same:

$$H^k(\mathrm{Hom}_\Gamma(F_*, M), \delta) = H^k(\mathrm{Hom}_\mathbb{C}((F_* \otimes_\Gamma N, \partial), \mathbb{C})).$$

As a vector space, $\mathbb{C}$ is an injective $\mathbb{C}$-module, which means that the functor $\mathrm{Hom}_\mathbb{C}(-, \mathbb{C})$ is exact. In particular, the functor translates homology groups into cohomology groups. Thus, we have a group isomorphism

$$H^k(\mathrm{Hom}_\Gamma(F_*, M), \delta) \cong \mathrm{Hom}_\mathbb{C}(H_k(F_* \otimes_\Gamma N, \partial), \mathbb{C}),$$

which is in fact $\mathbb{C}$-linear. Tracking down the definitions, we see that this isomorphism is induced by

$$\begin{aligned}\mathrm{Hom}_\Gamma(F_k, M) &\longrightarrow \mathrm{Hom}_\mathbb{C}(F_k \otimes_\Gamma N, \mathbb{C})\\ u &\longmapsto [x_k \otimes n \mapsto B(u(x_k) \otimes n)],\end{aligned}$$

but $B(u(x_k) \otimes n) = B(\langle u, x_k \otimes n\rangle_\Gamma)$, so by definition, this isomorphism is actually induced by $\tilde{\omega}^B$. □

3.2. Goldman's symplectic form

3.2.1. Definition and non-degeneracy of the form

Consider an arbitrary orientable compact surface S. We will abbreviate $X_G(S) = X_G(\pi_1(S))$ and accordingly for the other definitions. Let G be a reductive group and let $\mathfrak{g}$ be its Lie algebra. As before, we will turn $\mathfrak{g}$ into a $\pi_1(S)$-module via $\mathrm{Ad} \circ \rho$ and denote it by $\mathfrak{g}_{\mathrm{Ad}\,\rho}$. Let $\mathcal{A}$ be a local coefficient system with fiber $\mathfrak{g}_{\mathrm{Ad}\,\rho}$. As S is asperical, we can identify singular cohomology with local coefficients and group cohomology

$$H^k_{\mathrm{sing}}(S, \mathcal{A}) \cong H^k_{\mathrm{Grp}}(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho}),$$

see Appendix C.2 for details. We will abbreviate both of them by $H^k(S, \mathfrak{g}_{\mathrm{Ad}\,\rho})$.

Definition 3.3. A bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is called Ad-invariant, if it is invariant under the adjoint representation, i.e. $B(\mathrm{Ad}(g)x, \mathrm{Ad}(g)y) = B(x, y)$.

In particular, this means that $B : \mathfrak{g}_{\mathrm{Ad}\,\rho} \times \mathfrak{g}_{\mathrm{Ad}\,\rho} \to \mathbb{C}$ is $\pi_1(S)$-invariant.

Using a "nice" bilinear form B, we will now define a complex symplectic structure on the manifold $X^g_G(S)$. Let $\rho \in \mathrm{Hom}(\pi_1(S), G)$ be an arbitrary representation. We begin by defining a complex symplectic form on the cohomology group $H^1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho})$. Denote by $\cup$ the cup product, then

$$\omega_\rho^B : H^1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \times H^1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \xrightarrow{\cup} H^2(S, \mathfrak{g}_{\mathrm{Ad}\,\rho} \otimes \mathfrak{g}_{\mathrm{Ad}\,\rho}) \xrightarrow{B} H^2(S, \mathbb{C}) \cong \mathbb{C}$$

is obviously a bilinear form. Note that S is two-dimensional and orientable, so $H^2(S, \mathbb{Z}) \cong \mathbb{Z}$ and hence $H^2(S, \mathbb{C}) \cong \mathbb{C}$. ω_ρ^B can be identified to the $\tilde{\omega}^B$ from the previous section as we will see in the proof of the following Proposition 3.4. Let $[S] \in H_2(S, \mathfrak{g}_{\mathrm{Ad}\,\rho})$ be the fundamental class of the manifold. Poincaré duality with local coefficients [see Spa93] implies that we have isomorphisms

$$PD : H^k(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho}) \longrightarrow H_{2-k}(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho}), \quad u \mapsto u \cap [S]. \tag{21}$$

Using Poincaré duality, we will show the following result:

Proposition 3.4. *For a symmetric,* Ad*-invariant and non-degenerate* $\mathbb{C}$*-bilinear form* $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$*, the form* ω_ρ^B *is a symplectic form on* $H^1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho})$.

Proof. We follow the proof in [Sik09, Cor. 59].

A bilinear form on a vector space is called *symplectic* if it is skew-symmetric and induces an isomorphism between the space and its dual. The cup product fulfills $a \cup b = (-1)^{pq}(b \cup a)$, so on the level $p = q = 1$, it is skew-symmetric. Together with the symmetry of B, this shows that ω_ρ^B is skew-symmetric.

Hence, it only remains to be shown that ω_ρ^B induces an isomorphism between $H^1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho})$ and its dual. For $M = N = \mathfrak{g}_{\mathrm{Ad}\,\rho}$ and $\Gamma = \pi_1(S)$, ω_ρ^B is “isomorphic” to the form $\tilde{\omega}^B$ defined in the previous section. Indeed, consider the commutative diagram

$$\begin{array}{ccccc}
H^1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \times H^1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho}) & \xrightarrow{\quad\cup\quad} & H^2(S, \mathfrak{g}_{\mathrm{Ad}\,\rho} \otimes \mathfrak{g}_{\mathrm{Ad}\,\rho}) & \xrightarrow{\quad B\quad} & H^2(S, \mathbb{C}) \cong \mathbb{C} \\
\cong \Big\downarrow \mathrm{id} \times (\,\cdot \cap [S]) & & \cong \Big\downarrow \cdot \cap [S] & & \cong \Big\downarrow \cdot \cap [S] \\
H^1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \times H_1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho}) & \xrightarrow{\langle \cdot, \cdot \rangle} & H_0(S, \mathfrak{g}_{\mathrm{Ad}\,\rho} \otimes \mathfrak{g}_{\mathrm{Ad}\,\rho}) & \xrightarrow{\quad B\quad} & H_0(S, \mathbb{C}) \cong \mathbb{C}\,.
\end{array} \tag{22}$$

The columns are given by the Poincaré duality isomorphisms PD, see Eq. (21). The upper line is the form ω_ρ^B defined above, the lower line is the form $\tilde{\omega}^B$ from Section 3.1. The left cell is commutative by Proposition 3.1.1 and 3.1 .2, and the commutativity of the right cell is trivial. Therefore, the isomorphism between $H^1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho})$ and its dual is given according to Proposition 3.2. ∎

The following corollary is a direct consequence is a direct consequence of the non-degeneracy of $\tilde{\omega}^B$ or ω_ρ^B. Let V be a $\mathbb{C}$-vector, then we denote its dual vector space by $V^\vee = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$.

Corollary 3.5. *The diagram in Eq.* (22) *gives us the following isomorphisms:*

$$\begin{aligned}
H^1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho}) &\to H^1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho})^\vee, & u &\mapsto \omega_\rho^B(u, \cdot), \\
H^1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho}) &\to H_1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho})^\vee, & u &\mapsto \omega_\rho^B(u, PD^{-1}(\cdot)) = \tilde{\omega}^B(u, \cdot) \\
H_1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho}) &\to H^1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho})^\vee, & x &\mapsto \omega_\rho^B(\cdot, PD^{-1}(x)) = \tilde{\omega}^B(\cdot, x).
\end{aligned}$$

More generally, we have isomorphisms

$$H^k(M, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \to H_k(M, \mathfrak{g}_{\mathrm{Ad}\,\rho})^\vee, \quad u \mapsto \tilde{\omega}^B(u, \cdot)$$
$$H_k(M, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \to H^k(M, \mathfrak{g}_{\mathrm{Ad}\,\rho})^\vee, \quad x \mapsto \tilde{\omega}^B(\cdot, x)$$

for any orientable compact finite-dimensional manifold M.

Lemma 3.6. *Suppose that group (co)homology and singular (co)homology are isomorphic up to a certain level $k \leqslant N$. Then the isomorphisms from Corollary 3.5 are natural in the following sense: Let $S \to M$ be a smooth map. Furthermore, let $r^* : H^k(M, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \to H^k(S, \mathfrak{g}_{\mathrm{Ad}\,\rho})$ and $r_* : H_k(S, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \to H_k(M, \mathfrak{g}_{\mathrm{Ad}\,\rho})$ be the respective induced maps on (co)homology. Then, under the isomorphisms from Corollary 3.5, r_* corresponds to the dual map of r^*, i.e. $r_* = (r^*)^\vee$.*

Proof. How is the dual map of a vector space homomorphism defined? Let $\phi : V \to W$ be a linear map of finite-dimensional vector spaces, and let $V^\vee \times V \to \mathbb{C}$, $(u, x) \mapsto \langle u, x\rangle_{\mathrm{eval},V} = u(x)$ be the evaluation pairing. Then, the dual map $\phi^\vee : W^\vee \to V^\vee$ can simply be defined via $\langle \phi^\vee(u), x\rangle_{\mathrm{eval},V} = \langle u, \phi(x)\rangle_{\mathrm{eval},W}$.

Under the isomorphism $H^k(M, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \cong H_k(M, \mathfrak{g})^\vee$ described above, $\tilde{\omega}^B$ corresponds to the evaluation pairing, as $\langle \tilde{\omega}^B(u, \cdot), x\rangle_{\mathrm{eval},V} = \tilde{\omega}^B(u, x)$ per definition for $u \in H^k(S, \mathfrak{g}_{\mathrm{Ad}\,\rho})$ and $x \in H_k(S, \mathfrak{g}_{\mathrm{Ad}\,\rho})$ (and accordingly for M, of course). Now, take $u \in H^k(M, \mathfrak{g}_{\mathrm{Ad}\,\rho})$ and $x \in H_k(S, \mathfrak{g}_{\mathrm{Ad}\,\rho})$. According to Proposition 3.1.3, we have

$$\tilde{\omega}^{B,S}(r^*u, x) = B \circ \langle r^*u, x\rangle_{\pi_1(S)} = B \circ \langle u, r_*x\rangle_{\pi_1(M)} = \tilde{\omega}^{B,M}(u, r_*x),$$

so r^* is dual to r_* under the above isomorphisms. □

Remark 3.7.

1. *According to the Cartan-Killing criterion, the Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form on $\mathfrak{g}$ is non-degenerate. So, in the semisimple case, the Killing form can be used as B in Proposition 3.4.*
2. *In the case where $\mathfrak{g}$ is simple, it is a well-known fact that every symmetric, G-invariant, non-degenerate form on $\mathfrak{g}$ is a scalar multiple of the Killing form.*

The tangent spaces of the complex manifold $X_G^g(S)$ are $H^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho})$ according to Theorem 2.10. We define a 2-form $\omega^B \in \Omega^2(X_G^g(S))$ via $[\rho] \mapsto \omega_\rho^B$.

Theorem 3.8. *$(X_G^g(S), \omega^B)$ is a complex symplectic manifold.*

Proof. In Proposition 3.4, we showed that ω_ρ^B is a symplectic form on any tangent space, so it remains to be shown that ω^B is closed. Closedness will be proved in the Section 3.2.2 below. □

3.2.2. Closedness of the form

Proposition 3.9. *The 2-form ω^B on $X_G^g(S)$ is closed.*

Proof sketch. The closedness is proved by performing a symplectic reduction of a larger, infinite-dimensional space. We follow the sketch as it is provided in [Gol84, Sec. 1.8.] and [AB83, Sec. 9]. We will omit the discussion of smoothness and well-definedness of those infinite dimensional manifolds, maps and their quotients.

As we will explain in Appendix B.2, the space of all connections $\mathcal{A}(P)$ is an affine space, and thus a manifold with tangent space $T_A\mathcal{A}(P) = \Omega^1(S, \operatorname{ad} P)$ at a point $A \in \mathcal{A}(P)$. The gauge group $\mathcal{G}(P)$ is a Lie group with Lie algebra $\mathfrak{gau}(P) = \Omega^0(S, \operatorname{ad} P)$, which acts on $\mathcal{A}(P)$, see Appendix B.4 for details. Finally, we have the curvature function $F : \mathcal{A}(P) \to \Omega^2(S, \operatorname{ad} P)$. Furthermore, in Appendix B.3 we will see that the level set $\mathcal{F} = \mathcal{F}(P) = F^{-1}(0)$ of flat connections has tangent spaces $T_A\mathcal{F} = Z^1(S, \operatorname{ad} P)$.

Now, we have a 2-form $\bar{\omega}^B$ on $\mathcal{A}(P)$, which is defined by

$$\bar{\omega}^B_A : T_A\mathcal{A}(P) \times T_A\mathcal{A}(P) \to \Omega^2(S, \underline{\mathbb{C}}) \xrightarrow{\sim} \mathbb{C}, \quad (\eta, \theta) \mapsto \int_S B_*(\eta \wedge \theta),$$

where $B_*(\eta \wedge \theta)$ is defined as explained in Proposition C.1. The same proposition gives us

$$\Omega^2(S, \operatorname{ad} P) \times \Omega^0(S, \operatorname{ad} P) \to \mathbb{C}, \quad (\eta, \theta) \mapsto \int_S B_*(\eta \cdot \theta) = \int_S B_*(\eta \wedge \theta),$$

and this second pairing induces an isomorphism $\Omega^2(S, \operatorname{ad} P) \cong \Omega^0(S, \operatorname{ad} P)^\vee$.

We want to perform a symplectic reduction on $\mathcal{A}(P)$ in the sense of Marsden, Weinstein and Meyer [Sil08, Thm. 23.1]. For this, we will need a moment map $\mu : \mathcal{A}(P) \to \mathfrak{gau}(P)^\vee$. The curvature is a good candidate. We define

$$\mu : \mathcal{A}(P) \to \Omega^2(M, \operatorname{ad} P) \xrightarrow{\sim} \Omega^0(M, \operatorname{ad} P)^\vee, \quad A \mapsto \int_S B_*(F(A) \wedge \cdot\,).$$

For μ to be a moment map, we must show G-equivariance (where $\mathcal{G}(P)$ acts on $\mathfrak{gau}(P)^\vee$ via the coadjoint action) and the "Hamilton function property", i.e.

$$d\mu^X = \iota_{\tilde{X}}\bar{\omega}^B \text{ for any } X \in \Omega^0(M, \operatorname{ad} P), \tag{23}$$

where $\mu^X : \mathcal{A}(P) \to \mathbb{C}, \mu^X(A) = \mu(A)(X)$, and $\iota_{\tilde{X}}$ is the contraction along the fundamental vector field $\tilde{X}$ generated by X.

Step i: The Hamilton function property. We will now prove the property from Eq. (23). Let A_t be a curve in $\mathcal{A}(P)$ such that $\frac{d}{dt}|_{t=0}A_t = \eta \in \Omega^1(S, \operatorname{ad} P)$ and $A_0 = A$. Then,

$$\begin{aligned}(d\mu^X)_A(\eta) = \frac{d}{dt}\mu^X(A_t)\Big|_{t=0} &= \frac{d}{dt}\int_S B_*(F(A_t) \wedge X)\Big|_{t=0} \\ &= \int_S B_*(\frac{d}{dt}F(A_t)|_{t=0} \wedge X) \\ &= \int_S B_*(d_A\eta \wedge X),\end{aligned}$$

where the last line is a consequence of Eq. (33). Now, note that

$$\int_S dB_*(\eta \wedge X) = \int_S B_* d_A(\eta \wedge X) = \int_S B_*(d_A\eta \wedge X) + \int_S B_*(\eta \wedge d_A X).$$

The left-hand side is 0 as a consequence of Stokes' theorem for closed surfaces. Therefore,

$$(d\mu^X)_A(\eta) = \int_S B_*(-\eta \wedge d_A X) = \int_S B_*(d_A X \wedge \eta) = \iota_{\tilde{X}} \bar{\omega}^B(\eta),$$

where we applied that $\tilde{X}(A) = d_A X$, see Corollary B.14.

Step II: $\mathcal{G}(P)$-equivariance. The $\mathcal{G}(P)$-equivariance of the moment map is easy to see. We have an anti-isomorphism $\mathcal{G}(P) \cong C^\infty(P, G)^G$, which is given by

$$f = [P \to P, p \mapsto p.a_f(p)] \mapsto a_f(\cdot).$$

Under the $\mathcal{G}(P)$-action, the curvature varies by

$$F(f^*A) = \mathrm{Ad}^G(a_f^{-1}) \circ F(A)$$

[see Bau14, Satz 3.22]. B_* is invariant under the Ad^G action, simply because B is Ad-invariant. Therefore, we have

$$B_*(\mathrm{Ad}^G(a_f^{-1}) \circ F(A) \wedge X) = B_*(\mathrm{Ad}^G(a_f) \circ (\mathrm{Ad}^G(a_f^{-1}) \circ F(A) \wedge X)) = B_*(F(A) \wedge \mathrm{Ad}^G(a_f) \circ X)$$

for any $X \in \Omega^0(S, \mathrm{ad}\, P)$. However, $\mathrm{Ad}^G(a_f) \circ X = \mathrm{Ad}^{\mathcal{G}(P)}(f^{-1})(X)$, because $\mathcal{G}(P) \cong C^\infty(P, G)^G$ is an anti-isomorphism, as explained in Eq. (34). In other words,

$$\mu(f^*A)(X) = \mu(A)(\mathrm{Ad}^{\mathcal{G}(P)}(f^{-1}).X) = (\mathrm{Ad}^{\mathcal{G}(P)})^*(f).\mu(A)(X),$$

which shows that μ is indeed $\mathcal{G}(P)$-equivariant.

Contrary to the classical approach to symplectic reduction [see Sil08, Sec. 23], we are dealing with infinite dimensional manifolds in this case. Nevertheless, a proper treatment shows that the quotient $\mu^{-1}(0)/\mathcal{G}(P) = \mathcal{F}/\mathcal{G}(P)$ admits a symplectic structure coming from $\bar{\omega}_A^B$ as defined above. A more in-depth treatment of the theory of Sobolev spaces necessary for dealing with the issues of infinite dimensions can be found in [AB83, Sec. 14].

Now, any connected component of the character variety is isomorphic to a $\mathcal{A}(P)$ as explained in Proposition B.3. It can be shown that orbits $O_A \subseteq \mathcal{F}$ have tangent spaces $T_A O_A \cong B^1(S, \mathrm{ad}\, P)$, and $T_A \mathcal{F}/\mathcal{G}(P) \cong Z^1(S, \mathrm{ad}\, P)/B^1(S, \mathrm{ad}\, P) \cong H^1(S, \mathrm{ad}\, P)$. Furthermore, the Goldman symplectic form ω_ρ^B equals $\bar{\omega}_A^B$ from this proof. Therefore, this symplectic reduction proves the closedness ω^B. □

3.3. Character varieties of 3-manifolds as Lagrangian subspaces

In symplectic geometry, we have the following definition [see also Sil08, Sec. II.3].

