Phase space of billiards with and without magnetic field as symplectic quotient

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Abstract

The system considered is the motion of a charged point mass within a compact, connected subset of the euclidean plane with non-empty interior, bounded by a curve diffeomorphic to the unit circle, with constant magnetic field perpendicular to the plane and with reflection at the boundary according to the law of reflection. Two models are introduced, which both describe the motion in terms of the iteration of a smooth map preserving a symplectic stucture. Both models are discussed for zero and non-zero magnetic field. For the Birkhoff billiards model, the related concept of generating functions is discussed. The orbit dynamics model is generalized to higher dimensions. The equivalence of both models in the two-dimensional case is shown.

Zusammenfassung

Betrachtet wird die Bewegung eines geladenen Massepunktes innerhalb einer kompakten, zusammenhängenden Teilmenge der euklidischen Ebene mit nichtleerem Inneren, berandet durch eine Kurve welche diffeomorph zum Einheitskreis ist, mit einem konstanten Magnetfeld senkrecht zur Ebene, und mit Reflektion am Rand nach dem Reflektionsgesetz. Zwei Modelle werden vorgestellt, beide beschreiben die Bewegung in Form von der Iteration einer glatten Abbildung welche eine symplektische Struktur erhält. Für das Birkhoff-Billard-Modell wird das verwandte Konzept der erzeugenden Funktion diskutiert. Das Orbit-Dynamik-Modell wird auf höhere Dimensionen verallgemeinert. Die Äquivalenz beider Modelle im zweidimensionalen Fall wird gezeigt.

Contents

Introduction

The object of study in this text will be mathematical billiards. In two-dimensional space, this can be described as follows: Consider a non-empty compact connected subset $\Omega \subseteq \mathbb{R}^2$ whose boundary is a smooth connected manifold of codimension 1. The set Ω shall be called the *billiards table*, or just *table* for short. Consider the following flow within the billiards table, which can be imagined as the movement of a point mass: For an initial position $x \in \text{int}(\Omega)$ and an initial velocity $v \in \mathbb{R}^2$, the point mass moves along a straight line

$$t \mapsto x + tv$$

as long as it stays in the interior of the table. Here, t can be understood as the time. As soon as the point mass reaches the table boundary, the flow goes on by elastic collision of the point mass at the boundary, such that the angle of incidence is equal to the angle of reflection (see Fig. 1).

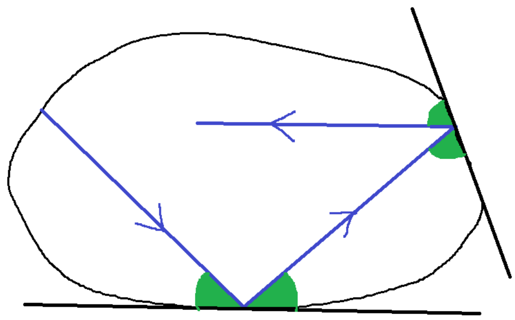


Figure 1: Billiards in 2 dimensions with straight line flow

The reflection law can also be described as follows: The velocity vector v_{in} before reflection can be split up into a component tangential to the edge curve of the table, and into an orthogonal component (with respect to the standard scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^2$):

$$v_{\text{in}} = v_{\|} + v_{\perp}.$$

The velocity v_{refl} after reflection can then be computed by reversing the sign of the orthogonal component:

$$v_{\text{refl}} = v_{\|} - v_{\perp}.$$

Using the outward pointing unit normal vector n of the boundary curve of the table at the point of reflection, the orthogonal component $v_{\perp}$ can be given as

$$v_{\perp} = \langle v_{\text{in}}, n \rangle n.$$

Then

$$v_{\text{refl}} = v_{\text{in}} - 2v_{\perp} = v_{\text{in}} - 2\langle v_{\text{in}}, n \rangle n$$

and thus

$$\|v_{\text{refl}}\|_2 = \|v_{\text{in}} - 2\langle v_{\text{in}}, n \rangle n\|_2 = \sqrt{\|v_{\text{in}}\|_2^2 - 2 \cdot 2\langle v_{\text{in}}, n \rangle \langle v_{\text{in}}, n \rangle + 4\langle v_{\text{in}}, n \rangle^2 \|n\|_2^2} = \|v_{\text{in}}\|_2,$$

so the absolute value of the velocity does not change by reflection. Using this description, the reflection law can be generalized to higher dimensions.

Apart from the flow of the point mass along straight lines, which shall be called *straight line flow*, or SLF for short, the *circular flow* or CF for short, is also considered: Here, after fixing a radius $R > 0$ and a rotational direction (i.e. clockwise or counterclockwise), the point mass moves along circular arcs of radius R in the fixed rotational direction, whereby the absolute value of the velocity remains constant. If the mass reaches the table boundary, it gets reflected according to the same reflection law as described above, and continues on a new circular arc with same radius R and same rotational direction. This radius R shall be called *Larmor radius*, following the nomenclature in [1, Section 1 and 2] or [2, Section 1.1]. The circular flow is depicted in Fig. 2.

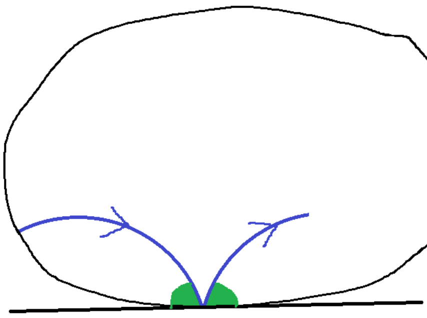


Figure 2: Billiards in 2 dimensions with circular flow

For the circular flow, the data of the radius and the rotational direction can be encoded in the *magnetic field* $B \neq 0$, where

$$R = \frac{\|v\|_2}{|B|},$$

and the rotational direction is given by the sign of B, where it is taken to be counterclockwise if $B > 0$ and clockwise if $B < 0$. In physics, the symbol B usually denotes magnetic flux density. This similarity of notation is intentional because the circular flow can be physically motivatied by considering the point mass to be a charged particle moving in the $\mathbb{R}^2$-plane while there is a constant magnetic field perpendicular to this plane. Details of the physical motivation can be found in section 2.3. Because of this, the CF case will also be called the *magnetic case* or *magnetic billiards*. The case $B = 0$ can then be considered to be the limit case $R \to \infty$ which corresponds to the straight line flow. Although such a limit process will not be explicitly considered in this text, it helps to keep this in mind to see the connection between SLF and CF.
Given an initial position in the interior of the table and an initial velocity, the resulting flow curve of the point mass shall be called *billiard trajectory* or *trajectory* for short. Observe that the absolute value of the velocity remains constant along the entire trajectory. If the initial velocity is zero, then the trajectory is for ever stationary; this case shall not be considered. Neither shall the trajectories of the circular flow be considered where no reflections with the boundary occur, i.e. trajectories described by a closed circle in the interior of the table.
Furthermore, to avoid the occurence of "degenrate" trajectories which become tangential to the table boundary (see Fig. 3), a certain *regularity condition* is imposed on the billiards table.

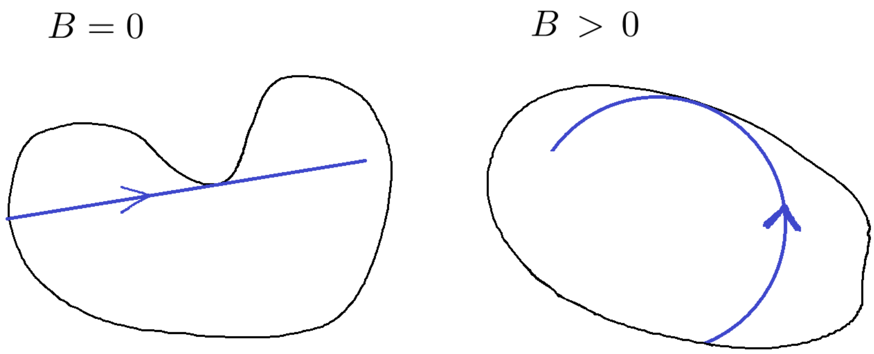


Figure 3: "degenerate" trajectories for SLF and CF

Fixing the absolute value of the velocity, the billiard trajectories can be described as a discrete dynamics of the positions and velocities at the table boundary. This is done in the *Birkhoff billiards model*, which is the topic of the first part of this text.
The second part of this text deals with an alternative description: The billiard trajectories are seen as a discrete dynamics of the trajectory segments between the reflections, or rather of the extensions of these segments: For SLF, a discrete dynamics of oriented lines is considered, and for CF, it is a discrete

dynamics of circles of radius R. This description shall be called the *orbit dynamics model*. Whereas the Birkhoff billiards model only refers to dimension 2, the orbit dynamics model is formulated in the setting of higher dimensions.

The starting point of the orbit dynamics model is the description of the billiards flow as a Hamiltonian flow in $T^*\mathbb{R}^N \cong \mathbb{R}^N \times \mathbb{R}^N$, which induces an $\mathbb{R}$-action in the case of SLF, or an $\mathbb{S}^1$-action in the case of CF. The orbit space is then obtained by a quotient construction, which is described in a more general setting in [6, Chapter 5.4]. This construction is detailed and applied to the specific case of the straight line flow and circular flow. Finally, the two-dimensional case of the orbit dynamics model is compared with Birkhoff billiards.

In both models, the discrete dynamics is decribed by a smooth invertible map T, called *billiards map*, from a smooth manifold to itself:

$$T : PS \longrightarrow PS.$$

Call this manifold PS the *phase space*. In both models, the phase space has a symplectic sturcture: It can be equipped with a *symplectic form*, i.e. a closed, non-degenerate 2-form. This form is preserved by the billiards map, which makes the map T a *symplectomorphism*. Moreover, the natural correspondence of the two models in dimension 2 preserves this symplectic structure.

1 Birkhoff billiards in dimension 2

1.1 Phase space and billiard map

Let the billiards table be a compact connected subset $\Omega \subseteq \mathbb{R}^2$ such that $\partial\Omega$ is a smooth 1-manifold diffeomorphic to $\mathbb{S}^1$, i.e. a smooth closed curve without self-intersections with a finite length $L > 0$. To simplify notation, write

$$\mathbb{S}^1_L := \mathbb{R}/L\mathbb{Z}$$

in the rest of this text. Describe the boundary $\partial\Omega$ by the curve

$$\gamma : \mathbb{S}^1_L \longrightarrow \mathbb{R}^2$$

which is parametrized by unit length, and runs along $\partial\Omega$ in counterclockwise orientation.
Fix the absolute value of the velocity of billiard trajectories to 1. Thus for the circular flow, the Larmor radius is simply given by

$$R = \frac{1}{|B|}.$$

For the straight line flow, use the notion that $B = 0$ and $R = +\infty$. The *radius of curvature* of γ is given by

$$\rho : \mathbb{S}^1_L \longrightarrow \mathbb{R} \cup \{+\infty\}, \quad \rho(l) := \frac{1}{\|\gamma''(l)\|_2}.$$

Here, the value of positive infinity is taken if the second derivative of γ vanishes. The following regularity condition is inspired by [2, Section 1.1].

Definition 1.1 (Regularity condition)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be a billiards table, i.e. a compact connected subset of $\mathbb{R}^2$ with boundary diffeomorphic to $\mathbb{S}^1$. Then Ω is said to fulfil the* regularity condition *if*

$$\max_{l \in \mathbb{S}^1_L} \rho(l) < R \quad \text{or} \quad \min_{l \in \mathbb{S}^1_L} \rho(l) > R.$$

Proposition 1.2. *For the case of straight line flow, the regularity condition is equivalent to asking that Ω be a convex set.*

Proof. Since $R = +\infty$, it must hold that

$$\rho(l) < +\infty \quad \forall l \in \mathbb{S}^1_L,$$

which is the same as asking that γ'' never vanishes. Now because γ is parametrized by unit length, it holds that

$$0 = \frac{\mathrm{d}}{\mathrm{d}l}\langle\gamma'(l), \gamma'(l)\rangle = 2\langle\gamma'(l), \gamma''(l)\rangle,$$

so γ'' is perpendicular to γ'. Considering that the curve γ bounds the connected convex set Ω, the second derivative γ'' can be geometrically described as "inwards pointing" or "outwards pointing". By continuity of γ'', transitioning from inwards to outwards pointing would require that γ'' vanishes at some value of $l \in \mathbb{S}^1_L$. But this is forbidden by the regularity condition. Thus, the regularity condition for straight line flow can be fulfilled if and only if γ'' is always inwards pointing, and this equivalent to the condition that Ω is a convex set. □

Because of following remark, degenerate trajectories as in Fig. 3 do not occur:

Remark 1.3. *Let $\Omega \subseteq \mathbb{R}^2$ be a billiards table fulfilling the regularity condition. For $B \neq 0$, for a circle with radius equal to the Larmor radius $R = 1/|B|$, exactly one of following three cases can occur:*

- *The circle does not have any common points with the table boundary $\partial\Omega$.*
- *The circle has exactly one common point with the table boundary $\partial\Omega$, and at this point the circle touches $\partial\Omega$, i.e. the intersection is not transversal.*
- *The circle has two distinct common points with $\partial\Omega$, and the intersection is transversal in both points.*

In particular, if a circle intersects $\partial\Omega$ transversally, then it intersects $\partial\Omega$ in exactly two points, and transversally in both points.
Similarly, if $B = 0$, the analogous statement is valid for straight lines instead of circles.

Idea of Proof. If $B = 0$ then Ω is a convex set, and this implies that the above statement is valid for straight lines. So consider the case $B \neq 0$.
Use the terminology that a circle *intersects* $\partial\Omega$ if, while going along the circle, there is a transition from inside $\partial\Omega$ to outside $\partial\Omega$, and that a circle *touches* $\partial\Omega$ if there is no such transition. Observe that at a touching point, the tangents of $\partial\Omega$ and the circle agree, but at an intersection point, in general, the tangents might or might not agree: The intersection might be transversal (i.e. the tangents do not agree) or non-transversal (i.e. the tangents agree). Also observe that every common point of a circle with $\partial\Omega$ must fall into one of these three categories: touching point, point of transversal or of non-transversal intersection.
Also use the terminology that two common points of a circle with $\partial\Omega$ are *consecutive* if, while following the circle in counterclockwise orientation, one intersection point comes directly after the other.
First, prove the fact that a circle of Larmor radius R cannot have a non-transversal intersection point with $\partial\Omega$. For this, let $m \in \mathbb{R}^2$ and consider the function

$$d_m : \mathbb{S}^1_L \longrightarrow \mathbb{R}, \quad l \mapsto \|\gamma(l) - m\|_2{}^2 - R^2 .$$

The first and second derivatives of this function are

$$d_m'(l) = 2\langle \gamma'(l) \, , \, \gamma(l) - m \rangle \quad \text{and} \quad d_m''(l) = 2(\langle \gamma''(l) \, , \, \gamma(l) - m \rangle + 1) .$$

If a circle with Larmor radius R centered at m has a non-transversal intersection with $\partial\Omega$ at $\gamma(l_0)$, then because it is an intersection, the function d_m has a zero with change of sign at l_0. Furthermore, because the intersection is non-transversal, $\gamma'(l_0) \perp \gamma(l_0) - m$. This in turn implies $d_m'(l_0) = 0$. But this means that d_m has a *saddle point* at l_0, i.e. $d_m'(l_0) = d_m''(l_0) = 0$. Now because γ is parametrized by unit length, $\gamma'' \perp \gamma'$, which implies $\gamma''(l_0) \parallel \gamma(l_0) - m$, thus $d_m''(l_0) = 0$ implies

$$\langle \gamma''(l_0) \, , \, \gamma(l_0) - m \rangle = -1 = -\|\gamma''(l_0)\|_2 \cdot \underbrace{\|\gamma(l_0) - m\|_2}_{=R} .$$

This can only be fulfilled if $\rho(l_0) = 1/\|\gamma''(l_0)\|_2 = R$, but this is a contradiction to the regularity condition.
Now, if a circle of Larmor radius has only one common point with $\partial\Omega$, then this cannot be an intersection, because an intersection with a transition from inside $\partial\Omega$ to outside $\partial\Omega$, or vice versa, implies that another transition in the other dircetion must occur, so there would be at least another intersection point.
Also, the case that there is an open interval $I \subseteq \mathbb{S}^1_L$ of common points with a circle of Larmor radius (i.e. for all $l \in I$, the point $\gamma(l)$ lies on the circle) can be excluded, because in this case, the part of $\partial\Omega$ described by $\gamma\big|_I$ would be a circular arc with Larmor radius, so the radius of curvature would be $\rho(l) = R$ for all $l \in I$, which contradicts the regularity condition.
The last case which remains to be considered is the case of three or more common points. The idea for this: Assume that a circle C of Larmor radius has three or more common points with $\partial\Omega$. Then by moving C such that "two of the common points move together", it should be possible to find two circles of Larmor radius C_{in} and C_{out} which touch $\partial\Omega$, but are locally inside / locally outside Ω, respectively: If all common points are transversal intersection points, if $\gamma(l_1), \gamma(l_2)$ and $\gamma(l_3)$ are three consecutive transversal intersection points, then C_{in} and C_{out} can be obtained by on the one hand "moving together" $\gamma(l_1)$ and $\gamma(l_2)$, and on the other hand by "moving together" $\gamma(l_2)$ and $\gamma(l_3)$. If one of the intersection points already is a touching point, say $\gamma(l_1)$, then C is one of the circles C_{in} and C_{out}, and if $\gamma(l_2)$ is the next common point after $\gamma(l_1)$, then the other circle is obtained by "moving $\gamma(l_1)$ and $\gamma(l_2)$ together".
But if C_{in} and C_{out} can be constructed, then at the points l_{in} and l_{out} where these circles touch $\partial\Omega$, the curvature fulfils

$$\rho(l_{\text{in}}) \geqslant R \quad \text{and} \quad \rho(l_{\text{out}}) \leqslant R ,$$

so there must be a point l_0 such that $\rho(l_0) = R$, which contradicts the regularity condition. $\square$

From this point on, assume that the billiards table Ω fulfils the regularity condition. Now, every trajectory can be described as a discrete dynamics of positions and velocities at the table boundary. This description of the billiard dynamics is called *Birkhoff billiards.* The *phase space of Birkhoff billiards* is given by

$$\mathrm{PS}_{\text{Birk}} := \mathbb{S}^1_L \times (0; \pi),$$

this is a smooth 2-manifold. The coordinates (l, α) describe the billiard trajectory leaving the point $\gamma(l)$ of the table boundary at an angle of α relative to the tangent vector $\gamma'(l)$, and then following the respective billiard flow (see Fig. 4).

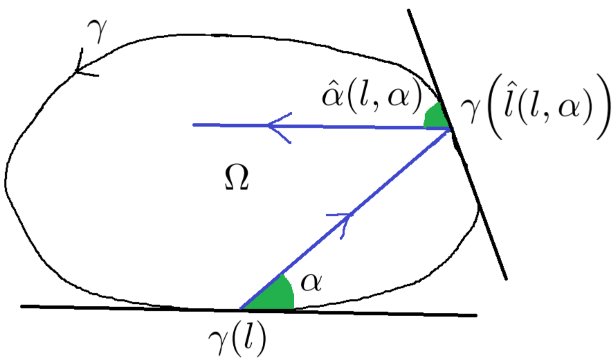


Figure 4: Birkhoff billiards

Now, fixing a value for B, the billiard map is defined on this phase space:

$$T_B : \mathrm{PS}_{\mathrm{Birk}} \longrightarrow \mathrm{PS}_{\mathrm{Birk}}, \quad (l,\alpha) \mapsto \left(\hat{l}(l,\alpha), \hat{\alpha}(l,\alpha)\right).$$

Here, $\hat{l}$ and $\hat{\alpha}$ are the component funtions of T_B, where $\hat{l}(l,\alpha)$ and $\hat{\alpha}(l,\alpha)$ describe the position and velocity at the next boundary point after (l,α), respectively (see Fig. 4).

Proposition 1.4. *Let $\Omega \subseteq \mathbb{R}^2$ be a billiards table fulfilling the regularity condition, and let T_B be the associated billiard map. Then T_B is a diffeomorphism.*

Proof. The billiard trajectory segment starting with position and velocity as described by $(l,\alpha) \in \mathrm{PS}_{\mathrm{Birk}}$ describes either a line segment (if $B = 0$) or a circular arc of a circle with Larmor radius (if $B \neq 0$). According to Remark 1.3, the intersection of this billiard trajectory segment with $\partial\Omega$ at the second point is transversal, thus the map T_B is well-defined. The continuity of T_B is apparent from the fact that "degenrate" trajectories (see Fig. 3) are not possible, and thus no discontinuities occur.
Smoothness of T_B is apparent, since $\partial\Omega$ is smooth and of finite length, and a smooth variation of the coordinates (l,α) leads to a smooth variation of the line segment (for $B = 0$) or the circular arc (for $B \neq 0$), and thus also the second point of intersection and the angle of intersection at this point smoothly vary, such that the component functions $\hat{l}$ and $\hat{\alpha}$ are smooth.
To see that T_B is a diffeomorphism, define the smooth involution

$$r : \mathrm{PS}_{\mathrm{Birk}} \longrightarrow \mathrm{PS}_{\mathrm{Birk}}, \quad (l,\alpha) \mapsto (l, \pi - \alpha),$$

then the inverse map can be described as

$${T_B}^{-1} = r \circ T_{-B} \circ r.$$

Note that the regularity condition depends only on the absolute value of B, thus if Ω fulfils the regularity condition with respect to a magnetic field B, then it also does this with respect to magnetic field $-B$. Thus T_{-B} is well-defined and smooth as well, so also ${T_B}^{-1}$ is smooth. This proves that T_B is a diffeomorphism. □

1.2 Symplectic structure and generating function

Define the 2-form

$$\omega_{\mathrm{Birk}} := \sin\alpha \, \mathrm{d}\alpha \wedge \mathrm{d}l$$

on $\mathrm{PS}_{\mathrm{Birk}}$, this is taken from [7, Section 1.2].

