Ruprecht-Karls University Heidelberg

Master's Thesis

On Volume Rigidity of Lattices

Yannick M. Krifka

December 2, 2015

Supervisor: Prof. Dr. A. Wienhard

Für meine Eltern

Erklärung zur eigenständigen Arbeit / Statement of Originality

Hiermit erkläre ich, dass ich der alleinige Autor dieser Arbeit bin, und keine weiteren Hilfsmittel oder Quellen als die von mir angegebenen verwendet habe.

I hereby confirm, that I am the sole author of this thesis and that I did not use any utilities or sources other than those cited in the text.

Heidelberg, ______________________________

Datum/Date, Yannick Krifka

Abstract

Our goal in this Master's thesis is to give a detailed proof of the volume rigidity theorem due to Bucher, Burger, and Iozzi, following the lines of the article [BBI13]. To every lattice embedding $i : \Gamma \hookrightarrow \mathrm{Isom}^+(\mathbb{H}^n)$ and any representation $\rho : \Gamma \to \mathrm{Isom}^+(\mathbb{H}^n)$ we may associate a real number $\mathrm{Vol}(\rho)$, the so called volume of ρ. The definition of $\mathrm{Vol}(\rho)$ relies on techniques from bounded cohomology and is reminiscent of the definition of the Toledo invariant for surface group representations as in [BIW03], [BI07]. If $n \geq 3$, the volume rigidity theorem asserts that $|\mathrm{Vol}(\rho)| \leq |\mathrm{Vol}(i)| = \mathrm{Vol}(M)$, where $M = i(\Gamma)\backslash\mathbb{H}^n$. Moreover equality holds if and only if ρ is conjugated to i by an isometry. This may be considered as a generalization of Mostow's rigidity theorem for finite volume hyperbolic manifolds of dimension at least three.

Along the way, background information on hyperbolic geometry and in particular on continuous (bounded) cohomology is provided, introducing the reader to the subject. We also prove a version of de Rham's theorem for relative de Rham cohomology in the appendix. Further a detailed discussion of Douady-Earle's barycenter construction for probability measures on $\partial\mathbb{H}^n$ with no atoms of mass $\geq 1/2$ is included.

Zusammenfassung

Das Ziel dieser Masterarbeit ist einen detaillierten Beweis des Volumenstarrheitssatzes von Bucher, Burger und Iozzi zu geben, wobei wir [BBI13] folgen. Für jede Gittereinbettung $i : \Gamma \hookrightarrow \mathrm{Isom}^+(\mathbb{H}^n)$ ordnen wir einer beliebigen Darstellung $\rho : \Gamma \to \mathrm{Isom}^+(\mathbb{H}^n)$ eine reelle Zahl $\mathrm{Vol}(\rho)$ zu, das sog. Volumen von ρ. Die Definition von $\mathrm{Vol}(\rho)$ benutzt hierbei beschränkte Kohomologie und ist ähnlich zu der Definition der Toledo-Invariante für Darstellunen von Flächengruppen wie in [BIW03], [BI07]. Für $n \geq 3$ besagt der Volumenstarrheitssatz nun, dass $|\mathrm{Vol}(\rho)| \leq |\mathrm{Vol}(i)| = \mathrm{Vol}(M)$ gilt, wobei $M = i(\Gamma)\backslash\mathbb{H}^n$. Darüber hinaus gilt Gleichheit genau dann, wenn ρ und i durch eine Isometrie konjugiert sind. Dieser Satz kann als eine Verallgemeinerung des Mostow'schen Starrheitssatzes für hyperbolische Mannigfaltigkeiten endlichen Volumens mit Dimension mindestens drei verstanden werden.

Es werden viele Hintergrundinformationen zu hyperbolischer Geometrie und insbesondere beschränkter Kohomologie bereitgestellt, welche den Leser an das Thema heranführen. Im Anhang beweisen wir zudem eine Version des Satzes von de Rham für relative de-Rham-Kohomologie. Darüber hinaus erläutern wir detailliert Douady-Earles Baryzenter Konstruktion für Wahrscheinlichkeitsmaße auf $\partial\mathbb{H}^n$, welche keine Atome der Masse $\geq 1/2$ besitzen.

Contents

Acknowledgement

I would like to thank Prof. Dr. Anna Wienhard for introducing me to the interesting subject of continuous bounded cohomology and its connection to geometry. Her support and frequent advise were much appreciated. I also owe my deep gratitude to Dr. Andreas Ott for his guidance, the inspiring discussions on all kinds of different topics and his insightful comments on my thesis. For her help in many respects I feel especially grateful to Prof. Dr. Alessandra Iozzi. Thank you for answering my e-mails with such a tremendous amount of patience and good will. Moreover I want to thank Dr. Daniele Alessandrini, Dr. Gye-Seon Lee and the rest of the research group differential geometry at the Ruprecht-Karls Universität Heidelberg for their help and the nice atmosphere they provided during my studies.

Last but not least I want to thank my family and friends for their constant support and patience. Thank you.

Introduction

The objective of this Master's thesis is to outline the paper by Bucher, Burger, and Iozzi [BBI13] in a way that is digestible for non-experts in the field of bounded cohomology. The main result of [BBI13] is the so called *volume rigidity theorem*, which can be regarded as a generalization of the Mostow rigidity theorem for finite volume hyperbolic manifolds.

Mostow's rigidity theorem is a remarkable result linking the topology and geometry of finite volume hyperbolic manifolds of dimension at least three. Specifically it asserts, that two such hyperbolic manifolds with isomorphic fundamental groups are already isometric. This is in rough contrast to the case of dimension two. In fact the study of Teichmüller spaces $\mathcal{T}_g$ for hyperbolic surfaces of genus $g \geq 2$ shows, that there are already uncountably many surfaces with isomorphic fundamental groups but different/non-isometric hyperbolic metrics. More precisely the Fenchel-Nielsen coordinates on $\mathcal{T}_g$ yield a bijection $\mathcal{T}_g \cong \mathbb{R}_+^{3(g-1)} \times \mathbb{R}^{3(g-1)}$; see e.g. [BP92].

By now there are several proofs of Mostow's rigidity theorem all using different techniques. Mostow's original proof – for compact hyperbolic manifolds of dimension at least three – uses pseudo-isometries in order to extend a given homotopy equivalence continuously to the boundary of hyperbolic n-space and then shows that this extension is in fact induced by an isometry; see e.g. [Thu]. For compact hyperbolic 3-manifolds Thurston [Thu] gives another proof making use of Gromov's l_1-homology and measure homology. More recently Besson, Courtois and Gallot [BCG96] gave a proof using entropy methods. A nice survey comparing the preceding approaches may be found in [Lü10].

However the starting point of [BBI13] is [BI09] where Burger and Iozzi succeed in applying the machinery of continuous bounded cohomology to prove Mostow's rigidity theorem for compact hyperbolic 3-manifolds. Their proof is along the lines of the Gromov-Thurston proof and may be regarded as a "dual version" of it, since the classical singular bounded cohomology can be interpreted as the dual theory to l_1-homology (cf. [Gro82]). The reason why [BI09] is limited to dimension three is that there is additional knowledge on the continuous bounded cohomology group $H^3_{cb}(\mathrm{Isom}^+(\mathbb{H}^3), \mathbb{R})$, which is unavailable in higher degress. Further effort has been made to remedy this in [BBI13], as we will see.

As we have already mentioned, Bucher, Burger, and Iozzi do not prove Mostow's rigidity theorem directly, but rather prove the following *volume rigidity theorem* for representations of hyperbolic lattices (cf. section I.4). It is formulated by means of the volume $\mathrm{Vol}(\rho)$ of a lattice representation $\rho : \Gamma < \mathrm{Isom}^+(\mathbb{H}^n) \to \mathrm{Isom}^+(\mathbb{H}^n)$, which we will define in section III.2. In fact the definition $\mathrm{Vol}(\rho)$ is very similar to the Toledo invariant for surface group representations; see e.g. [BIW03], [BI07], [BIW10], [Wie04].

Theorem (Volume Rigidity Theorem; [BBI13])**.** *Let $n \geq 3$. Let $i\colon \Gamma \hookrightarrow \mathrm{Isom}^+(\mathbb{H}^n)$ be a lattice embedding and let $\rho\colon \Gamma \to \mathrm{Isom}^+(\mathbb{H}^n)$ be any representation. Then:*

$$|\mathrm{Vol}(\rho)| \leq |\mathrm{Vol}(i)| = \mathrm{Vol}(M)$$

with equality, if and only if ρ is conjugated to i by an isometry. Here M denotes the quotient $i(\Gamma)\backslash\mathbb{H}^n$.

We will see that Mostow's rigidity theorem follows quite easily from the volume rigidity theorem in section III.1.

Note that Francaviglia and Klaff [FK06] were able to prove a similar volume rigidity theorem with a different definition of volume based on the notion of pseudo-developing maps following Dunfield [Dun99]. However [FK06] is also concerned with representations $\rho : \Gamma < \mathrm{Isom}^+(\mathbb{H}^k) \to \mathrm{Isom}^+(\mathbb{H}^n)$ where $3 \leq k \leq n$, i.e. k is not necessarily equal to n.

The proof of the volume rigidity theorem can be divided into two parts. Establishing the inequality is the first part. This will not be very difficult after we will have deduced some basic properties of $\mathrm{Vol}(\rho)$ by using transfer maps and applying ideas from relative cohomology. The second part is concerned with the case of equality and the construction of a conjugating isometry. We will do this in three steps following the usual strategy of proofs of Mostow rigidity. First, we will construct a boundary map $\varphi : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$, which is in some sense "compatible" with the action of Γ on $\partial\mathbb{H}^n$ via the representation ρ. In the second step we will use a very explicit version of Burger and Iozzi's useful formula [BI09] to deduce, that the constructed boundary map sends regular ideal simplices to regular ideal simplices preserving their orientation. Finally we will apply a general proposition about boundary maps, which asserts that such a boundary map is already induced by an isometry. This is infact the only step in the proof which requires $n \geq 3$. The compatibility of the boundary map with the ρ-action of Γ on $\partial\mathbb{H}^n$ then implies that this isometry conjugates i and ρ.

Outline of the thesis:

In our elaboration of [BBI13] we strive for a very detailed exposition. Therefore we give plenty of background information on hyperbolic geometry, continuous and bounded cohomology, measure theory and Douady-Earles's barycenter construction among others. Our declared goal is to provide an explanation of all relevant aspects for an audience with some background in differential geometry and algebraic topology.

Chapter I recalls some results in hyperbolic geometry. Sections I.1-I.4 are concerned with the basic notions of hyperbolic geometry and constitute the fundament of our further investigations. The most important among these is probably section I.4 where we introduce lattice subgroups and show how they relate to finite volume hyperbolic manifolds and the existence of invariant probability measures. In section I.5 we are interested in ergodicity phenomena and prove that every lattice $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ acts double ergodically on the boundary $\partial\mathbb{H}^n$. This double ergodicity is one of the central technical ingredients in our proof of the volume rigidity theorem and also in the Gromov-Thurston proof of Mostow's rigidity theorem. Section I.6 then introduces the thick-thin decomposition of finite volume hyperbolic manifolds. In particular the notion of a compact core will be important in the definition of $\mathrm{Vol}(\rho)$ later on. In section I.7 we consider (euclidean and) hyperbolic simplices, which are play a fundamental role in many respects. The final section I.8 is about boundary maps. We give a proof of the above mentioned proposition, which will be the key in the third step in the proof of the volume rigidity theorem.

Chapter II is about continuous cohomology and continuous bounded cohomology. We give a concise introduction to the main aspects of both theories in section II.1 and II.2 respectively. These sections follow a similar outline and we hope that this facilitates the comparison of both. In section II.3 we then apply these cohomology theories to the group of hyperbolic isometries $\mathrm{Isom}(\mathbb{H}^n)$ and their subgroups. We try to be very concrete and provide some isomorphisms at the cochain level. For example we give the van Est isomorphism in continuous cohomology at the cochain level, which is otherwise a bit hard to find in the literature. This is also where we introduce the volume class $\omega_n^{(b)} \in H^n_{c(b)}(\mathrm{Isom}(\mathbb{H}^n), \mathbb{R}_\varepsilon)$ and see some cocycles representing it in different resolutions. The volume class plays the central role in the definition of $\mathrm{Vol}(\cdot)$ as its name already suggests. Afterwards we show that the comparison map $c : \langle \omega_n^b \rangle \cong H^n_{cb}(\mathrm{Isom}(\mathbb{H}^n), \mathbb{R}_\varepsilon) \to H^n_c(\mathrm{Isom}(\mathbb{H}^n), \mathbb{R}_\varepsilon) \cong \langle \omega_n \rangle$ is an isomorphism in top degree. The latter is the key result to overcome the limitation of Burger and Iozzi's proof of Mostow's rigidity theorem in [BI09] to dimension three.

Chapter III uses the previously gathered results in hyperbolic geometry and continuous (bounded) cohomology to give a proof of the volume rigidity theorem following the above mentioned strategy. In section III.1 we restate the volume rigidity theorem and deduce two versions of Mostow's rigidity theorem from it. Section III.2 then gives a precise definition of the volume $\mathrm{Vol}(\rho)$ of a representation $\rho : \Gamma \to \mathrm{Isom}^+(\mathbb{H}^n)$ for a hyperbolic lattice $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ – as promised. Further properties are deduced from which the inequality in the volume rigidity theorem follows. Finally, we give a proof in the case of equality and construct a conjugating isometry following the described three steps in section III.3.

We added five appendices to this thesis, containing background knowledge on certain fundamental theories or constructions that did not seem to fit in the main body of the text.

Appendix A gives an introduction to measure theory with a view towards harmonic analysis on general locally compact groups. Many constructions and computations throughout the thesis depend on some knowledge on Haar measures, quotient measures, measure classes and so on. In particular we treat the canonical measure class on oriented smooth manifolds.

Appendix B is about G-modules and Banach G-modules, which are the underlying objects in the functorial framework of continuous cohomology resp. continuous bounded cohomology.

Appendix C gives a brief discussion of amenability, i.e. amenable groups as well as amenable actions. These are important notions in the theory of continuous bounded cohomology and form a subject themselves. We shall only state some necessary results and refer to the literature for proofs and a more thorough discussion.

Appendix D recalls the classical cohomology theories such as singular and de Rham cohomology very briefly fixing some notation. Further we introduce relative singular bounded cohomology and prove a relative version of de Rham's theorem, which will be needed in our discussion of the properties of $\mathrm{Vol}(\cdot)$ and the relative transfer maps.

Appendix E provides a proof of Douady-Earle's barycenter construction in the realm of hyperbolic geometry. First we compute the Busemann functions in the Poincaré ball model and the upper half space model explicitly, and investigate their transformation behaviour. Then we give a proof of the barycenter construction for probability measures on the boundary $\partial\mathbb{H}^n$ that have no atoms of mass $\geq 1/2$. At the end we include some graphics that visualize the barycenter construction and a python script that was used to plot the images.

I. Hyperbolic Geometry

In this first chapter we want to give an exposition of some results in hyperbolic geometry, that we will need in the context of the volume rigidity theorem. In section I.1 we just recall some fundamental terminology assuming that the reader is already familiar with the basic notions of hyperbolic geometry. Section I.2 deals with the isometries of the different models of hyperbolic space. We will also identify hyperbolic n-space and its boundary as homogeneous spaces, which will be important throughout the rest of this thesis. Investigating further the isometry group we will introduce the notion of elementary (sub-)groups in section I.3. Section I.4 then covers lattice subgroups and relates them to hyperbolic manifolds with finite volume using covering theory and some results about quotient measures (cf. appendix A). Staying in the realm of measure theory we will recall the basic notions of ergodic theory and give a proof of the double ergodicity of lattice actions on the boundary in section I.5. Section I.6 then introduces the thick-thin decomposition of hyperbolic manifolds with finite volume, which will be important in the definition of the volume of a representation later on. Coming back to hyperbolic n-space section I.7 is concerned with hyperbolic simplices, their volume and their reflection group. In section I.8 we will then see under which conditions a boundary map is already induced by an isometry. This is one of the central aspects of our proof of the volume rigidity theorem.

Throughout this chapter let $n \geq 2$.

I.1. Basics

We assume that the reader is already familiar with the basic notions of hyperbolic geometry as there are already many textbooks on the subject, e.g. [BP92], [Kap09], [Rat06], [Thu97]. Therefore we will just fix some notation and recall some basic facts here.

Definition I.1.1 (Hyperbolic n-space $\mathbb{H}^n$)**.** Let $n \in \mathbb{N}$. *Hyperbolic n-space* $\mathbb{H}^n$ is the unique simply connected and complete Riemannian manifold with constant sectional curvature $K = -1$ (cf. [dC92, Theorem 4.1, p. 163]).

We will further assume, that the reader is familiar with the *upper half space model* U^n, the *Poincaré ball model* B^n, the *hyperboloid model* H^n and the *projective disc model* D^n of hyperbolic n-space (cf. [Rat06]). Recall that from all the previous models the projective disc model is not conformal. However this gives us the advantage, that geodesics are the usual straight euclidean cords in the disc D^n, which is quite helpful whilst dealing with convex sets.

$\mathbb{H}^n$ carries a metric, which we will denote by $d(\cdot, \cdot)$ in the following.

Definition I.1.2 (Boundary $\partial\mathbb{H}^n$)**.** Consider the set $\mathcal{S}$ of all geodesic half-lines in $\mathbb{H}^n$ parametrized by arc length on $[0, \infty)$, and define an equivalence relation $\sim$ on $\mathcal{S}$ in the following way:

$$\gamma_1 \sim \gamma_2 \iff \sup_{t \geq 0} d(\gamma_1(t), \gamma_2(t)) < \infty$$

Set $\partial\mathbb{H}^n = \mathcal{S}/\sim$ and $\overline{\mathbb{H}}^n = \mathbb{H}^n \cup \partial\mathbb{H}^n$. We define a topology on $\mathbb{H}^n$ such that $\mathbb{H}^n \subset \overline{\mathbb{H}}^n$ is open and inherits its original topology, and a neighborhood of $p \in \partial\mathbb{H}^n$ is obtained in the following way:

choose γ in the class of p, and let x be its starting point, let V be a neighborhood of $\dot\gamma(0)$ in the unit sphere of $T_x\mathbb{H}^n$ and let $r > 0$; then we set

$$U(\gamma, V, r) = \{\gamma_1(t) : \gamma_1 \in \mathcal{S}, \gamma_1(0) = x, \dot\gamma_1(0) \in V, t > r\} \\ \cup \{[\gamma_1]_\sim : \gamma_1 \in \mathcal{S}, \gamma_1(0) = x, \dot\gamma_1(0) \in V\}$$

These fulfill the axioms of a fundamental system of neighborhoods of p.

It turns out, that $\partial\mathbb{H}^n$ is homeomorphic to S^{n-1} and $\overline{\mathbb{H}}^n$ is homeomorphic to $\overline{B}^n$. Moreover if we consider the Poincaré ball model B^n, $\overline{B}^n$ is canonically identified with the closure of B^n as a subset of $\mathbb{R}^n$ (cf. [BP92, Proposition A.5.10, p. 29]). In the very same way for the upper half space model U^n we can identify $\partial U^n \cong \mathbb{R}^{n-1} \times \{0\} \cup \{\infty\} \cong \mathbb{R}^{n-1} \cup \{\infty\}$, where the latter can be understood as the one-point-compactification of $\mathbb{R}^{n-1}$. $\partial\mathbb{H}^n$ is called the *boundary at infinity of* $\mathbb{H}^n$.

Instead of writing $[\gamma]$ for the class of a geodesic ray $\gamma : [0, \infty) \to \mathbb{H}^n$ we will often use the more suggestive notation $\lim_{t\to\infty} \gamma(t)$; being consistent with this notation we will also write $\lim_{t\to\infty} \exp_x(t \cdot v_x)$ for the boundary point represented by the class of the geodesic ray starting in the direction of the tangent vector $v_x \in T_x\mathbb{H}^n$ at $x \in \mathbb{H}^n$.

Definition I.1.3 ((Generalized) Hyperbolic Subspace)**.** A subset N of $\mathbb{H}^n$ is a *hyperbolic subspace* if and only if it contains the entire geodesic passing through any two of its points. One may now compactify N to a compact subset of $\overline{\mathbb{H}}^n$ by considering its closure in $\overline{\mathbb{H}}^n$. This is just the set we obtain by adding all the classes $[\gamma]_\sim$ of geodesics γ in N. A set of this form is then called a *generalized hyperbolic subspace*; or just a *hyperbolic subspace* if there is no ambiguity.

Hyperbolic subspaces take different shapes in the different models. In the ball model B^n these are just spheres meeting S^{n-1} orthogonally resp. linear subspaces (through 0). In the upper half space model U^n theses are also just spheres meeting the boundary $\mathbb{R}^{n-1} \cong \mathbb{R}^{n-1} \times \{0\}$ orthogonally resp. vertical affine subspaces. In the projective disc model D^n these are euclidean affine subspaces and in the hyperboloid model theses are intersections of H^n with linear subspaces in $\mathbb{R}^{n,1}$. In particular (generalized) hyperbolic subspaces are submanifolds (with boundary) of $\mathbb{H}^n$ (resp. $\overline{\mathbb{H}}^n$).

Further we get, that a p-dimensional hyperbolic subspace is isometric to $\mathbb{H}^p$ (cf. [BP92, Corollary A.5.7, p. 27]).

We now turn to a different geometric notion in hyperbolic n-space, namely horospheres and horoballs.

Definition I.1.4 (Horosphere)**.** Consider the upper half space model U^n. The *horosphere centered at* ∞ is a hypersurface of the form $\{x = (x_1, \ldots, x_n) \in U^n : x_n = c\}$ for some $c > 0$. A horosphere N centered at ∞ inherits a Euclidean structure by restrichting the hyperbolic metric to N (cf. [BP92, Theorem A.4.3., p. 24]). A *horosphere centered at a point* $\xi \in \mathbb{R}^{n-1} = \partial U^n - \{\infty\}$ is defined to be the image of a horosphere centered at ∞ under an isometry taking ∞ to ξ, i.e. euclidean spheres tangent to ξ with center in U^n.

Passing to the Poincaré ball model B^n a horosphere centered at $\xi \in \partial B^n$ is also a euclidean sphere properly contained in $\overline{B}^n$ and tangent to ξ.

Equivalently, a horosphere centered at $\xi \in \partial\mathbb{H}^n$ is a maximal hypersurface N in $\mathbb{H}^n$ such that for every point $x \in N$ the geodesic passing through x in the direction of ξ is perpendicular to N.

Definition I.1.5 (Horoball)**.** Let N be a horosphere in $\mathbb{H}^n$ centered at $\xi \in \partial\mathbb{H}^n$. Then $\mathbb{H}^n - N$ has two connected components. One of them contains every geodesic ray $\gamma|_{(0,\infty)}$ emanating from some point $\gamma(0) \in N$ and tending towards $\gamma(\infty) = \xi$; we will call any subset of this form a *horoball*.

In the Poincaré ball model horoballs are just the "interiors" of horospheres, whence the name.

I.2. Isometries of $\mathbb{H}^n$

Let us first conisder the hyperboloid model H^n. Let us denote by $O(H^n)$ the subgroup of $O(n,1)$ which leaves $H^n \subset \mathbb{R}^{n,1}$ invariant, and denote by $SO(H^n)$ the intersection of $SL(n+1,\mathbb{R})$ with $O(H^n)$. Note that $O(H^n)$ and $SO(H^n)$ are closed subgroups of $GL(n+1,\mathbb{R})$, and hence they are naturally endowed with a Lie group structure.

Proposition I.2.1. *$O(H^n)$ is generated by the reflections it contains.*

Proof. See [BP92, Proposition A.2.3, p. 5]. □

Theorem I.2.2 (Isometries in the hyperboloid model H^n)**.** $\mathrm{Isom}(H^n)$ *consists of the restrictions of the elements of $O(H^n)$, thus $O(H^n) \cong \mathrm{Isom}(H^n)$; in particular* $\mathrm{Isom}(H^n)$ *is generated by reflections. Similarly* $\mathrm{Isom}^+(H^n) \cong SO(H^n)$.

Proof. See [BP92, Theorem A.2.4, p. 6]. □

Identifying $H^n \cong \mathbb{H}^n$ we get that $\mathrm{Isom}(\mathbb{H}^n)$ resp. $\mathrm{Isom}^+(\mathbb{H}^n)$ are Lie groups. However this description of their Lie group structure is not intrinsic, since it depends on the chosen model. Therefore we want to mention that the isometry group of a connected Riemannian manifold M can be equipped with a Lie group structure by identifying it with a submanifold of the frame bundle $F(TM)$. Recall that, for any $p \in M$, an isometry φ is uniquely determined $\varphi(p)$ and $d\varphi_p$. It is easy to check that both Lie group structures coincide.

Observe further that the topology on $\mathrm{Isom}(\mathbb{H}^n)$ induced by its Lie group structure can be defined intrinsically as well. For that consider $\mathrm{Isom}(\mathbb{H}^n)$ as a topological subspace of all continuous mappings $C(\mathbb{H}^n, \mathbb{H}^n)$. The latter is now equipped naturally with the compact-open topology. Recall that a subbasis for the compact-open topology on $C(X,X)$, for any topological space X, is given by all the following subsets

$$W(K,U) := \{f \in C(X,X) : f(K) \subset U\}$$

where $K \subset X$ is compact and $U \subset X$ is open; hence the name "compact-open topology".

Now we turn to the more important models B^n and U^n.

Definition I.2.3 (Inversion $i_{x_0,\alpha}$)**.** Let $x_0 \in \mathbb{R}^n$ and $\alpha > 0$. We will call the mapping

$$i_{x_0,\alpha} : x \mapsto \alpha \cdot \frac{x - x_0}{|x - x_0|^2} + x_0$$

an *inversion at the sphere with centre x_0 and radius $\sqrt{\alpha}$*. This mapping is understood both as a mapping of $\mathbb{R}^n - \{x_0\}$ onto itself and as a mapping of the extended euclidean space $\hat{\mathbb{R}}^n \cong \mathbb{R}^n \cup \{\infty\}$ onto itself, where $i_{x_0,\alpha}$ exchanges x_0 and ∞.

Recall that the stereographic projection $\pi : H^n \to B^n$

$$\pi(x) = \frac{(x_1, \ldots, x_n)}{x_{n+1} + 1}, \qquad x = (x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n,1}$$

is an isometry and further that the inversion $i_{-e_n,2} : B^n \to U^n$ at the sphere of radius $\sqrt{2}$ centered at $-e_n$ is an isometry; the latter is sometimes called the (inverse) Cayley transform in complex analysis.

Proposition I.2.4. $\mathrm{Isom}(B^n)$ *resp.* $\mathrm{Isom}(U^n)$ *is generated by inversions at spheres orthogonal to $\partial B^n \cong S^{n-1}$ resp. $\partial U^n \cong \mathbb{R}^{n-1} \cup \{\infty\}$.*

Proof. It is easy to check, that under the above isometries $H^n \to B^n$ and $B^n \to U^n$ reflections in H^n correspond to inversions at spheres orthogonal to the boundary in the respective models. The assertion now follows from Proposition I.2.1 and Theorem I.2.2. □

Finally we have the following very helpful description of the isometries in the Poincaré ball model B^n and the upper half space model U^n.

Theorem I.2.5 (Isometries of B^n)**.** $\mathrm{Isom}(B^n)$ *consists of all and only the mappings of the form*

$$\varphi : x \mapsto A \cdot i(x)$$

where $A \in O(n)$ and i is either the identity or an inversion with respect to a sphere orthogonal to ∂B^n.

Further φ is orientation preserving, i.e. $\varphi \in \mathrm{Isom}^+(B^n)$, if and only if

$$(A \in SO(n), i = \mathrm{id}) \textit{ or } (A \notin SO(n), i \neq \mathrm{id})$$

Proof. This follows from [BP92, Theorem A.4.1, p. 22], [BP92, Theorem A.3.9, p. 21] and [BP92, Remark A.3.10, p. 21]. □

Theorem I.2.6 (Isometries of U^n)**.** $\mathrm{Isom}(U^n)$ *consists of all and only the mappings of the form*

$$\varphi : x \mapsto \lambda \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} i(x) + \begin{pmatrix} b \\ 0 \end{pmatrix}$$

where $\lambda > 0$, $A \in O(n)$, i is either the identity or an inversion with respect to a sphere orthogonal to $\mathbb{R}^{n-1}$ and $b \in \mathbb{R}^{n-1}$.

Further φ is orientation preserving, if and only if

$$(A \in SO(n), i = \mathrm{id}) \textit{ or } (A \notin SO(n), i \neq \mathrm{id})$$

Proof. This follows from [BP92, Theorem A.4.2, p. 22], [BP92, Theorem A.3.9, p. 21] and [BP92, Remark A.3.10, p. 21]. □

Observe that by the above description, the stabilizer $\mathrm{Isom}(B^n)_0$ of the origin $0 \in B^n$ is just $O(n)$. Further the stabilizer $\mathrm{Isom}(U^n)_\infty$ of $\infty \in \partial U^n$ is the set of all mappings of the form

$$\begin{pmatrix} x \\ t \end{pmatrix} \mapsto \lambda \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} b \\ 0 \end{pmatrix}$$

for every $(x, t) \in U^n$, where $\lambda > 0$, $A \in O(n)$ and $b \in \mathbb{R}^{n-1}$. Hence it can be identified with the group of euclidean similarities $S(\mathbb{R}^{n-1})$ via the identification $\mathbb{R}^{n-1} \cong \mathbb{R}^{n-1} \times \{0\}$.

From the concrete description of the isometries it is easy to deduce the following proposition.

Proposition I.2.7. *(i) All isometries of $\mathbb{H}^n$ extend to homeomorphisms of $\overline{\mathbb{H}}^n$, and hence they have some fixed point in $\overline{\mathbb{H}}^n$.*

(ii) $\mathrm{Isom}(\mathbb{H}^n)$ *and* $\mathrm{Isom}^+(\mathbb{H}^n)$ *(the group of orientation preserving isometries) act transitively on $\partial\mathbb{H}^n$ and the unit tangent bundle*

$$T^1\mathbb{H}^n = \{(x, v) : x \in \mathbb{H}^n, v \in T_x\mathbb{H}^n, \|v\|_x = 1\}$$

(iii) $\mathrm{Isom}(\mathbb{H}^n)$ *and* $\mathrm{Isom}^+(\mathbb{H}^n)$ *act transitively on the set of triples of distinct boundary points* $(\partial\mathbb{H}^n)^{(3)}$ *via the diagonal action, i.e.* $\mathrm{Isom}(\mathbb{H}^n)^{(+)}$ *acts* 3-transitively *on the boundary.*

(iv) Every isometry of $\mathbb{H}^n$ is uniquely determined by its action on $\partial\mathbb{H}^n$.

(v) If M is either the ball model B^n or the upper half space model U^n the restriction to the boundary is an isomorphism of $\mathrm{Isom}(M)$ *onto* $\mathrm{Conf}(\partial M)$ *(the group of conformal diffeomorphisms). In particular* $\mathrm{Isom}(M)$ *acts via diffeomorphisms on ∂M.*

Proof. This is [BP92, Proposition A.5.13, p. 31] except of (iii). However (iii) is obvious when we pass to the upper half space model U^n. Indeed, if we have a triple $(x, y, z) \in (\partial U^n)^{(3)}$ we may without loss of generality assume that $x = \infty$, since $\mathrm{Isom}^{(+)}(\mathbb{H}^n)$ acts transitively on $\partial\mathbb{H}^n$. Now the isometries fixing ∞ are just all euclidean similarities $S(\mathbb{R}^{n-1})$ and it is obvious that these act transitively on the set of pairs of distinct points. □

There are three different types of isometries depending on their fixed points. Recall that due to Brouwer's fixed point theorem every isometry has to fix at least one point of $\overline{\mathbb{H}}^n \cong \overline{B}^n$.

Proposition I.2.8. *If $\varphi \in \mathrm{Isom}(\mathbb{H}^n)$ the following mutually excluding possibilities are given:*

(i) φ has some fixed point in $\mathbb{H}^n$; in this case φ is called elliptic

(ii) φ has no fixed points in $\mathbb{H}^n$ and exactly on fixed point in $\partial\mathbb{H}^n$; in this case φ is called parabolic

(iii) φ has no fixed points in $\mathbb{H}^n$ and exactly two fixed points in $\partial\mathbb{H}^n$; in this case φ is called hyperbolic

Proof. See [BP92, Proposition A.5.14, p. 31]. □

Remark I.2.9. *Sometimes in the literature hyperbolic isometries are also called loxodromic.*

Because $G = \mathrm{Isom}(\mathbb{H}^n)$ acts transitively on $\mathbb{H}^n$ the mapping $p_x : G \to \mathbb{H}^n, g \mapsto gx$ is surjective for every $x \in \mathbb{H}^n$. It is not hard to see that p_x induces a diffeomorphism $\bar{p}_x : G/K \to \mathbb{H}^n, gK \mapsto gx$, where $K = G_x$, allowing us to identify $\mathbb{H}^n \cong G/K$. Note that any two stabilizers are conjugate, such that $K = G_x$ is conjugate to $O(n) = G_0$ and hence compact. In fact it is even a maximal compact subgroup of G as we will show in the next lemma.

Remark I.2.10. *The above statement is also true for the group of orientation preserving isometries $G^+ = \mathrm{Isom}^+(\mathbb{H}^n)$ and one may identify $G^+/K \cong \mathbb{H}^n$ where $K = G^+_x$ is the stabilizer of some point $x \in \mathbb{H}^n$.*

Definition I.2.11. Let G be a topological group and $K < G$ a subgroup. K is called a *maximal compact subgroup* if K is compact and for every other compact subgroup $H < G$ such that $H \subseteq K$, we have that $K = H$.

