A Basmajian-type inequality for the indefinite orthogonal group

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Chapter 1

Introduction

The purpose of this work is to prove a Basmajian-type inequality for the indefinite orthogonal group $\mathrm{O}(2,n)$. If you are wondering what *Basmajian-type* means, then probably you do not know the *Basmajian identity*: this beautiful equality was published by Basmajian in 1993 ([Bas93]) and it allows us to compute the length of the boundary of a compact hyperbolic surface in function of the length of some particular kind of geodesic contained in this surface.

The Basmajian-type inequality proved in this thesis is, instead, a generalization working in the context of the Hermitian symmetric space associated to the Lie group $\mathrm{SO}_0(2,n)$, for $n \geq 3$.

Before of starting with the proper work, let me explain more in details what this Basmajian identity states and why one should consider exactly $\mathrm{SO}_0(2,n)$ for a possible generalization.

1.1 Basmajian identity

Consider a compact hyperbolic surface Σ with nonempty geodesic boundary $\partial\Sigma$, where *geodesic boundary* means that every connected component of the boundary of Σ is a closed geodesic. An *orthogeodesic* in Σ is a geodesic that is orthogonal to the boundary of Σ in both its endpoints:

Definition 1.1. *A geodesic $\alpha : [a,b] \to \Sigma$ is called* orthogeodesic *if $\alpha(a), \alpha(b) \in \partial\Sigma$ and α is orthogonal to the boundary $\partial\Sigma$ in $\alpha(a)$ and $\alpha(b)$.*

Given a boundary component $c \subset \partial\Sigma$, we denote with $\mathcal{O}_\Sigma^{\mathbb{H}^2}(c)$ the set of all orthogeodesics orthogonal to c and with $\mathcal{O}_\Sigma^{\mathbb{H}^2}$ the set of all orthogeodesics contained in Σ; in particular,

$$\mathcal{O}_\Sigma^{\mathbb{H}^2} = \bigcup_{\substack{c \text{ boundary} \\ \text{component of } \Sigma}} \mathcal{O}_\Sigma^{\mathbb{H}^2}(c);$$

Let us consider the universal cover $\tilde{\Sigma}$ of the surface Σ as a subset of the hyperbolic plane $\mathbb{H}^2$. As the fundamental group $\pi_1(\Sigma)$ acts by isometries on $\tilde{\Sigma} \subset \mathbb{H}^2$, it acts by isometries also on $\mathbb{H}^2$ and, so, we have a representation $\pi_1(\Sigma) \to \mathrm{PSL}(2,\mathbb{R})$ of the fundamental group of Σ inside the isometry group of the hyperbolic plane.

The length of every boundary component $c \subset \Sigma$ is equal to the translational length of a peripheral element $\gamma \in \pi_1(\Sigma)$ translating along a lift $\tilde{c} \subset \mathbb{H}^2$ of the component c:

$$\ell(c) = d(z, \gamma \cdot z)$$

for every $z \in \tilde{c}$.

On the other hand, if $\tilde{d} \subset \mathbb{H}^2$ is a lift (different from $\tilde{c}$) of a boundary component $d \subset \partial\Sigma$ (d may also coincide with c), then it is possible to write

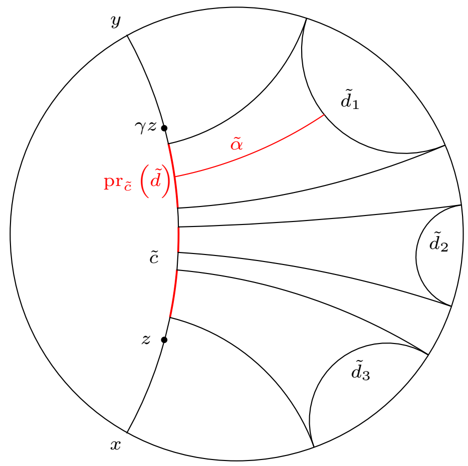

Figure 1.1: Universal cover $\tilde{\Sigma}$ as a subset of the hyperbolic plane $\mathbb{H}^2$ (represented via the Poincaré disk model).

the length of the projection of $\tilde{d}$ on the geodesic $\tilde{c}$ in function of the length of the geodesic segment $\tilde{\alpha}$, orthogonal in its endpoints to both $\tilde{c}$ and $\tilde{d}$:

$$\ell\left(\mathrm{pr}_{\tilde{c}}(\tilde{d})\right) = 2\log\coth\frac{\ell(\tilde{\alpha})}{2},$$

where $\mathrm{pr}_{\tilde{c}}$ is the orthogonal projection on $\tilde{c}$. Notice also that the geodesic segment $\tilde{\alpha}$ is a lift of an orthogeodesic $\alpha \subset \Sigma$ orthogonal to both c and d and it is uniquely determined by $\tilde{c}$ and $\tilde{d}$.

From these two equalities it follows (not easily) the identity

$$\ell(c) = 2\sum_{\alpha\in\mathcal{O}_\Sigma^{\mathbb{H}^2}(c)} \log\coth\frac{\ell(\alpha)}{2}$$

(the hard step is to prove that the set of points not contained in any projection of a lift of a boundary component has measure zero).

Finally, repeating the same reasoning for every boundary component, we have the Basmajian identity.

Theorem ([Bas93])**.** *Let Σ be an hyperbolic surface with nonempty geodesic boundary $\partial\Sigma$, then*

$$\ell(\partial\Sigma) = 4\sum_{\alpha\in\mathcal{O}_\Sigma^{\mathbb{H}^2}} \log\coth\frac{\ell(\alpha)}{2}.$$

1.2 The reason behind $\mathrm{SO}_0(2,n)$

As already observed in the previous section, the Basmajian identity works with the action of the fundamental group $\pi_1(\Sigma)$ on the hyperbolic plane $\mathbb{H}^2$, where $\mathbb{H}^2$ is exactly the symmetric space associated to $\mathrm{PSL}(2, \mathbb{R})$; in other words, we have a representation

$$h : \pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbb{R}).$$

Instead of $\mathrm{PSL}(2, \mathbb{R})$ and $\mathbb{H}^2$, one can try to consider a representation

$$\rho : \pi_1(\Sigma) \to G$$

in a more general Lie group G and study the action of $\rho(\pi_1(\Sigma))$ on the symmetric space $\mathcal{X} = G/K$ associated to G.

However, not any choice of such a Lie group G is suitable for a generalization: for example, in the proof of the Basmajian identity it is necessary to work with the orthogonal projection and, so, it is reasonable to require that something similar happens in the symmetric space $\mathcal{X}$; as the orthogonal protection is not always well defined, it may be convenient to consider only Lie groups of noncompact type, so that the associated symmetric space $\mathcal{X}$ is a nonpositively curved manifold.

Considerations like this may suggest taking into account only Hermitian symmetric spaces, that are symmetric spaces with an Hermitian structure preserved by the inversion symmetries.

In [Car35], Cartan proved that a symmetric space is Hermitian if and only if it is associated to one of the following Lie group:

- $\mathrm{SU}(n, m)$,
- $SO^*(2n)$,
- $\mathrm{Sp}(2n, \mathbb{R})$,
- $SO_0(2, n)$,
- exceptional Lie group $E_7(-25)$ and $E_6(-14)$.

Moreover, in [FP16] Pozzetti and Fanoni already proved four Basmajian type inequalities for the sympletic group $\mathrm{Sp}(2n, \mathbb{R})$ and so it makes sense to consider one of the other groups appearing in the Cartan classification. In particular, for this work, we choose to study the identity component of the indefinite orthogonal group $\mathrm{SO}_0(2, n)$.

More details about the Hermitian Lie group $\mathrm{SO}_0(2, n)$ can be found in Section 2.1, while a general description of the symmetric space associated is contained in Section 2.2.

1.3 Towards a generalization

As you may expect, it is not so easy to obtain our generalization and there are some problems that one has to deal with.

First of all, there is no obvious way to associate a peripheral element in $\pi_1(\Sigma)$ with a geodesic in $\mathcal{X}_{2,n}$ (the symmetric space associated to $\mathrm{SO}_0(2,n)$): in the proof of the Basmajian identity, given a peripheral element $\gamma \in \pi_1(\Sigma)$ corresponding to a boundary component c, we know that there exists a unique geodesic $\tilde{c} \subset \mathbb{H}^2$ (lift of the boundary component c) translated by the action of γ; in particular, the two endpoints (at infinity) of this geodesic $\tilde{c}$ are the unique points fixed by the action of γ on $\partial\mathbb{H}^2$; in other words, we can associate each peripheral element $\gamma \in \pi_1(\Sigma)$ with the geodesic whose endpoints in the boundary $\partial\mathbb{H}^2$ are fixed by γ.

With a representation $\rho : \pi_1(\Sigma) \to \mathrm{SO}_0(2,n)$ the situation is more complicated: once described the (visual) boundary $\partial\mathcal{X}_{2,n}$ of $\mathcal{X}_{2,n}$ and observed that there is a natural action of the group $\mathrm{SO}_0(2,n)$ on this boundary, it is not hard to notice that this action is not transitive (Section 2.3); as this property is fundamental in order to get our generalization, we need to replace $\partial\mathcal{X}_{2,n}$ with a compact orbit contained in there. We will see in Chapter 3 that a suitable subset of $\partial\mathcal{X}_{2,n}$ is given by the set of isotropic lines $\mathrm{Is}_1(\mathbb{R}^{2,n})$; in the same chapter it is also contained a description of this set.

We still have two more problems: the first one is that, given two points in $\mathrm{Is}_1(\mathbb{R}^{2,n})$, the geodesic between these two points, whenever exists, is not unique; for this reason we replace the notion of geodesic with the one of $\mathbb{R}$-*tube*, a kind of higher dimensional geodesic (Section 4.2).

The second problem is that, given a peripheral element $\gamma \in \pi_1(S)$, there is no guarantee that the action of $\rho(\gamma)$ fixes two points in $\mathrm{Is}_1(\mathbb{R}^{2,n})$ and so it is not possible to associate γ with an $\mathbb{R}$-tube. We will see in Chapter 5 that a possible solution is to consider only *Anosov maximal representations*.

Finally, in Chapter 6, we will prove our Basmajian-type inequality:

Theorem A. *Let Σ be an hyperbolic surface with nonempty geodesic boundary and $\rho : \pi_1(\Sigma) \to \mathrm{SO}_0(2,n)$ an Anosov maximal representation. For every peripheral element $\gamma \in \pi_1(\Sigma)$, it holds*

$$\ell^F(\rho(\gamma)) \geq \sum_{\alpha \in \mathcal{O}_\Sigma^{\mathcal{X}_{2,n}}(\gamma)} 2 \log \coth \frac{\ell^F(\alpha)}{2}.$$

In this inequality the superscript F means that we are considering a Finsler metric: in Chapter 4 we will see that there are several $\mathrm{SO}_0(2,n)$-invariant metric on the manifold $\mathcal{X}_{2,n}$; among the others, the Finsler one we consider is particularly suitable for our purpose.

Chapter 2

The Positive Grassmannian

The *indefinite orthogonal group* $\mathrm{O}(p,q)$ is the Lie Group of linear transformations of $\mathbb{R}^{p+q}$ leaving invariant a symmetric bilinear form of signature (p,q). This Lie group, endowed with the Euclidean topology, consists of four connected components, of which $\mathrm{SO}_0(p,q)$ is the one containing the identity. The symmetric space associated to $\mathrm{SO}_0(p,q)$ is called *positive Grassmannian* $\mathrm{Gr}_p^+(\mathbb{R}^{p,q})$.

In the case $q > p = 2$, a lot of interest arises from the fact that the symmetric space $\mathrm{Gr}_2^+(\mathbb{R}^{2,q})$ is, in particular, an Hermitian Symmetric space, that is a symmetric manifold with a *compatible* Hermitian structure.

In this chapter we introduce in more details the indefinite orthogonal group $\mathrm{O}(p,q)$ and the associated symmetric space. General definition and results about symmetric spaces can be found in [Hel79].

2.1 The orthogonal indefinite group

Given a symmetric matrix $A \in \mathrm{Sym}(n,\mathbb{R})$, the bilinear form associated to A is the function $h_A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$h_A(v,w) = v^T A w,$$

for every $v, w \in \mathbb{R}^n$. The signature of the bilinear form is the signature of the matrix A.

The *standard bilinear form $h_{p,q}$ of signature* (p,q) is the bilinear form associated to the matrix

$$\mathrm{Id}_{p,q} = \left(\begin{array}{c|c} \mathrm{Id}_p & \\ \hline & -\mathrm{Id}_q \end{array}\right);$$

explicitly, if $\{g_1, ..., g_{p+q}\}$ is the canonical basis of $\mathbb{R}^{p+q}$ and $v = \sum_{i=1}^{p+q} v_i g_i$, $w = \sum_{i=1}^{p+q} w_i g_i$ are two elements in $\mathbb{R}^{p+q}$, then

$$h_{p,q}(v,w) = \sum_{i=1}^{p} v_i w_i - \sum_{j=p+1}^{q} v_j w_j.$$

We denote with $\mathbb{R}^{p,q}$ the space $\mathbb{R}^{p+q}$ endowed with the bilinear form $h_{p,q}$:

$$\mathbb{R}^{p,q} = \left(\mathbb{R}^{p+q}, h_{p,q}\right).$$

The *indefinite orthogonal group* is the set of matrices preserving $h_{p,q}$, that coincides with the Lie group

$$\mathrm{O}(p,q) = \{g \in \mathrm{GL}(\mathbb{R}^{p+q}) | g^t \,\mathrm{Id}_{p,q}\, g = \mathrm{Id}_{p,q}\};$$

if a matrix in $\mathrm{O}(p,q)$ preserves the orientation of the space, then it is also an element of the *indefinite special orthogonal group*:

$$\mathrm{SO}(p,q) = \mathrm{O}(p,q) \cap \mathrm{SL}(\mathbb{R}^{p+q}).$$

If $p, q > 0$, then there exist two subspaces $V^+, V^- \subset \mathbb{R}^{p,q}$ such that $\mathbb{R}^{p,q} \cong V^+ \oplus V^-$ and the induced bilinear forms $h_{p,q}|_{V^+}$ and $h_{p,q}|_{V^-}$ are, respectively, positive and negative definite. Via this decomposition, it is possible to see that $\mathrm{SO}(p, q)$ is the union of two connected components: one preserving the orientations of V^+ and V^- and one reversing both.

To make this idea more precise, let us consider the projections

$$\mathrm{pr}_+ : \mathbb{R}^{p,q} \to V^+$$

$$\mathrm{pr}_- : \mathbb{R}^{p,q} \to V^-$$

of the space $\mathbb{R}^{p,q}$ on V^+, respectively V^- with respect to the decomposition $\mathbb{R}^{p,q} = V^+ \oplus V^-$.

Definition 2.1. *An element $g \in \mathrm{SO}(p, q)$* preserves the time orientation *if*

$$\det(\mathrm{pr}_+ \circ g|_{V^+}) > 0;$$

the element g reverse the time orientation *if*

$$\det(\mathrm{pr}_+ \circ g|_{V^+}) < 0.$$

Because elements in $\mathrm{SO}(p, q)$ preserve the orientation of the whole space $\mathbb{R}^{p,q}$, if $g \in \mathrm{SO}(p, q)$ preserves the time orientation, then we also have that $\det(\mathrm{pr}_- \circ g|_{V^-}) > 0$: in this sense, g preserves the orientation of both V^+ and V^-, even if it is not necessarily true that $gV^+ \subset V^+$ and $gV^- \subset V^-$.

Moreover, it is possible to see that the definition of *time orientation* does not depend on the choice of the decomposition $V^+ \oplus V^-$.

The connected component of $\mathrm{SO}(p, q)$ containing the identity consists of all those elements preserving time orientation:

$$\mathrm{SO}_0(p, q) = \{g \in \mathrm{SO}(p, q) | g \text{ preserves time orientation}\}.$$

The canonical basis is not always the most convenient one and often computations are easier when $h_{p,q}$ is expressed through a matrix different from $\mathrm{Id}_{p,q}$; for example, suppose $q \geq p$ and consider the matrix

$$J_{p,q} = \left(\begin{array}{c|c|c} 0 & 0 & \mathrm{Id}_p \\ \hline 0 & -\mathrm{Id}_{q-p} & 0 \\ \hline \mathrm{Id}_p & 0 & 0 \end{array}\right). \tag{2.1}$$

Because $J_{p,q}$ and $\mathrm{Id}_{p,q}$ have the same eigenvalues, there exists an orthogonal matrix P such that $\mathrm{Id}_{p,q} = P^t J_{p,q} P$; this implies that they are conjugate and so they represent the same bilinear form with respect to two different bases. We denote with $\{g_i\}_{i=1}^{p+q}$ the elements of the basis for which $h_{p,q}$ is represented by $\mathrm{Id}_{p,q}$ (that is the canonical basis) and with $\{e_i\}_{i=1}^{p+q}$

the ones for which $h_{p,q}$ is expressed through $J_{p,q}$. In order to avoid any confusion, we will use the notation

$$G_{p,q} = \{g \in \mathrm{SL}(\mathbb{R}^{p+q}) | g^t \, \mathrm{Id}_{p,q} \, g = \mathrm{Id}_{p,q}, \; g \text{ preserves time orientation}\}, \tag{2.2}$$

$$H_{p,q} = \{g \in \mathrm{SL}(\mathbb{R}^{p+q}) | g^t J_{p,q} g = J_{p,q}, \; g \text{ preserves time orientation}\}. \tag{2.3}$$

Notice also that these two groups are Lie groups and, so, it makes sense to talk about tangent space; in particular, the algebra

$$\begin{aligned} \mathfrak{g}_{p,q} := \mathrm{T}_{\mathrm{Id}} G_{p,q} &= \left\{ g \in \mathrm{Mat}(p+q, \mathbb{R}) | g^t \, \mathrm{Id}_{p,q} + \mathrm{Id}_{p,q} \, g = 0 \right\} = \\ &= \left\{ \left(\begin{array}{c|c} A & B \\ \hline B^T & D \end{array} \right) \in \mathrm{Mat}(p+q, \mathbb{R}) \middle| A, D \text{ skew-sym.} \right\}, \end{aligned} \tag{2.4}$$

is the tangent space in Id of the Lie group $G_{p,q}$, while

$$\begin{aligned} &\mathfrak{h}_{p,q} := \mathrm{T}_{\mathrm{Id}} H_{p,q} = \{ g \in \mathrm{Mat}(p+q, \mathbb{R}) | g^t J_{p,q} + J_{p,q} g = 0 \} = \\ &= \left\{ \left(\begin{array}{c|c|c} A & B & C \\ \hline D & E & B^T \\ \hline G & D^T & -A^T \end{array} \right) \in \mathrm{Mat}(p+(p-q)+p, \mathbb{R}) \middle| C, E, G \text{ skew-sym} \right\} \end{aligned} \tag{2.5}$$

is the tangent space in Id of the Lie group $H_{p,q}$.

The two different representations give raise to two different expressions of the bilinear form $h_{p,q}$, that we denote with $\langle , \rangle$ and $\langle\!\langle , \rangle\!\rangle$:

$$\langle v, w \rangle := v^t \, \mathrm{Id}_{p,q} \, w, \qquad \langle\!\langle v, w \rangle\!\rangle := v^t J_{p,q} w.$$

In the case we want to refer to the identity component of the indefinite orthogonal group, independently from the choice of the basis, we keep the notation $\mathrm{SO}_0(p,q)$ with bilinear form $h_{p,q}$ and tangent space $\mathfrak{so}(p,q)$.

2.2 Positive Grassmannian

The *positive Grassmannian* $\mathrm{Gr}_p^+(\mathbb{R}^{p,q})$ is a subset of the standard Grassmannian consisting of all those subspaces $V \subset \mathbb{R}^{p,q}$ on which the bilinear form $h_{p,q}$ restricts to a positive definite bilinear form $h_{p,q}|_V$,

$$\mathrm{Gr}_p^+(\mathbb{R}^{p,q}) = \{ V \subset \mathbb{R}^{p,q} | \dim V = p, h_{p,q}|_V \text{ pos. def.} \}.$$

A rich geometric structure on the positive Grassmannian comes from the fact that it is the symmetric space associate to the Lie group $\mathrm{SO}_0(p,q)$. The purpose of this section is to explore this symmetric structure.

2.2.1 Symmetric space

Let G be a connected Lie group and $K < G$ a closed subgroup; the pair (G, K) is called *Riemannian symmetric pair* if there exists an involutive automorphism

$$\theta : G \to G$$

such that $G_0^\theta \subset K \subset G^\theta$, where G^θ is the set of elements in G fixed by θ and G_0^θ is the connected component of G^θ containing the identity; In this context the quotient $\mathcal{X} = G/K$ is the *symmetric space associated* to G.

The name *Riemannian symmetric pair* is justified by the following result:

Theorem 2.2 [Hel79, Prop. 3.4]. *If (G, K) is a Riemannian symmetric pair, then there exists a G-invariant Riemmanian metric on the homogeneous space $\mathcal{X} := G/K$ and $\mathcal{X}$ is a symmetric space with respect to every such metric.*

Consider, now, the Lie group $G = G_{p,q}$ (defined in Equation (2.2)) with closed subgroup $K_{p,q} = \mathrm{S}\left(\mathrm{O(p)} \times \mathrm{O(q)}\right)$; then $(G_{p,q}, K_{p,q})$ is a Riemannian symmetric pair with respect to the involution

$$\begin{array}{rccc} \theta : & G_{p,q} & \to & G_{p,q} \\ & X & \mapsto & \left(X^{-1}\right)^T \end{array}$$

and the symmetric space associated is

$$\mathcal{X}_{p,q} = \frac{\mathrm{SO}_0(p, q)}{\mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q))}$$

In order to prove that the symmetric space $\mathcal{X}_{p,q}$ and the positive Grassmannian $\mathrm{Gr}_p^+(\mathbb{R}^{p,q})$ coincide, consider the action of $\mathrm{SO}_0(p, q)$ on $\mathrm{Gr}_p^+(\mathbb{R}^{p,q})$ given by

$$g \cdot \mathrm{Span}\langle v_1, ... \, v_p\rangle = \mathrm{Span}\langle g \cdot v_1, ... \, g \cdot v_p\rangle;$$

this action is transitive and, moreover,

$$\mathrm{Stab}_{\mathrm{SO}_0(p,q)}(\mathrm{Span}\langle g_1, ..., g_p\rangle) = \mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q)) = K_{p,q};$$

it follows

$$\mathcal{X}_{p,q} = \mathrm{Gr}_p^+(\mathbb{R}^{p,q}).$$

If we consider the Lie group $H_{p,q}$ instead of $G_{p,q}$, the resulting symmetric space is still $\mathcal{X}_{p,q}$: this is an immediate consequence of the fact that $H_{p,q}$ and $G_{p,q}$ are conjugate subgroups in the group $\mathrm{SL}(\mathbb{R}^{p+q})$.

Until now we worked on the group $G_{p,q}$, however many computations result to be easier with the group $H_{p,q}$.

2.2.2 Cartan decomposition

Let us go back to the general theory about symmetric spaces: the involution θ on the Lie group G induces the so-called *Cartan involution*

$$\sigma := d\theta,$$

that is an involutive automorphism on the tangent space $\mathfrak{g} = \mathrm{T}_e G$.

As $\sigma^2 = \mathrm{Id}$, the only two possible eigenvalues of this automorphism are 1 and -1; the associated eigenspaces in $\mathfrak{g}$ are often denoted with $\mathfrak{k} = \mathrm{E}_1(\sigma)$ and $\mathfrak{p} := \mathrm{E}_{-1}(\sigma)$ and they are, respectively, the tangent space of K and the tangent space of the symmetric space $\mathcal{X}$. The splitting $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ is called *Cartan decomposition.*

In $H_{p,q}$ the Cartan involution is given by

$$\begin{array}{rccc} \sigma : & \mathfrak{h}_{p,q} & \to & \mathfrak{h}_{p,q} \\ & h & \mapsto & -h^T \end{array}$$

and so the tangent space of $K_{p,q}$ is

$$\mathrm{T}_{\mathrm{Id}} K_{p,q} = \mathfrak{k}_{p,q} = \left\{ \left(\begin{array}{c|c|c} A & B & C \\ \hline -B^T & E & B^T \\ \hline -C^T & -B & -A^T \end{array} \right) \middle| \, A, C, E \text{ skew-sym} \right\},$$

while the tangent space of the symmetric space $\mathcal{X}_{p,q}$ is

$$\mathrm{T}_e \mathcal{X}_{p,q} = \mathfrak{p}_{p,q} = \left\{ \left(\begin{array}{c|c|c} A & B & C \\ \hline B^T & 0 & B^T \\ \hline C^T & B & -A^T \end{array} \right) \middle| \, \begin{array}{l} C \text{ skew-sym} \\ A \text{ sym} \end{array} \right\},$$

where $e \in \mathcal{X}_{p,q}$ is the image, in $\mathcal{X}_{p,q}$, of $\mathrm{Id} \in H_{p,q}$ via the quotient map.

From this it follows also that the dimension of $\mathcal{X}_{p,q}$ is pq.

An important tool used to distinguish between different types of Lie groups is given by the *Killing form*: this bilinear form is defined on every Lie algebra $\mathfrak{g}$ via

$$\begin{array}{rccc} \mathbb{B} : & \mathfrak{g} \times \mathfrak{g} & \to & \mathbb{R} \\ & (X, Y) & \mapsto & \mathrm{tr}(\mathrm{ad}\, X \,\mathrm{ad}\, Y) \end{array}$$

where the map ad is called *adjoint operator* and defined via

$$(\mathrm{ad}\, X)(Y) = [X, Y].$$

The Killing form is always negative definite on $\mathfrak{k}$, while the sign it takes on the vector space $\mathfrak{p}$ determines the geometry of the symmetric space associate to G; in particular, if $\mathbb{B}|_{\mathfrak{p}\times\mathfrak{p}}$ is negative definite, then the Lie algebra $\mathfrak{g}$ and the Lie group G are called of *non-compact type* and the symmetric space

$\mathcal{X} = G/K$ is an *Hadamard manifold* (with respect to the Riemannian metric defined in Theorem 2.2), that means that $\mathcal{X}$ is a complete, simply connected, nonpositively curved manifold.