Proposition 1.5. $(\mathrm{PS}_{\mathrm{Birk}}, \omega_{\mathrm{Birk}})$ *is a symplectic manifold.*

Proof. Because $\mathrm{PS}_{\mathrm{Birk}}$ is of dimension 2, it is sufficient to prove that ω_{Birk} does not vanish anywhere. But $\mathrm{d}\alpha \wedge \mathrm{d}l$ does not vanish anywhere, and because $\alpha \in (0;\pi)$, it holds that $\sin\alpha \neq 0$. □

The next aim is to prove that T_B preserves the symplectic structure, this can be done by means of a *generating function*.

Proposition 1.6. *Let*

$$T : \mathrm{PS_{Birk}} \longrightarrow \mathrm{PS_{Birk}}, \quad (l, \alpha) \mapsto \Big(\hat{l}(l, \alpha), \hat{\alpha}(l, \alpha)\Big)$$

be a smooth map with component functions $\hat{l}$ *and* $\hat{\alpha}$*, and define the diagonal* $\Delta := \big\{(l, l) \big|\, l \in \mathbb{S}^1_L\big\}$*. Let* $U \subseteq \big(\mathbb{S}^1_L \times \mathbb{S}^1_L\big)\backslash\Delta$ *be open such that*

$$\forall (l, \alpha) \in \mathrm{PS_{Birk}} : \quad \Big(l, \hat{l}(l, \alpha)\Big) \in U.$$

Then a function

$$G : U \longrightarrow \mathbb{R} \quad \textit{with} \quad \partial_2 G\Big(l, \hat{l}(l, \alpha)\Big) = -T^*\Big(\partial_1 G\Big(l, \hat{l}(l, \alpha)\Big)\Big) \quad \forall (l, \alpha) \in \mathrm{PS_{Birk}} \tag{1}$$

is called generating function *of* T*, and it fulfils the following property: Define*

$$F : \mathrm{PS_{Birk}} \longrightarrow \mathbb{R}, \quad (l, \alpha) \mapsto G\Big(l, \hat{l}(l, \alpha)\Big)$$

and

$$\lambda := \partial_1 G\Big(l, \hat{l}(l, \alpha)\Big)\, \mathrm{d}l \quad \textit{and} \quad \omega := \mathrm{d}\lambda = \frac{\partial}{\partial\alpha}\Big(\partial_1 G\Big(l, \hat{l}(l, \alpha)\Big)\Big)\, \mathrm{d}\alpha \wedge \mathrm{d}l.$$

Call λ *the associated 1-form of* G *and* ω *the associated 2-form. Then it holds that*

$$\lambda - T^*\lambda = \mathrm{d}F \quad \textit{and} \quad \omega - T^*\omega = \mathrm{d}^2 F = 0,$$

thus T *preserves* ω*.*

Proof. It only needs to be proven that $\lambda - T^*\lambda = \mathrm{d}F$. This follows directly from Eq. (1):

$$\begin{aligned}\lambda - T^*\lambda &= \partial_1 G\Big(l, \hat{l}(l, \alpha)\Big)\, \mathrm{d}l - T^*\Big(\partial_1 G\Big(l, \hat{l}(l, \alpha)\Big)\, \mathrm{d}l\Big) = \partial_1 G\Big(l, \hat{l}(l, \alpha)\Big)\, \mathrm{d}l - T^*\Big(\partial_1 G\Big(l, \hat{l}(l, \alpha)\Big)\Big)\, \mathrm{d}\hat{l} \\ &\overset{(1)}{=} \partial_1 G\Big(l, \hat{l}(l, \alpha)\Big)\, \mathrm{d}l + \partial_2 G\Big(l, \hat{l}(l, \alpha)\Big)\, \mathrm{d}\hat{l} = \mathrm{d}F.\end{aligned}$$

□

The name "generating function" is taken from [7, Section 1.4]. Using above proposition, the task of proving that T_B preserves ω_{Birk} amounts to finding a fitting generating function .For the case of $B = 0$, this is done in [7, Section 1.2], and that proof is reformulated here.

Proposition 1.7. *The function*

$$G_0 : \big(\mathbb{S}^1_L \times \mathbb{S}^1_L\big)\backslash\Delta \longrightarrow \mathbb{R}, \quad (l, l_1) \mapsto \|\gamma(l) - \gamma(l_1)\|_2$$

is a generating function of T_0*, with associated 2-form* ω_{Birk}*. Thus, by Prop. 1.6,* T_0 *preserves* ω_{Birk}*.*

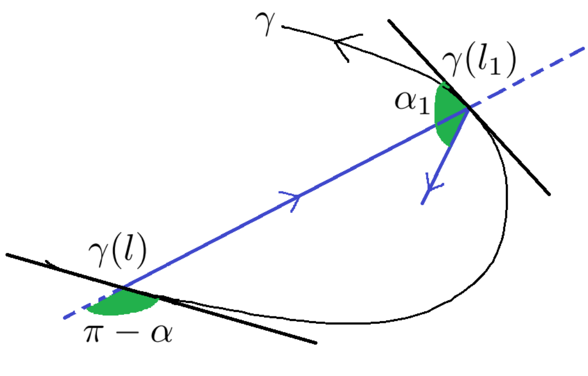


Figure 5: A trajectory segment $(l, \alpha) \overset{T_0}{\mapsto} (l_1, \alpha_1)$ of the straight line flow

Proof. Calculate the partial derivatives of G_0:

$$\partial_1 G_0(l, l_1) = \Bigg\langle \underbrace{\frac{\gamma(l) - \gamma(l_1)}{\|\gamma(l) - \gamma(l_1)\|_2}}_{\text{unit vector from } \gamma(l_1) \text{ to } \gamma(l)}, \gamma'(l) \Bigg\rangle \quad \text{and} \quad \partial_2 G_0(l, l_1) = \Bigg\langle \underbrace{\frac{\gamma(l_1) - \gamma(l)}{\|\gamma(l_1) - \gamma(l)\|_2}}_{\text{unit vector from } \gamma(l) \text{ to } \gamma(l_1)}, \gamma'(l_1) \Bigg\rangle.$$

Because γ is parametrized by unit length, these are scalar products of unit vectors, and their values are given by the cosine of the angle between them. Fix $(l, \alpha) \in \mathrm{PS}_{\mathrm{Birk}}$ and define

$$T_0(l, \alpha) = \Big(\hat{l}(l, \alpha), \hat{\alpha}(l, \alpha)\Big) =: (l_1, \alpha_1),$$

then it follows that

$$\begin{aligned} \partial_2 G_0\Big(l, \hat{l}(l, \alpha)\Big) &= \partial_2 G_0(l, l_1) = \cos \alpha_1 = \cos \hat{\alpha}(l, \alpha) \\ &= -{T_0}^*(-\cos \alpha) = -{T_0}^*(\cos(\pi - \alpha)) = -{T_0}^*\Big(\partial_1 G_0\Big(l, \hat{l}(l, \alpha)\Big)\Big), \end{aligned}$$

see Fig. 5. Thus, Eq. (1) is fulfilled, and so G_0 is a generating function of T_0, with associated 2-form

$$\frac{\partial}{\partial \alpha}\Big(\partial_1 G_0\Big(l, \hat{l}(l, \alpha)\Big)\Big)\, \mathrm{d}\alpha \wedge \mathrm{d}l = \frac{\partial}{\partial \alpha}(-\cos \alpha)\, \mathrm{d}\alpha \wedge \mathrm{d}l = \sin \alpha\, \mathrm{d}\alpha \wedge \mathrm{d}l = \omega_{\mathrm{Birk}}.$$

□

To find a generating function for the CF case, some considerations need to be made: To prove thet T_B preserves ω_{Birk}, a generating function G_B is required such that

$$\partial_1 G_B\Big(l, \hat{l}(l, \alpha)\Big) = -\cos \alpha.$$

Because G_B is a function of two "position coordinates" from $\mathbb{S}^1_L$, this condition implicitly requires that the angle α (or at least its cosine) can be determined as a function of two consecutive length coordinates l and $\hat{l}(l, \alpha)$. For SLF, this is true: Given $l, l_1 \in \mathbb{S}^1_L$ with $l \neq l_1$, these uniquely determine angles $\alpha, \alpha_1 \in (0; \pi)$ such that $T_0(l, \alpha) = (l_1, \alpha_1)$. For CF, it is not always possible to determine such angles, and if they can be determined, the angles are not always unique.

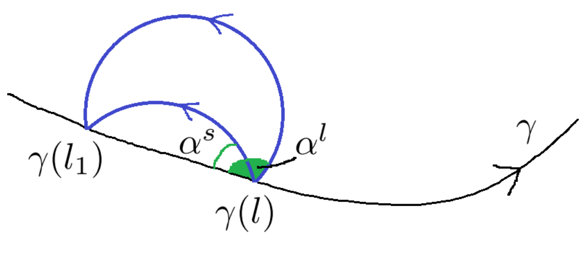


Figure 6: Two possibilities of connecting $\gamma(l)$ and $\gamma(l_1)$ by a billiard trajectory segment for $B > 0$

If the points $\gamma(l)$ and $\gamma(l_1)$ lie apart by a distance greater than $2R$, there is no way of connecting them by a circular arc with Larmor radius R, so no fitting value for α can be found. If the distance is smaller or equal to $2R$, then the points can only be connected if the circular arc from $\gamma(l)$ to $\gamma(l_1)$ following the rotational direction determined by the sign of B lies within the billiards table. If this is the case, then if the distance between $\gamma(l)$ and $\gamma(l_1)$ is exactly $2R$, there is a unique semi-circular arc connecting these points, but if the distance between them is smaller than $2R$, there might be two different possible choices of α, call them α^s and α^l (see Fig. 6).

Notation 1.8. *For $B \neq 0$, the short arc case / long arc case shall refer to situations concerning one iteration of the billiard map T_B, where the trajectory segment described by this iteration corresponds to a short / long arc, respectively. Here, a short arc refers to a circular arc of angle smaller or equal to π, and a long arc refers to a circular arc of angle greater or equal to π. Thus, these cases are not mutually exclusive, but occur at the same time when the circular arc has the angle of exactly π, i.e. when it describes a semicircle.*
The short arc case and the long arc case shall be denoted by a superscript "s" or "l" on all relevant quantities, respectively. To summarily describe both cases at the same time (in the same way that the symbol "±" describes "+" and "−"), the notation "s, l" as a superscript will be used, e.g. $\alpha^{s,l}$ for the two possible choices of an angle α, where α^s refers to the short arc case, and α^l to the long arc case.

Remark 1.9. *If $B \neq 0$ and the regularity condition is met by*

$$\max_{l \in \mathbb{S}^1_L} \rho(l) < R,$$

i.e. if "large radii" or "weak magnetic fields" are considered, then only the short arc case can occur.

Idea of Proof. The condition implies that

$$\rho(l) < R < +\infty \quad \forall l \in \mathbb{S}^1_L$$

which in turn implies that the billiards table is convex, as explained in Prop. 1.2. Furthermore, because the radius of curvature is smaller than R everywhere, Ω is contained in a circle or radius R, i.e. there exists

$$\overline{B_R(p_0)} := \left\{ q \in \mathbb{R}^2 \middle| \|p_0 - q\|_2 \leqslant R \right\} \quad \text{such that} \quad \Omega \subseteq \overline{B_R(p_0)}.$$

Now let $l, l_1 \in \mathbb{S}^1_L$ with $l \neq l_1$, such that a trajectory segment connects points $\gamma(l)$ and $\gamma(l_1)$, then this circular arc of Larmor radius R is also contained in $\overline{B_R(p_0)}$. But a circle of radius R can contain a circular arc of same radius R only if the arc is a short arc. □

In [1, Section 4, Prop. 2] the generating function

$$G = \mathcal{L} + \frac{1}{R}\mathcal{S} \tag{2}$$

is considered, where $\mathcal{L}$ is the length of the circular arc connecting two points on $\partial\Omega$, and $\mathcal{S}$ is the area of that portion of the billiards table which lies to the right of the trajectory segment, see Fig. 7.

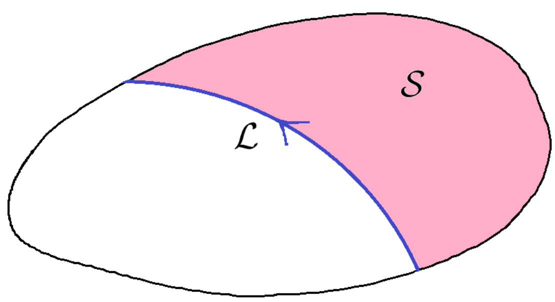


Figure 7: Quantities $\mathcal{L}$ and $\mathcal{S}$ used in the generating function for circular flow

It should be noted that in [1], only the case $B > 0$ is considered[1], and it is explicitly assumed that only the short arc case occurs[2].
In the following text, the possibility of the long arc case shall also be included, as well as the case $B < 0$.

[1]See [1, Section 2], in particular the sign convention $qB < 0$, which implies counterclockwise rotation.
[2]"Suppose that the shape of [the billiards table] is such that it cannot contain any arc of [Larmor radius] and angle larger than π. [...] In such a case, [the billiards map] is called a (symplectic) *twist map*.", see [1, Section 3] The condition that the billiards map is a twist map is one of the prerequisites of [1, Prop. 2 in Section 4].

Inspired by (2), a generating function which can informally be described as "a function of the trajectory segment / circular arc" given by

$$G_B = \mathcal{L} + B \cdot \mathcal{S} \tag{3}$$

is considered. Observe that in the limit case of $B = 0$, this coincides with the generating function G_0 defined in Prop. 1.7.

The next definition introduces some notation to describe the generating function G_B more formally, as a function of the coordinates in $\mathbb{S}^1_L$ of the starting point and ending point of a trajectory segment, while considering both the short arc case and the long arc case.

Definition 1.10. *Define*

$$\mathcal{L}, \mathcal{S} : \mathrm{PS}_{\mathrm{Birk}} \longrightarrow \mathbb{R}$$

as in Fig. 7 with respect to the points (l, α) and $T_B(l, \alpha)$, i.e. $\mathcal{L}$ is the length of the trajectory segment from (l, α) to $T_B(l, \alpha)$, and $\mathcal{S}$ is the area of that part of the billiards table which lies to the right of the trajectory segment from (l, α) to $T_B(l, \alpha)$. Define W^s, W^l as

$$W^{s,l} := \left\{(l, l_1) \in \left(\mathbb{S}^1_L \times \mathbb{S}^1_L\right)\backslash\Delta \,\middle|\, l \text{ and } l_1 \text{ can be connected by a short / long arc trajectory segment}\right\},$$

and define maps

$$\alpha^{s,l} : W^{s,l} \longrightarrow (0; \pi)$$

such that for $(l, l_1) \in W^{s,l}$, the trajectory segment defined by $\big(l, \alpha^{s,l}(l, l_1)\big)$ describes a short / long arc, respectively. Define functions

$${G_B}^{s,l} : W^{s,l} \longrightarrow \mathbb{R}, \quad (l, l_1) \mapsto \mathcal{L}\big(l, \alpha^{s,l}(l, l_1)\big) + B \cdot \mathcal{S}\big(l, \alpha^{s,l}(l, l_1)\big).$$

Furthermore, define maps

$${\alpha_1}^{s,l} : W^{s,l} \longrightarrow (0; \pi), \quad (l, l_1) \mapsto \hat{\alpha}\big(l, \alpha^{s,l}(l, l_1)\big).$$

The above definition can be summarized as follows: If $(l, l_1) \in W^{s,l}$ describe a pair of distinctive position coordinates which can be connected by a short / long arc, then the functions $\alpha^{s,l}$ and ${\alpha_1}^{s,l}$ describe the corresponding angles such that

$$T_B : (l, \alpha^{s,l}(l, l_1)) \mapsto (l_1, {\alpha_1}^{s,l}(l, l_1))$$

is the billiard dynamics, following a short / long arc.

Remark 1.11. *Define*

$$W^b := \big\{(l,l_1) \in \big(\mathbb{S}^1_L \times \mathbb{S}^1_L\big)\backslash\Delta \,\big|\, l \text{ and } l_1 \text{ can be connected by a semicircular trajectory segment}\big\},$$

then

$$W^b \subseteq W^s \cap W^l \quad \text{and} \quad \partial W^s = \partial W^l = \Delta \cup W^b \quad \text{and} \quad W^{s,l} = \operatorname{int}\big(W^{s,l}\big) \cup W^b.$$

The functions $\mathcal{L}$ and $\mathcal{S}$ are smooth, and the functions $\alpha^{s,l}, {\alpha_1}^{s,l}$ and ${G_B}^{s,l}$ are continuous, and are smooth on

$$\operatorname{int}\big(W^{s,l}\big) = W^{s,l}\backslash W^b.$$

Furthermore, following "boundary condition" holds:

$$\alpha^s\big|_{W^b} = \alpha^l\big|_{W^b}, \text{ and thus also } {G_B}^s\big|_{W^b} = {G_B}^l\big|_{W^b} \text{ and } {\alpha_1}^s\big|_{W^b} = {\alpha_1}^l\big|_{W^b}.$$

Consider the space

$$W^{s\cup l} := W^s \coprod_{W^b} W^l$$

which refers to the disjoint union of W^s and W^l "glued together" along the boundary W^b. The resulting space $W^{s\cup l}$ can be understood as a topological 2-manifold, the spaces W^s and W^l can then be understood as subspaces of $W^{s\cup l}$ with an additional differentiable structure.
Because of the boundary condition, the maps α^s and α^l can also be glued along W^b, such that the map

$$\alpha^{s\cup l} : W^{s\cup l} \longrightarrow (0;\pi), \quad w \mapsto \alpha^{s,l}(w) \text{ if } w \in W^{s,l}$$

is well-defined and continuous. The maps ${G_B}^{s\cup l}$ and ${\alpha_1}^{s\cup l}$, which can be constructed in the same manner, are also well-defined and continuous. In particular, all these maps are also defined on W^b.
Now let

$$\mathrm{pr}_1 : \mathbb{S}^1_L \times \mathbb{S}^1_L \longrightarrow \mathbb{S}^1_L, \quad (l,l_1) \mapsto l\ ,$$

then a map ${\mathrm{pr}_1}^{s\cup l}$ can be defined on $W^{s\cup l}$ by gluing $\mathrm{pr}_1\big|_{W^s}$ and $\mathrm{pr}_1\big|_{W^l}$ along W^b. The map

$$\eta_B := \big({\mathrm{pr}_1}^{s\cup l}, \alpha^{s\cup l}\big) : W^{s\cup l} \longrightarrow \mathrm{PS}_{\mathrm{Birk}}, \quad w \mapsto \big({\mathrm{pr}_1}^{s\cup l}(w), \alpha^{s\cup l}(w)\big)$$

is well-defined and continuous, and smooth on $\operatorname{int}\big(W^{s,l}\big)$, respectively. It is also a homeomorphism, with continuous inverse map

$$\eta_B{}^{-1} : \mathrm{PS}_{\mathrm{Birk}} \longrightarrow W^{s\cup l}, \quad (l,\alpha) \mapsto \Big(l, \hat{l}(l,\alpha)\Big) \in W^{s,l} \subseteq W^{s\cup l}.$$

This should be read as follows: If the trajectory segment from $\gamma(l)$ to $\gamma\Big(\hat{l}(l,\alpha)\Big)$ is a short / long arc, $\Big(l, \hat{l}(l,\alpha)\Big)$ is taken to be an element of $W^{s,l}$, respectively.
In the limit case of $B = 0$, following Remark 1.9, only the short arc case can occur. Then

$$W^b = W^l = \emptyset \quad \text{and} \quad W^{s\cup l} = W^s = \big(\mathbb{S}^1_L \times \mathbb{S}^1_L\big)\backslash\Delta,$$

so η_B simplifies to

$$\eta_0 : \big(\mathbb{S}^1_L \times \mathbb{S}^1_L\big)\backslash\Delta \longrightarrow \mathrm{PS}_{\mathrm{Birk}}.$$

By above remark, instead of describing the billiard dynamics as a map on $\mathrm{PS}_{\mathrm{Birk}}$, it can also be described as a map on the space $W^{s\cup l}$.
In the next Lemma, the partial derivatives of ${G_B}^{s,l}$ are computed.

Lemma 1.12. *On $\operatorname{int}\big(W^{s,l}\big)$, it holds that*

$$\mathrm{d}{G_B}^{s,l} = -\cos\alpha^{s,l}\,\mathrm{d}l + \cos{\alpha_1}^{s,l}\,\mathrm{d}l_1.$$

Proof. For $(l,l_1) \in W^{s,l}$, define A to be the area to the right of the line segment from $\gamma(l)$ to $\gamma(l_1)$, and let D be the area between the line segment from $\gamma(l)$ to $\gamma(l_1)$ and the circular arc, see Fig. 8.