Definition 3.10. Let (M, ω) be a symplectic manifold, and let $X \subset M$ be a submanifold, i.e. the inclusion $\iota : X \hookrightarrow M$ is a proper injective immersion. Then, X is called a *Lagrangian submanifold* of M if $\dim X = \frac{1}{2} \dim M$ and if it is an isotropic subspace, i.e. $\iota^*\omega = 0$.

Let M be an orientable compact 3-manifold with boundary $\partial M = S$, and let G be any reductive affine algebraic group. The embedding $S = \partial M \hookrightarrow M$ induces a homomorphism $r : \pi_1(S) \to \pi_1(M)$. In accordance with Lemma 1.3, we have G-equivariant morphism on the representation varieties. According to the universal property of the categorial quotient, we obtain a morphism $r^* : X_G(M) \to X_G(S)$. In Theorem 3.12, we will see that the smooth part of the image of $X_G(M)$ in $X_G^g(S)$ is a Lagrangian submanifold.

Proposition 3.11. *1. Consider the G-equivariant morphism $r^* : \mathrm{Hom}(\pi_1(M), G) \to \mathrm{Hom}(\pi_1(S), G)$. Its image is Zariski-closed in $\mathrm{Hom}(\pi_1(S), G)$.*

2. Assume that $\mathrm{Hom}^g(\pi_1(S), G)$ has a subset $\mathrm{Hom}^{gg}(\pi_1(S), G)$ which is non-empty, G-invariant and Zariski-open in $\mathrm{Hom}(\pi_1(S), G)$. Write $X_G^{gg}(\pi_1(S)) = \mathrm{Hom}^{gg}(\pi_1(S), G)/\!\!/G$. Consider the image $r^*X_G(M)$ of the induced morphism $r^* : X_G(M) \to X_G(S)$. Then, $r^*X_G(M) \cap X_G^{gg}(S)$ is Zariski-closed in $X_G^{gg}(S)$.

Proof. 1. Consider finite presentations of the group $\pi_1(S) = \langle \gamma_i | w_\lambda \rangle$ and $\pi_1(M) = \langle \gamma_j | w_\nu \rangle$ with finite index sets $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, N'\}$. Every curve in S is a curve in M. Therefore, we can assume that $\{\gamma_i\} \subset \{\gamma_j\}$ by adding the necessary relations to the w_ν. Under the embeddings $\mathrm{Hom}(\pi_1(M), G) \subseteq G^{N'}$ and $\mathrm{Hom}(\pi_1(S), G) \subseteq G^N$, the map $r^* : \mathrm{Hom}(\pi_1(M), G) \to \mathrm{Hom}(\pi_1(S), G)$ is therefore of a very simple form: It simply projects

$$(\rho(\gamma_i), \rho(\gamma_{j\neq i})) \mapsto (\rho(\gamma_i)).$$

We embed $\iota : G^N \hookrightarrow G^{N'}$, $(g_i) \mapsto (g_i, e, \dots, e)$. Thus, the image $r^* \mathrm{Hom}(\pi_1(M), G)$ is given by $\bigcap_\nu (w_\nu \circ \iota)^{-1}(e)$, which is a Zariski-closed subset of G^N. However, it already lies in $\mathrm{Hom}(\pi_1(S), G)$, so that it is closed in the subspace topology on $\mathrm{Hom}(\pi_1(S), G)$.

2. Assume that $\mathrm{Hom}^g(\pi_1(S), G)$ has a subset $\mathrm{Hom}^{gg}(\pi_1(S), G)$ which is non-empty, G-invariant and Zariski-open in $\mathrm{Hom}(\pi_1(S), G)$. Note that the inverse image

$$(r^*)^{-1}(\mathrm{Hom}^{gg}(\pi_1(S), G)) \subseteq \mathrm{Hom}(\pi_1(M), G)$$

is Zariski-open because r^* is continuous. Hence, we have a commutative diagram of quasi-affine algebraic sets

$$\begin{array}{ccc}
(r^*)^{-1}(\mathrm{Hom}^{gg}(\pi_1(S), G)) & \xrightarrow{r^*} & \mathrm{Hom}^{gg}(\pi_1(S), G) \\
\big\downarrow \pi & & \big\downarrow \pi \\
(r^*)^{-1}(\mathrm{Hom}^{gg}(\pi_1(S), G))/\!\!/G & \xrightarrow{r^*} & X^{gg}(\pi_1(S), G).
\end{array}$$

According to Lemma 2.11, all representations in $(r^*)^{-1}(\mathrm{Hom}^{gg}(\pi_1(S), G))$ are good. Therefore, all the G-orbits in $(r^*)^{-1}(\mathrm{Hom}^{gg}(\pi_1(S), G))$ and $\mathrm{Hom}^{gg}(\pi_1(S), G)$ are closed. This means that the categorical quotients $(r^*)^{-1}(\mathrm{Hom}^{gg}(\pi_1(S), G))/\!\!/G$ and $X^{gg}(\pi_1(S), G)$ are geometric quotients [see MFK94, Amp. 1.3]. In particular, π is surjective, and the topology on the quotients is the quotient topology. Hence, the fact that $r^*(X_G(M)) \cap X_G^{gg}(S)$ is closed in $X_G^{gg}(S)$ follows from the commutative diagram above and from 1. ◻

We are now coming to the main result of this section: the non-singular part of the image $r^*(X_G(M)) \cap X_G^{gg}(S)$ is a Lagrangian submanifold of $X_G^{gg}(S)$.

Theorem 3.12. *Let S be a compact connected closed orientable surface. Let M be a compact orientable 3-manifold with boundary $\partial M = S$, and let G be a reductive affine algebraic group. Let B be a symmetric non-degenerate* Ad*-invariant bilinear form on $\mathfrak{g}$. Consider the manifold $X_G^g(S)$ as a symplectic manifold via Goldman's symplectic form ω^B (as defined in Section 3.2).*

1. *Let $\rho \in \mathrm{Hom}(\pi_1(M), G)$ be a representation such that $\rho \circ r$ is a good representation. Then the image $dr^*(T_{[\rho]}\mathcal{X}_G(M))$ is a Lagrangian subspace of $T_{[\rho r]}X_G(S)$.*[14]

2. *Assume there is a G-equivariant, non-empty subset $\mathrm{Hom}^{gg}(\pi_1(S), G)$ of the good representations, which is Zariski-open in $\mathrm{Hom}^g(\pi_1(S), G)$. Let*

   $$Y_G(M) = \left[r^*(X_G(M)) \cap X_G^{gg}(S)\right]^{ns}$$

   be the non-singular part of the image in the good representations.[15] *Then, $Y_G(M)$ is a disjoint union of isotropic submanifolds of $X_G^g(S)$.*

   In particular, every connected component of $Y_G(M)$ with dimension $\frac{1}{2}\dim X_G(S)$ is a Lagrangian submanifold of $X_G(S)$.

3. *Consider a reduced representation $\rho : \pi_1(M) \to G$, such that $[\rho r] \in Y_G(M)$. Then, the connected component of $Y_G(M)$ that contains $[\rho r]$ is a Lagrangian submanifold of $X_G(S)$.*

Be aware that bullet point 2 doesn't directly follow from 1! One of the difficulties that need to be dealt with is the issue that $T_{[\rho r]}Y_G(M)$ could possibly be different from $dr^*T_{[\rho]}\mathcal{X}_G(M)$.

The main work in order prove the theorem above consists of proving the following statement on cohomology. Choose any representation $\rho \in \mathrm{Hom}(\pi_1(M), G)$. The group homomorphism $r : \pi_1(S) \to \pi_1(M)$ induces a homomorphism $r^* : H^1(\pi_1(M), \mathfrak{g}_{\mathrm{Ad}\,\rho}) \to H^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho r})$; this is a general property of group cohomology. This r^* corresponds to the differential $dr^* : T_{[\rho]}\mathcal{X}_G(M) \to T_{[\rho r]}\mathcal{X}_G(S)$. In order to keep the following equations neat and in order to avoid confusion with the boundary operator of cohomology, we will nevertheless denote it by r^*.

Theorem 3.13. *Let $\rho \in \mathrm{Hom}(\pi_1(S), G)$ be any representation. Consider the group homomorphism $r : \pi_1(S) \to \pi_1(M)$ and any non-degenerate,* Ad*-invariant, symmetric bilinear form B on $\mathfrak{g}$. Then, the image $r^*H^1(\pi_1(M), \mathfrak{g}_{\mathrm{Ad}\,\rho})$ is a Lagrangian subspace of the symplectic space $H^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho})$ equipped with the sympletic form ω^B.*

Proof. We follow the proof as it is laid out in [Sik09, Thm. 63]. In order to show that the subspace is Lagrangian, we need to verify two properties: $\dim_{\mathbb{C}} r^*H^1(\pi_1(M), \mathfrak{g}_{\mathrm{Ad}\,\rho}) = \frac{1}{2}\dim_{\mathbb{C}} H^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho})$, and isotropy, i.e. the property that the pullback of the symplectic form is trivial.

While the tangent spaces are naturally defined in terms of group cohomology, this cohomology theory is not sufficiently powerful to answer these questions. Fortunately, it is isomorphic

[14] From Proposition 2.9, it follows that $T_{[\rho r]}X_G(S) \cong T_{[\rho r]}\mathcal{X}_G(S)$.

[15] [Sik09, Thm. 61] uses $\mathrm{Hom}^{gg}(\pi_1(S), G) = \mathrm{Hom}^g(\pi_1(S), G)$. In my opinion, we need the fact that $\mathrm{Hom}^{gg}(\pi_1(S), G)$ is Zariski-open, which Sikora does not prove. As I laid out in the remark under Proposition 1.15, I believe that $\mathrm{Hom}^g(\pi_1(S), G)$ should at least have a subset that is Zariski-open.

to singular cohomology. In the continuation of this proof, we will make use of singular (co)homology groups instead.

Step i: Calculation of dimensions. For any compact orientable $d+1$-dimensional manifold M with boundary S, the Poincaré-Lefschetz duality theorem implies isomorphisms

$$PD : H^k(M, S, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \xrightarrow{\sim} H_{d+1-k}(M, S, \mathfrak{g}_{\mathrm{Ad}\,\rho}), \quad u \mapsto u \cap [M],$$

where $[M]$ is the fundamental class of M. Poincaré-Lefschetz duality with boundary is a direct generalization of Poincaŕe duality without boundary, since singular (co)homology relative to the boundary is equal to normal singular cohomology for manifolds without boundary. That said, we obtain the following commutative diagram

$$\begin{tikzcd}
H_2(M, S, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \arrow[r, "\partial"] \arrow[d, "\cong"', "\phi"] & H_1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho r}) \arrow[r, "r_*"] \arrow[d, "\cong"', "\eta"] & H_1(M, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \arrow[d, "\cong"'] \\
H^1(M, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \arrow[r, "r^*"] & H^1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho r}) \arrow[r, "\delta"] & H^2(M, S, \mathfrak{g}_{\mathrm{Ad}\,\rho})
\end{tikzcd} \tag{24}$$

with exact rows, where the isomorphisms in the columns are given by the inverse PD^{-1} of Poincaré-Lefschetz duality. The two rows are a parts of the long exact (co)homology sequences. In accordance with Corollary 3.5, we have $\mathbb{C}$-linear isomorphisms between the cohomology groups and the $\mathbb{C}$-duals of the homology groups:

$$\psi_S : H_1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \xrightarrow{\sim} H^1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho})^\vee, \quad x \mapsto \tilde{\omega}^B(\cdot, x),$$
$$\psi_M : H_1(M, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \xrightarrow{\sim} H^1(M, \mathfrak{g}_{\mathrm{Ad}\,\rho})^\vee, \quad x \mapsto \tilde{\omega}^B(\cdot, x).$$

Under these isomorphisms, $\psi_M r_* \psi_S^{-1}$ is the dual of r^* according to Lemma 3.6, i.e. $(r^*)^\vee = \psi_M r_* \psi_S^{-1}$. Hence Eq. (24) becomes

$$\begin{tikzcd}
H_2(M, S, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \arrow[r, "\tilde{\partial}"] \arrow[d, "\cong"', "\phi"] & H^1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho r})^\vee \arrow[r, "(r^*)^\vee"] \arrow[d, "\cong"', "\tilde{\eta}"] & H^1(M, \mathfrak{g}_{\mathrm{Ad}\,\rho})^\vee \arrow[d, "\cong"'] \\
H^1(M, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \arrow[r, "r^*"] & H^1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho r}) \arrow[r, "\delta"] & H^2(M, S, \mathfrak{g}_{\mathrm{Ad}\,\rho}),
\end{tikzcd} \tag{25}$$

which is a commutative diagram with exact rows, where $\tilde{\partial} = \psi_S \partial$ and $\tilde{\eta} = \eta \psi_S^{-1}$. Note that the rank of a linear map equals the rank of its dual.[16] It follows that

$$\begin{aligned}
\dim_{\mathbb{R}} r^* H^1(M, \mathfrak{g}_{\mathrm{Ad}\,\rho}) &= \operatorname{rank}_{\mathbb{R}} r^* \\
&= \operatorname{rank}_{\mathbb{R}} (r^*)^\vee && \text{by duality} \\
&= \operatorname{rank}_{\mathbb{R}} \delta && \text{by commutativity} \\
&= \dim_{\mathbb{R}} H^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho r}) - \dim_{\mathbb{R}} \ker \delta \\
&= \dim_{\mathbb{R}} H^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho r}) - \operatorname{rank}_{\mathbb{R}} r^* && \text{by exactness} \\
&= \dim_{\mathbb{R}} H^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho r}) - \dim_{\mathbb{R}} r^* H^1(M, \mathfrak{g}_{\mathrm{Ad}\,\rho}),
\end{aligned}$$

[16] Let $f : V \to W$ be a linear map of two finite-dimensional vector spaces over an arbitrary field. For any matrix representation of f, the column rank equals the row rank, so $\operatorname{rank} f = \operatorname{rank} f^*$.

in other words, $\dim_{\mathbb{R}} r^*H^1(\pi_1(M), \mathfrak{g}_{\mathrm{Ad}\,\rho}) = \frac{1}{2}\dim_{\mathbb{R}} H^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho r})$, which we wanted to show.

Step ii: Isotropy. We would like to show that $r^*H^1(\pi_1(M), \mathfrak{g}_{\mathrm{Ad}\,\rho})$ is an isotropic subspace of $H^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho r})$ with respect to ω^B. According to Corollary 3.5, we have

$$\begin{aligned}\tilde{\eta}^{-1} : H^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho r}) \to H^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho r})^\vee, \quad \alpha \mapsto \tilde{\eta}^{-1}(u) &= \psi_S \circ \eta^{-1}(u) \\ &= \tilde{\omega}^B(\cdot, PD(u)) \\ &= \omega^B_{\rho r}(\cdot, u).\end{aligned}$$

Accordingly, we have an isomorphism $H_1(\pi_1(M), \mathfrak{g}_{\mathrm{Ad}\,\rho}) \cong H^1(\pi_1(M), \mathfrak{g}_{\mathrm{Ad}\,\rho r})^\vee$, and we obtain

$$r_* \circ \tilde{\eta}^{-1}(\alpha) = (r^*)^\vee(\omega^B_{\rho r}(\cdot, \alpha)) = \omega^B_{\rho r}(r^*(\cdot), \alpha),$$

as $(r^*)^\vee$ and r^* are dual to each other. Now, assume that $u = r^*v$ is the image of some $v \in H^1(\pi_1(M), \mathfrak{g}_{\mathrm{Ad}\,\rho r})$. We have

$$(r^*)^\vee \circ \tilde{\eta}^{-1}(u) = (r^*)^\vee \circ \tilde{\eta}^{-1} \circ r^*(v) = (r^*)^\vee \circ \tilde{\partial} \circ \phi^{-1}(v) = 0$$

on account of commutativity and exactness of the diagram in Eq. (24). Let $v' \in H^1(\pi_1(M), \mathfrak{g}_{\mathrm{Ad}\,\rho r})$ be another element. Combining the last two equations, we obtain

$$\omega^B_{\rho r}(r^*v', r^*v) = [(r^*)^\vee \circ \tilde{\eta}^{-1}(u)](v') = 0,$$

which means that $r^*H^1(\pi_1(M), \mathfrak{g}_{\mathrm{Ad}\,\rho})$ is an isotropic subspace of $H^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho r})$. □

Proof of Theorem 3.12. We follow [Sik09, Thm. 61]. 1. According to Lemma 2.11, the subspace

$$dr^*T_{[\rho]}\mathcal{X}_G(M) \subseteq T_{[\rho r]}\mathcal{X}_G(S) = T_{[\rho r]}X_G(S)$$

corresponds to $r^*H^1(\pi_1(M), \mathfrak{g}_{\mathrm{Ad}\,\rho}) \subseteq H^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho r})$. However, this is a Lagrangian subspace according to Theorem 3.13, which proves this statement.

2. The proof of statement 2 will require several substeps: First, we will see how $Y_G(M)$ becomes a disjoint union of complex manifolds. In the complex topology, it is then sufficient to show that

$$T_{[\rho r]}Y_G(M) \subseteq T_{[\rho r]}X^g_G(S)$$

is an isotropic subspace on a dense subset of points $[\rho r] \in Y_G(M)$. Indeed, for $\iota : Y_G(M) \hookrightarrow X^g_G(S)$, the restricted form $\iota^*\omega^B : Y_G(M) \to \bigwedge^2 T^*X^g_G(S)$ is a continuous map into a Hausdorff space, so it must be constant whenever it is constant on a dense subset. Therefore, it suffices to find an appropriate dense subset of $Y_G(M)$.

We will construct such a dense subset in two further steps: We will see that $r^* : X_G(M) \to X_G(S)$ can be restricted to a holomorphic map of manifolds $r^* : X'_G(M) \to Y_G(M) \subseteq X^g_G(S)$ with dense image in $Y_G(M)$. Then, we will see that the differential dr^* is surjective on a dense subset of $X'_G(M)$. This will be sufficient to show that $Y_G(M)$ is isotropic.

Step 2.i: Manifold structure of $Y_G(M)$. As we saw in Proposition 3.11, $r^*X_G(M) \cap X^{gg}_G(M)$ is closed in $X^{gg}_G(S)$, so it naturally becomes an affine algebraic subvariety. As it bears a variety structure, the notion of its non-singular points is well-defined. Using the GAGA functor, a non-singular point has a neighborhood that is a complex manifold, see Proposition A.12, so that $Y_G(M) = [r^* X_G(M) \cap X_G^{gg}(M)]^{ns}$ is a disjoint union of complex manifolds.