In particular, if the Lie algebra $\mathfrak{g}$ is a subspece of $\mathfrak{sl}(n, \mathbb{R})$ (and this is the case for $\mathfrak{g} = \mathfrak{h}_{p,q}$), then the Killing form can be expressed more explicitly via

$$\mathbb{B}_{\mathfrak{sl}(n,\mathbb{R})}(X, Y) = 2n \operatorname{tr}(XY).$$

This bilinear form restricts to a positive definite bilinear form

$$\mathbb{B}_{\mathfrak{sl}(n,\mathbb{R})}|_{\mathfrak{p}_{p,q} \times \mathfrak{p}_{p,q}}$$

on $\mathfrak{p}_{p,q}$ and so $\mathrm{SO}_0(p,q)$ is of non-compact type, while the positive Grassmannian $\mathcal{X}_{p,q}$ is an Hadamard manifold.

For simplicity, we consider a scaled bilinear form on $\mathfrak{h}_{p,q}$, given by

$$B(X, Y) = \frac{1}{2} \operatorname{tr}(XY). \tag{2.6}$$

2.2.3 Weyl chamber

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian ($[X, Y] = 0$ for every $X, Y \in \mathfrak{a}$); a lienear functional $\alpha \in \mathfrak{a}^*$ is a *root* if $\alpha \neq 0$ and there exists an $X \in \mathfrak{g}$ such that

$$[H, X] = \alpha(H)X,$$

for all $H \in \mathfrak{a}$. A vector $X \in \mathfrak{a}$ is called *regular* if there is no root vanishing in X. If Δ denote the set of all regular vectors in $\mathfrak{a}$, then every connected component of $\mathfrak{a} \setminus \Delta$ is a *Weyl chamber*.

Weyl chambers represent useful tools in the study of the tangent space of a symmetric space: by definition, the Lie group G acts by isometries on the symmetric space $\mathcal{X} = G/K$; this action, at the same time, induces an action on the tangent space $\mathrm{T}\mathcal{X}$,

$$g \cdot X := \exp_{gx}^{-1} \left(g \exp_x(X) \right), \tag{2.7}$$

for every $x \in \mathcal{X}$, $X \in T_x\mathcal{X}$ and $g \in G$, that is not transitive; in this context, every *Weyl chamber* is a fundamental domain for the G-action on the tangent space $\mathrm{T}\mathcal{X}$.

In the case of the identity component of the indefinite special orthogonal group, a maximal abelian is given by

$$\mathfrak{a}_{p,q} = \left\{ \left(\begin{array}{c|c|c} A & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & -A \end{array} \right) \middle| \, A \text{ diag.} \right\}.$$

Moreover, a vector

$$X = \mathrm{diag}(\lambda_1, ..., \lambda_p, 0, ...0, -\lambda_1, ..., -\lambda_p) \in \mathfrak{a}$$

is regular if and only if $\lambda_i \neq \lambda_j$, for all $i \neq j$ and so

$$\mathfrak{w}_{p,q} = \left\{ \left(\begin{array}{c|c|c} A & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & -A \end{array} \right) \middle| A = \mathrm{diag}(\lambda_1, ..., \lambda_p) \text{ with } \lambda_1 > ... > \lambda_p > 0 \right\}$$

is a Weyl chamber.

Remark 2.3. In the case of $\mathrm{SO}_0(p, q)$, the action on the tangent space can be written explicitely via

$$g \cdot X := g^{-1} X g.$$

for every $g \in \mathrm{SO}_0(p, q)$ and $X \in \mathfrak{p}_{p,q}$.

Remark 2.4. Even if each $X \in \mathfrak{w}_{p,q}$ is a matrix of dimension $(p+q) \times (p+q)$, there is a correspondence between the Weyl chamber and the set

$$\{ (\lambda_1, ..., \lambda_p) \in \mathbb{R}^p | \, \lambda_1 > ... > \lambda_p > 0 \} .$$

Keeping in mind this correspondence, we will often refer to vectors of dimension p as elements in the Weyl chamber.

Remark 2.5. Every matrix $X \in \mathfrak{p}_{p,q}$ represents a vector in the tangent space $\mathrm{T}_{\mathrm{Id}}(\mathrm{SO}(p, q))$ and so it corresponds to a geodesic in the Lie group $\mathrm{SO}(p, q)$ via the exponential map. In particular, for every element

$$X = \mathrm{diag}(a_1, ..., a_p, 0, ..., 0, -a_1, ..., -a_p)$$

in the maximal abelian $\mathfrak{a}_{p,q}$, the expression of such geodesic is given by

$$\gamma(t) = \left(\begin{array}{c|c|c} \begin{matrix} e^{a_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{a_p t} \end{matrix} & 0 & 0 \\ \hline 0 & \mathrm{Id}_q & 0 \\ \hline 0 & 0 & \begin{matrix} e^{-a_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{-a_p t} \end{matrix} \end{array} \right).$$

2.3 The visual boundary

As specified above, $\mathcal{X}_{p,q}$ is an Hadamard manifold and, for this reason, a CAT(0) manifold. An interesting object associated to this category of manifolds is the *boundary at infinity*, also called *visual boundary*. In this section we present briefly definition and properties about this boundary; further details can be found in [BGS85] or [Ebe96].

Let $\mathcal{X}$ a CAT(0) manifold; two unit speed geodesic rays $\rho, \rho' : \mathbb{R}^+ \to \mathcal{X}$ are called *asymptotic* if $d(\rho(t), \rho'(t))$ is a bounded function in t. The *visual boundary* $\partial\mathcal{X}$ of $\mathcal{X}$ is the set of classes of asymptotic rays.

Let us assume $\mathcal{X}$ is also a complete space; then for every $x \in \mathcal{X}$ and $\xi \in \partial\mathcal{X}$ there exists a unique geodesic ray $\rho : \mathbb{R}^+ \to \mathcal{X}$ such that ρ represents the class ξ and $\rho(0) = x$; we denote such ray with $\xi_x(t)$.

Fix a point $x \in \mathcal{X}$; given two classes of rays $\xi, \eta \in \partial\mathcal{X}$, the distance $\angle_x$ between ξ and η is defined as the angle in x between $\xi_x(t)$ and $\eta_x(t)$, that is

$$\angle_x(\xi, \eta) = \angle_x\left(\xi_x(t), \eta_x(t)\right).$$

This metric, also called *visual metric* in x, generates a topology on $\partial\mathcal{X}$.

Moreover, it is possible, via the *cone topology*, to extend the topology on $\mathcal{X}$ to the entire space $\mathcal{X} \cup \partial\mathcal{X}$ in such a way that the induced topology on $\partial\mathcal{X}$ is exactly the one generated by the visual metric.

Although the space $\mathcal{X} \cup \partial\mathcal{X}$ with the cone topology is homeomorphic to the unit closed ball of the same dimension of $\mathcal{X}$, the structure described is useful to study the action of G on the set of geodesics of the symmetric space associated. Even more, the G-action can be extended on the boundary of the symmetric space: indeed G acts by isometries on G/K, so it sends geodesics to geodesics and it preserves the relation of being asymptotic.

Let us go back to $\mathcal{X}_{p,q}$: this is a symmetric space associated to a Lie group of non-compact type and, for this reason, it can be embedded (almost) isometrically in the symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$ as a totally geodesic submanifold ([Ebe96, Theorem 2.6.5]); the space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$, on the other hand, is the symmetric space associated to the Lie group $\mathrm{SL}(n,\mathbb{R})$, of non-compact type, and so it is an Hadamard manifold; in particular, $\partial\mathcal{X}_{p,q}$ is a subset of the visual boundary $\partial(\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R}))$.

Definition 2.6. *A* flag $F = (V_1, ...V_k)$ *in a finite dimensional vector space* V *is a strictly increasing sequence of subspaces* $\{0\} \subset V_1 \subset .... \subset V_k = V$.

The flag is called complete flag if $\dim V_i = i$ *and* $k = n$.

Theorem 2.7 [Ebe96, Section 2.13.8]. *There is a correspondence between points at infinity in* $\partial(\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R}))$ *and the set of pairs* (λ, F)*, where* $\lambda = (\lambda_1, ..., \lambda_k)$ *is a vector and* $F = (V_1, ..., V_k)$ *is a flag in* $\mathbb{R}^n$ *such that*

- $\lambda_1 > \lambda_2 > ... > \lambda_k$,
- $\sum_{i=1}^k m_i \lambda_i = 0$,
- $\sum_{i=1}^k m_i \lambda_i^2 = 1$,

where $m_1 = \dim V_1$ *and* $m_i = \dim V_i - \dim V_{i-1}$ *for* $i = 1, ..., k-1$.

We sketch briefly how to recover a pair (λ, F) starting from a point $\xi \in \partial(\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R}))$: for every point at infinity ξ there exists a unit tangent vector $X \in \mathfrak{p}$ such that the geodesic ray $\exp_{\mathrm{Id}}(tX)$ represents the point ξ; let $\lambda_1 < \lambda_2 < \ldots < \lambda_k$ be the eigenvalues of X and E_i the eigenspaces corresponding to λ_i; then, denoting $V_i = \bigcup_{j \le i} E_j$, $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $F = (V_1, \ldots. V_k)$, we have that the pair (λ, F) agrees with the three requests of Theorem 2.7 and it represents the point ξ.

Because $\partial(\mathcal{X}_{p,q})$ is a subset of $\partial(\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R}))$, each point in $\partial(\mathcal{X}_{p,q})$ can be identified with a pair (λ, F) corresponding to a matrix $X \in \mathfrak{p}_{p,q}$.

We give now an even more detailed explanation of how this correspondence works in the positive Grassmannian $\mathcal{X}_{2,n} = \mathrm{Gr}_2^+(\mathbb{R}^{2,n})$. Let us start by considering a unit vector $X \in \overline{\mathfrak{w}}_{p,q}$ in the closure of the Weyl chamber,

$$X = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -A \end{pmatrix} \quad \text{with } A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ and } \begin{cases} \lambda_1 \ge \lambda_2 \ge 0, \\ 2\left(\lambda_1^2 + \lambda_2^2\right) = 1; \end{cases}$$

then the correspondent geodesic is

$$\gamma(t) = \left(\begin{array}{cc|c|cc} e^{\lambda_1 t} & 0 & 0 & 0 & \\ 0 & e^{\lambda_2 t} & & & \\ \hline \multicolumn{2}{c|}{0} & \mathrm{Id}_{n-2} & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{0} & 0 & e^{-\lambda_1 t} & 0 \\ & & & 0 & e^{-\lambda_2 t} \end{array}\right).$$

The eigenvalues of A are

$$(\lambda_1, \lambda_2, 0, -\lambda_2, -\lambda_1),$$

but it suffices to consider only the pair (λ_1, λ_2), because the knowledge of these two values determines the values of the others. The corresponding eigenspaces are $E_{\lambda_1} = \mathrm{Span}(e_1)$, $E_{\lambda_2} = \mathrm{Span}(e_2)$ (if $\lambda_1, \lambda_2 \neq 0$). So, $\gamma(t)$ represents the point $\xi = (\lambda, F) \in \partial\mathcal{X}_{2,n}$, where

1. $\lambda = (\lambda_1, \lambda_2)$ and $F = (\mathrm{Span}(e_1), \mathrm{Span}(e_1, e_2))$ if $\lambda_1 > \lambda_2 > 0$;
2. $\lambda = (1/2)$ and $F = (\mathrm{Span}(e_1, e_2))$ if $\lambda_1 = \lambda_2 > 0$;
3. $\lambda = (\sqrt{2}/2)$ and $F = (\mathrm{Span}(e_1))$ if $\lambda_1 > \lambda_2 = 0$.

Notice that in each of these cases the flag F consists only of isotropic vector spaces, namely subspeces of $\mathbb{R}^{2,n}$ on which the bilinear form $h_{2,n}$ vanishes (see the next chapter for further details).

Now let Y any unit vector in $\mathfrak{p}_{2,n}$ corresponding to a unit speed geodesic $\delta(t)$; as the Weyl chamber is a fundamental domain for the $\mathrm{SO}_0(2,n)$-action on the tangent space of $\mathcal{X}_{2,n}$, there exists an element $g \in \mathrm{SO}_0(2,n)$ such that

$$g^{-1}Yg = X \in \overline{\mathfrak{w}}_{2,n};$$

moreover, if $\gamma(t)$ is the geodesic ray associated to X, then $g^{-1}\delta(t)g = \gamma(t)$; now, calling $\xi = (\lambda, F) \in \partial\mathcal{X}_{2,n}$ the class of asymptotic rays containing $\gamma(t)$, the point at infinity corresponding to the geodesic ray δ is $(\lambda, g \cdot F)$, where the action of g on a flag $F = (V_1, ..., V_k)$ is defined by $g \cdot F = (gV_1, ..., gV_k)$.

Chapter 3

Action on the set of isotropic lines

As mentioned in the introduction, the action of $\mathrm{SO}_0(2,n)$ is not transitive on the visual boundary $\partial(\mathcal{X}_{2,n})$: this is an immediate consequence of the fact that the Weyl chamber is a fundamental domain for the action of $\mathrm{SO}_0(2,n)$ on tangent space $\mathrm{T}\mathcal{X}$. For this reason we need to shift the focus on a subset of the visual boundary in such a way that the $\mathrm{SO}_0(2,n)$-action on $\partial\mathcal{X}_{2,n}$ restricts to a well-defined transitive action on this subset.

Thanks to Section 2.3, we know that every point in the visual boundary of $\mathcal{X}_{2,n}$ can be identified with a pair (λ, F), where λ is a vector and F a flag in $\mathbb{R}^{2,n}$ consisting only of isotropic subspeces; it turns out that a suitable subset for our purpose is given by the set of pairs (λ, F) in which $\lambda = \left(\sqrt{2}/2\right)$ and the flag $F = (V_1)$ is composed by a unique 1-dimensional isotropic vector space V_1. In particular, there is a natural correspondence between this subset and the set of isotropic lines in $\mathbb{R}^{2,n}$.

In the first part of this chapter we describe, in general, the set of isotropic lines contained in $\mathbb{R}^{p,q}$; in a second part we restrict our attention to the case $(p,q) = (2,n)$, for $n > 2$ and we study the action of $\mathrm{SO}_0(2,n)$ on the set of isotropic lines in $\mathbb{R}^{2,n}$.

3.1 Isotropic subspaces

Given a bilinear form h over a vector space V, a vector $v \in V \setminus \{0\}$ is said to be *isotropic* if $h(v,v) = 0$; if such a v exists, then the bilinear form is called *isotropic*.

A subspece $W \subset V$ is called *totally isotropic subspace*, or simply *isotropic subspace*, if the bilinear form h vanishes on it:

$$h(v,w) = 0 \qquad \forall v, w \in W.$$

We denote with $\mathrm{Is}_k(V)$ the set of k-dimensional isotropic subspaces in V:

$$\mathrm{Is}_k(V) = \{ W \in \mathrm{Gr}_k(V) | \, \forall v, w \in W, h(v,w) = 0 \} ,$$

where $\mathrm{Gr}_k(V)$ is the *Grassmannian* of k-dimensional subspeces in V; in particular, $\mathrm{Is}_1(V)$ is the set of isotropic lines, that are 1-dimensional subspaces of V generated by an isotropic vector. If $v \in V$ is an isotropic vector, we denote with $[v]$ the isotropic line generated by v.

The dimension of a maximal isotropic subspace in V is called *isotropy index* of the space V. If the signature of the bilinear form h is (p,q), then the isotropy index is the minimum between p and q.

Action of $\mathrm{SO}_0(p,q)$ *on* $\mathrm{Is}_1(\mathbb{R}^{p,q})$*:* Let $V = \mathbb{R}^{p+q}$ and $h = h_{p,q}$; as $\mathrm{SO}_0(p,q)$ preserves $h_{p,q}$, if $g \in \mathrm{SO}_0(p,q)$ and $v \in \mathbb{R}^{p,q}$ is an isotropic vector, then also $g \cdot v$ is an isotropic vector and so there is a well defined action of

$\mathrm{SO}_0(p,q)$ on the set of isotropic vectors contained in $\mathbb{R}^{p,q}$; by linearity, it descends an action on $\mathrm{Is}_1(\mathbb{R}^{p,q})$, defined by

$$g \cdot [v] := [g \cdot v]$$

for every isotropic vector v and $g \in \mathrm{SO}_0(p,q)$. Finally, this action extends to a diagonal action on $(\mathrm{Is}_1(\mathbb{R}^{p,q}))^k$:

$$g \cdot ([v_1], ..., [v_k]) := ([g \cdot v_1], ..., [g \cdot v_k]).$$

We are mostly interested in orbits of k-tuples $([v_1], ..., [v_k]) \in (\mathrm{Is}_1(\mathbb{R}^{p,q}))^k$ in which the vectors $v_1, ..., v_k$ are linearly independent; for this reason we define the set

$$(\mathrm{Is}_1(\mathbb{R}^{p,q}))^{k*} := \left\{ ([v_1], ..., [v_k]) \in (\mathrm{Is}_1(\mathbb{R}^{p,q}))^k \,\middle|\, v_1, ..., v_k \text{ lin. indip.} \right\}.$$

In order to study the action of $\mathrm{SO}_0(p,q)$ on the set of isotropic lines in $\mathbb{R}^{p,q}$ it is possible to consider the action of $G_{p,q}$ and $H_{p,q}$ on, respectively, the sets

$$\left\{ [v] \in \mathrm{Gr}_1\left(\mathbb{R}^{p+q}\right) \middle| \langle v, v \rangle = 0 \right\},$$
$$\left\{ [v] \in \mathrm{Gr}_1\left(\mathbb{R}^{p+q}\right) \middle| \langle\!\langle v, v \rangle\!\rangle = 0 \right\}.$$

Induced bilinear form on $\mathrm{Is}_1(\mathbb{R}^{p,q})$*:* One easy consequence of the fact that matrices in $\mathrm{SO}_0(p,q)$ preserve the bilinear form $h_{p,q}$ is that two k-tuples $(v_1, ..., v_k)$, $(w_1, ..., w_k) \in (\mathbb{R}^{p,q})^k$ are in the same orbit (with respect to the $\mathrm{SO}_0(p,q)$ action) only if $h_{p,q}(v_i, v_j) = h_{p,q}(w_i, w_j)$ for all $1 \leq i, j \leq n+2$.

In order to apply the same reasoning to the set of isotropic lines, we need to understand how to extend the function $h_{p,q}$ on $\mathrm{Is}_1(\mathbb{R}^{p,q})$: notice that, for every $[v], [w] \in \mathrm{Is}_1(\mathbb{R}^{p,q})$, the fact that $h_{p,q}(v, w)$ vanishes or not does not depend on the choice of the representatives v, w of $[v], [w]$; so, the function $h_{p,q} : \mathrm{Is}_1(\mathbb{R}^{p,q}) \times \mathrm{Is}_1(\mathbb{R}^{p,q}) \to \{0, 1\}$, given by

$$h_{p,q}([v], [w]) = \begin{cases} 1 & \text{if } h_{p,q}(v, w) \neq 0, \\ 0 & \text{if } h_{p,q}(v, w) = 0, \end{cases}$$

is well defined.

Remark 3.1. With an abuse of notation, we denote with $h_{p,q}$ both the standard bilinear form of signature (p,q) and the function just defined. However, it will be clear from the context which one we are using.

Same argument apply also to the bilinear forms $\langle \, , \rangle$, $\langle\!\langle , \rangle\!\rangle$.

Remark 3.2. For $q \geq p$, $\mathrm{SO}(q-p)$ is embedded in $H_{p,q}$ via

$$\begin{array}{cccc} j_1: & \mathrm{SO}(q-p) & \to & H_{p,q} \\ & A & \mapsto & \left(\begin{array}{c|c|c} \mathrm{Id}_p & 0 & 0 \\ \hline 0 & A & 0 \\ \hline 0 & 0 & \mathrm{Id}_p \end{array}\right) \end{array}$$

Thanks to this observation, we know that for every vector $v \in \mathbb{R}^{2,n}$, there exists an element $g \in \mathrm{Stab}_{H_{2,n}}(e_1, e_2, e_{n+1}, e_{n+2})$ such that

$$g \cdot v = w_1 e_1 + e_2 w_2 + e_3 w_3 + w_{n+1} e_{n+1} + w_{n+2} e_{n+2}.$$

Remark 3.3. $\mathrm{SO}(p) \times \mathrm{SO}(q)$ is embedded in $G_{p,q}$ via

$$\begin{array}{cccc} j_2: & \mathrm{SO}(p) \times \mathrm{SO}(q) & \to & G_{p,q} \\ & (A, B) & \mapsto & \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) \end{array}$$

From now on, we restrict our attention to the case $p = 2$ and $n := q > 2$ and we consider only the group $\mathrm{SO}_0(2, n)$.

3.2 Action of $\mathrm{SO}_0(2, n)$ on $\mathrm{Is}_1(\mathbb{R}^{2,n})$

First, we need to verify that $\mathrm{SO}_0(2, n)$ acts transitively on $\mathrm{Is}_1(\mathbb{R}^{2,n})$.

Lemma 3.4. *The action of* $\mathrm{SO}_0(2, n)$ *on* $\mathrm{Is}_1(\mathbb{R}^{2,n})$ *is transitive.*

Proof. We use the representation of $\mathrm{SO}_0(2, n)$ provided by $G_{2,n}$ and prove that every element in $\mathrm{Is}_1(\mathbb{R}^{2,n})$ is in the same orbit of $[g_1 + g_3]$ (see the notation introduced in Section 2.1).

Consider an isotropic vector $v \in \mathbb{R}^{2+n}$ that splits on $\mathbb{R}^2 \oplus \mathbb{R}^n$ as $v = w_1 + w_2$, where w_1 is the projection on the first two components and w_2 is the projection on the last n components, and let $\{g_1, g_2\}$, $\{g_3, ..., g_{n+2}\}$ the canonical bases of $\mathbb{R}^2$, $\mathbb{R}^n$.

By $\langle v, v\rangle = 0$, it follows $\|w_1\| = \|w_2\|$ (with $\|\cdot\|$ we mean the Euclidean norm, that coincide with $h_{2,n}|_{\mathbb{R}^2 \times \mathbb{R}^2}$ on $\mathbb{R}^2$ and with $-h_{2,n}|_{\mathbb{R}^n \times \mathbb{R}^n}$ on $\mathbb{R}^n$). As for $k > 1$ the action of $\mathrm{SO}(k)$ is transitive on $\mathbb{S}^{k-1}$ (the set of unit vectors of $\mathbb{R}^k$), there are two matrices $A \in \mathrm{SO}(2)$, $B \in \mathrm{SO}(n)$ such that

$$\begin{cases} Aw_1 = \|w_1\| \, g_1, \\ Bw_2 = \|w_2\| \, g_3. \end{cases}$$

From Remark 3.3, it follows that $X = j_2(A, B) \in G_{2,n}$ is such that

$$Xv = \|w_1\| \, (g_1 + g_3),$$

that is $X \cdot [v] = [g_1 + g_3]$. □

According to the notation introduced in Section 2.1, we have $[g_1 + g_3] = [e_1]$. With some easy computations, it is possible to find the stabilizer of this element:

Lemma 3.5. *If $g \in Stab_{H_{2,n}}([e_1])$, then it is written in the form*

$$g = \left(\begin{array}{cc|c|cc} \alpha & a_1 & b_1 & c_1 & c_2 \\ 0 & a_2 & b_2 & c_3 & c_4 \\ \hline 0 & d & E & f_1 & f_2 \\ \hline 0 & 0 & 0 & \alpha^{-1} & 0 \\ 0 & g_1 & h & i_1 & i_2 \end{array}\right) \in \mathrm{Mat}(2 + (n-2) + 2, \mathbb{R})$$

In the rest of the chapter it is used the representation of $\mathrm{SO}_0(2, n)$ provided by $H_{2,n}$.

3.3 Action of $\mathrm{SO}_0(2, n)$ on $(\mathrm{Is}_1(\mathbb{R}^{2,n}))^{2*}$

On the set of pairs in $\mathrm{Is}_1(\mathbb{R}^{2,n})$, the $\mathrm{SO}_0(2, n)$-action is no longer transitive: indeed a pair $(a, b) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^{2*}$ with $h_{2,n}(a, b) = 1$ lays in a different orbit of a pair $(c, d) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^{2*}$ in which $h_{2,n}(c, d) = 0$ (such a pair exists because the isotropy index of $\mathbb{R}^{2,n}$ is 2).

Lemma 3.6. *The action of $\mathrm{SO}_0(2, n)$ on $(\mathrm{Is}_1(\mathbb{R}^{2,n}))^{2*}$ has two orbits:*

$$\mathcal{A}_0 = \left\{ ([v], [w]) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^{2*} \middle| \, h_{2,n}([v], [w]) = 0 \right\},$$

$$\mathcal{A}_1 = \left\{ ([v], [w]) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^{2*} \middle| \, h_{2,n}([v], [w]) = 1 \right\}.$$

Proof. It is clear that elements in $\mathcal{A}_0$ and elements in $\mathcal{A}_1$ are contained in different orbits, so we only need to prove that $H_{2,n}$ acts transitively on the two sets.

For every $([v], [w]) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^{2*}$ there exists an $h \in H_{2,n}$ such that $h \cdot [v] = [e_1]$ (by Lemma 3.4), so that $(e_1, h \cdot w) = h \cdot (v, w)$. Then, it is enough to prove that the action of $\mathrm{Stab}_{H_{2,n}}([e_1])$ has two orbits on the set of pairs of the type $(e_1, [v]) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^{2*}$ or, simply, on $\mathrm{Is}_1(\mathbb{R}^{2,n}) \setminus \{[e_1]\}$.

By Remark 3.2, it is enough to consider only the case $n = 3$.