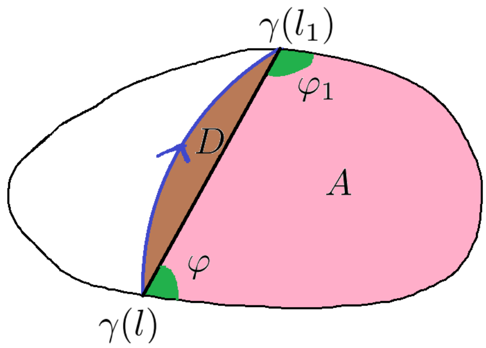


Figure 8: Areas A and D

The area $\mathcal{S}$ can then be described by

$$\mathcal{S} = A - \operatorname{sign}(B) \cdot D.$$

Define

$$C := \mathcal{L} - \frac{1}{R} D = \mathcal{L} - |B| \cdot D,$$

then

$$G_B{}^{s,l} = \mathcal{L} + B\mathcal{S} = \mathcal{L} + B(A - \operatorname{sign}(B) \cdot D) = \mathcal{L} + B \cdot A - |B| \cdot D = B \cdot A + C.$$

Observe that A does not depend on the magnetic field B, nor does it differ for the short arc and long arc cases. Furthermore, C only depends on R and on the distance between $\gamma(l)$ and $\gamma(l_1)$, although it differs for the short arc and long arc cases. Thus, A and C can be defined as functions

$$A : \left(\mathbb{S}^1_L \times \mathbb{S}^1_L\right) \backslash \Delta \longrightarrow \mathbb{R}$$

and

$$C_B{}^{s,l} : (0; 2R) \longrightarrow \mathbb{R}, \quad \text{dist} \mapsto C_B{}^{s,l}(\text{dist}).$$

Remember that the distance between $\gamma(l)$ and $\gamma(l_1)$ is given by $G_0(\gamma(l), \gamma(l_1))$, where G_0 is the generating function for the SLF case, see Prop. 1.7. Then

$$G_B{}^{s,l}(l, l_1) = B \cdot A(l, l_1) + C_B{}^{s,l} \circ G_0(l, l_1).$$

Observe that this is consistent with the case $B = 0$, beacuse in this case $C_0{}^{s,l}(\text{dist}) = \text{dist}$. The partial derivatives of $G_B{}^{s,l}$ are

$$\partial_i G_B{}^{s,l} = B \cdot \partial_i A(l, l_1) + \partial_i G_0 \cdot \left(C_B{}^{s,l}\right)' \circ G_0, \quad \text{for } i = 1, 2. \tag{4}$$

The partial derivatives $\partial_i G_0$ are known from the proof of Prop. 1.7, only the angles need to be named differently. Call the corresponding angles φ and φ_1 as in Fig. 8, then

$$\partial_1 G_0 = -\cos\varphi \quad \text{and} \quad \partial_2 G_0 = \cos\varphi_1. \tag{5}$$

The derivatives of A are

$$\partial_1 A = -\frac{1}{2} G_0 \sin\varphi \quad \text{and} \quad \partial_2 A = \frac{1}{2} G_0 \sin\varphi_1\,, \tag{6}$$

this can be proven using Stokes: Let $\gamma = (\gamma_x, \gamma_y)$ be the coordinate functions of γ. For simplicity, use $A(l, l_1)$ to denote the area as a value (i.e. a non-negative real number) and also as the geometric entity (i.e. a measurable subset of $\mathbb{R}^2$) at the same time. Then

$$\begin{aligned} A(l, l_1) = \int\limits_{A(l,l_1)} \mathrm{d}x \wedge \mathrm{d}y = \int\limits_{\partial A(l,l_1)} x\,\mathrm{d}y &= \int_l^{l_1} \gamma_x(s)\gamma_y'(s)\,\mathrm{d}s + \int_0^1 (t\gamma_x(l) + (1-t)\gamma_x(l_1))(\gamma_y(l) - \gamma_y(l_1))\,\mathrm{d}t \\ &= \int_l^{l_1} \gamma_x(s)\gamma_y'(s)\,\mathrm{d}s + \frac{1}{2}(\gamma_x(l) + \gamma_x(l_1))(\gamma_y(l) - \gamma_y(l_1))\,. \end{aligned}$$

Taking the derivative with respect to the first coordinate yields

$$\begin{aligned}\partial_1 A = \frac{\partial A(l,l_1)}{\partial l} &= -\gamma_x(l)\gamma_y'(l) + \frac{1}{2}\gamma_x'(l)(\gamma_y(l)-\gamma_y(l_1)) + \frac{1}{2}(\gamma_x(l)+\gamma_x(l_1))\gamma_y'(l) \\ &= \frac{1}{2}\langle i(\gamma(l_1)-\gamma(l))\,,\,\gamma'(l)\rangle = -\frac{1}{2}\|\gamma(l_1)-\gamma(l)\|_2\cos\Big(\varphi+\frac{\pi}{2}\Big) = -\frac{1}{2}G_0\sin\varphi\,.\end{aligned}$$

Here, multiplication with i denotes rotation by $\pi/2$ in the counterclockwise direction, i.e. $\mathbb{R}^2$ is identified with $\mathbb{C}$ via the standard complex structure, $i\cdot(x,y) := (-y,x)$.
Taking the derivative of $A(l,l_1)$ with respect to the second coordinate yields

$$\begin{aligned}\partial_2 A = \frac{\partial A(l,l_1)}{\partial l_1} &= +\gamma_x(l_1)\gamma_y'(l_1) + \frac{1}{2}\gamma_x'(l_1)(\gamma_y(l)-\gamma_y(l_1)) - \frac{1}{2}(\gamma_x(l)+\gamma_x(l_1))\gamma_y'(l_1) \\ &= -\frac{1}{2}\langle i(\gamma(l)-\gamma(l_1))\,,\,\gamma'(l_1)\rangle = -\frac{1}{2}\|\gamma(l)-\gamma(l_1)\|_2\cos\Big(\varphi_1+\frac{\pi}{2}\Big) = \frac{1}{2}G_0\sin\varphi_1\,.\end{aligned}$$

To compute the derivative of ${C_B}^{s,l}$, write

$${C_B}^{s,l} = {\mathcal{L}_B}^{s,l} - \frac{1}{R}{D_B}^{s,l}.$$

Here, ${\mathcal{L}_B}^{s,l}$ and ${D_B}^{s,l}$ are meant to refer to the same geometric quantities as defined by $\mathcal{L}$ and D, but here are viewed as functions only of the distance "dist" between the points $\gamma(l)$ and $\gamma(l_1)$, and depending on R. Then with

$$\delta := \arcsin\Big(\frac{\text{dist}}{2R}\Big),$$

see Fig. 9, it holds that

$${\mathcal{L}_B}^{s} = 2\delta R \quad\text{and}\quad {\mathcal{L}_B}^{l} = 2(\pi-\delta)R.$$

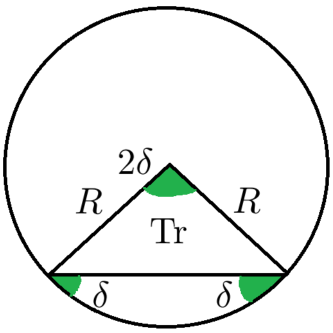


Figure 9: Angle δ and area of triangle Tr

Moreover, with the area of the triangle Tr as in Fig. 9, it holds that

$$\text{Tr} = \frac{R}{2}\text{dist}\cdot\sqrt{1-\Big(\frac{\text{dist}}{2R}\Big)^2}$$

and

$${D_B}^{s} = \delta R^2 - \text{Tr} \quad\text{and}\quad {D_B}^{l} = (\pi-\delta)R^2 + \text{Tr}.$$

Taking the expressions for ${\mathcal{L}_B}^{s,l}$ and ${D_B}^{s,l}$ together and inserting them in the expression for ${C_B}^{s,l}$, while using the expression for Tr, it follows that

$${C_B}^{s} = \delta R + \frac{\text{dist}}{2}\cdot\sqrt{1-\Big(\frac{\text{dist}}{2R}\Big)^2} \qquad\text{and}\qquad {C_B}^{l} = (\pi-\delta)R - \frac{\text{dist}}{2}\cdot\sqrt{1-\Big(\frac{\text{dist}}{2R}\Big)^2}\,.$$

With

$$\frac{\partial}{\partial\,\text{dist}}\delta = \frac{1}{\sqrt{1-\big(\frac{\text{dist}}{2R}\big)^2}}\cdot\frac{1}{2R},$$

it follows that

$$\left(C_B{}^{s,l}\right)'(\text{dist}) = \text{sign}^{s,l}\sqrt{1-\left(\frac{\text{dist}}{2R}\right)^2}\,, \tag{7}$$

where

$$\text{sign}^s = +1 \quad \text{and} \quad \text{sign}^l = -1.$$

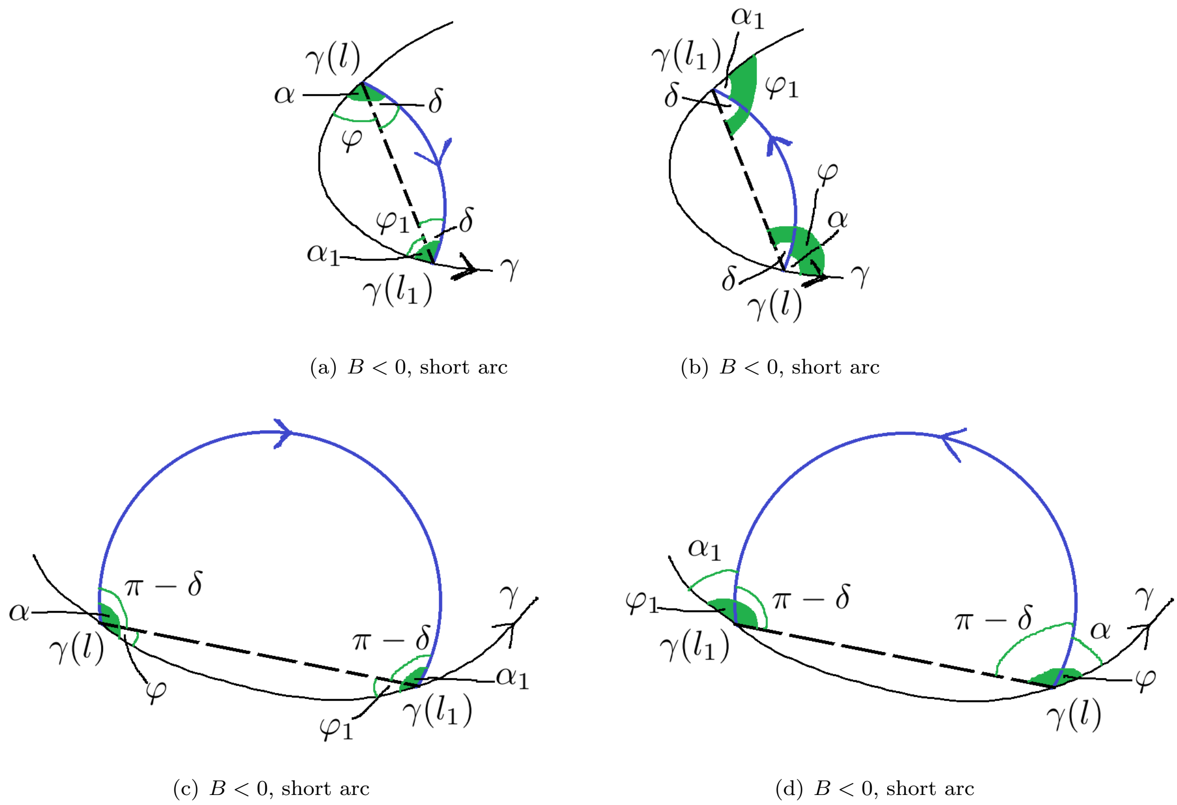
Using the fact that $B = \text{sign}(B)/R$ and that $\sqrt{1-\left(\frac{\text{dist}}{2R}\right)^2} = \cos\delta$, and inserting Eq. (5), (6) and (7) in (4), the derivatives of $G_B{}^{s,l}$ are

$$\partial_1 G_B{}^{s,l} = -\text{sign}(B)\sin\delta\sin\varphi - \text{sign}^{s,l}\cos\delta\cos\varphi \quad \text{and} \quad \partial_2 G_B{}^{s,l} = \text{sign}(B)\sin\delta\sin\varphi_1 + \text{sign}^{s,l}\cos\delta\cos\varphi_1\,.$$

By means of the cosine addition formula, these further simplify to

$$\partial_1 G_B{}^{s,l} = -\text{sign}^{s,l}\cos\bigl(\delta - \text{sign}^{s,l}\text{sign}(B)\varphi\bigr) \quad \text{and} \quad \partial_2 G_B{}^{s,l} = \text{sign}^{s,l}\cos\bigl(\delta - \text{sign}^{s,l}\text{sign}(B)\varphi_1\bigr)\,. \tag{8}$$

A certain "abuse of notation" has occured here: The angle "δ" refers to the same geometric quantity throughout, but where it was initially defined as a function of "dist", here in Eq. (8) it is taken to be a function of (l, l_1) via the identification "dist $= G_0(l, l_1)$". This is why the function G_0 does not appear here although it appears in Eq. (4).
The angles and signs in Eq. (8) can be further simplified: Consider the cases $B > 0$ and $B < 0$ separately, also consider the short arc case and long arc case separately for each of these, see Fig. 10.



(a) $B < 0$, short arc

(b) $B < 0$, short arc

(c) $B < 0$, short arc

(d) $B < 0$, short arc

Figure 10: Relations of the angles α, φ and α_1, φ_1 with δ

It follows that, for the short arc case and $B < 0$,

$$\alpha = \varphi + \delta \quad \text{and} \quad \alpha_1 = \varphi_1 + \delta$$

and for the short arc case and $B > 0$,

$$\alpha = \varphi - \delta \quad \text{and} \quad \alpha_1 = \varphi_1 - \delta.$$

For the long arc case and $B < 0$,

$$\alpha = \pi + \varphi - \delta \quad \text{and} \quad \alpha_1 = \pi + \varphi_1 - \delta$$

and for the long arc case and $B > 0$,

$$\alpha = \varphi + \delta - \pi \quad \text{and} \quad \alpha_1 = \varphi_1 + \delta - \pi.$$

Using these identities for the angles, and the cosine rules

$$\forall x \in \mathbb{R}: \ \cos x = \cos(-x) \quad \text{and} \quad \cos(\pi - x) = -\cos x,$$

the partial derivatives of ${G_B}^{s,l}$ simplify to

$$\partial_1 {G_B}^{s,l} = -\cos \alpha^{s,l} - \cos \quad \text{and} \quad \partial_1 {G_B}^{s,l} = \cos {\alpha_1}^{s,l}.$$

This concludes the proof. □

Now to the problem of proving that T_B preserves the symplectic form ω_{Birk}. The proof closely follows the method in Prop. 1.6, except that, instead of having a generating function on an open subset of $\left(\mathbb{S}^1_L \times \mathbb{S}^1_L\right)\backslash\Delta$, the function ${G_B}^{s\cup l}$ defined on $W^{s\cup l}$ is used. This finally justifies using the term "generating function" for ${G_B}^{s\cup l}$.

Proposition 1.13. *Define the function*

$$F_B := {G_B}^{s\cup l} \circ {\eta_B}^{-1} : \mathrm{PS}_{\text{Birk}} \longrightarrow \mathbb{R},$$

where

$${G_B}^{s\cup l} : W^{s\cup l} \longrightarrow \mathbb{R}$$

and η_B are defined as in Remark 1.11. Then

$$\mathrm{d}F_B = -\cos \alpha \, \mathrm{d}l + \cos \hat{\alpha} \, \mathrm{d}\hat{l}.$$

Furthermore, T_B preseves ω_{Birk}.

Proof. Observe that the restriction of ${\eta_B}^{-1}$ to $\operatorname{int}\left(W^{s,l}\right)$, given by

$${\eta_B}^{-1}\big|_{W^{s,l}} : \mathrm{PS}_{\text{Birk}} \longrightarrow W^{s,l}, \quad (l, \alpha) \mapsto \left(l, \hat{l}(l, \alpha)\right)$$

is smooth. Thus, for $(l, \alpha) \in \eta_B\left(\operatorname{int}\left(W^{s,l}\right)\right)$ the function F_B yields

$$F_B(l, \alpha) = G_B^{s,l}\left(l, \hat{l}(l, \alpha)\right),$$

and thus by Lemma 1.12 and chain rule, and using

$$\alpha^{s,l}\left(l, \hat{l}(l, \alpha)\right) = \alpha \quad \text{and} \quad {\alpha_1}^{s,l}\left(l, \hat{l}(l, \alpha)\right) = \hat{\alpha}(l, \alpha),$$

the exterior derivative of F_B at (l, α) is

$$\mathrm{d}F_B\big|_{(l,\alpha)} = -\cos \alpha \, \mathrm{d}l\big|_{(l,\alpha)} + \cos \hat{\alpha}(l, \alpha) \, \mathrm{d}\hat{l}\big|_{(l,\alpha)}. \tag{9}$$

What remains to be considered is the case $(l, \alpha) \in \eta_B\left(W^b\right)$, i.e. when the billiard trajectory segment from (l, α) to $T_B(l, \alpha)$ is a semicircular arc. First, observe that F_B itself is smooth: The functions F_B and ${G_B}^{s\cup l}$ describe the same geometric entity, namely the value of the generating function as described in Eq. (3). The difference is that F_B describes the trajectory segment by coordinates (l, α), and ${G_B}^{s\cup l}$ describes it by coordinates (l, l_1) as well as the information about whether the short arc case or long arc case is being considered. From the geometric construction, it is apparent that the value of F_B smoothly varies with l and α, therefore F_B is smooth.
Next, consider the local topological structure of W^b: By definition,

$$W^b \subseteq \left\{(l, l_1) \in \left(\mathbb{S}^1_L \times \mathbb{S}^1_L\right)\backslash\Delta \,\middle|\, \left\|\gamma(l) - \gamma(l_1)\right\|_2 = R\right\} =: \Delta_R.$$

Now, Δ_R is a 1-submanifold of $\left(\mathbb{S}^1_L \times \mathbb{S}^1_L\right)\backslash\Delta$, in particular, for any $(l,\alpha) \in \eta_B\left(W^b\right)$, there are sequences $\{(l_n,\alpha_n)\}_{n\in\mathbb{N}}$ in $\eta_B\left(\operatorname{int}\left(W^{s,l}\right)\right)$ such that

$$\lim_{n\longrightarrow\infty}(l_n,\alpha_n) = (l,\alpha)\,.$$

Considering that the component functions $\hat{l}$ and $\hat{\alpha}$ of T_B are smooth, by continuity of the various functions and 1-forms in Eq. (9) it follows that

$$\mathrm{d}F_B\big|_{(l,\alpha)} = \lim_{n\longrightarrow\infty}\mathrm{d}F_B\big|_{(l_n,\alpha_n)} = -\cos\alpha\,\mathrm{d}l\big|_{(l,\alpha)} + \cos\hat{\alpha}(l,\alpha)\,\mathrm{d}\hat{l}\big|_{(l,\alpha)}.$$

Now, taking the exterior derivative of the equation

$$\mathrm{d}F_B = -\cos\alpha\,\mathrm{d}l + \cos\hat{\alpha}\,\mathrm{d}\hat{l}$$

results in

$$0 = \mathrm{d}^2F_B = \sin\alpha\,\mathrm{d}\alpha\wedge\mathrm{d}l - \sin\hat{\alpha}\,\mathrm{d}\hat{\alpha}\wedge\mathrm{d}\hat{l} = \omega_{\mathrm{Birk}} - {T_B}^*\omega_{\mathrm{Birk}},$$

thus T_B preserves ω_{Birk}. □

2 Billiards as a dynamics of the orbit space

The regularity condition in Def. 1.1 guarantees that any straight line intersecting the billiards table boundary transversally has exactly two points of intersection, and both intersections are transversal. Similarly, for CF, any circle of Larmor radius R transversally intersecting the table boundary has exactly two intersections with it, both of which are transversal.
This makes it possible to think of the billiard dynamics as a discrete dynamics of an orbit space: These are oriented lines for SLF, and circles of Larmor radius R for CF, see Fig. 11.

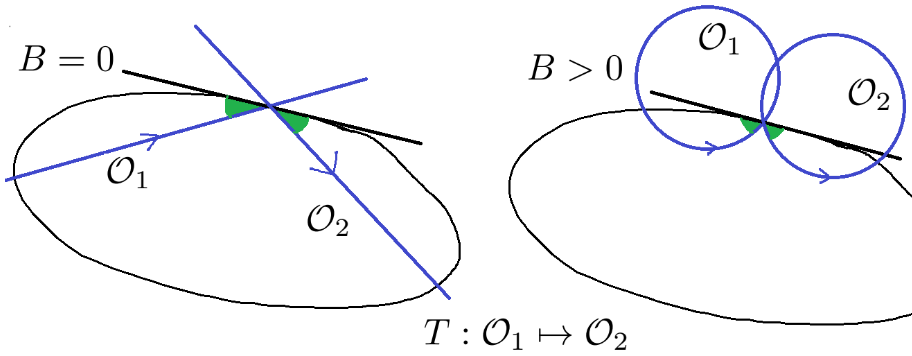


Figure 11: The billiard map as discrete dynamics of circles (for CF) or oriented lines (for SLF)

The aim of this section is to formalize this idea. For this, the flow in the surrounding space (i.e. not restricted to the billiards table) is described as a Hamiltonian flow of a Hamiltonian system, and then a quotient construction leads to the description of the orbit space of this flow. The billiard map then operates on an open subset of this space, namely the subset of all orbits which intersect the billiards table transversally. As in section 1, a regularity condition for the billiards table is formulated.