STEP 2.II: $r*$ CAN BE RESTRICTED TO $X'_G(M) \to Y_G(M)$ WITH DENSE IMAGE. Consider $(r^*)^{-1}(Y_G(M))$. It is the inverse image of a Zariski-open set, so it is Zariski-open in $X_G(M)$. In particular, it is a quasi-affine set, so that – again – it is appropriate to speak of its non-singular points. In accordance with Proposition A.10, we have a Zariski-open dense subset $X'_G(M) \subseteq (r^*)^{-1}(Y_G(M))$, which consists only of non-singular points. It follows that $r^* X'_G(M)$ is dense in $Y_G(M)$[17]. Thus, the regular map $r^* : X'_G(M) \to Y_G(M)$ has Zariski-dense image. As a consequence of a lemma of Chevalley, the image is also dense in the complex topology [Ser56, Lem. 2., Prop. 7]. Using [Ser56, Lem. 2] once again, we even see that the image contains an Zariski-open and dense subset, so that this subset is also open and dense in the complex topology. Hence, $r^* : X'_G(M) \to Y_G(M)$ is both a regular map with Zariski-dense image, and a holomorphic map between complex manifolds with dense image.

STEP 2.III: SURJECTIVE DIFFERENTIAL ON A DENSE SUBSET. In particular, r^* is a smooth map between two finite-dimensional manifolds, so we can apply Sard's Theorem, [Sar65, Thm. 3.1]. What does this mean? Denote by $A_d \subseteq Y_G(M)$ the set of critical values. A critical value is the image $r^*(\rho) \in Y_G(M)$ of a critical point $\rho \in X'_G(M)$, i.e. a point with $\operatorname{rank}_{\mathbb{R}} dr^*_\rho \leqslant d-1$. (Maybe we will need to split up $Y_G(M)$ and $X'_G(M)$ into several connected components for this to make sense.) Furthermore denote the d-dimensional Hausdorff measure by μ_d, where $d = \dim_{\mathbb{R}} Y_G(M)$ is the dimension of the manifold. Sard's Theorem states that the set of critical values is a null set, i.e. $\mu_d(A_d) = 0$. In consequence, the complement $Y_G(M) \backslash A_d$ is dense in $Y_G(M)$.[18] The intersection of a dense set with an open dense set is again dense. The image $r^* X'_G(M)$ contains an open dense set, so $r^* X'_G(M) \cap Y_G(M) \backslash A_d$ is dense in $Y_G(M)$. However, this intersection is exactly the set of points where

$$dr^* T_{[\rho]} X'_G(M) = T_{[\rho r]} Y_G(M).$$

Now, $X'_G(M)$ is open in $X_G(M)$, and therefore $T_{[\rho]} X'_G(M) = T_{[\rho]} X_G(M)$. Therefore, we have a dense subset of points $[\rho r] \in Y_G(M)$ with

$$dr^* T_{[\rho]} X_G(M) = T_{[\rho r]} Y_G(M).$$

In order to prove statement 2., it is sufficient to show that $dr^* T_{[\rho]} X_G(M) \subseteq T_{[\rho r]} X_G(S)$ is an isotropic subspace. However, $[\rho r]$ is a good point by definition of $Y_G(M)$, so $dr^* T_{[\rho]} \mathcal{X}_G(M)$ is isotropic by 1. $dr^* T_{[\rho]} X_G(M)$ is a subspace of $dr^* T_{[\rho]} \mathcal{X}_G(M)$, so it is isotropic as well.

3. Let $\rho : \pi_1(M) \to G$ be a reduced representation such that $[\rho r] \in Y_G(M)$. Let $C \subseteq Y_G(M)$ be the connected component containing $[\rho r]$; it is an isotropic submanifold of $X_G^g(S)$

[17]Let $f : X \to Y$ be a continuous map between two topological spaces and let $D \subseteq X$ be a dense subset, i.e. $\bar{D} = X$. Consider a point $y \in f(X)$, and an open neighborhood $y \in U \subseteq Y$. Then, $f^{-1}(U) \subseteq X$ is open and non-empty, so $f^{-1}(U) \cap D \neq \emptyset$. This means that $U \cap f(D)$ is non-empty, so $y \in \overline{f(D)}$. It follows that $\overline{f(D)} = f(X)$, so $f(D)$ is dense in the image.

[18]Let $d < \infty$. Let X be a d-dimensional manifold with Hausdorff measure μ_d and let $A \subseteq X$ be a null set. Assume that $X \backslash A$ were not dense in X. Then, an $a \in A$ with an open neighborhood $U \subseteq A$ would exist. In particular, there would be a chart in which U contains a ball of radius $\varepsilon > 0$, as balls form a basis of the topology. On d-dimensional manifolds, the d-Hausdorff measure coincides with Lebesgue measure [see Hau19, Sec. 7]. Therefore, the ε-ball is μ_d-positive, which implies that U is μ_d-positive. This implies that A is μ_d-positive, which stands in contradiction to A being a null set.

according to 2. We have $dr^*T_{[\rho]}X_G(M) = dr^*T_{[\rho]}\mathcal{X}_G(M)$, as ρ is a reduced point. Using bullet point 1, we know that $dr^*T_{[\rho]}\mathcal{X}_G(M)$ is a Lagrangian subspace, in particular $\dim_{\mathbb{C}} T_{[\rho]}X_G(M) = \frac{1}{2}\dim_{\mathbb{C}} T_{[\rho r]}X_G^g(S)$. C is a smooth manifold, so the dimension of tangent spaces is constant throughout C, i.e. $\dim_{\mathbb{C}} C = \frac{1}{2}\dim_{\mathbb{C}} X_G^g(S)$. Hence, the statement follows. □

3.4. A differential proof of isotropy

The theory of character varieties is particularly delightful as we can endue the identical object with a lot of different "flavors". To demonstrate this abundance of structure, we will show that the isotropy of $r^*H^1(\pi_1(M), \mathfrak{g}_{\mathrm{Ad}\,\rho})$ in $H^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho r})$ from Theorem 3.13 can also be proved with a differential argument using Stoke's theorem instead of the algebraic argument using exact sequences and Poincaré duality. I would like to thank Arnaud Maret from the University of Heidelberg for pointing this out to me during one of my seminar talks.

Let $E \longrightarrow M$ be a flat vector bundle, whose fiber is the $\pi_1(M)$-module $\mathfrak{g}_{\mathrm{Ad}\,\rho}$ and whose holonomy is given by the $\pi_1(M)$-action. Denote the pullback bundle with respect to the inclusion $\iota : S \hookrightarrow M$ by $\iota^*E \longrightarrow S$. Elements of twisted de Rham cohomology $H^1_{\mathrm{dR}}(S, \iota^*E)$ are represented by global sections $\Gamma(M, T^*S \otimes \iota^*E)$, i.e. smooth maps $p \mapsto u_p \otimes x_p$ with $u_p \in T^*S$ and $x_p \in E_{\iota(p)} \cong \mathfrak{g}_{\mathrm{Ad}\,\rho}$. Following de Rham's theorem, we have isomorphisms $H^k_{\mathrm{dR}}(S, E) \cong H^k_{\mathrm{sing}}(S, \mathcal{A})$ for a twisted coefficient system with fiber $\mathcal{A}_p = \mathfrak{g}_{\mathrm{Ad}\,\rho}$. Goldman's symplectic form can be written in twisted de Rham cohomology as follows:

$$\omega^B_{\mathrm{dR},\rho} : H^1_{\mathrm{dR}}(S, \iota^*E) \times H^1_{\mathrm{dR}}(S, \iota^*E) \xrightarrow{\wedge\otimes B} H^2_{\mathrm{dR}}(S, \mathbb{C}) \xrightarrow{\Phi} \mathbb{C},$$

where $\wedge \otimes B$ is locally given by $(u \otimes x, v \otimes y) \mapsto (u \wedge v) \otimes B(x, y)$ and Φ is the "top-level isomorphism" $u \otimes z \mapsto z \cdot \int_S u$. The induced map $r^* : H^1(M, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \to H^1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho r})$ in group cohomology now translates into $\iota^* : H^1_{\mathrm{dR}}(M, E) \to H^1_{\mathrm{dR}}(S, \iota^*E), u \mapsto \iota^*u$, where $\iota : \partial M \hookrightarrow M$ is the embedding and ι^*u is the pullback along ι. Equivalently to Theorem 3.13, Step II, we have:

Theorem 3.14. *$\iota^*H^1_{\mathrm{dR}}(M, E)$ is an isotropic subspace of $(H^1_{\mathrm{dR}}(S, \iota^*E), \omega^B_{\mathrm{dR},\rho})$.*

Proof. Let $u \otimes x$ and $v \otimes y$ locally represent two elements of $H^1_{\mathrm{dR}}(M, E)$. As u and v are cocycles, they must be closed, i.e. $du = dv = 0$. Slightly abusing our notation above, we would like to show that $\omega^B_{\mathrm{dR},\rho}(\iota^*u \otimes x, \iota^*v \otimes y) = 0$. We have

$$\begin{aligned}
\omega^B_{\mathrm{dR},\rho}(\iota^*u \otimes x, \iota^*v \otimes y) &= \Phi((u \wedge v) \otimes B(x, y)) \\
&= B(x, y) \cdot \int_S \iota^*u \wedge \iota^*v \\
&= B(x, y) \cdot \int_{\partial M} \iota^*(u \wedge v) \\
&= B(x, y) \cdot \int_M d(u \wedge v) && \text{according to Stoke's theorem,} \\
&= B(x, y) \cdot \int_M du \wedge v - u \wedge dv \\
&= 0 && \text{as } u \text{ and } v \text{ are closed.}
\end{aligned}$$

This shows that the subspace $\iota^*H^1_{\mathrm{dR}}(M, E)$ is indeed isotropic. □

4. Anti-holomorphic involutions on a Riemann surface

In the entire section, let S be a compact connected Riemann surface, and consider an anti-holomorphic involution $f : S \to S$, i.e. $z \mapsto f(\bar{z})$ is holomorphic and $f^2 = \mathrm{id}$.

4.1. Properties of anti-holomorphic involutions

Anti-holomorphic involutions have been of interest since as early as 1882, when Felix Klein studied them in the context of Klein surfaces [Kle82]. They naturally appear when studying real algebraic curves. Below, we will recall some of their basic properties that will be needed later on.

Definition 4.1. Let $U \subseteq \mathbb{C}^n$ and $V \subseteq \mathbb{C}^m$ be two open subsets. A map $f : U \to V$ is called *anti-holomorphic*, if $z \mapsto f(\bar{z})$ is a holomorphic map. Here, $\bar{z}$ is the coordinate-wise conjugate of z. A map $f : M \to N$ between two complex manifolds is called *anti-holomorphic* if it is anti-holomorphic in local coordinates.

The following result is specific for the case of complex dimension 1.

Proposition 4.2. *Let S be a Riemann surface. An anti-holomorphic involution $f : S \to S$ is orientation-reversing.*

Proof. Let $U \subseteq \mathbb{C}$ be an open subset. It is a well-known result from complex analysis that a a smooth function $U \to \mathbb{C}$ is orientation-preserving and angle-preserving (i.e. locally conformal) if and only if it is holomorphic with nowhere vanishing differential [BF09, Thm. I.5.15].

Consider the map $h : U \to \mathbb{C}, z \mapsto f(\bar{z})$. By definition, it is holomorphic, and it has an inverse map given by $z \mapsto \overline{f(z)}$. As the complex conjugation is a diffeomorphism, it follows that h is a diffeomorphism as well. According to the result quoted in the previous paragraph, this means that h is orientation-preserving. Now $f(z) = h(\bar{z})$. As we will explain in Appendix A.7, the complex conjugation is orientation-reversing, so f must be orientation-reversing as well. □

4.2. Induced automorphisms on the fundamental group

We will now see that an anti-holomorphic involution $f : S \to S$ induces an isomorphism on the fundamental group. As f is continuous and its own inverse, it is a homeomorphism. Therefore, we obtain a group isomorphism

$$f_* : \pi_1(S, z_0) \to \pi_1(S, f(z_0)), \quad \gamma \mapsto f(\gamma)$$

from the fundamental group with base point z_0 to that with base point $f(z_0)$. By selecting a path δ from z_0 to $f(z_0)$, we obtain another isomorphism

$$\phi_\delta : \pi_1(S, f(z_0)) \to \pi_1(S, z_0), \quad \gamma \mapsto \delta.\gamma.\delta^{-1}.$$

The composition $f_{*,\delta} = \phi_\delta \circ f_*$ is a group automorphism of $\pi_1(S, z_0)$ that has the following properties:

Proposition 4.3. *1. If δ' is another path from z_0 to $f(z_0)$, then $f_{*,\delta}$ and $f_{*,\delta'}$ differ by an inner automorphism.*

2. $f_{*,\delta}^2(\gamma) = h.\gamma.h^{-1}$, where h is the loop $\delta.f(\delta)$ based at z_0.

3. Assume that z_0 is a fixed point of f. Then, we can choose δ to be the constant curve. In this case, $f_{*,\delta}^2 = \mathrm{id}$, so $f_{*,\delta}$ is an involution on $\pi_1(S, z_0)$.

Proof. We will prove 1. Bullet points 2 and 3 can be shown by similarly straightforward calculations. Let $g = \gamma.(\gamma')^{-1}$. This is a loop based at z_0. It follows that

$$f_{*,\delta}(u) = \delta.f(u).\delta^{-1} = g.\delta'.f(u).\delta'^{-1}.g^{-1} = g.f_{*,\delta'}(u).g^{-1},$$

which was to be shown. □

1. and 2. show that $f_{*,\delta}$ is an involution that is independent of δ "modulo conjugations". Therefore, it is natural to consider the induced map on the character variety. We will do this in the following Section 4.3.

4.3. Induced maps on the character variety

Consider the representation variety $\mathrm{Hom}(\pi_1(S), G)$. The automorphism $f_{*,\delta}$ from Section 4.2 above induces a G-equivariant map

$$\mathrm{Hom}(\pi_1(S), G) \to \mathrm{Hom}(\pi_1(S), G), \quad \rho \mapsto \rho \circ f_{*,\delta}. \tag{26}$$

It is obviously G-equivariant, so it induces a map

$$\hat{f} : X_G(S) \to X_G(S)$$

on the character variety.

Proposition 4.4. *1. The map $\hat{f} : X_G(S) \to X_G(S)$ preserves the equivalence classes of good representations.*

2. The restricted map $\hat{f} : X_G^g(S) \to X_G^g(S)$ is independent of the choice of δ.

3. The restricted map $\hat{f} : X_G^g(S) \to X_G^g(S)$ is an involution, i.e. $\hat{f}^2 = \mathrm{id}$.

4. The restricted map $\hat{f} : X_G^g(S) \to X_G^g(S)$ is holomorphic and anti-symplectic, i.e. $\hat{f}^*\omega^B = -\omega^B$ for the symplectic form ω^B on $X_G^g(S)$.

Proof. 1. For the stabilizer of an arbitrary representation ρ and any automorphism ϕ of $\pi_1(S)$, the following stands true:

$$\begin{aligned} G_\rho &= \{g \in G, \quad g\rho(x)g^{-1} = \rho(x) \text{ for } x \in \pi_1(S)\} \\ &= \{g \in G, \quad g\rho(\phi(x))g^{-1} = \rho(\phi(x)) \text{ for } x \in \pi_1(S)\} = G_{\rho\circ\phi}. \end{aligned}$$

In particular, $G_\rho = G_{\rho\circ f_{*,\delta}} = G_{\hat{f}(\rho)}$, which shows that good representations are mapped to good representations.

2. The good character variety $X_G^g(\pi_1(S))$ is the "normal" orbit space $\mathrm{Hom}^g(\pi_1(S), G)/G$, so the independence of δ follows from Proposition 4.3.1.

3. The fact that $\hat{f}$ is an involution on the good character variety follows from Proposition 4.3.2, as two conjugated elements are equal in $X_G^g(S)$.

4. In accordance with Lemma 1.3, $\mathrm{Hom}(\pi_1(S), G) \to \mathrm{Hom}(\pi_1(S), G), \rho \mapsto \rho \circ f_{*,\delta}$ is a regular G-equivariant map, so the induced map $\hat{f}$ on the character varieties is regular as well. Hence, it is holomorphic on non-singular points. It remains to be shown that the restriction to the good points is anti-symplectic. Let u and v be two elements of $H^1_{\mathrm{sing}}(S, \mathfrak{g}_{\mathrm{Ad}\,\rho})$ for some ρ. Let f^* and f_* be the maps induced by f on singular cohomology and homology, respectively. Then, we have

$$f^*\omega^B_\rho(u, v) = \omega^B_\rho(f_*u, f_*v) = \int_S B_*(f_*u \cup f_*v) = \int_S f_*B_*(u \cup v) = \int_S -B_*(u \cup v) = -\omega^B_\rho(u, v).$$

The equation $f_*B_*(u \cup v) = -B_*(u \cup v)$ holds because f is orientation-reversing, see Proposition 4.2, and $B_*(u \cup v) \in H^2_{\mathrm{sing}}(M, \mathbb{C}) \cong \mathbb{C}$ is in the "top-level" of cohomology, see Appendix A.7. It follows that $\hat{f}^*\omega^B_\rho = f^*\omega^B_\rho = -\omega^B_\rho$. ◻

Let $\mathcal{L}_G$ be the fixed point set of $\hat{f} : X_G^g(S) \to X_G^g(S)$. We have:

Theorem 4.5. *If it is non-empty, $\mathcal{L}_G$ is a smooth Lagrangian submanifold of $X_G^g(S)$.*

In [BS14], this fixed point set is called the (A, B, A)-brane.

Proof. Assume that $\mathcal{L}_G$ is a manifold. As it is a set of fixed points, we have $\hat{f}^*\omega^B = \omega^B$ on $\mathcal{L}_G$. On the other hand, we have $\hat{f}^*\omega^B = -\omega^B$ according to Proposition 4.4.3. This proves that $\omega^B \equiv 0$ on $\mathcal{L}_G$; thus, it is an isotropic subspace.

It remains to be shown that $\mathcal{L}_G$ is indeed a smooth complex submanifold and that its dimension is

$$\dim_{\mathbb{R}} \mathcal{L}_G = \frac{1}{2} \dim_{\mathbb{R}} X_G^g(S).$$

This is proven in the following general lemma. ◻

Lemma 4.6. *Let M be a complex manifold of complex dimension n and $f : M \to M$ an anti-holomorphic involution. If the fixed point set of f is non-empty, it is a smooth analytic submanifold of real dimension n.*

Proof. Consider a fixed point p of f. A classical argument using a partition of unity and the Euklidean metric on each chart shows that a Riemannian metric exists on any manifold [see Lee13, Prop. 13.3]. Therefore, we can consider a metric g on M. Since M is a complex manifold, its charts are analytic. The Euklidean metric is analytic as well; thus, in a sufficiently small neighborhood of p, g will be analytic. Without loss of generality, we can assume that g is f-invariant by choosing $g + f^*g$ instead of g. This new metric is still analytic near p, and f is an isometry with respect to g.