Let $v = \sum_{i=1}^{5} v_i e_i$, with $[v] \in \mathrm{Is}(\mathbb{R}^{2,3})$ and $[v] \neq [e_1]$;

- If $\langle\!\langle [v], [e_1] \rangle\!\rangle = 0$ then $v_4 = 0$, so that $2v_2v_5 = v_3^2$. Consider

$$X_1 = \begin{pmatrix} 1 & v_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & v_3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & v_5 & v_3 & -v_1 & 1 \end{pmatrix}, X_2 = \begin{pmatrix} 1 & v_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -v_1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

with $X_1, X_2 \in \mathrm{Stab}_{H_{2,3}}([e_1])$. If $v_2 \neq 0$, we can suppose, without loss of generality, $v_2 = 1$ and then $X_1 \cdot [e_2] = [v]$; otherwise, $v = v_1 e_1 + v_5 e_5$, with $v_5 \neq 0$, and we can suppose $v_5 = 1$, so that $X_2 \cdot [e_2] = [v]$. This proves the transitivity of the action on $\mathcal{A}_0$.

- If $\langle\!\langle [v], [e_1] \rangle\!\rangle = 1$ then $v_4 \neq 0$ and so, without loss of generality, $v_4 = 1$. The matrix

$$X = \begin{pmatrix} 1 & -v_5 & v_3 & v_1 & -v_2 \\ 0 & 1 & 0 & v_2 & 0 \\ 0 & 0 & 1 & v_3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & v_5 & 1 \end{pmatrix} \in \mathrm{Stab} H_{2,3}([e_1])$$

is such that $X \cdot [e_4] = [v]$ and we have the transitivity of $H_{2,3}$ on the set $\mathcal{A}_1$.

$\square$

We are particularly interested in pairs of isotropic lines inside $\mathcal{A}_1$.

Definition 3.7. *Two isotropic lines l and n are called* transverse *and denoted with $l \pitchfork n$ if $h_{2,n}(l, n) = 1$.*

In particular l, n are transverse if and only if they do not lay in the same isotropic plane.

As the isotropic lines $[e_1], [e_{n+1}]$ are central in our study, it is useful to compute the stabilizer $\mathrm{Stab}_{H_{2,n}}([e_1], [e_{n+1}])$.

Lemma 3.8.

$$\mathrm{Stab}_{H_{2,n}}([e_1], [e_{n+1}]) =$$

$$= \left\{ \begin{array}{c} X = \left(\begin{array}{cc|c|cc} \alpha & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & c \\ \hline 0 & d & E & 0 & f \\ \hline 0 & 0 & 0 & \alpha^{-1} & 0 \\ 0 & g & h & 0 & j \end{array}\right) \\ \in \mathrm{Mat}(2 + (\mathrm{n}-2) + 2, \mathbb{R}) \end{array} \middle| \begin{array}{l} 2ag = d^T d \\ 2jc = f^T f \\ gb + ah = d^T E \\ ch + jb = f^T E \\ cg + aj = f^T d + 1 \\ b^T h + h^T b = E^t E - \mathrm{Id}_{n-2} \\ \alpha(a + c + g + j) > 0 \\ \det X = 1 \end{array} \right\}.$$

3.4 Action of $\mathrm{SO}_0(2,n)$ on $(\mathrm{Is}_1(\mathbb{R}^{2,n}))^{3*}$

Given a triple $t = ([v_1], [v_2], [v_3]) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^{3*}$, let us denote

$$x_{ij}(t) = \langle\!\langle [v_i], [v_j] \rangle\!\rangle.$$

For $\delta_i \in \{0,1\}$, $i = 1,2,3$, define

$$\mathcal{A}_{\delta_1,\delta_2,\delta_3} := \{t \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^{3*} | x_{1,2}(t) = \delta_1, x_{1,3}(t) = \delta_2, x_{2,3}(t) = \delta_3\}.$$

In order to avoid a huge number of cases and a lot of computations we make the following observations:

1. $\mathcal{A}_{0,0,0} = \emptyset$, because the isotropy index of $\mathbb{R}^{2,n}$ is 2.
2. Studying the orbit of $([v_1], ..., [v_k]) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^k$ is the same of studying the orbit of $\left(\left[v_{\sigma(1)}\right], ..., \left[v_{\sigma(k)}\right]\right)$, for every permutation σ (even if the two orbits might be different).

Then it suffices to study the action of $\mathrm{SO}_0(2,n)$ on $\mathcal{A}_{0,1,0}$, $\mathcal{A}_{0,1,1}$, $\mathcal{A}_{1,1,1}$.

Lemma 3.9. *The action of* $\mathrm{SO}_0(2,n)$ *is transitive on* $\mathcal{A}_{0,1,0}$.

Proof. As in the proof of Lemma 3.6, it is enough to prove the transitivity of the action of $\mathrm{Stab}_{H_{2,n}}(e_1, e_2)$ on elements of the type $([e_1], [e_2], [v]) \in \mathcal{A}_{0,1,0}$. Again, all cases can be reduced to $n = 3$.

Consider $([e_1], [e_2], [v]) \in \mathcal{A}_{0,1,0}$, with $v = \sum_{i=1}^5 v_i e_i$, then $v_4 \neq 0$, $v_5 = 0$. Without loss of generality $v_4 = 1$. The element

$$X = \begin{pmatrix} 1 & 0 & v_3 & v_1 & -v_2 \\ 0 & 1 & 0 & v_2 & 0 \\ 0 & 0 & 1 & v_3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Stab}_{H_{2,3}}([e_1], [e_2])$$

is such that $X \cdot [e_4] = [v]$; so all triples in $\mathcal{A}_{0,1,0}$ are in the same orbit of $([e_1], [e_2], [e_4])$. □

Lemma 3.10. *The action of* $\mathrm{SO}_0(2,n)$ *has two orbits on* $\mathcal{A}_{0,1,1}$.

Proof. $\mathrm{SO}(2,n)$ acts transitively on $\mathcal{A}_{0,1,1}$: as in the previous lemma, let $n = 3$ and consider the action of $\mathrm{Stab}_{\mathrm{SO}(2,3)}(e_1, e_2)$ on triples of the type $([e_1], [e_2], [v]) \in \mathcal{A}_{0,1,1}$. If $v = \sum_{i=1}^5 v_i e_i$ is such that $([e_1], [e_2], [v]) \in \mathcal{A}_{0,1,1}$, then $v_4 \neq 0$, $v_5 \neq 0$. Without loss of generality $v_4 = 1$. The element

$$X = \begin{pmatrix} 1 & 0 & v_3 & v_3^2/2 & v_1 - v_3^2/2 \\ 0 & v_5^{-1} & 0 & v_2 & 0 \\ 0 & 0 & 1 & v_3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & v_5 \end{pmatrix}$$

is such that $X \cdot ([e_1],[e_2],[e_4+e_5]) = ([e_1],[e_2],[v])$ and $X^T J_{2,3} X = J_{2,3}$, where $J_{p,q}$ is defined in Equation (2.1), so that $X \in \mathrm{SO}(2,3)$.

Because of this, $\mathrm{SO}_0(2,3)$ has at most two orbits in $\mathcal{A}_{0,1,1}$ (one for each connected component of $\mathrm{SO}(2,3)$). To verify that it has exactly two orbits, we prove that $([e_1],[e_2],[e_4+e_5])$, $([e_1],[e_2],[e_4-e_5])$ are not in the same orbit: indeed if $X \in \mathrm{Stab}_{H_{2,3}}([e_1],[e_2])$, then X is written as

$$\begin{pmatrix} \alpha & 0 & * & * & * \\ 0 & \beta & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & \alpha^{-1} & 0 \\ 0 & 0 & 0 & 0 & \beta^{-1} \end{pmatrix}$$

with α, $\beta \neq 0$ and $\alpha\beta > 0$ (because X preserves time orientation); this implies $X[e_4+e_5] \neq [e_4-e_5]$. □

On $\mathcal{A}_{1,1,1}$ the situation is slightly different: consider an orbit

$$[(l,m,n)] \in \mathcal{A}_{1,1,1}/\mathrm{SO}_0(2,n);$$

the bilinear form $h_{2,n}|_V$, restriction of $h_{2,n}$ on the vector space $V = l \oplus m \oplus n$, has signature $(2,1)$ or $(1,2)$; in particular, this signature does not depend on the choice of the representative (l,m,n) of the orbit $[(l,m,n)]$.

Moreover, if the triple (l,m,n) is such that $l \oplus m \oplus n \cong \mathbb{R}^{2,1}$, then it fixes also a time orientation, that can not be changed with the action of $\mathrm{SO}_0(2,n)$.

We have, then, at least three orbits contained in $\mathcal{A}_{1,1,1}$. More formally:

Lemma 3.11. *The action of* $\mathrm{SO}_0(2,n)$ *has three orbits on* $\mathcal{A}_{1,1,1}$*:*

$$\mathcal{O}_{1,2} = H_{2,n} \cdot ([e_1],[e_{n+1}],[e_1+e_2+e_{n+1}-e_{n+2}]),$$

$$\mathcal{O}_{2,1}^{+} = H_{2,n} \cdot ([e_1],[e_{n+1}],[e_1+e_2-e_{n+1}+e_{n+2}]),$$

$$\mathcal{O}_{2,1}^{-} = H_{2,n} \cdot ([e_1],[e_{n+1}],[e_1-e_2-e_{n+1}-e_{n+2}]).$$

Proof. As usual it is enough to study the case $n=3$. Consider $v = \sum_{i=1}^{5} v_i e_i$ such that $([e_1],[e_4],[v]) \in \mathcal{A}_{1,1,1}$, so that $v_1 \neq 0$, $v_4 \neq 0$. Let

$$X_1 = \begin{pmatrix} v_1 & 0 & 0 & 0 & 0 \\ 0 & v_2 & 0 & 0 & 0 \\ 0 & v_3 & 1 & 0 & 0 \\ 0 & 0 & 0 & v_4 & 0 \\ 0 & v_2^{-1}+v_5 & v_3 v_2^{-1} & 0 & v_2^{-1} \end{pmatrix},$$

$$X_2 = \begin{pmatrix} v_1 & 0 & 0 & 0 & 0 \\ 0 & v_5^{-1} & v_3 v_5^{-1} & 0 & v_5^{-1} + v_2 \\ 0 & 0 & 1 & 0 & v_3 \\ 0 & 0 & 0 & v_4 & 0 \\ 0 & 0 & 0 & 0 & v_5 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} v_1 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & \sqrt{2}/2 & 0 & 1/2 \\ 0 & -\sqrt{2}/2 & 0 & 0 & \sqrt{2}/2 \\ 0 & 0 & 0 & v_4 & 0 \\ 0 & 1/2 & -\sqrt{2}/2 & 0 & 1/2 \end{pmatrix},$$

where X_1 is defined only when $v_2 \neq 0$, while X_2 only when $v_5 \neq 0$.

If $\mathrm{Span}(e_1, e_4, v) \cong \mathbb{R}^{1,2}$, then $-2v_1v_4 = -v_3^2 + 2v_2v_5 < 0$ and we can assume, without loss of generality, $v_1v_4 = 1$; we have two possible cases:

- If $v_2 = v_5 = 0$, then assume also $v_1 + v_4 > 0$, so that

$$X_3 \cdot ([e_1 + e_2 + e_4 - e_5]) = [v]$$

  and $X_3 \in \mathrm{Stab}_{H_{2,3}}([e_1], [e_4])$;

- If one between v_2 and v_5 is different from 0 (and we can assume it is positive), then X_1 or X_2 (let say X) is such that

$$X \cdot ([e_1 + e_2 + e_4 - e_5]) = [v]$$

  and $X \in \mathrm{Stab}_{H_{2,3}}([e_1], [e_4])$.

This proves that $\mathrm{SO}_0(2,3)$ is transitive on $\mathcal{O}_{1,2}$.

With analogous computations it is possible to prove that $\mathrm{SO}(2,3)$ is transitive on

$$\mathcal{O}_{2,1} = \{(l_1, l_2, l_3) \in \mathcal{A}_{1,1,1} | l_1 \oplus l_2 \oplus l_3 \cong \mathbb{R}^{2,1}\}$$

(it suffices to change some sign in X_1 and in X_2).

On the other hand, the two quadruples

$$([e_1], [e_4], [e_1 + e_2 - e_4 + e_5]),\ ([e_1], [e_4], [e_1 - e_2 - e_4 - e_5]) \in \mathcal{O}_{2,1}$$

can not lay in the same $\mathrm{SO}_0(2,3)$-orbit: indeed if

$$X \cdot ([e_1], [e_4], [e_1 + e_2 - e_4 + e_5]) = ([e_1], [e_4], [e_1 - e_2 - e_4 - e_5])$$

then

$$\begin{cases} Xe_1 = \alpha e_1, \\ Xe_4 = \beta e_4, \\ X(e_1 + e_2 - e_4 + e_5) = \gamma(e_1 - e_2 - e_4 - e_5), \end{cases}$$

for some α, β, $\gamma \neq 0$; from Lemma 3.8 it follows that $\alpha = \beta = \gamma = \pm 1$ and so

$$\begin{aligned} X(e_1 + e_2) &= \pm(e_1 + e_4), \\ X(e_2 + e_5) &= X(e_1 + e_2 - e_4 + e_5) - Xe_1 + Xe_4 = \mp(e_2 + e_5); \end{aligned}$$

but this implies that X does not preserve time orientation.

It follows that $\mathrm{SO}_0(2,3)$ has exactly two orbits on $\mathcal{O}_{2,1}$ (one for each connected component of $\mathrm{SO}(2,3)$). $\square$

For simplicity, we introduce the following notation:

$$\begin{aligned} l_1 &:= [e_1], \\ l_2 &:= [e_2], \\ l_3 &:= [e_1 + e_2 - e_{n+1} + e_{n+2}]. \end{aligned}$$

Remark 3.12. Let

$$\mathcal{O}_{2,1} := \mathcal{O}_{2,1}^+ \cup \mathcal{O}_{2,1}^-$$

and consider a triple $([v_1],[v_2],[v_3]) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^{3*}$. It is possible to see whether this triple is in $\mathcal{A}_{0,1,0}$, $\mathcal{A}_{1,1,0}$, $\mathcal{O}_{2,1}$ or $\mathcal{O}_{1,2}$ just by looking at the set

$$\mathcal{S} = \{h_{2,n}(v_i, v_j) | 1 \leq i < j \leq 3\};$$

indeed it is clear how to distinguish between elements in $\mathcal{A}_{0,1,0}$, $\mathcal{A}_{1,1,0}$ or $\mathcal{A}_{1,1,1}$. Moreover, if the triple is inside $\mathcal{A}_{1,1,1}$ and k denotes the number of negative values inside $\mathcal{S}$, then the parity of k does not depend on the choice of the representatives v_1, v_2, v_3; because $\mathrm{SO}_0(2,n)$ preserves $h_{2,n}$, we conclude that $([v_1],[v_2],[v_3])$ is inside $\mathcal{O}_{1,2}$ if k is even, while it is contained in $\mathcal{O}_{2,1}$ if k is odd.

There exists also a nice way to see whether an element is inside $\mathcal{O}_{2,1}^+$ or $\mathcal{O}_{2,1}^-$, but we will discuss it in the following subsection.

Lemma 3.13.

$$\mathrm{Stab}_{H_{2,n}}(l_1, l_2, l_3) =$$

$$= \left\{ \left(\begin{array}{cc|c|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 1-a \\ \hline 0 & d & E & 0 & -d \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 1-a & -b & 0 & a \end{array}\right) \in \mathrm{SL}(\mathbb{R}^{2+n}) \left| \begin{array}{l} 2a(1-a) = d^T d \\ (1-2a)b = d^T E \\ 2b^T b + E^T E = \mathrm{Id}_{n-2} \end{array} \right. \right\} \cup$$

$$\left\{ \left(\begin{array}{cc|c|cc} -1 & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & -1-a \\ \hline 0 & d & E & 0 & -d \\ \hline 0 & 0 & 0 & -1 & 0 \\ 0 & -1-a & -b & 0 & a \end{array}\right) \in \mathrm{SL}(\mathbb{R}^{2+n}) \left| \begin{array}{l} 2a(-1-a) = d^T d \\ (-1-2a)b = d^T E \\ 2b^T b + E^T E = \mathrm{Id}_{n-2} \end{array} \right. \right\}.$$

Moreover, each component of $\mathrm{Stab}_{H_{2,n}}(l_1, l_2, l_3)$ *acts on* $\mathrm{Span}(l_1, l_2, l_3)^{\perp_{h_{2,n}}}$ *as* $\mathrm{SO}(n-1)$*.*

Remark 3.14. Let V be a subspace of $\mathbb{R}^{2,n}$; we denote with $V^{\perp_{h_{2,n}}}$ the subspace

$$V^{\perp_{h_{2,n}}} := \left\{ v \in \mathbb{R}^{2,n} \middle| \, \forall w \in V, h_{2,n}(v, w) = 0 \right\}.$$

3.4.1 Maximal triples

Between the others, triples inside $\mathcal{O}^+_{2,1}$ are the ones in which we are more interested.

Definition 3.15. *A k-tuple* $(l_1, ..., l_s) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^s$*, for* $s \geq 3$*, is called* maximal *if, for each* $1 \leq i < j < k \leq s$*, the triple* (l_i, l_j, l_k) *is contained in* $\mathcal{O}^+_{2,1}$*.*

In particular, a triple is maximal if and only if it is contained in $\mathcal{O}^+_{2,1}$.

Remark 3.16. The term *maximal* comes from the fact that these triples maximize the *generalized Maslov cocycle*

$$\beta : (\mathrm{Is}_1(\mathbb{R}))^{3*} \to \{-2, -1, 0, 1, 2\};$$

in particular, it is known that

$$\begin{cases} \beta(\mathcal{O}^+_{2,1}) = \{+2\}, \\ \beta(\mathcal{O}^-_{2,1}) = \{-2\}, \\ \beta(\mathcal{O}_{1,2}) = \{0\}. \end{cases}$$

Definition and some results about this map can be found in [DLP18] or [BIW10a].

Before of starting with the study of maximality, we introduce the concept of orientation on triples of isotropic lines. The main idea is to use the orientation on the unit circle $\mathbb{S}^1$ to induce an orientation on $\mathrm{Is}_1(\mathbb{R}^{2,n})$; however, because we are dealing with linear subspaces instead of vectors, to formalize this idea requires some work.

Let us denote

$$\begin{aligned} f_1 =& \frac{1}{2}(e_1 + e_{n+1}), \\ f_2 =& \frac{1}{2}(e_2 + e_{n+2}), \\ f_3 =& \frac{1}{2}(e_1 - e_{n+1}), \end{aligned}$$

the elements of the basis $\mathcal{B}_V := \{f_1, f_2, f_3\}$ of the vector space $V := l_1 \oplus l_2 \oplus l_3$ and let

$$\begin{aligned} V^+ &= \mathrm{Span}(f_1, f_2), \\ V^- &= \mathrm{Span}(f_3), \end{aligned}$$

so that $h_{2,n}|_{V^+} = h_{2,0}$ and $h_{2,n}|_{V^-} = h_{0,1}$; denote, also, with pr_{V^+}, pr_{V^-} the projections of V to the vector subspaces V^+, respectively V^- according to the splitting $V^+ \oplus V^-$.

With this notation, the set $\mathrm{Is}_1(V)$ of isotropic lines in V coincides with the cone

$$\left\{ [x] \in \mathrm{Gr}_1^+(V) \middle| \, h_{2,0}(\mathrm{pr}_{V^+}(x), \mathrm{pr}_{V^+}(x)) = -h_{0,1}\left(\mathrm{pr}_{V^-}(x), \mathrm{pr}_{V^-}(x)\right) \right\}.$$

For every isotropic element $l \in \mathrm{Is}_1(V)$, let us denote with $(l)_{V,f_3}$ the representative of l in V such that $\mathrm{pr}_{V^-}((l)_{V,f_3}) = f_3$.

Now, the map

$$\begin{array}{cccc} \varphi_{V,f_3}: & \mathrm{Is}_1(V) & \to & \mathbb{S}^1 \\ & l & \mapsto & \mathrm{pr}_{V^+}((l)_{V,f_3}) \end{array}$$

is a bijection between the set of isotropic lines in V and the unit circle

$$\mathbb{S}^1 = \left\{ x \in V^+ \middle| \, h_{2,n}(x, x) = 1 \right\}.$$

In particular, the orientation on $\mathbb{S}^1$ (counterclockwise according the basis $\{f_1, f_2\}$) induces an orientation on the set of isotropic lines in V.

Definition 3.17. *A triple* $(m_1, m_2, m_3) \in \mathrm{Is}(V)^3$ *is called* positively *(respectively* negatively*)* oriented *if* $(\varphi_{V,f_3}(m_1), \varphi_{V,f_3}(m_2), \varphi_{V,f_3}(m_3))$ *is a positively (negatively) oriented triple on* $\mathbb{S}^1$*.*

Example 3.18. The triple (l_1, l_2, l_3) is negatively oriented (Figure 3.1).

Proceeding in an analogous way, it is possible to extend the definition of positively oriented triple to the whole space $\mathbb{R}^{2,n}$: let

$$(m_1, m_2, m_3) \in \left(\mathrm{Is}(R^{2,n})\right)^{3*}$$

in such a way that the space

$$V' := m_1 \oplus m_2 \oplus m_3$$

is of signature $(2, 1)$ and consider a splitting $V' = V'^+ \oplus V'^-$ such that V'^+ is of signature $(2, 0)$ and V'^- of signature $(0, 1)$; let us also fix a vector $\nu \in V'^- \setminus \{0\}$. For every $l \in \mathrm{Is}(V')$, we denote with $(l)_{V',\nu} \in V'$ the representative of l such that $\mathrm{pr}_{V'^-}(l)_{V',\nu} = \nu$. As before, the map

$$\begin{array}{cccc} \varphi_{V',\nu}: & \mathrm{Is}(V') & \to & \mathbb{S}^1 \\ & m & \mapsto & \dfrac{\mathrm{pr}_{V^+}((m)_{V',\nu})}{\sqrt{h_{2,n}\left(\mathrm{pr}_{V^+}((m)_{V',\nu}), \mathrm{pr}_{V^+}((m)_{V',\nu})\right)}} \end{array}$$

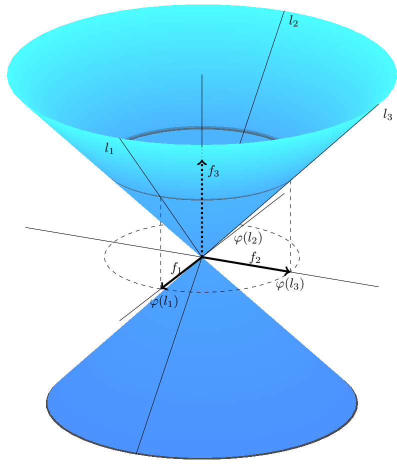

Figure 3.1: Cone of isotropic vectors in V: each isotropic line corresponds to a point on the circle $\mathbb{S}^1$.

is a bijection between $\mathrm{Is}(V')$ and $\mathbb{S}^1$ (notice that we are projecting on the unit cirle contained in V^+ and not in V'^+).

Definition 3.19. *Let* $(m_1, m_2, m_3) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^3$ *be such that the linear space* $V' := m_1 \oplus m_2 \oplus m_3$ *is of signature* $(2, 1)$. *Fix a splitting* $V' = V'^+ \oplus V'^-$, *where* V'^+ *is of signature* $(2, 0)$ *and* V'^- *of signature* $(0, 1)$, *and a vector* $\nu \in V'^-$. *The triple* (m_1, m_2, m_3) *is called* positively oriented *(respectively* negatively oriented*) if*

$$\big(\varphi_{V',\nu}(m_1), \varphi_{V',\nu}(m_2), \varphi_{V',\nu}(m_3)\big) \in \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$$

is positively oriented (negatively oriented) in $\mathbb{S}^1$.

Before of proceeding with the study of maximal triples, we make some remarks about this definition:

Remark 3.20. This definition is really intuitive, but still it is not clear whether it depends or not of the choices we made on the space V'.

First of all notice that, once fixed the subspace V'^-, the map $\varphi_{V',\nu}$ does depend on the choice of ν, but only in the sense that a isotropic line $m \in \mathrm{Is}(V')$ can be mapped in an element or in its opposite. More precisely:

$$\varphi_{V',\pm\lambda\nu}(m) = \pm\varphi_{V',\nu}(m),$$

for every $\lambda > 0$; however, for every triple $(p_1, p_2, p_3) \in (\mathbb{S}^1)^3$, the orientation of (p_1, p_2, p_3) is the same of its opposite $(-p_1, -p_2, -p_3)$. In particular, the definition of orientation of triples in $\mathrm{Is}(V')$ does not depend on the choice of ν.

Moreover, as we are projecting on the space V^+, the splitting $V'^+ \oplus V'^-$ does not affect the orientation of $(\varphi_{V',\nu}(m_1), \varphi_{V',\nu}(m_2), \varphi_{V',\nu}(m_3))$.

Remark 3.21. Instead of this construction, one could use the fact that $\mathrm{SO}_0(2,n)$ acts transitively on the set of planes in $\mathbb{R}^{2,n}$ of signature $(2,0)$ to induce an orientation on the unit circle inside V'^{+}; thanks to this it is possible to define a triple $(m_1,m_2,m_3) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^{3*}$ positively oriented if the projection of the triple $((m_1)_{V',\nu},(m_2)_{V',\nu},(m_3)_{V',\nu})$ on V'^{+} (with respect to the splitting $V' = V'^{+} \oplus V'^{-}$) is positively oriented. This definition and Definition 3.19 are equivalent.

Remark 3.22. Given $V' \subset \mathbb{R}^{2,1}$ subspace of signature $(2,1)$ and a triple $(m_1,m_2,m_3) \in (\mathrm{Is}(V'))^3$, the representatives $(m_1)_{V',\nu}$, $(m_2)_{V',\nu}$, $(m_3)_{V'\nu}$ are such that $h_{2,n}((m_i)_{V'},(m_j)_{V'}) < 0$, for every $1 \leq i < j \leq 3$.

Triples orientation is invariant under the action of $\mathrm{SO}_0(2,n)$.