2.1 Moment maps and symplectic quotients

The following statements about the theory of symplectic quotients, leading up to and including Prop. 2.2, except for Notation 2.1, are paraphrased and summarized from [6, Section 5.2 to 5.4].
Consider a symplectic manifold (M, ω) as well as a smooth Lie group action of G on M, then the action is called *symplectic* if, for every $g \in G$, the map

$$\psi_g : M \longrightarrow, \quad m \mapsto g \cdot m$$

is a symplectomorphism. Let $\mathfrak{g} := \mathrm{Lie}(G)$, the associated Lie algebra. The action of G on M defines a map

$$\mathfrak{g} \longrightarrow \mathfrak{X}(M, \omega), \quad \xi \mapsto X_\xi := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi_{\exp(t\xi)} \,,$$

where

$$\exp : \mathfrak{g} \longrightarrow G$$

is the Lie group exponential, and

$$\mathfrak{X}(M, \omega) := \{X \in \mathfrak{X}(M) \,|\, \omega(X, \cdot) \text{ is closed}\},$$

is the set of *symplectic vector fields.*
A symplectic group action is called *weakly Hamiltonian* if, for every $\xi \in \mathfrak{g}$, the 1-form $\omega(X_\xi, \cdot)$ is exact. If so, then each X_ξ admits a *Hamiltonian function* H_ξ, i.e. H_ξ is a real-valued function on M such that X_ξ is the associated *Hamiltonian vector field* fulfilling

$$\omega(X_\xi, \cdot) = \mathrm{d}H_\xi.$$

This equation defines the Hamiltonian functions H_ξ up to a constant, and the constants can be chosen in such a way that the map

$$\mathfrak{g} \longrightarrow C^\infty(M), \quad \xi \mapsto H_\xi$$

is linear. Choose constants for the functions H_ξ in this way.
A weakly Hamiltonian action is called *Hamiltonian* if, for $\xi \in \mathfrak{g}$ and $g \in G$, it holds that

$$H_\xi \circ \psi_g = H_{\mathrm{Ad}(g^{-1})\xi} \, .$$

Here,

$$\mathrm{Ad}(g^{-1}) : \mathfrak{g} \longrightarrow \mathfrak{g}, \qquad \xi \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g^{-1} \exp(t\xi) g$$

is the adjoint action of g^{-1}.
Now consider a Hamiltonian action of a Lie group G on a symplectic manifold (M, ω), then a *moment map* is a smooth map

$$\mu : M \longrightarrow \mathfrak{g}^*$$

which fulfills the condition that

$$\langle \mu(p), \xi \rangle = H_\xi(p),$$

where the brackets "$\langle \cdot, \cdot \rangle$" denote the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.
The coadjoint action of $g \in G$ is given by

$$\mathrm{Ad}^*(g) : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*, \qquad \xi^* \mapsto \big(\eta \mapsto \big\langle \xi^*, \mathrm{Ad}(g^{-1})\eta \big\rangle\big),$$

or, equivalently,

$$\langle \mathrm{Ad}^*(g)\xi^*, \eta \rangle := \big\langle \xi^*, \mathrm{Ad}(g^{-1})\eta \big\rangle \, .$$

Now, to be able to formulate a proposition about symplectic quotients more precisely, the following notation will be useful:

Notation 2.1. *Consider a family of isomorphic vector spaces $\{V_i\}_{i \in I}$, where I is an index set, as well as an associated family of linear isomorphisms $\{F_{ij} : V_i \longrightarrow V_j\}_{i,j \in I}$ such that*

$$F_{ii} = \mathrm{id}_{V_i} \quad \textit{and} \quad F_{jk} \circ F_{ij} = F_{ik} \quad \textit{for all } i, j, k \in I.$$

Call this family of vector spaces, together with the associated family of linear isomorphisms, an associative vector space family*.*
Call any other vector space V, together with linear isomorphisms $\{F_i : V \longrightarrow V_i\}_{i \in I}$ with the property

$$F_{ij} \circ F_i = F_j \quad \textit{for all } i, j \in I, \tag{10}$$

a representation *of the associative vector space family.*
Observe that each member V_{i_0} of the family, together with $\{F_{i_0 i} : V_{i_0} \longrightarrow V_i\}_{i \in I}$, is a representation of the associative vector space family.
A linear map to / from the associative vector space family *is understood as a linear map to / from a member of the family of vector spaces, or a map to / from a representation of the family. Using the fitting associated linear maps, such a map can be understood as a linear map to / from any other representation of the family.*

Proposition 2.2. *Consider a Hamiltonian group action of a Lie group G on a symplectic manifold (M, ω), and let $\mu : M \longrightarrow \mathfrak{g}^*$ be a moment map. Let $\xi^* \in \mathfrak{g}^*$ be a fixed point of the coadjoint action, i.e.*

$$\mathrm{Ad}(g)\xi^* = \xi^* \quad \text{for all } g \in G\,.$$

Then the G-action on M restricts to $Q := \mu^{-1}(\xi^*)$. If the G-action on Q is free and proper, then ξ^* is a regular value of μ, and Q is a coisotropic submanifold, i.e. for all $q \in Q$ it holds that $T_qQ^\omega \subseteq T_qQ$. Furthermore, the G-orbits

$$\mathcal{O}(p) := \{g \cdot p \,|\, g \in G\}$$

for $p \in Q$ are isotropic leaves which foliate Q. (This in particular implies that for $p \in Q$ the G-orbit $\mathcal{O}(p)$ is a submanifold of Q, with $T_q\mathcal{O}(p) = T_qQ^\omega$ for all $q \in \mathcal{O}(p)$.) Moreover, the quotient space*, or* orbit space

$$\overline{Q} := Q/G := \{\mathcal{O}(q) \,|\, q \in Q\}$$

is then a smooth manifold of dimension

$$\dim \overline{Q} = \dim M - 2\dim G$$

which inherits a symplectic structure $\overline{\omega}$ from M, uniquely determined by the condition that $\overline{\pi}^*\overline{\omega} = \omega\big|_Q$. Here,

$$\overline{\pi} : Q \longrightarrow \overline{Q}, \quad q \mapsto [q] := \mathcal{O}(q)$$

is the canonical projection of Q onto the quotient $\overline{Q}$, which is a surjective submersion.
The tangent space of the quotient at a point $[q]$ can be described by the associative vector space family

$$\left\{ \frac{T_rQ}{T_r\mathcal{O}(q)} \right\}_{r \in \mathcal{O}(q)}$$

with well-defined associated linear isomorphisms

$$\mathrm{Lin}_{rs} : \frac{T_rQ}{T_r\mathcal{O}(q)} \longrightarrow \frac{T_sQ}{T_s\mathcal{O}(q)}, \quad [X] \mapsto \left[\mathrm{d}\psi_g\big|_r(X)\right], \quad r, s \in \mathcal{O}(q),$$

where $g \in G$ is unique such that $g \cdot r = s$. The tangent space $T_{[q]}\overline{Q}$ is then taken to be any representation of this associative vector space family.
The differential of $\overline{\pi}$ at $q \in Q$ can be described by

$$\mathrm{d}\overline{\pi}\big|_q : T_qQ \longrightarrow \frac{T_qQ}{T_q\mathcal{O}(q)} \cong T_{[q]}\overline{Q}, \quad V \mapsto [V].$$

Proof. The complete proof will not be stated here, but only the parts which are not found in other sources. For the majority of the proof, see [6, Prop. 5.4.5, Prop. 5.4.13 and the text before it, Prop. 5.4.15]. The proofs of those propositions implicitly use the quotient manifold theorem, which can be found in [5, Theorem 21.10], for instance. From there, the fact that $\overline{\pi}$ is a smooth surjective submersion is taken.
The ideas for the description of the tangent spaces of $\overline{Q}$ are inspired by [6, Proof of Prop. 5.4.5], however, there the description is not as precise as it is possible with Notation 2.1. In particular, no mention of the consistency of the various representations of the tangent space is made, and no condition similar to Eq. (10) is checked. This is the only condition that remains to be checked here: Let $q \in Q$. It needs to be checked that Lin_{rs} are well-defined linear isomorphisms, and that

$$\mathrm{Lin}_{rr} = \mathrm{id}, \quad \mathrm{Lin}_{st} \circ \mathrm{Lin}_{rs} = \mathrm{Lin}_{rt} \quad \text{for } r, s, t \in \mathcal{O}(q)$$

are fulfilled.
Let $r, s \in \mathcal{O}(q)$. To prove that Lin_{rs} is well-defined, use the isomorphism theorem of linear algebra: Consider

$$\widetilde{\mathrm{Lin}_{rs}} := \mathrm{pr}_{q,s} \circ \mathrm{d}\psi_g\big|_r : T_rQ \longrightarrow \frac{T_sQ}{T_s\mathcal{O}(q)}, \quad X \mapsto \left[\mathrm{d}\psi_g\big|_r(X)\right],$$

where $\mathrm{pr}_{q,s} : T_sQ \longrightarrow \frac{T_sQ}{T_s\mathcal{O}(q)}$ is the canonical projection. Because ψ_g is a diffeomorphism, the differential at each point is a linear isomorphism, so $\widetilde{\mathrm{Lin}_{rs}}$ is a surjective map as a composition of surjective maps.

Compute the kernel: Let $X \in T_rQ$ such that $\left[\mathrm{d}\psi_g\big|_r(X)\right] = 0$, i.e. $\mathrm{d}\psi_g\big|_r(X) \in T_s\mathcal{O}(q)$. This is equivalent to

$$X = \mathrm{d}\psi_{g^{-1}}\big|_s \circ \mathrm{d}\psi_g\big|_r(X) \in \mathrm{d}\psi_{g^{-1}}\big|_s(T_s\mathcal{O}(q)),$$

since $\mathrm{d}\psi_g\big|_r$ and $\mathrm{d}\psi_{g^{-1}}\big|_s$ are inverse to each other. By definition, the G-orbit $\mathcal{O}(q)$ preserves the G-action, so $\psi_{g^{-1}}$ restricts to a map

$$\psi_{g^{-1}} : \mathcal{O}(q) \longrightarrow \mathcal{O}(q)\,,$$

and so the derivative also restricts as follows:

$$\mathrm{d}\psi_{g^{-1}}\big|_s : T_s\mathcal{O}(q) \longrightarrow T_r\mathcal{O}(q).$$

This map is surjective, thus $\mathrm{d}\psi_{g^{-1}}\big|_s(T_s\mathcal{O}(q)) = T_r\mathcal{O}(q)$, so $X \in T_r\mathcal{O}(q)$. Thus the kernel is computed: $\ker \widetilde{\mathrm{Lin}}_{rs} = T_r\mathcal{O}(q)$. This proves well-definedness of Lin_{rs}.
To understand that $\mathrm{Lin}_{rr} = \mathrm{id}$ for $r \in \mathcal{O}(q)$, it is sufficient to note that $\mathrm{d}\psi_e\big|_r = \mathrm{id}$, and this is true because $\psi_e = \mathrm{id}$.
The fact that $\mathrm{Lin}_{st} \circ \mathrm{Lin}_{rs} = \mathrm{Lin}_{rt}$ for $s, t \in \mathcal{O}(q)$ readily follows from the fact that if $g, h \in G$ are the unique elements such that $g \cdot r = s$ and $h \cdot s = t$, then $hg \in G$ is the unique element such that $hg \cdot r = t$, and from the fact that

$$\mathrm{d}\psi_h\big|_s \circ \mathrm{d}\psi_g\big|_r = \mathrm{d}\psi_{hg}\big|_r.$$

□

As stated in Prop. 2.2, the symplectic form on $\overline{Q}$ is given by

$$\overline{\pi}^*\overline{\omega} = \omega_Q, \quad \text{where} \quad \overline{\pi} : Q \longrightarrow \overline{Q} \quad \text{is a smooth surjective submersion.}$$

This property is in fact a defining property of the space $(\overline{Q}, \overline{\omega})$, which is to be understood as follows:

Proposition 2.3.

(i) Le $\overline{Q}$ and Q be smooth manifolds, and let $\overline{\pi} : Q \longrightarrow \overline{Q}$ be a smooth surjective submersion. Then, for another smooth manifold $\widetilde{Q}$, there exists a diffeomorphism $F : \overline{Q} \longrightarrow \widetilde{Q}$ if and only if there is a smooth surjective submersion $\widetilde{\pi} : Q \longrightarrow \widetilde{Q}$ such that $\widetilde{\pi}$ and $\overline{\pi}$ are constant on each other's fibers. Furthermore, the maps F and $\widetilde{\pi}$ uniquely determine each other via the following commutative diagram:

$$\begin{tikzcd} & Q \arrow[dl, "\overline{\pi}"'] \arrow[dr, "\widetilde{\pi}"] & \\ \overline{Q} \arrow[rr, "F"] & & \widetilde{Q} \end{tikzcd}$$

(ii) Let the situation be as in above diagram, i.e. $\overline{\pi}$ and $\widetilde{\pi}$ are smooth surjective submersions and F is a diffeomorphism and the diagram commutes. Let ω and $\overline{\omega}$ be differential k-forms on Q and $\overline{Q}$, respectively, with $\overline{\pi}^*\overline{\omega} = \omega$. Then, for a differential k-form $\widetilde{\omega}$ on $\widetilde{Q}$, the following equivalence holds:

$$F^*\widetilde{\omega} = \overline{\omega} \iff \widetilde{\pi}^*\widetilde{\omega} = \omega\,.$$

Furthermore, the above equivalence uniquely defines $\widetilde{\omega}$.

Proof. The proof of (i) can be found in [5, Theorem 4.31].
Prove (ii): Let $q \in Q$ and $X_1, \ldots, X_k \in T_qQ$. Let $\overline{q} := \overline{\pi}(q)$, and for $j = 1, \ldots, k$, let $\overline{X_j} := \mathrm{d}\overline{\pi}\big|_q(X_j)$. Then the condition $\overline{\pi}^*\overline{\omega} = \omega$ at q can be written as

$$\overline{\omega}\big|_{\overline{q}}(\overline{X_1}, \ldots, \overline{X_k}) = \omega\big|_q(X_1, \ldots, X_k). \tag{11}$$

Furthermore, it holds that

$$F^*\widetilde{\omega}\big|_{\overline{q}}(\overline{X_1}, \ldots, \overline{X_k}) = \widetilde{\pi}^*\widetilde{\omega}\big|_q(X_1, \ldots, X_k), \tag{12}$$

this can be seen as follows:

$$\begin{aligned} &F^*\widetilde{\omega}\big|_{\overline{q}}(\overline{X_1}, \ldots, \overline{X_k}) = \widetilde{\omega}\big|_{F(\overline{q})}\Big(\mathrm{d}F\big|_{\overline{q}}(\overline{X_1}), \ldots, \mathrm{d}F\big|_{\overline{q}}(\overline{X_k})\Big) \\ &= \widetilde{\omega}\big|_{\widetilde{\pi}(q)}\Big(\mathrm{d}(F \circ \overline{\pi})\big|_q(X_1), \ldots, \mathrm{d}(F \circ \overline{\pi})\big|_q(X_k)\Big) = \widetilde{\omega}\big|_{\widetilde{\pi}(q)}\Big(\mathrm{d}\widetilde{\pi}\big|_q(X_1), \ldots, \mathrm{d}\widetilde{\pi}\big|_q(X_k)\Big) \\ &= \widetilde{\pi}^*\widetilde{\omega}\big|_q(X_1, \ldots, X_k). \end{aligned}$$

Now, first prove " $\Longrightarrow$ ":
Let $F^*\widetilde{\omega} = \overline{\omega}$ be given. Then

$$\widetilde{\pi}^*\widetilde{\omega}\big|_q(X_1,\dots,X_k) \overset{(12)}{=} \underbrace{F^*\widetilde{\omega}}_{=\overline{\omega}}\big|_{\overline{q}}(\overline{X_1},\dots,\overline{X_k}) \overset{(11)}{=} \omega\big|_q(X_1,\dots,X_k).$$

Because this holds for any $q \in Q$ and $X_1,\dots,X_k \in T_qQ$, this proves $\widetilde{\pi}^*\widetilde{\omega} = \omega$. Now, prove " $\Longleftarrow$ ":
Let $\widetilde{\pi}^*\widetilde{\omega} = \omega$. Then

$$F^*\widetilde{\omega}\big|_{\overline{q}}(\overline{X_1},\dots,\overline{X_k}) \overset{(12)}{=} \underbrace{\widetilde{\pi}^*\widetilde{\omega}}_{=\omega}\big|_q(X_1,\dots,X_k) \overset{(11)}{=} \overline{\omega}\big|_{\overline{q}}(\overline{X_1},\dots,\overline{X_k}).$$

Now, because this is true for all $q \in Q$ and $X_1,\dots,X_k \in T_qQ$, and because $\overline{\pi}$ and $\mathrm{d}\overline{\pi}$ are surjective, this is true for all $\overline{q} \in \overline{Q}$ and $\overline{X_1},\dots,\overline{X_k} \in T_{\overline{q}}\overline{Q}$, so this proves $F^*\widetilde{\omega} = \overline{\omega}$. □

2.2 Orbit space of a Hamiltonian flow

Consider a symplectic manifold (M,ω) and a Hamiltonian function $H : M \longrightarrow \mathbb{R}$, and let X_H be the associated Hamiltonian vector field, i.e. the unique vector field determined by

$$\omega(X_H,\cdot) = \mathrm{d}H\,,$$

and assume that the Hamiltonian flow ϕ_t of X_H is global. Assume further that one of the following two cases hold: On all non-stationary points of the flow, either the flow is non-periodic everywhere, or it is periodic everywhere with a globally fixed minimum period $P > 0$ for every orbit. In both cases, this leads to a group action on M defined by the flow: In the first case, it is

$$G = (\mathbb{R},+) \curvearrowright M, \quad t\cdot m := \phi_t(m) \quad \text{for } t \in G,$$

and in the second case, it is

$$G = (\mathbb{R}/P\mathbb{Z},+) \curvearrowright M, \quad [t]\cdot m := \phi_t(m) \quad \text{for } [t] \in G,$$

whereby the well-definedness in the second case follows directly from the fact that the maps ϕ_t and ϕ_{t+zP} are identical for any integer z.
In both cases, the action is smooth. Moreover, on the set of all non-stationary points, the action is free. Because the maps ϕ_t are symplectomorphisms, the action is symplectic:

Proposition 2.4. *The Hamiltonian flow preserves the symplectic structure.*

Proof. Prove $\phi_s^*\omega = \omega$ for all $s \in \mathbb{R}$. For $s = 0$, this equation apparently holds. Furthermore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\phi_s^*\omega &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\phi_{s+t}^*\omega = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\phi_s^*\phi_t^*\omega = \phi_s^*\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\phi_t^*\omega\right) = \phi_s^*\mathcal{L}_{X_H}\omega\\ &= \phi_s^*(\iota_{X_H}\underbrace{\mathrm{d}\omega}_{=0} + \mathrm{d}\underbrace{\iota_{X_H}\omega}_{=\mathrm{d}H}) = \phi_s^*\mathrm{d}^2H = 0.\end{aligned}$$

This implies that $\phi_s^*\omega = \omega$ for all $s \in \mathbb{R}$. □

For both choices of G, it is abelian, and the associated Lie algebra is $\mathfrak{g} = \mathbb{R}$. For $G = \mathbb{R}$, the Lie exponential map $\exp : \mathfrak{g} \longrightarrow G$ is the identity on $\mathbb{R}$, and for $G = \mathbb{R}/P\mathbb{Z}$, it is the quotient map $\mathbb{R} \longrightarrow \mathbb{R}/P\mathbb{Z}$. Thus, the map $\mathbb{R} = \mathfrak{g} \longrightarrow \mathfrak{X}(M,\omega)$ is given by

$$\xi \mapsto X_\xi := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\phi_{t\xi} = \xi\cdot X_H.$$

Thus, the Hamiltonians for $\xi \in \mathfrak{g}$ can be chosen as $H_\xi = \xi H$. Because the group acting on M is abelian, the adjoint action $\mathrm{Ad}(g^{-1}) : \mathfrak{g} \longrightarrow \mathfrak{g}$ is the identity, and because $H_\xi = \xi H$, the condition for the action to be Hamiltonian can be written as

$$H \circ \phi_t = H \quad \forall t \in \mathbb{R},$$

prove this:

Proposition 2.5. *The Hamiltonian flow preserves the Hamiltonian function.*

Proof.
$$\frac{\mathrm{d}}{\mathrm{d}t}H(\phi_t(p)) = \mathrm{d}H\big|_{\phi_t(p)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(p)\right) = \mathrm{d}H(X_H)\big|_{\phi_t(p)} = \omega(X_H, X_H)\big|_{\phi_t(p)} = 0.$$
□