Consider the Riemannian exponential map $\exp : T_pM \to M$ with respect to g. It is a diffeomorphism on a small neighborhood of p. As f is anti-holomorphic, $df_p : T_pM \to T_pM$ is an anti-linear involution.[19] Because f is an isometry, we have $\exp \circ (df_p) = f \circ \exp$ [see Lee97, Prop. 5.9]. This implies that the behavior of f on a small neighborhood of p is entirely determined by df_p. By choosing a complex basis $v_1, \ldots, v_n$ of T_pM, we can consider T_pM a $2n$-dimensional real vector space:

$$\mathbb{R}^{2n} \to T_pM, (x_1, y_1, \ldots, x_n, y_n) \mapsto (x_1 + iy_1)v_1 + \ldots (x_n + iy_n)v_n.$$

Obviously, the maximal fixed subspace under the anti-linear involution df_p is the n-dimensional real vector space defined by $y_k = 0$ for all k. Via the exponential map, we obtain a small neighborhood of p in M in which all fixed points of f compose a n-dimensional real submanifold.

Therefore, the set of fixed points of f is a smooth submanifold of real dimension n. In fact, it is analytic. To show this, we only need to show that exp is analytic. exp maps each tangent vector to a geodesic. Geodesics are solutions of the geodesic equation, which is an analytic partial differential equation in our case, simply because g is analytic. Thus, the equation's solution can be seen to be analytic using the Cauchy-Kowalevski theorem [Fol95, Thm. 1.25]. □

4.4. Comparison of Lagrangian submanifolds

We have found two isotropic, and possibly Lagrangian, submanifolds of $X_G^g(S)$. One of them is $Y_G(M)$, the non-singular good part of $r^*(X_G(M))$, whenever $S = \partial M$ is the boundary of a three-manifold M, see Section 3.3. The other is the fixed point set $\mathcal{L}_G$ that was introduced in Section 4.3 above. In this section, we can see that, in the case of a special three-manifold M, these two submanifolds can be related.

We will begin by constructing the manifold M. Consider a compact connected closed Riemann surface S, and an anti-holomorphic involution $f : S \to S$. We define the three-dimensional product manifold $\Sigma = S \times [-1, 1]$. On Σ, we have a smooth involution

$$\sigma : \Sigma \to \Sigma, \quad (z, t) \mapsto (f(z), -t).$$

Via σ, the discrete group $\mathbb{Z}/2\,\mathbb{Z}$ acts on Σ. We denote the orbit space by $M = \Sigma/\sigma$.

We will use the following notations: Denote the natural projection by $\pi : \Sigma \to M$, and denote the image $\pi(z, t)$ by $[z, t]$. We have inclusions

$$i_0 : S \to \Sigma, z \mapsto (z, 0) \text{ and } i_1 : S \to \Sigma, z \mapsto (z, 1)$$

[19] Let $U \subseteq \mathbb{C}^n$ and $V \subseteq \mathbb{C}^m$ be open subsets. A smooth function $f : U \to V$ is holomorphic if and only if its differential is complex linear on its tangent spaces. In dimension $n = m = 1$, this is an immediate consequence of the Cauchy-Riemann differential equations: Indeed, the Cauchy-Riemann equations state that for $z_0 = x_0 + iy_0$, we have $(df)_{z_0} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for some $a, b \in \mathbb{R}$. However, the multiplication of this matrix with any vector $\begin{pmatrix} u \\ v \end{pmatrix}$ is the same as the complex multiplication $(a+ib)(u+iv)$, so that it is complex linear. The result can be generalized to several dimensions using Osgood's lemma [Osg99], so that smooth functions with complex linear differential are holomorphic. Now, the differential of the complex conjugation is $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, which is clearly anti-linear. Any anti-holomorphic map is the composition of complex conjugation with a holomorphic map, so it must have an anti-linear differential as well.

and their respective projections

$$j_0 = \pi \circ i_0 : S \longrightarrow M, z \mapsto [z, 0] \text{ and } j_1 = \pi \circ i_1 : S \longrightarrow M, z \mapsto [z, 1].$$

There is a classification theory of anti-holomorphic involutions on surfaces. It indicates that the fixed point set $F \subset S$ of the involution f is a disjoint union of circles [see BS14, Sec. 2.1]. We identify F with $F \times \{0\} \subseteq \Sigma$ and $F \times \{0\} \subseteq M$.

σ does not have any fixed points on $\Sigma \backslash F$, so the group action of $\mathbb{Z}/2\,\mathbb{Z}$ is free and proper. In particular, $M \backslash F = (\Sigma \backslash F)/\sigma$ bears a canonical quotient manifold structure. It is a three-manifold with boundary $\partial(M \backslash F) \cong S$ (the image of $S \times \{1\}$ or $S \times \{-1\}$).

Proposition 4.7. *The orbit space M carries the structure of a three-manifold with boundary $\partial M \cong S$. The natural projection $\pi : \Sigma \longrightarrow M$ is smooth with respect to this structure. The smooth structure is compatible with the canonical manifold structure on $M \backslash F$. The identification $\partial M \cong S$ is given by $j_1 : S \longrightarrow M, z \mapsto [z, 1]$.*

I would like to thank David Baraglia from the University of Adelaide for explaining the proof of this proposition in a private e-mail conversation.

The construction of the manifold structure involves the choice of a σ-invariant Riemannian metric g on Σ. Such a metric exists: Every manifold possesses a Riemannian metric g by the partition-of-unity argument [see Lee13, Prop. 13.3]. The metric $g + \sigma^* g$ is obviously σ-invariant as σ is an involution. It is not clear whether, or not, the structure of M is dependent on the choice of g.

Before we can turn to the proof of the proposition, we will need to prove the following lemma.

Lemma 4.8. *Consider one of the circles $S^1 \subset F$. We can find a neighborhood of S^1 in Σ diffeomorphic to $S^1 \times B^2_\varepsilon(0)$, on which σ is given by*

$$\sigma(x, y, t) = (x, -y, -t)$$

for $x \in S^1$ and $(y, t) \in B^2_\varepsilon(0) \subset \mathbb{R}^2$.

If we imagine the anti-holomorphic involution $f : S \longrightarrow S$ as a complex conjugation, then x is the "real part" that remains fixed and y is the "imaginary part" that gets flipped.

Proof of Lemma 4.8. We will prove this lemma in several steps. For $x \in S^1$, every tangent space splits into $T_x\Sigma = T_x S^1 \oplus N_x S^1$, where $N_x S^1$ is the orthogonal space to $T_x S^1$ with respect to g_x. The vector bundle $NS^1 \longrightarrow S^1$ with fibers $N_x S^1$ is called the orthogonal bundle. As S^1 is one-dimensional, NS^1 is a vector bundle of rank 2. As Σ is compact, the Riemannian exponential map $\exp : T_{(z,t)}\Sigma \longrightarrow \Sigma$ is defined on the entire tangent space. We want to see that the map

$$\exp : NS^1 \longrightarrow \Sigma, \quad (x, v) \mapsto \exp_x(v)$$

for $x \in S^1$ and $v \in N_x S^1$ can be restricted such that it is injective on a tubular neighborhood. $\Sigma = S \times [-1, 1]$ is orientable as a product of orientable manifolds. $S^1 \subset \Sigma$ is an orientable submanifold. In particular, the normal bundle NS^1 is orientable as well. Choose compatible orientations on Σ, $T\Sigma$, S^1, TS^1 and NS^1.

Step i: The normal bundle NS^1 is trivial. We want to show that $NS^1 \cong S^1 \times \mathbb{R}^2$. This follows from the classification theory of principal bundles [see Hus94, Cor. 8.3], but it can also be shown using elementary methods. In fact, any oriented vector bundle $E \to S^1$ of rank n is trivial. We will recall a brief proof: The oriented vector bundle E is associated to a $\mathrm{GL}_n^+(\mathbb{R})$-principal bundle, the oriented frame bundle,

$$E = P_{\mathrm{GL}^+}(E) \times_{\mathrm{GL}_n^+(\mathbb{R})} E,$$

where $\mathrm{GL}_n^+(\mathbb{R})$ is the general linear group with positive determinant. Hence, it is sufficient to prove that any $\mathrm{GL}_n^+(\mathbb{R})$-principal bundle $P \to S^1$ is trivial.

S^1 can be covered by two intervals $I_1, I_2 \subseteq \mathbb{R}$. Their intersection $I_1 \cap I_2 \subset S^1$ is a disjoint union $J_1 \sqcup J_2$ of two connected open sets. Since intervals are contractible, every principal bundle on I_1 and I_2 is trivial. A principal bundle $P \to S^1$ is now constructed by patching the principal bundles $I_1 \times \mathrm{GL}_n^+(\mathbb{R}) \to I_1$ and $I_2 \times \mathrm{GL}_n^+(\mathbb{R}) \to I_2$ in a smooth manner that is compatible with the $\mathrm{GL}_n^+(\mathbb{R})$-action [Bau14, Satz 2.5]. Since the bundles are trivial on J_1 and J_2, patching results to the choice of two elements A and $B \in \mathrm{GL}_n^+(\mathbb{R})$. The structure of $P = P_{A,B} \to S^1$ only depends on the choice of these two elements. $\mathrm{GL}_n^+(\mathbb{R})$ is path-connected, so there is a smooth path connecting each of A and B with $\mathrm{id} \in \mathrm{GL}_n^+(\mathbb{R})$. Therefore, $P_{A,B} \cong P_{\mathrm{id},\mathrm{id}}$. However, $P_{\mathrm{id},\mathrm{id}}$ is the trivial bundle, which proves that every $\mathrm{GL}_n^+(\mathbb{R})$-principal bundle over S^1 is trivial. Therefore, every orientable vector bundle over S^1 is trivial, and it follows that $NS^1 \cong S^1 \times \mathbb{R}^2$.

In particular, the map $S^1 \times \mathbb{R}^2 \to NS^1$, $(x, y, t) \mapsto (ye_1 + te_2)$ is a well-defined diffeomorphism, where (e_1, e_2) is an orthonormal basis of $N_x S^1$ that varies smoothly in x. Therefore, the map

$$\exp : S^1 \times B_\varepsilon^2(0) \to \Sigma, \quad (x, y, t) \mapsto \exp(x, y, t) = \exp_x(ye_1 + te_2)$$

is well-defined for any $\varepsilon > 0$, where $\exp_x$ is the Riemannian exponential function on $T_x\Sigma \to \Sigma$.

Step ii: exp is a local diffeomorphism around S^1. For a fixed x, we know that the differential of $\exp_x : T_x\Sigma \to \Sigma$ is given by $(d\exp_x)_0 = \mathrm{id} : T_x\Sigma \to T_x\Sigma$. By the inverse function theorem, $\exp_x$ is a diffeomorphism in a neighborhood of x. However, we need to see that $\exp(x, y, t)$ is still a local diffeomorphism when x varies. We will identify S^1 with $S^1 \times \{0\} \subset S^1 \times B_\varepsilon^2(0)$. We would like to show that the differential $d\exp_{(x,0,0)}$ is invertible for every $x \in S^1$. The differential is a $\mathbb{R}$-linear function $d\exp_{(x,0,0)} : T_x S^1 \times T_0\,\mathbb{R}^2 \to T_x\Sigma = T_x S^1 \oplus N_x S^1$. Choosing a basis vector e_0 for $T_x S^1$, the standard basis for $T_0\,\mathbb{R}^2$ and e_1, e_2 for $N_x S^1$ as before, we obtain a matrix representation

$$d\exp_{(x,0,0)} = \left(\partial_x \exp, \partial_y \exp, \partial_t \exp\right)_{(x,0,0)} \in \mathbb{R}^{3\times 3}.$$

In order to calculate $\partial_x \exp$, we can leave (y, t) fixed, so we obtain

$$\partial_x \exp|_{(x,0,0)} = \partial_x \exp_x(0) = \partial_x[x \mapsto x \in S^1 \subset \Sigma] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

where we applied that the Riemann exponential function at x maps $0 \in T_x\Sigma$ to x. In order to calculate $\partial_y \exp$ and $\partial_t \exp$, we can leave x fixed. This shows that

$$\partial_y \exp|_{(x,0,0)} = \partial_y \exp_x|_{(0,0,0)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \partial_t \exp|_{(x,0,0)} = \partial_t \exp_x|_{(0,0,0)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

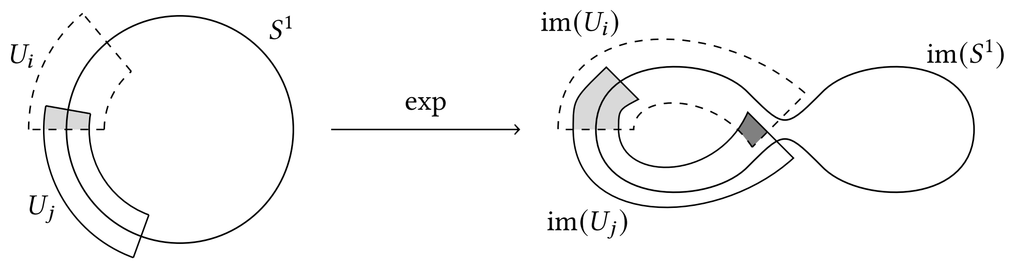

Figure 1: exp only becomes injective after shrinking U_i and U_j.

Taken together, we see that the $d\exp_{(x,0,0)}$ is the identity matrix with respect to the chosen bases, and therefore it is invertible.

Step iii: exp is injective for small ε. We already know that exp is injective when restricted to S^1, as it is merely the embedding of S^1 in Σ. We also know that exp is locally a diffeomorphism around S^1. We will now see that this suffices in order to "extend" injectivity onto a small neighborhood of S^1.

Cover the subset $S^1 \subseteq NS^1$ by small neighborhoods on whom exp is a diffeomorphism. Without loss of generality, we can assume that each neighborhood is of the form $U_i = B^1_{\delta_i}(x_i) \times B^2_{\varepsilon_i}(0) \subseteq S^1 \times \mathbb{R}^2 = NS^1$, i.e. they are shaped like small open tubes. As S^1 is compact, we may assume that there are only finitely many U_i. Now, exp is not necessarily a diffeomorphism on the union of the U_i, as they may have non-trivial intersections in the image, but it will be injective after shrinking the U_i appropriately, as Fig. 1 illustrates.

Take two different $U_i \neq U_j$. We can assume without loss of generality that $\overline{U_i}\backslash U_j \cap \overline{U_j}\backslash U_i = \emptyset$ for $U_i \neq U_j$.[20] If we can shrink ε_i and ε_j such that $\exp(U_i) \cap \exp(U_j) = \exp(U_i \cap U_j)$, we are done. (When exp is injective on every U_i and the images $\exp(U_i)$ only intersect where the U_i intersect, then exp is injective everywhere.) We will assume the opposite: Suppose that for every $\varepsilon_i > 0$ and $\varepsilon_j > 0$, there are points $p(\varepsilon_i, \varepsilon_j) \in U_i \backslash U_j$ and $q(\varepsilon_i, \varepsilon_j) \in U_j \backslash U_i$ such that

$$\exp(p(\varepsilon_i, \varepsilon_j)) = \exp(q(\varepsilon_i, \varepsilon_j)).$$

As $\overline{U_i}\backslash U_j$ and $\overline{U_j}\backslash U_i$ are compact, there must be convergent subseries $\{p_n\} \subseteq \{p(\varepsilon_i, \varepsilon_j)\}$ and $\{q_n\} \subseteq \{q(\varepsilon_i, \varepsilon_j)\}$ with

$$\exp(p_n) = \exp(q_n)$$

and $p_n \to p \in S^1$ as well as $q_n \to q \in S^1$, and $p \neq q$. Continuity of exp implies that $\exp(p) = \exp(q)$, which is a contradiction to exp being injective on S^1. Therefore we can choose radii ε_i and ε_j such that $\exp(U_i)$ and $\exp(U_j)$ only intersect on the intersection. As there are only finitely many U_i, the infimum $\varepsilon = \min \varepsilon_i$ is greater than 0. Therefore, exp must be injective on $S^1 \times B^2_\varepsilon(0)$.

Step iv: σ is given by $(x, y, t) \mapsto (x, -y, -t)$. We saw that exp is an injective local diffeomorphism, i.e. it is an open immersion of $S^1 \times B^2_\varepsilon(0)$ in Σ. It remains to be shown that σ maps (x, y, t) to $(x, -y, -t)$ on this tubular neighborhood.

[20] This is merely a technical requirement, which can be achieved by shrinking δ_i a little bit and possibly adding another small neighborhood at $x_i + \delta_i \in S^1$.

σ leaves $S^1 \subset \Sigma$ invariant. Furthermore, it is an isometry with respect to the metric g. Therefore, we have

$$\sigma \exp(x, y, t) = \sigma \exp_x(ye_1 + te_2) = \exp_x d\sigma_{(x,0,0)}(ye_1 + te_2).$$

Choose the basis $(e_0, e_1, e_2) \subset T_x\Sigma$ as before. $d\sigma$ preserves g, i.e. $d\sigma_{(x,0,0)} \in \mathrm{O}_3(\mathbb{R})$. N_xS^1 and T_xS^1 are preserved by $d\sigma_{(x,0,0)}$, so

$$d\sigma_{(x,0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix},$$

where $A \in \mathrm{O}_2(\mathbb{R})$. σ is orientation-preserving, which implies $\det A = 1$ and thereby $A \in \mathrm{SO}_2(\mathbb{R})$. $\sigma^2 = \mathrm{id}$ implies $A = \pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. σ is given by $(z, t) \mapsto (f(z), -t)$, so $d\sigma_{(x,0,0)}$ must have at least one negative eigenvalue. Therefore, $A = -\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, which shows that

$$\sigma \exp(x, y, t) = \exp_x d\sigma_{(x,0,0)}(ye_1 + te_2) = \exp_x(-ye_1 - te_2) = \exp(x, -y, -t).$$

This is exactly what remained to be shown. □

Proof of Proposition 4.7. Let $F \subset S$ be the fixed point set of f. It is composed of disjoint copies of the circle S^1. Select one of these circles S^1. We identify S^1 with $S^1 \subseteq S$, $S^1 \times \{0\} \subseteq \Sigma$ and $S^1 \times \{0\} \subseteq M$. We equip $M \backslash F$ with the canonical quotient manifold structure of $(\Sigma \backslash F)/\sigma$, and we see that $\pi : \Sigma \backslash F \to M \backslash F$ is smooth according to the quotient manifold theorem. It remains to be shown that we have smooth charts around $S^1 \subset M$, which are compatible with the smooth structure on $M \backslash F$.