Lemma 3.23. *If $X \in \mathrm{SO}_0(2,n)$ and $(m_1,m_2,m_3) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^{3*}$ is a positively (respectively negatively) oriented triple, then $X \cdot (m_1,m_2,m_3)$ is also positively (negatively) oriented.*

Proof. It is an easy consequence of the fact that matrices in $\mathrm{SO}_0(2,n)$ preserve the bilinear form $h_{2,n}$ (and, so, the signature of vector subspaces in $\mathbb{R}^{2,n}$) and time orientation. $\square$

We are now ready to characterize maximal triples.

Lemma 3.24. *A triple $(n_1,n_2,n_3) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^3$ is maximal if and only if it is negatively oriented.*

Proof. A triple $(m_1,m_2,m_3) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^3$ is maximal if it is contained in the same orbit of (l_1,l_2,l_3), that is if there exists an $X \in \mathrm{SO}_0(2,n)$ such that $(m_1,m_2,m_3) = X \cdot (l_1,l_2,l_3)$. As $\mathrm{SO}_0(2,n)$ preserves triples orientation and (l_1,l_2,l_3) is negatively oriented, we have that every maximal triple is negatively oriented.

Viceversa, let $(m_1,m_2,m_3) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^3$ be a negatively oriented triple. Because $V' := m_1 \oplus m_2 \oplus m_3$, with the bilinear form $h_{2,n}|_{V'}$, is of signature $(2,1)$, (m_1,m_2,m_3) lies either in $\mathcal{O}^{+}_{2,1}$ or in $\mathcal{O}^{-}_{2,1}$. If it was in $\mathcal{O}^{-}_{2,1}$, then it would have the same orientation of $([e_1],[e_{n+1}],[e_1 - e_2 - e_{n+1} - e_{n+2}])$, that is the positive one, and this is not possible. $\square$

The following result is an easy, but useful, consequence of the previous characterization.

Corollary 3.25. *If a triple $(n_1,n_2,n_3) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^3$ is maximal then also (n_2,n_3,n_1) is maximal, while (n_1,n_3,n_2) is not.*

Every cyclic permutation of a maximal k-tuple (for $k \geq 3$) is a maximal k-tuple.

3.5 Action of $\mathrm{SO}_0(2,n)$ on $(\mathrm{Is}_1(\mathbb{R}^{2,n}))^4$

The study of orbits inside $(\mathrm{Is}_1(\mathbb{R}^{2,n}))^{4*}$ could proceed exactly as before: it is possible to make a first distinction of orbits of quadruples $(m_1, m_2, m_3, m_4) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^{4*}$ by considering the values $h_{2,n}(m_i, m_j) = x_{ij}$, for $1 \leq i < j \leq 4$. In general, for every i, there exists at most one $j \neq i$ such that $x_{ij} = 0$; observations like these reduce a little the number of cases, that, anyway, is still a large quantity.

For simplicity, we consider quadruples $(m_1, m_2, m_3, m_4) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^4$ in which (m_1, m_2, m_3) is maximal (notice that we are not requiring that the isotropic lines m_1, m_2, m_3 and m_4 are generated by linearly independent vectors). On the other hand, in this specific kind of quadruples are also contained the maximal ones, in which we are more interested.

So consider (m_1, m_2, m_3, m_4) such a quadruple. Up to $\mathrm{SO}_0(2,n)$ action, we can assume $(m_1, m_2, m_3) = (l_1, l_2, l_3)$ and, so, study the action of $\mathrm{Stab}_{H_{2,n}}(l_1, l_2, l_3)$ on the element m_4 (exactly as in the previous section).

Let $V = l_1 \oplus l_2 \oplus l_3$, $U = V^{\perp_{h_{2,n}}}$ and $\mathrm{pr}_V, \mathrm{pr}_U$ the projection of $\mathbb{R}^{2,n}$ on V, respectively U, with respect to the decomposition $\mathbb{R}^{2,n} = V \oplus U$; notice also that $h_{2,n}|_U$ is the standard bilinear form of signature $(0, n-1)$.

Lemma 3.26. *Two isotropic lines $[v]$, $[w] \in \mathrm{Is}_1(\mathbb{R}^{2,n})$, both transverse to l_1, l_2, l_3, are in the same orbit with respect to the $\mathrm{Stab}_{H_{2,n}}(l_1, l_2, l_3)$-action if and only if $\mathrm{pr}_V(v) = \lambda\, \mathrm{pr}_V(w)$, for some $\lambda \neq 0$.*

Proof. If $\mathrm{pr}_V(v) = \lambda\, \mathrm{pr}_V(w)$, then

$$h_{2,n}\left(\mathrm{pr}_U(v), \mathrm{pr}_U(v)\right) = \lambda h_{2,n}\left(\mathrm{pr}_U(w), \mathrm{pr}_U(w)\right)$$

(because both v, w are isotropic) and, so, there exists an element $X \in \mathrm{SO}_0(2,n)$ such that X acts as the identity on V and $X\, \mathrm{pr}_U(v) = \lambda\, \mathrm{pr}_U(w)$ (this is an easy consequence of Remark 3.2). Then $X \cdot v = \lambda w$.

Viceversa, if the elements $[v]$, $[w]$ are in the same orbit, then there exists an element $X \in \mathrm{Stab}_{H_{2,n}}(l_1, l_2, l_3)$ such that $X[v] = [w]$. By Lemma 3.13, X acts on V as $\pm\, \mathrm{Id}_V$, so

$$\begin{aligned} \pm\, \mathrm{pr}_V(v) + X\, \mathrm{pr}_U(v) = X(\mathrm{pr}_V(v) + \mathrm{pr}_U(v)) = Xv = \\ = \lambda' w = \lambda'(\mathrm{pr}_V(w) + \mathrm{pr}_U(w)), \end{aligned}$$

for some λ', and it follows $\mathrm{pr}_V(v) = \lambda\, \mathrm{pr}_V(w)$ for some $\lambda = \pm\lambda' \neq 0$. □

3.5.1 Maximal quadruples

As in the previous section, we want to describe maximal quadruples: first, we find a subset of $V = l_1 \oplus l_2 \oplus l_3$ that is in bijection with the set of orbits of maximal quadruples; in a second part, instead, we select a particular representative for each one of these orbits.

We know that for every maximal quadruple (m_1, m_2, m_3, m_4) there exists an isotropic line $[w] \in \mathrm{Is}_1(\mathbb{R}^{2,n})$ such that (m_1, m_2, m_3, m_4) and $(l_1, l_2, l_3, [w])$ are contained in the same orbit. On the other hand two quadruples of the type $(l_1, l_2, l_3, [w])$ and $(l_1, l_2, l_3, [v])$ are contained in the same orbit if and only if $[v]$ and $[w]$ project on the same line on V.

Moreover, a quadruple (m_1, m_2, m_3, m_4) is maximal if and only if every subtriple in it is maximal. From the previous section, we know that the maximality of a triple depends on the projection of this triple on the unit circle in V^+; from this fact we have that the maximality of the triples (m_1, m_2, m_3) and (m_1, m_3, m_4) implies the maximality of (m_1, m_2, m_4), and, so, also of the quadruple (m_1, m_2, m_3, m_4).

So, in order to find a set that is in bijection with the set of orbits of maximal quadruples, it is enough to find the subset of V containing all the projections of vectors $w \in \mathbb{R}^{2,n}$ for which $(l_1, l_3, [w])$ is maximal (as we already know that (l_1, l_2, l_3) is maximal).

Let $n = [w] \in \mathrm{Is}_1(\mathbb{R}^{2,n})$ such that $(l_1, l_3, [w])$ is maximal; because we are interested in the projection of w on V, we write

$$w = w_1 f_1 + w_2 f_2 + w_3 f_3 + w',$$

where $w' \in V^{\perp_{h_{2,n}}}$, so that

$$h_{2,n}\left(\mathrm{pr}_{V^+} w, \mathrm{pr}_{V^+} w\right) \geq \left|h_{2,n}\left(\mathrm{pr}_{V^-} w, \mathrm{pr}_{V^-} w\right)\right|;$$

in particular, choose w such that $h_{2,n}\left(\mathrm{pr}_{V^+} w, \mathrm{pr}_{V^+} w\right) = 1$; in this way, every isotropic line is represented by a vector whose projection in V lies on the cylinder

$$\mathcal{C} = \left\{ w \in V \;\middle|\; \begin{array}{l} h_{2,n}\left(\mathrm{pr}_{V^+} w, \mathrm{pr}_{V^+} w\right) = 1, \\ \mathrm{pr}_{V^-} w = \lambda f_3, \text{ for some } -1 \leq \lambda \leq 1 \end{array} \right\}.$$

Notice, also, that every isotropic line n corresponds to two different vectors in the cylinder $\mathcal{C}$ (a vector w and its opposite $-w$).

As $(l_1, l_3, [w])$ is maximal, $l_1 \oplus l_3 \oplus [w]$ has signature $(2, 1)$ and, by Remark 3.12, the set

$$\mathcal{S} = \left\{h_{2,n}\left((l_1)_{V,f_3}, (l_3)_{V,f_3}\right), h_{2,n}\left((l_1)_{V,f_3}, w\right), h_{2,n}\left((l_3)_{V,f_3}, w\right)\right\}$$

contains an odd number of negative values (recall that for $l \in \mathrm{Is}_1(V)$, $(l)_{V,f_3}$ is the representative of l with $\mathrm{pr}_{V^-}(l)_{V,f_3} = f_3$). Because

$$h_{2,n}\left((l_1)_{V,f_3}, (l_3)_{V,f_3}\right) < 0,$$

then

$$\operatorname{sign} h_{2,n}\left((l_1)_{V,f_3}, w\right) = \operatorname{sign} h_{2,n}\left((l_3)_{V,f_3}, w\right).$$

So, we can choose the representative $w \in \mathcal{C}$ of the isotropic line $[w]$ in such a way that $h_{2,n}((l_1)_{V,f_3}, w)$ and $h_{2,n}((l_3)_{V,f_3}, w)$ are both negative; now, some easy computations yield

$$\begin{cases} w_3 - w_1 > 0, \\ w_3 - w_2 > 0. \end{cases}$$

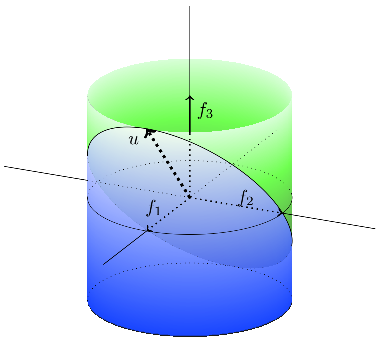

Figure 3.2: Set $\mathcal{C}$;
Green area: set of w such that $h(u,w) < 0$;
Blue area: set of w such that $h(u,w) > 0$.

Finally, the triple $(l_1, l_3, [w])$ is negatively oriented if and only if

$$w_1, w_2 > 0.$$

Summarizing:

Lemma 3.27. *There exists a correspondence between orbits of maximal quadruples and the set*

$$\left\{ w = w_1 f_1 + w_2 f_2 + w_3 f_3 \in \mathcal{C} \,\middle|\, \begin{array}{l} w_1, w_2 > 0 \\ w_3 - w_1 > 0 \\ w_3 - w_2 > 0 \end{array} \right\}.$$

It is also useful to choose a particular representative for each orbit of maximal quadruple.

Lemma 3.28. *A quadruple* $(a,b,c,d) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^4$ *is maximal if and only if there exists*

$$X = \lambda e_1 + \mu e_2 - \lambda^{-1} e_{n+1} + \mu^{-1} e_{n+2},$$

for $\lambda \in (1, \infty)$ *and* $\mu \in (\lambda^{-1}, \lambda)$*, such that* (a,b,c,d) *and* $(l_2, l_3, [X], l_1)$ *are contained in the same* $\mathrm{SO}_0(2,n)$*-orbit.*

Proof. Suppose there exists such X, then

$$(a,b,c,d) \text{ max.} \iff (l_2, l_3, [X], l_1) \text{ max.} \iff (l_1, l_2, l_3, [X]) \text{ max.}$$

But the last quadruple is maximal because of Lemma 3.27.

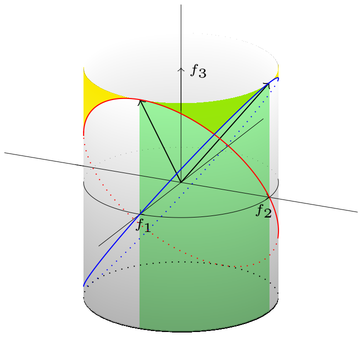

Figure 3.3: Set $\mathcal{C}$
Red line: set of vectors $w \in \mathcal{C}$ such that $h(w,(l_1)_{V,f_3}) = 0$;
Blue line: set of vectors $w \in \mathcal{C}$ such that $h(w,(l_3)_{V,f_3}) = 0$;
Green area: set of vectors $w \in \mathcal{C}$ such that the triple $((l_1)_{V,f_3},(l_3)_{V,f_3},w)$ project on a negatively oriented triple.
Yellow area: set of vectors $w \in \mathcal{C}$ such that $\operatorname{sign} h(w,(l_1)_{V,f_3}) = \operatorname{sign} h(w,(l_2)_{V,f_3}) = -1$.

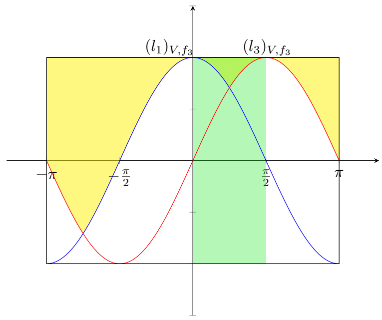

Figure 3.4: Net of Figure 3.3.

Viceversa, suppose $(a,b,c,d) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^4$ is a maximal quadruple. As (l_2,l_3,l_1) and (a,b,d) are maximal, there exists an element $g \in \mathrm{SO}_0(2,n)$ such that

$$g \cdot (l_2,l_3,l_1) = (a,b,d).$$

Moreover, as $\mathrm{Stab}_{\mathrm{SO}_0(2,n)}((l_1,l_2,l_3))$ acts transitively on $(l_1 \oplus l_2 \oplus l_3)^{\perp_{h_{2,n}}}$ (Lemma 3.13), we can choose g in such a way that $g \cdot c$ is represented by a vector in $\mathrm{Span}(e_1,e_2,e_{n+1},e_{n+2})$; let

$$X = x_1e_1 + x_2e_2 + x_{n+1}e_{n+1} + x_{n+2}e_{n+2}$$

such representative. Because X is isotropic, it holds

$$x_1x_{n+1} = -x_2x_{n+2}$$

and, from the maximality of the triple (l_2, gc, l_1), we have

$$l_1 \oplus l_2 \oplus gc \simeq \mathbb{R}^{2,1}$$

(Lemma 3.24); so, $x_2x_{n+2} > 0$. It follows that X can be written as

$$X = \lambda e_1 + \mu e_2 - \lambda^{-1}e_{n+1} + \mu^{-1}e_{n+2}$$

(up to scaling), for some λ and μ.

We still need to prove $\lambda \in (1,\infty)$ and $\mu \in (\lambda^{-1},\lambda)$, but these conditions can be obtained applying Lemma 3.27 and noticing that $(l_2,l_3,[X],l_1)$ is maximal if and only if $(l_1,l_2,l_3,[X])$ is maximal. □

Maximal quadruples might be represented by more than one quadruple of the type $(l_2, l_3, [X], l_1)$, with X as in the lemma; to avoid this problem, we have the following:

Lemma 3.29. *Let*

$$X_i = \lambda_i e_1 + \mu_i e_2 - \lambda_i^{-1} e_{n+1} + \mu_i^{-1} e_{n+2},$$

with $\lambda_i \in (1, \infty)$*,* $\mu_i \in (\lambda_i^{-1}, \lambda_i)$*, for* $i = 1, 2$*; then the two quadruples* $(l_2, l_3, [X_1], l_1)$ *and* $(l_2, l_3, [X_2], l_1)$ *are in the same orbit if and only if*

$$\begin{cases} \lambda_1 = \lambda_2, \\ \mu_1 = \mu_2, \end{cases} \qquad \textit{or} \qquad \begin{cases} \lambda_1 = \lambda_2, \\ \mu_1 = \mu_2^{-1}. \end{cases}$$

Proof. There exists an element $g \in \mathrm{SO}_0(2, n)$ such that

$$g \cdot (l_2, l_3, [X_1], l_1) = (l_2, l_3, [X_2], l_1)$$

if and only if there exists $g \in \mathrm{Stab}_{\mathrm{SO}_0(2,n)}(l_1, l_2, l_3)$ such that $g \cdot [X_1] = [X_2]$, but, from Lemma 3.13, this is equivalent to require $\lambda_1 = \lambda_2$, $\mu_1 = \mu_2$ or $\lambda_1 = \lambda_2$, $\mu_1 = \mu_2^{-1}$. □

Chapter 4

Metric on the positive Grassmannian

Let us consider the half-plane model for the Hyperbolic plane:

$$\mathbb{H}^2 = \{x + iy \in \mathbb{C} | y > 0\};$$

here geodesics are vertical lines or semicircles with center on the real line. Two geodesics intersecting in a point z are orthogonal if their tangent spaces in z are orthogonal.

Orthogonal geodesics have many applications and, between the others, they can be used to compute the distance between two points: more precisely, consider $z, w \in \mathbb{H}^2$, let $\gamma(t)$ be the unique geodesic through them and $\alpha(t), \beta(t)$ the two geodesics orthogonal to γ and passing through z, respectively w; if $\alpha^{\pm}, \beta^{\pm}, \gamma^{\pm} \in \partial\mathbb{H}^2$ are the endpoints of α, β, γ (see Figure 4.1), then the distance between z and w is the logarithm of the crossratio between the four points in the boundary $\gamma^+, \alpha^-, \beta^+, \gamma^-$:

$$d^{\mathbb{H}^2}(z, w) = \log \mathbb{B}(\gamma^+, \alpha^-, \beta^+, \gamma^-) = \log \frac{(\beta^+ - \gamma^+)(\alpha^- - \gamma^-)}{(\beta^+ - \gamma^-)(\alpha^- - \gamma^+)}.$$

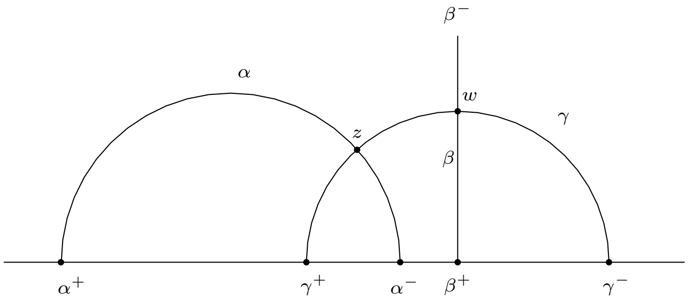

Figure 4.1: Half-plane model

We follow the same idea to construct distances in $\mathcal{X}_{2,n}$. However, we first need to find suitable analogous for some of the tools appearing above: we use the generalized crossratio, introduced in [Lab08], to construct a Weyl chamber-valued crossratio (sending maximal quadruples to pairs in $\mathbb{R}^2$) and we extend the notion of geodesic to the notion of $\mathbb{R}$-tube, a sort of "higher dimensional geodesic". Then, thanks to the Riemannian structure endowed in the symmetric space $\mathcal{X}_{2,n}$, it is possible to study orthogonal $\mathbb{R}$-tubes. Once done this, we are ready to define a projection of pairs of points to the closure of the Weyl chamber and use it to introduce two different $\mathrm{SO}_0(2, n)$-invariant distances in $\mathcal{X}_{2,n}$.

4.1 Crossratio

A really important tool in projective geometry is the so-called *projective crossratio*: this object turns out to characterize the geometry of projective lines in such a way that even the distance is expressed in terms of this function.

Labourie, in [Lab08], generalized this notion to every topological space S equipped with an action of a group Γ.

Definition 4.1. *Let S be a topological space equipped with an action of a group Γ. A* crossratio *on S is a real valued Γ-invariant continuous function $\mathcal{T}$ defined on a subset of the set $S^{4*} = \{(x, y, z, w) \in \mathcal{X}^4 | x \neq y,\ z \neq w\}$, which satisfies the following rules*

1. $\mathcal{T}(x, y, z, w) = \mathcal{T}(z, w, x, y)$,
2. $\mathcal{T}(x, y, z, w) = 0 \iff x = z$ *or* $y = w$,
3. $\mathcal{T}(x, y, z, w) = 1 \iff x = w$ *or* $y = z$,
4. $\mathcal{T}(x, y, z, w) = \mathcal{T}(x, y, z, t)\mathcal{T}(t, y, z, w)$,
5. $\mathcal{T}(x, y, z, w) = \mathcal{T}(x, t, z, w)\mathcal{T}(x, y, t, w)$.[1]

The properties required by Labourie allowed him to study distances and translational lengths in the context of maximal representations.

Example 4.2. The function

$$\begin{array}{rccc} T: & \left\{ (a, b, c, d) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^4 \middle| (a, b, c, d) \text{ max.} \right\} & \to & \mathbb{R} \\ & ([x], [y], [z], [w]) & \mapsto & \frac{h_{2,n}(x,z)h_{2,n}(w,y)}{h_{2,n}(x,y)h_{2,n}(w,z)} \end{array}$$

is a crossratio in the sense of Definition 4.1 on the set of istropic lines: indeed, T does not depend on the choice of the representatives x, y, z, w and it is $\mathrm{SO}_0(2, n)$-invariant.

About the function T:

Lemma 4.3. *For any maximal quadruple $(a, b, c, d) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^4$, it holds $T(a, b, c, d) \in (1, \infty)$.*

Proof. By Lemma 3.28, there exists a $g \in \mathrm{SO}_0(2, n)$ such that

$$g \cdot (a, b, c, d) = (l_2, l_3, \ell, l_1),$$

where

$$\ell = \left[\lambda e_1 + \mu e_2 - \lambda^{-1} e_{n+1} + \mu^{-1} e_{n+2}\right]$$

for some $\lambda > \mu \geq 1$. Now the claim follows from an easy computation and the fact that T is $\mathrm{SO}_0(2, n)$-invariant. $\square$

[1] This definition is not exactly the same of the one appearing in [Lab08]: indeed, Labourie gave a different order on the entries of the function $\mathcal{T}$ and he considered the entire set S^{4*} as a domain of $\mathcal{T}$ and not a subset of it.

When the space S is a subset of the visual boundary of a symmetric space, an even more useful function is a *Weyl chamber-valued* crossratio: this tool is already well known in the Siegel space $\mathcal{X}$ and it is used to describe a projection of pairs of points in $(\mathcal{X})^2$ to the closure of the Weyl chamber of $\mathcal{X}$; see, for instance, [Sie43].

In the rest of this section we define and study an analogous Weyl chamber-valued crossratio for the set of maximal quadruples of isotropic lines.

At the end of Chapter 3 it is stated that every orbit of maximal quadruples contains exactly one element of the type $(l_2, l_3, [X], l_1)$, where

$$X = \lambda e_1 + \mu e_2 - \lambda^{-1} e_{n+1} + \mu^{-1} e_{n+2},$$

with $\lambda > \mu \geq 1$; in particular, it is possible to associate each orbit of maximal quadruples with a pair of the type (λ, μ).

Definition 4.4. *Consider (a, b, c, d) a maximal quadruple and let the vector*

$$X = \lambda e_1 + \mu e_2 - \lambda^{-1} e_{n+1} + \mu^{-1} e_{n+2},$$

with $\lambda > \mu \geq 1$, be such that (a, b, c, d) and $(l_2, l_3, [X], l_1)$ are in the same orbit. Then the Weyl chamber-valued crossratio *on the quadruple (a, b, c, d) is defined by*

$$R(a, b, c, d) = (\lambda, \mu).$$

Remark 4.5. The function R can also be extended to those quadruples (a, b, c, d) contained in the orbit of some quadruples of the type $(l_2, l_3, [X], l_1)$, where

$$X = \lambda e_1 + \lambda e_2 - \lambda^{-1} e_{n+1} + \lambda^{-1} e_{n+2}$$

and $\lambda \geq 1$. In this case we set

$$R(a, b, c, d) = (\lambda, \lambda).$$

Remark 4.6. Despite the name, it is not true that the function R takes its values in the Weyl chamber $\mathfrak{w}_{2,n}$. However, if $R(a, b, c, d) = (\lambda, \mu)$, then the vector $(\log \lambda, \log \mu)$ is contained in the closure of the Weyl chamber (according to Remark 2.4).

Even if the definition of R is quite simple to understand, it is not very efficient from a computational view point. For this reason we use the function T (Example 4.2) in order to give an explicit expression of the function R on every maximal quadruple: consider a maximal quadruple (a, b, c, d) in the same orbit of $(l_2, l_3, [X], l_1)$ (where X is as above). Because T is $\mathrm{SO}_0(2, n)$-invariant, we have

$$\begin{aligned} T(a, b, c, d) &= T(l_2, l_3, [X], l_1) = \lambda^2, \\ T(d, a, b, c) &= T(l_1, l_2, l_3, [X]) = \frac{\lambda}{\lambda + \lambda^{-1} - \mu - \mu^{-1}}. \end{aligned}$$

So, denoting $T(a,b,c,d) = s$, $T(d,a,b,c) = t$ and $r = \frac{st+t-s}{t\sqrt{s}}$, we get

$$\begin{cases} \lambda = \sqrt{s}, \\ \mu = \frac{r+\sqrt{r^2-4}}{2}. \end{cases} \tag{4.1}$$

In order to obtain these expressions, we solved two equations of degree 2: the solution is unique because we are requiring $\lambda > \mu \geq 1$.

Using Expressions (4.1), it is easy to check that $r \geq 2$.

Lemma 4.7. *Let (a,b,c,d) be a maximal quadruple and consider the values $T(a,b,c,d) = s, T(d,a,b,c) = t$, then*

$$r = \frac{st+t-s}{t\sqrt{s}} \geq 2.$$

With the computations above, the following result is immediate;

Lemma 4.8. *Let (a,b,c,d) be a maximal quadruple and consider the values $T(a,b,c,d) = s, T(d,a,b,c) = t$ and $r = \frac{st+t-s}{t\sqrt{s}}$. Then the Weyl chamber-valued crossratio evaluated on (a,b,c,d) is*

$$R(a,b,c,d) = \left(\sqrt{s}, \frac{r+\sqrt{r^2-4}}{2}\right).$$

4.2 $\mathbb{R}$-tubes

The goal of this section is to find a suitable generalization of classical geodesics in the context of the Hermitian symmetric space $\mathcal{X}_{2,n}$. In [BP17] it is pointed out that a good generalization of the notion of geodesic in the setting of maximal representations is given by the notion of $\mathbb{R}$-tube.