With the identification $\mathfrak{g}^* = \mathbb{R}^* \cong \mathbb{R}$, a moment map can be chosen as identical with the Hamiltonian function:
$$\mu = H : M \longrightarrow \mathfrak{g}^* \cong \mathbb{R}.$$
Observe that, with the identification $\mathfrak{g}^* \cong \mathbb{R}$, the inner multiplication of $\mathfrak{g}^*$ and $\mathfrak{g}$ is the same as the standard multiplication in $\mathbb{R}$. Furthermore, note that, because $\mathrm{Ad}(g^{-1})$ is the identity, so is $\mathrm{Ad}^*(g)$:
$$\mathrm{Ad}^*(g) : \mathfrak{g}^* \cong \mathbb{R} \longrightarrow \mathfrak{g}^* \cong \mathbb{R}, \quad \xi^* \mapsto (\eta \mapsto \xi^* \cdot \eta) = \xi^*.$$
Thus, all coadjoint orbits of Ad^* in $\mathfrak{g}^*$ are stationary, i.e. any value $r \in \mathbb{R}$ is a fixed point of the coadjoint action.
Now let $E \in \mathbb{R}$ such that $Q = H^{-1}(E)$ doesn't contain any stationary points of the Hamiltonian flow. This is the case if and only if E is a regular value of H:

Proposition 2.6. *The zeroes of* $\mathrm{d}H$ *and* X_H *coincide, and these are the stationary points of the Hamiltonian flow.*

Proof. Let $p \in M$, then because ω_p is non-degenerate,
$$X_H\big|_p = 0 \iff \mathrm{d}H\big|_p = \omega(X_H, \cdot)\big|_p \equiv 0.$$
Assume that p fulfils this equivalent statements. The point p is stationary if and only if $\phi_s(p) = p$ for all times $s \in \mathbb{R}$. For $s = 0$, this is apparently true, and it is true for all times if and only if
$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\phi_s(p) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\phi_{s+t}(p) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\phi_s(\phi_t(p)) = \mathrm{d}\phi_s(X_H\big|_p),$$
and because $\mathrm{d}\phi_s$ is a linear isomorphism, this is the case if and only if $X_H\big|_p = 0$. □

Call Q an *energy hypersurface*. Now apply Prop. 2.2 to this setting: It implies that the group action restricts to Q. (This also follows directly from Prop. 2.5.) Because no stationary points lie in Q, the action on Q is free. In the case of the periodic flow, the group actiong on M is compact, so the action is proper, because continuous actions of topological groups on Hausdorff spaces are always proper, see [4, Prop. 12.22] for a proof of this.
For the case of $G = \mathbb{R}$, assume that the action is proper. Then, for both cases, Prop. 2.2 further implies that
$$\overline{Q} := Q/G$$
is a symplectic manifold of dimension $\dim M - 2$. This manifold can be called the *orbit space of the Hamiltonian flow*, since each orbit of the flow is a point in $\overline{Q}$.
The symplectic form is given by
$$\overline{\omega}_{[q]}([V],[W]) := \omega_q(V_q, W_q) \quad \text{for } q \in Q \text{ and } [V],[W] \in \frac{T_qQ}{T_q\mathcal{O}(q)},$$
and this is well-defined. The tangent spaces $T_q\mathcal{O}(q)$ are 1-dimensional, and are given by the span of the Hamiltonian vector field:
$$T_q\mathcal{O}(q) = \mathrm{span}\Big\{X_H\big|_q\Big\}.$$

2.3 Physical description of SLF and CF in dimension 2

The physical derivations in this section follow the formalism described in [3, Chapters 9 and 27], here applied to the specific situation of a charged particle in the euclidean plane.
Consider a negatively charged point mass moving in a 2-dimensional euclidean plane spanned by cartesian coordinates, let e_1, e_2 be the unit vectors in the coordinate directions. Now let e_3 be a third unit vector perpendicular to the 2-dimensional plane such that e_1, e_2, e_3 form a right-handed system, and let there be a homogenous magnetic field $-Be_3$ perpendicular to the plane. Assume the mass to be 1, and the charge to be -1. Use the variable $x = (x_1, x_2)$ to describe the position, and $v = (v_1, v_2)$ to describe the velocity of the point mass. Then the kinetic energy T is

$$T = \frac{1}{2}\|v\|_2{}^2$$

and the Lortenz force F exerted on the point mass by the magnetic field is given by

$$F = (F_1, F_2) = (-Bv_2, Bv_1).$$

According to [3, Eq. (3.90)], to determine the Lagrangian function of this system, a potential function $U = U(x, v, t)$ for the Lorentz function which fulfils

$$F_i = -\frac{\partial U}{\partial x_i} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial U}{\partial v_i} \quad \text{for } i = 1, 2$$

needs to be chosen. Here, t is the time variable. One choice of U which fulfils this condition is

$$U = \frac{B}{2}(x_1 v_2 - x_2 v_1).$$

Choose U this way. Then the Lagrangian function L is given by

$$L(x, v) = T - U = \frac{1}{2}(v_1{}^2 + v_2{}^2) + \frac{B}{2}(v_1 x_2 - v_2 x_1).$$

Notice that L is time-independant. Now, the generalized momenta p_1 and p_2 are given by

$$p_1 = \frac{\partial L}{\partial v_1} = v_1 + \frac{B}{2}x_2 \quad \text{and} \quad p_2 = \frac{\partial L}{\partial v_2} = v_2 - \frac{B}{2}x_1.$$

By means of the above expressions, the velocity can be written in terms of x and $p = (p_1, p_2)$, i.e.

$$v_i = v_i(x, p) \quad \text{for } i = 1, 2\,.$$

Using this, the Hamiltonian function H is given by

$$H(x, p) = v_1(x, p) \cdot p_1 + v_2(x, p) \cdot p_2 - L(x, v(x, p)) = \frac{1}{2}\left(\left(p_1 - \frac{B}{2}x_2\right)^2 + \left(p_2 + \frac{B}{2}x_1\right)^2\right).$$

The solutions of this Hamiltonian system are the solutions of the Hamiltonian differential equations

$$\frac{\partial H}{\partial p_i} = \dot{x}_i \quad \text{and} \quad \frac{\partial H}{\partial x_i} = -\dot{p}_i \quad \text{for } i = 1, 2\,.$$

The solutions of these differential equations can also be described as the Hamiltonian flow ϕ_t of a symplectic manifold $M = \mathbb{R}^2 \times \mathbb{R}^2$ with coordinates (x, p), symplectic form

$$\omega = \mathrm{d}x_1 \wedge \mathrm{d}p_1 + \mathrm{d}x_2 \wedge \mathrm{d}p_2$$

and the Hamiltonian function defined as above. The Hamiltonian differential equations are then equivalent to the equations

$$\omega(X_H, \cdot) = \mathrm{d}H \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\phi_t = X_H \circ \phi_t\,.$$

Changing the coordinates that ω and H are defined on from (x, p) to (x, v) can be accomplished by pulling back ω and H via the diffeomorphism

$$\Phi : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \times \mathbb{R}^2, \qquad (x, v) \mapsto (x, p(x, v)) = \left(x, v_1 + \frac{B}{2}x_2, v_2 - \frac{B}{2}x_1\right).$$

This results in

$$\Phi^* H = H \circ \Phi = \frac{1}{2}\|v\|_2{}^2 \quad \text{and} \quad \Phi^*\omega = \mathrm{d}x_1 \wedge \mathrm{d}v_1 + \mathrm{d}x_2 \wedge \mathrm{d}v_2 + B\mathrm{d}x_1 \wedge \mathrm{d}x_2.$$

The case $B = 0$ where there is no magnetic field and thus no force acting on the point mass describes the straight line flow. The case $B \neq 0$ describes the magnetic flow. These descriptions will be generalized to higher dimensions in section 2.4 and 2.6.
Describing the Hamiltonian system in the coordinates (x, v) has the property that the second component of the flow represents the time-derivative of the first component. This statement can be generalized:

Proposition 2.7. *Let $M := \mathbb{R}^N \times \mathbb{R}^N$ with coordinates (x, v) and with symplectic form*

$$\omega = \sum_{\nu=0}^{2n} \mathrm{d}x_\nu \wedge \mathrm{d}v_\nu + \sum_{\substack{i,j=0\\ i<j}}^{n} f_{ij}\, \mathrm{d}x_i \wedge \mathrm{d}x_j\,,$$

where f_{ij} are smooth functions on M. Let the Hamiltonian function be

$$H : M \longrightarrow \mathbb{R}, \qquad (x, v) \mapsto \frac{1}{2}\|v\|_2{}^2.$$

Then the Hamiltonian flow $\phi_t = (\phi_{x,t}, \phi_{v,t})$ fulfils

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_{x,t} = \phi_{v,t}.$$

Thus, the coordinate v can be viewed as the velocity coordinate*, and x as the* position coordinate*. Moreover, the functions f_{ij} do not depend on the velocity coordinate v.*

Proof. To prove the above stated property of the Hamiltonian flow, it is sufficient to show that the Hamiltonian vector field has the form

$$X_H\big|_{(x,v)} = (v, *),$$

i.e. the first coordinate of X_H is the projection onto the v-coordinate. Then the statement follows from the flow equation $\frac{\mathrm{d}}{\mathrm{d}t}\phi_t = X_H \circ \phi_t$.
Write the components of X_H as

$$X_H = (X_H^{x_1}, \ldots, X_H^{x_N}, X_H^{v_1}, \ldots, X_H^{v_N}).$$

Then

$$\sum_{\nu=0}^{N} v_\nu \mathrm{d}v_\nu = \mathrm{d}H = \omega(X_H, \cdot) = \sum_{\nu=0}^{2n}(X_H^{x_\nu}\mathrm{d}v_\nu - X_H^{v_\nu}\mathrm{d}x_\nu) + \sum_{\substack{i,j=0\\ i<j}}^{n} f_{ij}(X_H^{x_i}\mathrm{d}x_j - X_H^{x_j}\mathrm{d}x_i).$$

Comparing the coefficients of $\mathrm{d}v_\nu$ for $\nu = 1, \ldots, N$ yields

$$X_H^{x_\nu} = v_\nu \quad \text{for } \nu = 1, \ldots, N\,,$$

so X_H has the desired form.
The fact that tne functions f_{ij} only depend on x follows from the fact that $\mathrm{d}\omega = 0$: Let $i, j, k \in \{1, \ldots, n\}$, then

$$0 = \mathrm{d}\omega\left(\frac{\partial}{\partial v_k}, \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\partial f_{ij}}{\partial v_k},$$

thus the partial derivatives of f_{ij} with respect v_k for $k = 1, \ldots, n$ vanish. Since M is connected, this implies that the functions f_{ij} do not depend on v. □

2.4 Orbit space of SLF

For $N \in \mathbb{N}$, consider the smooth manifold $M := \mathbb{R}^N \times \mathbb{R}^N$ woth coordinates (x, v), the symplectic form

$$\omega = \mathrm{d}x \wedge \mathrm{d}v = \sum_{j=0}^{N} \mathrm{d}x_j \wedge \mathrm{d}v_j$$

and the Hamiltonian function

$$H : M \longrightarrow \mathbb{R}, \quad (x, v) \mapsto \frac{1}{2}\|v\|_2{}^2 .$$

Then the corresponding Hamiltonian vector field and Hamiltonian flow are given by

$$X_H\big|_{(x,v)} = (v, 0), \qquad \phi_t(x, v) = (x + tv, v).$$

Note that the case $N = 2$ corresponds to the description in section 2.3: Call this the *straight line flow* because the x-coordinate of the flow describes the flow along a straight line, and the v-coordinate describes its velocity, see Prop. 2.7.
The fixed points of the flow are all points with $v = 0$, and the flow is non-periodic everywhere outside these fixed points. Now, $E := \frac{1}{2}$ is a regular value of H, since

$$H(x, v) = E = \frac{1}{2} = \frac{1}{2}\|v\|_2{}^2 \implies v \neq 0 \implies X_H\big|_{(x,v)} = (v, 0) \neq 0 \implies \mathrm{d}H\big|_{(x,v)} \not\equiv 0,$$

see Prop. 2.6. Let

$$Q := H^{-1}(E) = \mathbb{R}^N \times \mathbb{S}^{N-1},$$

this energy hypersurface is a submanifold of M of codimension 1, and the Hamiltonian flow induces a smooth, free $(\mathbb{R}, +)$-action on Q. This action turns out to be proper. Use the following condition from [4, Prop. 12.23] to prove this:

Lemma 2.8. *For a continuous action of a topological group G on a Hausdorff space X, the action is proper if and only if, for every compact subset $K \subseteq X$, the set*

$$G_K := \{g \in G \,|\, g \cdot K \cap K \neq \emptyset\}$$

is compact. □

As a first step, consider the following statement:

Lemma 2.9. *Consider a continuous $(\mathbb{R}, +)$-action on a Hausdorff space X. For any compact $K \subseteq X$, the set*

$$\mathbb{R}_K := \{t \in \mathbb{R} \,|\, t \cdot K \cap K \neq \emptyset\}$$

is closed.

Proof. Fix $K \subseteq X$ compact, and define $W := \mathbb{R}\backslash\mathbb{R}_K = \{t \in \mathbb{R} \,|\, t \cdot K \cap K = \emptyset\}$. It needs to be proven that W is open. Let $t_0 \in W$. Then for every $k \in K$, it holds that $t_0 \cdot k \notin K$. Thus, considering the continuous map

$$f : \mathbb{R} \times X \longrightarrow X \quad (t, x) \mapsto t \cdot x,$$

it follows that $(t_0, k) \in f^{-1}(X\backslash K)$, which is an open set. Thus, for every $k \in K$, there is an open neighbourhood $U_k \subseteq X$ and $\varepsilon_k > 0$ such that

$$(t_0, k) \in B_{\varepsilon_k}(t_0) \times U_k \subseteq f^{-1}(X\backslash K),$$

here $B_{\varepsilon_k}(t_0)$ is the open ball of radius ε_k centered at t_0.
Now $\{U_k\}_{k\in K}$ is an open cover of K, so there exists a finite subcover $\{U_{k_1}, \dots, U_{k_l}\}$. But then, with $\varepsilon := \{\varepsilon_{k_1}, \dots, \varepsilon_{k_l}\}$, it follows that $B_\varepsilon(t_0) \times K \subseteq f^{-1}(X\backslash K)$. This implies that $B_\varepsilon(t_0) \subseteq W$. This proves that W is open, so $\mathbb{R}_K$ is closed. □

To prove that the $(\mathbb{R}, +)$-action on Q is proper, since $\mathbb{R}$ has the Heine-Borel property (i.e. every closed bounded subset of $\mathbb{R}$ is compact), it needs to be shown that for any $K \subseteq Q$, the set $\mathbb{R}_K$ as defined in Lemma 2.9 is bounded for any compact set K. Following Lemma gives a necessary condition for this:

Lemma 2.10. *Consider a continuous action of $(\mathbb{R}, +)$ on a metric space (X, dist) such that for any non-empty compact set $K \subseteq X$ it holds that*

$$f_K : \mathbb{R} \longrightarrow \mathbb{R}, \quad t \mapsto \min_{k\in K} \mathrm{dist}(t \cdot k, k)$$

diverges to $+\infty$ for $t \longrightarrow \pm\infty$. Then $\mathbb{R}_K$ defined as in Lemma 2.9 is bounded, and thus the action is proper.

Proof. Fix a non-empty, compact set $K \subseteq X$ such that f_K diverges to $+\infty$ for $t \longrightarrow \pm\infty$. This can also be written as follows:

$$\forall B > 0 \; \exists C(B) > 0 \; \forall t \in \mathbb{R} : \; (|t| > C(B) \implies \forall k \in K : \; \mathrm{dist}(t \cdot k, k) > B). \tag{13}$$

Now K, as a compact subset of a metric space, is bounded, let $B_K > 0$ be a bound of K such that

$$\forall k, \widetilde{k} \in K : \; \mathrm{dist}(k, \widetilde{k}) < B_K.$$

Then it turns out that $\mathbb{R}_K$ is bounded by $C(B_K)$, i.e.

$$\forall t \in \mathbb{R}_K : \; |t| \leqslant C(B_K).$$

This can be seen as follows: Assume that this is not true, then choose $t_0 \in \mathbb{R}_K$ with $|t_0| > C(B_K)$. Then, by condition (13), $\forall k \in K : \; \mathrm{dist}(t_0 \cdot k, k) > B_K$, which in turn implies that $\forall k \in K : \; t_0 \cdot K \notin K$. Thus, $t_0 \cdot K \cap K = \emptyset$; which contradicts $t_0 \in \mathbb{R}_K$. $\square$

Using Lemma 2.10, verify that $(\mathbb{R}, +)$-action on Q is proper: For any $(x, v) \in Q$ it holds that

$$\mathrm{dist}(t \cdot (x, v) \, , \, (x, v)) = \|(x + tv, v) - (x, v)\|_2 = |t| \underbrace{\|v\|_2}_{=1} = |t|.$$

Thus, for any compact set $K \subseteq \mathbb{R}$, it follows that $f_K(t) = |t|$, which diverges to $+\infty$ for $t \longrightarrow \pm\infty$. So the Hamiltonian flow induces a smooth, free and proper $(\mathbb{R}, +)$-action on Q, this implies that the orbit space $\overline{Q} = Q/\mathbb{R}$ is a symplectic manifold, as described in section 2.2.
In the following proposition, an equivalent description $\widetilde{Q}$ of the orbit space for SLF is described:

Proposition 2.11. *Define*

$$\widetilde{Q} := T\mathbb{S}^{N-1} = \big\{(v, x) \in \mathbb{R}^N \times \mathbb{R}^N \,\big|\, \langle v, v\rangle = 1, \; \langle x, v\rangle = 0\big\},$$

i.e. $\widetilde{Q}$ *consists of elements* (v, x) *with* $v \in \mathbb{S}^{N-1}$, *where* $\mathbb{S}^{N-1}$ *is understood as a submanifold of* $\mathbb{R}^N$, *and* $x \in T_v\mathbb{S}^{N-1} = (\mathbb{R}v)^\perp$. *Furthermore, let*

$$\widetilde{\pi} : Q \longrightarrow \widetilde{Q}, \quad (x, v) \mapsto (v, x - \langle x, v\rangle v).$$

Then $\widetilde{\pi}$ *is a smooth surjective submersion, and the equation*

$$\widetilde{\pi}^* \widetilde{\omega} = \omega\big|_Q = (\mathrm{d}x \wedge \mathrm{d}v)\big|_Q$$

defines a symplectic structure on $\widetilde{Q}$ *such that* $(\widetilde{Q}, \widetilde{\omega})$ *is symplectomorphic to* $(\overline{Q}, \overline{\omega})$.

Proof. By Prop. 2.3. it is sufficient to check that $\widetilde{\pi}$ is a smooth surjective submersion, and that the fibers of $\widetilde{\pi}$ are $\mathbb{R}$-Orbits (i.e. orbits of the Hamiltonian flow) in Q.
Smoothness is apparent. Surjectivity follows from the fact that, for $(v, x) \in \widetilde{Q}$, it holds that $\widetilde{\pi}(x, v) = (v, x)$, since X and V are othogonal by definition of $\widetilde{Q}$.
To see that $\widetilde{\pi}$ is a submersion, it is required to show that its differential is surjective at every point. So let $(x, v) \in Q$ and $(X, V) \in T_{(x,v)}Q$, then

$$\begin{aligned}
\mathrm{d}\widetilde{\pi}\big|_{(x,v)}(X, V) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \widetilde{\pi}(x + tX, v + tV) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (v + tV \, , \, x + tX - \langle x + tX, v + tV\rangle(v + tV)) \\
&= (V \, , \, X - \langle X, v\rangle v - \langle x, V\rangle v - \langle x, v\rangle V).
\end{aligned}$$

Here, the fact that $\widetilde{\pi}$ can be extended to a map from $\mathbb{R}^N \times \mathbb{R}^N$ to $\mathbb{R}^N \times \mathbb{R}^N$ was used. Now consider the kernel of this linear map $\mathrm{d}\widetilde{\pi}\big|_{(x,v)}$:

$$\mathrm{d}\widetilde{\pi}\big|_{(x,v)}(X, V) = 0 \iff (V = 0 \text{ and } X = \langle X, v\rangle v) \iff (X, V) \in \mathbb{R}v \times \{0\}.$$

Dimension formula yields

$$\begin{aligned}\dim \operatorname{im} \mathrm{d}\widetilde{\pi}\big|_{(x,v)} &= \dim T_{(x,v)}Q - \dim \ker \mathrm{d}\widetilde{\pi}\big|_{(x,v)}\\ &= \dim Q - \dim(\mathbb{R}v \times \{0\})\\ &= N-1-1 = N-2.\end{aligned}$$

But $N-2$ is the dimension of the codomain of $\mathrm{d}\widetilde{\pi}\big|_{(x,v)}$. Thus the differential of $\widetilde{\pi}$ is surjective at every point, i.e. $\widetilde{\pi}$ is a submersion. □

As a next step, describe the symplectic form $\widetilde{\omega}$ of $\widetilde{Q}$ in more detail. For this, the following definition of the *tautological 1-form* on the cotangent bundle, taken from [5, Chapter 22, text before Prop. 22.11], is necessary:

Definition 2.12. *Let S be a smooth manifold, $q \in S$ and $\varphi \in T_q^*S$. Then the* tautological 1-form λ *on T^*S is given by*

$$\lambda_{(q,\varphi)} : T_{(q,\varphi)}(T^*S) \longrightarrow \mathbb{R}, \quad v \mapsto \varphi\Big(\mathrm{d}\pi_S\big|_{(q,\varphi)}(v)\Big).$$

Here, $\pi_S : T^*S \longrightarrow S$ is the natural projection.