According to Lemma 4.8, we have a neighborhood $S^1 \subset V \subset \Sigma$, with $V \cong S^1 \times B^2_\varepsilon(0)$ on which σ looks like $\sigma(x, y, t) = (x, -y, -t)$. Consider its image $\pi(V) \subset M$. We have

$$[x, y, t] = [x, -y, -t] \quad \text{in } M$$

according to the definition of the equivalence relation. Consider $w = y + it$ as a complex number and consider the smooth map

$$V \to \mathbb{R}^3, \quad (x, w) \mapsto (x, w^2) = (x, y^2 - t^2, 2yt).$$

The map $w \mapsto w^2$ is open, as every non-constant holomorphic function is open (open mapping theorem). This implies that $(x, w) \mapsto (x, w^2)$ is open as well. Consider the image $\pi(V) \subseteq M$. Because of $w^2 = (-w)^2$, we have a commutative diagram

$$\begin{tikzcd} V \arrow[rr, "(x,w)\mapsto(x,w^2)"] \arrow[dr, "\pi"'] & & \mathbb{R}^3 \\ & \pi(V). \arrow[ur, "\phi:[x,w]\mapsto(x,w^2)"'] & \end{tikzcd}$$

The map ϕ is injective, since every non-zero complex number has exactly two square roots, which are identified by the equivalence relation. As M is equipped with the quotient topology coming from Σ, ϕ is also continuous and open, so it is a homeomorphism onto its image. Denote the image by $U = \phi(\pi(V))$.

We claim that ϕ is actually a diffeomorphism away from S^1. Indeed, the complex square root $w \mapsto +\sqrt{w}$ is smooth away from the real axis. Therefore, the map

$$\psi : U \to V, (x, w) \mapsto (x, +\sqrt{w})$$

is smooth away from the image of S^1 in U. In particular, the map

$$\pi \circ \psi : U \backslash \phi(S^1) \to \pi(V) \backslash S^1 \subseteq M \backslash F, \quad (x, w) \mapsto [x, \sqrt{w}]$$

is a smooth map of manifolds. The map $\pi \circ \psi$ is inverse to ϕ, so ϕ is a diffeomorphism away from S^1.

The idea is to choose exactly this $\phi : \pi(V) \to U$ to be the chart around $S^1 \subset M$. By construction, the projection $\pi : \Sigma \to M$ is smooth when restricted to $\pi|_V : V \to \pi(V)$. The fact that ϕ is diffeomorphic away from S^1 shows that ϕ is compatible with the manifold structure of $M \backslash F$.

Note that we chose exactly one chart around $S^1 \subset M$. The construction of this chart depends on the choice of a σ-invariant metric, so the manifold structure could possibly depend on the choice of this metric.

The same construction is performed for every circle in the fixed point set $F \subset M$. In particular, M becomes a smooth 3-manifold. □

Now that we have defined M, we will consider the image $Y_G(M) = [r^*(X_G(M)) \cap X_G^g(S)]^{ns}$ in $X_G^g(S)$. We have the following:

Theorem 4.9. *With the 3-manifold M being constructed as in Proposition 4.7, $Y_G(M)$ is contained in $\mathcal{L}_G$.*

Proof. Recall the maps $i_0 : S \to \Sigma, z \mapsto (z, 0)$ and $i_1 : S \to \Sigma, z \mapsto (z, 1)$ as well as their projections $j_0 : S \to M, z \mapsto [z, 0]$ and $j_1 : S \to M, z \mapsto [z, 1]$. We already know that $S \cong \partial M$ is given by j_1. Consider a representation $\rho : \pi_1(M) \to G$ such that $r^*(\rho)$ lies in $Y_G(M)$. We want to show that $r^*(\rho)$ must be a fixed point of the involution $\hat{f}$ on $X_G^g(S)$. Note that the embedding $S = \partial M \hookrightarrow M$ is given by j_1 as explained in Proposition 4.7. This means that

$$r^*(\rho) = \rho \circ j_1^*,$$

where $j_1^* : \pi_1(M) \to \pi_1(S)$ is the induced map on the fundamental groups. Note that the embedding j_1 is homotopic to the map j_0, where the homotopy is simply

$$H : S \times [0, 1] \to M, \quad (z, t) \mapsto [z, t]$$

with $H(z, 0) = j_0(z)$ and $H(z, 1) = j_1$. Thus, the induced maps on the fundamental groups are identical, i.e.

$$j_0^* = j_1^*.$$

Hence, it is sufficient to show that $\hat{f}(\rho \circ j_0^*) = \rho \circ j_0^*$. Using the definition $\hat{f}(\rho) = \rho \circ f_{*,\delta}$, this means that it is sufficient to prove

$$\rho \circ j_0^* \circ f_{*,\delta} \overset{!}{=} \rho \circ j_0^*. \tag{27}$$

Recall the definition of the automorphism $f_{*,\delta} : \pi_1(S, z_0) \longrightarrow \pi_1(S, z_0)$. It is defined by

$$f_{*,\delta}(\gamma) = \delta^{-1}.f(\gamma).\delta$$

for some curve δ that connects z_0 with $f(z_0)$. The definition of $f_{*,\delta}$ depends on the choice of δ, but the induced map on the character variety does not. We have the base point $[z_0, 0] = j_0^*(z_0) \in M$. Let $\tau = j_0^*(\delta) = j_0 \circ \delta$ be the image of the path in M. As $[z_0, 0] = [f(z_0), 0]$, τ is actually a closed path at $[z_0, 0]$ and therefore represents an element of the fundamental group $\pi_1(M, [z_0, 0])$.

Note that for $i_0 : S \longrightarrow \Sigma, z \mapsto (z, 0)$, we have $i_0 \circ f = \sigma \circ i_0$. Because of $j_0 = \pi \circ i_0$, this means that

$$j_0 \circ f = j_0. \tag{28}$$

All of this said, we are ready to calculate $j_0^* \circ f_{*,\delta}$.

$$(j_0^* \circ f_{*,\delta})(\gamma) = j_0^*(\delta^{-1}.f(\gamma).\delta) = j_0^*(\delta^{-1}).(j_0^* \circ f)(\gamma).j_0^*(\delta) = \tau^{-1}.(j_0 \circ f \circ \gamma).\tau \overset{(28)}{=} \tau^{-1}.j_0^*(\gamma).\tau.$$

In particular, we have

$$(\rho \circ j_0^* \circ f_{*,\delta})(\gamma) = \rho(\tau)^{-1}\rho(j_0^*(\gamma))\rho(\tau),$$

in other words

$$\rho \circ j_0^* \circ f_{*,\delta} \equiv \rho \circ j_0^* \mod G,$$

which is precisely Eq. (27), which remained to be shown. □

5. Generalizations of the Lagrangian submanifold theorem

5.1. The Lagrangian submanifold theorem for non-connected surfaces

Let S be a compact closed surface that is not necessarily connected. This means that $S = S_1 \sqcup \ldots \sqcup S_l$ is a disjoint union of compact connected closed surfaces. We can define $\pi_1(S) = \pi_1(S_1) \times \ldots \times \pi_1(S_l)$. Then, we have a canonical isomorphism

$$\mathrm{Hom}(\pi_1(S), G) \cong \mathrm{Hom}(\pi_1(S_1), G) \times \ldots \times \mathrm{Hom}(\pi_1(S_l), G),$$

which is given by $\rho \mapsto (\rho|_{\pi_1(S_1)}, \ldots, \rho|_{\pi_1(S_l)})$. Accordingly, we can see that

$$\mathcal{H}om(\pi_1(S), G) \cong \mathcal{H}om(\pi_1(S_1), G) \times_{\mathbb{C}} \ldots \times_{\mathbb{C}} \mathcal{H}om(\pi_1(S_l), G)$$

as well as

$$X_G(S) \cong X_G(S_1) \times_{\mathbb{C}} \ldots \times_{\mathbb{C}} X_G(S_l).$$

This shows that Theorem 3.12.1 holds word-for-word for the non-connected case as well.

The proofs of Theorem 3.12.2 and 3.12.3 are not as easy to generalize, as we need to check under what circumstances good representations in $X_G(S)$ can be restricted to good representations in the $X_G(S_i)$: As the Zariski topology on the project is finer than the project topology, there may be closed orbits in $\mathrm{Hom}(\pi_1(S), G)$ which are not the product of closed orbits in the $\mathrm{Hom}(\pi_1(S_i), G)$.

5.2. The Goldman symplectic form on compact Kähler manifolds

In Section 3.2, we constructed a symplectic form on the character variety of a compact surface S, defined by

$$\omega_\rho^B : H^1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \times H^1(S, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \xrightarrow{\cup} H^2(S, \mathfrak{g}_{\mathrm{Ad}\,\rho} \otimes \mathfrak{g}_{\mathrm{Ad}\,\rho}) \xrightarrow{B} H^2(S, \mathbb{C}) \cong \mathbb{C}\,.$$

Let us instead consider any compact connected Kähler manifold K of finite dimension $\dim_{\mathbb{R}} K = d = 2n$. Yael Karshon [Kar92, Thm. 5] developed a way of generalizing Goldman's symplectic form to Kähler manifolds. Let's start naively in the same way, for a representation $\rho : \pi_1(K) \to G$. We will formulate the definition in de Rham cohomology for a reason which will become clear later. Let $E \to K$ be a flat vector bundle with fiber $\mathfrak{g}_{\mathrm{Ad}\,\rho}$. We define the following form:

$$H^1_{\mathrm{dR}}(K, E) \times H^1_{\mathrm{dR}}(K, E) \xrightarrow{\wedge} H^2_{\mathrm{dR}}(K, E \otimes E) \xrightarrow{B} H^2_{\mathrm{dR}}(K, \mathbb{C}).$$

However, $d = \dim_{\mathbb{R}} K$ can be greater than 2, so $H^2_{\mathrm{dR}}(K, \mathbb{C})$ is not necessarily isomorphic to $\mathbb{C}$. Fortunately, the Hard Lefschetz theorem [GH78, Sec. 0.7] provides us with the necessary bridge. There are isomorphisms

$$H^{n-k}_{\mathrm{dR}}(K, E) \xrightarrow{\sim} H^{n+k}_{\mathrm{dR}}(K, E) \tag{29}$$

induced by $[u] \mapsto [u \wedge \Omega^k]$, where Ω is the symplectic form on K that comes from the Kähler structure. We obtain a commutative diagram

$$\begin{tikzcd} H^1_{\mathrm{dR}}(K,E) \times H^1_{\mathrm{dR}}(K,E) \arrow[r, "\wedge"] \arrow[d, "\cong"', "\mathrm{id}\times(\,\cdot\,\wedge[\Omega]^{n-1})"] & H^2_{\mathrm{dR}}(K,E\otimes E) \arrow[r, "B"] \arrow[d, "\cong"', "(\,\cdot\,\wedge[\Omega]^{n-1})"] & H^2_{\mathrm{dR}}(K,\mathbb{C}) \arrow[d, "\cong"', "(\,\cdot\,\wedge[\Omega]^{n-1})"] \\ H^1_{\mathrm{dR}}(K,E) \times H^{d-1}_{\mathrm{dR}}(K,E) \arrow[r, "\wedge"] & H^d_{\mathrm{dR}}(K,E\otimes E) \arrow[r, "B"] & H^d_{\mathrm{dR}}(K,\mathbb{C}). \end{tikzcd}$$

In the lower right corner, we have $H^d_{\mathrm{dR}}(K,\mathbb{C}) \cong \mathbb{C}$. Therefore, the diagram defines a bilinear form $\omega^B_{\mathrm{dR},\rho}$. In order to show that this is a form on the character variety, we need to show that the diagram still commutes when passing over to group cohomology. This is non-trivial, as K is generally no longer aspherical (i.e. $\pi_2(K)$ could possibly not be equal to 0), but this problem can be circumvented by passing to a bigger topological space $\tilde{K}$. Karshon [Kar92] shows that this process defines a non-degenerate closed form on the character variety $X_G(K)$ (possibly after making some restrictions on the topology and points of $X_G(K)$). This symplectic form is in fact compatible with the cup product of group cohomology: This means that we have a commutative diagram

$$\begin{tikzcd} H^1_{\mathrm{Grp}}(K,E) \times H^1_{\mathrm{Grp}}(K,E) \arrow[r, "\cup"] \arrow[d, "\cong"'] & H^2_{\mathrm{Grp}}(K,E\otimes E) \arrow[r, "B"] \arrow[d] & H^2_{\mathrm{Grp}}(K,\mathbb{C}) \arrow[d] \\ H^1_{\mathrm{dR}}(K,E) \times H^1_{\mathrm{dR}}(K,E) \arrow[r, "\wedge"] & H^2_{\mathrm{dR}}(K,E\otimes E) \arrow[r, "B"] & H^2_{\mathrm{dR}}(K,\mathbb{C}). \end{tikzcd}$$

5.3. The Lagrangian submanifold theorem on compact Kähler manifolds

Consider a compact connected $(d+1)$-dimensional manifold M with boundary $\partial M = K$, where K is a compact connected Kähler manifold. As in the surface case, we have a morphism $r^* : \mathcal{X}_G(M) \to \mathcal{X}_G(K)$ of the character varieties.

Let Ω be the symplectic form defined on K that is defined by the Kähler structure. First, we would like to prove isotropy, i.e. the generalization of Theorem 3.14. As we will see, we will need to impose a new condition on M with respect to Ω.

Theorem 5.1. *Suppose that M is a compact manifold with compact connected Kähler boundary K. Denote the identification $\partial M = K$ by $\iota : K \hookrightarrow M$. Furthermore, assume that the symplectic form Ω on K can be extended to a closed differential form $\tilde{\Omega} \in \Omega^2(M)$ with $\Omega = \iota^*\tilde{\Omega}$. Then, $\iota^* H^1_{\mathrm{dR}}(M,E)$ is an isotropic subspace of $(H^1_{\mathrm{dR}}(K,\iota^*E), \omega^B_{\mathrm{dR},\rho})$.*

Proof. Let $u \otimes x$ and $v \otimes y$ locally represent two elements of $H^1_{\mathrm{dR}}(M,E)$. As u and v are cocycles, they must be closed, i.e. $du = dv = 0$. We want to show that $\omega^B_{\mathrm{dR},\rho}(\iota^*u \otimes x, \iota^*v \otimes y) = 0$. We have

$$\begin{aligned}
\omega^B_{\mathrm{dR},\rho}(\iota^* u \otimes x, \iota^* v \otimes y) &= \Phi((u \wedge v) \otimes B(x,y)) \\
&= B(x,y) \cdot \int_S (\iota^* u \wedge \iota^* v) \wedge \Omega^{n-1} \\
&= B(x,y) \cdot \int_{\partial M} \iota^*(u \wedge v \wedge \tilde{\Omega}^{n-1}) \\
&= B(x,y) \cdot \int_M d(u \wedge v \wedge \tilde{\Omega}^{n-1}) && \text{according to Stoke's theorem,} \\
&= 0 && \text{as } u, v \text{ and } \tilde{\Omega} \text{ are closed.}
\end{aligned}$$

This shows that the subspace $\iota^* H^1_{\mathrm{dR}}(M, E)$ is indeed isotropic. □

Unfortunately, we cannot prove that the dimension of the subspace is half of the full dimension in the same way as we did for Riemann surfaces in the proof of Theorem 3.13. In the following theorem, we will formulate a general condition under which the theorem could be generalized.

Theorem 5.2. *Consider a compact manifold M with Kähler boundary $\partial M = K$ of dimension $\dim_{\mathbb{R}} M = d + 1 = 2n + 1$. Suppose we have isomorphisms $\psi_K : H_{d-1}(K, \mathfrak{g}_{\mathrm{Ad}\,\rho r}) \longrightarrow H^1(K, \mathfrak{g}_{\mathrm{Ad}\,\rho r})^\vee$ and $\psi_M : H_{d-1}(M, \mathfrak{g}_{\mathrm{Ad}\,\rho r}) \longrightarrow H^1(M, \mathfrak{g}_{\mathrm{Ad}\,\rho r})^\vee$ such that the diagram*

$$\begin{tikzcd}
H_{d-1}(K, \mathfrak{g}_{\mathrm{Ad}\,\rho r}) \arrow[r, "r_*"] \arrow[d, "\cong"', "\psi_K"] & H_{d-1}(M, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \arrow[d, "\cong"', "\psi_M"] \\
H^1(K, \mathfrak{g}_{\mathrm{Ad}\,\rho r})^\vee \arrow[r, "(r^*)^\vee"] & H^1(M, \mathfrak{g}_{\mathrm{Ad}\,\rho})^\vee
\end{tikzcd} \tag{30}$$

commutes. Then the image $r^* H^1(M, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \subseteq H^1(K, \mathfrak{g}_{\mathrm{Ad}\,\rho r})$ is a subspace of "half dimension", i.e. $\dim_{\mathbb{R}} r^* H^1(M, \mathfrak{g}_{\mathrm{Ad}\,\rho}) = n$.

Proof. First note that for a manifold, the first singular cohomology group and the first group cohomology group are always isomorphic. Hence, we can prove the isotropy on singular cohomology. The argument is completely analogous to the proof of Theorem 3.13. Again, according to the Poincaré-Lefschetz duality theorem [Spa93], we obtain a commutative diagram

$$\begin{tikzcd}
H_d(M, K, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \arrow[r, "\partial"] \arrow[d, "\cong"', "\phi"] & H_{d-1}(K, \mathfrak{g}_{\mathrm{Ad}\,\rho r}) \arrow[r, "r_*"] \arrow[d, "\cong"', "\eta"] & H_{d-1}(M, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \arrow[d, "\cong"'] \\
H^1(M, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \arrow[r, "r^*"] & H^1(K, \mathfrak{g}_{\mathrm{Ad}\,\rho r}) \arrow[r, "\delta"] & H^2(M, K, \mathfrak{g}_{\mathrm{Ad}\,\rho}),
\end{tikzcd}$$

the same way as we saw in Eq. (24). According to the condition in Eq. (30), we obtain

$$\begin{tikzcd}
H_d(M, K, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \arrow[r, "\psi_K \partial"] \arrow[d, "\cong"', "\phi"] & H^1(K, \mathfrak{g}_{\mathrm{Ad}\,\rho r})^\vee \arrow[r, "(r^*)^\vee"] \arrow[d, "\cong"', "\eta \psi_K^{-1}"] & H^1(M, \mathfrak{g}_{\mathrm{Ad}\,\rho})^\vee \arrow[d, "\cong"'] \\
H^1(M, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \arrow[r, "r^*"] & H^1(K, \mathfrak{g}_{\mathrm{Ad}\,\rho r}) \arrow[r, "\delta"] & H^2(M, K, \mathfrak{g}_{\mathrm{Ad}\,\rho}).
\end{tikzcd}$$

It follows that

$$\begin{aligned}
\dim_{\mathbb{R}} r^* H^1(M, \mathfrak{g}_{\operatorname{Ad}\rho}) &= \operatorname{rank}_{\mathbb{R}} r^* \\
&= \operatorname{rank}_{\mathbb{R}} (r^*)^\vee && \text{by duality} \\
&= \operatorname{rank}_{\mathbb{R}} \delta && \text{by commutativity} \\
&= \dim_{\mathbb{R}} H^1(K, \mathfrak{g}_{\operatorname{Ad}\circ\rho r}) - \dim_{\mathbb{R}} \ker \delta \\
&= \dim_{\mathbb{R}} H^1(K, \mathfrak{g}_{\operatorname{Ad}\circ\rho r}) - \operatorname{rank}_{\mathbb{R}} r^* && \text{by exactness} \\
&= \dim_{\mathbb{R}} H^1(K, \mathfrak{g}_{\operatorname{Ad}\circ\rho r}) - \dim_{\mathbb{R}} r^* H^1(M, \mathfrak{g}_{\operatorname{Ad}\rho}),
\end{aligned}$$

and therefore $\dim_{\mathbb{R}} r^* H^1(M, \mathfrak{g}_{\operatorname{Ad}\rho}) = \frac{1}{2} \dim_{\mathbb{R}} H^1(K, \mathfrak{g}_{\operatorname{Ad}\rho})$. □

5.4. Outlook

Suppose we have a compact connected manifold M with compact connected Kähler boundary K, such that the symplectic Kähler form $\Omega \in \Omega^2(K)$ can be extended to a form $\tilde{\Omega} \in \Omega^2(M)$ with $\iota^*\tilde{\Omega} = \Omega$. Furthermore suppose that isomorphisms Ψ_K and Ψ_M as described in Theorem 5.2 exist. Then, Theorem 5.1 and Theorem 5.2 show that

$$r^* H^1(M, \mathfrak{g}_{\operatorname{Ad}\rho}) \subseteq H^1(K, \mathfrak{g}_{\operatorname{Ad}\circ\rho r})$$

is a Lagrangian subspace. The next step in generalizing the Lagrangian Submanifold Theorem 3.12 would be to understand which points of $X_G^g(K)$ are non-singular, i.e. the Kähler counterpart of Proposition 2.7. Afterwards, a theorem similar to Theorem 3.12.2 would need to be proven in order to show that there actually is a Lagrangian submanifold $Y_G(M) \subseteq X_G^g(K)$.