Lemma I.2.12. *Let $x \in \mathbb{H}^n$ and $K = G_x^{(+)}$. Then $K < G^{(+)} = \mathrm{Isom}^{(+)}(\mathbb{H}^n)$ is a maximal compact subgroup.*

Proof. Let us first consider the case of arbitrary isometries $G = \mathrm{Isom}(\mathbb{H}^n)$. By conjugation we may assume without loss of generality, that $x = 0$ in the ball model B^n. Hence $K = O(n)$. Now let $H < G = \mathrm{Isom}(B^n)$ be another compact subgroup containing K.

We claim that H fixes a point $y \in \mathbb{H}^n$. For that let ν be a Haar measure on H. Since H is compact ν is finite. We now pass to the hyperboloid model and set

$$\hat{y} = \int_H h \cdot o \, d\nu(h) \in \mathbb{R}^{n,1}$$

where o is some point in $H^n \subset \mathbb{R}^{n,1}$ and the elements of $H \subset O(H^n)$ act via linear transformation on $\mathbb{R}^{n,1}$. In general $\hat{y}$ is not in H^n anymore, but still a time-like vector (cf. [Rat06, §3.1, p. 54]). Hence by normalizing we get

$$y := \frac{\hat{y}}{|||y|||} \in H^n$$

where $|||z||| = \sqrt{-\langle z|z\rangle_{n,1}}$ for every time-like vector $z \in \mathbb{R}^{n,1}$. It is easy to check that $y \in H^n$ is indeed H-invariant, i.e. a fixed point of the H-action.

Back in the ball model B^n, if y is a fixed point of H, then it is also a fixed point of $O(n) = K \subset H$. $O(n)$ fixes no other point but 0, whence $y = 0$. This in turn implies that $H \subset G_0 = O(n) = K$, i.e. $H = K$.

Our proof also works in the case of orientation preserving isometries $G^+ = \mathrm{Isom}^+(\mathbb{H}^n)$, if one replaces $O(n)$ by $SO(n)$ etc. □

So far we have described $\mathbb{H}^n$ as a quotient of G by a maximal compact subgroup. It is also possible to – in a sense – reverse this point of view, and identify $G \cong B^n \times O(n)$ in the ball model B^n. Here the key ingredient are the so called hyperbolic translations.

Definition I.2.13 (hyperbolic translation)**.** Let $b \in B^n$. Then there is a unique isometry τ_b taking 0 to b whose jacobian is a positive multiple of the identity; in particular $\tau_b \in G^+$. Considering the reflection ρ_b at the hyperplane $\langle b\rangle^\perp$ and the unique inversion i_b at a sphere perpendicular to S^n and the line through 0 and b taking 0 to b gives the formula $\tau_b = i_b \circ \rho_b$, or explicitly

$$\tau_b(x) = \frac{(1-|b|^2)x + (|x|^2 + 2\langle x, b\rangle + 1)b}{|b|^2|x|^2 + 2\langle x, b\rangle + 1}$$

for every $x \in B^n$ (cf. [Rat06, (4.5.5), p. 124] and [BP92, p. 135]).

τ_b is called the *hyperbolic translation by b.*

Theorem I.2.14. *The map $B^n \times O(n) \to G, (b, A) \mapsto \tau_b A$ is a diffeomorphism.*

Proof. All that needs to be said is that the map $\eta : B^n \to G, b \mapsto \tau_b$ is a smooth global section of the bundle map $\pi : G \to G/K \cong B^n$. Then we get two maps $\Phi : B^n \times O(n) \to G, (b, A) \mapsto \tau_b A$ and $\Psi : G \to B^n \times O(n), g \mapsto (\pi(g), \eta(\pi(g))^{-1}g)$ which are smooth and inverse to each other, i.e. they are diffeomorphisms. We leave out the easy verifactions.

Compare also [BP92, p. 136] and [Rat06, Theorem 5.2.8, p. 154]. □

We can also identify the boundary $\partial\mathbb{H}^n$ as a homogeneous space. If $\xi \in \partial\mathbb{H}^n$, then the map $p_\xi : G = \mathrm{Isom}(\mathbb{H}^n) \to \partial\mathbb{H}^n, g \mapsto g\xi$ is a surjective because G acts transitively on $\partial\mathbb{H}^n$. It is easy to deduce that p_ξ induces therefore a diffeomorphism $\bar{p}_\xi : G/P \to \partial\mathbb{H}^n, gP \mapsto g\xi$, where $P = G_\xi$ is the stabilizer of $\xi \in \partial\mathbb{H}^n$, allowing us to identify $G/P \cong \partial\mathbb{H}^n$. This time the stabilizer is not compact, but still amenable (cf. Definition C.1.1).

Lemma I.2.15. *Let $\xi \in \partial\mathbb{H}^n$ and $P = G_\xi$ its stabilizer. Then P is the compact extension of a solvable group. Hence P is amenable (cf. Definition C.1.1).*

Proof. We pass to the upper half space model U^n and assume without loss of generality that $\xi = \infty$. Hence P can be identified with the group of euclidean similarities $S(\mathbb{R}^{n-1})$. Every euclidean similarity can be uniquely written as

$$p(x) = \lambda Ax + b \qquad \forall x \in \mathbb{R}^{n-1}$$

where $\lambda > 0$, $A \in O(n)$ and $b \in \mathbb{R}^{n-1}$. Clearly the map

$$f : P \to O(n), (p : x \mapsto \lambda Ax + b) \mapsto A$$

is a surjective group homomorphism. Hence we get the short exact sequence

$$1 \to \ker f \to P \to O(n) \to 1$$

Because $O(n)$ is compact it will suffice to show, that $\ker f$ is solvable. Now every $p \in \ker f$ is of the form

$$p(x) = \lambda x + b \qquad \forall x \in \mathbb{R}^{n-1}$$

where $\lambda > 0$ and $b \in \mathbb{R}^{n-1}$. Let $p_i : x \mapsto \lambda_i x + b_i$ $(i = 1, 2)$ be two elements of $\ker f$. Then for every $x \in \mathbb{R}^{n-1}$

$$\begin{aligned}[p_1, p_2](x) &= p_1 p_2 p_1^{-1} p_2^{-1}(x) = p_1 p_2 p_1^{-1}(\lambda_2^{-1} x - \lambda_2^{-1} b_2) = p_1 p_2(\lambda_1^{-1}\lambda_2^{-1} x - \lambda_1^{-1}\lambda_2^{-1} b_2 - \lambda_1^{-1} b_1) \\ &= p_1(\lambda_1^{-1} x - \lambda_1^{-1} b_2 - \lambda_2 \lambda_1^{-1} b_1 + b_2) = x + (1 - \lambda_2) b_2 + (\lambda_1 - 1) b_1\end{aligned}$$

Thus $(\ker f)^{(1)} \leq (\mathbb{R}^{n-1}, +)$ which is abelian, i.e. $(\ker f)^{(2)} = \{0\}$.

As we have seen P is a compact extension of a solvable group. Since compact groups and solvable groups are amenable so is P (cf. Proposition C.1.5). $\square$

We will now construct an explicit section of the map $p : G \to \partial\mathbb{H}^n \cong G/P, g \mapsto g\xi_0$, for some fixed boundary point $\xi_0 \in \partial\mathbb{H}^n$, i.e. a map $\eta : \partial\mathbb{H}^n \to G$ such that $p(\eta(\xi)) = \xi$ for all $\xi \in \partial\mathbb{H}^n$. We define $\eta : \partial\mathbb{H}^n \to G$ via

$$\eta(\xi) = \begin{cases} \mathrm{id}, & \text{if } \xi = \xi_0 \\ \rho_\xi, & \text{else} \end{cases}, \qquad \forall \xi \in \partial\mathbb{H}^n$$

where $\rho_\xi \in \mathrm{Isom}(\mathbb{H}^n)$ is the unique euclidean reflection at a hyperplane through 0 in the Poincaré ball model B^n, that takes ξ_0 to ξ. We can give an explicit formula for $\rho_\xi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\rho_\xi(x) = x - 2\frac{\langle x, \xi_0 - \xi\rangle}{|\xi_0 - \xi|^2} \cdot (\xi_0 - \xi), \qquad \forall x \in \mathbb{R}^n$$

We now get the following lemma.

Lemma I.2.16. *The above mapping $\eta : \partial\mathbb{H}^n \to G$ is (Borel) measurable, fulfills $p(\eta(\xi)) = \xi$ for all $\xi \in \partial\mathbb{H}^n$, i.e. it is a section, and its image $\eta(G/P)$ is relatively compact in G, i.e. $\overline{\eta(G/P)} \subset G$ is compact.*

Proof. It is immediate, that $\eta : \partial\mathbb{H}^n \cong G/P \to G$ is continuous on $\partial\mathbb{H}^n - \{\xi_0\}$. In particular it is (Borel) measurable on all of $\partial\mathbb{H}^n \cong G/P$.

We check that $\eta : \partial\mathbb{H}^n \to G$ is indeed a section. First $p(\eta(\xi_0)) = p(\mathrm{id}) = \mathrm{id}(\xi_0) = \xi_0$. Second

$$p(\eta(\xi)) = p(\rho_\xi) = \rho_\xi(\xi_0) = \xi$$

for every $\xi \in \partial\mathbb{H}^n - \{\xi_0\}$ by construction.

Further its image $\eta(G/P)$ is contained in $O(n)$, since id as well as every ρ_ξ fix the origin $0 \in B^n$ for every $\xi \in \partial B^n$. Because $O(n)$ is compact and $\overline{\eta(G/P)} \subset O(n)$ is closed by definition, the latter is also compact. $\square$

It is important to note, that $G = \mathrm{Isom}(\mathbb{H}^n)$ is unimodular (cf. Definition A.4.5).

Proposition I.2.17. $\mathrm{Isom}(\mathbb{H}^n)$ *is a unimodular Lie group.*

Proof. See [BP92, Proposition C.4.11, p. 111].

Alternatively we shall see later, that $\mathrm{Isom}(\mathbb{H}^n)$ contains lattices from which it also easily follows, that it is unimodular (see for example [Bou04b, Corollary 3, VII.44 §2]). $\square$

I.3. Elementary Groups

There are particularly simple subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ called elementary groups. We want to recall some of their characteristic properties here as they will play a role in the proof of the volume rigidity theorem later on.

Our main reference for this section is [Rat06, §5.5].

Definition I.3.1 (Elementary Group)**.** A subgroup G of $\mathrm{Isom}(\mathbb{H}^n)$ is *elementary* if and only if G has a finite orbit in $\overline{\mathbb{H}}^n$.

We shall divide the elementary subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ into three types. Let G be an elementary subgroup of $\mathrm{Isom}(\mathbb{H}^n)$.

(i) The group G is said to be of *elliptic type* if and only if G has a finite orbit in $\mathbb{H}^n$.

(ii) The group G is said to be of *parabolic type* if and only if G fixes a boundary point in $\partial\mathbb{H}^n$ and has no other finite orbits in $\overline{\mathbb{H}}^n$.

(iii) The group G is said to be of *hyperbolic type* if and only if G is neither of elliptic type nor of parabolic type.

Remark I.3.2. *It is plain to see, that the type of an elementary group depends only on its conjugacy class within* $\mathrm{Isom}(\mathbb{H}^n)$*.*

Elementary groups of elliptic type are characterized by the following theorem.

Theorem I.3.3. *Let G be an elementary subgroup of* $\mathrm{Isom}(\mathbb{H}^n)$*. Then the following are equivalent:*

(i) The group G is of elliptic type.

(ii) The group G fixes a point in $\mathbb{H}^n$.

(iii) The group G is conjugate in $\mathrm{Isom}(B^n)$ *to a subgroup of $O(n)$.*

Proof. See [Rat06, Theorem 5.5.1, p. 177]. ☐

Elementary groups of parabolic type are characterized by the following theorem.

Theorem I.3.4. *Let G be an elementary subgroup of* $\mathrm{Isom}(\mathbb{H}^n)$*. Then the following are equivalent:*

(i) The group G is of parabolic type.

(ii) The group G has a unique fixed point in $\partial\mathbb{H}^n$.

(iii) The group G is conjugate in $\mathrm{Isom}(U^n)$ *to a subgroup of $S(\mathbb{R}^{n-1})$ (the group of euclidean similarities) that fixes no point of $\mathbb{R}^{n-1}$.*

Proof. See [Rat06, Theorem 5.5.3, p. 178]. ☐

Elementary groups of hyperbolic type are characterized by the following theorem.

Theorem I.3.5. *Let G be an elementary subgroup of* $\mathrm{Isom}(\mathbb{H}^n)$*. Then the following are equivalent:*

(i) The group G is of hyperbolic type.

(ii) The union of all the finite orbits of G in $\overline{U}^n$ consists of two points in $\mathbb{R}^{n-1} \cup \{\infty\}$.

(iii) The group G is conjugate in $\mathrm{Isom}(U^n)$ *to a subgroup of $S(\mathbb{R}^{n-1})_*$ that fixes no point of the positive n-th axis.*

Here $S(\mathbb{R}^{n-1})_*$ *is the subgroup of* $\mathrm{Isom}(U^n)$ *that fixes the set* $\{0, \infty\}$.

Proof. See [Rat06, Theorem 5.5.6, p. 179]. $\square$

There is also an interesting relation between limit sets and elementary subgroups.

Definition I.3.6. A point $a \in \partial\mathbb{H}^n$ is a *limit point* of a subgroup G of $\mathrm{Isom}(\mathbb{H}^n)$ if there is a point $x \in \mathbb{H}^n$ and a sequence $\{g_i\}_{i\in\mathbb{N}}$ of elements in G such that$\{g_i x\}_{i\in\mathbb{N}}$ converges to a. The *limit set* of G is the set $L(G)$ of all limit points of G.

There is also the following useful characterization of the limit set.

Theorem I.3.7. *If* G *is a subgroup of* $\mathrm{Isom}(\mathbb{H}^n)$*, then for each point* $x \in \mathbb{H}^n$*, we have* $L(G) = \overline{Gx} \cap \partial\mathbb{H}^n$.

Proof. This is [Rat06, Theorem 12.1.2, p. 601]. $\square$

Theorem I.3.8. *Let* G *be a subgroup of* $\mathrm{Isom}(\mathbb{H}^n)$*. Then* $L(G)$ *is empty if and only if* G *is elementary of elliptic type.*

Proof. See [Rat06, Theorem 12.1.4, p. 602]. $\square$

Proposition I.3.9. *If* $G < \mathrm{Isom}(\mathbb{H}^n)$ *is non-elementary then* $L(G)$ *is infinite and the fixed points of hyperbolic elements of* G *are dense in* $L(G)$.

Proof. This is [Kap09, Corollary 3.26, p. 42]. $\square$

I.4. Hyperbolic Manifolds and Lattices

In this section we will introduce lattice subgroups and see how they relate to finite volume hyperbolic manifolds. In the following we will only be concerned with complete and connected hyperbolic manifolds, whence the following definition.

Definition I.4.1. A *hyperbolic manifold* M is a complete connected Riemannian manifold of constant sectional curvature $K = -1$.

Because $\mathbb{H}^n$ is the unique complete simply connected Riemannian manifold of constant sectional curvature $K = -1$, basic covering theory yields that the universal cover of M is $\mathbb{H}^n$ and we get a covering map $\pi : \mathbb{H}^n \to M$. If we set $\Gamma = \mathrm{Deck}(\pi) < \mathrm{Isom}(\mathbb{H}^n)$, we can identify $M \cong \Gamma\backslash\mathbb{H}^n$ (cf. [dC92, Chap. 8, 4. Space forms, pp. 162]).

One can reverse the above process starting with a subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ and construct a hyperbolic manifold as the quotient $\Gamma\backslash\mathbb{H}^n$, such that the quotient map $\pi : \mathbb{H}^n \to \Gamma\backslash\mathbb{H}^n$ is a covering. Of course this does not work for any arbitrary subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$. A necessary and sufficient condition is given by the following proposition, which works for even more general topological spaces than $\mathbb{H}^n$.

Proposition I.4.2. *Let X be a connected locally compact (Hausdorff [1]) topological space, and let Γ be a group of homeomorphisms of X. Then the following are equivalent.*

(i) Γ *acts freely and properly discontinuously on X.*

(ii) $\Gamma\backslash X$ *is a Hausdorff space and the quotient projection $\pi : X \to \Gamma\backslash X$ is a covering.*

Proof. This is [BP92, Proposition B.1.6, p. 49]. □

Recall that a group Γ is said to act ...

...*freely* on a topological space X if $\gamma \in \Gamma$, $x \in X$ and $\gamma(x) = x$ implies $\gamma = \mathrm{id}$.

...*properly discontinuously* on a topological space X if for every $K \subset X$ compact the number of $\gamma \in \Gamma$ such that $\gamma K \cap K \neq \emptyset$ is finite. Note that if Γ is discrete, it acts properly discontinuously if and only if it acts properly.

However we would like to have a set of more intrinsic and group theoretic conditions for which $\pi : \mathbb{H}^n \to \Gamma\backslash\mathbb{H}^n$ is a covering of hyperbolic manifolds. These are given by the following two propositions.

Proposition I.4.3. *A subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ acts properly discontinuously on $\mathbb{H}^n$ if and only if Γ is discrete.*

Proof. See [Rat06, Theorem 5.3.5, p. 164]. □

Proposition I.4.4. *A discrete subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ acts freely on $\mathbb{H}^n$ if and only if Γ is torsion-free.*

Proof. See [Rat06, Theorem 8.2.1, p. 341]. □

If we drop the assumption of Γ being torsion-free, i.e. Γ does not act freely on $\mathbb{H}^n$, we will not get a smooth manifold anymore. However the quotient $\Gamma\backslash\mathbb{H}^n$ is still Hausdorff and carries what is called a $(\mathbb{H}^n, \mathrm{Isom}(\mathbb{H}^n))$-orbifold structure (cf. [Rat06, Example 1, p. 692]). Such spaces are called *(complete) hyperbolic n-orbifolds.*

[1] We adopt Bourbaki's definition of a locally compact topological space, in which it is always assumed to be Hausdorff (cf. [Bou89, Definition 4,p. 90]).

We are now ready to introduce the notion of hyperbolic lattices. Let us therefore consider the classical case of euclidean geometry first. In $\mathbb{R}^n$ a discrete subgroup $\Lambda < \mathbb{R}^n$ generated by n-linearly independent vectors $v_1, \ldots, v_n \in \mathbb{R}^n$ is called a lattice. The volume of a fundamental set for the action Λ on $\mathbb{R}^n$ by translation is $\det(v_1, \ldots, v_n)$ and hence finite. In other words the quotient measure λ/μ_Γ is finite, where λ is the Lebesgue measure on $\mathbb{R}^n$ and μ_Γ is the normalized Haar measure on Γ (cf. Theorem A.4.20); for further details on measure theoretic constructions we refer to appendix A. We take this as an inspiration for our definition of a (hyperbolic) lattice.

From now on we abbreviate $G := \mathrm{Isom}(\mathbb{H}^n)$.

Definition I.4.5 (Lattice)**.** Let $\Gamma < G$ be a discrete subgroup, let ν be the hyperbolic volume measure on $\mathbb{H}^n$ (cf. Example A.3.1) and μ_Γ the normalized Haar measure on Γ (i.e. the counting measure). Γ is called a *lattice*, if the quotient measure $\mu := \nu/\mu_\Gamma$ is finite, i.e. if $\mu(\Gamma\backslash\mathbb{H}^n) < \infty$.

Remark I.4.6. *Note that $\nu \in \mathcal{M}(\mathbb{H}^n)$ is invariant under G, in particular under Γ, and Γ is unimodular (cf. Proposition A.4.8), such that the quotient measure ν/μ_Γ exists (cf. Proposition A.4.9 and Remark A.4.18).*

However one may also have a different look at the previously described euclidean situation. $\mathbb{R}^n$ is a Lie group and $\Lambda < \mathbb{R}^n$ a discrete (hence closed) subgroup. The quotient $\Lambda\backslash\mathbb{R}^n$ is therefore a smooth manifold and in this particular case a flat torus. The flat structure induces a Riemannian volume form on the torus, which in turn induces a measure. The right action of $\mathbb{R}^n$ on itself descends to a right action of $\mathbb{R}^n$ on $\Lambda\backslash\mathbb{R}^n$ by isometries. In summary $\Lambda\backslash\mathbb{R}^n$ admits a finite $\mathbb{R}^n$-invariant measure. Thus one may define for an arbitrary Lie group G and a discrete subgroup $\Gamma < G$, that Γ is a lattice if the quotient $\Gamma\backslash G$ admits a finite G-invariant measure (cf. Remark I.4.7 below). This is exactly the definition, that occurs in texts concerning more general Lie groups. However this is not really a different definition and we will soon see, that both definitions coincide in our case. Intuitively speaking the reason for this is, that $\mathbb{H}^n$ is the quotient of $G = \mathrm{Isom}(\mathbb{H}^n)$ by a compact subgroup.

Remark I.4.7. *Note that, if G is an arbitrary Lie group and $\Gamma < G$ a discrete (and hence closed) subgroup, then Γ acts on G freely and properly discontinuously via left translation. Indeed, the action is clearly free. The action is proper since for any two sequences $(\gamma_n) \subset \Gamma$, $(g_n) \subset G$ such that $\gamma_n g_n \to h$ and $g_n \to g$ as n tends to infinity, we get $\gamma_n = (\gamma_n g_n) g_n^{-1} \to hg^{-1} \in \Gamma$ as $n \to \infty$.*

Therefore the canonical quotient map $\pi : G \to \Gamma\backslash G$ is a covering map and the quotient $\Gamma\backslash G$ admits a unique smooth structure such that π is smooth. Also the right action of G on itself via right translation descends to a smooth right action on $\Gamma\backslash G$.

The next proposition is a very useful result from general topology, that will be needed to justify some constructions in the following (e.g. lifting measures, taking quotient measures). Basically it asserts that all sorts of maps corresponding to a continuous group action by a compact group are proper.

Proposition I.4.8 (Compact groups act properly)**.** *Let K be a compact group operating continuously on a Hausdorff space X. Then:*

(i) K operates properly on X.

(ii) The mapping $K \times X \to X, (k, x) \mapsto kx$ is proper.

(iii) The canonical quotient map $X \to K\backslash X$ is proper.

Proof. This is [Bou89, Proposition 2, III §4.2, p. 252]. □

Remark I.4.9. *There is nothing special about left actions, such that Proposition I.4.8 also holds for right actions. In fact we will mainly use it for right actions in the following.*

In an intermediate step we want to see, that a discrete subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^n) = G$ is a lattice if and only if the quotient measure μ_G/μ_Γ is finite. We will need the following lemma.

Lemma I.4.10. *Let $\Gamma < \mathrm{Isom}(\mathbb{H}^n) = G$ be a discrete subgroup, and $\nu \in \mathcal{M}(\mathbb{H}^n) = \mathcal{M}(G/K)$ the hyperbolic volume measure. Then the lifted measure $\nu^\sharp \in \mathcal{M}(G)$ corresponding to ν via $\pi : G \to G/K \cong \mathbb{H}^n$ is a Haar measure on G.*

Proof. First, note that the lifted measure $\nu^\sharp$ exists, since K is compact and hence acts properly on G (cf. Proposition I.4.8).

Let $\mu_K \in \mathcal{M}(K)$ be the normalized Haar measure on K. Then:

$$\begin{aligned}\int_G f(g)\, d\nu^\sharp(g) &= \int_{G/K}\int_K f(gk)\, d\mu_K(k)\, d\nu(gK)\\ &= \int_{G/K}\int_K f(g'gk)\, d\mu_K(k)\, d\nu(g'gK)\\ &= \int_{G/K}\int_K f(g'gk)\, d\mu_K(k)\, d\nu(gK)\\ &= \int_G f(g'g)\, d\nu^\sharp(g)\end{aligned}$$

for every $f \in C_c(G)$ and every $g' \in G$. Hence $\nu^\sharp$ is invariant and therefore a Haar measure on G. $\square$

Proposition I.4.11. *Let $\Gamma < G$ be a discrete subgroup. Further let ν be the hyperbolic volume measure on $\mathbb{H}^n$, μ_G a Haar measure on G and μ_Γ the normalized Haar measure on Γ. Choose $x \in \mathbb{H}^n$ and set $K = G_x$ as usual.*

Then there is $\alpha > 0$ such that

$$\frac{\mu_G}{\mu_\Gamma} = \alpha \cdot \bar{\nu}$$

where $\bar{\nu} \in \mathcal{M}(\Gamma\backslash G)$ is the lift of ν/μ_Γ along the quotient map $p : \Gamma\backslash G \to \Gamma\backslash G/K = M$.

In particular ν/μ_Γ is finite if and only if μ_G/μ_Γ is finite, and $A \subset M$ is a null set if and only if $p^{-1}(A) \subset \Gamma\backslash G$ is a null set.

Proof. Again, the lift $\bar{\nu}$ exists due to Proposition I.4.8. Let $\mu_K \in \mathcal{M}(K)$ be the normalized Haar measure on K. We may lift the measure $\bar{\nu} \in \mathcal{M}(\Gamma\backslash G)$ one more time to a measure $\bar{\nu}^\sharp \in \mathcal{M}(G)$. Recall that the unique lift of ν/μ_Γ to $G/K = \mathbb{H}^n$ is $\nu \in \mathcal{M}(G/K)$ and that the lift $\nu^\sharp$ of ν to G is a Haar measure on G. First we shall see that $\nu^\sharp = \bar{\nu}^\sharp \in \mathcal{M}(G)$. For that let $f \in C_c(G)$. Then

$$\begin{aligned}\int_G f(g)\, d\nu^\sharp(g) &= \int_{G/K}\int_K f(gk)\, d\mu_K(k)\, d\nu(gK)\\ &= \int_{\Gamma\backslash G/K}\int_\Gamma\int_K f(\gamma gk)\, d\mu_K(k)\, d\mu_\Gamma(\gamma)\, d(\nu/\mu_\Gamma)(\Gamma gK)\\ &= \int_{\Gamma\backslash G/K}\int_K\int_\Gamma f(\gamma gk)\, d\mu_\Gamma(\gamma)\, d\mu_K(k)\, d(\nu/\mu_\Gamma)(\Gamma gK)\\ &= \int_{\Gamma\backslash G}\int_\Gamma f(\gamma g)\, d\mu_\Gamma(\gamma)\, d\bar{\nu}(\Gamma g)\\ &= \int_G f(g)\, d\bar{\nu}^\sharp(g)\end{aligned}$$

where we have used Fubini's theorem in the third line. This proves that $\nu^\sharp = \bar{\nu}^\sharp$. Because $\nu^\sharp$ is a Haar measure on G there is a positive real number $\alpha > 0$ such that $\alpha \cdot \nu^\sharp = \mu_G$. Therefore

$$\frac{\mu_G}{\mu_\Gamma} = \frac{\alpha \cdot \nu^\sharp}{\mu_\Gamma} = \alpha \cdot \frac{\bar{\nu}^\sharp}{\mu_\Gamma} = \alpha \cdot \bar{\nu}$$

by Lemma A.4.11.

By Proposition A.4.12 we get

$$\begin{aligned}
\frac{\mu_G}{\mu_\Gamma}(\Gamma\backslash G) &= \int_{\Gamma\backslash G} 1\, d(\mu_G/\mu_\Gamma)(\Gamma g) \\
&= \alpha \cdot \int_{\Gamma\backslash G} 1\, d\bar{\nu}(\Gamma g) \\
&= \alpha \cdot \int_{\Gamma\backslash G/K} \int_K 1\, d\mu_K(k)\, d\nu(\Gamma g K) \\
&= \alpha \cdot \underbrace{\mu_K(K)}_{=1} \cdot \nu(\Gamma\backslash G/K) \\
&= \alpha \cdot \nu(\Gamma\backslash G/K)
\end{aligned}$$

and thus ν/μ_Γ is finite if and only if μ_G/μ_Γ is finite.

By Proposition A.4.13 a subset $A \subset M$ is a null set if and only if $p^{-1}(A) \subset \Gamma\backslash G$. □

It is now easy to deduce that both of the proposed definitions for a lattice subgroup in $G = \mathrm{Isom}(\mathbb{H}^n)$ coincide.

Corollary I.4.12. *Let $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ be a discrete subgroup. Then Γ is a lattice if and only if the quotient $\Gamma\backslash G$ admits a finite G-invariant measure.*

Proof. First assume that $\Gamma < G$ is a lattice. Then for any Haar measure μ_G on G the quotient measure μ_G/μ_Γ is finite. It is easy to see, that μ_G/μ_Γ is G-invariant, since G is unimodular, such that μ_G/μ_Γ is indeed a finite invariant measure on $\Gamma\backslash G$.

Conversely if μ is a finite G-invariant measure on $\Gamma\backslash G$ we can show by the same method as in Lemma I.4.10 that $\mu^\sharp$ is a Haar measure on G such that $\mu^\sharp/\mu_\Gamma = \mu$. Therefore Γ is a lattice by Proposition I.4.11. □

Our next objective is to make sense of the quotient measure ν/μ_Γ geometrically. It will turn out, that this measure can be realized as the integral of ν over a fundamental region for Γ in $\mathbb{H}^n$ (cf. Theorem A.4.20).

Definition I.4.13 (fundamental region)**.** A subset R of $\mathbb{H}^n$ is called a *fundamental region* for a group $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ if and only if

(i) the set R is open in $\mathbb{H}^n$;

(ii) the members of $\{gR : g \in \Gamma\}$ are mutually disjoint; and

(iii) $\mathbb{H}^n = \cup\{g\overline{R} : g \in \Gamma\}$.

There is a nice relation between fundamental sets (cf. Definition A.4.19) and fundamental regions in $\mathbb{H}^n$.

Theorem I.4.14. *An open subset $R \subset \mathbb{H}^n$ is a fundamental region for a group $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ if and only if there is a fundamental set F for Γ such that $R \subset F \subset \overline{R}$.*

Proof. This is [Rat06, Theorem 6.6.11, p. 242]. □

Definition I.4.15 (proper fundamental region)**.** A fundamental region R for a group $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ is called *proper* if and only if $\mathrm{vol}(\partial R) = 0$, that is ∂R is a null set in $\mathbb{H}^n$.

It follows immediately from the definition and Theorem I.4.14, that for a fundamental region R the fundamental set $R \subset F \subset \overline{R}$ is measurable as it is equal to R up to a null set.

Thus we get by Theorem A.4.20 the following:

Proposition I.4.16. *Let R be a proper fundamental region for $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ and $k : \Gamma\backslash\mathbb{H}^n \to \mathbb{R}$ a function. Denote by $\pi : \mathbb{H}^n \to \Gamma\backslash\mathbb{H}^n$ the canonical quotient map and we set $\lambda = \nu/\mu_\Gamma$ where ν denotes the hyperbolic volume measure on $\mathbb{H}^n$ and μ_Γ the normalized Haar measure on Γ.*

Then for k to be measurable (resp. λ-integrable) it is necessary and sufficient that $\chi_R \cdot (k \circ \pi)$ be measurable (resp. ν-integrable); and if k is λ-integrable then

$$\int_{\Gamma\backslash\mathbb{H}^n} k \, d\lambda = \int_R (k \circ \pi) \, d\mu$$

Proof. The necessary and sufficient conditions for k to be measurable (resp. λ-integrable) follow at once from Theorem A.4.20 and our observation that there is a fundamental set $R \subset F \subset \overline{R}$ which differs from R by a null set; recall that $\mathbb{H}^n$ is a complete measure space.

As for the formula we may replace F by R since they differ by a null set. In view of Theorem A.4.20 we are now left to prove that $n(x) = |\Gamma_x| = 1$ for every $x \in R$. For that let $x \in R$ and $\gamma \in \Gamma_x$. Then by definition $\gamma x = x$ such that $x \in \gamma R \cap R$, which can only be the case if $\gamma = \mathrm{id}$ by definition of a fundamental region for Γ. Hence $\Gamma_x = \{\mathrm{id}\}$ for every $x \in R$ and the assertion follows. □

Because of the above proposition we will sometimes write vol instead of ν/μ_Γ.

Although we have now characterized the quotient measure on $\Gamma\backslash\mathbb{H}^n$ quite nicely by means of proper fundamental regions, we still do not know, whether such regions exist. In order to see that we need the following theorem.

Theorem I.4.17. *Let $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ be discrete. Then there is a point x of $\mathbb{H}^n$ whose stabilizer Γ_x is trivial.*

Proof. This follows from [Rat06, Theorem 6.6.12, p. 243] and the fact that $\mathbb{H}^n$ is a rigid metric space. □

We will now encounter a particularly nice kind of fundamental region; the so called Dirichlet domain.

Definition I.4.18 (Dirichlet domain)**.** Let $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ be discrete and let $a \in \mathbb{H}^n$ be a point whose stabilizer Γ_a is trivial. For each $\gamma \neq \mathrm{id}$ in Γ define

$$H_\gamma(a) = \{x \in \mathbb{H}^n : d(x, a) < d(x, \gamma a)\}$$

Note that the set $H_\gamma(a)$ is an open (and convex) half-space of $\mathbb{H}^n$ containing the point a whose boundary is the perpendicular bisector of every geodesic segment joining a to γa.

The *Dirichlet domain* $D(a)$ for Γ, with center a, is either $\mathbb{H}^n$ if Γ is trivial or

$$D(a) = \bigcap \{H_\gamma(a) : \gamma \in \Gamma - \{\mathrm{id}\}\}$$

if Γ is non-trivial.