Definition 4.9. *Let a, b two isotropic lines; the* $\mathbb{R}$-tube $\mathcal{Y}_{a,b}$ *with endpoints a, b is the parallel set of Riemannian singular geodesics whose endpoints in the visual boundary of $\mathcal{X}_{2,n}$ are precisely the points a and b.*

Even if every pair of points in $\mathcal{X}_{2,n}$ can be connected with a unique geodesic, the same does not hold true in its visual boundary and, so, it might happen that some $\mathbb{R}$-tube is empty. However, from symmetric space theory it is known that, if $a, b \in \mathrm{Is}_1(\mathbb{R}^{2,n})$ are the two endpoints of a geodesic, then a and b are transverse; viceversa, if $a \pitchfork b$, then there exists a geodesic whose endpoints are exactly a and b:

$$\mathcal{Y}_{a,b} \neq \emptyset \qquad \Longleftrightarrow \qquad h_{2,n}(a,b) \neq 0;$$

on the other hand, also in [BP17] the authors considered only $\mathbb{R}$-tubes whose endpoints a, b are *transverse* in the respective boundary.

Remark 4.10. For every pair $(a,b) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^{2*}$ of transverse isotropic lines and every $g \in \mathrm{SO}_0(2,n)$, it holds

$$g\mathcal{Y}_{a,b} = \mathcal{Y}_{ga,gb};$$

moreover, $\mathrm{SO}_0(2,n)$ acts transitively on the set of $\mathbb{R}$-tubes. These are consequences of the fact that $\mathrm{SO}_0(2,n)$ acts by isometries on $\mathcal{X}_{2,n}$ and transitively on the set of pairs of transverse isotropic lines.

Let a,b be two transverse isotropic lines in $\mathbb{R}^{2,n}$ and denote with

$$P_a = \mathrm{Stab}_{\mathrm{SO}_0(2,n)}(a),$$
$$P_b = \mathrm{Stab}_{\mathrm{SO}_0(2,n)}(b),$$

the parabolic subgroups of $\mathrm{SO}_0(2,n)$ fixing a, respectively b. Notice that an element $g \in \mathrm{SO}_0(2,n)$ preserves the $\mathbb{R}$-tube $\mathcal{Y}_{a,b}$ if and only if

$$g \cdot \{a,b\} = \{a,b\};$$

in particular, one of the two connected components of the stabilizer of $\mathcal{Y}_{a,b}$ is given by the set of matrices in $\mathrm{SO}_0(2,n)$ preserving both a and b, that is the intersection

$$P_{a,b} = P_a \cap P_b;$$

so, for every $x \in \mathcal{Y}_{a,b}$ we have

$$\mathcal{Y}_{a,b} = \{g \cdot x | g \in P_{a,b}\}.$$

The strategy to get a "nice" expression for every $\mathbb{R}$-tube $\mathcal{Y}_{a,b}$ is to compute it for our preferred isotropic lines $l_1 = [e_1], l_2 = [e_2]$ and then use the transitivity of the action of $\mathrm{SO}_0(2,n)$ on the set of $\mathbb{R}$-tubes in order to compute the others.

Let us denote with

$$V_0 = \mathrm{Span}(e_1 + e_{n+1}, e_2 + e_{n+2})$$

the base point in $\mathcal{X}_{2,n}$. The subgroup $P_{l_1,l_2} = \mathrm{Stab}_{\mathrm{SO}_0(2,n)}(l_1,l_2)$ has been computed in Lemma 3.8, so $\mathcal{Y}_{l_1,l_2}$ is readily given by $(P_{l_1,l_2}) \cdot V_0$:

$$\mathcal{Y}_{l_1,l_2} = \left\{ \mathrm{Span}(v,w) \middle| \begin{matrix} v \in \mathrm{Span}(e_1,e_{n+1}), & h_{2,n}(v,v) > 0 \\ w \in \mathrm{Span}(e_1,e_{n+1})^{\perp_{h_{2,n}}}, & h_{2,n}(w,w) > 0 \end{matrix} \right\}, \quad (4.2)$$

that is, indeed, a totally geodesic submanifold of dimension n in $\mathcal{X}_{2,n}$.

Our next goal is to describe the metric structure on $\mathcal{Y}_{l_1,l_2}$.

Proposition 4.11. *The submanifold $\mathcal{Y}_{l_1,l_2} \subset \mathcal{X}_{2,n}$ is isometric to $\mathbb{R} \times \mathbb{H}^{n-1}$.*

Proof. Consider the splitting

$$\begin{array}{rcccc} \varphi: \mathcal{Y}_{l_1,l_2} & \to & \mathrm{Gr}_1^+(l_1 \oplus l_2) & \times & \mathrm{Gr}_1^+\left((l_1 \oplus l_2)^{\perp_{h_{2,n}}}\right) \\ V & \mapsto & \big(V \cap (l_1 \oplus l_2) & , & V \cap (l_1 \oplus l_2)^{\perp_{h_{2,n}}}\big) \end{array}$$

and the two maps

$$\begin{array}{rccc} \alpha: & \mathrm{Gr}_1^+(l_1 \oplus l_2) & \to & \mathbb{R} \\ & \left[\frac{1}{\sqrt{2}}\left(e^{\lambda}e_1 + e^{-\lambda}e_{n+1}\right)\right] & \mapsto & \lambda \end{array}$$

$$\begin{array}{rccc} \beta: & \mathrm{Gr}_1^+\left((l_1 \oplus l_2)^{\perp_{h_{2,n}}}\right) & \to & \mathbb{H}^{n-1} \\ & [v] & \mapsto & \pm\frac{v}{\sqrt{h_{2,n}(v,v)}} \end{array}$$

About these definitions:

- the bilinear forms on $l_1 \oplus l_2$ and $(l_1 \oplus l_2)^{\perp_{h_{2,n}}}$ are respectively $h_{2,n}|_{l_1\oplus l_2}$, of signature $(1,1)$, and $h_{2,n}|_{(l_1\oplus l_2)^{\perp_{h_{2,n}}}}$, of signature $(1, n+1)$; so it makes sense to talk about positive Grassmannian of these two spaces;
- from Equality (4.2) it follows that the intersection $V \cap (l_1 \oplus l_2)$ and $V \cap (l_1 \oplus l_2)^{\perp}$ are nonempty;
- the map α is well defined: every line in $\mathrm{Gr}_1^+(l_1 \oplus l_2)$ can be represented by a unique unit vector $v \in l_1 \oplus l_2$ such that $h_{2,n}(v, e_1) > 0$ and this vector can be written as

$$\frac{1}{\sqrt{2}}(e^{\lambda}e_1 + e^{-\lambda}e_{n+1});$$

- consider the hyperboloid model of $\mathbb{H}^{n-1}$: every element $x \in \mathbb{H}^{n-1}$ is a vector in $\mathbb{R}^{1,n}$ such that $h_{1,n}(x,x) = 1$ and $x_1 > 0$. In the definition of β, $\mathbb{R}^{1,n}$ is identified with $(l_1 \oplus l_2)^{\perp}$. Every line $[v] \in \mathrm{Gr}_1^+((l_1 \oplus l_2)^{\perp})$ can be represented by a unique vector

$$x = \pm v/\sqrt{h_{2,n}(v,v)}$$

such that $h_{2,n}|_{(l_1\oplus l_2)^{\perp}}(x,x) = 1$ and $h_{2,n}(x, e_2 + e_{n+2}) > 0$ and, so, an element in $\mathbb{H}^{n-1}$.

The three maps α, β, φ are bijective.

We still need to prove $(\alpha, \beta)\circ\varphi$ is an isometry: because the Weyl chamber $\mathfrak{w}_{2,n} \subset \mathfrak{a}_{2,n}$ is a fundamental domain for the $\mathrm{SO}_0(2,n)$-action on the tangent space (see Section 2.2.3), it is enough to consider vectors of the type

$$X = \mathrm{diag}\,(\lambda, \mu, 0, ..., 0, -\lambda, -\mu) \in \mathfrak{a}_{2,n}.$$

for some $\lambda, \mu \in \mathbb{R}$. In this case, the corresponding geodesic is

$$\exp(tX) = \gamma(t) = \operatorname{Span}\left\langle e^{\lambda t}e_1 + e^{-\lambda t}e_{n+1}, e^{\mu t}e_2 + e^{-\mu t}e_{n+2}\right\rangle,$$

sent, via the map $(\alpha, \beta) \circ \varphi$, to

$$((\alpha, \beta) \circ \varphi)(\gamma(t)) = \left(\lambda t, \frac{1}{\sqrt{2}}\left(e^{\mu t}e_2 + e^{-\mu t}e_{n+2}\right)\right),$$

whose tangent vector in $((\alpha, \beta) \circ \varphi)(\gamma(0)) = \left(0, \frac{1}{\sqrt{2}}\left(e_2 + e_{n+1}\right)\right)$ is

$$\left(\lambda, \frac{1}{\sqrt{2}}\left(\mu e_2 - \mu e_{n+2}\right)\right).$$

This proves that $(\alpha, \beta) \circ \varphi$ is an isometry (recall that the bilinear form on $\mathrm{T}_{V_0}\mathcal{X}_{2,n}$ is given by Expression (2.6)). $\square$

Let us compute the tangent space of $\mathcal{Y}_{l_1,l_2}$ in V_0: first of all, notice that the maximal abelian $\mathfrak{a}_{2,n}$ (studied in Section 2.2.3) is a subspace of $\mathrm{T}_{V_0}\mathcal{Y}_{l_1,l_2}$; moreover, for every $X \in \mathrm{T}_{V_0}\mathcal{Y}_{l_1,l_2}$ and $g \in P_{l_1,l_2} \cap \mathrm{Stab}_{\mathrm{SO}_0(2,n)}(V_0)$, the vector $g^{-1}Xg$ is still contained in the tangent space $\mathrm{T}_{V_0}\mathcal{Y}_{l_1,l_2}$; now, some computations yield

$$\mathrm{T}_{V_0}\mathcal{Y}_{l_1,l_2} = \left\{ \left(\begin{array}{cc|c|cc} a_1 & & & & \\ & a_2 & b^T & & \\ \hline & b & 0_{n-2} & & b \\ \hline & & & -a_1 & \\ & & b^T & & -a_2 \end{array}\right) \middle| \begin{array}{l} a_1, a_2 \in \mathbb{R} \\ b \in \mathbb{R}^{n-2} \end{array} \right\}, \tag{4.3}$$

that is, indeed, a vector space of dimension n.

We are now ready to write the general expression for an $\mathbb{R}$-tube:

Proposition 4.12. *Given two transverse isotropic lines $a, b \in \mathrm{Is}_1(\mathbb{R}^{2,n})$, the $\mathbb{R}$-tube relative to a, b is*

$$\mathcal{Y}_{a,b} = \left\{ \operatorname{Span}(v, w) \middle| \begin{array}{ll} v \in \operatorname{Span}(a, b), & h_{2,n}(v, v) > 0 \\ w \in \operatorname{Span}(a, b)^{\perp_{h_{2,n}}}, & h_{2,n}(w, w) > 0 \end{array} \right\}.$$

An immediate consequence of Lemma 4.11 is that every $\mathbb{R}$-tube is isometric to $\mathbb{R} \times \mathbb{H}^{n-1}$.

Corollary 4.13. *Let a, b two transverse isotropic lines, then the $\mathbb{R}$-tube $\mathcal{Y}_{a,b}$ is isometric to $\mathbb{R} \times \mathbb{H}^{n-1}$.*

In analogy with geodesics, given two points in $\mathcal{X}_{2,n}$, there exixts an $\mathbb{R}$-tube through them.

Lemma 4.14. *For each pair of points $V, W \in \mathcal{X}_{2,n}$, there exists at least one $\mathbb{R}$-tube $\mathcal{Y}$ containing both V, W.*

Proof. Let $V, W \in \mathcal{X}_{2,n}$. As $\mathcal{X}_{2,n}$ is an Hadamard manifold, there exists $Y \in \mathrm{T}_V\mathcal{X}_{2,n}$ such that

$$\exp_V(Y) = W.$$

Because the Weyl chamber is a fundamental domain for the $\mathrm{SO}_0(2,n)$-action on the tangent space, there are $X \in \overline{\mathfrak{w}}_{2,n}$ and $g \in \mathrm{SO}_0(2,n)$ such that

$$\begin{cases} g \cdot Y = X, \\ g \cdot V = V_0; \end{cases}$$

in particular $g \cdot W = \exp_{V_0}(X) \in \mathcal{Y}_{l_1,l_2}$. Now

$$g^{-1} \cdot \mathcal{Y}_{l_1,l_2} = \mathcal{Y}_{g^{-1}l_1, g^{-1}l_2}$$

gives the statement. $\square$

However, the $\mathbb{R}$-tube through X and Y may not be unique: for example the maximal flat $\exp_{V_0} \mathfrak{a}_{2,n}$ is contained in both $\mathcal{Y}_{l_1,l_2}$ and $\mathcal{Y}_{[e_2],[e_{n+2}]}$.

4.3 Orthogonal $\mathbb{R}$-tubes

Two geodesics, contained in a Riemannian manifold M and intersecting in a point $x \in M$, are said to be orthogonal if their velocity vectors are orthogonal in T_xM. In analogy, two $\mathbb{R}$-tubes, contained in the Hermitian symmetric space $\mathcal{X}_{2,n}$ and intersecting in a point $x \in \mathcal{X}_{2,n}$, are orthogonal in x if their tangent spaces in x are orthogonal subspaces of $\mathrm{T}_x\mathcal{X}_{2,n}$.

Definition 4.15. *Two $\mathbb{R}$-tubes $\mathcal{Y}, \mathcal{Y}' \subset \mathcal{X}_{2,n}$ are called* orthogonal $\mathbb{R}$-tubes *in x if they are orthogonal submanifolds of $\mathcal{X}_{2,n}$ in x with respect to the Riemannian metric. In this case we write $\mathcal{Y} \perp_x \mathcal{Y}'$ or simply $\mathcal{Y} \perp \mathcal{Y}'$.*

The scalar product on the tangent space $\mathfrak{p}_{2,n}$ is given by the Killing form, that is positive definite since $\mathcal{X}_{2,n}$ is of non-compact type; we recall, from Chapter 2, that the expression of the (scaled) Killing form in $\mathrm{SO}_0(2,n)$ is:

$$B(X,Y) = \frac{1}{2}\,\mathrm{tr}(XY).$$

Using this, it is easy to check that the n-dimensional vector space orthogonal to $\mathrm{T}_{V_0}\mathcal{Y}_{l_1,l_2}$ is

$$(\mathrm{T}_{V_0}\mathcal{Y}_{l_1,l_2})^{\perp} = \left\{ \left(\begin{array}{cc|c|cc} & a & b^T & & c \\ a & & & -c & \\ \hline b & & 0_{n-2} & b & \\ \hline & -c & b^T & & -a \\ c & & & -a & \end{array} \right) \middle| \begin{array}{l} a, c \in \mathbb{R} \\ b \in \mathbb{R}^{n-2} \end{array} \right\}. \tag{4.4}$$

In particular, when $a = c \neq 0$ and $b = 0$, we have a matrix with eigenvalues

$$a + c, 0, -a - c,$$

whose multiplicities are, respectively, 1, n, 1. The eigenlines corresponding to $a + c$ and $-a - c$ are

$$\begin{aligned} l_3 =& [e_1 + e_2 - e_{n+1} + e_{n+2}], \\ l_4 :=& [-e_1 + e_2 + e_{n+1} + e_{n+2}]. \end{aligned}$$

So, $(\mathrm{T}_{V_0}\mathcal{Y}_{l_1,l_2})^\perp$ is the tangent space of $\mathcal{Y}_{l_3,l_4}$ and, moreover,

$$\mathcal{Y}_{l_1,l_2} \perp_{V_0} \mathcal{Y}_{l_3,l_4}.$$

More generally:

Lemma 4.16. *Let $a, b, c, d \in \mathrm{Is}_1(\mathbb{R}^{2,n})$, with (a, b, c) maximal and $b \pitchfork d$. The $\mathbb{R}$-tubes $\mathcal{Y}_{a,c}, \mathcal{Y}_{b,d}$ are orthogonal if and only if (a, b, c, d) is maximal and $R(a, b, c, d) = (2, 1)$. Moreover, if this is the case, then $d \in a \oplus b \oplus c$.*

Proof. Claim 1: given a point $x \in \mathcal{Y}_{a,c}$, there exists $g \in \mathrm{SO}_0(2, n)$ such that

$$\begin{cases} g \cdot \mathcal{Y}_{l_1,l_2} = \mathcal{Y}_{a,c}, \\ g \cdot V_0 = x; \end{cases}$$

indeed $\mathrm{SO}_0(2, n)$ acts transitively on the set of $\mathbb{R}$-tubes, so there exists a matrix $g_1 \in \mathrm{SO}_0(2, n)$ such that $g_1 \cdot \mathcal{Y}_{l_1,l_2} = \mathcal{Y}_{a,c}$; moreover, because the set $\mathrm{Stab}_{\mathrm{SO}_0(2,n)}(l_1, l_2)$ acts transitively on $\mathcal{Y}_{l_1,l_2}$ (by construction), also $\mathrm{Stab}_{\mathrm{SO}_0(2,n)}(a, c)$ acts transitively on $\mathcal{Y}_{a,c}$ and there exists another matrix $g_2 \in \mathrm{Stab}_{\mathrm{SO}_0(2,n)}(a, c)$ such that $g_2 \cdot (g_1 \cdot V_0) = x$. Now $g = g_2 g_1$ gives the claim.

Claim 2: the element $\ell = l_3$ is the unique isotropic line contained in $l_1 \oplus l_2 \oplus l_4$ such that

$$R(l_1, l_4, l_2, \ell) = (2, 1);$$

this is just a computation.

Claim 3: given $x \in \mathcal{Y}_{a,c}$, there exists at most one $\mathbb{R}$-tube $\mathcal{Y}$ orthogonal to $\mathcal{Y}_{a,c}$ in x. Indeed $\mathcal{Y}_{a,c}$ is a totally geodesic submanifold of dimension n; this means that the tangent space in x is a vector space of dimension n and its orthogonal vector space in $\mathrm{T}_x\mathcal{X}_{2,n}$ is still a vector space of dimension n (because $\dim \mathrm{T}_x\mathcal{X}_{2,n} = 2n$), that is the tangent space in x of at most one totally geodesic submanifold of dimension n.

Suppose $\mathcal{Y}_{a,c} \perp_x \mathcal{Y}_{b,d}$. There exists $g \in \mathrm{SO}_0(2, n)$ such that

$$\begin{cases} g \cdot (a, c) = (l_1, l_2), \\ g \cdot x = V_0. \end{cases}$$

Because $\mathrm{SO}_0(2,n)$ acts by isometries, $\mathcal{Y}_{l_1,l_2} \perp_{V_0} \mathcal{Y}_{gb,gd}$; by claim 3,

$$\mathcal{Y}_{gb,gd} = \mathcal{Y}_{l_3,l_4}$$

and so

$$\begin{cases} gb = l_3, \\ gd = l_4, \end{cases} \quad \text{or} \quad \begin{cases} gd = l_3, \\ gb = l_4. \end{cases}$$

As the triple (a,b,c) is maximal, (ga,gb,gc) is also maximal and so we have $gb = l_4$ and $gd = l_3$. We deduce that (a,b,c,d) is a maximal quadruple and

$$R(a,b,c,d) = R(l_1,l_4,l_2,l_3) = (2,1).$$

Viceversa, suppose that (a,b,c,d) is a maximal quadruple and that $R(a,b,c,d) = (2,1)$. By Lemma 3.28, there exists an $h \in \mathrm{SO}_0(2,n)$ such that

$$h \cdot (a,b,c,d) = (l_2,l_3,[X],l_1),$$

where

$$X = \lambda e_1 + \mu e_2 - \lambda^{-1} e_{n+1} + \mu^{-1} e_{n+2},$$

for some $\lambda > \mu \geq 1$. But now, by definition of R, we have

$$(\lambda,\mu) = R(l_2,l_3,[X],l_1) = R(a,b,c,d) = (2,1)$$

and so $d \subset a \oplus b \oplus c$ (because $l_1 \subset l_2 \oplus l_3 \oplus [X]$).

By Lemma 3.11, the action of $\mathrm{SO}_0(2,n)$ is transitive on the set of maximal triples and, in particular, there exists $g \in \mathrm{SO}_0(2,n)$ such that

$$g \cdot (a,b,c) = (l_1,l_4,l_2).$$

Because $d \in a \oplus b \oplus c$ also $gd \in l_1 \oplus l_4 \oplus l_2$; moreover

$$R(l_1,l_4,l_2,gd) = R(a,b,c,d) = (2,1).$$

By claim 2, $gd = l_3$. Now we have $\mathcal{Y}_{ga,gc} \perp \mathcal{Y}_{gb,gd}$ and so $\mathcal{Y}_{a,c} \perp \mathcal{Y}_{b,d}$. □

4.4 Orthogonality to $\mathcal{Y}_{l_1,l_2}$

The transitivity of $\mathrm{SO}_0(2,n)$ on the set of $\mathbb{R}$-tubes allow us to move always one of the $\mathbb{R}$-tubes we are considering to $\mathcal{Y}_{l_1,l_2}$ and, for this reason, it is useful to study properties of this $\mathbb{R}$-tube.

Let us start with some notation:

Definition 4.17. *Given an isotropic line $a \in \mathrm{Is}_1(\mathbb{R}^{2,n})$ such that one between (l_1,a,l_2) and (l_2,a,l_1) is maximal, we denote with $\bar{a}$ the unique isotropic line such that $\mathcal{Y}_{l_1,l_2} \perp \mathcal{Y}_{a,\bar{a}}$. We call $\bar{a}$ the* conjugate isotropic line *of a with respect to the pair (l_1,l_2).*

Given a as in the definition above, the existence and uniqueness of its conjugated is assured by the following result:

Lemma 4.18. *Let*

$$a = \left[\sum_{i=1}^{n+2} a_i e_i\right] \in \mathrm{Is}_1(\mathbb{R}^{2,n})$$

be an isotropic line such that one between (l_1, a, l_2) *and* (l_2, a, l_1) *is maximal, then*

$$\bar{a} = \left[\sum_{\substack{i=2 \\ i \neq n+1}}^{n+2} a_i e_i - a_1 e_1 - a_{n+1} e_{n+1}\right].$$

Moreover

$$\mathcal{Y}_{l_1,l_2} \cap \mathcal{Y}_{a,\bar{a}} = \left\{ \mathrm{Span}\left(a_1 e_1 - a_{n+1} e_{n+1}, \sum_{\substack{i=2 \\ i \neq n+1}}^{n+2} a_i e_i \right) \right\}.$$

Proof. It is an application of Lemma 4.16. □

This lemma justifies also the term *conjugate*: let us consider an isotropic line $a \in \mathrm{Is}_1(\mathbb{R}^{2,n})$ as in the lemma; the set

$$\mathcal{B} = \left\{ e_1 + e_{n+1}, \sum_{\substack{i=2 \\ i \neq n+1}}^{n+2} a_i e_i, \ e_1 - e_{n+1} \right\}$$

is a basis of $l_1 \oplus l_2 \oplus a$. Then, denoting with f_1, f_2 and f_3 the elements of $\mathcal{B}$ and proceeding as in Section 3.4.1, we can associate the isotropic lines $l_1, l_2, a, \bar{a}$ with four points on $\mathbb{S}^1$ (recall from Lemma 4.16, that $\bar{a} \in l_1 \oplus l_2 \oplus a$). In this way the isotropic line $\bar{a}$ is represented by the conjugate point associated to a (see Figure 4.4).

On the other hand, given a point $V \in \mathcal{Y}_{l_1,l_2}$, it is also useful to have an expression for $a \in \mathrm{Is}_1(\mathbb{R}^{2,n})$ such that $\mathcal{Y}_{l_1,l_2} \perp_V \mathcal{Y}_{a,\bar{a}}$.

Lemma 4.19. *Let* $V \in \mathcal{Y}_{l_1,l_2}$, *and consider the* 1*-dimensional subspaces*

$$[v] \subset V^{\perp_{h_{2,n}}} \cap (l_1 \oplus l_2),$$
$$[w] \subset V \cap (l_1 \oplus l_2)^{\perp_{h_{2,n}}};$$

then the isotropic lines a *and* $\bar{a}$ *such that* $\mathcal{Y}_{l_1,l_2} \perp_x \mathcal{Y}_{a,\bar{a}}$ *are the only two isotropic lines contained in* $\mathrm{Span}(v, w)$.

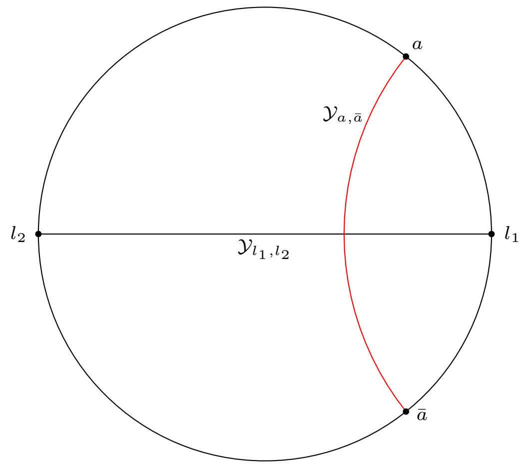

Figure 4.2: The isotopic line $\bar{a}$ is represented by the conjugate point of a with respect to the line connecting l_1 and l_2.