Proposition 2.13. *Let $\widetilde{Q} = T\mathbb{S}^{N-1}$ be identified with the cotangent bundle $T^*\mathbb{S}^{N-1}$ via the induced Riemannian metric on $\mathbb{S}^{N-1}$ which it naturally inherits as a submanifold of $\mathbb{R}^N$ with the standard metric:*

$$\Phi : \widetilde{Q} = T\mathbb{S}^{N-1} \longrightarrow T^*\mathbb{S}^{N-1}, \quad (v,x) \mapsto (v, \langle x, \cdot\rangle_{\mathbb{R}^N}).$$

Let λ be the tautological 1-form on $T^*\mathbb{S}^{N-1}$. Then the symplectic form $\widetilde{\omega}$ from Prop. 2.11 is identical with the differential of λ pulled back via Φ, and is given by

$$\widetilde{\omega} = \Phi^*\mathrm{d}\lambda = \mathrm{d}(\Phi^*\lambda) = (-\mathrm{d}v \wedge \mathrm{d}x)\big|_{\widetilde{Q}}.$$

Proof. Start by computing $\widetilde{\omega}$, which is given by the condition $\widetilde{\pi}^*\widetilde{\omega} = \omega\big|_Q = (\mathrm{d}x \wedge \mathrm{d}v)\big|_Q$. Let $(v,x) \in \widetilde{Q}$. The tangent space of $\widetilde{Q}$ at this point is given by

$$T_{(v,x)}\widetilde{Q} = \Big\{(\widetilde{V}, \widetilde{X}) \in \mathbb{R}^N \times \mathbb{R}^N \;\Big|\; \Big\langle v, \widetilde{V}\Big\rangle = 0,\ \Big\langle \widetilde{X}, v\Big\rangle + \Big\langle x, \widetilde{V}\Big\rangle = 0\Big\}.$$

Let $(\widetilde{V}, \widetilde{X})\,,\,(\widetilde{W}, \widetilde{Y}) \in T_{(v,x)}\widetilde{Q}$. Now $\widetilde{\pi}(x,v) = (v,x)$ and, using the expression for $\mathrm{d}\widetilde{\pi}\big|_{(x,v)}$ computed in the proof of Prop. 2.11, and remembering that $\langle x, v\rangle = 0$ and $\langle v, v\rangle = 1$, it follows that

$$\begin{aligned}&\mathrm{d}\widetilde{\pi}\big|_{(x,v)}\Big(\widetilde{X} - \Big\langle x, \widetilde{V}\Big\rangle v\,,\,\widetilde{V}\Big)\\ &= \Big(\widetilde{V}\,,\,\widetilde{X} - \Big\langle x, \widetilde{V}\Big\rangle v - \Big\langle \widetilde{X} - \Big\langle x, \widetilde{V}\Big\rangle v\,,\,v\Big\rangle v - \Big\langle x, \widetilde{V}\Big\rangle v - \langle x, v\rangle \widetilde{V}\Big)\\ &= \Big(\widetilde{V}\,,\,\widetilde{X} - \Big\langle x, \widetilde{V}\Big\rangle v - \Big\langle \widetilde{X}, v\Big\rangle v + \Big\langle x, \widetilde{V}\Big\rangle v - \Big\langle x, \widetilde{V}\Big\rangle v\Big)\\ &= \Big(\widetilde{V}\,,\,\widetilde{X} - \Big(\Big\langle \widetilde{X}, v\Big\rangle + \Big\langle x, \widetilde{V}\Big\rangle\Big)v\Big)\\ &= (\widetilde{V}, \widetilde{X}).\end{aligned}$$

In the same way, it follows that $\mathrm{d}\widetilde{\pi}\big|_{(x,v)}\Big(\widetilde{Y} - \Big\langle x, \widetilde{W}\Big\rangle v\,,\,\widetilde{W}\Big) = (\widetilde{W}, \widetilde{Y})$.

Now,

$$
\begin{aligned}
&\widetilde{\omega}_{(v,x)}\Big((\widetilde{V},\widetilde{X})\,,\,(\widetilde{W},\widetilde{Y})\Big)\\
&=\widetilde{\omega}_{\widetilde{\pi}(x,v)}\Big(\mathrm{d}\widetilde{\pi}\big|_{(x,v)}\Big(\widetilde{X}-\Big\langle x,\widetilde{V}\Big\rangle v\,,\,\widetilde{V}\Big)\,,\,\mathrm{d}\widetilde{\pi}\big|_{(x,v)}\Big(\widetilde{Y}-\Big\langle x,\widetilde{W}\Big\rangle v\,,\,\widetilde{W}\Big)\Big)\\
&=\underbrace{(\widetilde{\pi}^*\widetilde{\omega})\big|_{(x,v)}}_{=(\mathrm{d}x\wedge\mathrm{d}v)|_Q}\Big(\Big(\widetilde{X}-\Big\langle x,\widetilde{V}\Big\rangle v\,,\,\widetilde{V}\Big)\,,\,\Big(\widetilde{Y}-\Big\langle x,\widetilde{W}\Big\rangle v\,,\,\widetilde{W}\Big)\Big)\\
&=\Big\langle\widetilde{X}-\Big\langle x,\widetilde{V}\Big\rangle v\,,\,\widetilde{W}\Big\rangle-\Big\langle\widetilde{Y}-\Big\langle x,\widetilde{W}\Big\rangle v\,,\,\widetilde{V}\Big\rangle\\
&=\Big\langle\widetilde{X},\widetilde{W}\Big\rangle-\Big\langle x,\widetilde{V}\Big\rangle\underbrace{\Big\langle v,\widetilde{W}\Big\rangle}_{=0}-\Big\langle\widetilde{Y},\widetilde{V}\Big\rangle+\Big\langle x,\widetilde{W}\Big\rangle\underbrace{\Big\langle v,\widetilde{V}\Big\rangle}_{=0}\\
&=\Big\langle\widetilde{X},\widetilde{W}\Big\rangle-\Big\langle\widetilde{Y},\widetilde{V}\Big\rangle\\
&=\mathrm{d}x\wedge\mathrm{d}v\big|_{\widetilde{Q}}\Big((\widetilde{V},\widetilde{X})\,,\,(\widetilde{W},\widetilde{Y})\Big)\,.
\end{aligned}
$$

Thus, the result of the calculation is

$$\widetilde{\omega}=(-\mathrm{d}v\wedge\mathrm{d}x)\big|_{\widetilde{Q}}\,.$$

Now compute λ. The projection

$$\pi_{\mathbb{S}^{N-1}}:T^*\mathbb{S}^{N-1}\longrightarrow\mathbb{S}^{N-1},\quad(v,\alpha)\mapsto v$$

can be understood as the restriction of the projection

$$\pi_{\mathbb{R}^N}:T^*\mathbb{R}^N\longrightarrow\mathbb{R}^N,\quad(v,\alpha)\mapsto v$$

and thus, for $(v,\alpha)\in T^*\mathbb{S}^{N-1}$ and $(V,A)\in T_{(v,\alpha)}T^*\mathbb{S}^{N-1}$ it holds that

$$\mathrm{d}\pi_{\mathbb{S}^{N-1}}\big|_{(v,\alpha)}(V,A)=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\pi_{\mathbb{R}^N}(v+tV,\alpha+tA)=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}v+tV=V.$$

Thus, the tautological form λ at (v,α) is given by

$$\lambda_{(v,\alpha)}(V,A)=\alpha\big(\mathrm{d}\pi_{\mathbb{S}^{N-1}}\big|_{(v,\alpha)}(V,A)\big)=\alpha(V).$$

Now compute the pullback $\Phi^*\lambda$. The map Φ can be understood as the restriction of the smooth map

$$\Phi_{\mathbb{R}^N}:T\mathbb{R}^N\longrightarrow T^*\mathbb{R}^N,\quad(v,x)\mapsto(v,\langle x,\cdot\rangle_{\mathbb{R}^N})\,.$$

Thus, for $(\widetilde{V},\widetilde{X})\in T_{(v,x)}\widetilde{Q}$, it holds that

$$\mathrm{d}\Phi\big|_{(v,x)}(\widetilde{V},\widetilde{X})=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Phi_{\mathbb{R}^N}\Big(v+t\widetilde{V}\,,\,x+t\widetilde{X}\Big)=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Big(v+t\widetilde{V}\,,\,\Big\langle x+t\widetilde{X}\,,\,\cdot\Big\rangle\Big)=\Big(\widetilde{V}\,,\,\Big\langle\widetilde{X}\,,\,\cdot\Big\rangle\Big)\,.$$

It follows that

$$(\Phi^*\lambda)\big|_{(v,x)}(\widetilde{V},\widetilde{X})=\lambda_{\Phi(v,x)}\Big(\mathrm{d}\Phi\big|_{(v,x)}(\widetilde{V},\widetilde{X})\Big)=\lambda_{(v,\langle x,\cdot\rangle)}\Big(\widetilde{V}\,,\,\Big\langle\widetilde{X}\,,\,\cdot\Big\rangle\Big)=\Big\langle x,\widetilde{V}\Big\rangle.$$

This implies that $\Phi^*\lambda=(x\,\mathrm{d}v)\big|_{\widetilde{Q}}$. Taking the exterior derivative of this equation yields

$$\mathrm{d}(\Phi^*\lambda)=\mathrm{d}(x\,\mathrm{d}v)\big|_{\widetilde{Q}}=(-\mathrm{d}v\wedge\mathrm{d}x)\big|_{\widetilde{Q}}=\widetilde{\omega}.$$

□

2.5 Complex projective space as an orbit space

The construction in this section is motivated by [6, Exercise 5.1.3].
For $n \in \mathbb{N}$, consider the smooth manifold $M := \mathbb{R}^n \times \mathbb{R}^n$ with coordinates (x, y), the symplectic form

$$\omega = \mathrm{d}x \wedge \mathrm{d}y = \sum_{j=0}^{n} \mathrm{d}x_j \wedge \mathrm{d}y_j$$

and the Hamiltonian function

$$H : M \longrightarrow \mathbb{R}, \quad (x, y) \mapsto -\frac{1}{2}\|(x, y)\|_2^{\,2}.$$

The corresponding Hamiltonian vector field is given by

$$X_H\big|_{(x,y)} = (-y, x).$$

The resulting Hamiltonian flow can better be described in complex coordinates: Identify M with $\mathbb{C}^n$, using complex coordinates $z = (z_1, \ldots, z_n)$ with

$$z_j = x_j + iy_j \quad \text{for } \ j = 1, \ldots, n\,.$$

Then the Hamiltonian vector field and Hamiltonian flow are given by

$$X_H\big|_z = iz, \qquad \phi_t(z) = e^{it}z.$$

This flow is 2π-periodic everywhere, except at the only stationary point $z = 0$.
Choose $E := -\frac{1}{2}$. This is a regular value of H, since

$$H(z) = E = -\frac{1}{2} = -\frac{1}{2}|z|^2 \implies z \neq 0 \implies X_H\big|_z = iz \neq 0 \implies \mathrm{d}H\big|_z \not\equiv 0,$$

see Prop. 2.6. The energy hypersurface

$$Q := H^{-1}(E) = \mathbb{S}^{2n-1}$$

is a submanifold of M of codimension 1. The 2π-periodic Hamiltonian flow induces a smooth, free and proper action of $(\mathbb{R}/2\pi\mathbb{Z}, +) \cong \mathbb{S}^1$ on Q, as described in section 2.2. Thus,

$$\mathbb{CP}^{n-1} := Q/\mathbb{S}^1 = \mathbb{S}^{2n-1}/\mathbb{S}^1,$$

is a symplectic manifold, called the *complex projective space* of complex dimension $n-1$, i.e. the dimension as a smooth real manifold is $2n-2$. The symplectic form $\omega_{\mathbb{CP}^{n-1}}$ is uniquely determined by the condition $\Pi^*\omega_{\mathbb{CP}^{n-1}} = \omega\big|_Q$, where Π is the projection

$$\Pi : \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^{2n-1}/\mathbb{S}^1 = \mathbb{CP}^{n-1} \quad z \mapsto [z]\,. \tag{14}$$

Call $\omega_{\mathbb{CP}^{n-1}}$ the *standard symplectic form on* $\mathbb{CP}^{n-1}$.

2.6 Orbit space of CF

For $n \in \mathbb{N}$ and $B \in \mathbb{R}\backslash\{0\}$, define $M := \mathbb{R}^{2n} \times \mathbb{R}^{2n}$, a smooth manifold with coordinates (x, v) and symplectic form

$$\omega = \sum_{j=1}^{2n} \mathrm{d}x_j \wedge \mathrm{d}v_j + B \sum_{j=1}^{n} \mathrm{d}x_j \wedge \mathrm{d}x_{n+j} = \mathrm{d}x \wedge \mathrm{d}v + B\,\mathrm{d}\widehat{x} \wedge \mathrm{d}\widehat{y}\,,$$

where $x = (\widehat{x}, \widehat{y})$ are a break-down of the coordinates of the first factor $\mathbb{R}^{2n}$ of M, i.e.

$$\widehat{x} = (x_1, \ldots, x_n) \quad \text{and} \quad \widehat{y} = (x_{n+1}, \ldots x_{2n}).$$

Let the Hamiltonian function be

$$H : M \longrightarrow \mathbb{R}, \quad (x, v) \mapsto \frac{1}{2}\|v\|_2^{\,2}.$$

Note that the case $n = 1$ corresponds to the case described in section 2.3. Also note that, if $B = 0$, this corresponds to the straight line flow for even dimensions $N = 2n$ as described in section 2.4.
Describe the corresponding Hamiltonian vector field and Hamiltonian flow in complex coordinates: Identify M with $\mathbb{C}^n \times \mathbb{C}^n$, using complex coordinates (z, w) given by

$$z_j = x_j + ix_{n+j} \quad \text{and} \quad w_j = v_j + iv_{n+j} \quad \text{for } j = 1, \dots, n\,.$$

In these coordinates, the Hamiltonian vector field and Hamiltonian flow are given by

$$X_H\big|_{(z,w)} = (w, B \cdot iw), \qquad \phi_t(z, w) = \left(z + \frac{1}{B} i(1 - e^{iBt})w\,,\, e^{iBt}w\right).$$

Call this flow the *circular flow*, because the z-coordinate describes the flow along circles of radius $\frac{\|w\|_2}{|B|}$ with constant speed, i.e. absolute value of the velocity $\|w\|_2$. These circles lie in the affine complex plane parallel to $\mathbb{C}w$ containing z. The w-coordinate describes the velocity, see Prop. 2.7.
Define $P := \frac{2\pi}{|B|} > 0$. The flow is P-periodic everywhere except in the stationary points, which are exactly the points with $w = 0$. Now fix $E := \frac{1}{2}$, this is a regular value of H, since

$$H(z, w) = E = \frac{1}{2} = \frac{1}{2}|w|^2 \implies w \neq 0 \implies X_H\big|_{(z,w)} \neq 0 \implies \mathrm{d}H\big|_{(z,w)} \not\equiv 0,$$

see Prop. 2.6. Thus, the energy hypersurface

$$Q := H^{-1}(E) = \mathbb{C}^n \times \mathbb{S}^{2n-1}$$

is a submanifold of M of codimension 1. The Hamiltonian flow induces a smooth, free and proper action of $(\mathbb{R}/P\mathbb{Z}, +) \cong \mathbb{S}^1$ on Q, as described in section 2.2. Thus, the orbit space

$$\overline{Q} = Q/\mathbb{S}^1$$

is a symplectic manifold. The following proposition describes an equivalent description $\widetilde{Q}$ of this orbit space:

Proposition 2.14. *Let*

$$\widetilde{Q} := \mathbb{C}^n \times \mathbb{CP}^{n-1}\,,$$

and let

$$\widetilde{\pi} : Q \longrightarrow \widetilde{Q}, \quad (z, w) \mapsto \left(z + \frac{1}{B} iw\,,\, \Pi(w)\right).$$

Here, Π is the projection defined in Eq. (14) in section 2.5. Then $\widetilde{\pi}$ is a smooth surjective submersion, and the equation $\widetilde{\pi}^*\widetilde{\omega} = \omega\big|_Q$ defines a symplectic structure on $\widetilde{Q}$ such that it is symplectomorphic to $(\overline{Q}, \overline{\omega})$. Furthermore, this symplectic form can be described as a direct sum of the standard symplectic forms on $\mathbb{R}^{2n}$ and $\mathbb{CP}^{n-1}$ with prefactors depending on B:

$$\widetilde{\omega} = B\, \mathrm{d}\widehat{x} \wedge \mathrm{d}\widehat{y} \oplus \left(-\frac{1}{B}\omega_{\mathbb{CP}^{n-1}}\right).$$

Here, $(\widehat{x}, \widehat{y}) = (\mathrm{Re}(z), \mathrm{Im}(z))$ are the real coordinates of $\mathbb{C}^n \cong \mathbb{R}^n \times \mathbb{R}^n$.

Proof. First, show that $\widetilde{\pi}$ is a smooth surjective submersion. Smoothness is clear because both component functions are apparently smooth. Surjectivity can be seen by the fact that Π is surjective, so every element in $\widetilde{Q}$ can be written in the form $(c, \Pi(w))$, and

$$\widetilde{\pi}\left(c - \frac{1}{B} iw\,,\, w\right) = (c, \Pi(w))\,.$$

To check that it is a submersion, consider the differential of $\widetilde{\pi}$: Let $(Z, W) \in T_{(z,w)}Q \cong \mathbb{C}^n \times T_w\mathbb{S}^{2n-1}$. Then

$$\mathrm{d}\widetilde{\pi}\big|_{(z,w)}(Z, W) = \left(Z + \frac{1}{B} iW, \mathrm{d}\Pi\big|_w(W)\right),$$

and this map is surjective because $\mathrm{d}\Pi\big|_w$ is surjective, and for $\big(C, \mathrm{d}\Pi\big|_w(W)\big) \in T_{\widetilde{\pi}(z,w)}\widetilde{Q}$ it holds that

$$\mathrm{d}\widetilde{\pi}\big|_{(z,w)}\left(C - \frac{1}{B}iW\,,\, W\right) = \big(C, \mathrm{d}\Pi\big|_w(W)\big).$$