Proposition I.4.19. *Let $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ be discrete. Then every Dirichlet domain for Γ is a proper fundamental region for Γ.*

Proof. This follows from [Rat06, Corollary 1, p. 247] and [Rat06, Theorem 6.6.13, p. 244]. However it is also not hard to deduce this from Theorem I.4.14 and the fact, that the sets $\{x \in \mathbb{H}^n : d(x,a) = d(x,\gamma a)\}$ are null sets in $\mathbb{H}^n$ for every $\gamma \in \Gamma - \{\mathrm{id}\}$ and a the centre of a Dirichlet domain for Γ. $\square$

Theorem I.4.20. *Let $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ be discrete and $\Gamma' < \Gamma$ a subgroup. Then*

$$\mathrm{vol}(\Gamma'\backslash\mathbb{H}^n) = |\Gamma' : \Gamma| \cdot \mathrm{vol}(\Gamma\backslash\mathbb{H}^n)$$

Proof. This is [Rat06, Theorem 6.7.3, p. 248]. The idea is to consider a Dirichlet domain D for Γ and set

$$R = \bigcup\{g_i D : i \in I\}$$

where $\{g_i : i \in I\}$ is a system of representatives for the coset space $\Gamma'\backslash\Gamma$. One may now show, that R is a proper fundamental region for Γ'. Thus by Proposition I.4.16 we have

$$\mathrm{vol}(\Gamma'\backslash\mathbb{H}^n) = \mathrm{vol}(R) = \sum_i \mathrm{vol}(g_i D) = [\Gamma' : \Gamma] \cdot \mathrm{vol}(D) = [\Gamma' : \Gamma] \cdot \mathrm{vol}(\Gamma\backslash\mathbb{H}^n)$$

where we have also written vol for the hyperbolic volume measure on $\mathbb{H}^n$. $\square$

Now if $\Gamma < G$ is not only discrete but also torsion-free, Γ acts freely and properly discontinuously on $\mathbb{H}^n$ (cf. Proposition I.4.4). In particular $\pi : \mathbb{H}^n \to \Gamma\backslash\mathbb{H}^n$ is a covering and $M := \Gamma\backslash\mathbb{H}^n$ admits a (unique) smooth manifold structure such that π is smooth. If additionally $\Gamma < G^+$, i.e. Γ consists only of orientation preserving isometries, then also the quotient $M = \Gamma\backslash\mathbb{H}^n$ inherits an orientation from $\mathbb{H}^n$ such that π is orientation preserving. We thus get another measure μ_ω on M by considering the Riemannian volume form ω on M (cf. section A.3). Note that $\pi^*\omega = \omega_n$ where the latter is the hyperbolic volume form on $\mathbb{H}^n$.

However both measures ν/μ_Γ and μ_ω coincide. Indeed, let $f \in C_c(\mathbb{H}^n)$ and recall the definition of $f^\flat \in C_c(M)$ via $f^\flat(\pi(x)) = \sum_\gamma f(\gamma x)$ (cf. section A.4.3). Then by [Lee13, Proposition 16.8, p. 408] we have

$$\int_{\Gamma\backslash\mathbb{H}^n} f^\flat \, d\mu_\omega = \int_M f^\flat \cdot \omega = \int_F \pi^*(f^\flat \cdot \omega) = \int_F f^\flat \circ \pi \cdot \omega_n$$

where F is a measurable fundamental set for Γ in $\mathbb{H}^n$. Further

$$\begin{aligned} \int_F f^\flat \circ \pi \cdot \omega_n &= \int_F f^\flat(\pi(x)) \, d\nu(x) \\ &= \int_F \sum_\gamma f(\gamma x) \, d\nu(x) \\ &= \sum_\gamma \int_{\gamma F} f(x) \, d\nu(x) \\ &= \int_{\mathbb{H}^n} f(x) \, d\nu(x) \end{aligned}$$

such that by the uniqueness of quotient measures $\nu/\mu_\Gamma = \mu_\omega$ (cf. Proposition A.4.9). Thus if Γ is a (torsion-free) lattice in $\mathrm{Isom}^+(\mathbb{H}^n)$ the quotient manifold M has finite volume.

Conversely if M is a finite volume hyperbolic manifold and $\Gamma = \mathrm{Deck}(\pi) \cong \pi_1(M)$ its group of Deck transformations, then under the identification $M = \Gamma\backslash\mathbb{H}^n$ we get that the measure $\mu_\omega = \nu/\mu_\Gamma$ is finite, such that Γ is a lattice. Therefore torsion-free lattices occur naturally whilst considering finite volume hyperbolic manifolds.

Finally the next proposition yields, that every quotient $\Gamma\backslash\mathbb{H}^n$ by a lattice subgroup $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ is at least finitely covered by a finite volume hyperbolic manifold.

Proposition I.4.21. *Let* $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ *be a lattice. Then there is a torsion-free subgroup* $\Gamma' < \Gamma$ *of finite index; in particular* Γ' *is again a lattice (cf. Theorem I.4.20).*

Proof. This is an application of the celebrated Selberg's Lemma (cf. [Rat06, Corollary 5 §7.6, p. 331]), which states that every finitely generated subgroup Γ of $\mathrm{Isom}(\mathbb{H}^n)$ has a torsion-free normal subgroup of finite index. Hence it will be enough to show, that the lattice Γ is finitely generated. By [Bow93, Proposition 4.7, p. 297] Γ is geometrically finite if $\Gamma\backslash\mathbb{H}^n$ has finite volume. Hence by [Bow93, Proposition 3.1.6] Γ is finitely generated. $\square$

I.5. Ergodic Theory

We will now investigate some ergodicity phenomena in hyperbolic geometry. First we will give some basic definitions and characterizations of ergodicity. Our goal is to deduce that every lattice acts double ergodically on the boundary of hyperbolic n-space. For some results and basic definitions of measure theory we refer to appendix A.

Definition I.5.1 (Ergodic Action)**.** Let $(X, \mathfrak{A}, \mu)$ be a σ-finite measure space and G a locally compact second countable group, that acts measurably on X from the left such that the measure class of μ is preserved, i.e. $g_*\mu$ is equivalent to μ for every $g \in G$.

The action of G is called *ergodic* if there are no non-trivial invariant subsets of X, that is, if the following holds: if $A \subset X$ is measurable and G-invariant, then A is either null or conull, i.e. $\mu(A) = 0$ or $\mu(X - A) = 0$.

Remark I.5.2. *It is apparent from the above definition that ergodicity only depends on the measure class of a measure and not on the measure itself. This will be important in the following as we will be concerned with smooth manifolds (with or without boundary) equipped with their canonical measure class (cf. section A.3).*

The next theorem gives a quite useful characterization of ergodic actions. Before that we need the notion of essentially G-invariant functions.

Definition I.5.3. Let G be a group acting on a measure space $(X, \mathfrak{A}, \mu)$. A measurable function $f : X \to \mathbb{R}$ is called *essentially G-invariant* if, for any $g \in G$, one has $f(x) = f(gx)$ for μ-almost every $x \in X$.

A function $f : X \to \mathbb{R}$ is called *G-invariant* if, for any $g \in G$, one has $f(gx) = f(x)$ for all $x \in X$.

Theorem I.5.4 (Characterization of Ergodic Actions)**.** *Let G be a locally compact second countable group acting on a σ-finite measure space $(X, \mathfrak{A}, \mu)$. Then the following are equivalent:*

(i) The action of G is ergodic.

(ii) If $f : X \to \mathbb{R}$ is measurable and G-invariant, then f is constant almost everywhere.

(iii) If $f : X \to \mathbb{R}$ is measurable and essentially G-invariant, then f is constant almost everywhere.

Proof. The equivalence of (i) and (iii) is [BM00, 1.3 Theorem, p. 3]. The implication of (iii) to (ii) is clear by definition, since every G-invariant function is also essentially G-invariant. Conversely the implication (ii) to (iii) follows from [BM00, 1.2 Lemma, p. 2] as in the proof of [BM00, 1.3 Theorem, p. 3]. □

The next lemma will be useful later.

Lemma I.5.5. *Let G be a locally compact second countable group and let M be a smooth manifold equipped with its canonical measure class. Further let G act continuously and transitively on M and let $H < G$ be a dense subgroup.*

Then the induced action of H on M is ergodic.

Proof. Let $f : M \to \mathbb{R}$ be an H-invariant measurable function and let μ be a probability measure in the canonical measure class of M (the existence of such a probability measure is easily verifed using a partition of unity and appropriate local measures). We may now apply Lusin's theorem and get for every $n \geq 3$ a compact set K_n such that f restricted to K_n is continuous and

$$\mu(M - K_n) = 1 - \mu(K_n) \leq 1/n \leq 1/3.$$

Let $n \geq 3$. From the invariance of f and the continuity of the action it follows easily that f is also continuous on $HK_n = \bigcup_{h \in H} hK_n$. We claim that f is already constant on K_n.

Let $k, k' \in K_n$. Because G acts transitively on M there is a $g \in G$ such that $gk = k'$. Because $H \subset G$ is dense, there is a sequence $(h_j) \subset H$ converging to g. This in turn implies that $h_j k \to gk = k'$ as $j \to \infty$. However f is continuous on all of HK_n such that $f(k') = \lim_{j \to \infty} f(h_j k) = f(k)$. Thus f is constant on K_n.

Observe that for every $n, m \geq 3$ we have that $\mu(K_n), \mu(K_m) \geq 2/3$ such that

$$1 = \mu(M) \geq \mu(K_n \cup K_m) = \mu(K_n) + \mu(K_m) - \mu(K_n \cap K_m) \geq \frac{4}{3} - \mu(K_n \cap K_m).$$

Hence $\mu(K_n \cap K_m) \geq 1/3$ and in particular $K_n \cap K_m \neq \emptyset$. Therefore f is equal to the same constant on every K_n, i.e. there is a constant $c \in \mathbb{R}$ such that $f(x) = c$ for every $x \in K_n$ and every $n \geq 3$. This implies, that f is constant on $A := \bigcup_{n \geq 3} K_n$. Finally A has full measure, since

$$\mu(A) \geq \mu(K_n) \geq 1 - \frac{1}{n}$$

for every $n \geq 3$. □

Similar to ergodic group actions one may also define when a flow is called ergodic.

Definition I.5.6 (Ergodic Flow)**.** Let M be a smooth manifold with or without boundary and $\Phi : \mathbb{R} \times M \to M$ a smooth (global) flow (cf. [Lee13, p. 211]). A set $A \subset M$ is said to be Φ-*invariant* if $\Phi_t(A) = A$ for every $t \in \mathbb{R}$, where $\Phi_t(x) = \Phi(t, x)$ for every $t \in \mathbb{R}$ and every $x \in M$.

The flow Φ is said to be *ergodic* or *act ergodically on* M if every Φ-invariant measurable set $A \subset M$ is either null or conull.

Now let M be a Riemannian manifold. We denote by TM its tangent bundle and by

$$T^1M = \{v \in T_xM : x \in M, \|v\|_x = 1\} \subset TM$$

its unit tangent bundle. Recall that we have the geodesic flow Φ on T^1M, which is global (i.e. defined on all of $\mathbb{R}$) if M is complete.

Proposition I.5.7. *Let M be a finite volume hyperbolic manifold. Then the geodesic flow $\Phi : \mathbb{R} \times T^1M \to T^1M$ on the unit tangent bundle is ergodic.*

Proof. See [BM00, 4.29 Corollary]. □

This implies that every lattice acts double ergodically on the boundary as the next corollary states.

Corollary I.5.8. *Let $\Gamma < G^+ = \mathrm{Isom}^+(\mathbb{H}^n)$ be a lattice. Then Γ acts* double ergodically *on $\partial\mathbb{H}^n$, i.e. the diagonal action of Γ on $\partial\mathbb{H}^n \times \partial\mathbb{H}^n$ is ergodic.*

The following proof is inspired by the remark following up [Thu, Chapter 5, Theorem 5.9.10, p. 112].

Proof. We may assume that Γ is torsion-free without loss of generality. Indeed, by Proposition I.4.21 there is a torsion-free lattice $\Gamma' < \Gamma$ of finite index. Now if $A \subset \partial\mathbb{H}^n \times \partial\mathbb{H}^n$ is Γ-invariant, it is also Γ'-invariant, such that Γ acts ergodically if Γ' acts ergodically. Thus we may assume that Γ is torsion-free and hence acts freely on $\mathbb{H}^n$ such that its quotient $M = \Gamma\backslash\mathbb{H}^n$ is a finite volume hyperbolic manifold.

Now observe that Γ acts freely and properly discontinuously on $T^1\mathbb{H}^n$ via $\gamma \cdot v_x = d\gamma_x(v_x)$, for every $v_x \in T^1_x\mathbb{H}^n$, such that we can identify $T^1M \cong \Gamma\backslash T^1\mathbb{H}^n$.

Further consider the map $\Psi : T^1\mathbb{H}^n \to \partial\mathbb{H}^n \times \partial\mathbb{H}^n$ given by

$$\begin{aligned}\Psi(v_x) &= (\lim_{t\to\infty} \exp_x(t\cdot v_x), \lim_{t\to-\infty} \exp_x(t\cdot v_x))\\ &= (\lim_{t\to\infty} p(\Phi(t,v_x)), \lim_{t\to-\infty} p(\Phi(t,v_x)))\end{aligned}$$

for every $v_x \in T^1\mathbb{H}^n$, where $p : T^1\mathbb{H}^n \to \mathbb{H}^n$ is the bundle projection assigning each tangent vector its base point. One easily checks that Ψ is smooth and surjective.

Further $\Psi : T^1\mathbb{H}^n \to \partial\mathbb{H}^n \times \partial\mathbb{H}^n$ is Γ-equivariant with respect to the previously discussed action of Γ on the unit tangent bundle and the diagonal action of Γ on $\partial\mathbb{H}^n \times \partial\mathbb{H}^n$. Indeed, let $v_x \in T^1_x\mathbb{H}^n$ and $\gamma \in \Gamma$, then

$$\begin{aligned}\gamma \cdot \lim_{t\to\pm\infty} \exp_x(t\cdot v_x) &= \lim_{t\to\pm\infty} \gamma(\exp_x(t\cdot v_x))\\ &= \lim_{t\to\pm\infty} \exp_{\gamma(x)}(t\cdot d\gamma_x(v_x))\end{aligned}$$

and therefore $\gamma \cdot \Psi(v_x) = \Psi(\gamma \cdot v_x)$ as asserted.

In addition Ψ is also invariant under the geodesic flow $\Phi : \mathbb{R} \times T^1\mathbb{H}^n \to T^1\mathbb{H}^n$, since for every $v_x \in T^1\mathbb{H}^n$ and every $s \in \mathbb{R}$, one has

$$\lim_{t\to\pm\infty} p(\Phi(t,\Phi(s,v_x))) = \lim_{t\to\pm\infty} p(\Phi(t+s,v_x)) = \lim_{t\to\pm\infty} p(\Phi(t,v_x))$$

Finally observe that the geodesic flow on $T^1\mathbb{H}^n$ induces the geodesic flow $\bar{\Phi} : \mathbb{R} \times T^1M \to T^1M$ of $M = \Gamma\backslash\mathbb{H}^n$ by passing to the quotient via $\pi : T^1\mathbb{H}^n \to \Gamma\backslash T^1\mathbb{H}^n \cong T^1M$, since Φ is clearly Γ-equivariant, i.e. $\bar{\Phi}$ is defined by $\pi \circ \Phi = \bar{\Phi} \circ \pi$.

Now let $A \subset \partial\mathbb{H}^n \times \partial\mathbb{H}^n$ be a measurable Γ-invariant subset. Then by the Γ-equivariance of Ψ also $\Psi^{-1}(A) \subset T^1\mathbb{H}^n$ is Γ-invariant. Passing to the quotient we consider $\pi(\Psi^{-1}(A))$ and claim, that it is $\bar{\Phi}$-invariant. Indeed,

$$\bar{\Phi}_t(\pi(\Psi^{-1}(A))) = \pi(\Phi_t(\Psi^{-1}(A))) = \pi((\Psi \circ \Phi_{-t})^{-1}(A)) = \pi(\Psi^{-1}(A))$$

for every $t \in \mathbb{R}$, since Ψ is invariant under the geodesic flow. Thus $\pi(\Psi^{-1}(A))$ is $\bar{\Phi}$-invariant and hence is either null or conull by ergodicity.

Let us first assume that $\pi(\Psi^{-1}(A))$ is null. Then also $\pi^{-1}(\pi(\Psi^{-1}(A))) = \Gamma\Psi^{-1}(A) = \Psi^{-1}(A)$ is null by Proposition A.3.6, since π is a covering map and thus in particular a smooth submersion. But then also $\Psi(\Psi^{-1}(A)) = A \subset \partial\mathbb{H}^n \times \partial\mathbb{H}^n$ is a null set by Theorem A.3.4.

Now let us assume that $\pi(\Psi^{-1}(A))$ is conull. Then again by Proposition A.3.6 $\pi^{-1}(\pi(\Psi^{-1}(A))) = \Gamma\Psi^{-1}(A) = \Psi^{-1}(A)$ is conull. Now

$$\partial\mathbb{H}^n \times \partial\mathbb{H}^n - A = \Psi(T^1\mathbb{H}^n) - \Psi(\Psi^{-1}(A)) = \Psi(T^1\mathbb{H}^n - \Psi^{-1}(A))$$

such that by Theorem A.3.4 also A is conull. $\square$

I.6. The Thick-Thin Decomposition

We will now see that finite volume hyperbolic manifolds admit a particular nice decomposition in some *thick* part and some *thin* part, where the thick part is compact and the thin part is "not very complicated". We will follow here in essence [BP92, Chapter D, pp. 133], which in turn elaborates on the more intuitive treatment in [Thu97, 4.5. The Thick-Thin Decomposition, pp. 253]. There are also other approaches to this kind of decomposition which arise out of the study of geometrical finiteness and uses extensively limit sets, which can be found for example in [Rat06, Chapter 12, pp. 600].

Let M be a Riemannian manifold. If σ is a piecewise differentiable path in M, we shall denote by $L(\sigma)$ its length. Remark that each loop in M is homotoptic to a piecewise differentiable loop based at the same point, so that we can think of $\pi_1(M)$ as the set of all piecewise differentiable loops up to homotopy. For $\varepsilon > 0$ we set

$$M_{(0,\varepsilon]} = \{x \in M : \exists[\sigma] \in \pi_1(M,x) - \{1\} \text{ s.t. } L(\sigma) \leq \varepsilon\}$$
$$M_{[\varepsilon,\infty)} = \{x \in M : \forall[\sigma] \in \pi_1(M,x) - \{1\}, L(\sigma) \geq \varepsilon\}$$

Of course if M is a compact manifold we have $M_{(0,\varepsilon]} = \emptyset$ whenever ε is small enough.

We shall say that $M_{(0,\varepsilon]}$ is the *ε-thin part* of M, and $M_{[\varepsilon,\infty)}$ is the *ε-thick part* of M; when a constant ε is fixed we will omit its specification, so we shall speak of the thin and the thick part of M.

With the above definitions the thick part of a hyperbolic manifold with finite volume is always compact as the following proposition asserts.

Proposition I.6.1. *Let M be a finite volume hyperbolic manifold. Then its ε-thick part $M_{[\varepsilon,\infty)}$ is compact for every $\varepsilon > 0$.*

Proof. See [BP92, Proposition D.2.6., p. 142]. □

We want to investigate the so called ε-ends:

Definition I.6.2 (ε-end)**.** Let M be a hyperbolic n-manifold and $0 < \varepsilon \leq \varepsilon_n$, where $\varepsilon_n > 0$ is the n-th Margulis constant (cf. [BP92, Theorem D.1.1 Margulis' Lemma, p. 134]).

We shall call the closure of a connected component of $M - M_{[\varepsilon,\infty)}$ an ε-end of M.

We get the following classification theorem for the ends of a hyperbolic n-manifold.

Theorem I.6.3. *Let M be a hyperbolic n-manifold and $0 < \varepsilon \leq \varepsilon_n$, where $\varepsilon_n > 0$ is the n-th Margulis constant. Then the ε-thin part $M_{(0,\varepsilon]}$ of M is the union of pieces homeomorphic to one of the following types:*

(i) $\overline{D^{n-1}} \times S^1$*, where D^{n-1} is the $(n-1)$-dimensional unit disk;*

(ii) $V \times [0, \infty)$*, where V is a smooth oriented $(n-1)$-manifold without boundary supporting a Euclidean structure;*

(iii) S^1*.*

Moreover:

- *these pieces have positive distance from each other;*
- *the ε-ends of M are the pieces of type (i) and (ii);*

- *the pieces of type (iii) are closed geodesics of length precisely* ε*;*
- *if* M *has finite volume the pieces of type (i) and (ii) are finitely many and in those of type (ii) the manifold* V *is compact.*

We will call an ε*-end of* M *a* tube*, if it is of type (i), and a* cusp*, if it is of type (ii).*

Proof. This is [BP92, Theorem D.3.3., p. 145]. □

For the rest of this section we will fix the hypothesis of the above theorem.

Lemma I.6.4. *If* M *has finite volume and* C *is a cusp of* M*, then there is a subset* C' *of* C *such that:*

(i) $C - C'$ *is compact in* M*;*

(ii) C' *is diffeomorphic to* $V \times \mathbb{R}$ *where* V *is a compact oriented smooth euclidean* $(n-1)$*-manifold;*

(iii) the inverse image of C' *under the universal covering* $\pi : \mathbb{H}^n \to M \cong \Gamma \backslash \mathbb{H}^n$ *is a horoball in* $\mathbb{H}^n$*;*

(iv) $M - C$ *is a (strong) deformation retract of* $M - C'$*.*

Due to (iii) such a C' *is called a* horocusp region for C *or simply a* horocusp *of* M*.*

Proof. (i) and (ii) follow directly from [BP92, Proposition D.3.12, pp. 151] when we replace C' by its interior. (iii) and (iv) are easy consequences of the proof of [BP92, Proposition D.3.12, pp. 151]. □

The next corollary summarizes the above results in a more qulitative statement about the overall topology of a finite volume hyperbolic manifold.

Corollary I.6.5. *Let* M *be a finite volume hyperbolic* n*-manifold. Then there is a compact embedded* n*-dimensional submanifold* $N \subset M$ *with (possibly empty) boundary* ∂N *such that:*

(i) $M - N$ *is the disjoint union of finitely many horocusps* E_i $(i = 1, \ldots, k)$*;*

(ii) each connected component of ∂N *is diffeomorphic to a compact oriented smooth euclidean* $(n-1)$*-manifold;*

(iii) N *is a (strong) deformation retract of* M*.*

We will call every such $N \subset M$ *a* compact core of M*. By (iii) any two compact cores are homotopy equivalent.*

Proof. This follows directly from the previous lemma. It is in fact a slight elaboration on [BP92, Corollary D.3.14, p. 156]. □

Remark I.6.6. *Note that, for a finite volume hyperbolic* n*-manifold* M *one may choose* $0 < \varepsilon \leq \varepsilon_n$ *so small, that the* ε*-ends of* M *are only cusps. In this case a compact core* N*, which contains* $M_{[\varepsilon,\infty)}$*, deformation retracts to* $M_{[\varepsilon,\infty)}$*.*

Finally we want to recall one of Bieberbach's Theorems from which we will deduce that every compact euclidean manifold is finitely coverd by a torus. This result will be important in the study of the volume of lattice representations.

Theorem I.6.7 (Bieberbach)**.** *Let Γ be a discrete subgroup of isometries of $\mathbb{R}^n$. Then Γ is crystallographic (i.e. $\Gamma\backslash\mathbb{R}^n$ is compact) if and only if the subgroup T of translations of Γ is of finite index and has rank n.*

Proof. This is [Rat06, Theorem 7.5.2, p. 311]. See also [Thu97, Theorem 4.2.2, p. 222]. □

Corollary I.6.8. *Every compact euclidean n-manifold is finitely covered by a torus.*

Proof. Let M be a compact euclidean n-manifold. Basic covering theory asserts, that M can be identified with the quotient $\Gamma\backslash\mathbb{R}^n$ where $\Gamma < \mathrm{Isom}(\mathbb{R}^n)$ is a discrete subgroup acting freely on $\mathbb{R}^n$. Thus Γ is crystallographic and therefore its subgroup of translations T is of finite index and has rank n by the previous theorem. Again by covering theory we get a covering map $p : T\backslash\mathbb{R}^n \to \Gamma\backslash\mathbb{R}^n$ and it is obvious, that the quotient $T\backslash\mathbb{R}^n$ is an n-dimensional flat torus. □

I.7. Simplices

Simplices will become very important in our discussion of boundary maps later on, which in turn are a key ingredient in the final step of the volume rigidity theorem. We will recall some basic notions and properties of simplices in the first subsection. In the second we will investigate the volume of simplices. In particular we will see that the volume of a simplex depends continuously on its vertices and that a simplex has maximal volume if and only if it is regular and ideal. In the last subsection we will investigate the reflection groups of regular n-simplices both euclidean and hyperbolic. The upshot here is, that the simplex reflection group of a regular ideal n-simplex is dense in $\operatorname{Isom}(\mathbb{H}^n)$ for $n \geq 4$. This will be very handy in the next section on boundary maps.

I.7.1. Regular Simplices

Definition I.7.1. Let $X = \mathbb{R}^n$ or $X = \overline{\mathbb{H}}^n$, and $v_0, \ldots, v_n \in X$. Then the (geodesically) convex hull of these points $\Delta^n = \operatorname{conv}(v_0, \ldots, v_n)$ is called the *n-simplex in X with vertices $v_0, \ldots, v_n$*; a 4-simplex is also called a *tetrahedron*. Δ^n is called *degenerate* if all its vertices lie in a proper (generalized) totally geodesic subspace of X. A non-degenerate n-simplex in X is called *regular* if every permutation of its vertices can be obtained by applying a suitable isometry.

If $X = \overline{\mathbb{H}}^n$ a simplex Δ^n in X is called *ideal* if all its vertices lie on the boundary $\partial\mathbb{H}^n$.

The following lemma gives us some information about the shape of regular ideal simplices in $\overline{\mathbb{H}}^n$ and $\mathbb{R}^m$.

Lemma I.7.2. *If Δ^n is an ideal n-simplex in U^n with vertices $\infty, v_1, \ldots, v_n$ then Δ^n is regular if and only if the euclidean $(n-1)$-simplex with vertices $v_1, \ldots, v_n \in \mathbb{R}^{n-1}$ is regular. Moreover an m-simplex in $\mathbb{R}^m$ is regular if and only if all its edges have the same length.*

Proof. See [BP92, Lemma C.2.4, p. 96]. □

Let us set $T = \{(\xi_0, \ldots, \xi_n) \in (\partial\mathbb{H}^n)^{n+1} : \xi_0, \ldots, \xi_n$ are vertices of a regular ideal simplex in $\overline{\mathbb{H}}^n\}$. It is not hard to see, that $T \subset (\partial\mathbb{H}^n)^{n+1}$ is an embedded submanifold. Note that the action of G on T is smooth and transitive. The next proposition will show, that one may even identify G with T after choosing some base simplex.

Proposition I.7.3. *Let $\bar{\eta} = (\eta_0, \ldots, \eta_n) \in T$ and define a map $\Phi_{\bar{\eta}} : G \to T$ by*

$$\Phi_{\bar{\eta}}(g) = g \cdot \bar{\eta} = (g(\eta_0), \ldots, g(\eta_n)).$$

Then $\Phi_{\bar{\eta}} : G \to T$ is a G-equivariant diffeomorphism. Further the following formula holds

$$\Phi_{\bar{\xi}}(g) = \Phi_{\bar{\eta}}(g\Phi_{\bar{\eta}}^{-1}(\bar{\xi}))$$

for every $\bar{\xi} \in T$ and every $g \in G$.

Proof. Let $\bar{\eta} = (\eta_0, \ldots, \eta_n) \in T$. By definition it is clear that $\Phi_{\bar{\eta}}$ is smooth and G-equivariant. Further it is surjective, since the action of G on T is transitive.

We claim that $\Phi_{\bar{\eta}}$ is also injective. Let $g, h \in G$ such that $\Phi_{\bar{\eta}}(g) = \Phi_{\bar{\eta}}(h)$, that is $h(\eta_i) = g(\eta_i)$ for every $i = 0, \ldots, n$. Without loss of generality we may assume that $\eta_0 = \infty$ in the upper half space model; otherwise conjugate g and h by some isometry sending η_0 to ∞. Then $\eta_1, \ldots, \eta_n \in \mathbb{R}^{n-1}$ are the vertices of a regular euclidean simplex and are fixed by $h^{-1}g \in G$. Because $h^{-1}g(\infty) = \infty$, $h^{-1}g$ is a euclidean similarity and it is easy to see, that every euclidean similarity fixing the vertices of some regular simplex is the identity. Hence $h^{-1}g = \mathrm{id}$ and thus $g = h$. This shows that $\Phi_{\bar{\eta}}$ is indeed injective.

Finally, we claim that $\Phi_{\bar{\eta}}$ has constant rank. For $g, h \in G$ we have

$$d_g(\Phi_{\bar{\eta}} \circ L_h) = d_{hg}\Phi_{\bar{\eta}} \cdot d_g L_h$$

where $L_h : G \to G, g \mapsto hg$ is the diffeomorphism given by left translation. Thus $\operatorname{rank} d_g\Phi_{\bar{\eta}} = \operatorname{rank} d_{hg}\Phi_{\bar{\eta}}$, since $d_g L_h : T_g G \to T_{hg} G$ is non-singular. This shows, that the rank of $\Phi_{\bar{\eta}}$ is constant.

By the Global Rank Theorem (cf. [Lee13, Theorem 4.14 (Global Rank Theorem), p. 83] $\Phi_{\bar{\eta}} : G \to T$ is a diffeomorphism.

The asserted formula follows from a simple computation. Let $\bar{\xi} \in T$ and $h = \Phi_{\bar{\eta}}^{-1}(\bar{\xi})$, i.e. $\bar{\xi} = h\bar{\eta}$. Then

$$\Phi_{\bar{\xi}}(g) = g\bar{\xi} = gh\bar{\eta} = \Phi_{\bar{\eta}}(gh) = \Phi_{\bar{\eta}}(g\Phi_{\bar{\eta}}^{-1}(\bar{\xi}))$$

for every $g \in G$. □

I.7.2. Volume

The next theorem is very important in hyperbolic geometry characterizing the simplices of maximal volume.

Theorem I.7.4. *A n-simplex in $\overline{\mathbb{H}}^n$ has maximal volume if and only if it is regular and ideal.*

Proof. See [Rat06, Theorem 11.4.1, p. 539]. □

Let us now turn to a proof, that the volume of a n-simplex is continuous. This will be helpful in the definition of the volume cocycle later on. The next proposition is the initial step of an induction argument.

Proposition I.7.5. *Let $\xi_0, \dots, \xi_3 \in \partial\mathbb{H}^3$ and $T = \operatorname{conv}(\xi_0, \dots, \xi_3)$ the ideal tetrahedron spanned by these points. Then the volume of T depends continuously on its vertices $\xi_0, \dots, \xi_3$.*

Proof. Obviously the three dihedral angles α, β, γ of the edges incident to a vertex of T depend continuously on the vertices of T. By [see Rat06, Theorem 10.4.10., p. 475] the volume of T is given by

$$\operatorname{vol}(T) = L(\alpha) + L(\beta) + L(\gamma)$$

where $L : \mathbb{R} \to \mathbb{R}$ is the so called Lobachevsky function (cf. [Rat06, pp. 465]). Because L is continuous (cf. [see Rat06, Theorem 10.4.3., p. 468]), $\operatorname{vol}(T)$ depends continuously on its vertices as asserted. □

Lemma I.7.6. *The volume of an ideal n-simplex Δ^n in U^n, with vertices $v_0, \dots, v_n$ such that $v_0 = \infty$ and $v_1, \dots, v_n$ are in S^{n-2}, is given by*

$$\operatorname{vol}(\Delta^n) = \frac{1}{n-1} \int_{p(\Delta^n)} \frac{dx_1 \dots dx_n}{(1 - |x|^2)^{(n-1)/2}}$$

where $p : U^n \to \mathbb{R}^{n-1}$ denotes the standard vertical projection.

Proof. See [Rat06, Lemma 1, p. 532]. □

Theorem I.7.7. *Let $n \geq 3$ and let $v_0, \dots, v_n \in \overline{D}^n \cong \mathbb{H}^n \cup \partial\mathbb{H}^n$. Then the volume of the generalized simplex $T = \operatorname{conv}(v_0, \dots, v_n)$ depends continuously on its vertices $v_0, \dots, v_n \in \overline{D}^n$.*

Our proof is based on the proof of [Rat06, Theorem 11.4.2., p. 541], which apparently only works in a situation where the vertices are not contained in a proper hyperbolic subspace. However we are specifically interested in this case later on such that we have to adapt the proof.

Proof. Let $\{(v_{0j},\ldots,v_{nj})\}_{j\in\mathbb{N}}$ be a sequence in $(\overline{D}^n)^{n+1}$ converging to $(v_0,\ldots,v_n)$. Denote $\Delta^n_j = \operatorname{conv}(v_{0j},\ldots,v_{nj})$ and accordingly $\Delta^n = \operatorname{conv}(v_0,\ldots,v_n)$. We have to prove that

$$\lim_{j\to\infty} \operatorname{vol}(\Delta^n_j) = \operatorname{vol}(\Delta^n)$$

Assume first that Δ^n_j is ideal for each j, i.e. $\{(v_{0j},\ldots,v_{nj})\}_{j\in\mathbb{N}} \subset (S^{n-1})^{n+1}$. Therefore also $(v_0,\ldots,v_n) \in (S^{n-1})^{n+1}$ and Δ^n is ideal. This part of the proof is by induction on the dimension n. By the above Proposition I.7.5 the initial step of the induction for dimension $n=3$ is settled. For the induction step we distinguish two more cases. Let $(S^{n-1})^{[n+1]}$ denote the set of all $(n+1)$-tuples of points, that are not contained in a proper (generalized) hyperbolic subspace. It is easy to see, that

$$(S^{n-1})^{[n+1]} = \{(\xi_0,\ldots,\xi_n) \in (S^{n-1})^{n+1} : D(\xi_0,\ldots,\xi_n) \neq 0\}$$

where

$$D(\xi_0,\ldots,\xi_n) = \det \begin{pmatrix} | & & | \\ \xi_1-\xi_0 & \cdots & \xi_n-\xi_0 \\ | & & | \end{pmatrix}$$

is the determinant of the matrix containing the differences $\xi_i - \xi_0$ as column vectors. Therefore $(S^{n-1})^{[n+1]} \subset (S^{n-1})^{n+1}$ is (relatively) open.