Proof. This is clearly true for $V = V_0$. The general case follows from the transitivity of $\mathrm{Stab}(l_1, l_2)$ on $\mathcal{Y}_{l_1,l_2}$: indeed, consider $g \in \mathrm{Stab}(l_1, l_2)$ with $gV_0 = V$, so that $g \cdot (l_3, l_4) = (a, \bar{a})$; as

$$[v] \subset V_0^{\perp_{h_{2,n}}} \cap (l_1 \oplus l_2),$$
$$[w] \subset V_0 \cap (l_1 \oplus l_2)^{\perp_{h_{2,n}}},$$

are such that $l_3, l_4 \in \mathrm{Span}(v, w)$, then

$$[gv] \subset (gV_0)^{\perp_{h_{2,n}}} \cap g(l_1 \oplus l_2) = V^{\perp_{h_{2,n}}} \cap (l_1 \oplus l_2),$$
$$[gw] \subset gV_0 \cap g(l_1 \oplus l_2)^{\perp_{h_{2,n}}} = V \cap (l_1 \oplus l_2)^{\perp_{h_{2,n}}},$$

are such that $a, \bar{a} \in \mathrm{Span}(gv, gw)$.

Moreover, since $v \in V_0$ and $w \in V_0^{\perp_{h_{2,n}}}$, then $\mathrm{Span}(v, w)$ (as well as $\mathrm{Span}(gv, gw)$) is of signature $(1, 1)$ and, so, $a, \bar{a}$ are the only two isotropic lines in $\mathrm{Span}(gv, gw)$. □

4.5 Projection to the Weyl chamber

Now we have all the tools needed to define a projection from $(\mathcal{X}_{2,n})^2$ to the closure of the Weyl chamber $\overline{\mathfrak{w}}_{2,n}$.

As $\mathcal{X}_{2,n}$ is an Hadamard manifold, given two points $(V, W) \in (\mathcal{X}_{2,n})^2$, there exists a unique vector Y in the tangent space $\mathrm{T}_V\mathcal{X}_{2,n}$ satisfying the relation

$$W = \exp_V(tY);$$

moreover, as the Weyl chamber is a fundamental domain for the $\mathrm{SO}_0(2,n)$-action on the tangent space $\mathrm{T}\mathcal{X}_{2,n}$, there exists a matrix $g \in \mathrm{SO}_0(2,n)$ such that

$$g \cdot Y = X \in \overline{\mathfrak{w}}_{2,n}.$$

Now, a vector contained in a maximal abelian is said to be *regular* if there is no root vanishing at that vector; in our case, a vector contained in $\mathfrak{a}_{2,n}$ (maximal abelian described in Section 2.2.3) is regular if and only if it has two non-zero distinct eignevalues. By definition of Weyl chamber, we know that if $X \in \overline{\mathfrak{w}}_{2,n}$ is regular, then it is contained in the interior of this Weyl chamber and, so, it is uniquely determined by Y.

On the other hand, if X is not regular, then it has a unique eigenvalue $\lambda > 0$ with geometric multiplicity 1 or 2. In the first case the endpoints of the geodesic $\gamma(t) := \exp_V(tY)$ are two isotropic lines; in the second case, when the geometric multiplicity of λ is 2, the endpoints of γ are two isotropic planes. It is now clear that the two cases can not lay in the same orbit. This means that the vector $X \in \overline{\mathfrak{w}}_{2,n}$, regular or not, is uniquely determined by the vector Y and, so, by the pair (V, W).

Remark 4.20. In other words, the Weyl chamber $\overline{\mathfrak{w}}_{2,n}$ is a *strict fundamental domain*: this is a more general fact, that holds for every symmetric space.

We say that the vector X is the projection of the pair $(V, W) \in (\mathcal{X}_{2,n})^2$ to the closure of the Weyl chamber $\overline{\mathfrak{w}}_{2,n}$.

Definition 4.21. *Let $\mathcal{X} = G/K$ be the symmetric space associated to the Lie group G and $\mathfrak{w}$ a Weyl chamber of $\mathcal{X}$. The function*

$$d^{\overline{\mathfrak{w}}} : (\mathcal{X})^2 \to \overline{\mathfrak{w}}$$

sends each pair of points $(V, W) \in (\mathcal{X})^2$ to the unique vector $X \in \overline{\mathfrak{w}}$ contained in the same G-orbit of the vector $Y \in \mathrm{T}_V\mathcal{X}$ that satisfies $\exp_V(Y) = W$. The function $d^{\overline{\mathfrak{w}}}$ is called vectorial distance *or, also,* projection to the closure of the Weyl chamber.

Recall, from Remark 2.4, that each vector

$$X = \mathrm{diag}(\lambda, \mu, 0, ..., 0, -\lambda, -\mu) \in \overline{\mathfrak{w}}_{2,n},$$

in the closure of the Weyl chamber of our Hermitian symmetric space $\mathcal{X}_{2,n}$, is identified with the pair (λ, μ).

Lemma 4.22. *For every $V, W \in \mathcal{X}_{2,n}$ and $g \in \mathrm{SO}_0(2,n)$, it holds:*

1. $d^{\overline{\mathfrak{w}}_{2,n}}(gV, gW) = d^{\overline{\mathfrak{w}}_{2,n}}(V, W)$;
2. $d^{\overline{\mathfrak{w}}_{2,n}}(V, W) = (0, 0) \iff V = W$;
3. $d^{\overline{\mathfrak{w}}_{2,n}}(V, W) = d^{\overline{\mathfrak{w}}_{2,n}}(W, V)$.

Proof. Consider $V, W \in \mathcal{X}_{2,n}$ and $g \in \mathrm{SO}_0(2,n)$ as in the lemma;

1. Let $Y \in \mathrm{T}_V\mathcal{X}_{2,n}$ be such that $\exp_V(Y) = W$ and $h \in \mathrm{SO}_0(2,n)$ such that $h \cdot Y = X \in \overline{\mathfrak{w}}_{2,n}$; then

$$\exp_{gV}(g \cdot Y) = gW,$$

$$\left(hg^{-1}\right) \cdot (g \cdot Y) = X \in \overline{\mathfrak{w}}_{2,n},$$

that is

$$d^{\overline{\mathfrak{w}}_{2,n}}(V, W) = X = d^{\overline{\mathfrak{w}}_{2,n}}(gV, gW).$$

2. It follows immediately by the definition of $d^{\overline{\mathfrak{w}}_{2,n}}$.

3. if γ is the geodesic with $\gamma(0) = V$ and $\gamma(1) = W$, then the *opposite involution in* $\gamma(1/2)$, defined for every $x = \exp_{\gamma(1/2)}(X) \in \mathcal{X}_{2,n}$ by

$$\sigma\left(\exp_{\gamma(1/2)}(X)\right) = \exp_{\gamma(1/2)}(-X),$$

is an isometry contained in $\mathrm{SO}_0(2,n)$ and satisfying

$$\begin{cases} \sigma(V) = W, \\ \sigma(W) = V; \end{cases}$$

now the assertion follows from the $\mathrm{SO}_0(2,n)$-invariance.

□

Remark 4.23. Despite the name, the *vectorial distance* is not $\mathbb{R}$-valued and, therefore, neither a distance. However, it is possible to define a partial ordering on the Weyl chamber in such a way that this vectorial distance satisfies a kind of triangular inequality (see [Par]); this last observation, together with Proposition 4.22, allows us to talk about distance.

The projection to the closure of the Weyl chamber can also be expressed using the Weyl chamber-valued crossratio. The goal for the rest of this section is to find such expression.

First, we need the following lemma:

Lemma 4.24. *Consider $(V, W) \in \mathcal{X}_{2,n}$, $V \neq W$, and suppose that the endpoints, in $\partial\mathcal{X}_{2,n}$, of the geodesic through these two points are not isotropic planes. Then there exist six isotropic lines $a, b, b', c, c', d \in \mathrm{Is}_1(\mathbb{R}^{2,n})$ such that $\mathcal{Y}_{a,d} \perp_V \mathcal{Y}_{b,b'}$, $\mathcal{Y}_{a,d} \perp_W \mathcal{Y}_{c,c'}$ and (a, b, c, d) is a maximal quadruple.*

Notice that these six isotropic lines may not be unique.

Proof. Let $Y \in \mathrm{T}_V\mathcal{X}_{2,n}$ such that $\exp_V(Y) = W$. Proceeding as at the beginning of this section, we know of the existence of a matrix $g \in \mathrm{SO}_0(2,n)$ such that

$$\begin{cases} g \cdot Y = X \in \overline{\mathfrak{w}}_{2,n}, \\ g \cdot V = V_0; \end{cases}$$

As $\mathrm{SO}_0(2,n)$ acts by isometry,

$$g \cdot W = \exp_{V_0}(X) \in \mathcal{Y}_{l_1,l_2}.$$

Let $\ell \in \mathrm{Is}_1(\mathbb{R}^{2,n})$ such that $\mathcal{Y}_{\ell,\bar{\ell}} \perp_{gW} \mathcal{Y}_{l_1,l_2}$.

Claim: One between (l_2, l_3, ℓ, l_1) and $(l_2, l_3, \bar{\ell}, l_1)$ is a maximal quadruple; indeed, as X is a vector contained in the closure of the Weyl chamber, there exist $\lambda > \mu \geq 1$ such that

$$g \cdot W = \exp_{V_0}(X) = \mathrm{Span}\left(\lambda e_1 + \lambda^{-1} e_{n+1}, \mu e_2 + \mu^{-1} e_{n+2}\right)$$

($\lambda \neq \mu$ because, otherwise, the geodesic through V_0 and gW would have one of the endpoints coinciding with $l_1 \oplus l_2$, that is an isotropic plane). Now the claim is a consequence of Lemma 4.19 and Lemma 3.28.

Suppose, without loss of generality, that (l_2, l_3, ℓ, l_1) is maximal. Then $a := g^{-1}l_2, b := g^{-1}l_3, b' := g^{-1}l_4, c := g^{-1}\ell, c' := g^{-1}\bar{\ell}$, and $d := g^{-1}l_1$ give the assertion. $\square$

Finally, we have:

Proposition 4.25. *Let $V, W \in \mathcal{X}_{2,n}$, with $V \neq W$, such that the endpoints of the geodesic through them are not isotropic planes, and consider the $\mathbb{R}$-tubes $\mathcal{Y}_{a,d}$, $\mathcal{Y}_{b,b'}$ and $\mathcal{Y}_{c,c'}$ such that (a,b,c,d) is a maximal quadruple and $\mathcal{Y}_{a,d} \perp_V \mathcal{Y}_{b,b'}, \mathcal{Y}_{a,d} \perp_W \mathcal{Y}_{c,c'}$. If $R(a,b,c,d) = (\lambda, \mu)$, then*

$$d^{\overline{\mathfrak{w}}_{2,n}}(V,W) = (\log \lambda, \log \mu).$$

Proof. Let $g \in \mathrm{SO}_0(2,n)$ such that $g \cdot (a,b,c,d) = (l_2, l_3, \ell, l_1)$, with

$$\ell := \left[\lambda e_1 + \mu e_2 - \lambda^{-1} e_{n+1} + \mu^{-1} e_{n+2}\right]$$

for some $\lambda > \mu \geq 1$ (existence of such a g is given by Lemma 3.28). In this case we have:

- $gb' = l_4$, because $g\mathcal{Y}_{b,b'} = \mathcal{Y}_{l_3,gb'}$ is orthogonal to $\mathcal{Y}_{l_1,l_2}$;
- $gV = V_0$, as gV is the intersection point between $g\mathcal{Y}_{a,d}$ and $g\mathcal{Y}_{b,b'}$;
- $gc' = \bar{\ell}$, because $g\mathcal{Y}_{c,c'} = \mathcal{Y}_{\ell,gc'}$ is orthogonal to $\mathcal{Y}_{l_1,l_2}$;
- $gW = \mathrm{Span}\left(\lambda e_1 + \lambda^{-1} e_{n+1}, \mu e_2 + \mu^{-1} e_{n+2}\right)$, by Lemma 4.18.

So

$$R(a,b,c,d) = R(l_2, l_3, \ell, l_1) = (\lambda, \mu),$$

by the definition of R, and the vector

$$X = \operatorname{diag}(\log\lambda, \log\mu, 0, ..., 0, -\log\lambda, -\log\mu)$$

is such that $\exp_{V_0}(X) = gW$. $\square$

From this result, the analogy between the half-plane model, presented at the beginning of this chapter, and $\mathcal{X}_{2,n}$ is evident.

Remark 4.26. If the endpoints of the geodesic through V, W are isotropic planes, then the isotropic lines a, b, c, d, constructed in the proof of Lemma 4.24, are not contained in a maximal quadruple; even more, the isotropic lines b and c are not transverse. However, if we are in this case, we already know that

$$R(a,b,c,d) = (\lambda, \lambda),$$

for some $\lambda > 1$ (Remark 4.5) and, repeating the proof above, we have

$$d^{\overline{\mathfrak{w}}_{2,n}}(V, W) = (\log\lambda, \log\lambda).$$

Also if $V = W$, the same equalities hold, but with $\lambda = 1$.

Let

$$\mathrm{pr}_{a,b} : \left\{ \ell \in \mathrm{Is}_1(\mathbb{R}^{2,n}) \middle| (a,\ell,b) \text{ max.} \right\} \to \mathcal{Y}_{a,b}$$

be the orthogonal projection onto $\mathcal{Y}_{a,b}$ (that is: for each isotropic line ℓ such that (a,ℓ,b) is maximal, $\mathrm{pr}_{a,b}(\ell)$ is the point of intersection between $\mathcal{Y}_{a,b}$ and the $\mathbb{R}$-tube $\mathcal{Y}_{\ell,\ell'}$ orthogonal to $\mathcal{Y}_{a,b}$); then the previous result can be stated as:

Lemma 4.27. *Let (a,b,c,d) be a maximal quadruple, then*

$$d^{\overline{\mathfrak{w}}_{2,n}}(\mathrm{pr}_{a,d}(b), \mathrm{pr}_{a,d}(c)) = (\log\lambda, \log\mu),$$

where $(\lambda,\mu) = R(a,b,c,d)$.

4.6 Distances

As the Positive Grassmannian is the symmetric space associated to $\mathrm{SO}_0(2,n)$, it makes sense to consider only $\mathrm{SO}_0(2,n)$-invariant distances; because the Weyl chamber is a fundamental domain for the $\mathrm{SO}_0(2,n)$-action on $\mathrm{T}\mathcal{X}_{2,n}$, every such a distance can be obtained composing the projection $d^{\overline{\mathfrak{w}}_{2,n}}$ with a suitable function.

In analogy with [FP16], we define the distances d^R and d^F:

Definition 4.28. *The* Riemannian distance d^R *on* $\mathcal{X}_{2,n}$ *is the composition of the vectorial distance* $d^{\overline{\mathfrak{w}}_{2,n}}$ *with the function*

$$\begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R} \\ (\lambda,\mu) & \mapsto & \sqrt{\lambda^2+\mu^2} \end{array}$$

From symmetric space theory, we know that there exists a unique (up to scaling) $\mathrm{SO}_0(2,n)$-invariant Riemannian metric on $\mathcal{X}_{2,n}$ and that this metric restricts to the Killing form on the tangent space $\mathfrak{p}_{2,n}$. On the other hand, the Killing form on $\mathfrak{p}_{2,n}$ defines exactly the metric d^R.

Definition 4.29. *The* Finsler distance d^F *on* $\mathcal{X}_{2,n}$ *is the composition of the vectorial distance* $d^{\overline{\mathfrak{w}}_{2,n}}$ *with the function*

$$\begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R} \\ (\lambda,\mu) & \mapsto & \lambda \end{array}$$

It is possible to verify that d^F is, indeed, a distance: thanks to Lemma 4.22, the only non obvious property is the triangular inequality, but this is a consequence of *Kostant convexity theorem.*

Remark 4.30. The two distances induce the same topology: indeed, for every $V, W \in \mathcal{X}_{2,n}$,

$$d^R(V,W) \geq d^F(V,W) \geq \frac{1}{\sqrt{2}} d^R(V,W).$$

An immediate consequence of the previous sections is that the distance d^F can easily be expressed using the function T, defined in Example 4.2.

Lemma 4.31. *Let* $(a,b,c,d) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^4$ *be a maximal quadruple, then*

$$d^F(p_{a,d}(b), p_{a,d}(c)) = \frac{1}{2}\log T(a,b,c,d).$$

Between the interesting properties of d^F we have also additivity along maximal 5-tuples;

Lemma 4.32. *Let* $(a,b,c,d,e) \in (\mathrm{Is}_1(\mathbb{R}^{2,n}))^5$ *be a maximal* 5*-tuple, then*

$$d^F(p_{a,e}(b), p_{a,e}(c)) + d^F(p_{a,e}(c), p_{a,e}(d)) = d^F(p_{a,e}(b), p_{a,e}(d)).$$

Proof. Up to $\mathrm{SO}_0(2,n)$-action, we can suppose

$$(a,b,c,d,e) = (l_2, l_3, [X], [Y], l_1),$$

where

$$\begin{aligned} X =& \lambda e_1 + \mu e_2 - \lambda^{-1} e_{n+1} + \mu^{-1} e_{n+2}, \\ Y =& \sum_{i=1}^{n+2} y_i, \end{aligned}$$

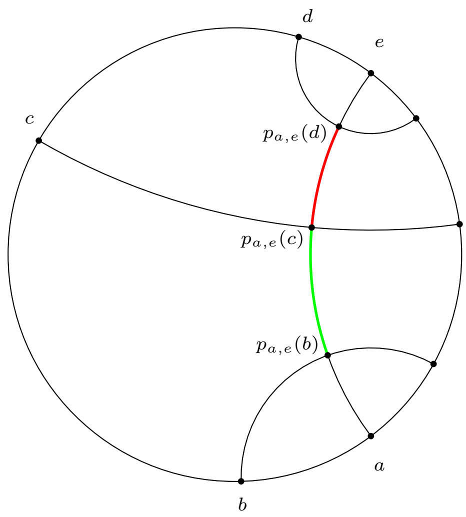

Figure 4.3: Representation of Lemma 4.32.

with $\lambda > \mu \geq 1$. From maximality of $(l_2, l_3, [Y], l_1)$, we get

$$T(l_2, l_3, [Y], l_1) = -y_1/y_{n+1} > 1$$

(Lemma 4.3). Using Lemma 4.31, we have

$$d^F(p_{l_2,l_1}(l_3), p_{l_2,l_1}([X])) = \frac{1}{2}\log T(l_2, l_3, [X], l_1) = \frac{1}{2}\log\left(\lambda^2\right),$$

$$d^F(p_{l_2,l_1}([X]), p_{l_2,l_1}([Y])) = \frac{1}{2}\log T(l_2, [X], [Y], l_1) = \frac{1}{2}\log\left(-\lambda^{-2}\frac{y_1}{y_4}\right),$$

$$d^F(p_{l_2,l_1}(l_3), p_{l_2,l_1}([Y])) = \frac{1}{2}\log T(l_2, l_3, [Y], l_1) = \frac{1}{2}\log\left(-\frac{y_1}{y_4}\right).$$

The assertion is now an easy computation. □

Chapter 5

Anosov maximal representations

Let Σ be an oriented surface with negative Euler characteristic. Let us denote with $\partial\Sigma$ the boundary (possibly empty) of Σ and with $\pi_1(\Sigma) = \Gamma$ the fundamental group. If G is a connected, adjoint, semisimple Lie group, then a *representation* of Γ in the group G is an homomorphism

$$\rho : \Gamma \to G.$$

If the group G is the identity component $\mathrm{Isom}_0(\mathcal{X})$ of the isometry group of a symmetric space $\mathcal{X}$, then the representation ρ induces an action of Γ on $\mathcal{X}$, defined by

$$\gamma \cdot x := \rho(\gamma) \cdot x,$$

for every $\gamma \in \Gamma$ and $x \in \mathcal{X}$. In this way, the representation ρ can be studied through the induced action of Γ on $\mathcal{X}$.

Of course, we are interested in the case in which the symmetric space $\mathcal{X}$ is the positive Grassmannian $\mathcal{X}_{2,n}$ and the Lie group G is $\mathrm{SO}_0(2, n)$.

In this chapter we study a specific kind of representations, the *Anosov maximal representations*, particularly suitable for our purpose: if a representation ρ is Anosov, then the image $\rho(\gamma)$ of every peripheral element $\gamma \in \Gamma$ is “nice” enough to fix exactly one $\mathbb{R}$-tube $\mathcal{Y}_\gamma$ in $\mathcal{X}^{2,n}$; with the further hypothesis that ρ is maximal, we can also say something about the mutual position of two different $\mathbb{R}$-tubes fixed by two peripheral elements in Γ; in particular, with these two assumptions, we will prove that for every two peripheral elements $\gamma, \delta \in \Gamma$ there exists a unique $\mathbb{R}$-tube, called *orthoube*, orthogonal to both the $\mathbb{R}$-tubes $\mathcal{Y}_\gamma, \mathcal{Y}_\delta$ fixed by $\rho(\gamma)$ and $\rho(\delta)$.

5.1 Maximal representations

In this section we present briefly the notion of *maximal representation* in the general context of Hermitian symmetric spaces. We refer to [BIW10a, Section 5] for further details.

Consider an Hermitian symmetric space $\mathcal{X}$, with $G = \mathrm{Isom}_0(\mathcal{X})$ the identity component of its isometry group and $\omega_{\mathcal{X}}$ the associated Kähler form. The *Toledo invariant* is the function

$$\begin{array}{ccc} \mathrm{Hom}(\gamma, G) & \to & \mathbb{R} \\ \rho & \mapsto & \frac{1}{2\pi}\int_S f^*(\omega_X) \end{array}$$

where f is a ρ-equivariant map from the universal cover $\tilde{\Sigma}$ of the surface Σ to the space $\mathcal{X}$. A well-known property about the Toledo invariant is the inequality

$$T(\rho) \leq \mathrm{rank}_{\mathcal{X}} |\chi(\Sigma)|,$$

where $\mathrm{rank}_{\mathcal{X}}$ is the rank of the symmetric space $\mathcal{X}$.

A representation is said to be maximal if it maximizes the Toledo invariant.

Definition 5.1. *A representation* $\rho : \Gamma \to G$ *is called* maximal *if*

$$T(\rho) = \mathrm{rank}_{\mathcal{X}} |\chi(\Sigma)|.$$

Let us fix a finite area hyperbolization $h : \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ of Σ; this map h induces an action of Γ on the hyperbolic plane $\mathbb{H}^2$; as this action is by isometries, it can be extended to the boundary $\mathbb{S}^1 = \partial\mathbb{H}^2$.

Let us denote with $\check{S} \subset \partial\mathcal{X}$ the Shilov boundary of $\mathcal{X}$ (in the case $\mathcal{X} = \mathcal{X}_{2,n}$, the Shilov boundary coincides with the set of isotropic lines). A central result in the study of maximal representations is the following:

Theorem 5.2 [BIW10b, Theorem 8]. *A representation* $\rho : \Gamma \to G$ *is maximal if and only if there exists a left continuous map* $\varphi : \mathbb{S}^1 \to \check{S}$ *such that*

- φ *is strictly* $\rho \circ h^{-1}$*-equivariant;*
- φ *is a* maximal framing*, that is a function mapping positively oriented triples in* $\mathbb{S}^1$ *to maximal triples in* $\check{S}$*.*

In particular, a representation

$$\rho : \Gamma \to \mathrm{SO}_0(2, n)$$

is maximal if and only if there exists a right continuous ρ-equivariant map $\varphi : \mathbb{S}^1 \to \mathrm{Is}_1(\mathbb{R}^{2,n})$ sending positively oriented triples in the circle to maximal triples of isotropic lines.

Another important result about maximal representations is the following:

Theorem 5.3 [BIW10b, Theorem 5]. *Maximal representations are injective and have discrete image.*

In the rest of this work we use the following notation:

- Σ oriented surface with negative Euler characteristic and nonempty boundary $\partial\Sigma$,
- $h : \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ finite area hyperbolization of Σ, including an action of Γ on $\mathbb{S}^1 = \partial\mathbb{H}^2$,
- $\rho : \Gamma \to \mathrm{SO}_0(2, n)$ maximal representation,
- φ a right continuous ρ-equivariant maximal framing of ρ.

We also fix an orientation on the boundary components of Σ in such a way that the surface lies to the right of each of them. Notice that this choice determines also an orientation on the boundary of the universal cover $\tilde{\Sigma} \subset \mathbb{H}^2$.

Remark 5.4. Recall that an element $\gamma \in \Gamma$ is called *peripheral* if it is conjugate in $\pi_1(\Sigma)$ to an element representing a boundary component of Σ. In this case its representation $h(\gamma) \in \mathrm{PSL}(2,\mathbb{R})$ is conjugate to one between the two matrices

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

with $\lambda \neq 0$.

In the rest of this text, we will always choose peripheral elements γ agreeing with the choice of the orientation on the boundary.

5.2 Shilov hyperbolic isometries

An element $g \in \mathrm{PSL}(2,\mathbb{R})$ is called hyperbolic if it fixes exactly two points g^+, g^- in the boundary $\partial\mathbb{H}^2$. It is well known that every element of this type has two distinct real eigenvalues and, so, it is conjugate to a matrix of the type

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

for $\lambda \neq 0$. From this observation it is easy to deduce that an hyperbolic element $g \in \mathrm{PSL}(2,\mathbb{R})$ acts as a transvection along the geodesic whose endpoints are g^+, g^-. Moreover, g (respectively g^{-1}) acts contracting a neighborhood of g^+ (of g^-). An element $\gamma \in \Gamma$ whose representation $h(\gamma) \in \mathrm{PSL}(2,\mathbb{R})$ is hyperbolic is also called *hyperbolic*.

Analogously, an element g in $\mathrm{SO}_0(2,n)$ fixing two isotropic lines is called Shilov hyperbolic.