Thus, by Prop. 2.3, $(\widetilde{Q}, \widetilde{\omega})$ is symplectomorphic to $(\overline{Q}, \overline{\omega})$.
Now calculate $\widetilde{\omega}$: Let $(c, \Pi(w)) \in \widetilde{Q}$, and $(C, \overline{Z}), (D, \overline{W}) \in T_{(c,\Pi(w))}\widetilde{Q} \cong \mathbb{C}^n \times T_{\Pi(w)}\mathbb{CP}^{n-1}$. Set $z = c - \frac{1}{B}iw$, then $\widetilde{\pi}(z, w) = (c, \Pi(w))$. Choose $Z, W \in T_w\mathbb{S}^{2n-1}$ such that $\mathrm{d}\Pi\big|_w(Z) = \overline{Z}$ and $\mathrm{d}\Pi\big|_w(W) = \overline{W}$. For $C, D \in \mathbb{C}^n$, write the real and imaginary components as $C = (C_{\widehat{x}}, C_{\widehat{y}})$ and $D = (D_{\widehat{x}}, D_{\widehat{y}})$, and use the same notation for $Z, W \in T_w\mathbb{S}^{2n-1} \subseteq \mathbb{C}^n$. Let the notation "$\langle \cdot, \cdot \rangle$" refer to the real scalar product of appropriate dimension. Then

$$\begin{aligned}
&\widetilde{\omega}\big|_{(c,\Pi(w))}\big((C, \overline{Z})\,,\,(D, \overline{W})\big) = \widetilde{\omega}\big|_{\widetilde{\pi}(z,w)}\left(\mathrm{d}\widetilde{\pi}\big|_{(z,w)}\left(C - \frac{1}{B}iZ\,,\,Z\right)\,,\, \mathrm{d}\widetilde{\pi}\big|_{(z,w)}\left(D - \frac{1}{B}iW\,,\,W\right)\right)\\
&= \underbrace{\widetilde{\pi}^*\widetilde{\omega}}_{=\omega|_Q}\big|_{(z,w)}\left(\left(C - \frac{1}{B}iZ\,,\,Z\right)\,,\,\left(D - \frac{1}{B}iW\,,\,W\right)\right)\\
&= \left\langle C - \frac{1}{B}iZ\,,\,W\right\rangle - \left\langle D - \frac{1}{B}iW\,,\,Z\right\rangle\\
&\qquad + B\left(\left\langle C_{\widehat{x}} + \frac{1}{B}Z_{\widehat{y}}\,,\,D_{\widehat{y}} + \frac{1}{B}W_{\widehat{x}}\right\rangle - \left\langle D_{\widehat{x}} + \frac{1}{B}W_{\widehat{y}}\,,\,C_{\widehat{y}} + \frac{1}{B}Z_{\widehat{x}}\right\rangle\right)\\
&= \langle C, W\rangle - \langle D, Z\rangle + \frac{1}{B}(\langle iW, Z\rangle - \langle iZ, W\rangle)\\
&\qquad + B(\langle C_{\widehat{x}}, D_{\widehat{y}}\rangle - \langle D_{\widehat{x}}, C_{\widehat{y}}\rangle) + \underbrace{(\langle Z_{\widehat{y}}, D_{\widehat{y}}\rangle - \langle C_{\widehat{x}}, W_{\widehat{x}}\rangle - \langle W_{\widehat{y}}, C_{\widehat{y}}\rangle + \langle D_{\widehat{x}}, Z_{\widehat{x}}\rangle)}_{=\langle D,Z\rangle - \langle C,W\rangle}\\
&\qquad + \frac{1}{B}(\langle W_{\widehat{y}}, Z_{\widehat{x}}\rangle - \langle Z_{\widehat{y}}, W_{\widehat{x}}\rangle)\\
&= B(\langle C_{\widehat{x}}, D_{\widehat{y}}\rangle - \langle D_{\widehat{x}}, C_{\widehat{y}}\rangle) + \frac{1}{B}(\langle -W_{\widehat{y}}, Z_{\widehat{x}}\rangle + \langle W_{\widehat{x}}, Z_{\widehat{y}}\rangle - \langle -Z_{\widehat{y}}, W_{\widehat{x}}\rangle - \langle Z_{\widehat{x}}, W_{\widehat{y}}\rangle)\\
&\qquad + \frac{1}{B}(\langle W_{\widehat{y}}, Z_{\widehat{x}}\rangle - \langle Z_{\widehat{y}}, W_{\widehat{x}}\rangle)\\
&= B(\langle C_{\widehat{x}}, D_{\widehat{y}}\rangle - \langle D_{\widehat{x}}, C_{\widehat{y}}\rangle) + \frac{1}{B}(\langle W_{\widehat{x}}, Z_{\widehat{y}}\rangle - \langle Z_{\widehat{x}}, W_{\widehat{y}}\rangle)\\
&= B\,(\mathrm{d}\widehat{x} \wedge \mathrm{d}\widehat{y})\big|_c(C, D) - \frac{1}{B}\,\underbrace{(\mathrm{d}\widehat{x} \wedge \mathrm{d}\widehat{y})\big|_w}_{=\Pi^*\omega_{\mathbb{CP}^{n-1}}|_w}(Z, W)\\
&= B\,(\mathrm{d}\widehat{x} \wedge \mathrm{d}\widehat{y})\big|_c(C, D) - \frac{1}{B}\,\omega_{\mathbb{CP}^{n-1}}\big|_{\Pi(w)}(\overline{Z}, \overline{W})\,.
\end{aligned}$$

Thus,

$$\widetilde{\omega} = B\,\mathrm{d}\widehat{x} \wedge \mathrm{d}\widehat{y} \oplus \left(-\frac{1}{B}\omega_{\mathbb{CP}^{n-1}}\right)$$

is proven. □

2.7 Billiard map in higher dimensions

Consider $M := \mathbb{R}^N \times \mathbb{R}^N$ with coordinates (x, v) as well as a generalized representation of the symplectic form which is applicable to both the SLF and CF case: Let

$$\omega = \sum_{k=0}^{N} \mathrm{d}x_k \wedge \mathrm{d}v_k + \sum_{\substack{i,j=0\\ i<j}} f_{ij}\,\mathrm{d}x_i \wedge \mathrm{d}x_j$$

be a symplectic form on M, where f_{ij} are smooth functions on M. Let the Hamiltonian function be

$$H : M \longrightarrow \mathbb{R}, \quad (x, v) \mapsto \frac{1}{2}\|v\|_2{}^2.$$

By Prop. 2.7, the functions f_{ij} only depend on x, and the second coordinate of the Hamiltonian flow describes the time-derivative of the first coordinate.
Now assume that the prerequisites are met such that the symplectic quotient construction described in section 2.2 can be applied here, as it was applied to the case of SLF (see section 2.4) and CF (see section 2.6), i.e. assume that the Hamiltonian flow induces a G-action on M, with $G = (\mathbb{R}, +)$ or $G = \mathbb{S}^1$, fix $E := \frac{1}{2}$ of H and assume that the G-action, which restricts to the energy hypersurface

$$Q := H^{-1}(E) = \mathbb{R}^N \times \mathbb{S}^{N-1}$$

acts smoothly, freely and properly on Q. Then the orbit space Q/G is a symplectic manifold of dimension $\dim M - 2$, the projection

$$\overline{\pi} : Q \longrightarrow Q/G =: \overline{Q}, \qquad (x, v) \mapsto [(x, v)] =: \mathcal{O}(x, v)$$

is a smooth surjective submersion, and the symplectic form $\overline{\omega}$ on $\overline{Q}$ is uniquely determined by the condition $\overline{\pi}^*\overline{\omega} = \omega\big|_Q$.
In this general case, consider the billiard map: Let $\Omega \subseteq \mathbb{R}^N$, the billiards table, be compact, connected and with smooth boundary, i.e. $\partial\Omega$ is a smooth submanifold of codimension 1. Let

$$\nu : \partial\Omega \longrightarrow \mathbb{R}^N$$

be the outward pointing unit normal vector field, and define

$$P^{\text{in}} := \big\{(x, v) \in \partial\Omega \times \mathbb{S}^{N-1} \,\big|\, \langle v, \nu(x)\rangle < 0\big\} \subseteq Q$$

and

$$P^{\text{out}} := \big\{(x, v) \in \partial\Omega \times \mathbb{S}^{N-1} \,\big|\, \langle v, \nu(x)\rangle > 0\big\} \subseteq Q.$$

These are open subsets of $\partial\Omega \times \mathbb{S}^{N-1}$. In particular, these are smooth manifolds of dimension $2N - 2$. Geometrically, if $(x, v) \in P^{\text{in}}$, then $x \in \partial\Omega$ and v is a transversally inward pointing unit vector with respect to Ω. Similarly, if $(x, v) \in P^{\text{in}}$, then $x \in \partial\Omega$ and v is a transversally outward pointing unit vector with respect to Ω. Note that P^{in} and P^{out} are disjoint.
The regularity condition can be formulated as follows:

Definition 2.15 (Regularity condition)**.** *For every $(x, v) \in Q$, if the component in position space of the orbit $\mathcal{O}(x, v)$ (i.e. the set $\{\phi_{x,t}(x, v) \,|\, t \in \mathbb{R}\}$) intersects $\partial\Omega$ transversally, then there are exactly two points of intersection with $\partial\Omega$, and both intersections are transversal. Moreover, if $(x^{\text{in}}, v^{\text{in}})$ and $(x^{\text{refl}}, v^{\text{refl}})$ describe these two distinct points of intersection, then*

$$(x^{\text{in}}, v^{\text{in}}) \in P^{\text{in}} \quad \textit{and} \quad (x^{\text{refl}}, v^{\text{refl}}) \in P^{\text{out}}\,.$$

Remark 2.16. *If the regularity condition is fulfilled, then the restrictions $\overline{\pi}\big|_{P^{\text{in}}}$ and $\overline{\pi}\big|_{P^{\text{out}}}$ are injective, and diffeomorphic onto their image. Moreover, the images of these maps agree:*

$$\overline{\pi}\big(P^{\text{in}}\big) = \overline{\pi}\big(P^{\text{out}}\big) =: \overline{\mathcal{O}}.$$

Furthermore, $\overline{\mathcal{O}} \subseteq \overline{Q}$ is open, because the orbits in $\overline{\mathcal{O}}$ are exactly those which transversally intersect $\partial\Omega$, and transversal intersection is an open condition.

Given these prerequisites, define a billiard map

$$T_P : P^{\text{in}} \longrightarrow P^{\text{out}}, \quad (x, v) \mapsto (x\,,\, v - 2\langle v, \nu(x)\rangle\nu(x))\,,$$

this geometrically describes the reflection law at the boundary $\partial\Omega$ of the billiard table.
Call $\overline{\mathcal{O}}$ the *phase space of the orbit dynamics model.* The billiards map T on $\overline{\mathcal{O}}$ is defined by following commutative diagram:

$$\begin{tikzcd} P^{\text{in}} \arrow[r, "T_P"] \arrow[d, "\overline{\pi}|_{P^{\text{in}}}"'] & P^{\text{out}} \arrow[d, "\overline{\pi}|_{P^{\text{out}}}"] \\ \overline{\mathcal{O}} \arrow[r, "T"] & \overline{\mathcal{O}} \end{tikzcd}$$

The symplectic form $\overline{\omega}$ defined on $\overline{Q}$ restricts to a symplectic form $\overline{\mathcal{O}} \subseteq \overline{Q}$, because this is an open subset.
To simplify notation, in the following considerations $\overline{\omega}$ instead of $\overline{\omega}\big|_{\overline{\mathcal{O}}}$ is written.
To prove that T preserves $\overline{\omega}$, following Lemma is used:

Lemma 2.17. *Let $x \in \partial\Omega$ and $X_1, X_2 \in T_x(\partial\Omega)$, then*

$$\left\langle X_1\,,\, \mathrm{d}\nu\big|_x(X_2)\right\rangle = \left\langle X_2\,,\, \mathrm{d}\nu\big|_x(X_1)\right\rangle .$$

Proof. Start by describing the normal vector ν differently: Because $\partial\Omega$ is a submanifold of $\mathbb{R}^N$ of codimension 1, it can be locally described as the preimage of a regular value: Fix $x \in \partial\Omega$ and an open neighbourhood $U \subseteq \mathbb{R}^N$ of x. Then there is the smooth map

$$F : U \longrightarrow \mathbb{R}$$

such that $U \cap \partial\Omega = F^{-1}(0)$, with $0 \neq \nabla F(y) \perp T_y\partial\Omega$ for all $y \in \partial\Omega \cap U$. Because $\partial\Omega$ is orientable, $\nabla F\big|_{\partial\Omega\cap U}$ is always outward pointing or always inwards pointing. Without loss of generality, assume that it is outwards pointing. (Otherwise, multiply F with -1.) Define

$$g : U \longrightarrow \mathbb{R}, \quad y \mapsto \frac{1}{\|\nabla F(y)\|_2}\,,$$

then for $y \in \partial\Omega \cap U$ it holds that $\nu(y) = g(y)\nabla F(y)$. Now define

$$h := g \cdot F : U \longrightarrow \mathbb{R}, \quad y \mapsto g(y) \cdot F(y)\,.$$

Observe that

$$\nabla h = \nabla(gF) = g\nabla F + F\nabla g.$$

In particular, for $y \in \partial\Omega \cap U$, it holds that $F(y) = 0$, so $\nabla h(y) = g(y)\nabla F(y) = \nu(y)$.
Fix $X_1, X_2 \in T_x(\partial\Omega)$, and let ν be a curve on $\partial\Omega \cap U$ such that $\nu(0) = x$ and $\nu'(0) = X_2$. Then

$$\mathrm{d}\nu\big|_x(X_2) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \nu(\eta(t)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \nabla h(\eta(t)) = \mathrm{Hess}(h)(\eta(t))\eta'(t)\big|_{t=0} = \mathrm{Hess}(h)\big|_x \cdot X_2,$$

similarly it follows that $\mathrm{d}\nu\big|_x(X_1) = \mathrm{Hess}(h)\big|_x \cdot X_1$. Now, because the Hessian is symmetric, it holds that

$$\left\langle X_1\,,\, \mathrm{d}\nu\big|_x(X_2)\right\rangle = \left\langle X_1\,,\, \mathrm{Hess}\big|_x \cdot X_2\right\rangle = \left\langle X_2\,,\, \mathrm{Hess}\big|_x \cdot X_1\right\rangle = \left\langle X_2\,,\, \mathrm{d}\nu\big|_x(X_1)\right\rangle .$$

□

Proposition 2.18. *The billiard map T preserves the symplectic structure $\overline{\omega}$.*

Proof. Remember the commutative diagram:

$$\begin{array}{ccc} P^{\mathrm{in}} & \xrightarrow{T_P} & P^{\mathrm{out}} \\ {\scriptstyle \overline{\pi}|_{P^{\mathrm{in}}}}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\pi}|_{P^{\mathrm{out}}}} \\ \overline{\mathcal{O}} & \xrightarrow{T} & \overline{\mathcal{O}} \end{array}$$

To prove that $T^*\overline{\omega} = \overline{\omega}$, use the fact that $\overline{\omega}$ is uniquely determined by $\pi^*\overline{\omega} = \omega\big|_Q$, so it suffices to prove that $\left(\overline{\pi}\big|_{P^{\mathrm{in}}}\right)^* T^*\overline{\omega} = \omega\big|_{P^{\mathrm{in}}}$. Now

$$\left(\overline{\pi}\big|_{P^{\mathrm{in}}}\right)^* T^*\overline{\omega} = \left(T \circ \overline{\pi}\big|_{P^{\mathrm{in}}}\right)^*\overline{\omega} = \left(\overline{\pi}\big|_{P^{\mathrm{out}}} \circ T_P\right)^*\overline{\omega} = {T_P}^*\left(\overline{\pi}\big|_{P^{\mathrm{out}}}\right)^*\overline{\omega} = {T_P}^*\omega\big|_{P^{\mathrm{out}}}\,.$$

Therefore, what remains to be proven is ${T_P}^*\omega\big|_{P^{\mathrm{out}}} = \omega\big|_{P^{\mathrm{in}}}$.
Write $\omega = \omega_{\mathrm{std}} + \tau$ with

$$\omega_{\mathrm{std}} := \sum_{k=0}^{N} \mathrm{d}x_k \wedge \mathrm{d}v_k \quad \text{and} \quad \tau := \sum_{\substack{i,j=0\\ i<j}} f_{ij}\mathrm{d}x_i \wedge \mathrm{d}x_j\,.$$

Now because the functions f_{ij} only depend on x, and because T_P preserves x, it directly follows that T_P preserves τ, so it remains to be proven that ${T_P}^*\omega_{\mathrm{std}}\big|_{P^{\mathrm{out}}} = \omega_{\mathrm{std}}\big|_{P^{\mathrm{in}}}$.
So let $(x, v) \in P^{\mathrm{in}}$ and $(X_1, V_1)\,,\,(X_2, V_2) \in T_{(x,v)}P^{\mathrm{in}} = T_x(\partial\Omega) \times (\mathbb{R}v)^{\perp}$. Then for $i = 1, 2$ it holds that

$$\begin{aligned} \mathrm{d}T_P\big|_{(x,v)}(X_i, V_i) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (x + tX_I\,,\, v + tV_i - 2\langle v + tV_i\,,\, \nu(x + tX_i)\rangle\nu(x + tX_i)) \\ &= \left(X_i\,,\, V_i - 2\langle V_i\nu(x)\rangle\nu(x) - 2\left\langle v, \mathrm{d}\nu\big|_x(X_i)\right\rangle\nu(x) - 2\langle v, \nu(x)\rangle\mathrm{d}\nu\big|_x(X_i)\right). \end{aligned}$$

So

$$
\begin{aligned}
&T_P{}^*\omega_{\mathrm{std}}\big|_{(x,v)}((X_1,V_1)\,,\,(X_2,V_2)) = \omega_{\mathrm{std}}\big|_{T_P(x,v)}\Big(\mathrm{d}T_P\big|_{(x,v)}(X_1,V_1)\,,\,\mathrm{d}T_P\big|_{(x,v)}(X_2,V_2)\Big)\\
&= \big\langle X_1\,,\,V_2 - 2\langle V_2\nu(x)\rangle\nu(x) - 2\big\langle v, \mathrm{d}\nu\big|_x(X_2)\big\rangle\nu(x) - 2\langle v,\nu(x)\rangle\mathrm{d}\nu\big|_x(X_2)\big\rangle\\
&\quad - \big\langle X_2\,,\,V_1 - 2\langle V_1\nu(x)\rangle\nu(x) - 2\big\langle v, \mathrm{d}\nu\big|_x(X_1)\big\rangle\nu(x) - 2\langle v,\nu(x)\rangle\mathrm{d}\nu\big|_x(X_1)\big\rangle.
\end{aligned}
$$

The fact that $\nu(x) \perp T_x(\partial\Omega)$ and $X_1, X_2 \in T_x(\partial\Omega)$ simplifies the scalar products, such that

$$
\begin{aligned}
&T_P{}^*\omega_{\mathrm{std}}\big|_{(x,v)}((X_1,V_1)\,,\,(X_2,V_2))\\
&= \big\langle X_1\,,\,V_2 - 2\langle v,\nu(x)\rangle\mathrm{d}\nu\big|_x(X_2)\big\rangle - \big\langle X_2\,,\,V_1 - 2\langle v,\nu(x)\rangle\mathrm{d}\nu\big|_x(X_1)\big\rangle\\
&= \langle X_1\,,\,V_2\rangle - \langle X_2\,,\,V_1\rangle - 2\langle v,\nu(x)\rangle\big(\big\langle X_1, \mathrm{d}\nu\big|_x(X_2)\big\rangle - \big\langle X_2, \mathrm{d}\nu\big|_x(X_1)\big\rangle\big).
\end{aligned}
$$

Using Lemma 2.17, this further simplifies to

$$
T_P{}^*\omega_{\mathrm{std}}\big|_{(x,v)}((X_1,V_1)\,,\,(X_2,V_2)) = \langle X_1\,,\,V_2\rangle - \langle X_2\,,\,V_1\rangle = \omega_{\mathrm{std}}|_{(x,v)}((X_1,V_1)\,,\,(X_2,V_2)).
$$

This concludes the proof. □

2.8 Comparison of orbit dynamics with Birkhoff billiards

Remember the description of Birkhoff billiards in section 1: It is a discrete dynamical system of billiards in 2-dimensional space, and the phase space $\mathrm{PS}_{\mathrm{Birk}}$ with coordinates (l,α) describes the segment of the billiard trajectory starting at $\gamma(l)$ on the edge of the table, and leaving the table at an angle α relative to the tangent vector $\gamma'(l)$, until the next intersection with the boundary, where reflection according to the reflection law happens. Here

$$
\gamma : \mathbb{S}^1_L \longrightarrow \mathbb{R}^2
$$

is a smooth curve of length L parametrized by unit length, and runs along $\partial\Omega$ in counterclockwise orientation, and $\Omega \subseteq \mathbb{R}^2$ is the billiards table, a compact connected set, which further fulfils the regularity condition in Def. 1.1. As a consequence, by Remark 1.3, every orbit of the corresponding Hamiltonian flow (which in the case of CF is a circle of Larmor radius, in the case of SLF is a straight line) which intersects the table boundary $\partial\Omega$ transversally has excatly two points of intersection, and both these intersections are transversal. This means that the regularity condition stated in Def. 2.15 is fulfilled.
Now consider the phase space of the orbit dynamics model for the 2-dimensional case, as described in section 2.7. Instead of considering these open subsets of the quotient spaces $\overline{Q} = Q/G$, consider the corresponding open subsets of the alternative descriptions $\widetilde{Q}$ for the orbit space: For the SLF case, as described in Prop. 2.11, for the CF case, as described in Prop. 2.14.

$$
\begin{aligned}
\widetilde{\mathcal{O}}_{\mathrm{SLF}} &:= \big\{(v,x) \in T\mathbb{S}^1 \,\big|\, \text{The orbit represented by } (v,x) \text{ intersects } \partial\Omega \text{ transversally}\big\} \subseteq T\mathbb{S}^1\,,\\
\widetilde{\mathcal{O}}_{\mathrm{CF},B} &:= \{z \in \mathbb{C} \,|\, \text{The orbit represented by } z \text{ intersects } \partial\Omega \text{ transversally}\} \subseteq \mathbb{C}\,.
\end{aligned}
$$

Here, the "B" in the index of $\widetilde{\mathcal{O}}_{\mathrm{CF},B}$ indicates that for fixed Ω, this space looks different for every value of B. Remember that $\widetilde{\mathcal{O}}_{\mathrm{CF},B}$ is only defined for $B \neq 0$, see section 2.6. Also, let ω_{SLF} and ω_{CF} represent the symplectic forms restricted to the open subsets $\widetilde{\mathcal{O}}_{\mathrm{SLF}}$ and $\widetilde{\mathcal{O}}_{\mathrm{CF},B}$ of the orbit space, respectively.
Now consider the natural maps F_{SLF} and F_{CF} which map the phase space of Birkhoff billiards $\mathrm{PS}_{\mathrm{Birk}}$ onto the corresponding phase space of the orbit dynamics model, i.e. the sets $\widetilde{\mathcal{O}}_{\mathrm{SLF}}$ and $\widetilde{\mathcal{O}}_{\mathrm{CF},B}$, respectively, see Fig. 12.