Assume now, that $\{(v_{0j},\ldots,v_{nj})\}_{j\in\mathbb{N}} \subset (S^{n-1})^{n+1} - (S^{n-1})^{[n+1]}$, i.e. Δ^n_j is contained in a proper hyperbolic subspace. Then also its limit $(v_0,\ldots,v_n)$ is in $(S^{n-1})^{n+1} - (S^{n-1})^{[n+1]}$, because the set is closed. Thus also Δ^n is contained in a proper hyperbolic subspace and we have

$$\lim_{j\to\infty} \operatorname{vol}(\Delta^n_j) = 0 = \operatorname{vol}(\Delta^n)$$

since degenerate simplices have no volume.

So we are left with the case that $\{(v_{0j},\ldots,v_{nj})\}_{j\in\mathbb{N}} \subset (S^{n-1})^{[n+1]}$, i.e. Δ^n_j is non-degenerate. Note that its limit Δ^n may still be degenerate. For each $j\in\mathbb{N}$ let A_j be the rotation of E^n that rotates v_{0j} to v_0 with no other nonzero angles of rotation. As $v_{0j}\to v_0$, we have that $A_j \to \mathrm{Id}$ in $O(n)$. Hence $(A_j v_{0j},\ldots,A_j v_{nj}) \to (v_0,\ldots,v_n)$. As

$$\operatorname{vol}(A_j(\Delta^n_j)) = \operatorname{vol}(\Delta^n_j)$$

we may replace Δ^n_j by $A_j(\Delta^n_j)$. Thus, we may assume, without loss of generality, that $v_{0j} = v_0$ for all j.

We now pass to the upper half space model U^n of hyperbolic space and assume, without loss of generality, that $v_0 = \infty$ and $v_1,\ldots,v_n$ lie on the unit sphere S^{n-2} in $\mathbb{R}^{n-1}$. For each j, the vertices $v_{1j},\ldots,v_{nj}$ lie on an $(n-2)$-sphere $S(a_j,r_j)$ in $\mathbb{R}^{n-1}$; here we need, that Δ^n_j is not contained in a proper hyperbolic subspace. Now as $(v_{1j},\ldots,v_{nj}) \to (v_1,\ldots,v_n)$, we have that $a_j\to 0$ and $r_j \to 1$. Let

$$\phi_j = -r_j^{-1}a_j + r_j^{-1}\mathrm{Id}$$

Then ϕ_j maps $S(a_j,r_j)$ onto S^{n-2}. Moreover $\phi_j\to\mathrm{Id}$ in $S(\mathbb{R}^{n-1})$. Hence $(\phi_j(v_{1j}),\ldots,\phi_j(v_{nj})) \to (v_1,\ldots,v_n)$. As

$$\operatorname{vol}(\phi_j(\Delta^n_j)) = \operatorname{vol}(\Delta^n_j)$$

we may replace Δ^n by $\phi_j(\Delta^n_j)$. Thus, we may assume, without loss of generality, that the vertices $v_1,\ldots,v_n$ lie on the sphere S^{n-2} for all j. By the above lemma, we have

$$\operatorname{vol}(\Delta^n) = \frac{1}{n-1}\int_{p(\Delta^n)} \frac{dx_1\ldots dx_{n-1}}{(1-|x|^2)^{(n-1)/2}}$$

For each j, let χ_j be the characteristic function of the set $p(\Delta_j^n)$ and let χ be the characteristic function of $p(\Delta^n)$. Then the sequence χ_j converges pointwise almost everywhere to χ. Thinking of $p(\Delta^n)$ as an ideal $(n-1)$-simplex in the projective disk model D^{n-1} we have for its volume

$$\mathrm{vol}_{D^{n-1}}(p(\Delta^n)) = \int_{p(\Delta^n)} \frac{dx_1 \ldots dx_{n-1}}{(1-|x|^2)^{n/2}} = \int_{D^{n-1}} \frac{\chi(x) dx_1 \ldots dx_{n-1}}{(1-|x|^2)^{n/2}}$$

(cf. [Rat06]). Hence by our induction hypothesis we have that

$$\begin{aligned}\lim_{j\to\infty} \int_{D^{n-1}} \frac{\chi_j(x) dx_1 \ldots dx_{n-1}}{(1-|x|^2)^{n/2}} &= \lim_{j\to\infty} \mathrm{vol}_{D^{n-1}}(p(\Delta_j^n)) \\ &= \mathrm{vol}_{D^{n-1}}(p(\Delta^n)) \\ &= \int_{D^{n-1}} \frac{\chi(x) dx_1 \ldots dx_{n-1}}{(1-|x|^2)^{n/2}}\end{aligned}$$

We have that

$$\frac{\chi_j(x)}{(1-|x|^2)^{n/2}} \to \frac{\chi(x)}{(1-|x|^2)^{n/2}} \quad \text{and} \quad \frac{\chi_j(x)}{(1-|x|^2)^{(n-1)/2}} \to \frac{\chi(x)}{(1-|x|^2)^{(n-1)/2}} \qquad (j\to\infty)$$

pointwise almost everywhere. Because $(1-|x|^2) \leq 1$ for every $x \in \overline{D}^{n-1}$, we have that $(1-|x|^2)^{n/2} \leq (1-|x|^2)^{(n-1)/2}$ and thus

$$\frac{|\chi_j(x)-\chi(x)|}{(1-|x|^2)^{(n-1)/2}} \leq \frac{|\chi_j(x)-\chi(x)|}{(1-|x|^2)^{n/2}}$$

for every $x \in D^{n-1}$. We may now apply the general dominated convergence theorem A.1.5, which yields

$$\begin{aligned}\lim_{j\to\infty} \mathrm{vol}(\Delta_j^n) &= \lim_{j\to\infty} \frac{1}{n-1} \int_{D^{n-1}} \frac{\chi_j(x) dx_1 \ldots dx_{n-1}}{(1-|x|^2)^{(n-1)/2}} \\ &= \frac{1}{n-1} \int_{D^{n-1}} \frac{\chi(x) dx_1 \ldots dx_{n-1}}{(1-|x|^2)^{(n-1)/2}} = \mathrm{vol}(\Delta^n)\end{aligned}$$

and the first part of the proof is finished.

We now return to the general case. Without loss of generality, we may assume that 0 is the centroid of Δ^n. As the vertices of Δ_j^n converge to the vertices of Δ^n, the centroid $c_j = (v_{0j} + \ldots + v_{nj})/(n+1)$ of Δ_j^n converges to 0. Let τ_j be the hyperbolic translation of D^n by $-c_j$ (cf. Definition I.2.13). Then $\tau_j \to \mathrm{Id}$ in $\mathrm{Isom}(\mathbb{H}^n)$ and hence

$$(\tau_j(v_{0j}), \ldots, \tau_j(v_{nj})) \to (v_0, \ldots, v_n) \qquad (j\to\infty)$$

As $\mathrm{vol}(\tau_j(\Delta_j^n)) = \mathrm{vol}(\Delta_j^n)$, we may replace Δ_j^n by $\tau_j(\Delta_j^n)$. Then 0 is in Δ_j^n for each j. Let $\hat{\Delta}_j^n$ be the ideal n-simplex with vertices $\hat{v}_{0j}, \ldots, \hat{v}_{nj}$, where $\hat{v}_{ij} = v_{ij}/|v_{ij}|$ for each j, and let $\hat{\Delta}^n$ be the ideal n-simplex with vertices $\hat{v}_0, \ldots, \hat{v}_n$, where $\hat{v}_i = v_i/|v_i|$. Then

$$(\hat{v}_{0j}, \ldots, \hat{v}_{nj}) \to (\hat{v}_0, \ldots, \hat{v}_n) \qquad (j\to\infty)$$

Let $\chi_j, \hat{\chi}_j, \chi, \hat{\chi}$ be the characteristic functions for the sets Δ_j^n, $\hat{\Delta}_j^n$, Δ^n, $\hat{\Delta}^n$, resp. Then $\chi_j \to \chi$ and $\hat{\chi}_j \to \hat{\chi}$ almost everywhere. Now as $\Delta_j^n \subset \hat{\Delta}_j^n$, we have that $\chi_j \leq \hat{\chi}_j$ for each j.

Denote by μ the measure of hyperbolic volume in the projetive disk model, i.e.

$$\frac{d\mu}{d\lambda}(x) = \frac{1}{(1-|x|^2)^{(n+1)/2}}$$

where λ denotes the Lebesgue measure on D^n and $d\mu/d\lambda$ is the Radon-Nikodym derivative (cf. [Rat06]). By the first case, we have

$$\lim_{j\to\infty}\int_{D^n}\hat{\chi}_j\,d\mu=\int_{D^n}\hat{\chi}\,d\mu<\infty$$

Again by the general version of Lebesgue's dominated convergenc theorem, we deduce that

$$\lim_{j\to\infty}\int_{D^n}\chi_j\,d\mu=\int_{D^n}\chi\,d\mu$$

Therefore, we have

$$\lim_{j\to\infty}\mathrm{vol}(\Delta_j^n)=\mathrm{vol}(\Delta^n)$$

□

Observe that the above theorem fails in dimension 2. In fact any two ideal 2-simplices are congruent, since $\mathrm{Isom}(\mathbb{H}^2)$ acts 3-transitively on the boundary. Therefore $\mathrm{vol}(\mathrm{conv}(\xi_0,\xi_1,\xi_2))$ is constant on the subset of triples of distinct points $(\xi_0,\xi_1,\xi_2)\in(\partial\mathbb{H}^2)^3$ and is 0 on its complement. As a consequence $\mathrm{vol}(\mathrm{conv}(\xi_0,\xi_1,\xi_2))$ is not continuous on all of $(\partial\mathbb{H}^2)^3$, but still on the subset $(\partial\mathbb{H}^2)^{(3)}$ of triples of distinct boundary points (and its complement). However we get the following theorem.

Theorem I.7.8. *The function* $\mathrm{vol}(\mathrm{conv}(v_0,v_1,v_2))$ *is continuous on* $(\mathbb{H}^2)^3\cup(\overline{\mathbb{H}}^2)^{[3]}$*, where* $(\overline{\mathbb{H}}^2)^{[3]}$ *denotes the set of triples* $(v_0,v_1,v_2)\in(\overline{\mathbb{H}}^2)^3$ *such that* v_0,v_1,v_2 *are not contained in a proper hyperbolic subspace.*

Remark I.7.9. *Note that* $(\partial\mathbb{H}^2)^{[3]}=(\partial\mathbb{H}^2)^{(3)}$*.*

Proof. We will work in the projective disk model and identify without furhter notice $\overline{\mathbb{H}}^2\cong\overline{D}^2$. Let $(v_0,v_1,v_2)\in(\mathbb{H}^2)^3\cup(\overline{\mathbb{H}}^2)^{(3)}$ and $\{(v_{0j},v_{1j},v_{2j})\}_{j\in\mathbb{N}}\subset(\mathbb{H}^2)^3\cup(\overline{\mathbb{H}}^2)^{(3)}$ a sequence converging to it.

Assume first, that $(v_0,v_1,v_2)\in(\mathbb{H}^2)^3$. Then also $(v_{0j},v_{1j},v_{2j})\in\mathbb{H}^2$ for j large enough. There is clearly a $\delta>0$ such that $B_\delta(x_i)\subset D^2$ for every $i=0,1,2$. Consider the convex hull $V=\mathrm{conv}(B_\delta(v_0)\cup B_\delta(v_1)\cup B_\delta(v_2))$. Then V is open in D^2, has compact closure and $\mathrm{conv}(v_{0j},v_{1j},v_{2j})\subset V$ for j large enough. Denote by χ, χ_j, χ_V the characteristic functions of $\mathrm{conv}(v_0,v_1,v_2)$, $\mathrm{conv}(v_{0j},v_{1j},v_{2j})$, V respectively. We now have

$$0\le\chi_j\le\chi_V$$

for j large enough. Further denote by μ the measure of hyperbolic volume in the projective disk model as in the proof of the previous theorem. As V has compact closure in D^2 we have that

$$\int_{D^2}\chi_V\,d\mu=\mu(V)=\mathrm{vol}(V)<\infty$$

Additionally $\chi_j\to\chi$ pointwise as j tends to ∞. Thus by Lebesgue's dominated convergence theorem

$$\lim_{j\to\infty}\mathrm{vol}(\mathrm{conv}(v_{0j},v_{1j},v_{2j}))=\lim_{j\to\infty}\int_{D^2}\chi_j\,d\mu=\int_{D^2}\chi\,d\mu=\mathrm{vol}(\mathrm{conv}(v_0,v_1,v_2)$$

Now assume that $(v_0,v_1,v_2)\in(\overline{\mathbb{H}}^2)^{[3]}$. As in the proof of the previous theorem we may assume that $\mathrm{conv}(v_{0j},v_{1j},v_{2j})$ contains 0 without loss of generality. Then we can choose j large enough, such that $(v_{0j}, v_{1j}, v_{2j}) \in (\overline{\mathbb{H}}^2)^{[3]}$, v_{0j}, v_{1j}, v_{2j} are all different from 0 and $(\hat{v}_{0j}, \hat{v}_{1j}, \hat{v}_{2j}) \in (\partial\mathbb{H}^2)^{[3]}$ where $\hat{v}_{ij} = v_{ij}/|v_{ij}|$ for $i = 0, 1, 2$. Consider also $\hat{v}_i = v_i/|v_i|$. Then $(\hat{v}_{0j}, \hat{v}_{1j}, \hat{v}_{2j}) \to (\hat{v}_0, \hat{v}_1, \hat{v}_2)$ as j tends to ∞. Denote by χ_j, $\hat{\chi}_j$, χ, $\hat{\chi}$ the characteristic functions of $\mathrm{conv}(v_{0j}, v_{1j}, v_{2j})$, $\mathrm{conv}(\hat{v}_{0j}, \hat{v}_{1j}, \hat{v}_{2j})$, $\mathrm{conv}(v_0, v_1, v_2), \mathrm{conv}(\hat{v}_0, \hat{v}_1, \hat{v}_2)$ respectively. As

$$\mathrm{conv}(v_{0j}, v_{1j}, v_{2j}) \subset \mathrm{conv}(\hat{v}_{0j}, \hat{v}_{1j}, \hat{v}_{2j})$$

we have that $\chi_j \leq \hat{\chi}_j$ for every j. Observe that $\mathrm{conv}(\hat{v}_{0j}, \hat{v}_{1j}, \hat{v}_{2j})$, $\mathrm{conv}(\hat{v}_0, \hat{v}_1, \hat{v}_2)$ and $\mathrm{conv}(v_0, v_1, v_2)$ are all regular ideal triangles in D^2 and therefore have the same maximal volume. Because $\hat{\chi}_j$ converges to $\hat{\chi}$ and χ_j converges to χ as j tends to ∞, we may apply the generalized dominated convergence theorem A.1.5 and get

$$\lim_{j\to\infty} \mathrm{vol}(\mathrm{conv}(v_{0j}, v_{1j}, v_{2j})) = \lim_{j\to\infty} \int_{D^2} \chi_j \, d\mu = \int_{D^2} \chi \, d\mu = \mathrm{vol}(\mathrm{conv}(v_0, v_1, v_2))$$

□

I.7.3. Simplex Reflection Groups

In this final section on simplices we will show that the reflection group of a regular ideal n-simplex is dense in $\mathrm{Isom}(\mathbb{H}^n)$ for $n \geq 4$. This will significantly facilitate the proof of Proposition I.8.3. Let us first see this result for euclidean n-simplices with $n \geq 3$.

Proposition I.7.10. *Let $n \geq 3$. Then the reflection group of a regular euclidean n-simplex is dense in* $\mathrm{Isom}(\mathbb{R}^n)$.

Proof. Let $v_0, \ldots, v_n \in \mathbb{R}^n$ denote the vertices of a regular euclidean n-simplex. The first step is to compute the dihedral angles of a regular euclidean n-simplex. Denote by F_i the i-th face of the simplex not adjacent to v_i, i.e. the convex hull of all the vertices but v_i, and denote by H_i the $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$ containing F_i. Consider the barycenter of each face F_i

$$b_i = \frac{1}{n} \sum_{j \neq i} v_j$$

By symmetry one easily checks that the vectors

$$n_i = v_i - b_i = v_i - \frac{1}{n} \sum_{j \neq i} v_j = \frac{1}{n} \sum_{j \neq i} (v_i - v_j) = \frac{1}{n} \sum_{j=0}^{n} (v_i - v_j)$$

are orthogonal to H_i, i.e. $y \in H_i \iff \langle y - v_j, n_i \rangle = 0$ for some $j \neq i$. Hence the dihedral angle α is just

$$\arccos\left(\frac{|\langle n_0, n_1 \rangle|}{|n_0|^2}\right)$$

Note that they are all the same again by symmetry.

We calculate explicitly

$$\begin{aligned}
\frac{\langle n_0, n_1\rangle}{|n_0|^2} &= \frac{\langle v_0 - b_0, v_1 - b_1\rangle}{|v_0 - b_0|^2} = \frac{\frac{1}{n^2}\sum_{i,j=0}^n \langle v_0 - v_i, v_1 - v_j\rangle}{\frac{1}{n^2}\sum_{i,j=0}^n \langle v_0 - v_i, v_0 - v_j\rangle} \\
&= \frac{\sum_{i,j=0}^n \langle v_0 - v_i, v_1 - v_0\rangle + \sum_{i,j=0}^n \langle v_0 - v_i, v_0 - v_j\rangle}{\sum_{i,j=0}^n \langle v_0 - v_i, v_0 - v_j\rangle} \\
&= 1 - \frac{\sum_{i,j=0}^n \langle v_0 - v_i, v_0 - v_1\rangle}{\sum_{i,j=0}^n \langle v_0 - v_i, v_0 - v_j\rangle} \\
&= 1 - \frac{(n+1)\sum_{i=0}^n \langle v_0 - v_i, v_0 - v_1\rangle}{\sum_{i\neq j} \langle v_0 - v_i, v_0 - v_j\rangle + n|v_0 - v_1|^2} \\
&= 1 - \frac{(n+1)(|v_0 - v_1|^2 + \sum_{i=2}^n \langle v_0 - v_i, v_0 - v_1\rangle)}{\sum_{i=1}^n \sum_{j\neq i} \langle v_0 - v_i, v_0 - v_j\rangle + n|v_0 - v_1|^2} \\
&= 1 - \frac{(n+1)(|v_0 - v_1|^2 + (n-1)\langle v_0 - v_2, v_0 - v_1\rangle)}{n(n-1)\langle v_0 - v_2, v_0 - v_1\rangle + n|v_0 - v_1|^2} \\
&= 1 - \frac{n+1}{n} = -\frac{1}{n}
\end{aligned}$$

where we have used that for distinct i, j

$$|v_i - v_j| = |v_0 - v_1|$$

and for distinct i, j, k

$$\langle v_i - v_j, v_i - v_k\rangle = \langle v_0 - v_1, v_0 - v_2\rangle$$

Thus the dihedral angle is

$$\arccos\left(\frac{1}{n}\right)$$

Now clearly

$$\frac{2\pi}{\arccos\left(\frac{1}{n}\right)} \to 4 \quad (n \to \infty)$$

in a strictly monotonically decreasing fashion and

$$\frac{2\pi}{\arccos\left(\frac{1}{3}\right)} \approx 5.1043 \qquad , \qquad \frac{2\pi}{\arccos\left(\frac{1}{4}\right)} \approx 4.7668$$

Hence the dihedral angle is no submultiple of 2π for $n \geq 3$.

If we denote by ρ_i the reflection in the affine subspace H_i then by definition the simplex reflection group Λ of the regular simplex $(v_0, \dots, v_n)$ is the subgroup of $\mathrm{Isom}(\mathbb{R}^n)$ generated by these reflections, i.e. $\Lambda = \langle \rho_i : i = 0, \dots, n\rangle$. We shall now consider the subgroups $\Lambda_i = \langle \rho_j : j \neq i\rangle < \Lambda$ generated by all reflections in the faces adjacent to the vertex v_i. Further let $\tau_b : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto x+b$ denote the translation by $b \in \mathbb{R}^n$. Recall that every $\varphi \in \mathrm{Isom}(\mathbb{R}^n)$ has the form $\varphi(x) = \tau_b(Ax) = Ax + b$ for some $b \in \mathbb{R}^n$ and $A \in O(n)$.

We claim that each subgroup $L_i := \tau_{v_i}^{-1}\Lambda_i\tau_{v_i}$ is dense in $O(n)$. By symmetry it is sufficient to show this for v_0. Clearly L_0 is generated by the reflections $r_i = \tau_{v_0}^{-1}\rho_i\tau_{v_0}$ which are just the reflections at the subvectorspaces $V_i := \tau_{v_0}^{-1}H_i = \langle n_i\rangle^\perp$ $(i = 1, \dots, n)$.

We know that $O(n)$ is generated by all reflections. Hence in order to show that L_0 is dense in $O(n)$ it suffices to show that any reflection can be approximated arbitrarily good by elements of L_0. It is clear that for a reflection $r(x) = x - 2\langle x, \nu\rangle\nu$ and a sequence of reflections $r^{(k)}(x) = x - 2\langle x, \nu^{(k)}\rangle\nu^{(k)}$ with respective normal vectors ν and $\nu^{(k)}$ $(k \in \mathbb{N})$

$$r^{(k)} \to r \quad \text{in } O(n) \qquad \Longleftrightarrow \qquad \nu^{(k)} \to \nu \quad \text{in } S^{n-1}$$

Taking the reflection $r_1(x) = x - 2\langle x, \hat{n}_1\rangle \hat{n}_1$ in L_0 ($\hat{n}_1 := n_1/|n_1|$) and conjugating it by some other element $\psi \in L_0$ we get a reflection $\psi r_1 \psi^{-1}(x) = x - 2\langle x, \psi(\hat{n}_1)\rangle \psi(\hat{n}_1)$ with the normal vector $\psi(\hat{n}_1)$. Hence it suffices to see that the orbit $L_0\hat{n}_1$ is dense in S^{n-1}.

This will follow easily by our discussion of the dihedral angles. Every iterated reflection $r_i r_j$ in L_0 is a rotation by the dihedral angle in the 2-dimensional subvectorspace generated by n_i and n_j ($i, j \in \{1, \dots, n\}, i \neq j$). Because the dihedral angle is no submultiple of 2π the element $r_i r_j$ does not have finite-order and any rotation in the n_i-n_j-plane can be approximated arbitrarily good by powers of $r_i r_j$. By regularity of the simplex the vectors $n_1, \dots, n_n$ form a basis of $\mathbb{R}^n$ which is not orthonormal though. However it is still easy to see, that one can reach every point $\xi \in S^{n-1}$ by succesively rotating $\hat{n}_1$ in some n_i-n_j-plane. Approximating each rotation by a power of $r_i r_j$ we can hence approximate ξ by an element of $L_0\hat{n}_1$ arbitrarily good. Hence $L_0\hat{n}_1$ is dense in S^{n-1} and L_0 is dense in $O(n)$.

We have now seen that each $L_i = \tau_{v_i}^{-1}\Lambda_i\tau_{v_i}$ is dense in $O(n)$ and hence $\Lambda_i = \tau_{v_i}L_i\tau_{v_i}^{-1}$ is dense in $\tau_{v_i}O(n)\tau_{v_i}^{-1}$. In order to conclude the proof it will be enough to show that $\mathrm{Isom}(\mathbb{R}^n)$ is generated by elements of $G_0 = \tau_{v_0}O(n)\tau_{v_0}^{-1}$ and $G_1 = \tau_{v_1}O(n)\tau_{v_1}^{-1}$, i.e. $\mathrm{Isom}(\mathbb{R}^n) = \langle \tau_{v_0}O(n)\tau_{v_0}^{-1}, \tau_{v_1}O(n)\tau_{v_1}^{-1}\rangle =: G$. Indeed if $\varphi = s_1 \cdots s_m \in G$ for some $s_k \in G_0 \cup G_1$ we can approximate each by a sequence $s_k^{(l)}$ in Λ_0 resp. Λ_1, i.e. $s_k^{(l)} \to s_k$ for $l \to \infty$. By continuity of the group action we then get

$$s_1^{(l)} \cdots s_m^{(l)} \to s_1 \cdots s_m = \varphi \quad (l \to \infty)$$

and hence

$$\overline{\Lambda} \supseteq \mathrm{Isom}(\mathbb{R}^n)$$

Let $\varphi \in \mathrm{Isom}(\mathbb{R}^n)$. Then

$$\varphi \in \tau_{v_0}O(n)\tau_{v_0}^{-1} = G_0 \iff \tau_{v_0}^{-1}\varphi\tau_{v_0} \in O(n) \iff \tau_{v_0}^{-1}\varphi\tau_{v_0}(0) = 0 \iff \varphi(v_0) = v_0$$

If we can find $\psi \in G$, such that $\psi(v_0) = \varphi(v_0)$, then $\psi^{-1}\varphi(v_0) = v_0$ and hence $\psi^{-1}\varphi \in G_0 \subset G$. This implies that $\varphi \in \psi \cdot G = G$ and we are done.

Indeed G acts transitively on $\mathbb{R}^n$. G_0 and G_1 are just orthogonal transformations based at v_0 and v_1 respectively. It is now easy to see that by iteratively rotating around v_0 or v_1 one can send v_0 to any point in $\mathbb{R}^n$.

□

The result for regular ideal hyperbolic simplices will follow quite easily from the next observation.

Lemma I.7.11. *Let $n \geq 3$ and $i : \mathbb{H}^n \to \mathbb{H}^n$ an inversion at a sphere orthogonal to $\mathbb{R}^{n-1} \cong \mathbb{R}^{n-1} \times \{0\} \subset \partial U^n$ in the upper half space model U^n. Then* $\mathrm{Isom}(\mathbb{H}^n)$ *is generated by i and* $\mathrm{Isom}(\mathbb{R}^{n-1})$ *where we regard the latter as a subgroup of* $\mathrm{Isom}(\mathbb{H}^n)$ *as usual.*

Proof. Let L denote the group generated by $\mathrm{Isom}(\mathbb{R}^{n-1})$ and i. We already know, that $\mathrm{Isom}(\mathbb{H}^n)$ is generated by all reflections at half spaces through ∞ and inversions at spheres centered on $\mathbb{R}^{n-1}$, i.e. inversions at (generalized) spheres orthogonal to the boundary. Since all reflections in half spaces through ∞ are already contained in $\mathrm{Isom}(\mathbb{R}^{n-1})$, we only need to show that also every inversion at an arbitrary sphere orthogonal to $\mathbb{R}^{n-1}$ is in L. Observe that for $\psi \in \mathrm{Isom}(\mathbb{R}^{n-1}) \subset \mathrm{Isom}(\mathbb{H}^n)$ and an inversion $i \in L$ with center $m \in \mathbb{R}^{n-1}$ and radius r also $\psi \circ i \circ \psi^{-1} \in L$ is an inversion at a sphere of radius r but with center $\psi(m)$, i.e. we can move inversions at spheres around by conjugating with elements in $\mathrm{Isom}(\mathbb{R}^{n-1})$. Thus we only need to see that there is for every $R > 0$ an inversion at a sphere with radius R in L.

We can construct a new inversion from the given one i in the following way. Let m be the center of the inversion i and r its radius, i.e. i is the inversion at the sphere $S(m, r)$. Consider a reflection ρ in an affine half space H at distance $d \in (0, r)$ from m. That is, H intersects $S(m, r)$ in more than one point and does not contain the center m. Since i and ρ are both inversions at generalized spheres, $i\rho i^{-1} = i\rho i$ is also an inversion at a generalized sphere. Observe that $\mathrm{Fix}(\rho) = H$, $\mathrm{Fix}(i) = S(m, r)$ and $\mathrm{Fix}(i\rho i) \supset \mathrm{Fix}(\rho) \cap \mathrm{Fix}(i)$. Also

$$i\rho i(m) = i\rho(\infty) = i(\infty) = m$$

and thus $\mathrm{Fix}(i\rho i) \supset (\mathrm{Fix}(\rho) \cap \mathrm{Fix}(i)) \cup \{m\}$. Because $m \notin \mathrm{Fix}(\rho) = H$ the union is disjoint. Therefore $i\rho i$ must be an inversion at a proper sphere $S(m', R)$ orthogonal to $\mathbb{R}^{n-1}$ through $(\mathrm{Fix}(\rho) \cap \mathrm{Fix}(i)) \cup \{m\}$. In particular m' lies on the line through m meeting H orthogonally in some point ξ. Note that ξ realizes the distance d between H and m in $\mathbb{R}^{n-1}$. By construction all points in $H \cap S(m, r)$ have the same distance from ξ - say h. Thus the following two equalities hold

$$h^2 + d^2 = r^2 \tag{I.1}$$

$$h^2 + (R - d)^2 = R^2 \tag{I.2}$$

Eliminating h^2 and solving for R yields

$$R = \frac{r^2}{2d}$$

Because $0 < d < r$ was arbitrary we can use the above construction to get an inversion at a sphere of arbitrary radius $R \in \left(\frac{r}{2}, \infty\right)$. Hence we can indeed construct by iteration an inversion to every given radius $R > 0$. □

Proposition I.7.12. *Let $n \geq 4$. Then the reflection group of a regular ideal hyperbolic n-simplex is dense in* $\mathrm{Isom}(\mathbb{H}^n)$.

Proof. Let $(\xi_0, \dots, \xi_n) \in (\partial\mathbb{H}^n)^{n+1}$ be the vertices of a regular ideal n-simplex. Without loss of generality we may assume that $\xi_0 = \infty$ in the upper half space model. Then $\xi_1, \dots, \xi_n \in \mathbb{R}^{n-1}$ are the vertices of a regular euclidean $(n-1)$-simplex. The reflection group Λ of the simplex $(\xi_0, \dots, \xi_n)$ is now generated by the reflections in the codimension 1 half spaces H_i through $\{\xi_0, \dots, \xi_n\} - \{\xi_i\}$ $(i = 1, \dots, n)$ and the inversion i at the sphere through $\xi_1, \dots, \xi_n$.

Denote by Λ' the subgroup generated by the reflections in the half spaces H_i. Clearly Λ' corresponds to the reflection group of the regular euclidean $(n-1)$-simplex $(\xi_1, \dots, \xi_n)$ and is hence by Proposition I.7.10 dense in $\mathrm{Isom}(\mathbb{R}^{n-1})$. Because $\mathrm{Isom}(\mathbb{H}^n)$ is generated by i and $\mathrm{Isom}(\mathbb{R}^{n-1})$ we have that

$$\overline{\Lambda} = \overline{\langle i, \Lambda' \rangle} \supseteq \langle i, \overline{\Lambda'} \rangle = \langle i, \mathrm{Isom}(\mathbb{R}^{n-1}) \rangle = \mathrm{Isom}(\mathbb{H}^n)$$

□

I.8. Boundary Maps

As in the classical proof of the Mostow Rigidity Theorem outlined in [Thu, Chapter 5, §9, pp. 106] boundary maps will play an important role in our proof of the volume rigidity theorem too. The reason for their use is, that boundary maps can be constructed quite easily and under certain conditions they are induced by an isometry of hyperbolic n-space; we will then also say, that the boundary map is "equal" to an isometry. The objective of this section is to prove Proposition I.8.3, which gives a condition for when a boundary map is essentially an isometry. This fact was used by Thurston in his revision of Gromov's proof of the Mostow Rigidity Theorem in [Thu, Chapter 6, §3, pp. 133]. It is worth noting, that the results of this section only work in dimenson $n \geq 3$. They will be needed the final step of the proof of the volume rigidity theorem and they are in fact the only places where we require $n \geq 3$.

We will again use the notation of the previous section and denote by T the set of $(n+1)$-tuples in $\partial\mathbb{H}^n$ which are vertices of a regular ideal n-simplex. Recall Proposition I.7.3 which states, that the map $\Phi_{\bar\eta} : G \to T, g \mapsto g\bar\eta$ is a diffeomorphism for every $\bar\eta \in T$. We will call an $(n+1)$-tuple in T simply a regular simplex. Note that the order of the vertices $\xi_0, \dots, \xi_n$ induces an orientation on the simplex $\bar\xi$. For $\bar\xi \in T$, denote by $\Lambda_{\bar\xi} < \mathrm{Isom}(\mathbb{H}^n)$ the reflection group generated by the reflections in the faces of the simplex $\bar\xi$.

Lemma I.8.1 (cf. [BBI13, Lemma 7, p. 26])**.** *Let $n \geq 3$. Let $\bar\xi = (\xi_0, \dots, \xi_n) \in T$. Suppose that $\varphi\colon \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ is a map such that for every $\gamma \in \Lambda_{\bar\xi}$ the simplex with vertices $(\varphi(\gamma\xi_0), \dots, \varphi(\gamma\xi_n))$ is regular and of the same orientation as $(\gamma\xi_0, \dots, \gamma\xi_n) \in T$.*

Then there exists a unique isometry $h \in \mathrm{Isom}^+(\mathbb{H}^n)$ such that $h(\xi) = \varphi(\xi)$ for every $\xi \in \cup_{i=0}^n \Lambda_{\bar\xi}\xi_i$. In particular the isometry is given by $\Phi_{\bar\xi}^{-1}(\varphi(\bar\xi))$ and we have the formula

$$\Phi_{\bar\xi}^{-1}(\varphi(\bar\xi)) = \Phi_{\bar\eta}^{-1}(\varphi(\bar\xi)) \cdot \left(\Phi_{\bar\eta}^{-1}(\bar\xi)\right)^{-1}$$

for any $\bar\eta \in T$.