Before performing this generalization, we need to know whether the conditions for Theorem 5.1 and Theorem 5.2 are reasonable. Are there manifolds K and M such that the form Ω can be extended to $\tilde{\Omega}$? Are there isomorphisms Ψ_K and Ψ_M? The natural candidate for Ψ_K is

$$\Psi_K : H_{d-1}(K, \mathfrak{g}_{\operatorname{Ad}\rho r}) \xrightarrow{\sim} H^1(K, \mathfrak{g}_{\operatorname{Ad}\rho r})^\vee, \quad x \mapsto \tilde{\omega}^B(\cdot, x)$$

from Corollary 3.5, but it is not clear how this could be extended to

$$H_{d-1}(M, \mathfrak{g}_{\operatorname{Ad}\rho}) \to H^1(M, \mathfrak{g}_{\operatorname{Ad}\rho})^\vee.$$

A. Appendix: Geometry without an adjective

In modern mathematics, a multitude of different geometric theories exists, each coming with its intrinsic definitions. Classical algebraic geometry deals with (quasi-projective) varieties; modern algebraic geometry defines schemes; analytic geometry defines analytic varieties; and differential geometry defines manifolds.

While these definitions may seem separate when encountering them for the first time, a student of mathematics soon discovers that they are in fact deeply intertwined, and the more advanced masters of the subject elegantly dance back and forth between the different concepts.

This section aims to provide a brief overview over the different concepts and to highlight their connections. None of the work is new, so for the proofs and more detailed descriptions of each concept, we will refer to the most common literature in the respective field. This section is aimed at graduate students like me, who have a basic understanding of both algebraic and differential geometry, but struggle to see it all in a larger context. I hope this helps to see algebraic and differential geometry not only as two separate concepts, but also as constituents of a more general geometry theory: "geometry without an adjective".

A.1. Spaces in geometry

Every geometric theory begins with a topological space equipped with some extra structure. Classical algebraic geometry deals with varieties, modern algebraic geometry with schemes, differential geometry with manifolds and analytic geometry with analytic spaces. All of these spaces have appeared in this thesis at some point. Therefore, we will have a review of their definitions.

Note that a *locally ringed space* is a topological space X together with a sheaf of rings $\mathcal{O}_X$ such that the stalk $\mathcal{O}_{X,x}$ is a local ring (i.e. a ring with exactly one maximal ideal) at any point $x \in X$.

Definition A.1.

1. An *affine algebraic set over* $\mathbb{C}$ is a locally ringed space $(X, \mathcal{O}_X)$, where $X = V(S)$ is the zero locus of some set of polynomials $S \subseteq \mathbb{C}[T_1, \dots, T_n]$ equipped with the Zariski topology, and $\mathcal{O}_X$ is the sheaf of regular functions on X. If X is an irreducible topological space, it is called an *affine variety over* $\mathbb{C}$.

2. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ that is locally isomorphic to $(\mathrm{spec}(A), \mathcal{O}_{\mathrm{spec}(A)})$, i.e. the spectrum of a ring together with its structure sheaf. A scheme that is globally isomorphic to the spectrum of a ring is called *affine scheme*. A scheme together with a morphism $X \to S$ to another scheme S is called a *scheme over* S. If $S = \mathrm{spec}(\mathbb{C})$, then we call X a *scheme over* $\mathbb{C}$.

3. A *smooth manifold of dimension n* is a locally ringed space (X, C_X^∞) that is locally isomorphic to (U, C_U^∞), where $U \subseteq \mathbb{R}^n$ is an open subset and C_U^∞ is the sheaf of smooth functions on U. In addition, we require the topological space X to be Hausdorff and fulfill the second countability axiom.

4. A *complex manifold of dimension d* is a locally ringed space $(X, \mathcal{H}_X)$ that is locally isomorphic to $(U, \mathcal{H}_U)$, where $U \subseteq \mathbb{C}^d$ is an open subset and $\mathcal{H}_U$ is the sheaf of holomorphic

functions on U. In addition, we require the topological space X to be Hausdorff and fulfill the second countability axiom.

5. An *analytic subset of* $\mathbb{C}^n$ is a locally ringed space $(U, \mathcal{H}_U)$, where $U = V(S)$ is the zero locus of some set of holomorphic functions $S \subseteq \mathrm{Hol}(W, \mathbb{C})$, where $U \subseteq W \subseteq \mathbb{C}^n$ is an open neighborhood, and $\mathcal{H}_U$ is the sheaf of holomorphic functions on $\mathbb{C}^n$ restricted to U. An *analytic space*$(X, \mathcal{H}_X)$ is a locally ringed space that is locally isomorphic to such $(U, \mathcal{H}_U)$. We require the topological space X to be Hausdorff.

A morphism between two spaces is a continuous function that preserves the structure sheaves. A morphism between algebraic sets is called *regular function*, *smooth function* between smooth manifolds, and *holomorphic function* between complex manifolds.

Remark A.2. *In most classical textbooks, including the ones I used during my graduate studies, manifolds are defined using local charts with smooth (or holomorphic) transition maps. This classical definition of a manifold coincides with the one presented here. This is a straightforward consequence of the fact that the structure sheaves of two different local charts must be preserved.*

These different notions of a "space with some geometrical structure" are by no means unrelated. The following two theorems illustrate how we can switch back and forth between the different definitions.

Theorem A.3 (Relating algebraic sets and schemes)**.** *Let* $(X, \mathcal{O}_X)$ *be an affine algebraic set,*[21] *and let* $A = \mathcal{O}_X(X)$ *be the global ring of regular functions.*

1. *As a consequence of Hilbert's Nullstellensatz, there is a homeomorphism between* X *and the maximal spectrum* $\mathrm{specm}(A)$ *[Har77, Cor. I.1.4].*

2. *There is a fully faithful functor* $\mathrm{sch} : \mathbf{Aff}_{\mathbb{C}} \to \mathbf{Sch}_{\mathbb{C}}$ *from the category of affine algebraic sets to the category of* $\mathbb{C}$*-schemes, which is given by*

$$X \cong \mathrm{specm}(A) \mapsto X^{\mathrm{sch}} = \mathrm{spec}(A).$$

   The topological space X *consists of the closed points of* X^{sch}. X^{sch} *is a reduced, separated scheme of finite type over* $\mathbb{C}$ *[Har77, Prop. II.2.6, Prop. II.4.10].*

This shows that every affine algebraic set can be considered a scheme. Sometimes, we will mark this difference by using both X and X^{sch}. However, whenever the context is obvious, we will just use X for both the algebraic set and the scheme.

The next important link is the connection between algebraic varieties and analytic spaces. In 1956, Jean-Pierre Serre published his influential article "Géométrie algébrique et géométrie analytique" – affectionately dubbed GAGA [Ser56]. Its results are an important key to connecting algebraic geometry with complex geometry.

Theorem A.4 (Relating algebraic sets and analytic sets)**.** *Let* $(X, \mathcal{O}_X)$ *be an affine algebraic set.*[21]

[21] More generally, this theorem is true for quasi-projective algebraic sets as well, but we will not need this.

1. *Let $U \subseteq X$ be an open chart, i.e. we have a Zariski-homeomorphism $\phi : U \to V \subseteq \mathbb{A}^n_{\mathbb{C}}$. There is a unique structure of an analytic space on X, denoted by X^{an}, in which all subsets U as above are also open and ϕ is a homeomorphism with respect to the complex topology on V, too [Ser56, Prop.2].*

2. *Regular functions $f : X \to Y$ are also holomorphic functions with respect to the analytic topology defined before, which shows that the GAGA functor* $\mathrm{an} : \mathbf{Aff}_{\mathbb{C}} \to \mathbf{Ana}_{\mathbb{C}} : X \mapsto X^{\mathrm{an}}$ *is a faithful functor from the category of affine algebraic sets to the category of analytic sets [Ser56, Sec. 2.5].*

We will see that this theorem also provides us with a number of tools to determine when an affine algebraic set bears the structure of a complex manifold. First, we will need to discuss several notions of tangent spaces and smoothness in geometry.

A.2. Tangent spaces in algebraic and differential geometry

Let S be a base scheme and $X \to S$ an S-scheme. The *Zariski tangent space* can already be defined in this general setting. However, in the case $S = \operatorname{spec} \mathbb{C}$ and $X = \operatorname{spec} A$, where A is a finitely generated $\mathbb{C}$-algebra, we have a multitude of equivalent definitions of this complex vector space. In the following definition, we denote the $\mathbb{C}$-algebra of *dual numbers* by $\mathbb{C}[\varepsilon] = \mathbb{C}[X]/(X^2)$. ε is the element that corresponds to $X \mod (X^2)$. In other words, $\varepsilon^2 = 0$.

Definition A.5. Let $X = \operatorname{spec} A$ as above, and let $\mathfrak{m}$ be a closed point, i.e. a maximal ideal of A. Let $r_{\mathfrak{m}} : A \to A/\mathfrak{m} \cong \mathbb{C}$ be the natural projection. Then the following definitions of the *Zariski tangent space* are canonically isomorphic:

1. $T_{\mathfrak{m}}X = \{f : A \to \mathbb{C}[\varepsilon], \quad f \text{ is a } \mathbb{C}\text{-algebra homomorphism with } \pi \circ f = r_{\mathfrak{m}}\}$, where $\pi : \mathbb{C}[\varepsilon] \to \mathbb{C}$ is the projection defined by $\varepsilon \mapsto 0$. The elements of the $\mathbb{C}$-algebra $\mathbb{C}[\varepsilon]$ are also called the *dual numbers.*

2. $T_{\mathfrak{m}}X = \operatorname{Hom}_{\mathbb{C}}(\mathfrak{m}/\mathfrak{m}^2, \mathbb{C})$ is the dual vector space of the Zariski cotangent space $\mathfrak{m}/\mathfrak{m}^2$.

3. $T_{\mathfrak{m}}X = \{D : A \to \mathbb{C}, \quad D \text{ is } \mathbb{C}\text{-linear and } D(ab) = r_{\mathfrak{m}}(a)D(b) + r_{\mathfrak{m}}(b)D(a)\}$. Such a D is called a *derivation.*

All definitions are also valid using the stalk $A_{\mathfrak{m}}$ and embedding the maximal ideal $\mathfrak{m} \subset A_{\mathfrak{m}}$ instead of A. This local definition is the better option when working with schemes that are not necessarily affine.

Similarly, different equivalent definitions for the tangent space of a smooth (or complex) manifold exist.

Definition A.6. Let M be a smooth (or complex) manifold, and let $k = \mathbb{R}$ (or $k = \mathbb{C}$). The following definitions of the *tangent space* T_pM at a point p are canonically isomorphic:

1. T_pM is the space of equivalence classes of smooth curves γ with $\gamma(0) = p$. Two curves γ, δ are equivalent when their velocity vectors at 0 are identical, i.e. $\gamma'(0) = \delta'(0)$.

2. $T_pM = \mathrm{Hom}_k(\mathfrak{m}_p/\mathfrak{m}_p^2, k)$ is the dual vector space of $\mathfrak{m}_p/\mathfrak{m}_p^2$, where $\mathfrak{m}_p \subset C^\infty(M)$ (or $\mathfrak{m}_p \subset \mathcal{H}(M)$) is the maximal ideal of functions vanishing at p.

3. $T_pM = \{D : A \to k, \quad D \text{ is } k\text{-linear and } D(fg) = f(p)D(g) + g(p)D(f)\}$ is the space of derivations at p on $A = C^\infty(M)$ (or $A = \mathcal{H}(M)$).

Definitions A.6.2 and A.6.3 also work when using the stalk $C_p^\infty(M)$ (or $\mathcal{H}_p(M)$) instead. A good starting point for these equivalences is the reference [Lee13, Chap. 3].

Note that Definition A.6.2 is also well-defined for an analytic space $(X, \mathcal{H}_X)$. As explained in Theorem A.4, GAGA draws a connection between analytic space and affine algebraic sets. It is very handy that the tangent spaces remain the same under the GAGA functor.

Proposition A.7 (Relating Zariski and complex tangent spaces). *Let $(X, \mathcal{O}_X)$ be an affine algebraic set, and let $(X^{\mathrm{an}}, \mathcal{H}_{X^{\mathrm{an}}})$ be the corresponding analytic space. Then, there is a canonical isomorphism between the tangent spaces $T_pX \cong T_pX^{\mathrm{an}}$ for every point $p \in X$.*

Proof. Consider the stalks $\mathcal{O}_{X,p}$ and $\mathcal{H}_{X^{\mathrm{an}},p}$ and the respective maximal ideals $\mathfrak{m} = \mathfrak{m}_p$ and $\mathfrak{m}^{\mathrm{an}} = \mathfrak{m}_p^{\mathrm{an}}$ inside them. We will use the definitions $T_pX = (\mathfrak{m}/\mathfrak{m}^2)^\vee$ and $T_pX^{\mathrm{an}} = (\mathfrak{m}^{\mathrm{an}}/(\mathfrak{m}^{\mathrm{an}})^2)^\vee$. In [Ser56, Prop. 3], Serre showed that the stalks' completions with respect to the $\mathfrak{m}$-adic and $\mathfrak{m}^{\mathrm{an}}$-adic topology are canonically isomorphic:

$$\hat{\mathcal{O}}_{X,p} \cong \hat{\mathcal{H}}_{X^{\mathrm{an}},p}.$$

In particular, we have an isomorphism

$$\hat{\mathfrak{m}}/\hat{\mathfrak{m}}^2 \cong \hat{\mathfrak{m}}^{\mathrm{an}}/(\hat{\mathfrak{m}}^{\mathrm{an}})^2.$$

Hence, it is sufficient to show that $\mathfrak{m}/\mathfrak{m}^2 \cong \hat{\mathfrak{m}}/\hat{\mathfrak{m}}^2$ and $\mathfrak{m}^{\mathrm{an}}/(\mathfrak{m}^{\mathrm{an}})^2 \cong \hat{\mathfrak{m}}^{\mathrm{an}}/(\hat{\mathfrak{m}}^{\mathrm{an}})^2$ in order to prove that $T_pX \cong T_pX^{\mathrm{an}}$. This isomorphism is proved in [AM69, Prop. 10.15], but I will include a short proof here. An element of $\hat{\mathfrak{m}} = \varprojlim \mathfrak{m}/\mathfrak{m}^k$ is a sequence

$$(\ldots, a_3, a_2, a_1, 0) \quad \text{with } a_k \in \mathfrak{m}/\mathfrak{m}^k \text{ and } \pi_k(a_{k+1}) = a_k,$$

where $\pi_k : \mathfrak{m}/\mathfrak{m}^{k+1} \to \mathfrak{m}/\mathfrak{m}^k$ is the canonical projection. $\mathcal{O}_X$ is Noetherian as a consequence of Hilbert's basis theorem, and $\mathcal{H}_{X^{\mathrm{an}}}$ is Noetherian as well [Ser56, Sec. 4]. Hence, the stalks are Noetherian as well. For Noetherian rings, we have $(\mathfrak{m}^2)\hat{} = \hat{\mathfrak{m}}^2$ [AM69, Prop. 10.15]. Therefore, an element of $\hat{\mathfrak{m}}^2 = \varprojlim \mathfrak{m}^2/\mathfrak{m}^k$ is a sequence

$$(\ldots, b_3, b_2, 0, 0) \quad \text{with } b_k \in \mathfrak{m}^2/\mathfrak{m}^k \text{ and } \pi_k(b_{k+1}) = b_k.$$

Since inverse limits are left-exact, we have an exact sequence

$$0 \to \hat{\mathfrak{m}}^2 \to \hat{\mathfrak{m}} \to \varprojlim(\mathfrak{m}/\mathfrak{m}^k)/(\mathfrak{m}^2/\mathfrak{m}^k). \tag{31}$$

The inverse limit $\varprojlim(\mathfrak{m}/\mathfrak{m}^k)/(\mathfrak{m}^2/\mathfrak{m}^k)$ consists of sequences

$$(\ldots, c_3, c_2, c_1, 0) \quad \text{with } c_k \in (\mathfrak{m}/\mathfrak{m}^k)/(\mathfrak{m}^2/\mathfrak{m}^k) \text{ and } \bar{\pi}_k(c_{k+1}) = c_k,$$

with the projection $\bar{\pi}_k : (\mathfrak{m}/\mathfrak{m}^{k+1})/(\mathfrak{m}^2/\mathfrak{m}^{k+1}) \to (\mathfrak{m}/\mathfrak{m}^k)/(\mathfrak{m}^2/\mathfrak{m}^k)$. There is a canonical isomorphism $(\mathfrak{m}/\mathfrak{m}^k)/(\mathfrak{m}^2/\mathfrak{m}^k) \cong \mathfrak{m}/\mathfrak{m}^2$ under which $\bar{\pi}_k$ simply becomes the identity. Therefore, the sequence $(\ldots, c_3, c_2, c_1, 0)$ is in fact a constant sequence

$$(\ldots, c, c, c, 0),$$

i.e. an element of $\mathfrak{m}/\mathfrak{m}^2$. In other words, $\mathfrak{m}/\mathfrak{m}^2 \cong \varprojlim (\mathfrak{m}/\mathfrak{m}^k)/(\mathfrak{m}^2/\mathfrak{m}^k)$. However, all constant sequences are contained in $\hat{\mathfrak{m}}$, so that Eq. (31) becomes

$$0 \to \hat{\mathfrak{m}}^2 \to \hat{\mathfrak{m}} \to \mathfrak{m}/\mathfrak{m}^2 \to 0$$

Thereby, we have a canonical isomorphism $\mathfrak{m}/\mathfrak{m}^2 \cong \hat{\mathfrak{m}}/\hat{\mathfrak{m}}^2$ as remained to be shown. □

A.3. Singularities of varieties and analytic sets

Let $(X, \mathcal{O}_X)$ be an affine algebraic set over $\mathbb{C}$. Now that we have established the notion of a tangent space in the algebraic and in the analytic case, we will talk about non-singular and singular points of algebraic varieties. Furthermore, we will see that non-singular points of X coincide with the non-singular points of X^{an} and that the non-singular points of X^{an} are actually a complex manifold. A basic reference for this section is [Har77, Sec. I.5].