Definition 5.5 ([Str15]). *An isometry $g \in \mathrm{SO}_0(2,n)$ is called* Shilov hyperbolic *if its action on $\mathrm{Is}_1(\mathbb{R}^{2,n})$ fixes two transverse isotropic lines, denoted by $g^+, g^- \in \mathrm{Is}_1(\mathbb{R}^{2,n})$, in such a way that there exists an open and dense neighbourhood U of g^+ (respectively of g^-) such that powers g^n (powers g^{-n}) converge uniformly to the constant map g^+ (g^-).*

If $g \in \mathrm{SO}_0(2,n)$ is Shilov hyperbolic, then g is contained in both the parabolic subgroups P_{g^+}, P_{g^-} and, so, in the stabilizer of the $\mathbb{R}$-tube $\mathcal{Y}_{g^+,g^-}$.

In general, if $\eta \in \mathbb{C}$ is an eigenvalue of $g \in \mathrm{SO}_0(2,n)$ relative to an eigenvector $v \in \mathbb{R}^{2,n}$, then $|\eta| = 1$ or v is an isotropic vector.

For Shilov hyperbolic elements it is also possible conclude something more:

Lemma 5.6. *If an element $g \in \mathrm{SO}_0(2,n)$ is Shilov hyperbolic, then it has two real eigenvalues λ, λ^{-1}, for $|\lambda| > 1$, with g^+ and g^- as associated eigenlines.*

Proof. If $g \in \mathrm{SO}_0(2,n)$ is Shilov hyperbolic, then g fixes two isotropic lines $g^+ = [v]$, $g^- = [w]$; in other words, $v, w \in \mathbb{R}^{2,n}$ are two eigenvectors of g with

$$h_{2,n}(v,v) = h_{2,n}(w,w) = 0;$$

as $\mathrm{SO}_0(2,n)$ is transitive on the set of pairs of transverse isotropic lines, there exists an element $h \in \mathrm{SO}_0(2,n)$ such that

$$h \cdot ([v],[w]) = (l_1, l_2);$$

then hgh^{-1} fixes l_1, l_2 and so it can be written as

$$hgh^{-1} = \left(\begin{array}{cc|c|cc} \lambda & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & c \\ \hline 0 & d & E & 0 & f \\ \hline 0 & 0 & 0 & \lambda^{-1} & 0 \\ 0 & g & h & 0 & j \end{array}\right),$$

with $\lambda \neq 0$ (see Lemma 3.8). As hgh^{-1} acts as a contraction on a neighborhood of l_1, we deduce $|\lambda| > 1$. In particular, λ, λ^{-1} are two real eigenvalues of hgh^{-1} with eigenlines l_1, respectively l_2. We deduce that λ, λ^{-1} are two real eigenvalues of g with eigenlines $h^{-1}l_1 = g^+$ and $h^{-1}l_2 = g^-$. □

If an element $\gamma \in \Gamma$ is hyperbolic and γ^+, γ^- are the two fixed elements in the boundary $\partial\mathbb{H}^2$, then, since φ is ρ-equivariant, also $\rho(\gamma)$ fixes two transverse isotropic lines $\varphi(\gamma^+), \varphi(\gamma^-) \in \mathrm{Is}_1(\mathbb{R}^{2,n})$, one attractive and one repulsive. In other words, $\rho(\gamma)$ is Shilov hyperbolic.

However, with the only hypothesis of maximality on the representation ρ, it is not possible to conclude the same on peripheral elements that are not hyperbolic. For this reason we need some more hypotheses about our representation ρ.

Definition 5.7. *A maximal representation $\rho : \Gamma \to \mathrm{SO}_0(2,n)$ is called* Anosov maximal representation *if, for every peripheral element $\gamma \in \Gamma$, the isometry $\rho(\gamma)$ is Shilov hyperbolic.*

The expert reader may have noticed that this is not the classical definition: indeed, *Anosov representation* is a concept much more general that extends also to other Lie groups G. More details can be found in [GW12].

Some more notation: From now on we assume that our representation $\rho : \Gamma \to \mathrm{SO}_0(2,n)$ is an Anosov maximal representation. In this way, for every $\gamma \in \Gamma$ peripheral, $\rho(\gamma)$ is Shilov hyperbolic and, so, it stabilizes exactly one $\mathbb{R}$-tube; we denote with $\mathcal{Y}_\gamma$ this $\mathbb{R}$-tube and with Λ_γ^+, $\Lambda_\gamma^- \in \mathrm{Is}_1(\mathbb{R}^{2,n})$ the two endpoints of $\mathcal{Y}_\gamma$ in such a way that Λ_γ^+ is the attractive point for the $\rho(\gamma)$-action while Λ_γ^- is the repulsive one. We also denote with pr_γ the orthogonal projection on $\mathcal{Y}_\gamma$:

$$\mathrm{pr}_\gamma : \left\{ \ell \in \mathrm{Is}_1(\mathbb{R}^{2,n}) \,\middle|\, \left(\Lambda_\gamma^-, \ell, \Lambda_\gamma^+\right) \text{ max.} \right\} \to \mathcal{Y}_\gamma$$

sending each isotropic line ℓ (such that $\left(\Lambda_\gamma^-, \ell, \Lambda_\gamma^+\right)$ is maximal) to the intersection between the $\mathbb{R}$-tube $\mathcal{Y}_\gamma$ and the unique $\mathbb{R}$-tube orthogonal to $\mathcal{Y}_\gamma$ and with one of the endpoints corresponding to ℓ.

5.3 Orthotubes

Let $\gamma, \delta \in \Gamma$ be two peripheral elements; as ρ is an Anosov representation, the isometries $\rho(\gamma)$, $\rho(\delta)$ are Shilov hyperbolic, both fixing two isotropic lines $\Lambda_\gamma^-, \Lambda_\gamma^+$ and, respectively, $\Lambda_\delta^+, \Lambda_\delta^-$; because the representation ρ is maximal, the quadruple $\left(\Lambda_\gamma^-, \Lambda_\delta^+, \Lambda_\delta^-, \Lambda_\gamma^+\right)$ is maximal and so it makes sense to talk about $\mathbb{R}$-tubes orthogonal to both $\mathcal{Y}_\gamma$, $\mathcal{Y}_\delta$.

Definition 5.8. *Let $\gamma, \delta \in \Gamma$ two peripheral elements. The $\mathbb{R}$-tube orthogonal to both $\mathcal{Y}_\gamma, \mathcal{Y}_\delta$ is called* Orthotube *relative to (γ, δ) and it is denoted with $\mathcal{O}_\Sigma^{\gamma,\delta}$.*

For every pair (γ, δ) of peripheral elements $\gamma, \delta \in \pi_1(\Sigma)$ there exists a unique orthotube; in order to prove this claim, we need the following lemma:

Lemma 5.9. *Each orbit of maximal quadruples in $\mathrm{Is}_1(\mathbb{R}^{2,n})$ contains a quadruple of the type $(l_3, a, \bar{a}, l_4)$, where $a \in \mathrm{Is}_1(\mathbb{R}^{2,n})$ is an isotropic line contained in $l_1 \oplus l_2 \oplus l_3 \oplus l_4$.*

Proof. From Chapter 3 we know that every orbit of maximal quadruples can be represented by a quadruple of the type $(l_1, l_2, l_3, [X])$, for some

$$X = \lambda e_1 + \mu e_2 - \lambda^{-1} e_{n+1} + \mu^{-1} e_{n+2}$$

and $\lambda > \mu \geq 1$. Then the matrix

$$g = \left(\begin{array}{cc|c|cc} -\lambda^{-1/2}/2 & -\mu^{-1/2}/2 & 0 & \lambda^{1/2}/2 & -\mu^{1/2}/2 \\ \lambda^{-1/2}/2 & -\mu^{-1/2}/2 & 0 & \lambda^{1/2}/2 & \mu^{1/2}/2 \\ \hline 0 & 0 & \mathrm{Id} & 0 & 0 \\ \hline \lambda^{-1/2}/2 & -\mu^{-1/2}/2 & 0 & -\lambda^{1/2}/2 & -\mu^{1/2}/2 \\ \lambda^{-1/2}/2 & \mu^{-1/2}/2 & 0 & \lambda^{1/2}/2 & -\mu^{1/2}/2 \end{array}\right) \in H_{2,n}$$

is such that

$$g \cdot (l_1, l_2, l_3, [X]) = (l_4, l_3, a, \bar{a})$$

for

$$a = g l_3 \subset l_1 \oplus l_2 \oplus l_3 \oplus l_4.$$

□

Proposition 5.10. *Given two peripheral elements $\gamma, \delta \in \Gamma$, there exists a unique orthotube $\mathcal{Y}$ corresponding to the pair (γ, δ).*

Proof. Existence: Let γ, $\delta \in \Gamma$ be two peripheral elements; as ρ is Anosov maximal, the four fixed points of the Shilov hyperbolic isometries $\rho(\gamma)$, $\rho(\delta)$ form a maximal quadruple $\left(\Lambda_\gamma^-, \Lambda_\delta^+, \Lambda_\delta^-, \Lambda_\gamma^+\right)$; from the previous lemma, there exists a $g \in \mathrm{SO}_0(2, n)$ such that

$$g \cdot \left(\Lambda_\gamma^-, \Lambda_\delta^+, \Lambda_\delta^-, \Lambda_\gamma^+\right) = (l_3, a, \bar{a}, l_4),$$

for some isotropic line a contained in $l_1 \oplus l_2 \oplus l_3 \oplus l_4$. As $\mathcal{Y}_{l_1,l_2}$ is orthogonal to both $\mathcal{Y}_{l_4,l_3}$ and $\mathcal{Y}_{a,\bar{a}}$, we deduce $g^{-1}\mathcal{Y}_{l_1,l_2}$ is orthogonal to $\mathcal{Y}_{\Lambda_\gamma^-,\Lambda_\gamma^+} = \mathcal{Y}_\gamma$ and to $\mathcal{Y}_{\Lambda_\delta^+,\Lambda_\delta^-} = \mathcal{Y}_\delta$.

Uniqueness: Suppose $\mathcal{Y}$ is an orthotube corresponding to the pair of peripheral elements (γ, δ); up to $\mathrm{SO}_0(2, n)$-action, we can suppose $\mathcal{Y}_\gamma = \mathcal{Y}_{l_1,l_2}$ and $\mathcal{Y}_\delta = \mathcal{Y}_{l_3,\ell}$, where

$$\ell = [X] = [\lambda e_1 + \mu e_2 - \lambda^{-1} e_{n+1} + \mu^{-1} e_{n+2}]$$

for some $\lambda > \mu \geq 1$ (this is a consequence of Lemma 3.28 and the fact that $\left(\Lambda_\gamma^-, \Lambda_\delta^+, \Lambda_\delta^-, \Lambda_\gamma^+\right)$ is maximal). From $\mathcal{Y} \perp \mathcal{Y}_{l_1,l_2}$, we have $\mathcal{Y} = \mathcal{Y}_{a,\bar{a}}$ for some isotropic line a such that (l_2, a, l_1) is maximal.

Suppose, by contradiction, that a is not contained in $l_1 \oplus l_2 \oplus l_3 \oplus l_4$, then we can assume, up to $\mathrm{SO}_0(2, n)$-action, that

$$[a] = [Y] = [a_1 e_1 + a_2 e_2 + a_3 e_3 + a_{n+1} e_{n+1} + a_{n+2} e_{n+2}]$$

for $a_1, a_3, a_{n+1} \neq 0$; moreover, from the fact that $\mathcal{Y}_{a,\bar{a}} \perp Y_{l_3,\ell}$, it holds $\bar{a} \subset l_3 \oplus \ell \oplus a$ (see Lemma 4.16); now some computations yield

$$Y = (\lambda - 1)e_1 + a_2 e_2 + a_3 e_3 + (1 - \lambda^{-1})e_{n+1} + a_{n+2} e_{n+2}$$

(up to scaling Y). Using again Lemma 4.16, we have $R(l_3, a, \ell, \bar{a}) = (2, 1)$, or, equivalently,

$$\begin{cases} T(l_3, a, \ell, \bar{a}) - 4 = 0, \\ T(a, \ell, \bar{a}, l_3) - 4 = 0, \end{cases}$$

that implies easily $a_{n+2} = -a_2$. The contradiction comes from the fact that the set

$$\{h_{2,n}(X, Y), h_{2,n}(X, e_1 + e_2 - e_{n+1} + e_{n+2}), h_{2,n}(Y, e_1 + e_2 - e_{n+1} + e_{n+2})\}$$

contains two negative values and a positive one (see Remark 3.12).

We deduce $a \subset \mathrm{Span}(e_1, e_2, e_{n+1}, e_{n+2})$ and now Lemma 4.16 gives a unique possible solution.

$\square$

For every peripheral element $\gamma \in \Gamma$, let

$$\mathcal{O}_\Sigma^{\mathcal{X}_{2,n}}(\gamma) := \left\{ \mathcal{O}_\Sigma^{\gamma,\sigma} \middle| \sigma \in \Gamma \text{ peripheral} \right\} / \langle \gamma \rangle$$

be the set of classes of orthotubes orthogonal to $\mathcal{Y}_\gamma$, where two orthotubes $\mathcal{O}_\Sigma^{\gamma,\sigma}$ and $\mathcal{O}_\Sigma^{\gamma,\tau}$ are contained in the same class if and only if σ is obtained by conjugating τ by an element in $\langle \gamma \rangle$.

Remark 5.11. The definition of the set $\mathcal{O}_\Sigma^{\mathcal{X}_{2,n}}(\gamma)$ is in analogy with the definition of the set $\mathcal{O}_\Sigma^{\mathbb{H}^2}(c)$ of orthogeodesics inside the surface Σ and orthogonal to a given boundary component c; see [FP16, Section 4.1] for more details.

We define the Riemannian (respectively Finsler, vectorial) length of an orthotube $\mathcal{O}_\Sigma^{\gamma,\delta}$ as the Riemannian (Finsler, vectorial) distance between the two intersection points of $\mathcal{O}_\Sigma^{\gamma,\delta}$ with the $\mathbb{R}$-tubes $\mathcal{Y}_\gamma, \mathcal{Y}_\delta$; We denote such lenghts with $\ell^R\left(\mathcal{O}_\Sigma^{\gamma,\delta}\right)$, $\ell^F\left(\mathcal{O}_\Sigma^{\gamma,\delta}\right)$ and $\ell^{\overline{\mathfrak{w}}_{2,n}}\left(\mathcal{O}_\Sigma^{\gamma,\delta}\right)$ respectively. In formula:

$$\begin{aligned}
\ell^R\left(\mathcal{O}_\Sigma^{\gamma,\delta}\right) &= \quad d^R\left(\mathcal{Y}_\gamma \cap \mathcal{O}_\Sigma^{\gamma,\delta}, \mathcal{Y}_\delta \cap \mathcal{O}_\Sigma^{\gamma,\delta}\right), \\
\ell^F\left(\mathcal{O}_\Sigma^{\gamma,\delta}\right) &= \quad d^F\left(\mathcal{Y}_\gamma \cap \mathcal{O}_\Sigma^{\gamma,\delta}, \mathcal{Y}_\delta \cap \mathcal{O}_\Sigma^{\gamma,\delta}\right), \\
\ell^{\overline{\mathfrak{w}}_{2,n}}\left(\mathcal{O}_\Sigma^{\gamma,\delta}\right) &= d^{\overline{\mathfrak{w}}_{2,n}}\left(\mathcal{Y}_\gamma \cap \mathcal{O}_\Sigma^{\gamma,\delta}, \mathcal{Y}_\delta \cap \mathcal{O}_\Sigma^{\gamma,\delta}\right).
\end{aligned}$$

Remark 5.12. The Riemannian length of an orthotube $\mathcal{O}_\Sigma^{\gamma,\delta}$ is the Riemannian distance between the two $\mathbb{R}$-tubes $\mathcal{Y}_\gamma$ and $\mathcal{Y}_\delta$:

$$\ell^R\left(\mathcal{O}_\Sigma^{\gamma,\delta}\right) = d^R(\mathcal{Y}_\gamma, \mathcal{Y}_\delta) := \inf\left\{ d^R(x,y) \middle| x \in \mathcal{Y}_\gamma, y \in \mathcal{Y}_\delta \right\}.$$

Analogous equalities hold for $\ell^{\overline{\mathfrak{w}}_{2,n}}\left(\mathcal{O}_\Sigma^{\gamma,\delta}\right)$ and $\ell^F\left(\mathcal{O}_\Sigma^{\gamma,\delta}\right)$.

Remark 5.13. Consider a peripheral element $\gamma \in \Gamma$ and a class of orthotubes $\alpha \in \mathcal{O}_\Sigma(\gamma)$ associated to γ. Using the fact that $\rho(\Gamma)$ acts by isometries on $\mathcal{X}_{2,n}$ and that all the representatives of the class α are orthotubes associated to conjugate peripheral elements, it is not hard to prove that all the orthotubes representing α have the same Riemannian, Finsler and vectorial length. In particular, $\ell^{\overline{\mathfrak{w}}_{2,n}}(\alpha), \ell^R(\alpha)$ and $\ell^F(\alpha)$ are well defined.

5.4 Translational length

Distances defined in Section 4.5 and Section 4.6 can be used to define translational lengths of elements in $\mathrm{SO}_0(2,n)$.

As d^R and d^F are distances in a classical sense, it is clear how the associated translational lengths ℓ^R and ℓ^F are defined: for every $g \in \mathrm{SO}_0(2,n)$

$$\begin{aligned}
\ell^R(g) &= \inf\left\{ d^R(x,gx) \middle| x \in \mathcal{X}_{2,n} \right\}, \\
\ell^F(g) &= \inf\left\{ d^F(x,gx) \middle| x \in \mathcal{X}_{2,n} \right\}.
\end{aligned}$$

More attention is required on the translational length associated to the vectorial distance: the partial order defined on the Weyl chamber in [Par] is, indeed, only partial; so, to extend the classical definition of translational length to the vectorial distance, one should, before, prove that the infimum is well defined on the set

$$\left\{ d^{\overline{\mathfrak{w}}_{2,n}}(x, gx) \middle| x \in \mathcal{X}_{2,n} \right\},$$

for every $g \in \mathrm{SO}_0(2,n)$. For this reason we prefer to keep the definition of *translational vector*, already existing in [Par12].

Theorem 5.14 [Par12, Prop. 4.1]. *Let $\mathcal{X}$ be a symmetric space and $g \in \mathrm{Isom}(\mathcal{X})$. The closure of the set*

$$\left\{ d^{\overline{\mathfrak{w}}}(x, gx) \middle| x \in \mathcal{X}_{2,n} \right\}$$

contains a unique segment $\ell^{\overline{\mathfrak{w}}}(g) \in \overline{\mathfrak{w}}$ of minimal Riemannian length. The vector $\ell^{\overline{\mathfrak{w}}}(g)$ is called translational vector *of g.*

In particular, if $g \in \mathrm{Isom}(\mathcal{X})$ and $x \in \mathrm{MIN}^R(g)$, where

$$\mathrm{MIN}^R(g) = \left\{ x \in \mathcal{X}_{2,n} \,\middle|\, d^R(x, gx) = \ell^R(g) \right\}$$

is the *minset* of g according to the Riemmanian metric on the symmetric space $\mathcal{X}$, then the translational vector of g is $\ell^{\overline{\mathfrak{w}}}(g) = d^{\overline{\mathfrak{w}}}(x, gx)$.

Applying the theorem above to $\mathcal{X}_{2,n}$, we know that the Riemannian translantional length of an element g can easily be expressed through the translational vector:

$$\ell^R(g) = \left\| \ell^{\overline{\mathfrak{w}}_{2,n}}(g) \right\|_R = \sqrt{\left(\ell^{\overline{\mathfrak{w}}_{2,n}}(g)_1\right)^2 + \left(\ell^{\overline{\mathfrak{w}}_{2,n}}(g)_2\right)^2}.$$

Also the Finsler translational length can be expressed using the translational vector:

$$\ell^F(g) = \ell^{\overline{\mathfrak{w}}_{2,n}}(g)_1; \tag{5.1}$$

in words, the Finsler translational length is the projection on the first component of the translational vector.

Lemma 5.15. *Let $g \in \mathrm{SO}_0(2,n)$ be a Shilov hyperbolic element and $\mathcal{Y}_g$ the $\mathbb{R}$-tube preserved by g, then*

$$\ell^R(g) = \inf \left\{ d^R(x, gx) \middle| x \in \mathcal{Y}_g \right\},$$

$$\ell^F(g) = \inf \left\{ d^F(x, gx) \middle| x \in \mathcal{Y}_g \right\}.$$

Moreover, the translational vector $\ell^{\overline{\mathfrak{w}}_{2,n}}(g)$ is contained in the closure of

$$\left\{ d^{\overline{\mathfrak{w}}_{2,n}}(x, gx) \middle| x \in \mathcal{Y}_g \right\}.$$

Proof. Let $x \in \mathcal{X}_{2,n} \setminus \mathcal{Y}_g$, pr_g the orthogonal projection of $\mathcal{X}_{2,n}$ to $\mathcal{Y}_g$ (well defined as $\mathcal{Y}_g$ is a totally geodesic submanifold in a nonpositively curved manifold) and α the unique geodesic containing both x and $\mathrm{pr}_g(x)$. The geodesic α is orthogonal to $\mathcal{Y}_g$ in $\mathrm{pr}_g(x)$, that means that the tangent vector of α in $\mathrm{pr}_g(x)$ is orthogonal to the tangent space $\mathrm{T}_{\mathrm{pr}_g(x)}\mathcal{Y}_g$ and, in particular,

$$\alpha \cap \mathcal{Y}_g = \{\mathrm{pr}_g(x)\}.$$

As g is an isometry preserving $\mathcal{Y}_g$, $g \cdot \alpha$ is still a geodesic orthogonal to $\mathcal{Y}_g$ and so

$$g \cdot \mathrm{pr}_g(x) = \mathrm{pr}_g(g \cdot x).$$

Moreover, if we define the distance between the two geodesics α and $g \cdot \alpha$ as

$$d^R(\alpha, g \cdot \alpha) = \inf \left\{ d^R(u,v) \middle| \, u \in \alpha, v \in g \cdot \alpha \right\},$$

then it is well known that such a distance is given by the length of every geodesic segment (if it exists) orthogonal to both α and $g \cdot \alpha$ and with endpoints on them; as the geodesic segment with endpoints $\mathrm{pr}_g(gx)$ and $\mathrm{pr}_g(x)$ is contained in $\mathcal{Y}_g$, it is orthogonal to α and $g \cdot \alpha$ and so

$$d^R(\alpha, g \cdot \alpha) = d^R\left(\mathrm{pr}_g(x), \mathrm{pr}_g(g \cdot x)\right).$$

We deduce

$$d^R(x, gx) \geq d^R(\alpha, g \cdot \alpha) = d^R\left(\mathrm{pr}_g(x), g \cdot \mathrm{pr}_g(x)\right);$$

by definition of $\ell^R(g)$, we conclude

$$\ell^R(g) = \inf \left\{ d^R(x, gx) \middle| \, x \in \mathcal{X}_{2,n} \right\} = \inf \left\{ d^R(x, gx) \middle| \, x \in \mathcal{Y}_g \right\}.$$

The other two assertions follow immediately from this one: indeed, by the definition of translational vector, we have that $\ell^{\overline{\mathfrak{w}}_{2,n}}(g)$ is contained in the closure of

$$\left\{ d^{\overline{\mathfrak{w}}_{2,n}}(x, gx) \middle| \, x \in \mathcal{Y}_g \right\}$$

and, by Equality (5.1), also

$$\ell^F(g) = \inf \left\{ d^F(x, gx) \middle| \, x \in \mathcal{Y}_g \right\}.$$

$\square$

Chapter 6

Inequalities

We are finally ready to prove our inequality: the goal is to bound from below the translational length of $\rho(\gamma)$ ($\gamma \in \Gamma$ peripheral element), using the length of the orthotubes contained in $\mathcal{O}_{\Sigma}^{\mathcal{X}_{2,n}}(\gamma)$. A good plane to obtain this inequality is given by the proof of the Basmajian identity (presented briefly in the introduction), where we need to substitute the environment of the hyperbolic plane with the one of the Positive Grasmannian: We divide this proof in 3 parts:

Section 6.1: given two peripheral elements $\gamma, \delta \in \Gamma$, we prove an inequality between the Finsler length of the orthotube $\mathcal{O}_{\Sigma}^{\gamma,\delta}$ orthogonal to both $\mathcal{Y}_\gamma, \mathcal{Y}_\delta$ and the Finsler distance $d^F\left(\mathrm{pr}_\gamma \Lambda_\delta^+, \mathrm{pr}_\gamma \Lambda_\delta^-\right)$ between the projections on $\mathcal{Y}_\gamma$ of the two endpoints of $\mathcal{Y}_\delta$.

Section 6.2: given a peripheral element $\gamma \in \Gamma$, we find particular points inside $\mathcal{Y}_\gamma$ on which $\rho(\gamma)$ realizes its Finsler translational length.

Section 6.3: given a peripheral element γ and one of the points x found in Section 6.2, we prove that the Finsler distance between x and γx is larger than the sum of the distances $d^F\left(\mathrm{pr}_\gamma \Lambda_\delta^+, \mathrm{pr}_\gamma \Lambda_\delta^-\right)$ over all possible classes

$$[\delta] \in \{\sigma \in \Gamma |\, \sigma \text{ peripheral}\} / \langle\gamma\rangle.$$

Our Basmajian-type inequality for $\mathrm{SO}_0(2,n)$ is an easy consequence of Section 6.1 and Section 6.3.

6.1 An upper bound for the length of an orthotube

The goal for this section is to find an inequality between $\ell^F\left(\mathcal{O}_{\Sigma}^{\gamma,\delta}\right)$ and $d^F\left(\mathrm{pr}_\gamma \Lambda_\delta^+, \mathrm{pr}_\gamma \Lambda_\delta^-\right)$, for every peripheral elements $\gamma, \delta \in \Gamma$.

The Weyl chamber-valued crossratio plays a fundamental role in this step.