Remark I.8.2. *Note that this lemma fails for n = 2. Indeed, any triple of distinct boundary points in $\partial\mathbb{H}^2$ are the vertices of a regular ideal simplex. Thus any orientation preserving homeomorphism of $\partial\mathbb{H}^2$ would satisfy the hypothesis. However not every orientation preserving homeomorphism is already induced by an isometry as one readily checks.*

Proof. Let $\bar\xi = (\xi_0, \dots, \xi_n) \in T$. Then $(\varphi(\xi_0), \dots, \varphi(\xi_n)) \in T$ and it has the same orientation as $\bar\xi$. Hence there is a unique isometry $h \in \mathrm{Isom}^+(\mathbb{H}^n)$ such that $h(\xi_i) = \varphi(\xi_i)$ for $i = 0, \dots, n$; namely $\Phi_{\bar\xi}^{-1}(\varphi(\bar\xi))$.

It remains to check that

$$h(\gamma\xi_i) = \varphi(\gamma\xi_i) \tag{I.3}$$

for every $\gamma \in \Lambda_{\bar\xi}$. Every $\gamma \in \Lambda_{\bar\xi}$ is a product $\gamma = r_k \cdot \ldots \cdot r_1$, where r_j is a reflection in a face of the regular simplex $r_{j-1} \cdot \ldots \cdot r_1(\bar\xi)$. We prove the equality (I.3) by induction on k, the case $k = 0$ being true by assumption. Set $\eta_i = r_{k-1} \cdot \ldots \cdot r_1(\xi_i)$. Our induction hypothesis is, that $h(\eta_i) = \varphi(\eta_i)$. The induction step will be proven, if we show that $h(r_k\eta_i) = \varphi(r_k\eta_i)$ for all $i = 0, \dots, n$. The simplex $(\eta_0, \dots, \eta_n)$ is regular and r_k is a reflection in one of its faces, say without loss of generality the face containing $\eta_1, \dots, \eta_n$. Since $r_k\eta_i = \eta_i$ for $i = 1, \dots, n$ and by the induction hypothesis $h(\eta_i) = \varphi(\eta_i)$, we obtain

$$h(r_k\eta_i) = h(\eta_i) = \varphi(\eta_i) = \varphi(r_k\eta_i) \quad \forall i = 1, \dots, n$$

and it just remains to show, that $h(r_k\eta_0) = \varphi(r_k\eta_0)$. Now the simplex $(r_k\eta_0, r_k\eta_1, \ldots, r_k\eta_n) = (r_k\eta_0, \eta_1, \ldots, \eta_n)$ is regular with opposite orientation to $(\eta_0, \ldots, \eta_n)$. Since h is orientation preserving, this implies that the simplex $(h(r_k\eta_0), h(\eta_1), \ldots, h(\eta_n))$ is regular with opposite orientation to $(h(\eta_0), h(\eta_1), \ldots, h(\eta_n))$. By assumption $(\varphi(r_k\eta_0), \varphi(r_k\eta_1), \ldots, \varphi(r_k\eta_n)) = (\varphi(r_k\eta_0), \varphi(\eta_1), \ldots, \varphi(\eta_n))$ is regular with opposite orientation to $(\varphi(\eta_0), \ldots, \varphi(\eta_n))$. Because $(h(\eta_0), \ldots, h(\eta_n)) = (\varphi(\eta_0), \ldots, \varphi(\eta_n))$ and in dimension $n \geq 3$ there is only one regular simplex with face $h(\eta_1), \ldots, h(\eta_n)$ and opposite orientation to $(h(\eta_0), \ldots, h(\eta_n))$ it follows that $h(r_k\eta_0) = \varphi(r_k\eta_0)$ and the induction step is proven.

The formula follows from Proposition I.7.3 with $h = \Phi_{\bar{\xi}}^{-1}(\varphi(\bar{\xi}))$ and

$$\varphi(\bar{\xi}) = \Phi_{\bar{\xi}}(h) = \Phi_{\bar{\eta}}(h \cdot \Phi_{\bar{\eta}}^{-1}(\bar{\xi})) \iff h = \Phi_{\bar{\eta}}^{-1}(\varphi(\bar{\xi})) \cdot \left(\Phi_{\bar{\eta}}^{-1}(\bar{\xi})\right)^{-1}$$

□

Proposition I.8.3 (cf. [BBI13, Proposition 6, p. 27])**.** *Let $n \geq 3$. Let $\varphi\colon \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ be a measurable map sending the vertices of almost every positively, resp. negatively, oriented regular ideal simplex to the vertices of a positively, resp. negatively, oriented regular ideal simplex with the same orientation. Then φ is essentially equal to an isometry (up to a null set).*

We shall first prove this result for dimensions $n \geq 4$ since the proof is easier. This is where Proposition I.7.12 comes in.

Proof in the case of $n \geq 4$. Let $T^\varphi \subset T$ be the set of regular simplices $\bar{\xi} = (\xi_0, \ldots, \xi_n) \in T$ such that $(\varphi(\xi_0), \ldots, \varphi(\xi_n))$ is also in T and has the same orientation as $(\xi_0, \ldots, \xi_n)$. By assumption T^φ has full measure in T. Now consider the subset

$$T_\Lambda^\varphi \coloneqq \{\bar{\xi} \in T \,|\, \gamma\bar{\xi} \in T^\varphi \quad \forall \gamma \in \Lambda_{\bar{\xi}}\} \subset T^\varphi$$

of those regular simplices for which all reflections by the reflection group $\Lambda_{\bar{\xi}}$ are in T^φ. We claim that T_Λ^φ has full measure in T.

Again we use the identification $\Phi_{\bar{\eta}} : G \to T$ as before, where $\bar{\eta} \in T$ is some reference point. The subset T^φ is mapped to a subset $G^\varphi \coloneqq \Phi_{\bar{\eta}}^{-1}(T^\varphi) \subset G$ via this correspondence. Observe that a regular simplex $\bar{\xi} = g(\bar{\eta})$ is in T_Λ^φ if and only if, $\gamma\bar{\xi} = \gamma g\bar{\eta}$ is in T^φ for every $\gamma \in \Lambda_{\bar{\xi}}$. One readily checks that $\Lambda_{\bar{\xi}} = g\Lambda_{\bar{\eta}}g^{-1}$, so the latter condition is equivalent to $g\gamma_0\bar{\eta} \in T^\varphi$ for every $\gamma_0 \in \Lambda_{\bar{\eta}}$, or in other words $g \in G^\varphi\gamma_0^{-1}$. Hence the subset T_Λ^φ is mapped to

$$G^\varphi = \Phi_{\bar{\eta}}^{-1}(T_\Lambda^\varphi) = \bigcap_{\gamma_0 \in \Lambda_\eta} G^\varphi\gamma_0^{-1} \subset G$$

Since a countable intersection of full measure subsets has full measure, the claim is proved.

For every $\bar{\xi} \in T_\Lambda^\varphi$ and hence almost every $\bar{\xi} \in T$ there exists by Lemma I.8.1 a unique isometry $h_{\bar{\xi}}$ such that $h_{\bar{\xi}}(\xi) = \varphi(\xi)$ on the orbit points $\xi \in \cup_{i=0}^n \Lambda_{\bar{\xi}}\xi_i$. By the uniqueness of the isometry, it is immediate that $h_{\gamma\bar{\xi}} = h_{\bar{\xi}}$ for every $\gamma \in \Lambda_{\bar{\xi}}$. We have thus a measurable map $h\colon T \to \mathrm{Isom}(\mathbb{H}^n)$ given by

$$\bar{\xi} \mapsto h_{\bar{\xi}} = \Phi_{\bar{\xi}}^{-1}(\varphi(\bar{\xi})) = \Phi_{\bar{\eta}}^{-1}(\varphi(\bar{\xi})) \cdot \left(\Phi_{\bar{\eta}}^{-1}(\bar{\xi})\right)^{-1}$$

defined on a full measure subset of T. Precomposing h by $\Phi_{\bar{\eta}}$ it is straightforward that the left $\Lambda_{\bar{\xi}}$-invariance of h on $\Lambda_{\bar{\xi}}\bar{\xi}$ naturally translates to a global right invariance of $h \circ \Phi_\eta$ on G. Indeed, let $g \in G$ and $\gamma_0 \in \Lambda_{\bar{\eta}}$. We compute

$$h \circ \Phi_{\bar{\eta}}(g \cdot \gamma_0) = h_{g\gamma_0\bar{\eta}} = h_{g\gamma_0 g^{-1} g\bar{\eta}} = h_{g\bar{\eta}} = h \circ \Phi_{\bar{\eta}}(g)$$

where we have used the left $\Lambda_{g\bar{\eta}}$-invariance of h on the reflections of $g\bar{\eta}$ in the third equality (recall $g\gamma_0 g^{-1} \in g\Lambda_{\bar{\eta}}g^{-1} = \Lambda_{g\bar{\eta}}$). Thus $h \circ \Phi_{\bar{\eta}}\colon G \to G$ is invariant under the right action of $\Lambda_{\bar{\eta}}$ (and measurable). Since the latter group is dense in G (cf. Proposition I.7.12) , it acts ergodically on G (cf. Lemma I.5.5) and $h \circ \Phi_{\bar{\eta}}$ is essentially constant (cf. Theorem I.5.4). This means that also h is essentially constant. Thus for almost every regular simplex $\bar{\xi} \in T$ the evaluation of φ on any orbit point of the vertices of $\bar{\xi}$ under the reflection group $\Lambda_{\bar{\xi}}$ is equal to h. In particular for almost every $\bar{\xi} = (\xi_0, \ldots, \xi_n) \in T$ and also for almost every $\xi_0 \in \partial\mathbb{H}^n$, we have $\varphi(\xi_0) = h(\xi_0)$. □

Now we turn to the proof for $n = 3$. We give essentially the proof of Dunfield in [Dun99, pp. 30] which is a more rigorous outline of Thurston's idea in [Thu, Chapter 6, §3, pp. 133].

Proof in the case of $n = 3$. As above let $T^\varphi \subset T$ be the set of regular simplices $\bar{\xi} = (\xi_0, \ldots, \xi_3) \in T$ such that $(\varphi(\xi_0), \ldots, \varphi(\xi_3))$ is also in T and has the same orientation as $(\xi_0, \ldots, \xi_3)$. By assumption T^φ has full measure in T. Let $\bar{\xi} = (\xi_0, \ldots, \xi_3) \in T^\varphi \subset T$ be such a simplex. By composing and precomposing φ with suitable isometries we may assume without loss of generaltiy that $\xi_0 = \infty$ and $\varphi(\xi_0) = \infty$ in the upper half space model of $\mathbb{H}^3$. As we have already seen all oriented regular simplices with one vertex at infinity can be identified with the (oriented) equilateral triangles in $\mathbb{C} \cong \mathbb{R}^2$. For almost all lines l through 0, almost all equilateral triangles with the edge between the first and second vertices parallel to l define tetrahedra which are in T^φ. We can assume without loss of generatliy that one such line is the real axis (apply a suitable isometry). Let $\mathcal{S}$ denote the set of regular simplices (tetrahedra) with first vertex at ∞ and such that the edge between the second and third vertices (the first and second vertices of the corresponding triangle) is parallel to the real axis.

We know that $\mathcal{S}^\varphi := \mathcal{S} \cap T^\varphi$ has full measure in $\mathcal{S}$. Let ω be the $\sqrt[3]{-1}$ which has positive imaginary part. Then $\{0, 1, \omega\}$ is an oriented equilateral triangle. Let L_0 be all equilateral triangles in the tiling of $\mathbb{C}$ by the triangle $\{0, 1, \omega\}$. Let L_k denote the same set of triangles scaled by 2^{-k}. Let $L = \bigcup_{k \in \mathbb{Z}} L_k$ be the nested family of equitriangular lattices.

We claim there is an $r \in \mathbb{R}$ such that for almost all $z \in \mathbb{C}$ the entire countable set of triangles $z + rL$ are in $\mathcal{S}^\varphi$. Consider the submersion $\pi\colon \mathbb{C} \times \mathbb{R} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \to \mathcal{S}$ which sends (z, r, k, n, m) to the equilateral triangle with vertices

$$\Big(z + r2^{-k}(n + m\omega), z + r2^{-k}(n + 1 + m\omega), z + r2^{-k}(n + (m + 1)\omega)\Big)$$

in $z + rL_k$. We will think of $\mathbb{Z}$ as having a finite measure ν, say $\nu(\{q\}) = 1/q^2$. As π is a submersion, $\pi^{-1}(\mathcal{S}^\varphi)$ has full measure. Thus by Fubini, for almost all r and z, we have $\pi^{-1}(\mathcal{S}^\varphi) \cap (\{r\} \times \{z\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})$ has full measure and is hence equal to $\{r\} \times \{z\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, as desired. Without loss of generality assume $r = 1$ has this property (again apply a suitable isometry). So for almost all $z \in \mathbb{C}$ all triangles in $z + L$ are in $\mathcal{S}^\varphi$. This forces $\varphi(z + L)$ to be a family of nested equitriangular lattices. Indeed one easily checks this fact by picking a triangle in $z + L$ and applying its reflection group as in our discussion of the proof for $n \geq 4$. Hence there is for each z a complex number $h(z)$ such that:

$$\varphi\Big(z + 2^{-k}(n + m\omega)\Big) = \varphi(z) + h(z)2^{-k}(n + m\omega) \tag{I.4}$$

for all $n, m, k \in \mathbb{Z}$. We claim that the function h is invariant under the group of translations of the form $z \mapsto z + 2^{-j}(a + b\omega)$ where $j, a, b \in \mathbb{Z}$. Let $z' = z + 2^{-j}(a + b\omega)$. We have by (I.4)

$$\varphi(z') = \varphi(z) + h(z)2^{-j}(a + b\omega) \tag{I.5}$$

Now at $z' + 2^{-j}$ we have by (I.4)

$$\varphi(z' + 2^{-j}) = \varphi(z') + h(z')2^{-j}$$

Since $z' + 2^{-j} = z + 2^{-j}(a + 1 + b\omega)$ we also have that

$$\varphi(z' + 2^{-j}) = \varphi(z) + h(z)2^{-j}(a + 1 + b\omega)$$

Putting these together we get

$$\varphi(z') + h(z')2^{-j} = \varphi(z) + h(z)2^{-j}(a + 1 + b\omega) = \varphi(z' + 2^{-j}) \tag{I.6}$$

By subtracting equation (I.5) from (I.6) and dividing by 2^{-j} we get $h(z) = h(z')$ as desired. Because our group of translations is dense, and so acts ergodically, h is constant almost everywhere. But then $\varphi(z') = \varphi(z) + h2^{-j}(a + b\omega)$ almost everywhere which implies that $\varphi(z) - h \cdot z$ is invariant under our group of transformations. So there is a constant c such that $\varphi(z) - h \cdot z = c$ almost everywhere and thus $\varphi(z) = c + hz$ almost everywhere.

Therefore φ is essentially a euclidean similarity and hence an isometry of $\mathbb{H}^3$. $\square$

II. Cohomology

Continuous bounded cohomology is at the core of our study of volume rigidity. Hence we want to cover the most important results of both continuous and continuous bounded cohomology, and apply them to the isometry group of hyperbolic n-space in this chapter. Both theories allow a functorial characterization which provide us with plenty of resolutions to compute the respective cohomology from. We will present these characterizations for continuous and continuous bounded cohomology together with some resolutions in section II.1 and in section II.2 respectively. They both follow a similar outline underlining the similarities and the differences between the two theories. In favour of a more concise exposition and in order to emphasize the theoretical constructions we do not give any examples in these sections. Instead we apply the developed theory to the case of $G = \mathrm{Isom}(\mathbb{H}^n)$ in section II.3. We do so in a way that is geared towards the proof of the volume rigidity theorem.

If not otherwise mentioned G denotes a locally compact second countable topological group and $n \geq 2$ in this chapter.

II.1. Continuous Cohomology

Let us present the key features of continuous cohomology in this section. Subsection II.1.1 gives a first hands-on definition of continuous cohomology omitting completely a functorial point of view and without any homological algebra. In subsection II.1.2 we then introduce some important notions of homological algebra and present the functorial characterization of continuous cohomology in terms of strong resolutions by relatively injective G-modules. Based on this functorial characterization we will then give some resolutions in subsection II.1.3, which will be important in the context of the volume rigidity theorem later on.

Our main reference for this section is [Gui80]. We will make extensive use of the terminology of G-modules as discussed in section B.1 of the appendix.

II.1.1. Naive Definition

Let (π, E) be a G-module. We shall frequently omit the actual representation π, if there is no ambiguity, i.e. $\pi(g)v = gv$ for all $g \in G$ and $v \in E$.

Let $q \geq 0$. Consider the spaces of continuous functions on G^{q+1} with values in E

$$C(G^{q+1}, E) := \{f : G^{q+1} \to E : f \text{ is continuous}\}.$$

$C(G^{q+1}, E)$ itself becomes a G-module via the left regular representation

$$(\lambda_\pi(g)f)(g_0, \ldots, g_q) := \pi(g)f(g^{-1}g_0, \ldots, g^{-1}g_q) \tag{II.1}$$

for $f \in C(G^{q+1}, E)$ and $g, g_0, \ldots, g_q \in G$. If it is clear from the context we shall simply write $g \cdot f$ instead of $\lambda_\pi(g)f$ in the following.

We can consider the subspaces of G-invariant functions

$$C(G^{q+1}, E)^G := \{f : G^{q+1} \to E : f \text{ is continuous and } g \cdot f = f\}$$

Note that $g \cdot f = f$ reads

$$gf(g_0, \dots g_q) = f(gg_0, \dots, gg_q) \qquad \forall g, g_0, \dots, g_q \in G$$

For that reason elements of $C(G^{q+1}, E)^G$ will be called *G-equivariant*. Moreover if (π, E) is a trivial G-module, i.e. $\pi(g)v = gv = v$ for all $g \in G, v \in E$, an element of $C(G^{q+1}, E)^G$ will be called *G-invariant*.

We now define the *homogeneous* coboundary operator $d^q : C(G^{q+1}, E) \to C(G^{q+2}, E)$ via

$$d^{q+1}(f)(g_0, \dots, g_{q+1}) = \sum_{i=0}^{q+1} (-1)^i f(g_0, \dots, \hat{g}_i, \dots, g_{q+1})$$

where the hat indicates omission of the variable underneath. We shall simply write d if q is understood. It is easy to check, that d is indeed a G-morphism and hence restricts to the subspace of invariants

$$d : C(G^{q+1}, E)^G \to C(G^{q+2}, E)^G$$

A standard calculation shows that $d \circ d = 0$ which yields the following cochain complex

$$0 \to C(G, E)^G \to C(G^2, E)^G \to \dots \to C(G^{q+1}, E)^G \to C(G^{q+2}, E)^G \to \dots$$

The *continuous cohomology* $H_c^\bullet(G, E)$ of G with coefficients in E is the cohomology of the above cochain complex

$$H_c^q(G, E) = \frac{\ker\{d^{q+1} : C(G^{q+1}, E)^G \to C(G^{q+2}, E)^G\}}{\operatorname{im}\{d^q : C(G^q, E)^G \to C(G^{q+1}, E)^G\}}.$$

Pullback

Let H be another locally compact second countable topological group and $\rho : H \to G$ a continuous homomorphism. Pulling back functions on G^{q+1} via ρ yields the so-called *pullback* $\rho^* : C(G^{q+1}, E) \to C(H^{q+1}, E)$

$$(\rho^* f)(h_0, \dots, h_q) := f(\rho(h_0), \dots, \rho(h_q)) \tag{II.2}$$

for $f \in C(G^{q+1}, E)$ and $h_0, \dots, h_q \in H$. Precomposing the G-representation π on E with ρ turns the G-module (π, E) into the H-module $(\pi\rho, E)$; or shorter $\rho^* E$.

Observe that for a G-equivariant function $f \in C(G^{q+1}, E)^G$ the image $\rho^*(f)$ is H-equivariant with respect to the induced representation $\pi\rho : H \to G \to \operatorname{Aut}(E)$. Indeed,

$$\begin{aligned}
(\lambda_{\pi\circ\rho}(h)(\rho^* f))(h_0, \dots, h_q) &= \pi(\rho(h))(\rho^* f)(h^{-1}h_0, \dots, h^{-1}h_q) && \text{(II.3)}\\
&= \pi(\rho(h)) f(\rho(h^{-1}h_0), \dots, \rho(h^{-1}h_q)) && \text{(II.4)}\\
&= \pi(\rho(h))\pi(\rho(h^{-1})) f(\rho(h_0), \dots, \rho(h_q)) && \text{(II.5)}\\
&= \rho^* f(h_0, \dots, h_q) && \text{(II.6)}
\end{aligned}$$

Hence $\rho^* : C(G^{q+1}, E) \to C(H^{q+1}, E)$ restricts to a map on the subspaces of invariants

$$\rho^* : C(G^{q+1}, E)^G \to C(H^{q+1}, \rho^* E)^H$$

One immediately checks that it also commutes with the homogeneous coboundary operator d, i.e. $\rho^* \circ d = d \circ \rho^*$, and thus induces a pullback on cohomology

$$\rho^* : H_c^\bullet(G, E) \to H_c^\bullet(H, \rho^* E)$$

II.1.2. Functorial Characterization

Basic Definitions

We will start with some notions from ordinary homological algebra in the category of LCTVS and G-modules. For the basics on LCTVS and G-modules we refer to section B.1 in the appendix.

A *complex* $(E^\bullet, d^\bullet)$ *of G-modules*, or *complex* for short, is a $\mathbb{Z}$-indexed sequence

$$\cdots \longrightarrow E^{n-1} \xrightarrow{d^n} E^n \xrightarrow{d^{n+1}} E^{n+1} \longrightarrow \cdots$$

of G-modules E^n and G-morphisms d^n such that $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$. The G-morphisms d^n are called *differentials* or *coboundary operators*, and elements of E^n are referred to as *cochains* of *degree n*. A *right complex* is a complex $(E^\bullet, d^\bullet)$ such that $E^n = 0$ for all $n < 0$, and will also be considered as a $\mathbb{N}_0$-indexed sequence. A complex $(E^\bullet, d^\bullet)$ is also said to *start at degree $k \in \mathbb{Z}$* if $E^n = 0$ for all $n < k$.

Remark II.1.1. *As in our "naive definition" in the previous section we will most of the time omit the superscript and simply write d instead of d^n. In diagrams we will often drop this label altogether and content ourselves with a horizontal arrow. Accordingly, we denote the comples $(E^\bullet, d^\bullet)$ simply by $E^\bullet$.*

A complex $(E^\bullet, d^\bullet)$ is said to be *exact at degree $k \in \mathbb{Z}$* if $\ker(d^{k+1}) = \mathrm{im}(d^k)$. If a complex is exact at every degree, we simply call it *exact* or sometimes an *exact sequence*.

A complex $E^\bullet$ is said to have a property $\mathcal{P}$ whenever all E^n $(n \in \mathbb{Z})$ share the property $\mathcal{P}$.

By $E^{\bullet G}$ we denote the subcomplex

$$\cdots \longrightarrow (E^{n-1})^G \longrightarrow (E^n)^G \longrightarrow (E^{n+1})^G \longrightarrow \cdots$$

of G-invariants.

A *morphism of complexes* $\alpha^\bullet : E^\bullet \to F^\bullet$ is a sequence α^n $(n \in \mathbb{Z})$ of morphisms $E^n \to F^n$ such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & E^{n-1} \arrow[r] \arrow[d, "\alpha^{n-1}"'] & E^n \arrow[r] \arrow[d, "\alpha^n"'] & E^{n+1} \arrow[r] \arrow[d, "\alpha^{n+1}"'] & \cdots \\
\cdots \arrow[r] & F^{n-1} \arrow[r] & F^n \arrow[r] & F^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

commutes. A *G-morphism of complexes* is a morphism of complexes consisting of G-morphisms. The identity and zero morphisms of complexes are simply denoted by Id and 0 respectively.

If $\alpha^\bullet$ and $\beta^\bullet$ are two morphisms of complexes from $(E^\bullet, d^\bullet)$ to $(F^\bullet, \partial^\bullet)$, a *homotopy* $\sigma^\bullet$ from $\alpha^\bullet$ to $\beta^\bullet$ is a sequence of morphisms $\sigma^n : E^n \to F^{n-1}$ $(n \in \mathbb{Z})$ such that

$$\partial^n \sigma^n + \sigma^{n+1} d^{n+1} = \beta^n - \alpha^n$$

for all $n \in \mathbb{Z}$, as depicted in the diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & E^{n-1} \arrow[r] \arrow[dl] \arrow[d, bend right] \arrow[d, bend left] & E^{n} \arrow[r] \arrow[dl] \arrow[d, bend right] \arrow[d, bend left] & E^{n+1} \arrow[r] \arrow[dl] \arrow[d, bend right] \arrow[d, bend left] & \cdots \arrow[dl] \\
\cdots \arrow[r] & F^{n-1} \arrow[r] & F^{n} \arrow[r] & F^{n+1} \arrow[r] & \cdots
\end{tikzcd}
$$

When such a homotopy exists, then $\alpha^\bullet$ is said to be *homotopic* to $\beta^\bullet$. This definition is an equivalence relation since the definition of homotopies is additive. Notice that by definition $\alpha^\bullet$ is homotopic to $\beta^\bullet$ if and only if the zero morphism of complexes is homotopic to $\beta^\bullet - \alpha^\bullet$. A morphism of complexes is said to be *null homotopic* if it is homotopic to zero.

Remark II.1.2. *Given three complexes $A^\bullet$, $B^\bullet$, $C^\bullet$ and morphisms $\alpha : A^\bullet \to B^\bullet$, $\beta_1^\bullet, \beta_2^\bullet : B^\bullet \to C^\bullet$ where $\beta_1^\bullet$ and $\beta_2^\bullet$ are supposed to be homotopic via $h^\bullet : B^\bullet \to C^{\bullet-1}$, it is easy to check that*

$$H^\bullet := h^\bullet \circ \alpha^\bullet : A^\bullet \to C^{\bullet-1}$$

is a homotopy between $\beta_1^\bullet \circ \alpha^\bullet$ and $\beta_2^\bullet \circ \alpha^\bullet$, i.e. they are homotopic morphisms.

A morphism of complexes $\alpha^\bullet : E^\bullet \to F^\bullet$ is called a *homotopy equivalence* if there is a morphism of complexes $\beta^\bullet : F^\bullet \to E^\bullet$ such that $\alpha^\bullet\beta^\bullet$ and $\beta^\bullet\alpha^\bullet$ are homotopic to the identity morphism of the respective complexes.

A complex $E^\bullet$ is said to admit a *contracting homotopy* $h^\bullet$ if $h^\bullet$ is a homotopy from $0 : E^\bullet \to E^\bullet$ to $\mathrm{id} : E^\bullet \to E^\bullet$. We want to emphasize, that $h^\bullet$ is not necessarily a G-morphism. We call a complex $E^\bullet$ *strong* if it admits a contracting homotopy.

The n-th cohomology space of a complex $(E^\bullet, d^\bullet)$ is by definition the quotient

$$H^n(E^\bullet) = H^n(E^\bullet, d^\bullet) = \ker(d^{n+1})/\mathrm{im}(d^n)$$

Any morphism of complexes $\alpha^\bullet : E^\bullet \to F^\bullet$ induces a sequence of continuous linear maps

$$\alpha^n : H^n(E^\bullet) \to H^n(F^\bullet)$$

as is known from usual homological algebra. The morphism of complexes $\alpha^\bullet$ is called a *homologism* if the induced map α^n is an isomorphism of topological vector spaces for all $n \in \mathbb{Z}$; *G-homologisms* are defined accordingly.

It follows from the definition of homotopies that homotopic morphisms of complexes induce identical maps in cohomology, so that in particular any homotopy equivalence is a homologism.

Remark II.1.3. *Observe that G-morphisms of complexes $E^\bullet \to F^\bullet$ as well as G-homotopies restrict to the continuous subcomplexes and restrict further to morphisms of complexes and homotopies $(E^\bullet)^G \to (F^\bullet)^G$ of the subcomplex of invariants. In particular, a G-homologism $E^\bullet \to F^\bullet$ induces a homologism $(E^\bullet)^G \to (F^\bullet)^G$.*

Let E be a G-module, $(E^\bullet, d^\bullet)$ a right complex of G-modules and $\mathfrak{a} : E \to E^0$ a G-morphism, such that

$$0 \longrightarrow E \xrightarrow{\ \mathfrak{a}\ } E^0 \xrightarrow{\ d^1\ } E^1 \xrightarrow{\ d^2\ } E^2 \xrightarrow{\ d^3\ } \cdots$$

is an exact complex (starting at degree -1). Then the latter complex is called an *(augmented) resolution of* E and is denoted by $(\mathfrak{a}, E^\bullet)$. The G-morphism $\mathfrak{a} : E \to E^0$ is then called the *augmentation.*

Definition II.1.4. Let E, F be LCTVS. An injective morphism $\iota : E \to F$ is called *strongly injective* if it admits a left-inverse morphism, i.e. a continuous linear map $\sigma : F \to E$ such that $\sigma \circ \iota = \mathrm{id}$.

Definition II.1.5. A G-module E is *relatively injective* if for every strongly injective G-morphism $\iota : A \to B$ of G-modules A, B and every G-morphism $\alpha : A \to E$ there is a G-morphism $\beta : B \to E$ satisfying $\beta\iota = \alpha$.

$$\begin{tikzcd} A \arrow[rr, hook, "\iota"'] \arrow[dr, "\alpha"'] & & B \arrow[ll, bend right, "\sigma"'] \arrow[dl, dotted, "\exists\beta"] \\ & E & \end{tikzcd}$$

If there is any ambiguity as to the group, we say that E is G-relatively injective.

Statement of the Functorial Characterization

We are now in a position to state the key theorem and some important lemmas in view of the functorial characterization of continuous cohomology in terms of strong resolutions by relatively injective G-modules.

Lemma II.1.6. *Let A and B be G-modules. Further let*

$$0 \longrightarrow A \xrightarrow{\mathfrak{a}} A^0 \xrightarrow{d^1} A^1 \xrightarrow{d^2} A^2 \xrightarrow{d^3} \cdots$$

be a strong resolution of A and

$$0 \longrightarrow B \xrightarrow{\mathfrak{b}} B^0 \xrightarrow{\partial^1} B^1 \xrightarrow{\partial^2} B^2 \xrightarrow{\partial^3} \cdots$$

a complex of relatively injective G-modules beginning at degree -1. Then for any G-morphism $\alpha : A \to B$ there exists a G-morphism of complexes $\alpha^\bullet : A^\bullet \to B^\bullet$ such that the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\mathfrak{a}"] \arrow[d, "\alpha^{-1} = \alpha"'] & A^0 \arrow[r, "d^1"] \arrow[d, "\alpha^0"'] & A^1 \arrow[r, "d^2"] \arrow[d, "\alpha^1"'] & A^2 \arrow[r, "d^3"] \arrow[d, "\alpha^2"'] & \cdots \\ 0 \arrow[r] & B \arrow[r, "\mathfrak{b}"'] & B^0 \arrow[r, "\partial^1"'] & B^1 \arrow[r, "\partial^2"'] & B^2 \arrow[r, "\partial^3"'] & \cdots \end{tikzcd}$$

Proof. See [Gui80, Proposition 1.1 (i),p. 176]. □

Definition II.1.7. In the situation of Lemma II.1.6, one says that the G-morphism α *extends* to a G-morphism of complexes, and $\alpha^\bullet$ is called an *extension* of α.

Lemma II.1.8. *Keep the notation of Lemma II.1.6 and Definition II.1.7. Then any two extensions of α are G-homotopic.*

Proof. See [Gui80, Proposition 1.1 (ii),p. 176]. □

Putting these together we get a lemma similar to one from standard homological algebra.

Lemma II.1.9. *Let $(\mathfrak{a}, E^\bullet)$ and $(\mathfrak{b}, F^\bullet)$ be two strong resolutions of a G-module E by relatively injective G-modules. Then there is a G-homotopy equivalence $E^\bullet \to F^\bullet$ which induces a canonical isomorphism of topological vector spaces*

$$H^n(E^{\bullet G}) \cong H^n(F^{\bullet G})$$

for all $n \geq 0$.

In particular this canonical isomorphism is given by a restriction to the subcomplexes of invariants $E^{\bullet G}$ and $F^{\bullet G}$ of an extension of the identity morphism $E \to E$ to the strong augmented resolutions $(\mathfrak{a}, E^\bullet)$ and $(\mathfrak{b}, F^\bullet)$.

Proof. The proof is standard and an immediate consequence of Lemma II.1.6 and Lemma II.1.8. We shall give a similar proof in the setting of bounded cohomology later (cf. Lemma II.2.15). Of course there is also a reference for this result: [Gui80, Corollaire 1.1, p. 177]. □

We now have to fit our previous definition of continuous cohomology as the cohomology of the cochain complex

$$0 \longrightarrow C(G,E)^G \xrightarrow{d^1} C(G^2,E)^G \xrightarrow{d^2} C(G^3,E)^G \xrightarrow{d^3} \cdots$$

into the new more abstract framework of strong resolutions by relatively injective G-modules. The next proposition establishes this link.

Proposition II.1.10. *Let E be a G-module. Then*

$$0 \longrightarrow E \xrightarrow{\epsilon} C(G,E) \xrightarrow{d^1} C(G^2,E) \xrightarrow{d^2} C(G^3,E) \xrightarrow{d^3} \cdots$$

is a strong augmented resolution of E by relatively injective G-modules, where the augmentation $\epsilon : E \to C(G,E)$ is given by

$$\epsilon(v)(g) := v$$

for all $v \in E$, $g \in G$ and $d^\bullet$ is the usual homogeneous coboundary operator.