Definition A.8. A point $p \in X$ is called *non-singular*, if the local ring $\mathcal{O}_{X,p}$ is regular, i.e. $\dim_{\mathbb{C}} \mathfrak{m}_p/\mathfrak{m}_p^2 = \dim \mathcal{O}_{X,p}$. Otherwise, p is called *singular*.

Proposition A.9. *A point P of an affine algebraic set X is non-singular if and only if it is contained in one unique irreducible component C of X and* $\dim_{\mathbb{C}} T_pX = \dim C$ *[Mil17a, Cor. 4.45].*

Proposition A.10. *Every open subset of an affine algebraic set X has an open subset of non-singular points, which is dense in the Zariski topology.*[22]

Proof. If X is irreducible, i.e. an affine variety, refer to [Har77, Cor. II.8.16] for a proof. If not, consider a decomposition $X = \bigcup_i U_i$ into a finite union of affine varieties using the Lasker-Noether primary decomposition theorem, the argument can then be generalized by a simple argument [see Mil17a, Thm. 4.37]. □

Accordingly, let $(X, \mathcal{H}_X)$ be any analytic space. Again, a point $p \in X$ is called *non-singular*, if the local ring $\mathcal{H}_{X,p}$ is regular. We have the following very intuitive characterization of non-singular points:

Proposition A.11 (Relating non-singular analytic spaces and complex manifolds)**.** *A point $p \in X$ of an analytic space $(X, \mathcal{H}_X)$ is non-singular if and only if it has a neighborhood $(U, \mathcal{H}_X|_U)$ isomorphic to an open subspace of $(\mathbb{C}^n, \mathcal{H}_{\mathbb{C}^n})$ [Ser56, Sec. 4].*

This means that X is locally a complex manifold at its non-singular points.

Proposition A.12 (Relating non-singular points in algebraic sets and analytic spaces)**.** *Let X be an affine algebraic set and let X^{an} be the corresponding analytic space. A point p of X is non-singular if and only if its image in X^{an} is non-singular.*

[22] An open subset of an affine algebraic set is called *quasi-affine algebraic set.*

Proof. It has been proven in [Ser56, Cor. 2] that $\mathcal{O}_{X,p}$ and $\mathcal{H}_{X^{\mathrm{an}},p}$ have the same dimension. In Proposition A.7, we saw that the tangent spaces T_pX and T_pX^{an} are isomorphic, so they have the same dimension as well. Therefore, the equivalence of non-singularity in the algebraic and analytic space follows directly from the definitions. $\square$

Combining Propositions A.11 and A.12, this means that the set of non-singular points of an algebraic variety naturally bears the structure of a complex manifold (as the dimension remains the same throughout all points of an irreducible variety). More generally, if we have a not necessarily irreducible algebraic set, then its non-singular points are the disjoint union of several complex manifolds.

A.4. Lie groups and algebraic groups

Let G be a group with group operation $\mu : G \times G \to G$ and inversion $\iota : G \to G$. G is called a smooth (or complex) *Lie group*, if G is a smooth (or complex) manifold and both μ and ι are smooth (or holomorphic) functions.

Analogously, G is called an *affine algebraic group* if G is an affine algebraic variety and both μ and ι are regular maps.

Finally, a *group scheme* over $\mathbb{C}$ is a group object in the category of schemes over $\mathrm{spec}(\mathbb{C})$. In other words, a group scheme over $\mathbb{C}$ is a scheme G with structure morphism $G \to \mathrm{spec}(\mathbb{C})$ together with the following $\mathbb{C}$-morphisms: the group operation $\mu : G \times_{\mathrm{spec}\,\mathbb{C}} G \to G$, the neutral element $e : \mathrm{spec}(\mathbb{C}) \to G$ and the inversion $\iota : G \to G$, such that associativity and the defining properties for the neutral element and the inversion are fulfilled.

For any group scheme G over $\mathbb{C}$ and any $\mathbb{C}$-algebra A, the set of A-valued points $G(A)$ actually is a group [see Mil17b, Sec. 1.4].

These three definitions – Lie group, algebraic group and group scheme – are interrelated. Every affine algebraic group G can be considered a group scheme under the fully faithful functor $\mathrm{sch} : \mathbf{Aff}_{\mathbb{C}} \to \mathbf{Sch}_{\mathbb{C}}$. Indeed, if A is the ring of regular functions of $G = \mathrm{specm}(A)$, then the ring of regular functions of $G \times G$ is given by $A \otimes_{\mathbb{C}} A$. In particular, the functor sch maps $\mu : G \times G \to G$ to $\mu^{\mathrm{sch}} : G^{\mathrm{sch}} \times_{\mathrm{spec}(\mathbb{C})} G^{\mathrm{sch}} \to G^{\mathrm{sch}}$, and accordingly for ι. A point has $\mathbb{C}$ as its ring of regular functions, so $e \hookrightarrow G$ turns into $e : \mathrm{spec}(\mathbb{C}) \to G^{\mathrm{sch}}$ as required. This shows that an affine algebraic group is in fact a group scheme over $\mathbb{C}$.

Furthermore, it can be seen that every affine algebraic group G is non-singular [see Mil17b, Prop. 1.26]. In particular, its analytification G^{an} is a complex Lie group. These two observations enable us to tackle affine algebraic groups with both differential and algebraic methods, has been useful throughout the thesis.

A.5. The adjoint action of an algebraic group

The Lie algebra $\mathfrak{g}$ of the Lie group G is isomorphic to the tangent space T_eG at the identity. We have a representation of G in $\mathfrak{g}$, the *adjoint representation*

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}).$$

It is defined as follows: Let $\operatorname{conj}_g : G \to G, x \mapsto gxg^{-1}$ be an inner automorphism, then define Ad_g as

$$\operatorname{Ad}_g = (d \operatorname{conj}_g)_e : \mathfrak{g} \to \mathfrak{g},$$

and we obtain an action of G on its Lie algebra. In Definition A.5, we have seen several isomorphic characterizations of the Zariski tangent space, so let us consider the case that G is an algebraic group as well.

For the proof of Lemma 1.22, it is useful to know what the adjoint representation looks like in the characterization $T_e G \cong \{f : \mathcal{O}_G(G) \to \mathbb{C}[\varepsilon], \pi \circ f = r_e\}$ from Definition A.5.1.

Consider the points functor $G(\cdot)$. The isomorphism $G \cong G(\mathbb{C})$ is given by

$$G \to \operatorname{Hom}_{\mathbf{Alg}_{\mathbb{C}}}(\mathcal{O}_G(G), \mathbb{C}), \quad g \mapsto r_g,$$

where r_g is the projection such that $\ker r_g$ is the maximal ideal representing the closed point g in $\operatorname{spec} \mathcal{O}_G(G)$. Thereby, $T_e G$ is simply

$$\ker\left(G(\pi) : G(\mathbb{C}[\varepsilon]) \to G(\mathbb{C})\right),$$

where π is defined by $\varepsilon \mapsto 0$, and $G(\pi)$ is the induced group homomorphism. On the other hand, we have an embedding $\iota : \mathbb{C} \hookrightarrow \mathbb{C}[\varepsilon]$, which induces an injective group homomorphism $G(\iota) : G(\mathbb{C}) \to G(\mathbb{C}[\varepsilon])$, so G is in fact a subgroup of $G(\mathbb{C}[\varepsilon])$. This subgroup acts on the subgroup $T_e G \subseteq G(\mathbb{C}[\varepsilon])$ by conjugation,

$$g.f = gfg^{-1} \quad \text{for any } f \in T_e G = \ker(G(\pi)), g \in G = G(\mathbb{C}). \tag{32}$$

Note that we write all groups multiplicatively (even though $\ker G(\pi) \cong T_e G$ is abelian). This action is precisely the adjoint action, as can be seen in [Mil17b].

A.6. Luna's étale slice theorem for algebraic groups

Definition A.13. Let X and Y be two affine algebraic sets. A morphism $\phi : X \to Y$ is called *étale* at a point $x \in X(\mathbb{C})$, if the induced map on tangent spaces $(d\phi)_x : T_x(X) \to T_{\phi(x)} Y$ is an isomorphism.

Let G be a reductive affine algebraic group and X an affine algebraic set. Let $\sigma : G \times X \to X$ be a group action and let $\pi_X : X \to X /\!\!/ G$ be the projection onto the categorical quotient.

Theorem A.14. *Let $x \in X(\mathbb{C})$ be a point of the variety of which the orbit $O_x \subseteq X$ is closed. Let $G_x \subseteq G$ be the stabilizer subgroup. A subvariety $V \subseteq X$ with the following properties exists:*

1. *V is an affine variety and contains the point x.*
2. *V is preserved by the action of G_x.*
3. *G_x has a left action on $G \times V$ via $h.(g, y) = (gh^{-1}, hy)$, and G has a left action on $G \times V$ via $h.(g, y) = (hg, y)$. The G-equivariant morphism $\psi : (G \times V) /\!\!/ G_x \to X$ is étale.*
4. *The image U of ψ is open and affine in X. Furthermore, it is π_X-saturated, i.e. $U = \pi_X^{-1}(\pi_X(U))$.*

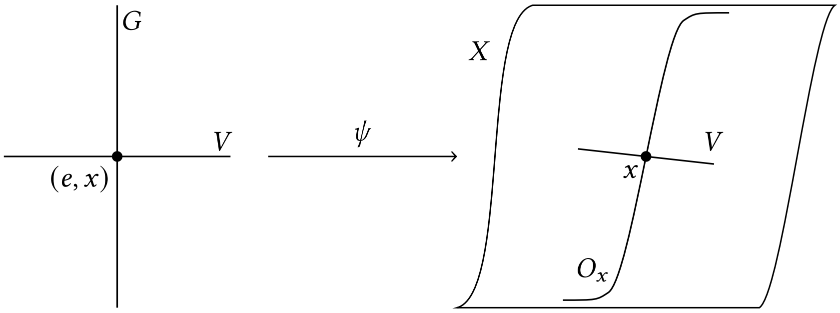

Figure 2: Luna's étale slice theorem: When $G_x = \{e\}$, then $T_xX \cong T_eG \oplus T_xV$.

5. *The map* $V /\!\!/ G_x \cong ((G \times V) /\!\!/ G_x) /\!\!/ G \xrightarrow{\bar{\psi}} U /\!\!/ G$ *is étale at* x.
6. *The maps* ψ *and* $(G \times V) /\!\!/ G_x \to ((G \times V) /\!\!/ G_x) /\!\!/ G \cong V /\!\!/ G_x$ *induce an isomorphism*

$$(G \times V) /\!\!/ G_x \cong U \times_{U /\!\!/ G} (V /\!\!/ G_x).$$

The proof for this theorem can be found in [Lun73, Sec. III.1]. When the stabilizer is trivial, Luna's étale slice theorem implies that $T_xX \cong T_eG \oplus T_xV$, and accordingly for all points in the image $\psi(G \times V) \subseteq X$, as Fig. 2 illustrates.

Definition A.15. A subscheme V as described in the above theorem is called an *étale slice at* x.

Proposition A.16. *The slices in Theorem A.14 additionally fulfill the following properties:*

1. *The embedding* $U \hookrightarrow X$ *is an open immersion, and so is the quotient map* $U /\!\!/ G \to X /\!\!/ G$. *Therefore, both are étale morphisms, and the map in bullet point 5 of the theorem can be extended to an étale map* $V /\!\!/ G_x \to X /\!\!/ G$.
2. *Compose* $G \to (G \times V) /\!\!/ G_x$ *with* ψ *from bullet point 3, then the image of* T_eG *is precisely* $T_xO_x \subseteq T_xX$.

Proof. 1. According to the theorem, $U \subseteq X$ is open, so the natural restriction of the structure sheaf makes $U \hookrightarrow X$ an open immersion. In fact, the statement is implicitly proved in [Lun73], so we will simply sketch how different parts of the reference [Lun73] have to be combined in order to obtain the statement.

The theorem is first proved for the case that x is a smooth point, using [Lun73, Lem. II.2.3]. Consider the map $\psi : (G \times V) /\!\!/ G_x \to X$. Its image is called U. According to this lemma, $U /\!\!/ G$ is the image of the quotient map $\bar{\psi} : ((G \times V) /\!\!/ G_x) /\!\!/ G \to X /\!\!/ G$. The lemma already shows that $U /\!\!/ G$ is open (and affine) in $X /\!\!/ G$, so $U /\!\!/ G \to X /\!\!/ G$ is an open immersion.

For the general case, assume that X is embedded in some larger smooth affine variety $X \subseteq \tilde{X}$. The theorem is then proved for the smooth $\tilde{X}$ and the map $\psi : (G \times \tilde{V}) /\!\!/ G_x \to \tilde{X}$ as before. Then, a base change to X is performed. The proof of the lemma after [Lun73, Lem. fondamentale] shows that a base change $X \to \tilde{X}$ corresponds to a base change $X /\!\!/ G \to \tilde{X} /\!\!/ G$ on the quotients.

As open immersions are invariant under base change, [see Sta19, Lemma 01JU], this proves that the image of the changed maps ψ_X and $\bar{\psi}_{X/\!\!/G}$ will be open in X and $X/\!\!/G$, respectively.

Let $\mathcal{O}_X$ be the structure sheaf of X, then $\mathcal{O}_U = \mathcal{O}_X|_U$. The Zariski cotangent space of X at x is $\mathfrak{m}_x/\mathfrak{m}_x^2$ for the maximal ideal $\mathfrak{m}_x$ of the stalk $\mathcal{O}_{X,x}$. We have an isomorphism $\mathcal{O}_{U,x} \cong \mathcal{O}_{X,x}$, which shows that the Zariski tangent spaces of X and U at x coincide. This means that open immersions are étale.

2. The orbit O_x is the image of the map $\sigma_x : G \to X, g \mapsto gx$. We have a commutative diagram

$$\begin{tikzcd} G \arrow[rr, "\sigma_x"] \arrow[dr] & & X \\ & (G \times V)/\!\!/G_x \arrow[ur, "\psi"'] & \end{tikzcd}$$

and in particular, the image of T_eG is T_xO_x. □

A.7. Orientation

A multitude of equivalent definitions exist as to when a smooth manifold M can be called *orientable*. A survey of different definitions can be found in [Kre13]. For our purposes, there are two relevant definitions.

The first definition defines orientability via the manifold's atlas. A manifold is called orientable, if an atlas exists for which the differentials of all transition functions have a positive determinant. In that case, choosing an orientation simply amounts to choosing a maximal atlas with that property. This definition is handy: For example, it immediately shows that the complex conjugation on $\mathbb{C}$ is an orientation-reversing map on $\mathbb{R}^2 \cong \mathbb{C}$ as its Jacobian determinant is -1.

The second definition involves the singular homology with coefficients in $\mathbb{Z}$. An n-dimensional smooth manifold is called orientable if we have a "top-level" isomorphism

$$H_{n,\mathrm{sing}}(M, \mathbb{Z}) \cong \mathbb{Z}\,.$$

Choosing an orientation then simply amounts to choosing a generator of that free abelian group, i.e. 1 or -1. The inverse image of 1 respectively -1 under the isomorphism is also called *fundamental class*, written $[M]$. This definition is useful as well, as it gives us a very simple characterization of orientation-reversing maps: A diffeomorphism $f : M \to M$ is orientation-reversing if its induced map

$$f_* : H_{n,\mathrm{sing}}(M, \mathbb{Z}) \to H_{n,\mathrm{sing}}(M, \mathbb{Z}), \quad \alpha \mapsto f \circ \alpha$$

is given by $f_*(\alpha) = -\alpha$, or under the isomorphism $\mathbb{Z} \to \mathbb{Z}, x \mapsto -x$.

B. Appendix: Principal G-bundles and representations

B.1. Flat principal G-bundles and group representations

Definition B.1. Two G-principal bundles $\pi : P \to M$ and $\pi' : P' \to M$ are called *isomorphic* (or, more precisely, isomorphic over the identity), if a G-equivariant diffeomorphism $\Phi : P \to P'$ exists such that the diagram

$$\begin{tikzcd} P \arrow[r, "\Phi"] \arrow[d, "\pi"] & P' \arrow[d, "\pi'"] \\ M \arrow[r, "\mathrm{id}"] & M \end{tikzcd}$$

commutes.

It is a well-known fact that isomorphism classes of flat principal G-bundles stand in one-to-one correspondence to conjugacy classes of G-representations of the fundamental group. In this section, we will present a short survey of the definitions that are necessary for the establishment of this correspondence. For technical details and a complete proof see [Bau14, Satz 4.7].

Let M be a smooth connected manifold and let G be a Lie group. Consider the fundamental group $\pi_1(M) = \pi_1(M, m_0)$ at the base point m_0. Let $\rho : \pi_1(M) \to G$ be a group homomorphism. We will now define a principal bundle $P_\rho \to M$ associated to the representation ρ. Fix a universal covering $\tilde{M} \to M$. The group $\pi_1(M)$ acts on $\tilde{M}$ via deck transformations. Thereby, we have an action on the product $\tilde{M} \times G$ via $\gamma.(\tilde{x}, g) = (\gamma\tilde{x}, \rho(\gamma)g)$. Then, we define the orbit space

$$P_\rho = (\tilde{M} \times G)/\pi_1(M)$$

of this action. The canonical flat connection on $\tilde{M} \times G$ induces a flat connection on P_ρ.