Lemma 6.1. *Let $x, y \in \mathrm{Is}_1(\mathbb{R}^{2,n})$ be such that (l_2, x, y, l_1) is maximal; if*

$$R(l_2, x, y, l_1) = (\lambda, \mu),$$

then

$$R(\bar{y}, \bar{x}, x, y) = \left(\frac{(\lambda/\mu + 1)(\lambda\mu + 1)}{(\lambda/\mu - 1)(\lambda\mu - 1)}, \frac{(\lambda/\mu + 1)(\lambda\mu - 1)}{(\lambda/\mu - 1)(\lambda\mu + 1)}\right).$$

Proof. Up to $\mathrm{SO}_0(2,n)$-action, we can suppose $x = l_3$ and $y = [Y]$, where

$$Y = \lambda e_1 + \mu e_2 - \lambda^{-1} e_{n+1} + \mu^{-1} e_{n+2}$$

for some $\lambda > \mu \geq 1$. By definition of R, we already know

$$R(l_2, x, y, l_1) = (\lambda, \mu).$$

The assertion is now a consequence of Lemma 4.8. $\square$

As the crossratio can be used to compute both the distance between two points and the length of an orthotube, we have that the previous lemma, togheter with the following one, acts as a bridge between the length of the orthotube $\mathcal{O}_\Sigma^{\gamma,\delta}$ and the distance between the points $\mathrm{pr}_\gamma \Lambda_\delta^+$ and $\mathrm{pr}_\gamma \Lambda_\delta^-$.

Lemma 6.2. *Given two peripheral elements $\gamma, \delta \in \Gamma$, it holds*

$$2 \log \coth \frac{\ell^F\left(\mathcal{O}_\Sigma^{\gamma,\delta}\right)}{2} \leq d^F\left(\mathrm{pr}_\gamma \Lambda_\delta^+, \mathrm{pr}_\gamma \Lambda_\delta^-\right).$$

Proof. As ρ is Anosov maximal, the quadruple $\left(\Lambda_\gamma^-, \Lambda_\delta^+, \Lambda_\delta^-, \Lambda_\gamma^+\right)$ is maximal. Let $a, b \in \mathrm{Is}_1(\mathbb{R}^{2,n})$ be the endpoints of the orthotube $\mathcal{O}_\Sigma^{\gamma,\delta}$ in such a way that the triples $\left(b, \Lambda_\gamma^+, a\right)$ and $\left(b, \Lambda_\delta^-, a\right)$ are maximal, so that $\left(b, \Lambda_\gamma^+, \Lambda_\delta^-, a\right)$ is also maximal; up to $\mathrm{SO}_0(2,n)$-action, we can suppose

$$\left(b, \Lambda_\gamma^+, \Lambda_\delta^-, a\right) = (l_2, l_3, [X], l_1),$$

for some

$$[X] = [\lambda e_1 + \mu e_2 - \lambda^{-1} e_{n+1} + \mu^{-1} e_{n+2}];$$

because $\mathcal{O}_\Sigma^{\gamma,\delta}$ is the orthotube relative to the pair (γ, δ), we deduce $\Lambda_\gamma^- = l_4$ and $\Lambda_\delta^+ = \overline{[X]}$, so that $\mathcal{Y}_\gamma = \mathcal{Y}_{l_3,l_4}, \mathcal{Y}_\delta = \mathcal{Y}_{[X],\overline{[X]}}$ and $\mathcal{O}_\Sigma^{\gamma,\delta} = \mathcal{Y}_{l_1,l_2}$. From Lemma 4.27 and the definition of the function R we have

$$\begin{aligned}\ell^{\overline{\mathfrak{w}}_{2,n}}\left(\mathcal{O}_\Sigma^{\gamma,\delta}\right) &= d^{\overline{\mathfrak{w}}_{2,n}}\left(\mathrm{pr}_{l_1,l_2}(l_3), \mathrm{pr}_{l_1,l_2}([X])\right) = \\ &= \log R(l_2, l_3, [X], l_1) = (\log \lambda, \log \mu),\end{aligned}$$

where pr_{l_1,l_2} is the orthogonal projection on the $\mathbb{R}$-tube $\mathcal{Y}_{l_1,l_2}$. On the other hand,

$$\begin{aligned}d^{\overline{\mathfrak{w}}_{2,n}}\left(\mathrm{pr}_\gamma \Lambda_\delta^+, \mathrm{pr}_\gamma \Lambda_\delta^-\right) &= \log R(l_3, [X], \overline{[X]}, l_4) = \\ &= \left(\log \frac{(\lambda/\mu + 1)(\lambda\mu + 1)}{(\lambda/\mu - 1)(\lambda\mu - 1)}, \log \frac{(\lambda/\mu + 1)(\lambda\mu - 1)}{(\lambda/\mu - 1)(\lambda\mu + 1)}\right),\end{aligned}$$

where the first equality is a consequence of Lemma 4.27 an the second one of Lemma 6.1. An easy computation gives now

$$\log \coth \frac{\ell^F\left(\mathcal{O}_\Sigma^{\gamma,\delta}\right)}{2} = \log \coth\left(\frac{\log \lambda}{2}\right),$$

$$d^F\left(\mathrm{pr}_\gamma \Lambda_\delta^+, \mathrm{pr}_\gamma \Lambda_\delta^-\right) = \log \coth \frac{\log \lambda + \log \mu}{2} + \log \coth \frac{\log \lambda - \log \mu}{2}.$$

The statement is now a consequence of Jensen's inequality (the function $f(x) = \log \coth(x)$ is, indeed, a strictly convex function). $\square$

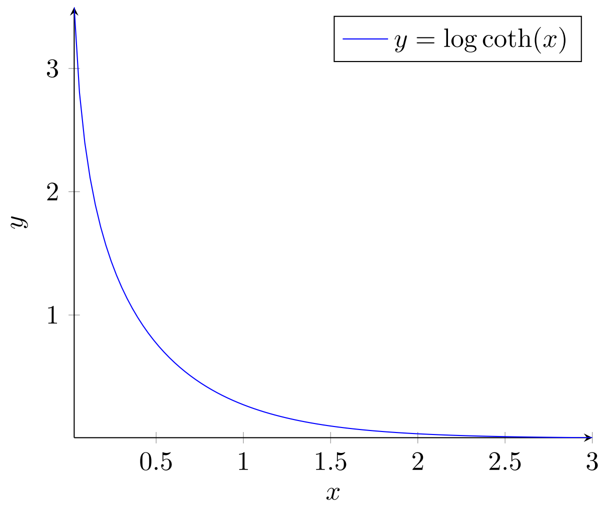

Figure 6.1: Graph of $\log\coth(x)$.

6.2 An identity for the translational length of a Shilov hyperbolic element

Given a peripheral element $\gamma \in \Gamma$ we need to find some points x in $\mathcal{Y}_\gamma$ on which $\rho(\gamma)$ realizes its Finsler translational length, that is

$$\ell^F(\rho(\gamma)) = d^F(x, \rho(\gamma)x).$$

First, we find a lower bound for the translational length of $\rho(\gamma)$.

Lemma 6.3. *Let g be Shilov hyperbolic; if λ, λ^{-1}, with $|\lambda| > 1$, are the two real eigenvalues associated to the attractive and repulsive isotropic eigenlines fixed by g, then*

$$\ell^F(g) \geq \log|\lambda|.$$

Proof. Up to conjugate by an element in $\mathrm{SO}_0(2, n)$, we can assume that l_1, l_2 are the attractive, respectively repulsive isotropic eigenlines fixed by g.

Consider an element $V \in \mathcal{Y}_{l_1,l_2}$; the goal is to prove

$$d^F(g \cdot V, V) \geq \log\lambda.$$

From Section 4.2, we know that the set $P_{l_1,l_2} := \mathrm{Stab}(l_1, l_2)$ is transitive on the $\mathbb{R}$-tube $\mathcal{Y}_{l_1,l_2}$ and, moreover,

$$\mathrm{Stab}_{P_{l_1,l_2}}(V_0) \cdot \mathfrak{a}_{2,n} = \mathrm{T}_{V_0}\mathcal{Y}_{l_1,l_2},$$

where $\mathfrak{a}_{2,n}$ is a maximal abelian contained in $\mathrm{T}_{V_0}\mathcal{X}_{l_1,l_2}$ and defined in Section 2.2.3; in particular, there exists an $h \in P_{l_1,l_2}$ such that $hV = V_0$ and

$$Y = \exp_{V_0}^{-1}(hgV) = \exp_{V_0}^{-1}(hgh^{-1}V_0) \in \mathfrak{a}_{2,n}.$$

As hgh^{-1} is conjugate to g with respect to an element in P_{l_1,l_2}, we know that λ, λ^{-1} are also two eigenvalues of hgh^{-1} with eigenlines l_1, l_2 and we can write

$$hgh^{-1} = \left(\begin{array}{cc|c|cc} \lambda & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & c \\ \hline 0 & d & E & 0 & f \\ \hline 0 & 0 & 0 & \lambda^{-1} & 0 \\ 0 & g & h & 0 & j \end{array}\right),$$

and so

$$Y = \mathrm{diag}(\log|\lambda|, \log\mu, 0, ...0, -\log|\lambda|, -\log\mu)$$

for some $\mu > 0$ (Notice that we only know $Y \in \mathfrak{a}_{2,n}$, so it is not necessary true that $\mu < \lambda$). Now it is easy to check that

$$\begin{aligned} d^{\overline{\mathfrak{w}}_{2,n}}(V, gV) = d^{\overline{\mathfrak{w}}_{2,n}}\left(V_0, hgh^{-1}V_0\right) = \\ = (\max\{\log|\lambda|, |\log\mu|\}, \min\{\log|\lambda|, |\log\mu|\}) \end{aligned}$$

and so

$$d^F(V, gV) \geq \log|\lambda|.$$

□

Thanks to the previous lemma, we have a lower bound on $\ell^F(\rho(\gamma))$, for $\gamma \in \Gamma$ peripheral element; so, in order to find a point on which $\rho(\gamma)$ realizes its translational length, it is enough to find a point x for which $d^F(x, \gamma x)$ reaches this lower bound.

Lemma 6.4. *For every peripheral element $\gamma \in \pi_1(\Sigma)$ and $x \in \partial\mathbb{H}^2$ such that (γ^-, x, γ^+) is maximal, the Finsler translation length $\ell^F(\rho(\gamma))$ is obtained on the projection of $\varphi(x)$ on the $\mathbb{R}$-tube $\mathcal{Y}_\gamma$:*

$$\ell^F(\rho(\gamma)) = d^F\left(\mathrm{pr}_\gamma(\varphi(x)), \rho(\gamma)\cdot \mathrm{pr}_\gamma(\varphi(x))\right).$$

Proof. Because the representation is Anosov, $\rho(\gamma)$ is Shilov hyperbolic and, up to $\mathrm{SO}_0(2,n)$-action, we can suppose it fixes the isotropic lines $\Lambda_\gamma^- = l_2$ and $\Lambda_\gamma^+ = l_1$. If λ is the eigenvalue of $\rho(\gamma)$ relative to the eigenline Λ_γ^+, then

$$\rho(\gamma) = \left(\begin{array}{cc|c|cc} \lambda & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & c \\ \hline 0 & d & E & 0 & f \\ \hline 0 & 0 & 0 & \lambda^{-1} & 0 \\ 0 & g & h & 0 & j \end{array}\right),$$

and, by the previous lemma, we have

$$\ell^F(\rho(\gamma)) \geq \log|\lambda|.$$

On the other hand, $\varphi(x) \in \mathrm{Is}_1(\mathbb{R}^{2,n})$ is such that $(l_2, \varphi(x), l_1)$ is maximal (because φ is a maximal framing of ρ) and so

$$\rho(\gamma) \cdot \mathrm{pr}_\gamma(\varphi(x)) = \mathrm{pr}_\gamma(\varphi(\gamma x)) :$$

indeed, as $\rho(\gamma)$ is an isometry, it preserves orthogonality between $\mathbb{R}$-tubes and so

$$\begin{aligned}\{\rho(\gamma)\,\mathrm{pr}_\gamma(\varphi(x))\} = \rho(\gamma) \cdot \left(\mathcal{Y}_\gamma \cap \mathcal{Y}_{\varphi(x),\overline{\varphi(x)}}\right) = \mathcal{Y}_\gamma \cap \mathcal{Y}_{\rho(\gamma)\varphi(x),\rho(\gamma)\overline{\varphi(x)}} = \\ = \{\mathrm{pr}_\gamma(\rho(\gamma)\varphi(x))\} = \{\mathrm{pr}_\gamma(\varphi(\gamma x))\}.\end{aligned}$$

As the representation ρ is Anosov maximal and φ is ρ-equivariant, the quadruple $(l_2, \varphi(x), \varphi(\gamma x), l_1)$ is maximal; up to $\mathrm{SO}_0(2,n)$-action, we can suppose

$$(l_2, \varphi(x), \varphi(\gamma x), l_1) = (l_2, l_3, [X], l_1)$$

for

$$[X] = \lambda' e_1 + \mu e_2 - \lambda'^{-1} e_{n+1} + \mu^{-1} e_{n+2}.$$

with $\lambda' > \mu \geq 1$; because $[X] = \rho(\gamma) \cdot l_3$, we deduce $\lambda' = \lambda$.

Using Lemma 4.31, we get

$$\begin{aligned}d^F\left(\mathrm{pr}_\gamma\left(\varphi(x)\right), \rho(\gamma) \cdot \mathrm{pr}_\gamma\left(\varphi(x)\right)\right) = \frac{1}{2} \log T(l_2, \varphi(x), \rho(\gamma)\varphi(x), l_1) = \\ = \frac{1}{2} \log\left(\lambda^2\right),\end{aligned}$$

that concludes the proof. □

6.3 A sum over orthotubes

Let $\gamma \in \Gamma$ be a peripheral element and $x \in \partial\mathbb{H}^2$ be such that the triple (γ^-, x, γ^+) is maximal.

Thanks to the previous section, we know that the Finsler translational length $\ell^F(\rho(\gamma))$ is equal to the Finsler distance between the two points $\mathrm{pr}_\gamma(\varphi(x))$ and $\mathrm{pr}_\gamma(\varphi(\gamma x))$; On the other hand, it is possible to prove, using the additivity of the Finsler distance along maximal k-tuples, that this last distance is larger than the sum of the distances $d^F\left(\mathrm{pr}_\gamma \Lambda_\delta^+, \mathrm{pr}_\gamma \Lambda_\delta^-\right)$ over all possible classes of peripheral elements

$$[\delta] \in \{\sigma \in \Gamma |\, \sigma \text{ peripheral}\} / \langle\gamma\rangle.$$

Lemma 6.5. *Let $\gamma \in \pi_1(\Sigma)$ be a peripheral element, then*

$$\ell^F(\rho(\gamma)) \geq \sum_{\alpha \in \mathcal{O}_\Sigma^{\mathcal{X}_{2,n}}(\gamma)} d^F\left(\mathrm{pr}_\gamma\left(\Lambda_{\delta_\alpha}^-\right), \mathrm{pr}_\gamma\left(\Lambda_{\delta_\alpha}^+\right)\right)$$

where δ_α is the peripheral element associated to the orthotube α different from γ.

Proof. Let us fix a peripheral element $\gamma \in \Gamma$.

Notice that the set $\mathcal{O}_{\Sigma}^{\mathcal{X}_{2,n}}(\gamma)$ contains at most countably many elements and so it is enough to prove that for every nonempty, finite subset

$$S \subset \mathcal{O}_{\Sigma}^{\mathcal{X}_{2,n}}(\gamma)$$

it holds

$$\ell^F(\rho(\gamma)) \geq \sum_{\alpha \in S} d^F\left(\mathrm{pr}_\gamma\left(\Lambda^-_{\delta_\alpha}\right), \mathrm{pr}_\gamma\left(\Lambda^+_{\delta_\alpha}\right)\right).$$

Fix a class ξ in S and choose a peripheral element $\delta \in \Gamma$ such that the orthotube $\mathcal{O}_{\Sigma}^{\gamma,\delta}$ is a representative of ξ; from Lemma 6.4, we have

$$\ell^F(\rho(\gamma)) = d^F\left(\mathrm{pr}_\gamma\left(\Lambda^+_\delta\right), \mathrm{pr}_\gamma\left(\gamma \cdot \Lambda^+_\delta\right)\right).$$

Now, for every $\alpha \in S \setminus \{\xi\}$, there exists a peripherial element $\delta_\alpha \in (\Gamma \setminus \{\gamma\})$ such that the class α contains the orthotube $\mathcal{O}_{\Sigma}^{\gamma,\delta_\alpha}$ and the quadruple $\left(\Lambda^-_\delta, \Lambda^+_{\delta_\alpha}, \Lambda^-_{\delta_\alpha}, \rho(\gamma)\cdot\Lambda^+_\delta\right)$ is maximal.

Moreover, if $\alpha, \beta \in S \setminus \{\xi\}$ are distinct, then one between the two quadruples $\left(\Lambda^-_\delta, \Lambda^+_{\delta_\alpha}, \Lambda^-_{\delta_\alpha}, \Lambda^+_{\delta_\beta}\right)$ and $\left(\Lambda^-_\delta, \Lambda^+_{\delta_\beta}, \Lambda^-_{\delta_\beta}, \Lambda^+_{\delta_\alpha}\right)$ is maximal; so, because the set S is finite, it is possible to find an enumeration

$$S = \{\alpha_0, \alpha_1, \alpha_2, ..., \alpha_n\},$$

in such a way that $\xi = \alpha_0$ and the $(2n+3)$-tuple

$$\left(\Lambda^-_\gamma, \Lambda^+_{\delta_{\alpha_0}}, \Lambda^-_{\delta_{\alpha_0}}, \Lambda^+_{\delta_{\alpha_1}}, \Lambda^-_{\delta_{\alpha_1}}, ..., \Lambda^+_{\delta_{\alpha_n}}, \Lambda^-_{\delta_{\alpha_n}}, \rho(\gamma)\cdot\Lambda^+_{\delta_{\alpha_0}}, \Lambda^+_\gamma\right)$$

is maximal (see Figure 6.2).

Finally, by the additivity of d^F along maximal k-tuples (Lemma 4.32), we have

$$\begin{aligned}
&\ell^F(\rho(\gamma)) = d^F\left(\mathrm{pr}_\gamma\left(\Lambda^-_{\delta_\alpha}\right), \mathrm{pr}_\gamma\left(\Lambda^+_{\delta_\alpha}\right)\right) = \\
&= \left(\sum_{i=0}^{n-1} d^F\left(\mathrm{pr}_\gamma\left(\Lambda^+_{\delta_{\alpha_i}}\right), \mathrm{pr}_\gamma\left(\Lambda^-_{\delta_{\alpha_i}}\right)\right) + d^F\left(\mathrm{pr}_\gamma\left(\Lambda^-_{\delta_{\alpha_i}}\right), \mathrm{pr}_\gamma\left(\Lambda^-_{\delta_{\alpha_{i+1}}}\right)\right)\right) \\
&\quad d^F\left(\mathrm{pr}_\gamma\left(\Lambda^+_{\delta_{\alpha_n}}\right), \mathrm{pr}_\gamma\left(\Lambda^-_{\delta_{\alpha_n}}\right)\right) + d^F\left(\mathrm{pr}_\gamma\left(\Lambda^-_{\delta_{\alpha_n}}\right), \mathrm{pr}_\gamma\left(\rho(\gamma)\cdot\Lambda^-_{\delta_{\alpha_0}}\right)\right) \\
&\qquad \geq \sum_{i=0}^{n} d^F\left(\mathrm{pr}_\gamma\left(\Lambda^+_{\delta_{\alpha_i}}\right), \mathrm{pr}_\gamma\left(\Lambda^-_{\delta_{\alpha_i}}\right)\right).
\end{aligned}$$

□

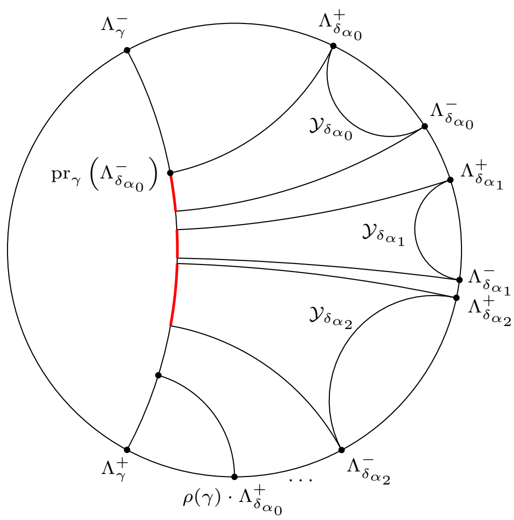

Figure 6.2: Representation of the proof of Lemma 6.5.

6.4 A Basmajian-type inequality

Finally we can state our last result.

Theorem A. *Let Σ be an hyperbolic surface with nonempty geodesic boundary and $\rho : \pi_1(\Sigma) \to \mathrm{SO}_0(2,n)$ an Anosov maximal representation. For every peripheral element $\gamma \in \pi_1(\Sigma)$ it holds*

$$\ell^F(\rho(\gamma)) \geq \sum_{\alpha \in \mathcal{O}_\Sigma^{\mathcal{X}_{2,n}}(\gamma)} 2 \log \coth \frac{\ell^F(\alpha)}{2}.$$

Proof. It follows immediately from Lemma 6.5 and Lemma 6.2. $\square$

Notice the analogy with the Basmajian identity: let $c \subset \partial\Sigma$ be a boundary component of Σ and $\gamma \in \Gamma$ a corresponding peripheral element; on the left of our inequality we have the translational length of $\rho(\gamma)$, as in the Basmajian identity there is the length of the boundary component c (that is also the transaltional length of γ acting on the universal cover $\tilde{\Sigma} \subset \mathbb{H}^2$); on the right, instead, there is a sum over all possible orthotubes orthogonal to $\mathcal{Y}_\gamma$, exactly as in the Basmajian identity with the orthogeodesics orthogonal to c.

Appendix A

Utility functions

Many proofs in this work require a large amount of computations and, for this reason, I would recommend the use of a software.

I decided to use *Python* with the support of the library *sympy*. So, to help the interested reader who wants to check proofs and computations here claimed, I attached some utility functions that I personally used in my programs.

```
from sympy import *
import numpy

def T(a,b,c,d):
"""Crossratio"""
    return ((a.T *J*c)*(b.T*J*d)/
            ((a.T*J*b)*(c.T*J*d)))[0]

def R(a,b,c,d):
"""Weyl chamber-valued crossratio"""
    s=T(a,b,c,d)
    t=T(b,c,d,a)
    r=(s*t+t-s)/(sqrt(s)*t)
    return Matrix([sqrt(s),
                   (r+sqrt(r**2-4))/2] )

def conj(x):
"""Conjugatation of an isotropic line
      w.r.t. l1,l2"""
    I=diag(-1,1,1,-1,1)
    return I*x


def iom(M):
"""Check matrix contained in H_{2,3}"""
    v = M * Matrix([1, 0, 0, 1, 0])
    w = M * Matrix([0, 1, 0, 0, 1])
    d1 = simplify(M.det())
    d2=simplify(Matrix([[v.T*v, v.T*w],
                        [w.T*v, w.T*w]]).det())
    X = simplify(M.T * J * M)
    print('condition for the matrix to be in H_{2,n}:')
    if X != J :
        print('* ',
              repr(simplify(M.T * J * M)),
              '= J')
    if d1 != 1 :
```

```
        print('* ', d1,  ' = 1')
    print('* ', d2, ' > 0\n')
    return

#standard isotropic lines
l,m = symbols('l m',
              positive = True, real = True)
l1 = Matrix([1 , 0, 0, 0   , 0  ])
l2 = Matrix([0 , 0, 0, 1   , 0  ])
l3 = Matrix([1 , 1, 0, -1  , 1  ])
l4 = Matrix([-1, 1, 0, 1   , 1  ])
ell= Matrix([l , m, 0, -1/l, 1/m])

#canonical basis
e1=Matrix([1, 0, 0, 0, 0])
e2=Matrix([0, 1, 0, 0, 0])
e3=Matrix([0, 0, 1, 0, 0])
e4=Matrix([0, 0, 0, 1, 0])
e5=Matrix([0, 0, 0, 0, 1])

J = Matrix([[e4.T],
            [e5.T],
            [-e3.T],
            [e1.T],
            [e2.T]])

#base point
V0=Matrix([[(e1+e4).T],
           [(e2+e5).T]]).T
```

All the objects/functions defined here are referred to the case $n = 3$, in the space $\mathbb{R}^{2,3}$; on the other hand, from Remark 3.2, we know that many cases can be reduced to this one.

About the code:

- The function

  ```
  T(a,b,c,d)
  ```

  takes four vectors a, b, c, d in $\mathbb{R}^5$ and returns $T([a],[b],[c],[d])$, where T is the crossratio defined in Example 4.2, while $[a],[b],[c],[d]$ are the four isotropic lines generated by a, b, c, d;

- The function

```
R(a,b,c,d)
```

takes four vectors a, b, c, d in $\mathbb{R}^5$ and returns $R([a],[b],[c],[d])$, where R is the Weyl chamber-valued crossratio defined in Definition 4.4, while $[a],[b],[c],[d]$ are the four isotropic lines generated by a, b, c, d;

- The function

```
conj(x)
```

takes a vector x in $\mathbb{R}^5$ and returns a vector representing the conjugate isotropic line $\overline{[x]}$ of the line $[x]$, where the conjuate line is defined in Definition 4.18;

- The function

```
iom(M)
```

takes a matrix $M \in \mathrm{Mat}(5, \mathbb{R})$ and returns the conditions for M to be inside $H_{2,n}$;

- the objects

```
l1
l2
l3
l4
```

are the vectors representing the isotropic lines l_1, l_2, l_3, $l_4 \in \mathrm{Is}_1(\mathbb{R}^{2,3})$, defined in Section 3.4 and Section 4.3

- the object

```
ell
```

is the vector

$$X = le_1 + me_2 - l^{-1}e_4 + m^{-1}e_5 \in \mathbb{R}^5$$

for $l, m > 0$;

- the objects

```
e1
e2
e3
e4
e5
```

are the element $e_1, e_2, e_3, e_4, e_5 \in \mathbb{R}^{2,3}$ defined in Section 2.1;

- the object

```
J
```

is the matrix $J_{2,3}$ defined in Section 2.1;

- the object

```
V0
```

is the point $V_0 \in \mathcal{X}_{2,3}$ defined in Section 4.2.

Appendix B

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