Proof. See [Gui80, Proposition 1.2, p. 179]. □

Definition II.1.11. The resolution $(\epsilon, C(G^\bullet, E))$ appearing in Proposition II.1.10 is called the *homogenoeus standard resolution* and the map $\epsilon : E \to C(G,E)$ the *standard coefficient inclusion* or *standard augmentation.*

Summarizing the previous results we get the following functorial characterization of continuous cohomology.

Theorem II.1.12. *Let E be a G-module. Then:*

(i) There exists a strong resolution of E by relatively injective G-modules.

(ii) For any strong resolution $(\mathfrak{a}, E^\bullet)$ of E by relatively injective Banach G-modules, the cohomology $H^n(E^{\bullet G})$ of the complex $E^{\bullet G}$ of invariants is canonically isomorphic, as a topological vector space, to the continuous cohomology $H^n_c(G,E)$ for all $n \geq 0$.

Proof. Proposition II.1.10 establishes (i). Lemma II.1.9 implies (ii) since we may take for $(\mathfrak{b}, F^\bullet)$ the homogeneous standard resolution $(\epsilon, C(G^\bullet, E))$. $\square$

As an immediate consequence we get the following corollary.

Corollary II.1.13. *Let E be a relatively injective G-module. Then*

$$H_c^\bullet(G, E) = 0$$

Proof. We can simply consider the following trivial strong augmented resolution of E by relatively injective G-modules

$$0 \longrightarrow E \xrightarrow{\text{id}} E \longrightarrow 0 \longrightarrow \cdots$$

Clearly the cohomology induced by this resolution vanishes and hence by Theorem II.1.12

$$H_c^\bullet(G, E) = 0$$

as asserted. $\square$

II.1.3. More Resolutions

The advantage of a functorial characterization of a cohomology theory is, that one usually gets many resolutions and hence many complexes to compute the cohomology with. We are going to present some of them here. For the sake of simplicity, and because we are not going to need any other resolutions later on, we content ourselves with resolutions of real G-modules $(\pi, \mathbb{R})$. In the following we will often omit the concrete representation π and simply write $\mathbb{R}$. If we want to stress that $\mathbb{R}$ might not be the trivial G-module we write $\mathbb{R}_\pi$.

Remark II.1.14. *Note that for a G-module $(\pi, \mathbb{R})$ we have $\mathrm{Aut}(\mathbb{R}) \cong \mathbb{R}^\times$ and via this isomorphism the action $\pi(g)t$ can be understood as multiplication $\pi(g) \cdot t$ for all $g \in G$, $t \in \mathbb{R}$.*

The Resolution $(\epsilon, C((G/K)^\bullet, \mathbb{R}_\pi))$

Our first example of a resolution is a generalization of our standard homogeneous resolution in some sense.

Let $K < G$ be a compact subgroup. Denote $X = G/K$ and $p : G \to X = G/K$ the canonical projection given by $p(g) = gK$ for $g \in G$. Then G acts canonically from the left on X via

$$g \cdot (hK) = ghK$$

for $g, h \in G$. Consider the space of continuous functions on X^{n+1}

$$C(X^{n+1}, \mathbb{R}_\pi) := \{f : X^n \to \mathbb{R} \text{ continuous}\}$$

which becomes a LCTVS via the topology of uniform convergence on compact subsets of X^n $(n \in \mathbb{N}_0)$. We can endow $C(X^{n+1}, \mathbb{R}_\pi)$ with a G-module structure via the so-called left regular representation λ_π

$$(\lambda_\pi(g)f)(x_0, \ldots, x_n) := \pi(g)f(g^{-1}x_0, \ldots, g^{-1}x_n)$$

for $f \in C(X^{n+1}, \mathbb{R}_\pi)$, $x_0, \ldots, x_n \in X$ and $g \in G$.

In great analogy to our naive definition of continuous cohomology we also define a homogeneous coboundary operator $d^n : C(X^{n+1}, \mathbb{R}_\pi \to C(X^{n+2}, \mathbb{R}_\pi)$ given by the formula

$$(d^n f)(x_0, \ldots, x_n + 1) = \sum_{i=0}^{n+1} (-1)^i f(x_0, \ldots, \hat{x_i}, \ldots, x_{n+1})$$

for $f \in C(X^{n+1}, \mathbb{R}_\pi)$, $x_0, \ldots, x_{n+1} \in X$, where the hat again indicates omission of the variable underneath.

Finally we define an augmentation $\epsilon : \mathbb{R}_\pi \to C(X, \mathbb{R}_\pi)$

$$\epsilon(t)(x_0) = t$$

for $t \in \mathbb{R}, x_0 \in X$ and get a strong resolution of $\mathbb{R}_\pi$ by relatively injective G-modules:

Proposition II.1.15. *Let $K < G$ be a compact subgroup. Then the complex*

$$0 \longrightarrow \mathbb{R}_\pi \xrightarrow{\epsilon} C(G/K, \mathbb{R}_\pi) \xrightarrow{d^1} C((G/K)^2, \mathbb{R}_\pi) \xrightarrow{d^2} C((G/K)^3, \mathbb{R}_\pi) \xrightarrow{d^3} \cdots$$

is a strong augmented resolution of $\mathbb{R}_\pi$ by relatively injective G-modules.

Moreover the cohomology of the complex

$$0 \longrightarrow C(G/K, \mathbb{R}_\pi)^G \xrightarrow{d^1} C((G/K)^2, \mathbb{R}_\pi)^G \xrightarrow{d^2} C((G/K)^3, \mathbb{R}_\pi)^G \xrightarrow{d^3} \cdots$$

is canonically isomorphic to $H_c^\bullet(G, \mathbb{R}_\pi)$.

Proof. This is [Gui80, Proposition 2.3, p. 187], although they adopt a slightly different notation. The G-modules under consideration in [Gui80] are the spaces $C(G^{n+1}, \mathbb{R}_\pi)_K$ of continuous functions $f \in C(G^{n+1}, \mathbb{R})$ satisfying

$$f(g_0 k_0, \ldots, g_n k_n) = f(g_0, \ldots, g_n)$$

for all $g_0, \ldots, g_n$ and $k_0, \ldots, k_n \in K$.

However it is easy to see that the maps

$$\begin{aligned} \varphi^n : C((G/K)^{n+1}, \mathbb{R}_\pi) &\to C(G^{n+1}, \mathbb{R}_\pi) \\ f &\mapsto ((g_0, \ldots, g_n) \mapsto f(g_0 K, \ldots, g_n K)) \end{aligned}$$

constitute a G-isomorphism $\varphi^\bullet$ of complexes. □

The Resolution $(\epsilon, \Omega^\bullet(G/K, \mathbb{R}_\pi))$

Now let G be a Lie group with a finite number of connected components and $K < G$ a maximal compact subgroup. Then the homogeneous space $M = G/K$ is a smooth manifold on which G acts via diffeomorphisms. We can consider the complex of differential forms $(\Omega^\bullet(M, \mathbb{R}_\pi), d^\bullet)$ where the coboundary operator is given as usual by exterior derivative. For every $n \in \mathbb{N}_0$ the vector space $\Omega^n(M, \mathbb{R}_\pi)$ becomes a LCTVS much like the space of smooth functions by using local coordinates and uniform convergence of every derivative on compact subsets.

A G-module structure is now given by

$$(g \cdot \alpha) = \pi(g) \left[(g^{-1})^* \alpha\right]$$

for $g \in G$, $\alpha \in \Omega^n(M, \mathbb{R}_\pi)$, where the application of $\pi(g) \in \mathrm{Aut}(\mathbb{R}) \cong \mathbb{R}^\times$ is thought of as multiplication and $(g^{-1})^*$ denotes the pullback (cf. [Gui80, § E.3, p. 364]).

By defining an augmentation $\epsilon : \mathbb{R}_\pi \to \Omega^0(M, \mathbb{R}_\pi) \cong C^\infty(M, \mathbb{R})$ via

$$\epsilon(t)(x_0) = t$$

for every $t \in \mathbb{R}$ and $x_0 \in M$, we get a strong augmented resolution of $\mathbb{R}_\pi$ by relatively injective G-modules:

Proposition II.1.16. *Let G be a Lie group with a finite number of connected components and let $K < G$ be a maximal compact subgroup. Then the complex*

$$0 \longrightarrow \mathbb{R}_\pi \xrightarrow{\epsilon} \Omega^0(G/K, \mathbb{R}_\pi) \xrightarrow{d^1} \Omega^1(G/K, \mathbb{R}_\pi) \xrightarrow{d^2} \Omega^2(G/K, \mathbb{R}_\pi) \xrightarrow{d^3} \cdots$$

is a strong augmented resolution of $\mathbb{R}_\pi$ by relatively injective G-modules.

Moreover the cohomology of the complex

$$0 \longrightarrow \Omega^0(G/K, \mathbb{R}_\pi)^G \xrightarrow{d^1} \Omega^1(G/K, \mathbb{R}_\pi)^G \xrightarrow{d^2} \Omega^2(G/K, \mathbb{R}_\pi)^G \xrightarrow{d^3} \cdots$$

is canonically isomorphic to $H_c^\bullet(G, \mathbb{R}_\pi)$.

Proof. See [Gui80, Proposition 7.2, p. 224]. □

Connection to Singular Cohomology

There is a nice relation between the continuous cohomology of a discrete group and covering theory, which we want to present here briefly. For a discussion of singular cohomology see section D.2 in the appendix.

Let X be a contractible topological space and Γ a group (with the discrete topology). Assume that Γ acts on X freely and denote by $Y = \Gamma\backslash X$ the resulting quotient space with the quotient map $\pi : X \to Y$. Finally let us assume that $\pi : X \to Y$ is a covering and hence the universal covering of Y. Basic covering theory asserts that we may now identify Γ with the fundamental group of Y resp. the group of Deck transformations of $\pi : X \to Y$; $\Gamma \cong \pi_1(X) \cong \mathrm{Deck}(\pi)$.

Thus one may ask for the relation between the continuous cohomology of Γ and the singular cohomology of Y. The next proposition gives a very satisfying answer.

Proposition II.1.17. *We keep the above notation.*

Then $H_c^\bullet(\Gamma, \mathbb{R}) \cong H^\bullet(Y)$, where $\mathbb{R}$ denotes the trivial Γ-module.

Proof. See [Gui80, Proposition 14.1., p. 93]. □

II.2. Continuous Bounded Cohomology

Let us now turn to continuous **bounded** cohomology. Our treatment of continuous bounded cohomology will be in great analogy to the previous section II.1 on continuous cohomology facilitating a direct comparison of both theories. Therefore we will again start with a naive definition of continuous bounded cohomology and its important features such as the pullback map and the comparison map in subsection II.2.1. After that we give the functorial characterization of bounded cohomology in terms of strong resolutions by relatively injective **Banach** G-modules in subsection II.2.2 and present some resolutions to compute it from in subsection II.2.3. Finally we revisit the naive definitions of the pullback map and the comparison map and put them in the functorial framework in subsections II.2.4/II.2.5 and II.2.6 respectively. Especially the realizations of the pullback via equivariant boundary maps as discussed in subsection II.2.5 will play a prominent role in the proof of the volume rigidity theorem.

Our main reference for this section is [Mon01]. However to us every Banach space will be over $\mathbb{R}$ and not $\mathbb{C}$ as in [Mon01]. Gladly this makes no significant difference for the theory as one sees in the associated papers [BM02] and [BI02], or in the classical treatment of bounded cohomology of discrete groups in [Iva87].

We will use the notion of Banach G-modules extensively in this section and refer to section B.2 in the appendix for more details.

II.2.1. Naive Definition

For the rest of this section let E be a Banach G-module and $q \in \mathbb{N}_0$. We may now similarly to the definition of continuous cohomology consider the space of bounded continuous functions from G^{q+1} to E

$$C_b(G^{q+1}, E) := \{f : G^{q+1} \to E : f \text{ continuous and bounded } \} \subset C(G^{q+1}, E).$$

Putting the sup-norm on $C_b(G^{q+1}, E)$

$$\|f\| := \sup\{\|f(g_0, \dots, g_q)\| : g_0, \dots, g_q \in G\} \qquad \forall f \in C_b(G^{q+1}, E)$$

it is easy to check, that this space becomes a Banach space. Evidently the left regular representation λ_π in (II.1) of $C(G^{q+1}, E)$ restricts to an isometric action on $C_b(G^{q+1}, E)$ and hence induces a Banach G-module structure on the latter.

Moreover also the homogeneous coboundary operator $d^{q+1} : C(G^{q+1}, E) \to C(G^{q+2}, E)$ restricts to $d^{q+1} : C_b(G^{q+1}, E) \to C_b(G^{q+1}, E)$. Again d^{q+1} is a G-morphism and hence gives a map between the invariant spaces

$$d^{q+1} : C_b(G^{q+1}, E)^G \to C_b(G^{q+2}, E)^G$$

and we get the (sub-)complex of cochains

$$0 \to C_b(G, E)^G \to C_b(G^2, E)^G \to \dots \to C_b(G^{q+1}, E)^G \to C_b(G^{q+2}, E)^G \to \dots$$

Now the *continuous bounded cohomology* $H^\bullet_{cb}(G, E)$ of G with coefficients in E is the cohomology of this cochain complex

$$H^q_{cb}(G, E) = \frac{\ker\{d^{q+1} : C_b(G^{q+1}, E)^G \to C_b(G^{q+2}, E)^G\}}{\operatorname{im}\{d^q : C_b(G^q, E)^G \to C_b(G^{q+1}, E)^G\}}$$

Since it is a quotient of Banach spaces, $H^\bullet_{cb}(G, E)$ carries a semi-norm

$$\|\alpha\| := \inf\{\|f\| : f \in \alpha\} \qquad \forall \alpha \in H^q_{cb}(G, E)$$

This is indeed only a semi-norm and not a norm, since $\operatorname{im}(d^q) \subset C_b(G^{q+1}, E)^G$ is in general not closed.

Remark II.2.1. *In other literature on bounded cohomology the q-th bounded cohomology group of a* discrete *group* Γ *with coefficients in a Banach* Γ*-module* E *is sometimes denoted by* $H_b^q(\Gamma, E)$ *omitting the additional "c" in the subscript. Of course this makes sense if one is only interested in discrete groups as in that case every function on* Γ^{q+1} *is continuous. However we are going to deal with non-discrete topological groups later on, e.g.* $\mathrm{Isom}(\mathbb{H}^n)$*. Therefore we want to be consistent in our notation and keep the "c" in the subscript for discrete groups as well.*

Pullback

Given another locally compact second countable topological group H and a continuous homomorphism $\rho : H \to G$ the pullback map $\rho^* : C(G^{q+1}, E) \to C(H^{q+1}, E)$ from (II.2) restricts to $\rho^* : C_b(G^{q+1}, E) \to C_b(H^{q+1}, E)$.

As for continuous cohomology we can endow the Banach G-module (π, E) with a Banach H-module structure via precomposition by ρ. Indeed, this is a Banach module since π ranges in the linear isometries of E and therefore $\pi \circ \rho$ does as well. We shall denote the resulting Banach H-module by $(\pi\rho, E)$, $\rho^* E$ or simply E as well (cf. section B.2.2 in the appendix).

By the very same computation as in (II.3) we verify that $\rho^*(C_b(G^{q+1}, E)^G) \subset C_b(H^{q+1}, E)^H$ such that the pullback map induces a map at the cochain level

$$\rho^* : C_b(G^{q+1}, E)^G \to C_b(H^{q+1}, E)^H$$

One easily verifies that it commutes with the homogeneous coboundary operator as for continuous cohomology. Thus ρ^* induces a map at the cohomology level

$$\rho^* : H_{cb}^\bullet(G, E) \to H_{cb}^\bullet(H, \rho^* E)$$

Further ρ^* is does not increase the semi-norm. Indeed, at the cochain level

$$|\rho^* f(h_0, \dots, h_q)| = |f(\rho(h_0), \dots, \rho(h_q))| \leq \|f\|$$

for all $f \in C_b(G^{q+1}, E)^G$ and $h_0, \dots, h_q \in H$. Hence at the cohomology level

$$\begin{aligned} \|\rho^* \alpha\| &= \inf\{\|f\| : f \in \rho^* \alpha\} \\ &\leq \inf\{\|\rho^* f'\| : f' \in \alpha\} \\ &\leq \inf\{\|f'\| : f' \in \alpha\} = \|\alpha\| \end{aligned}$$

for every cohomology class $\alpha \in H_{cb}^q(G, E)$.

The Comparison Map $c : H_{cb}^\bullet(G, E) \to H_c^\bullet(G, \mathcal{C}E)$

First observe that any *continuous* Banach G-module, such as $\mathcal{C}E$ (cf. Definition B.2.11), is also a G-module in the sense of continuous cohomology (cf. Remark B.2.9). Therefore $H_c^\bullet(G, \mathcal{C}E)$ is well defined.

Further the following holds.

Lemma II.2.2. $C_b(G^{\bullet+1}, E)^G = C_b(G^{\bullet+1}, \mathcal{C}E)^G$

Proof. Let $q \in \mathbb{N}_0$ and $f \in C_b(G^{q+1}, E)^G$. It is sufficient to show, that f ranges in $\mathcal{C}E$. We want to apply Lemma B.2.10.

Let $x \in G^{q+1}$ and let $(g_\alpha)_{\alpha \in A}$ be a net in G converging to the neutral element $e \in G$. Then

$$\|\pi(g_\alpha) f(x) - f(x)\| = \|\pi(g_\alpha) f(x) - \pi(g_\alpha) f(g_\alpha^{-1} x)\| = \|f(x) - f(g_\alpha^{-1} x)\| \to 0$$

which shows that $f(x)$ is indeed in $\mathcal{C}E$. $\square$

Therefore we can consider the inclusion

$$C_b(G^{q+1},E)^G = C_b(G^{q+1},\mathcal{C}E)^G \hookrightarrow C(G^{q+1},\mathcal{C}E)^G$$

Since both cochain complexes use the very same homogeneous coboundary operator d the inclusion is actually a map between cochain complexes, i.e. commutes with d.

The induced map in cohomology is called the *comparison map* $c : H^\bullet_{cb}(G,E) \to H^\bullet_c(G,\mathcal{C}E)$.

Remark II.2.3. *Although $C_b(G^{q+1},\mathcal{C}E)^G \hookrightarrow C(G^{q+1},\mathcal{C}E)^G$ is injective, this does **not** imply, that also $c : H^\bullet_{cb}(G,E) \to H^\bullet_c(G,E)$ is injective!*

Remark II.2.4. *Observe that one could also apriori consider the cochain complex*

$$0 \to C(G,E)^G \to C(G^2,E)^G \to \cdots$$

and its cohomology $H^\bullet(C(G^{\bullet+1},E)^G)$. The advantage is, that one gets immediately a map $H^\bullet_{cb}(G,E) \to H^\bullet(C(G^{\bullet+1},E)^G)$ induced by the inclusion $C_b(G^{\bullet+1},E)^G \hookrightarrow C(G^{\bullet+1},E)^G$. However one now has to prove that $H^\bullet(C(G^{\bullet+1},E)^G) = H^\bullet_c(G,\mathcal{C}E)$. This can be achieved by a lemma similar to Lemma II.2.2 (cf. [Mon01, Proposition 9.1.3., p.120]). [Mon01] takes this approach and even gives a "new" definition of continuous cohomology by the above cochain complex. Nevertheless this appears to be somewhat unnatural and thus we chose our slightly different treatment of the comparison map. It is easy to see that both approaches yield the same comparison map in cohomology.

Finally we want to note that all these distinctions become completely irrelevant when we consider an apriori continuous Banach G-module, since then $\mathcal{C}E = E$. This will be the case in our application of the theory later on, where we will only be concerned with continuous (Banach) G-modules $(\pi,\mathbb{R})$.

II.2.2. Functorial Characterization

As for continuous cohomology there is also a functorial characterization of continuous bounded cohomology as developed by Burger and Monod in [BM02] and [Mon01]. This will provide us with easier to compute cochain complexes.

We follow here essentially section III.7 in [Mon01].

Basic Definitions

We want to use the same terminology of homological algebra as for continuous cohomology. By replacing G-modules by Banach G-modules in the definitions of section II.1.2 we get the notions for complexes, morphisms between those, resolutions etc. Instead of repeating every definition and terminology with Banach G-modules instead of G-modules, we will only point out the differences.

Most of these differences arise for two reasons. First, in our definition of Banach G-modules we had no continuity assumption on the action whatsoever. In order to remedy this one considers maximal continuous submodules instead (cf. Definition B.2.11). Second, a good theory of bounded cohomology has to take care of the semi-norm and hence some of the occuring morphisms should at least not increase the norm. This leads to a slightly different definition of relative injectivity for Banach G-modules. Let us now delve into the details.

Let $(E^\bullet, d^\bullet)$ by a complex of Banach G-modules. By Lemma B.2.13 the coboundary operators restrict to the continuous submodules $\mathcal{C}E^\bullet$ and we get a continuous subcomplex

$$\cdots \longrightarrow \mathcal{C}E^{n-1} \longrightarrow \mathcal{C}E^{n} \longrightarrow \mathcal{C}E^{n+1} \longrightarrow \cdots$$

The coboundary operator restricts even further to the sub-sub-complex $E^{\bullet G}$ of invariants

$$\cdots \longrightarrow (E^{n-1})^G \longrightarrow (E^{n})^G \longrightarrow (E^{n+1})^G \longrightarrow \cdots$$

Note that by Lemma B.2.14 we have $(\mathcal{C}E^\bullet)^G = (E^\bullet)^G$ such that it is not important, whether we take the invariants of the original complex or its maximal continuous subcomplex.

As before the cohomology of a complex $(E^\bullet, d^\bullet)$ is defined as the quotient spaces

$$H^n(E^\bullet) = \ker(d^{n+1})/\mathrm{im}(d^n)$$

However this time we may equip this quotient space with the usual quotient semi-norm

$$\|\alpha\| := \inf\{\|v\| : v \in \alpha\} \qquad \forall \alpha \in H^n(E^\bullet)$$

Recall that a complex of G-modules $E^\bullet$ is said to admit a contracting homotopy, if there is a homotopy $h^\bullet$ from $\mathrm{id} : E^\bullet \to E^\bullet$ to $0 : E^\bullet \to E^\bullet$. In the case of Banach G-modules we require more, namely that additionally $\|h^n\| \leq 1$ for every $n \in \mathbb{Z}$.

Now a complex of Banach G-modules $E^\bullet$ is called *strong*, if its maximal continuous subcomplex $\mathcal{C}E^\bullet$ admits a contracting homotopy. This terminology applies now to resolutions $(\mathfrak{a}, E^\bullet)$, i.e. such a resolution is strong if the restricted resolution $(\mathfrak{a}, \mathcal{C}E^\bullet)$ admits a contracting homotopy (in the sense of Banach G-modules).

Remark II.2.5. *Although the notion of a complex with a contracting homotopy does not depend on the group G considered, the concept of strong complex does, because $\mathcal{C}E^\bullet$ depends on the group.*

Remark II.2.6. *The definition of a strong resolution in [Iva87] appears to be more restrictive than ours, since a resolution $(\mathfrak{a}, E^\bullet)$ is only strong in the sense of [Iva87], if the whole complex admits a contracting homotopy and not only the subcomplex $(\mathfrak{a}, \mathcal{C}E^\bullet)$ of continuous Banach G-modules (cf. [Iva87, p. 1099]). However [Iva87] is only concerned with discrete groups, such that $E^\bullet = \mathcal{C}E^\bullet$ and the definitions in fact coincide.*

Also the notion of strongly injective morphisms is different and even replaced by the notion of admissible morphisms.

Definition II.2.7. A morphism $\eta : A \to B$ of Banach spaces is *admissible* if there is a morphism $\sigma : B \to A$ with $\|\sigma\| \leq 1$ and $\eta\sigma\eta = \eta$.

A G-morphism of Banach G-modules is said to be admissible if the underlying morphism is so.

This fits also well into the context of G-modules as in the theory of continuous cohomology, as we regain our definition of a strongly injective morphism by simply dropping the norm requirement.

Remark II.2.8. *Observe that if η is an injective G-morphism, it is admissible, if and only if it admits a left inverse morphism σ satisfying $\|\sigma\| \leq 1$. This being said an injective admissible G-morphism η is strongly injective as defined in [Iva87].*

As we mentioned before the definition of relatively injective modules changes in order to take care of the semi-norm.

Definition II.2.9. A Banach G-module E is *relatively injective* if for every injective admissible G-morphism $\iota : A \to B$ of continuous Banach G-modules A, B and every G-morphism $\alpha : A \to E$ there is a G-morphism $\beta : B \to E$ satisfying $\beta\iota = \alpha$ and $\|\beta\| \leq \|\alpha\|$.

$$\begin{tikzcd}
A \arrow[rr, hook, "\iota"'] \arrow[dr, "\alpha"'] & & B \arrow[ll, bend right, "\sigma"'] \arrow[dl, dotted, "\exists\beta"] \\
& E &
\end{tikzcd}$$

If there is any ambiguity as to the group, we say that E is G-relatively injective.

Observe that by replacing "injective admissible" by "strongly injective" – in accordance with what we have said before – and dropping the norm requirement "$\|\beta\| \leq \|\alpha\|$" we get back the definition of relative injectivity in the sense of continuous cohomology.

Remark II.2.10. *This definition coincides in view of Remark II.2.8 with the definition of relatively injective G-modules in [Iva87].*

Note that there is a typo in [Iva87] in the corresponding definition. Instead of "[...] and $\|\beta\| \leq \|\sigma\|$ [...]" (p. 1098) it should say $\|\beta\| \leq \|\alpha\|$. It is clear, that this is what was meant here in view of the proof of the following up Lemma (3.2.2).

Lemma II.2.11. *A Banach G-module E is relatively injective if and only if $\mathcal{C}E$ is so.*

Proof. See [Mon01, Lemma 4.1.5, p. 32]. $\square$

Statement of the Functorial Characterization

We are now in a position to state the key theorem and some important lemmas in view of the functorial characterization of continuous bounded cohomology by strong resolutions of relatively injective Banach G-modules. The analogies to continuous cohomology are evident.

Lemma II.2.12. *Let A and B be Banach G-modules. Further let*

$$0 \longrightarrow A \xrightarrow{\mathfrak{a}} A^0 \xrightarrow{d^1} A^1 \xrightarrow{d^2} A^2 \xrightarrow{d^3} \cdots$$

be a strong resolution of A and

$$0 \longrightarrow B \xrightarrow{\mathfrak{b}} B^0 \xrightarrow{\partial^1} B^1 \xrightarrow{\partial^2} B^2 \xrightarrow{\partial^3} \cdots$$

a complex of relatively injective Banach G-modules beginning at degree -1. Then for any G-morphism $\alpha : A \to B$ there exists a G-morphism of complexes $\alpha^\bullet : A^\bullet \to B^\bullet$ such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\mathfrak{a}"] \arrow[d, "\alpha^{-1} = \alpha"'] & A^0 \arrow[r, "d^1"] \arrow[d, "\alpha^0"'] & A^1 \arrow[r, "d^2"] \arrow[d, "\alpha^1"'] & A^2 \arrow[r, "d^3"] \arrow[d, "\alpha^2"'] & \cdots \\
0 \arrow[r] & B \arrow[r, "\mathfrak{b}"'] & B^0 \arrow[r, "\partial^1"'] & B^1 \arrow[r, "\partial^2"'] & B^2 \arrow[r, "\partial^3"'] & \cdots
\end{tikzcd}$$

Proof. This is [Mon01, Lemma 7.2.4, p. 70]. □

Definition II.2.13. In the situation of Lemma II.2.12, one says that the G-morphism α *extends* to a G-morphism of complexes, and $\alpha^\bullet$ is called an *extension* of α.

Lemma II.2.14. *Keep the notation of Lemma II.2.12 and Definition II.2.13. Then any two extensions of α are G-homotopic.*

Proof. See [Mon01, Lemma 7.2.6, p. 71]. □

Putting these together we get a lemma familiar to one from standard homological algebra.

Lemma II.2.15. *Let $(\mathfrak{a}, E^\bullet)$ and $(\mathfrak{b}, F^\bullet)$ be two strong resolutions of a Banach G-module E by relatively injective Banach G-modules. Then there is a G-homotopy equivalence $\mathcal{C}E^\bullet \to \mathcal{C}F^\bullet$ which induces a canonical isomorphism of topological vector spaces*

$$H^n(E^{\bullet G}) \cong H^n(F^{\bullet G})$$

for all $n \geq 0$.

In particular this canonical isomorphism is given by a restriction to the subcomplexes of invariants $E^{\bullet G}$ and $F^{\bullet G}$ of an extension of the identity morphism $\mathcal{C}E \to \mathcal{C}E$ to the strong augmented resolutions $(\mathfrak{a}, \mathcal{C}E^\bullet)$ and $(\mathfrak{b}, \mathcal{C}F^\bullet)$.

Proof. As we have already mentioned before we get via restriction (cf. Lemma B.2.13) strong resolutions $(\mathfrak{a}, \mathcal{C}E^\bullet)$, $(\mathfrak{b}, \mathcal{C}F^\bullet)$ of $\mathcal{C}E$. Since $(\mathfrak{a}, E^\bullet)$ and $(\mathfrak{b}, F^\bullet)$ are strong augmented resolutions of E by relatively injective Banach G-modules, also $(\mathfrak{a}, \mathcal{C}E^\bullet)$ and $(\mathfrak{b}, \mathcal{C}F^\bullet)$ are strong augmented resolutions of $\mathcal{C}E$ by relatively injective Banach G-modules (cf. Lemma II.2.11).

Hence by Lemma II.2.12 there is a G-morphism of complexes $\alpha^\bullet : \mathcal{C}E^\bullet \to \mathcal{C}F^\bullet$ extending the identity morphism $\mathrm{id} : \mathcal{C}E \to \mathcal{C}E$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}E \arrow[r, "\mathfrak{a}"] \arrow[d, "\alpha^{-1} = \mathrm{id}"'] & \mathcal{C}E^0 \arrow[r] \arrow[d, "\alpha^0"'] & \mathcal{C}E^1 \arrow[r] \arrow[d, "\alpha^1"'] & \mathcal{C}E^2 \arrow[r] \arrow[d, "\alpha^2"'] & \cdots \\ 0 \arrow[r] & \mathcal{C}E \arrow[r, "\mathfrak{b}"'] & \mathcal{C}F^0 \arrow[r] & \mathcal{C}F^1 \arrow[r] & \mathcal{C}F^2 \arrow[r] & \cdots \end{tikzcd}$$

Exchanging the roles of the two resolutions we get another extension $\beta^\bullet : \mathcal{C}F^\bullet \to \mathcal{C}E^\bullet$ of the identity morphism $\mathrm{id} : \mathcal{C}E \to \mathcal{C}E$.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}E \arrow[r, "\mathfrak{b}"] \arrow[d, "\beta^{-1} = \mathrm{id}"'] & \mathcal{C}F^0 \arrow[r] \arrow[d, "\beta^0"'] & \mathcal{C}F^1 \arrow[r] \arrow[d, "\beta^1"'] & \mathcal{C}F^2 \arrow[r] \arrow[d, "\beta^2"'] & \cdots \\ 0 \arrow[r] & \mathcal{C}E \arrow[r, "\mathfrak{a}"'] & \mathcal{C}E^0 \arrow[r] & \mathcal{C}E^1 \arrow[r] & \mathcal{C}E^2 \arrow[r] & \cdots \end{tikzcd}$$

Therefore the composed G-morphism of complexes $\beta^\bullet\alpha^\bullet : \mathcal{C}E^\bullet \to \mathcal{C}E^\bullet$ extends the identity G-morphism $\mathrm{id} : \mathcal{C}E \to \mathcal{C}E$ to the strong augmented resolution of $\mathcal{C}E$ given by $(\mathfrak{a}, \mathcal{C}E^\bullet)$. Clearly the identity morphism of complexes $\mathrm{id}^\bullet : \mathcal{C}E^\bullet \to \mathcal{C}E^\bullet$ extends the identity as well and hence $\beta^\bullet\alpha^\bullet$ is G-homotopic to $\mathrm{id}^\bullet$ by Lemma II.2.14. In particular, $\beta^\bullet\alpha^\bullet$ restricted to the (non-augmented) complex of invariants $E^{\bullet G}$ is G-homotopic to the identity morphism of complexes and hence induces the identity $H^n(E^{\bullet G}) \to H^n(E^{\bullet G})$ for all $n \geq 0$. Likewise, $\alpha^\bullet\beta^\bullet$ is G-homotopic to the identity, such that $\alpha^\bullet$ and $\beta^\bullet$ are G-homotopy equivalences.

In particular, $\alpha^\bullet$ and $\beta^\bullet$ restrict to homotopy equivalences between $E^{\bullet G}$ and $F^{\bullet G}$ and thus induce topological isomorphisms $H^n(E^{\bullet G}) \cong H^n(F^{\bullet G})$ for all $n \geq 0$. These isomorphisms are canonical because by Lemma II.2.14 any choice of extensions $\alpha^\bullet$ and $\beta^\bullet$ would amount to the same maps in cohomology.

□

We now have to fit our previous definition of continuous bounded cohomology as the cohomology of the cochain complex

$$0 \longrightarrow C_b(G, E)^G \xrightarrow{d^1} C_b(G^2, E)^G \xrightarrow{d^2} C_b(G^3, E)^G \xrightarrow{d^3} \cdots$$

into the new more abstract framework of strong resolutions by relatively injective G-modules. The next proposition establishes this link.