On the other hand, consider any principal G-bundle P with a flat connection A. For any $[\gamma] \in \pi_1(M)$, the parallel transport of A along γ is a G-right invariant map

$$\mathcal{P}^A_\gamma : P_{m_0} \to P_{m_0}.$$

Furthermore, the fibers are G-equivariantly diffeomorphic to G, so after choosing such a diffeomorphism $P_{m_0} \cong G$, we obtain

$$\mathcal{P}^A_\gamma : G \to G, \quad g \mapsto \mathrm{hol}(\gamma)g,$$

where $\mathrm{hol}(\gamma) \in G$ depends on the choice of the diffeomorphism $P_{m_0} \cong G$. The algebraic properties of $\mathcal{P}^A$ show that $\mathrm{hol} : \pi_1(M) \to G$ is a group homomorphism and changing the diffeomorphism varies hol by conjugation in G. hol is called the *holonomy representation* of $\pi_1(M)$. The representation hol determines P "uniquely modulo conjugation":

Theorem B.2. *Let* $\mathrm{Hom}(\pi_1(M), G)/G$ *be the orbit space of the conjugation of representations. There is a one-to-one correspondence of sets*

$$\{(P, A) \textit{ flat principal } G\textit{-bundle}\}/\!\sim \; \xleftrightarrow{1:1} \; \mathrm{Hom}(\pi_1(S), G)/G$$

between the isomorphism classes of flat principal bundles and group representations modulo conjugation in G. The maps are given by

$$P \mapsto \mathrm{hol}$$

and

$$P_\rho \hookleftarrow \rho.$$

Consider a connected component of $\mathrm{Hom}(\pi_1(S), G)$. Then, all corresponding flat principal bundles are in fact isomorphic as principal bundles. This means that we can consider them as different flat structures of the same principal bundle, as the following proposition states.

Proposition B.3. *Let $C \subseteq \mathrm{Hom}(\pi_1(M), G)/G$ be a connected component. Then, all flat principal bundles corresponding to elements of C are isomorphic as principal bundles (forgetting the connection). In other words, the bijection from Theorem B.2 induces a bijection*

$$\mathcal{F}(P_{\rho_0}) \overset{1:1}{\longleftrightarrow} C$$

for any $\rho_0 \in C$, where $\mathcal{F}(P_{\rho_0}) \subset \mathcal{A}(P_{\rho_0})$ is the space of flat connections on P_{ρ_0} as defined in Definition B.9.

B.2. The space of connections

This section is based on [AB83, Sec. 3] and [Bau14, Chap. 3]. Let M be a finite-dimensional manifold, and let G be a Lie group. Let $\pi : P \to M$ be a smooth principal G-bundle. For an element $m \in M$, we will write $P_m = \pi^{-1}(m)$ for the fiber, and we abbreviate $P_p = F_{\pi(p)}$ for an element $p \in P$. The pullback bundle $\pi^*TM = \{(p, v), p \in P, v \in T_{\pi(p)}M\}$ is a vector bundle over P. We have a G-equivariant short exact sequence

$$0 \to T_F P \xrightarrow{\iota} TP \xrightarrow{d\pi} \pi^*TM \to 0$$

of vector bundles over P, where $T_F P = \{(p, v), p \in P, v \in T_p P_p \cong \mathfrak{g}\}$ is the tangent bundle along the fiber.

Definition B.4. A *connection* A on P is defined as one of the following equivalent data:

1. a G-equivariant smooth section $v_A : TP \to T_F P$ of the above sequence, i.e. $v_A \circ \iota = \mathrm{id}_{T_F P}$. The subspace $VP = T_F P \subseteq TP$ is also called the *vertical bundle*, and v_A the *vertical projection*.

2. a G-equivariant smooth section $s_A : \pi^*TM \to TP$, i.e. $d\pi \circ s_A = \mathrm{id}_{\pi^*TM}$. The subspace $HP = s_A(\pi^{-1}TM) \cong \pi^*TM$ is also called the *horizontal bundle* and $h_A = s_A \circ d\pi : TP \to HP$ the *horizontal projection*.

3. a G-equivariant splitting $TP \cong T_F P \oplus \pi^*TM$ into a direct sum of vector bundles, such that ι and $d\pi$ become the natural embedding and projection, respectively.

4. a so-called *connection form* A, i.e. a 1-form $A \in \Omega^1(P, \mathfrak{g})$ with $R_g^* A = \mathrm{Ad}(g^{-1}) \circ A$ for all $g \in G$ and $A(\tilde{X}) = X$ for all $X \in \mathfrak{g}$.[23]

Let V be a $\mathbb{R}$-vector space, and consider the graded algebra $\Omega^\bullet(P, V)$ of V-valued differential forms. For any connection A, the pullback $h_A^* : \Omega^\bullet(P, V) \longrightarrow \Omega^\bullet(P, V)$ maps differential forms to *horizontal differential forms*, i.e. forms that vanish on vertical vector fields (vector fields with values in VP).

Definition B.5. For any connection A, the *absolute differential* $D_A : \Omega^q(P, V) \longrightarrow \Omega^{q+1}(P, V)$ is defined to be $D_A = h_A^* \circ d$, where d is the exterior derivative. It induces the *covariant derivative* $d_A : \Omega^q(M, E) \longrightarrow \Omega^{q+1}(M, E)$, where $E = P \times_G V$ is the associated vector bundle with respect to a representation $\rho : G \longrightarrow \mathrm{GL}(V)$.

We have $d_A d_A \omega = F_A \wedge \omega$ [see Bau14, Satz 3.16]. Therefore, $d_A \circ d_A = 0$ if and only if $F_A = 0$. In other words, flat vector bundles E as above turn the $\Omega^\bullet(M, E)$ into a cochain complex.

Definition B.6. The cohomology groups $H^k_{\mathrm{dR}}(M, E)$ corresponding to $\Omega^\bullet(M, E)$ are called *de Rham-cohomology with twisted coefficients.*

The *adjoint bundle* is the vector bundle $\mathrm{ad}\, P = P \times_G \mathfrak{g}$, which is defined as follows: Let G act on $P \times \mathfrak{g}$ via $g.(p, x) = (pg^{-1}, \mathrm{Ad}_g\, x)$. The orbit space $(P \times \mathfrak{g})/G$ is a vector bundle, which we call the adjoint bundle.

Proposition B.7. *Let M be a compact finite-dimensional manifold. The set $\mathcal{A}(P)$ of all connections is an affine space[24] with transformation space $\Omega^1(M, \mathrm{ad}\, P)$, where $\mathrm{ad}(P)$ is the adjoint vector bundle.*

This is proved in [AB83, Sec. 3] or [Bau14, Folg. 3.1]. In consequence, we obtain a manifold structure on $\mathcal{A}(P)$.

Corollary B.8. *$\mathcal{A}(P)$ is an infinite dimensional manifold with tangent space*

$$T_A \mathcal{A}(P) \cong \Omega^1(M, \mathrm{ad}\, P)$$

at any connection $A \in \mathcal{A}(P)$.

Proof. Fix any point $A \in \mathcal{A}(P)$. Then, any other point can be written $A' = A + \eta$ for some $\eta \in \Omega^1(M, \mathrm{ad}\, P)$ according to Proposition B.7. We define those $\phi_A : \mathcal{A}(P) \longrightarrow \Omega^1(M, \mathrm{ad}\, P), A' \mapsto A' - A = \eta$ to be the charts of the manifold. □

[23] For an element $X \in \mathfrak{g}$ of the Lie algebra, the fundamental vector field $\tilde{X}$ is defined via $\tilde{X}(u) = \frac{d}{dt}(u \cdot \exp(tX))|_{t=0} \in (T_F P)_u$.

[24] By *affine space*, we mean a set with a transitive and free vector space action, and not an affine algebraic set as defined in algebraic geometry.

B.3. Curvature and flat connections

Definition B.9. The *curvature* is the function

$$F : \mathcal{A}(P) \to \Omega^2(P, \mathfrak{g}), \quad A \mapsto F(A) = D_A A.$$

The level set $\mathcal{F}(P) = F^{-1}(0) \subseteq \mathcal{A}(P)$, which consists the set of connections with vanishing curvature, is called the set of *flat connections.*

We will simply define the tangent space to be $T_A\mathcal{F} = \ker(dF)_A$ for $A \in \mathcal{F}$.

Proposition B.10. *For any $A \in \mathcal{F}$, the tangent space $T_A\mathcal{F}$ is given by all elements η of $T_A\mathcal{A}(P) \cong \Omega^1(M, \operatorname{ad} P)$ with $d_A\eta = 0$, where d_A is the covariant derivative. With the definition of twisted de Rham cohomology, see Definition B.6, we obtain $T_A\mathcal{F} = Z^1(M, \operatorname{ad} P)$.*

Proof. Consider the line $A_t = A + t\eta$ for $t \in \mathbb{R}$ in $\mathcal{A}(P)$. The curvature along this line is calculated in [AB83, Sec. 4]; it is

$$F(A_t) = F(A) + td_A\eta + \frac{1}{2}t^2[\eta, \eta]. \tag{33}$$

The differential of F at A is given by

$$(dF)_A\eta = \frac{d}{dt}F(A_t)|_{t=0} = \frac{d}{dt}F(A) + td_A\eta + \frac{1}{2}t^2[\eta, \eta]|_{t=0} = d_A\eta,$$

which implies that $T_A\mathcal{F}$ consists of the η with $d_A\eta = 0$. □

B.4. The gauge group action

Definition B.11. A *gauge transformation* $f : P \to P$ of the principal bundle P is a G-equivariant, fiber-preserving diffeomorphism. The group $\mathcal{G}(P)$ of gauge transformations is also called *gauge group.*

For any connection form A and any gauge transformation f, the pullback f^*A is again a connection form. In particular, the gauge group $\mathcal{G}(P)$ acts on the space $\mathcal{A}(P)$ of all connections.

Proposition B.12. *The gauge group can be identified with the space of invariant smooth functions, $C^\infty(P, G)^G$. Hence, it is an infinite-dimensional Lie group with corresponding Lie algebra $\mathfrak{gau}(P) \cong \Omega^0(M, \operatorname{ad} P) \cong C^\infty(P, \mathfrak{g})^G$.*

Let $f \in \mathcal{G}(P)$. Its corresponding element in $C^\infty(P, G)^G$ is found as follows: Simply write

$$p \mapsto f(p) = p.a_f(p),$$

then $a_f \in C^\infty(P, G)^G$, see [Bau14, Sec. 3.5] for a more detailed discussion. Note that $(fg)(p) = p.a_g(p)a_f(p)$, so the map $\mathcal{G}(P) \to C^\infty(P, G)^G$ is in fact an anti-isomorphism of Lie groups. Therefore, we must be cautious when switch between identifications. For example, the adjoint action of $C^\infty(P, G)^G$ on its Lie algebra $C^\infty(P, \mathfrak{g})^G$ is given by

$$\operatorname{Ad}^G(a_f) \circ X = \operatorname{Ad}^{C^\infty(P,G)^G}(a_f)(X) = \operatorname{Ad}^{\mathcal{G}(P)}(f^{-1})(X), \tag{34}$$

where $X \in C^\infty(P, \mathfrak{g})^G \cong \mathfrak{gau}(P)$ is a Lie algebra element, and $\mathrm{Ad}^G : G \to \mathrm{GL}(\mathfrak{g})$ is the adjoint representation of G.

Consider the evaluation map $\mathrm{eval}_A : \mathcal{G}(P) \to \mathcal{A}(P)$ defined by $f \mapsto f^*A$. The following lemma will turn out useful:

Lemma B.13. *The differential* $(d\,\mathrm{eval}_A)_e : \mathfrak{gau}(P) \to T_A\mathcal{A}(P)$ *is given by*

$$d_A : \Omega^0(M, \mathrm{ad}\, P) \to \Omega^1(M, \mathrm{ad}\, P)$$

under the identifications $\mathfrak{gau}(P) \cong \Omega^0(M, \mathrm{ad}\, P)$ *and* $T_A\mathcal{A}(P) \cong \Omega^1(M, \mathrm{ad}\, P)$.

Consider an element $X \in \mathfrak{gau}(P)$ of the Lie algebra. Its fundamental vector field $\tilde{X} \in \Gamma(\mathcal{A}(P), T\mathcal{A}(P))$ is defined by

$$\tilde{X}(A) = \frac{d}{dt}|_{t=0} \exp(tX).A = (d\,\mathrm{eval}_A)_e(X) \in T_A\mathcal{A}(P).$$

In particular, we can conclude:

Corollary B.14. *Under the identifications of Lemma B.13, we have*

$$\tilde{X}(A) = d_A(X),$$

for all $A \in \mathcal{A}(P)$*, where* $\tilde{X}$ *is the fundamental vector field and* d_A *is the covariant derivative.*

C. Appendix: Remarks on cohomology

C.1. A bilinear map on differential forms with coefficients

Let M be a manifold and let $E \xrightarrow{\pi_E} M$ be a smooth vector bundle with a $\mathbb{C}$-vector space V as its fiber. Let $(U_\alpha \subseteq M, \phi_\alpha : \pi_E^{-1}(U_\alpha) \to V \times U_\alpha)$ be a trivialization such that the transition maps $\phi_\beta \circ \phi_\alpha^{-1}$ are locally constant on $U_\alpha \cap U_\beta \times V$. Such a vector bundle is called a *flat vector bundle*. This notion of flatness is actually related to the flatness of connections on principal bundles from Appendix B.3 above, but we will not need this.

The space of *differential k-forms with coefficients in E* is the set of smooth sections $\Omega^k(M, E) = \Gamma(M, \bigwedge^k T^*M \otimes_{\mathbb{R}} E)$.

Proposition C.1. *Consider a* $\mathbb{C}$*-bilinear form* $B : V \times V \to \mathbb{C}$ *on the vector space* V*. It induces a* $C^\infty(M)$*-bilinear form*

$$\Omega^1(M, E) \times \Omega^1(M, E) \to \Omega^2(M, \mathbb{C}), \quad (\eta, \theta) \mapsto B_*(\eta \wedge \theta).$$

Proof. B induces a $\mathbb{C}$-linear map $B : V \otimes_{\mathbb{C}} V \to \mathbb{C}$. This induces a vector bundle homomorphism $E \otimes E \to \underline{\mathbb{C}}$, where $\underline{\mathbb{C}}$ denotes the trivial bundle with fiber $\mathbb{C}$ on M. Indeed, locally we have

$$\pi_{E\otimes E}^{-1}(U_\alpha) \xrightarrow{\phi_\alpha} U_\alpha \times V \otimes V \xrightarrow{\mathrm{id} \times B} U_\alpha \times \mathbb{C} = \pi_{\underline{\mathbb{C}}}^{-1}(U_\alpha).$$

This is obviously smooth. It is well-defined because E (and thus also $E \otimes E$) is flat: a transition to another chart U_β will be constant on every connected component of $(U_\alpha \cap U_\beta) \times V \otimes V$.

Thereby, we obtain a $C^\infty(M)$-bilinear map

$$\begin{aligned}
\Gamma(M, T^*M \otimes_{\mathbb{R}} E) \times \Gamma(M, T^*M \otimes_{\mathbb{R}} E) &\to \Gamma(M, T^*M \otimes_{\mathbb{R}} T^*M \otimes_{\mathbb{R}} E \otimes_{\mathbb{R}} E) \\
&\xrightarrow{\sim} \Gamma(M, T^*M \otimes_{\mathbb{R}} T^*M) \otimes_{C^\infty(M)} \Gamma(M, E \otimes_{\mathbb{R}} E) \\
&\xrightarrow{B} \Gamma(M, T^*M \otimes_{\mathbb{R}} T^*M) \otimes_{C^\infty(M)} \Gamma(M, \underline{\mathbb{C}}) \\
&\xrightarrow{\wedge} \Gamma(M, \bigwedge^2 T^*M) \otimes_{C^\infty(M)} \Gamma(M, \underline{\mathbb{C}}) \\
&\cong \Gamma(M, \bigwedge^2 T^*M \otimes_{\mathbb{R}} \underline{\mathbb{C}}).
\end{aligned}$$

This is precisely the form we wanted. $\square$

C.2. Comparing singular, de Rham and group cohomology

In this section, let M be any arc-connected topological space. Let Γ be an arbitrary group.

Definition C.2. An arc-connected topological space M is called *aspherical* or *Eilenberg-MacLane space* of type $(\Gamma, 1)$, if

$$\begin{aligned}
\pi_k(M) &= 0 \quad \text{for all } k > 1, \\
\pi_1(M) &\cong \Gamma.
\end{aligned}$$

More generally, we say that M is *aspherical up to r* if $\pi_k(M) = 0$ for all $1 < k < r$.

Aspherical spaces are quite remarkable, as their singular cohomology (as well as homology) groups are determined by Γ only. In other words, the singular cohomology does not decode any topological information. Historically, the cohomology theory of groups was developed in order to provide a topology-independent description of the singular cohomology of aspherical spaces, and more generally Eilenberg-MacLane spaces.

We will state this idea more precisely in the following theorem. Let A be a Γ-module, the module of coefficients.

Theorem C.3. *Suppose that the Γ-action on A is trivial and let M be aspherical up to r. Then, the group (co)homology is naturally isomorphic to the singular (co)homology with coefficents in A. In particular, we have isomorphisms*

$$\begin{aligned}
H^k_{\text{sing}}(M, A) &\cong H^k_{\text{Grp}}(\pi_1(M), A) \\
H_{k,\text{sing}}(M, A) &\cong H_{k,\text{Grp}}(\pi_1(M), A)
\end{aligned}$$

for $k < r$.

This trivial case is explained in [Die89, Sec. 3.V.1.D, Eq. 38]. It has the following generalization: Consider a $\pi_1(M)$-module A (with a possibly non-trivial $\pi_1(M)$-action). As M is arc-connected, a locally constant sheaf on M has the same fiber everywhere. Now, consider a locally constant sheaf $\mathcal{A}$ with constant fiber $\mathcal{A}_p = A$ for all $p \in M$. Such a sheaf is called a *local coefficient system*. Then, we have the following statement for comparing singular and group cohomology.

Theorem C.4. *Let M be aspherical up to r. Then, the group (co)homology and the singular (co)homology with local coefficients are isomorphic up to r. This means that*

$$H^k_{\mathrm{sing}}(M, \mathcal{A}) \cong H^k_{\mathrm{Grp}}(\pi_1(M), A)$$
$$H_{k,\mathrm{sing}}(M, \mathcal{A}) \cong H_{k,\mathrm{Grp}}(\pi_1(M), A)$$

for all $k < r$.

Unfortunately, I could not find a complete reference for this theorem, but [Eil47, Thm. 16.1] might be a start. A proper treatment of this topic, perhaps using the general approach via sheaf cohomology, might make up an interesting thesis on its own.

Finally, singular cohomology with local coefficients can also be compared to the de Rham-cohomology.

Theorem C.5 (Generalized de Rham-Theorem)**.** *Let M be a compact connected smooth manifold. Then, de Rham-cohomology with coefficients is isomorphic to the singular cohomology with local coefficients. More precisely, let $E \longrightarrow M$ be a flat vector bundle with fiber A, and $\mathcal{A}$ a local coefficient system as before. Then, we have isomorphisms*

$$H^k_{\mathrm{dR}}(M, E) \cong H^k_{\mathrm{sing}}(M, \mathcal{A})$$

for all k.

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