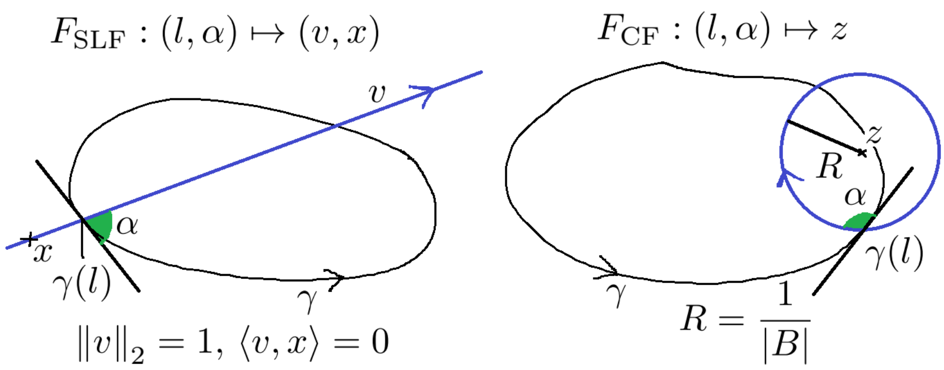


Figure 12: The mapping F from $\mathrm{PS}_{\mathrm{Birk}}$ to the phase space of the orbit dynamics model

Following definition will be useful later:

Definition 2.19. *The* signed curvature κ *of the curve* γ *at* $l \in \mathbb{S}^1_L$ *is given by*

$$\kappa(l) := \|\gamma''(l)\|_2 \mathrm{sign}(\langle \gamma''(l)\,,\, i\gamma'(l)\rangle)\,.$$

Because γ *is parametrized by unit length,* $\gamma''(l) \perp \gamma'(l)$ *for all* $l \in \mathbb{S}^1_L$*, and because* γ *goes along* $\partial\Omega$ *in counterclockwise orientation, the sign of* $\kappa(l)$ *is positive if* $\gamma''(l)$ *is inwards pointing, and clockwise if* $\gamma''(l)$ *is outwards pointing.*

The signed curvature is well-defined and smooth. In particular, the case where the scalar product $\langle \gamma''(l)\,,\, i\gamma'(l)\rangle$ becomes zero does not pose a problem even though no sign is defined for zero, because this expression only becomes zero when $\gamma''(l)$ is zero, and then $\kappa(l)$ is zero as well.
The next proposition compares the symplectic structures of the Birkhoff billiards and the orbit dynamics model.

Proposition 2.20. *Let*

$$F_{\mathrm{SLF}} : \mathrm{PS}_{\mathrm{Birk}} \longrightarrow \widetilde{\mathcal{O}}_{\mathrm{SLF}} \quad \textit{and} \quad F_{\mathrm{CF}} : \mathrm{PS}_{\mathrm{Birk}} \longrightarrow \widetilde{\mathcal{O}}_{\mathrm{CF}}$$

be the maps which map (l, α) *to the extension of the billiard trajectory segment from* (l, α) *to* $T_B(l, \alpha)$ *as represented in the corresponding orbit space* $\widetilde{\mathcal{O}}_{\mathrm{SLF}}$ *or* $\widetilde{\mathcal{O}}_{\mathrm{CF},B}$*. These maps are diffeomorphisms and preserve the symplectic structure, i.e.*

$$(F_{\mathrm{SLF}})^* \omega_{\mathrm{SLF}} = (F_{\mathrm{CF}})^* \omega_{\mathrm{CF}} = \omega_{\mathrm{Birk}}.$$

Proof. Start with the straight line flow. For $(l, \alpha) \in \mathrm{PS}_{\mathrm{Birk}}$, the point $(\gamma(l), e^{i\alpha}\gamma'(l))$ lies in the corresponding orbit in $Q = \mathbb{R}^2 \times \mathbb{S}^1 \subseteq \mathbb{R}^2 \times \mathbb{R}^2$, where $\mathbb{R}^2$ is identified with $\mathbb{C}$ via the standard complex structure. To determine the point in $\widetilde{\mathcal{O}}_{\mathrm{SLF}}$ corresponding to this orbit, follow the projection map $\widetilde{\pi}$ defined in Prop. 2.11, this leads to following description of F_{SLF}:

$$F_{\mathrm{SLF}} : \mathrm{PS}_{\mathrm{Birk}} \longrightarrow \widetilde{\mathcal{O}}_{\mathrm{SLF}}, \quad (l, \alpha) \mapsto \big(e^{i\alpha}\gamma'(l)\,,\, \gamma(l) - \big\langle \gamma(l), e^{i\alpha}\gamma'(l)\big\rangle e^{i\alpha}\gamma'(l)\big).$$

This map is apparently smooth, and surjectivity follows by construction, since by the regularity condition in Def. 2.15, every orbit in $\widetilde{\mathcal{O}}_{\mathrm{SLF}}$ has an intersection with $\partial\Omega$ which lies in P^{out}, and this point is represented in $\mathrm{PS}_{\mathrm{Birk}}$ in the coordinates (l, α). Injectivity follows from the fact that every element $(l, \alpha) \in \mathrm{PS}_{\mathrm{Birk}}$ uniquely represents an orbit in $\widetilde{\mathcal{O}}_{\mathrm{SLF}}$, by construction.
Now let $(l, \alpha) \in \mathrm{PS}_{\mathrm{Birk}}$ and $(L, A) \in T_{(l,\alpha)}\mathrm{PS}_{\mathrm{Birk}}$, then

$$\begin{aligned}
\mathrm{d}F_{\mathrm{SLF}}\big|_{(l,\alpha)}(L, A) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} F_{\mathrm{SLF}}(l + tL, \alpha + tA) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Big(e^{i(\alpha+tA)}\gamma'(l+tL)\,,\, \gamma(l+tL) - \Big\langle \gamma(l+tL)\,,\, e^{i(\alpha+tA)}\gamma'(l+tL)\Big\rangle e^{i(\alpha+tA)}\gamma'(l+tL)\Big) \\
&= \big(A \cdot ie^{i\alpha}\gamma'(l) + L \cdot e^{i\alpha}\gamma''(l)\,, L\gamma'(l) - \big\langle L\gamma'(l)\,,\, e^{i\alpha}\gamma'(l)\big\rangle e^{i\alpha}\gamma'(l) \\
&\quad - \big\langle \gamma(l)\,,\, A \cdot ie^{i\alpha}\gamma'(l) + L \cdot e^{i\alpha}\gamma''(l)\big\rangle e^{i\alpha}\gamma'(l) - \big\langle \gamma(l)\,,\, e^{i\alpha}\gamma'(l)\big\rangle \big(A \cdot ie^{i\alpha}\gamma'(l) + L \cdot e^{i\alpha}\gamma''(l)\big)\big)\,.
\end{aligned}$$

The following simplifications can be made:

$$\gamma''(l) = \kappa(l) \cdot i\gamma'(l) \quad \text{and} \quad \big\langle \gamma'(l) \,,\, e^{i\alpha}\gamma'(l) \big\rangle = \cos\alpha \,.$$

Here, κ is the signed curvature of γ. To simplify notation, omit the l-dependancy of all functions (e.g. write "γ'" instead of "$\gamma'(l)$"). Then the expression for the differential of F_{SLF} simplifies to

$$\begin{aligned}
&\mathrm{d}F_{\mathrm{SLF}}\big|_{(l,\alpha)}(L, A) \\
&= \big((A+\kappa L)ie^{i\alpha}\gamma' \,,\, L\gamma' - L\cos\alpha \cdot e^{i\alpha}\gamma' - \big\langle \gamma \,,\, (A+\kappa L)ie^{i\alpha}\gamma' \big\rangle e^{i\alpha}\gamma' - \big\langle \gamma \,,\, e^{i\alpha}\gamma' \big\rangle (A+\kappa L)ie^{i\alpha}\gamma' \big) \,.
\end{aligned}$$

Now, this differential is injective: Let $\mathrm{d}F_{\mathrm{SLF}}\big|_{(l,\alpha)}(L, A) = 0$, this implies $(A+\kappa L)ie^{i\alpha}\gamma' = 0$, and these two statements together imply $L\gamma' - L\cos\alpha \cdot e^{i\alpha}\gamma' = 0$. Because $\gamma' \neq 0$ and $0 < \alpha < \pi$, it holds that $\gamma' \neq \cos\alpha \cdot e^{i\alpha}\gamma'$, so $L = 0$. Together with $A + \kappa L = 0$, this implies that $A = 0$.
For dimension reasons, the differential is then invertible, and thus by the inverse function theorem, F_{SLF} is a local diffeomorphism. Because F_{SLF} is bijective, this makes it a diffeomorphism.
Now let $(l, \alpha) \in \mathrm{PS}_{\mathrm{Birk}}$ and $(L, A), (\widetilde{L}, \widetilde{A}) \in T_{(l,\alpha)}\mathrm{PS}_{\mathrm{Birk}}$, then

$$\begin{aligned}
&(F_{\mathrm{SLF}})^*\omega_{\mathrm{SLF}}\big|_{(l,\alpha)}\Big((L, A) \,,\, (\widetilde{L}, \widetilde{A})\Big) \\
&= \underbrace{\omega_{\mathrm{SLF}}|_{(l,\alpha)}}_{=(-\mathrm{d}v\wedge\mathrm{d}x)\big|_{\widetilde{Q}}} \Big(\mathrm{d}F_{\mathrm{SLF}}\big|_{(l,\alpha)}(L, A) \,,\, \mathrm{d}F_{\mathrm{SLF}}\big|_{(l,\alpha)}(\widetilde{L}, \widetilde{A})\Big) \\
&= \Big\langle L\gamma' - L\cos\alpha \cdot e^{i\alpha}\gamma' - \big\langle \gamma \,,\, (A+\kappa L)ie^{i\alpha}\gamma' \big\rangle e^{i\alpha}\gamma' - \big\langle \gamma \,,\, e^{i\alpha}\gamma' \big\rangle (A+\kappa L)ie^{i\alpha}\gamma' \,,\, \Big(\widetilde{A}+\kappa\widetilde{L}\Big)ie^{i\alpha}\gamma' \Big\rangle \\
&\quad - \Big\langle \widetilde{L}\gamma' - \widetilde{L}\cos\alpha \cdot e^{i\alpha}\gamma' - \Big\langle \gamma \,,\, (\widetilde{A}+\kappa\widetilde{L})ie^{i\alpha}\gamma' \Big\rangle e^{i\alpha}\gamma' - \big\langle \gamma \,,\, e^{i\alpha}\gamma' \big\rangle (\widetilde{A}+\kappa\widetilde{L})ie^{i\alpha}\gamma' \,,\, (A+\kappa L)ie^{i\alpha}\gamma' \Big\rangle \,.
\end{aligned}$$

Using the fact that

$$\langle \gamma' \,,\, i\gamma' \rangle = 0 \quad \text{and} \quad \big\langle e^{i\alpha}\gamma' \,,\, ie^{i\alpha}\gamma' \big\rangle = 0 \quad \text{and} \quad \big\langle \gamma' \,,\, ie^{i\alpha}\gamma' \big\rangle = -\sin\alpha \,,$$

the pullback simplifies to

$$\begin{aligned}
&(F_{\mathrm{SLF}})^*\omega_{\mathrm{SLF}}\big|_{(l,\alpha)}\Big((L, A), (\widetilde{L} \,,\, \widetilde{A})\Big) \\
&= -\sin\alpha \cdot L\Big(\widetilde{A}+\kappa\widetilde{L}\Big) - \big\langle \gamma \,,\, e^{i\alpha}\gamma' \big\rangle \Big\langle (A+\kappa L)ie^{i\alpha}\gamma' \,,\, \Big(\widetilde{A}+\kappa\widetilde{L}\Big)ie^{i\alpha}\gamma' \Big\rangle \\
&\quad + \sin\alpha \cdot \widetilde{L}(A+\kappa L) + \big\langle \gamma \,,\, e^{i\alpha}\gamma' \big\rangle \Big\langle \Big(\widetilde{A}+\kappa\widetilde{L}\Big)ie^{i\alpha}\gamma' \,,\, (A+\kappa L)ie^{i\alpha}\gamma' \Big\rangle \\
&= \sin\alpha\Big(A\widetilde{L} - \widetilde{A}L\Big) \\
&= \sin\alpha \,\mathrm{d}\alpha \wedge \mathrm{d}l\big|_{(l,\alpha)}\Big((L, A), (\widetilde{L} \,,\, \widetilde{A})\Big) \,.
\end{aligned}$$

This shows that $(F_{\mathrm{SLF}})^*\omega_{\mathrm{SLF}} = \omega_{\mathrm{Birk}}$.
Next, consider the circular flow. In $\widetilde{\mathcal{O}}_{\mathrm{CF},B}$, the circular orbit is described by its midpoint. For $(l, \alpha) \in \mathrm{PS}_{\mathrm{Birk}}$, the billiard trajectory segment from (l, α) to $T_B(l, \alpha)$ is a circular arc, and F_{CF} maps (l, α) to the midpoint of the circle corresponding to the extension of this circular arc. Because the velocity vector of this trajectory segment at $\gamma(l)$ is given by $e^{i\alpha}\gamma(l)$, follow the projection map $\widetilde{\pi}$ defined in Prop. 2.14 for the case $n = 1$ to determine the corresponding orbit in $\widetilde{\mathcal{O}}_{\mathrm{CF},B}$. Thus the map F_{CF} is given by

$$F_{\mathrm{CF}} : \mathrm{PS}_{\mathrm{Birk}} \longrightarrow \widetilde{\mathcal{O}}_{\mathrm{CF},B} \quad (l, \alpha) \mapsto \gamma(l) + \frac{1}{B} \cdot ie^{i\alpha}\gamma'(l).$$

This map is apparently smooth and, exactly as for the SLF case, surjectivity follows by construction because by the regularity condition in Def. 2.15, every orbit in $\widetilde{\mathcal{O}}_{\mathrm{CF},B}$ has an intersection with $\partial\Omega$ which lies in P^{out}, and this point is represented in $\mathrm{PS}_{\mathrm{Birk}}$ in the coordinates (l, α), and injectivity follows because every (l, α) uniquely represents an orbit in $\widetilde{\mathcal{O}}_{\mathrm{CF},B}$.

Now let $(l, \alpha) \in \mathrm{PS}_{\mathrm{Birk}}$ and $(L, A) \in T_{(l,\alpha)}\mathrm{PS}_{\mathrm{Birk}}$, then

$$\begin{aligned}
\mathrm{d}F_{\mathrm{CF}}\big|_{(l,\alpha)}(L, A) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} F_{\mathrm{CF}}(l + tL\,,\, \alpha + tA) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \gamma'(l + tL) + \frac{1}{B} \cdot ie^{i(\alpha+tA)}\gamma'(l + tL) \\
&= L\gamma'(l) - \frac{A}{B}e^{i\alpha}\gamma'(l) + \frac{L}{B}ie^{i\alpha}\gamma''(l) \\
&= L\gamma'(l) - \frac{1}{B}(A + \kappa L)e^{i\alpha}\gamma'(l)\,.
\end{aligned}$$

This differential is injective: Let $\mathrm{d}F_{\mathrm{CF}}\big|_{(l,\alpha)}(L, A) = 0$. Because $\gamma'(l)$ and $e^{i\alpha}\gamma'(l)$ are linearly independant, this implies $L = 0$ and $\frac{1}{B}(A + \kappa L) = 0$, which implies $A = 0$.
As above, dimension reasons imply that the differential is invertible, and thus by implicit function theorem, F_{CF} is a local diffeomorphism, which makes it a diffeomorphism because F_{CF} is bijective.
Now let $(l, \alpha) \in \mathrm{PS}_{\mathrm{Birk}}$ and $(L, A), (\widetilde{L}, \widetilde{A}) \in T_{(l,\alpha)}\mathrm{PS}_{\mathrm{Birk}}$, then

$$\begin{aligned}
&(F_{\mathrm{CF}})^*\omega_{\mathrm{CF}}\big|_{(l,\alpha)}\Big((L, A)\,,\,(\widetilde{L}, \widetilde{A})\Big) \\
&= \omega_{\mathrm{CF}}|_{(l,\alpha)}\Big(\mathrm{d}F_{\mathrm{CF}}\big|_{(l,\alpha)}(L, A)\,,\,\mathrm{d}F_{\mathrm{CF}}\big|_{(l,\alpha)}(\widetilde{L}, \widetilde{A})\Big) \\
&= B\,\mathrm{d}x \wedge \mathrm{d}y\bigg(L\gamma' - \frac{1}{B}(A + \kappa L)e^{i\alpha}\gamma'\,,\, \widetilde{L}\gamma' - \frac{1}{B}\Big(\widetilde{A} + \kappa\widetilde{L}\Big)e^{i\alpha}\gamma'\bigg)\,.
\end{aligned}$$

Here, $(x, y) = (\mathrm{Re}(z), \mathrm{Im}(z))$ are the real coordinates of $\mathbb{C} \cong \mathbb{R}^2$. The differential form $\mathrm{d}x \wedge \mathrm{d}y$ can instead be described using the complex coordinate z and its complex conjugate $\overline{z}$ as follows:

$$\mathrm{d}\overline{z} \wedge \mathrm{d}z = (\mathrm{d}x - i\,\mathrm{d}y) \wedge (\mathrm{d}x + i\,\mathrm{d}y) = \mathrm{d}x \wedge \mathrm{d}x + 2i\,\mathrm{d}x \wedge \mathrm{d}y - i^2\mathrm{d}y \wedge \mathrm{d}y = 2i\,\mathrm{d}x \wedge \mathrm{d}y\,.$$

Thus

$$\begin{aligned}
&(F_{\mathrm{CF}})^*\omega_{\mathrm{CF}}\big|_{(l,\alpha)}\Big((L, A)\,,\,(\widetilde{L}, \widetilde{A})\Big) \\
&= \frac{B}{2i}\,\mathrm{d}\overline{z} \wedge \mathrm{d}z\bigg(L\gamma' - \frac{1}{B}(A + \kappa L)e^{i\alpha}\gamma'\,,\, \widetilde{L}\gamma' - \frac{1}{B}\Big(\widetilde{A} + \kappa\widetilde{L}\Big)e^{i\alpha}\gamma'\bigg) \\
&= \frac{B}{2i}\bigg[\bigg(L\overline{\gamma'} - \frac{1}{B}(A + \kappa L)e^{-i\alpha}\overline{\gamma'}\bigg)\bigg(\widetilde{L}\gamma' - \frac{1}{B}\Big(\widetilde{A} + \kappa\widetilde{L}\Big)e^{i\alpha}\gamma\bigg) \\
&\quad - \bigg(L\gamma' - \frac{1}{B}(A + \kappa L)e^{i\alpha}\gamma\bigg)\bigg(\widetilde{L}\overline{\gamma'} - \frac{1}{B}\Big(\widetilde{A} + \kappa\widetilde{L}\Big)e^{-i\alpha}\overline{\gamma'}\bigg)\bigg] \\
&= \frac{B}{2i}\bigg(-\frac{1}{B}\bigg)\overline{\gamma'}\gamma'(e^{-i\alpha} - e^{i\alpha})\Big(\widetilde{L}(A + \kappa L) - L\Big(\widetilde{A} + \kappa\widetilde{L}\Big)\Big) \\
&= \underbrace{\overline{\gamma'}\gamma}_{=1}\,\frac{1}{2i}\underbrace{(e^{i\alpha} - e^{-i\alpha})}_{=2i\sin\alpha}\Big(A\widetilde{L} - \widetilde{A}L\Big) = \sin\alpha\,\mathrm{d}\alpha \wedge \mathrm{d}l\big|_{(l,\alpha)}\Big((L, A)\,,\,(\widetilde{L}, \widetilde{A})\Big)\,.
\end{aligned}$$

This shows that $(F_{\mathrm{CF}})^*\omega_{\mathrm{CF}} = \omega_{\mathrm{Birk}}$, and concludes the proof. $\square$

Conclusion

The Birkhoff billiards model describes the billiard dynamics in the two-dimensional case, whereby the phase space $\mathrm{PS}_{\mathrm{Birk}}$ and the symplectic form ω_{Birk} are identical for all values of magnetic field strength B. Thus, the billiard maps T_B constitute a whole family of symplectomorphisms, and the case $B = 0$ can in every way be viewed as a limit case of the magnetic billiards for $B \longrightarrow 0$. This is even true for the generating functions which are used to prove that the maps T_B are symplectomorphisms.
The symplectic form ω_{Birk} depends on Ω, since the coordinates l and α by which it is described are bound to the geometry of Ω.
In contrast, the orbit dynamics model describes a symplectic structure which is defined on the whole orbit space irrespective of Ω, and is then restricted to the open set of all orbits transversally intersecting Ω. Thus, here the symplectic structure is independant of the geometry.
However, even though for $B \neq 0$, limit case $B \longrightarrow 0$ of the Hamiltonian function, symplectic form and Hamiltonian flow on the total space $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ coincides with the case $B = 0$, in the resulting orbit space, the phase spaces of the magnetic and non-magnetic cases differ completley, and for the magnetic case, the symplectic form varies with B.
The symplectic structure on the orbit space is in some sense natural, since in dimension 2, for a fitting choice of coordinates, it directly emerges from the physical derivation. The fact that the symplectic stuctures defined on the two models are preserved by the natural correspondence in dimension 2 shows that the choice of ω_{Birk} is also natural.
The orbit dynamics in higher dimensions defined here leads to a description of higher-dimensional billiards, magnetic and non-magnetic, as a discrete dynamical system of a symplectic map, which can be further studied. Another avenue for further thought is the question whether higher-dimensional magnetic billiards has a description as a physical system, as is the case in dimension 2.

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