Proposition II.2.16. *Let E be a Banach G-module. Then*

$$0 \longrightarrow E \xrightarrow{\epsilon} C_b(G, E) \xrightarrow{d^1} C_b(G^2, E) \xrightarrow{d^2} C_b(G^3, E) \xrightarrow{d^3} \cdots$$

is a strong augmented resolution of E by relatively injective G-modules, where the augmentation $\epsilon : E \to C_b(G, E)$ is given by

$$\epsilon(v)(g) := v$$

for all $v \in E$, $g \in G$ and $d^\bullet$ is the usual homogeneous coboundary operator.

Proof. See [Mon01, Corollary 7.4.7, p. 80].

□

Definition II.2.17. The resolution $(\epsilon, C_b(G^\bullet, E))$ appearing in Proposition II.2.16 is called the *homogenoeus standard resolution* and the map $\epsilon : E \to C_b(G, E)$ the *standard coefficient inclusion* or *standard augmentation.*

Summarizing the previous results we get the following functorial characterization of continuous bounded cohomology.

Theorem II.2.18. *Let E be a Banach G-module. Then:*

(i) There exists a strong resolution of E by relatively injective Banach G-modules.

(ii) For any strong resolution $(\mathfrak{a}, E^\bullet)$ of E by relatively injective Banach G-modules, the cohomology $H^n(E^{\bullet G})$ of the complex $E^{\bullet G}$ of invariants is canonically isomorphic, as a topological vector space, to the continuous bounded cohomology $H^n_{cb}(G, E)$ for all $n \geq 0$.

Proof. Proposition II.2.16 establishes (i). Lemma II.2.15 implies (ii) since we may take for $(\mathfrak{b}, F^\bullet)$ the homogeneous standard resolution $(\epsilon, C_b(G^\bullet, E))$.

□

As for continuous cohomology we get the following immediate corollary.

Corollary II.2.19. *Let E be a relatively injective Baanch G-module. Then*

$$H^\bullet_{cb}(G, E) = 0$$

Proof. The proof given for continuous cohomology works verbatim (cf. Corollary II.1.13). □

With some further investigation of relative injectivity it is possible to show, that the trivial Banach G-module $\mathbb{R}$ is relatively injective, if G is amenable (cf. Definition C.1.1). This observation is crucial to us and one of the ingredients of bounded cohomology, that enable us to even define the volume of a representation later on.

Corollary II.2.20. *If G is amenable, then the trivial Banach G-module $\mathbb{R}$ is relatively injective. In particular*

$$H^\bullet_{cb}(G, \mathbb{R}) = 0.$$

Proof. See for example [Mon01, Corollary 5.4.1, p. 46] or for a more versatile approach dealing with coefficient G-modules [Mon01, Theorem 5.6.1, p. 55] resp. [BM02, Theorem 2.2.4, p. 31]. For the latter the assertion follows putting $E = \mathbb{R}$ and observing that a mean $L^\infty(G, \mathbb{R}) \to \mathbb{R}$ exists simply by the invariant mean property of amenable groups (cf. Definition C.1.1). □

We want to emphasize that the canonical isomorphism of Theorem II.1.12 (ii) is purely topological and does not take into account the semi-norms on $H^n(E^\bullet)$ and $H^n_{cb}(G, E)$. The next theorem shows, that this canonical isomorphism does not increase the semi-norm.

Theorem II.2.21. *Let E be a Banach G-module and $(\mathfrak{a}, E^\bullet)$ a strong augmented resolution of E by relatively injective Banach G-modules. Then the canonical isomorphism*

$$H^n(E^{\bullet G}) \to H^n_{cb}(G, E)$$

granted by Theorem II.1.12 does not increase the norm for all $n \geq 0$.

Proof. This is [Mon01, Theorem 7.3.1,p. 74]. Actually [Mon01] shows this for a different standard resolution than ours (cf. Remark II.2.23). This difference however is rendered irrelevant by [Mon01, Corollary 7.4.7, p. 80] and our next lemma. Also compare Remark II.2.23. □

The fact that we get only a semi-norm non-increasing isomorphism out of the general theory might by disappointing at first. However we will encounter in the next subsection several resolutions, which actually guarantee a canonical isometric isomorphism. The next lemma will state a trivial but nonetheless important relation between such resolutions.

Lemma II.2.22. *Let E be a Banach G-module. Let $(\mathfrak{a}, A^\bullet)$, $(\mathfrak{b}, B^\bullet)$ and $(\mathfrak{s}, S^\bullet)$ be strong augmented resolutions by relatively injective Banach G-modules. Denote by $\alpha^\bullet : H^\bullet(A^\bullet) \to H^\bullet(S^\bullet)$, $\beta^\bullet : H^\bullet(B^\bullet) \to H^\bullet(S^\bullet)$ and $\gamma^\bullet : H^\bullet(A^\bullet) \to H^\bullet(B^\bullet)$ the canonical isomorphisms in cohomology induced by extensions of the identity* $\mathrm{id} : \mathcal{C}E \to \mathcal{C}E$. *If $\alpha^\bullet$ and $\beta^\bullet$ are isometric, then also $\gamma^\bullet$ is isometric.*

Proof. Denote by

$$\begin{aligned} a^\bullet &: \mathcal{C}A^\bullet \to \mathcal{C}S^\bullet \\ b^\bullet &: \mathcal{C}B^\bullet \to \mathcal{C}S^\bullet \\ c^\bullet &: \mathcal{C}A^\bullet \to \mathcal{C}B^\bullet \end{aligned}$$

the extensions of $\mathrm{id} : \mathcal{C}E \to \mathcal{C}E$ to the according resolutions inducing the canonical isomorphisms $\alpha^\bullet, \beta^\bullet$ and $\gamma^\bullet$ respectively. Therefore $b^\bullet \circ c^\bullet : \mathcal{C}A^\bullet \to \mathcal{C}S^\bullet$ is also an extension of the identity, hence G-homotopic to $a^\bullet : \mathcal{C}A^\bullet \to \mathcal{C}S^\bullet$ and thus induces the same map in cohomology, i.e. $\alpha^\bullet = \beta^\bullet \circ \gamma^\bullet$.

Let $n \geq 0$ and $\omega \in H^n(A^\bullet)$. We then get

$$\|\gamma^n(\omega)\| = \|\beta^n(\gamma^n(\omega))\| = \|\alpha^n(\omega)\| = \|\omega\|$$

This concludes the proof. $\square$

Remark II.2.23. *Most of the functorial characterization summarized in this section also works for general topological groups. However care must be taken concerning our standard resolution. It is still an augmented resolution of E by relatively injective Banach G-modules, but it is unclear whether it is also a* strong *one. Instead one considers the inductively defined Banach G-modules $C_b^\bullet(G, E)$*

$$C_b^0(G,E) := C_b(G,E), \qquad C_b^n(G,E) = C_b\left(G, C_b^{n-1}(G,E)\right) \quad (n \in \mathbb{N})$$

which we can equip in an evident way with a Banach G-module structure. This is exactly the approach taken in [Mon01] and they show in fact, that the resulting augmented resolution is strong.

It is tempting to "identify" $C_b^n(G,E)$ with $C_b(G^n, E)$ via the map

$$A^n : C_b^n(G,E) \to C_b(G^{n+1}, E)$$

defined by

$$(A^n f)(x_0, \ldots, x_n) = (\cdots (f(x_0)(x_1) \cdots)(x_n)$$

Although this gives indeed an isometric G-morphism, it is in general not surjective!

II.2.3. More Resolutions

We will now investigate some resolutions of a Banach G-module $(\pi, \mathbb{R})$, since that is the only important application of continuous bounded cohomology to us. Sometimes we will write $\mathbb{R}_\pi$ in order to emphasize, that $\mathbb{R}$ is not necessarily a trivial Banach G-module. All of these resolutions work in a more abstract framework of general Banach G-modules (in case of the C_b-spaces) or coefficient G-modules (in case of the L^∞-spaces). However we will only work with the resolutions of $\mathbb{R}$ later on. Although the notion of coefficient G-modules is important to the general theory of continuous bounded cohomology, it is somewhat cumbersome and we omit it in favour of a more concise exposition. We shall just note that any real Banach G-module $(\pi, \mathbb{R})$ is in particular a coefficient G-module. For details we refer to [Mon01].

Remark II.2.24. *Note that for a Banach G-module $(\pi, \mathbb{R})$ we have $\mathrm{Iso}(\mathbb{R}) \cong \{\pm 1\}$ and via this isomorphism the action $\pi(g)t$ can be understood as multiplication $\pi(g) \cdot t$ for all $g \in G$, $t \in \mathbb{R}$.*

Resolutions by Function Spaces

In this section we will encounter spaces of functions as Banach G-modules such as $C_b(X^n,\mathbb{R})$, $L^\infty(G/H,\mathbb{R})$, ...etc. The coboundary operator of the occuring resolutions will always be in a sense the same and a natural generalization of our previous homogeneous coboundary operator:

Let X be a set, $n \in \mathbb{N}$ and denote by $\mathrm{Map}(X^n,\mathbb{R}) = \{f : X^n \to \mathbb{R}\}$ the set of all maps from X^n to $\mathbb{R}$. We define the homogeneous coboundary operator $d^n : \mathrm{Map}(X^n,\mathbb{R}) \to \mathrm{Map}(X^{n+1},\mathbb{R})$ via the familiar formula

$$(d^n f)(x_0,\dots,x_n) := \sum_{i=0}^{n}(-1)^i f(x_0,\dots,\hat{x_i},\dots,x_n), \qquad f \in \mathrm{Map}(X^n,\mathbb{R}); x_0,\dots,x_n \in X$$

where the hat over a variable means, that it is omitted. Note that this definition coincides with our previous definition of the homogeneous coboundary operator in the standard resolution when restricted to $C_b(G^{n+1},\mathbb{R})$.

We also generalize the standard augmentation in a similar fashion.

$$\begin{aligned} \epsilon : \mathbb{R} &\to \mathrm{Map}(X,\mathbb{R}) \\ t &\mapsto (x \mapsto t) \end{aligned}$$

In the following whenever we encounter a resolution by function spaces we mean by ϵ and $d^\bullet$ the above augmentation resp. coboundary operators restricted to the respective space of functions.

The first resolution by such function spaces is a generalization of our standard resolution. Let X be a locally compact topological space with a continuous G-action. Then G acts via the usual diagonal action also on X^n $(n \in \mathbb{N})$. Consider the Banach space

$$C_b(X^n,\mathbb{R}) := \{f : X^n \to \mathbb{R} : f \text{ is continuous and bounded}\}$$

with the common supremum norm. We can endow it with a Banach G-module structure via the *left regular representation* λ_π

$$(\lambda_\pi(g)f)(x_1,\dots,x_n) = \pi(g)f(g^{-1}x_1,\dots,g^{-1}x_n)$$

for $f \in C_b(X^n,\mathbb{R}_\pi)$, $g \in G$ and $x_1,\dots,x_n \in X$. Under more restrictive assumptions on the G-action and its quotient we get the following Theorem.

Theorem II.2.25. *Let X be a locally compact topological space with continuous proper G-action such that the quotient $G\backslash X^{n+1}$ is paracompact for all $n \geq 0$. Let $(\pi,\mathbb{R})$ be a Banach G-module. Then*

$$0 \longrightarrow \mathbb{R}_\pi \xrightarrow{\epsilon} C_b(X,\mathbb{R}_\pi) \xrightarrow{d^1} C_b(X^2,\mathbb{R}_\pi) \xrightarrow{d^2} C_b(X^3,\mathbb{R}_\pi) \xrightarrow{d^3} \cdots$$

is a strong augmented resolution of $\mathbb{R}_\pi$ by relatively injective Banach G-modules.

Moreover the cohomology of the complex

$$0 \longrightarrow C_b(X,\mathbb{R}_\pi)^G \xrightarrow{d^1} C_b(X^2,\mathbb{R}_\pi)^G \xrightarrow{d^2} C_b(X^3,\mathbb{R}_\pi)^G \xrightarrow{d^3} \cdots$$

is canonically isometrically isomorphic to $H^\bullet_{cb}(G,\mathbb{R}_\pi)$.

Proof. See [Mon01, Theorem 7.4.5, p. 77]. ∎

Corollary II.2.26. *Let $H < G$ be a closed subgroup and $K < G$ a compact subgroup. Let $(\pi, \mathbb{R})$ be a Banach H-module. Then*

$$0 \longrightarrow \mathbb{R}_\pi \xrightarrow{\epsilon} C_b(G/K, \mathbb{R}_\pi) \xrightarrow{d^1} C_b((G/K)^2, \mathbb{R}_\pi) \xrightarrow{d^2} C_b((G/K)^3, \mathbb{R}_\pi) \xrightarrow{d^3} \cdots$$

is a strong augmented resolution of $\mathbb{R}_\pi$ by relatively injective Banach H-modules.
Moreover the cohomology of the complex

$$0 \longrightarrow C_b(G/K, \mathbb{R}_\pi)^H \xrightarrow{d^1} C_b((G/K)^2, \mathbb{R}_\pi)^H \xrightarrow{d^2} C_b((G/K)^3, \mathbb{R}_\pi)^H \xrightarrow{d^3} \cdots$$

is canonically isometrically isomorphic to $H^\bullet_{cb}(H, \mathbb{R}_\pi)$.

Proof. See [Mon01, Corollary 7.4.10, p. 81]. □

The next important examples of resolutions by function spaces are the ones arising from L^∞-spaces. Let S be an amenable G-space (see Definition C.2.4). Then the space $L^\infty(S, \mathbb{R})$ of all essentially bounded function classes is a Banach space with the usual essential supremum norm. Recall that we have an invariant measure class on S by definition of an amenable regular G-space such that the notion of function classes up to null sets is well-defined. We get again a diagonal G-action on S^n for $n \in \mathbb{N}$ (which is also amenable by Proposition C.2.8). Hence $L^\infty(S^n, \mathbb{R}_\pi)$ becomes a Banach G-module with the left regular representation λ_π

$$(\lambda_\pi(g)f)(x_1, \ldots, x_n) = \pi(g)f(g^{-1}x_1, \ldots, g^{-1}x_n)$$

for $f \in L^\infty(S^n, \mathbb{R}_\pi)$, $g \in G$ and $x_1, \ldots, x_n \in S$.
We then get:

Theorem II.2.27. *Let S be an amenable regular G-space and $(\pi, \mathbb{R})$ a Banach G-module. Then*

$$0 \longrightarrow \mathbb{R}_\pi \xrightarrow{\epsilon} L^\infty(S, \mathbb{R}_\pi) \xrightarrow{d^1} L^\infty(S^2, \mathbb{R}_\pi) \xrightarrow{d^2} L^\infty(S^3, \mathbb{R}_\pi) \xrightarrow{d^3} \cdots$$

is a strong augmented resolution of $\mathbb{R}_\pi$ by relatively injective Banach G-modules.
Moreover the cohomology of the complex

$$0 \longrightarrow L^\infty(S, \mathbb{R}_\pi)^G \xrightarrow{d^1} L^\infty(S^2, \mathbb{R}_\pi)^G \xrightarrow{d^2} L^\infty(S^3, \mathbb{R}_\pi)^G \xrightarrow{d^3} \cdots$$

is canonically isometrically isomorphic to $H^\bullet_{cb}(G, \mathbb{R}_\pi)$.

Proof. See [Mon01, Theorem 7.5.3, p. 83]. □

Corollary II.2.28. *Let $G', H < G$ be closed subgroups and $(\pi, \mathbb{R})$ a Banach G-module. If H is amenable, then*

$$0 \longrightarrow \mathbb{R}_\pi \xrightarrow{\epsilon} L^\infty(G/H, \mathbb{R}_\pi) \xrightarrow{d^1} L^\infty((G/H)^2, \mathbb{R}_\pi) \xrightarrow{d^2} L^\infty((G/H)^3, \mathbb{R}_\pi) \xrightarrow{d^3} \cdots$$

is a strong augmented resolution of $\mathbb{R}_\pi$ by relatively injective Banach G'-modules and the cohomology of the complex

$$0 \longrightarrow L^\infty(G/H, \mathbb{R}_\pi)^{G'} \xrightarrow{d^1} L^\infty((G/H)^2, \mathbb{R}_\pi)^{G'} \xrightarrow{d^2} L^\infty((G/H)^3, \mathbb{R}_\pi)^{G'} \xrightarrow{d^3} \cdots$$

is canonically isomerically isomorphic to $H^\bullet_{cb}(G', \mathbb{R}_\pi)$.

Proof. By Proposition C.2.7 the space $S = G/H$ is an amenable regular G-space. By Lemma C.2.9 it is also an amenable regular G'-space. Finally by Proposition C.2.8 the productspaces S^n $(n \in \mathbb{N})$ are also amenable regular G'-spaces. The corollary now follows from Theorem II.2.27. See also [Mon01, Corollary 7.5.9, p. 87]. □

By setting $H = \{1\}$ and $G' = G$ we get that $H_{cb}(G, \mathbb{R}_\pi)$ can be computed by an L^∞-resolution as well. This is the assertion of the following corollary.

Corollary II.2.29. *Let* $(\pi, \mathbb{R})$ *a Banach* G*-module. Then*

$$0 \longrightarrow \mathbb{R}_\pi \xrightarrow{\epsilon} L^\infty(G, \mathbb{R}_\pi) \xrightarrow{d^1} L^\infty(G^2, \mathbb{R}_\pi) \xrightarrow{d^2} L^\infty(G^3, \mathbb{R}_\pi) \xrightarrow{d^3} \cdots$$

is a strong augmented resolution of $\mathbb{R}_\pi$ *by relatively injective Banach* G*-modules and the cohomology of the complex*

$$0 \longrightarrow L^\infty(G, \mathbb{R}_\pi)^G \xrightarrow{d^1} L^\infty(G^2, \mathbb{R}_\pi)^G \xrightarrow{d^2} L^\infty(G^3, \mathbb{R}_\pi)^G \xrightarrow{d^3} \cdots$$

is canonically isometrically isomorphic to $H^\bullet_{cb}(G, \mathbb{R}_\pi)$.

Because L^∞-spaces consist of function classes up to null sets which are hence not well-defined at an arbitrarily chosen point, they are sometimes not particularly handy to work with in a geometric situation. It is therefore natural to consider for a measurable space X with a measurable G-action the spaces

$$\mathcal{B}^\infty(X^n, \mathbb{R}) := \{f : X^n \to \mathbb{R} : f \text{ is measurable and bounded}\}, \qquad (n \in \mathbb{N})$$

without taking equivalence classes of functions up to null sets. Clearly $\mathcal{B}^\infty(X^n, \mathbb{R})$ becomes a Banach space with the supremum norm, since already the space of all bounded functions is a Banach space and every pointwise limit of a measurable function is again measurable. As for the function spaces before we equip $\mathcal{B}^\infty(X^n, \mathbb{R}_\pi)$ via the left regular representation λ_π with a Banach G-module structure.

It is not clear whether the spaces $\mathcal{B}^\infty(X^n, \mathbb{R}_\pi)$ are relatively injective Banach G-modules, but at least they form a strong augmented resolution as the following proposition asserts.

Proposition II.2.30. *Let* $(\pi, \mathbb{R})$ *be a Banach* G*-module,* X *a measurable space with a measurable* G*-action. Then the complex*

$$0 \longrightarrow \mathbb{R}_\pi \xrightarrow{\epsilon} \mathcal{B}^\infty(X, \mathbb{R}_\pi) \xrightarrow{d^1} \mathcal{B}^\infty(X^2, \mathbb{R}_\pi) \xrightarrow{d^2} \mathcal{B}^\infty(X^3, \mathbb{R}_\pi) \xrightarrow{d^3} \cdots$$

is a strong augmented resolution of $\mathbb{R}_\pi$ *by (not necessarily relatively injective) Banach* G*-modules.*

Proof. See [BI02, Proposition 3.1, p. 6]. □

Connection to Singular Bounded Cohomology

As for continuous cohomology there is a relation between singular bounded cohomology and continuous bounded cohomology for discrete groups given by covering theory. For a discussion of singular bounded cohomology see section D.3 in the appendix.

Recall that the singular bounded cohomology of a topological space X is the cohomology of the cochain complex

$$0 \longrightarrow S_b^0(X) \xrightarrow{\delta^1} S_b^1(X) \xrightarrow{\delta^2} S_b^2(X) \xrightarrow{\delta^3} \cdots$$

where $S_b^n(X)$ is the Banach space of singular bounded cochains with the norm given by

$$\|f\| := \sup\{f(\sigma) \,|\, \sigma : \Delta^n \to \mathbb{R} \text{ continuous}\}, \qquad f \in S_b^n(X)$$

and δ^n is the restriction of the usual singular coboundary operator $\delta^n : S^{n-1}(X) \to S^n(X)$ to the bounded singular cochains ($n \in \mathbb{N}$). We want to emphasize here, that we are only concerned with real coefficients and hence omit their explicit notation.

Let $\pi : \tilde{X} \to X$ be the universal covering and $\Gamma := \pi_1(X) \cong \mathrm{Deck}(\pi)$ the fundamental group of X identified with the (discrete) group of Deck transformations as usual. The pullback of singular bounded cochains by elements of $\Gamma = \mathrm{Deck}(\pi)$ equips $S_b^n(X)$ with a Banach Γ-module structure

$$\begin{aligned} \Gamma \times S_b^n(X) &\to S_b^n(X) \\ (\gamma, f) &\mapsto (\gamma^{-1})^* f \end{aligned}$$

Together with the usual augmentation

$$\begin{aligned} \epsilon : \mathbb{R} &\to S_b^0(\tilde{X}) \\ x &\mapsto (\sigma \mapsto x) \end{aligned}$$

we get a strong augmented resolution $(\epsilon, S_b^\bullet(\tilde{X}))$ of the trivial Banach Γ-module $\mathbb{R}$ as the following proposition asserts.

Proposition II.2.31. *Let X be a countable CW-complex, $\pi : \tilde{X} \to X$ its universal covering and $\Gamma = \pi_1(X) \cong \mathrm{Deck}(\pi)$. Then*

$$0 \longrightarrow \mathbb{R} \xrightarrow{\epsilon} S_b^0(\tilde{X}) \xrightarrow{\delta^1} S_b^1(\tilde{X}) \xrightarrow{\delta^2} S_b^2(\tilde{X}) \xrightarrow{\delta^3} \cdots$$

is a strong augmented resolution of $\mathbb{R}$ (as the trivial Γ-module) by relatively injective Banach Γ-modules.

Moreover the cohomology of the complex

$$0 \longrightarrow S_b^0(\tilde{X})^\Gamma \xrightarrow{\delta^1} S_b^1(\tilde{X})^\Gamma \xrightarrow{\delta^2} S_b^2(\tilde{X})^\Gamma \xrightarrow{\delta^3} \cdots$$

is canonically isometrically isomorphic to $H_{cb}^\bullet(\Gamma, \mathbb{R})$.

Proof. This is [Iva87, Theorem (4.1), p. 1104]. Observe that the slightly different definitions of relatively injective Γ-modules and strong resolutions in [Iva87] are compatible with ours as we have already pointed out in Remark II.2.8, Remark II.2.10 and Remark II.2.6.

□

The following lemma provides a more geometric insight on the bounded cohomology of Γ.

Lemma II.2.32. *Let X be a countable CW-complex, $\pi : \tilde{X} \to X$ its unversal covering and $\Gamma = \mathrm{Deck}(\pi)$. Then the pullback*

$$\pi^* : S_b^n(X) \to S_b^n(\tilde{X}), \qquad (n \in \mathbb{N}_0)$$

induces an isomorphism of complexes $\pi^* : S_b^\bullet(X) \to S_b^\bullet(\tilde{X})^\Gamma$. In particular it induces an isometric isomorphism in cohomology

$$\pi^* : H_b^\bullet(X) = H^\bullet(S_b^\bullet(X)) \longrightarrow H^\bullet(S_b^\bullet(\tilde{X})^\Gamma) \cong H_{cb}^\bullet(\Gamma, \mathbb{R})$$

Proof. Let $n \in \mathbb{N}_0$, $f \in S_b^n(X)$ and $\gamma \in \Gamma = \mathrm{Deck}(\pi)$. Then

$$\gamma^*(\pi^* f) = (\pi \circ \gamma)^* f = \pi^* f$$

Hence $\pi^*(S_b^n(X)) \subseteq S_b^n(\tilde{X})^\Gamma$

It is well known, that we can lift any continuous map $\sigma : \Delta^n \to X$ to some continuous map $\tilde{\sigma} : \Delta^n \to \tilde{X}$. Any two such lifts differ by some element of $\Gamma = \mathrm{Deck}(\pi)$. Thus we can define an inverse map $\varphi : S_b^n(\tilde{X})^\Gamma \to S_b^n(X)$. For $f \in S_b^n(\tilde{X})^\Gamma$ we set

$$\varphi(f)(\sigma) := f(\tilde{\sigma})$$

where $\sigma : \Delta^n \to X$ is a singular chain and $\tilde{\sigma} : \Delta^n \to \tilde{X}$ any lift of it along π. This is well-defined, since f is Γ-invariant. As usual we define $\varphi(f)$ by linear continuation on $S_n(X)$. It is easy to check, that φ is indeed the inverse of π^*.

By the above lifting argument it is also immediate, that π^* is isometric. $\square$

II.2.4. A Functorial View on the Pullback $\rho^* : H^\bullet_{cb}(G,E) \to H^\bullet_{cb}(H,\rho^*E)$

Let H be another locally compact second countable topological group and $\rho : H \to G$ a continuous homomorphism. In our naive definition of continuous bounded cohomology we have already defined a pullback map in cohomology $\rho^* : H^n_{cb}(G,E) \to H^n_{cb}(H,\rho^*E)$ for all $n \geq 0$. We now want to understand this construction at a more abstract level and deduce a way to compute the pullback map for some of the other resolutions. From a cohomological point of view this is what we have done in our naive definition:

Consider the Banach G-module (π, E). By pulling back the structure we may regard it as an H-module $(\pi\rho, E)$ (cf. section B.2.2). Now the Banach H-module structure on $C_b(H^{n+1}, E)$ is given by the left regular representation $\lambda_{\pi\rho}$. However we can also equip $C_b(G^{n+1}, E)$ with a Banach H-module structure and get the module $(\lambda_\pi\rho, C_b(G^{n+1}, E))$. With respect to these structures the map $\rho^* : C_b(G^{n+1}, E) \to C_b(H^{n+1}, E)$ becomes in fact H-equivariant. Indeed

$$\begin{aligned}
\rho^*(\lambda_\pi(\rho(h))f)(h_0,\dots,h_n) &= (\lambda_\pi(\rho(h))f)(\rho(h_0),\dots,\rho(h_n)) \\
&= \pi(\rho(h))f(\rho(h)^{-1}\rho(h_0),\dots,\rho(h)^{-1}\rho(h_n)) \\
&= \pi(\rho(h))f(\rho(h^{-1}h_0),\dots,\rho(h^{-1}h_n)) \\
&= \pi(\rho(h))(\rho^*f)(h^{-1}h_0,\dots,h^{-1}h_n) \\
&= \lambda_{\pi\rho}(h)(\rho^*f)(h_0,\dots,h_n)
\end{aligned}$$

for all $f \in C_b(G^{n+1}, E)$ and $h, h_0, \dots, h_n \in H$. Hence ρ^* is an H-morphism of complexes extending the identity $E \to E$.

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r, "\epsilon"] \arrow[d, "\mathrm{id}"'] & C_b(G,E) \arrow[r] \arrow[d, "\rho^*"'] & C_b(G^2,E) \arrow[r] \arrow[d, "\rho^*"'] & C_b(G^3,E) \arrow[r] \arrow[d, "\rho^*"'] & \cdots \\
0 \arrow[r] & E \arrow[r, "\epsilon"'] & C_b(H,E) \arrow[r] & C_b(H^2,E) \arrow[r] & C_b(H^3,E) \arrow[r] & \cdots
\end{tikzcd}$$

Restricting to the H-invariants we get $\rho^* : C_b(G^{n+1}, E)^H \to C_b(H^{n+1}, E)^H$. With respect to the pullback H-module structure on $C_b(G^{n+1}, E)$ the space of invariants is nothing but $C_b(G^{n+1}, E)^{\rho(H)}$ when we view $C_b(G^{n+1}, E)$ again as a G-module. Because $\rho(H) < G$ and hence $C_b(G^{n+1}, E)^G \subset C_b(G^{n+1}, E)^{\rho(H)}$ we can restrict the pullback map further and get our original $\rho^* : C_b(G^{n+1}, E)^G \to C_b(H^{n+1}, E)^H$, which induces $\rho^* : H^n_{cb}(G,E) \to H^n_{cb}(H, \rho^*E)$ in cohomology.

We can mimic this construction for arbitrary resolutions. Let $(\mathfrak{a}, A^\bullet)$ be a strong resolution of (π, E) by G-relatively injective Banach G-modules and let $(\mathfrak{b}, B^\bullet)$ be a strong resolution of $(\pi\rho, E)$ by H-relatively injective Banach H-modules. We may again consider $(\mathfrak{a}, A^\bullet)$ as a complex of Banach H-modules via the pullback structure (cf. Lemma B.2.15). This complex is now not necessarily a strong resolution of Banach H-modules anymore!

However the subcomplex of maximal continuous G-modules $(\mathfrak{a}, \mathcal{C}_\pi A^\bullet)$ is both G-strong and H-strong. In order to check that $(\mathfrak{a}, \mathcal{C}_\pi A^\bullet)$ is also H-strong, we have to check that the subcomplex of maximal continuous H-modules $(\mathfrak{a}, \mathcal{C}_{\pi\rho}\mathcal{C}_\pi A^\bullet)$ admits a contracting homotopy. Recall that by Lemma B.2.16 we have for any Banach G-module (π, E) that $\mathcal{C}_\pi E \subset \mathcal{C}_{\pi\rho} E$. Now

$$\mathcal{C}_\pi A^n = \mathcal{C}_\pi \mathcal{C}_\pi A^n \subset \mathcal{C}_{\pi\rho}\mathcal{C}_\pi A^n$$

and since clearly $\mathcal{C}_{\pi\rho}\mathcal{C}_\pi A^n \subset \mathcal{C}_\pi A^n$ we have

$$\mathcal{C}_{\pi\rho}\mathcal{C}_\pi A^n = \mathcal{C}_\pi A^n$$

for all $n \geq 0$.

Because $(\mathfrak{a}, A^\bullet)$ is G-strong, $(\mathfrak{a}, \mathcal{C}_\pi A^\bullet)$ admits a contracting homotopy. By the above equality of continuous modules this homotopy is also a contracting homotopy for $(\mathfrak{a}, \mathcal{C}_{\pi\rho}\mathcal{C}_\pi A^\bullet)$, such that $(\mathfrak{a}, \mathcal{C}_\pi A^\bullet)$ is H-strong.

We can now again by Lemma B.2.16 consider the inclusion $\mathcal{C}_\pi E \hookrightarrow \mathcal{C}_{\pi\rho} E$, which is clearly an H-morphism regarding $\mathcal{C}_\pi E$ as an H-module via $\pi\rho$. Because $(\mathfrak{a}, \mathcal{C}_\pi A^\bullet)$ is H-strong we can extend this inclusion to an H-morphism $i^\bullet$ between the complexes of Banach H-modules $(\mathfrak{a}, \mathcal{C}_\pi A^\bullet)$ and $(\mathfrak{b}, B^\bullet)$ by Lemma II.2.12.

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{C}_\pi E \arrow[r, "\mathfrak{a}"] \arrow[d, "i"'] & \mathcal{C}_\pi A^0 \arrow[r] \arrow[d, dotted, "i^0"'] & \mathcal{C}_\pi A^1 \arrow[r] \arrow[d, dotted, "i^1"'] & \mathcal{C}_\pi A^2 \arrow[r] \arrow[d, dotted, "i^2"'] & \cdots \\
0 \arrow[r] & \mathcal{C}_{\pi\rho} E \arrow[r, "\mathfrak{b}"'] & \mathcal{C}_{\pi\rho} B^0 \arrow[r] & \mathcal{C}_{\pi\rho} B^1 \arrow[r] & \mathcal{C}_{\pi\rho} B^2 \arrow[r] & \cdots
\end{tikzcd}
$$

As usual we may restrict this map to $i^\bullet : (\mathcal{C}_\pi A^\bullet)^H \to (\mathcal{C}_{\pi\rho} B^\bullet)^H = (B^\bullet)^H$. Observe that $(\mathcal{C}_\pi A^\bullet)^H = (\mathcal{C}_\pi A^\bullet)^{\rho(H)} \supset (\mathcal{C}_\pi A^\bullet)^G = (A^\bullet)^G$ when considered as a Banach G-module, such that we can restrict $i^\bullet$ even further to a morphism of complexes

$$i^\bullet : (A^\bullet)^G \to (B^\bullet)^H$$

This induces a map in cohomology

$$i^\bullet : H^\bullet(A^{\bullet G}) \to H^\bullet(B^{\bullet H})$$

We know that $H^\bullet(A^{\bullet G}) \cong H^\bullet_{cb}(G, E)$ and $H^\bullet(B^{\bullet H}) \cong H^\bullet_{cb}(H, \rho^* E)$ from Theorem II.1.12. Recall that these isomorphisms are given by a G-extension $\alpha^\bullet : \mathcal{C}_\pi A^\bullet \to \mathcal{C}_\pi C_b(G^\bullet, E)$ and an H-extension $\beta^\bullet : \mathcal{C}_{\pi\rho} C_b(H^\bullet, E) \to \mathcal{C}_{\pi\rho} B^\bullet$ of the identity $\mathrm{id} : \mathcal{C}_\pi E \to \mathcal{C}_\pi E$ and $\mathrm{id} : \mathcal{C}_{\pi\rho} E \to \mathcal{C}_{\pi\rho} E$ respectively (cf. Lemma II.2.15). Note that $\alpha^\bullet$ is also an H-morphism (cf. Lemma B.2.15). Further recall that $\rho^* : C_b(G^\bullet, E) \to C_b(H^\bullet, E)$ is an H-extension of the identity $E \to E$ and hence restricts to an H-extension of the inclusion $\mathcal{C}_\pi E \to \mathcal{C}_{\pi\rho} E$.

Considering the H-morphism of complexes

$$\beta^\bullet \circ \rho^* \circ \alpha^\bullet : \mathcal{C}_\pi A^\bullet \to \mathcal{C}_\pi C_b(G^\bullet, E) \to \mathcal{C}_{\pi\rho} C_b(H^\bullet, E) \to \mathcal{C}_{\pi\rho} B^\bullet$$

this is clearly an H-extension of

$$\mathrm{id} \circ i \circ \mathrm{id} = i : \mathcal{C}_\pi E \to \mathcal{C}_\pi E \to \mathcal{C}_{\pi\rho} E \to \mathcal{C}_{\pi\rho} E$$

and is thus H-homotopic to the previously defined extension $i^\bullet : \mathcal{C}_\pi A^\bullet \to \mathcal{C}_{\pi\rho} B^\bullet$. Thus both restrictions

$$\beta^\bullet \circ \rho^* \circ \alpha^\bullet : (A^\bullet)^G \to (B^\bullet)^H$$

and

$$i^\bullet : (A^\bullet)^G \to (B^\bullet)^H$$

are homotopic morphisms of complexes and hence induce the same map in cohomology. Indeed, both maps restricted to the subcomplex $(A^\bullet)^G$ of G-invariants are homotopic, because they are already homotopic when restricted to the subcomplex $(A^\bullet)^H$ of H-invariants (cf. Remark II.1.2).

Summarizing our previous discussion we get the following proposition.

Proposition II.2.33. *Let $(\mathfrak{a}, A^\bullet)$ be a strong resolution of (π, E) by G-relatively injective Banach G-modules and $(\mathfrak{b}, B^\bullet)$ be a strong resolution of $(\pi\rho, E)$ by H-relatively injective Banach H-modules. Consider the former as a complex of H-modules.*

Then the inclusion $\mathcal{C}_\pi E \to \mathcal{C}_{\pi\rho} E$ extends to an H-morphism of the augmented complexes and moreover for all $n \geq 0$ the map

$$i^\bullet : H^n(A^{\bullet G}) \to H^n(B^{\bullet H})$$

induced by any such extension $i^\bullet$ is conjugated to ρ^* by the canonical isomorphisms

$$H^n_{cb}(G, E) \cong H^n(A^{\bullet G}) \qquad \text{and} \qquad H^n_{cb}(H, \rho^* E) \cong H^n(B^{\bullet H})$$

given by Theorem II.2.18.

Note that Proposition II.2.33 is actually [Mon01, Proposition 8.4.2, p. 102], although we have used our slightly different standard resolution here (cf. Remark II.2.23).

Remark II.2.34. *A similar result holds for continuous cohomology too. We chose not to give the result in our section on continuous cohomology in favour of a more concise exposition and due to the fact, that it is not needed in our proof of the volume rigidity theorem. The proof is along the same lines as the one for bounded cohomology and even a bit easier, since no maximal continuous submodules have to be considered.*

This enables us to identify the pullback map in other resolutions than the standard one.

If $H < G$ is a closed subgroup we can due to Corollary II.2.28 compute $H^\bullet_{cb}(H, \mathbb{R}_{\pi i})$ also as $H^\bullet(L^\infty(G^{\bullet+1}, \mathbb{R}_{\pi i})^H)$. In this case we get from Proposition II.2.33, that the pullback corresponding to the canonical inclusion $i : H \to G$ is simply given by the inclusion $L^\infty(G^{\bullet+1}, \mathbb{R}_\pi)^G \to L^\infty((G)^{\bullet+1}, \mathbb{R}_{\pi i})^H$.

Corollary II.2.35. *Let $H < G$ be a closed subgroup and $i : H \to G$ the canonical inclusion. Then the pullback*

$$i^* : H^\bullet_{cb}(G, \mathbb{R}_\pi) \to H^\bullet_{cb}(H, \mathbb{R}_{\pi i})$$

along the canonical inclusion is given at the cochain level by the inclusion

$$\iota : L^\infty(G^{\bullet+1}, \mathbb{R}_\pi)^G \to L^\infty(G^{\bullet+1}, \mathbb{R}_{\pi i})^H$$

i.e. i^* is conjugated to ι with respect to the canonical isomorphisms $H^\bullet_{cb}(G, \mathbb{R}_\pi) \cong H^\bullet(L^\infty(G^{\bullet+1}, \mathbb{R}_\pi)^G)$ and $H^\bullet_{cb}(H, \mathbb{R}_{\pi i}) \cong H^\bullet(L^\infty(G^{\bullet+1}, \mathbb{R}_{\pi i})^H)$.

Proof. Observe that we have for the pullback of the left regular representation $\lambda_\pi \circ i = \lambda_{\pi i}$. Thus the inclusion

$$\iota : \mathcal{C}_{\lambda_\pi i} L^\infty(G^{\bullet+1}, \mathbb{R}_\pi) \to L^\infty(G^{\bullet+1}, \mathbb{R}_{\pi i})$$

is clearly an extension of the inclusion $\mathcal{C}_\pi \mathbb{R} \to \mathcal{C}_{\pi i} \mathbb{R}$. The assertion now follows from Proposition II.2.33. □

Moreover we can use this understanding of the pullback to realize it geometrically in a certain situation as the following corollary shows.

Corollary II.2.36. *Let X be a countable CW-complex, $\overline{\pi} : \tilde{X} \to X$ its universal cover and $\overline{\Lambda} := \pi_1(X) \cong \mathrm{Deck}(\pi)$. Further let $\Lambda < \overline{\Lambda}$ be a subgroup. Denote by $p : \Lambda \backslash \tilde{X} \to \overline{\Lambda} \backslash \tilde{X}$ the induced covering map and by $i^* : H^\bullet_{cb}(\overline{\Lambda}, \mathbb{R}) \to H^\bullet_{cb}(\Lambda, \mathbb{R})$ the pullback map induced by the canonical inclusion $i : \Lambda \hookrightarrow \overline{\Lambda}$. Then the following diagram commutes*

$$\begin{tikzcd}
H^\bullet_{cb}(\overline{\Lambda},\mathbb{R}) \arrow[r, "i^*"] & H^\bullet_{cb}(\Lambda,\mathbb{R}) \\
H^\bullet_b(\overline{\Lambda}\backslash\tilde{X}) \arrow[u, "\cong"] \arrow[r, "p^*"] & H^\bullet_b(\Lambda\backslash\tilde{X}) \arrow[u, "\cong"]
\end{tikzcd}$$

where both vertical arrows are the isomorphisms given by Lemma II.2.32.

Proof. As we have seen before in Proposition II.2.31 the bounded cohomology of both Λ and $\overline{\Lambda}$ can be computed by means of the strong augmented resolution $(\epsilon, S^\bullet_b(\tilde{X}))$. Since Λ and $\overline{\Lambda}$ are discrete, every module over them is continuous, such that we do not have to work with maximal continuous submodules at all. Therefore an extension $i^\bullet$ as in Proposition II.2.33 is simply given by the identity in every degree and the pullback $i^* : H^\bullet_{cb}(\overline{\Lambda},\mathbb{R}) \to H^\bullet_{cb}(\Lambda,\mathbb{R})$ is conjugated to the map induced by the inclusion of subcomplexes

$$S^\bullet_b(\tilde{X})^{\overline{\Lambda}} \hookrightarrow S^\bullet_b(\tilde{X})^{\Lambda}$$

By Lemma II.2.32 we have isomorphisms of complexes

$$\overline{\pi}^* : S^\bullet_b(\overline{\Lambda}\backslash\tilde{X}) \to S^\bullet_b(\tilde{X})^{\overline{\Lambda}}$$

and

$$\pi^* : S^\bullet_b(\Lambda\backslash\tilde{X}) \to S^\bullet_b(\tilde{X})^{\Lambda}$$

where $\overline{\pi} : \tilde{X} \to \overline{\Lambda}\backslash\tilde{X}$ and $\pi : \tilde{X} \to \Lambda\backslash\tilde{X}$ are the canonical covering maps.

These fit into the commutative diagram of covering maps

$$\begin{tikzcd}
\tilde{X} \arrow[d, "\pi"'] \arrow[dr, "\overline{\pi}"] & \\
\Lambda\backslash\tilde{X} \arrow[r, "p"] & \overline{\Lambda}\backslash\tilde{X}
\end{tikzcd}$$

which induces the commutative diagram of complexes

$$\begin{tikzcd}
S^\bullet_b(\tilde{X})^{\Lambda} & S^\bullet_b(\tilde{X})^{\overline{\Lambda}} \arrow[l, hook'] \\
S^\bullet_b(\Lambda\backslash\tilde{X}) \arrow[u, "\pi^*"] & S^\bullet_b(\overline{\Lambda}\backslash\tilde{X}) \arrow[l, "p^*"'] \arrow[u, "\overline{\pi}^*"']
\end{tikzcd}$$

where the upper horizontal map is the inclusion of subcomplexes.

Therefore we get at the cohomology level the asserted commutativity. $\square$

II.2.5. The Pullback via Equivariant Maps

We keep the notation of the previous subsection, i.e. $H < G$ is a second countable locally compact group and $\rho : H \to G$ a continuous homomorphism. Our main reference for this section is [BI02].

Proposition II.2.37. *Let (π, E) be a Banach G-module. Let $(\mathfrak{c}, C^\bullet)$ and $(\mathfrak{d}, D^\bullet)$ be strong resolutions of E by Banach G-modules and let $\alpha^\bullet : \mathcal{C}D^\bullet \to \mathcal{C}C^\bullet$ be a G-morphism of complexes extending the identity $\mathrm{id} : \mathcal{C}_\pi E \to \mathcal{C}_\pi E$. Then for any resolution $(\mathfrak{a}, A^\bullet)$ of $(\pi\rho, E)$ by relatively injective Banach H-modules the following diagram in cohomology commutes*

$$\begin{tikzcd}
H^\bullet(A^{\bullet H}) & H^\bullet(D^{\bullet G}) \arrow[l, "\gamma^\bullet"'] \arrow[d, "\alpha^\bullet"] \\
 & H^\bullet(C^{\bullet G}) \arrow[ul, "i^\bullet"]
\end{tikzcd}$$

where $i^\bullet$ is as in Proposition II.2.33 induced by an extension of the inclusion $\mathcal{C}_\pi E \hookrightarrow \mathcal{C}_{\pi\rho} E$ and $\gamma^\bullet$ is the map induced by any H-morphism of complexes $\mathcal{C}_\pi D^\bullet \to A^\bullet$ extending the inclusion H-morphism $\mathcal{C}_\pi E \hookrightarrow E$.

Proof. The proof is basically diagram chasing and using Lemma II.2.12 similar to our discussion of Proposition II.2.33. For details we refer to [BI02, Proposition 2.2, p. 4]. □

This proposition enables us to understand the pullback map via equivariant maps in certain situations.

Definition II.2.38. Let (B, ν) be a measure space with a measurable H-action and let X be a measurable space with a measurable G-action (H and G are here understood as measurable spaces via their Haar σ-algebras as usual). A measurable map $\varphi : B \to X$ is called *a.e.-ρ-equivariant* if

$$\varphi(hx) = \rho(h)\varphi(x)$$

for all $h \in H$ and almost every $x \in B$.

Lemma II.2.39. *We keep the notation of Definition II.2.38. Let $\varphi : B \to X$ be an a.e.-ρ-equivariant map. Then φ induces maps*

$$\varphi^* : \mathcal{B}^\infty(X^{n+1}, \mathbb{R}_\pi) \to L^\infty(B^{n+1}, \mathbb{R}_{\pi\rho})$$

given via precomposition

$$(\varphi^* f)(x_0, \dots, x_n) = f(\varphi(x_0), \dots, \varphi(x_n))$$

for every $f \in \mathcal{B}^\infty(X^{n+1}, \mathbb{R}_\pi)$, for all $x_0, \dots, x_n \in B$ and $n \in \mathbb{N}_0$. These maps constitute an H-morphism of complexes $\varphi^* : \mathcal{B}^\infty(X^{\bullet+1}, \mathbb{R}_\pi) \to L^\infty(B^{\bullet+1}, \mathbb{R}_{\pi\rho})$ which extends the inclusion $\mathcal{C}_\pi\mathbb{R} \hookrightarrow \mathbb{R}_{\pi\rho}$ and is norm non-increasing.

Proof. One immediately checks, that φ^* is norm non-increasing and extends the inclusion $\mathcal{C}_\pi\mathbb{R} \hookrightarrow \mathbb{R}_{\pi\rho}$. It remains to check, that it is indeed an H-morphism of complexes. For that recall that the H-action on $L^\infty(B^{n+1}, \mathbb{R}_{\pi\rho})$ is given by

$$(h \cdot f)(x_0, \dots, x_n) = \pi(\rho(h)) f(h^{-1}x_0, \dots, h^{-1}x_n)$$

for every $f \in L^\infty(B^{n+1}, \mathbb{R}_\pi)$, $h \in H$, $x_0, \dots, x_n \in B$, and that $\mathcal{B}^\infty(X^{n+1}, \mathbb{R}_\pi)$ becomes a Banach H-module via the pullback structure $\lambda_\pi\rho$, i.e.

$$(h \cdot f)(x_0, \dots, x_n) = \pi(\rho(h))f(\rho(h^{-1})x_0, \dots, \rho(h^{-1})x_n)$$

for every $f \in \mathcal{B}^\infty(X^{n+1}, \mathbb{R}_\pi)$, $h \in H$, $x_0, \dots, x_n \in X$.

With that in mind we compute

$$\begin{aligned}(\varphi^*(h \cdot f))(x_0, \dots, x_n) &= (h \cdot f)(\varphi(x_0), \dots, \varphi(x_n))\\ &= \pi(\rho(h))f(\rho(h^{-1})\varphi(x_0), \dots, \varphi(x_n))\\ &= \pi(\rho(h))f(\varphi(h^{-1}x_0), \dots, \varphi(h^{-1}x_n))\\ &= \pi(\rho(h))(\varphi^* f)(h^{-1}x_0, \dots h^{-1}x_n)\\ &= (h \cdot (\varphi^* f))(x_0, \dots, x_n)\end{aligned}$$

for every $f \in \mathcal{B}^\infty(X^{n+1}, \mathbb{R}_\pi)$, $h \in H$ and almost every $x_0, \dots, x_n \in B$, i.e. φ^* is indeed an H-morphism. ☐

Corollary II.2.40. *Let $\mathbb{R}_\pi$ be a Banach G-module, S be an amenable regular H-space, X be a measurable space with measurable G-action, $\varphi : S \to X$ an a.e.-ρ-equivariant map and $(\mathfrak{c}, C^\bullet)$ any strong resolution of $\mathbb{R}_\pi$ by relatively injective Banach G-modules. Then we have the following commutative diagram in cohomology*

$$\begin{tikzcd}
H^\bullet_{cb}(H, \mathbb{R}_{\pi\rho}) & H^\bullet(L^\infty(S^{\bullet+1}, \mathbb{R}_{\pi\rho})^H) \arrow[l, "\cong"'] & H^\bullet(\mathcal{B}^\infty(X^{\bullet+1}, \mathbb{R}_\pi)^G) \arrow[l, "\varphi^*"'] \arrow[d, "\alpha^\bullet"] \\
& H^\bullet_{cb}(G, \mathbb{R}_\pi) \arrow[ul, "\rho^*"] & H^\bullet(C^{\bullet G}) \arrow[l, "\cong"] \arrow[ul, "i^\bullet"']
\end{tikzcd}$$

where $\alpha^\bullet$ is induced by a G-morphism extending the identity $\mathrm{id} : \mathcal{C}\mathbb{R}_\pi \to \mathcal{C}\mathbb{R}_\pi$, *$i^\bullet$ is induced by an H-morphism extending the inclusion $\mathcal{C}_\pi\mathbb{R} \hookrightarrow \mathcal{C}_{\pi\rho}\mathbb{R}$ and both isomorphisms are the canonical ones given by Theorem II.2.27 and Theorem II.2.18.*

Proof. This is Proposition II.2.37 with

$$\begin{aligned}A^\bullet &= L^\infty(S^{\bullet+1}, \mathbb{R}_{\pi\rho})\\ D^\bullet &= B^\infty(X^{\bullet+1}, \mathbb{R}_\pi)\end{aligned}$$

and the functorial characterization of the pullback given in Proposition II.2.33. ☐

This corollary has the following two important special cases.

Corollary II.2.41. *Let $\mathbb{R}_\pi$ be a Banach G-module, S be an amenable regular G-space and $\varphi : S \to S$ an a.e.-ρ-equivariant map. Then the following diagram commutes*

$$\begin{tikzcd}
H^\bullet_{cb}(H, \mathbb{R}_{\pi\rho}) & H^\bullet(L^\infty(S^{\bullet+1}, \mathbb{R}_{\pi\rho})^H) \arrow[l, "\cong"'] & H^\bullet(\mathcal{B}^\infty(S^{\bullet+1}, \mathbb{R}_\pi)^G) \arrow[l, "\varphi^*"'] \arrow[d] \\
& H^\bullet_{cb}(G, \mathbb{R}_\pi) \arrow[ul, "\rho^*"] & H^\bullet(L^\infty(S^{\bullet+1}, \mathbb{R}_\pi)^G) \arrow[l, "\cong"] \arrow[ul, "i^\bullet"']
\end{tikzcd}$$

where the vertical arrow on the right is induced by the inclusion of complexes $\mathcal{B}^\infty(S^{\bullet+1}, \mathbb{R}_\pi) \hookrightarrow L^\infty(S^{\bullet+1}, \mathbb{R}_\pi)$ *and* $i^\bullet$ *is induced by an* H*-morphism extending the inclusion* $\mathcal{C}_\pi\mathbb{R} \hookrightarrow \mathcal{C}_{\pi\rho}\mathbb{R}$

Proof. Note that if S is an amenable regular G-space, then it is also an amenable regular H-space via the restricted action (cf. Lemma C.2.9).

Further it is immediate that the inclusion of complexes $\mathcal{B}^\infty(S^{\bullet+1}, \mathbb{R}_\pi) \hookrightarrow L^\infty(S^{\bullet+1}, \mathbb{R}_\pi)$ is a G-morphism of complexes extending $\mathrm{id} : \mathcal{C}\mathbb{R}_\pi \to \mathcal{C}\mathbb{R}_\pi$. ☐

Corollary II.2.42. *Let* $\mathbb{R}_\pi$ *be a Banach* G*-module. Then the map* $\varphi = \rho : H \to G$ *is an a.e.-*ρ*-equivariant map and the following diagram commutes*

$$\begin{tikzcd}
H^\bullet_{cb}(H, \mathbb{R}_{\pi\rho}) & H^\bullet(L^\infty(H^{\bullet+1}, \mathbb{R}_{\pi\rho})^H) \arrow[l, "\cong"'] & H^\bullet(\mathcal{B}^\infty(G^{\bullet+1}, \mathbb{R}_\pi)^G) \arrow[l, "\varphi^* = \rho^*"'] \arrow[d] \\
& H^\bullet_{cb}(G, \mathbb{R}_\pi) \arrow[ul, "\rho^*"] & H^\bullet(L^\infty(G^{\bullet+1}, \mathbb{R}_\pi)^G) \arrow[l, "\cong"] \arrow[ul, "i^\bullet"']
\end{tikzcd}$$

where the vertical arrow on the right is induced by the inclusion of complexes $\mathcal{B}^\infty(S^{\bullet+1}, \mathbb{R}_\pi) \hookrightarrow L^\infty(S^{\bullet+1}, \mathbb{R}_\pi)$ *and* $i^\bullet$ *is induced by an* H*-morphism extending the inclusion* $\mathcal{C}_\pi\mathbb{R} \hookrightarrow \mathcal{C}_{\pi\rho}\mathbb{R}$

Proof. Note that G is an amenable regular G-space and H is an amenable regular H-space with their respective Haar measures.

Further it is immediate that the inclusion of complexes $\mathcal{B}^\infty(G^{\bullet+1}, \mathbb{R}_\pi) \hookrightarrow L^\infty(G^{\bullet+1}, \mathbb{R}_\pi)$ is a G-morphism of complexes extending $\mathrm{id} : \mathcal{C}\mathbb{R}_\pi \to \mathcal{C}\mathbb{R}_\pi$. ☐

II.2.6. A Functorial View on the Comparison Map $c : H^\bullet_{cb}(G, E) \to H^\bullet_c(G, \mathcal{C}E)$

In this section we want to give a more functorial explanation of the comparison map similar to our functorial discussion of the pullback map. As for the pullback map we first want to investigate our naive definition from the beginning and put it in a more functorial framework.

Let (π, E) be a Banach G-module. Recall that the comparison map $c : H^\bullet_{cb}(G, E) \to H^\bullet_c(G, \mathcal{C}E)$ is given by the inclusion

$$C_b(G^{\bullet+1}, E)^G = C_b(G^{\bullet+1}, \mathcal{C}E)^G \hookrightarrow C(G^{\bullet+1}, \mathcal{C}E)^G$$

By Proposition II.2.16 the resolution $(\epsilon, C_b(G^{\bullet+1}, E))$ is a strong augmented resolution (by relatively injective Banach G-modules), i.e. the complex of continuous Banach G-modules

$$0 \longrightarrow \mathcal{C}E \longrightarrow \mathcal{C}C_b(G, E) \longrightarrow \mathcal{C}C_b(G^2, E) \longrightarrow \cdots$$

admits a contracting homotopy and is hence also a strong resolution of G-modules in the sense of continuous cohomology. Considering the standard resolution $(\epsilon, C(G^{\bullet+1}, \mathcal{C}E))$ of $\mathcal{C}E$ in continuous cohomology we know that there must be an extension of the identity $\mathrm{id} : \mathcal{C}E \to \mathcal{C}E$ from $(\epsilon, \mathcal{C}C_b(G^{\bullet+1}, E))$ to $(\epsilon, C(G^{\bullet+1}, \mathcal{C}E))$.

Now we have the following lemma.

Lemma II.2.43. $\mathcal{C}C_b(G^{\bullet+1}, E) \subseteq C_b(G^{\bullet+1}, \mathcal{C}E)$

Proof. Let $q \in \mathbb{N}_0$ and $f \in \mathcal{C}C_b(G^{\bullet+1}, E)$. We want to apply Lemma B.2.10.

Let $x \in G^{q+1}$ and let $(g_\alpha)_{\alpha \in A}$ be a net converging to the neutral element $e \in G$. Then

$$\begin{aligned}\|\pi(g_\alpha)f(x) - f(x)\| &\leq \|\pi(g_\alpha)f(x) - \pi(g_\alpha)f(g_\alpha^{-1}x)\| + \|\pi(g_\alpha)f(g_\alpha^{-1}x) - f(x)\| \\ &\leq \|f(x) - f(g_\alpha^{-1}x)\| + \|(\lambda_\pi(g_\alpha)f)(x) - f(x)\| \to 0\end{aligned}$$

where the first term tends to 0 because f is continuous and the second term tends to 0 because f is contained in the maximal continuous submodule. $\square$

Therefore we can consider the inclusion

$$\iota^\bullet : \mathcal{C}C_b(G^{\bullet+1}, E) \subseteq C_b(G^{\bullet+1}, \mathcal{C}E) \hookrightarrow C(G^{\bullet+1}, \mathcal{C}E)$$

It is immediate, that this is an extension of $\mathrm{id} : \mathcal{C}E \to \mathcal{C}E$.

At the level of invariants this is just the inclusion

$$C_b(G^{\bullet+1}, E)^G = C_b(G^{\bullet+1}, \mathcal{C}E)^G \hookrightarrow C(G^{\bullet+1}, \mathcal{C}E)^G$$

from above and hence induces the comparison map $c : H^\bullet_{cb}(G, E) \to H^\bullet_c(G, \mathcal{C}E)$ in cohomology.

In the case of arbitrary resolutions we can mimic the above approach to the comparison map. Let $(\mathfrak{a}, A^\bullet)$ be a strong augmented resolution of E by relatively injective Banach G-modules and let $(\mathfrak{b}, B^\bullet)$ be a strong augmented resolution of $\mathcal{C}E$ by (continuous) relatively injective G-modules.

Again the complex of continuous Banach G-modules

$$0 \longrightarrow \mathcal{C}E \xrightarrow{\mathfrak{a}} \mathcal{C}A^0 \longrightarrow \mathcal{C}A^1 \longrightarrow \cdots$$

is a strong resolution of $\mathcal{C}E$ by (continuous) G-modules. Therefore there is an extension $\psi^\bullet : \mathcal{C}A^\bullet \to B^\bullet$ of $\mathrm{id} : \mathcal{C}E \to \mathcal{C}E$.

Our objective is to show that the map in cohomology induced by such an extension $\psi^\bullet$ is conjugated to the comparison map $c : H^\bullet_{cb}(G, E) \to H^\bullet_c(G, \mathcal{C}E)$. For that let $\alpha^\bullet : \mathcal{C}A^\bullet \to \mathcal{C}C_b(G^{\bullet+1}, E)$ and $\beta^\bullet : C(G^{\bullet+1}, \mathcal{C}E) \to B^\bullet$ be extensions of $\mathrm{id} : \mathcal{C}E \to \mathcal{C}E$ inducing the respective canonical isomorphisms $H^\bullet(A^{\bullet G}) \cong H^\bullet_{cb}(G, E)$ and $H^\bullet_c(G, \mathcal{C}E) \cong H^\bullet(B^{\bullet G})$.

These fit into the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}E \arrow[r, "\mathfrak{a}"] \arrow[d, "\mathrm{id}"'] & \mathcal{C}A^0 \arrow[r] \arrow[d, "\alpha^0"'] & \mathcal{C}A^1 \arrow[r] \arrow[d, "\alpha^1"'] & \cdots \\
0 \arrow[r] & \mathcal{C}E \arrow[r, "\epsilon"] \arrow[d, "\mathrm{id}"'] & \mathcal{C}C_b(G, E) \arrow[r] \arrow[d, "\iota^0"'] & \mathcal{C}C_b(G^2, E) \arrow[r] \arrow[d, "\iota^1"'] & \cdots \\
0 \arrow[r] & \mathcal{C}E \arrow[r, "\epsilon"] \arrow[d, "\mathrm{id}"'] & C(G, \mathcal{C}E) \arrow[r] \arrow[d, "\beta^0"'] & C(G^2, \mathcal{C}E) \arrow[r] \arrow[d, "\beta^1"'] & \cdots \\
0 \arrow[r] & \mathcal{C}E \arrow[r, "\mathfrak{b}"] & B^0 \arrow[r] & B^1 \arrow[r] & \cdots
\end{tikzcd}$$

with the previous inclusion

$$\iota^\bullet : \mathcal{C}C_b(G^{\bullet+1}, E) \hookrightarrow C(G^{\bullet+1}, \mathcal{C}E)$$

in the middle. Now their composition $\beta^\bullet \circ \iota^\bullet \circ \alpha^\bullet : \mathcal{C}A^\bullet \to B^\bullet$ is also an extension of $\mathrm{id} : \mathcal{C}E \to \mathcal{C}E$ just as $\psi^\bullet : \mathcal{C}A^\bullet \to B^\bullet$. By Lemma II.1.8 these are G-homotopic and hence induce the same map in cohomology. Thus we get the following commutative diagram in cohomology

$$\begin{tikzcd} H^\bullet_{cb}(G,E) \arrow[r, "c"] & H^\bullet_c(G,\mathcal{C}E) \arrow[d, "\cong"'] \\ H^\bullet(A^{\bullet G}) \arrow[u, "\cong"] \arrow[r, "\psi^\bullet"] & H^\bullet(B^{\bullet G}) \end{tikzcd}$$

where the vertical arrows are the canonical isomorphisms induced by $\alpha^\bullet$ and $\beta^\bullet$.

The following proposition summarizes what we have proved so far.

Proposition II.2.44. *Let (π, E) be a Banach G-module. Further let $(\mathfrak{a}, A^\bullet)$ be a strong resolution of E by relatively injective Banach G-modules and let $(\mathfrak{b}, B^\bullet)$ a strong resolution of $\mathcal{C}E$ by (continuous) relatively injective G-modules.*

Then the identity morphism $\mathrm{id} : \mathcal{C}E \to \mathcal{C}E$ *extends to a G-morphism $\psi^\bullet : \mathcal{C}A^\bullet \to B^\bullet$ and moreover for all $n \geq 0$ the map*

$$\psi^n : H^n(A^{\bullet G}) \to H^n(B^{\bullet G})$$

induced by any such extension $\psi^\bullet$ is conjugated to the comparison map $c : H^n_{cb}(G,E) \to H^n_c(G,\mathcal{C}E)$ by the canonical isomorphisms

$$H^n_{cb}(G,E) \cong H^n(A^{\bullet G}) \qquad \textit{and} \qquad H^n_c(G,\mathcal{C}E) \cong H^n(B^{\bullet G})$$

given by Theorem II.2.18 and Theorem II.1.12.

The following corollary is immediate.

Corollary II.2.45. *Let $(\pi, \mathbb{R})$ be a Banach G-module and let $K < G$ be a compact subgroup. Then the inclusion*

$$i^\bullet : C_b((G/K)^{\bullet+1}, \mathbb{R}_\pi) \hookrightarrow C((G/K)^{\bullet+1}, \mathcal{C}\mathbb{R}_\pi)$$

restricts to an extension of $\mathrm{id} : \mathcal{C}\mathbb{R}_\pi \to \mathcal{C}\mathbb{R}_\pi$.

Therefore the comparison map $c : H^n_{cb}(G,\mathbb{R}_\pi) \to H^n_c(G,\mathcal{C}\mathbb{R}_\pi)$ is conjugated via the canonical isomorphisms to the map $i^\bullet : H^\bullet(C_b((G/K)^{\bullet+1}, \mathbb{R}_\pi)^G) \to H^\bullet(C((G/K)^{\bullet+1}, \mathcal{C}\mathbb{R}_\pi)^G)$.

II.3. Applications to $G = \mathrm{Isom}(\mathbb{H}^n)$

Let us apply the previously gathered results from continuous and continuous bounded cohomology to the isometry group of hyperbolic n-space now. Again we will first treat continuous cohomology and show how one may compute the continuous cohomology of $\mathrm{Isom}(\mathbb{H}^n)$ from different resolutions in subsection II.3.1. In order to facilitate switching between the different resolutions we give **concrete** isomorphisms at the cochain level. A particularly important example of such an isomorphism will be given by the van Est isomorphism, which we will present here in some detail. Similarly we will see how the continuous bounded cohomology may be computed from different resolutions in subsection II.3.2. We also give here some concrete isomorphisms at the cochain level. In subsection II.3.3 we will then introduce the key cohomology class in the context of volume rigidity, namely the volume class. The definition of the volume of a representation will be by means of this class. After these preparations we will finally compute some cohomology groups in subsection II.3.4. In particular we compute $H_c^n(G, \mathbb{R}_\varepsilon)$ and show that the comparison map $c : H_{cb}^\bullet(G, \mathbb{R}_\varepsilon) \to H_c^\bullet(G, \mathbb{R}_\varepsilon)$ is an isomorphism, where $\varepsilon : G \to \mathrm{Iso}(\mathbb{R}) \cong \{\pm 1\} \cong \mathrm{Iso}(\mathbb{R})$ is a special representation.

For the rest of this chapter we put $G = \mathrm{Isom}(\mathbb{H}^n)$, $G^+ = \mathrm{Isom}^+(\mathbb{H}^n)$ and fix $n \geq 2$.

II.3.1. Continuous Cohomology and Hyperbolic Geometry

Recall that $\mathbb{H}^n \cong G/K$, resp. $\mathbb{H}^n \cong G^+/K^+$, where K and K^+ are the stabilizers of a point in $\mathbb{H}^n$ by G and G^+ respectively. We have that K and K^+ are maximal compact subgroups of G and G^+ respectively (cf. Lemma I.2.12). By Proposition II.1.15 we get the following resolution identifying $\mathbb{H}^n \cong G/K$ resp. $\mathbb{H}^n \cong G^+/K^+$ as above.

Corollary II.3.1. *Let $H = G^+$ or G and let $(\pi, \mathbb{R})$ be an H-module. Then the complex*

$$0 \longrightarrow \mathbb{R}_\pi \xrightarrow{\epsilon} C(\mathbb{H}^n, \mathbb{R}_\pi) \xrightarrow{d^1} C((\mathbb{H}^n)^2, \mathbb{R}_\pi) \xrightarrow{d^2} C((\mathbb{H}^n)^3, \mathbb{R}_\pi) \xrightarrow{d^3} \cdots$$

is a strong augmented resolution of $\mathbb{R}_\pi$ by relatively injective H-modules.

Moreover the cohomology of the complex

$$0 \longrightarrow C(\mathbb{H}^n, \mathbb{R}_\pi)^H \xrightarrow{d^1} C((\mathbb{H}^n)^2, \mathbb{R}_\pi)^H \xrightarrow{d^2} C((\mathbb{H}^n)^3, \mathbb{R}_\pi)^H \xrightarrow{d^3} \cdots$$

is canonically isomorphic to $H_c^\bullet(H, \mathbb{R}_\pi)$.

As G and G^+ are also Lie groups (with a finite number of connected components) we get yet another resolution by Proposition II.1.16.

Corollary II.3.2. *Let $H = G^+$ or G and let $(\pi, \mathbb{R})$ be an H-module. Then the complex*

$$0 \longrightarrow \mathbb{R}_\pi \xrightarrow{\epsilon} \Omega^0(\mathbb{H}^n, \mathbb{R}_\pi) \xrightarrow{d^1} \Omega^1(\mathbb{H}^n, \mathbb{R}_\pi) \xrightarrow{d^2} \Omega^2(\mathbb{H}^n, \mathbb{R}_\pi) \xrightarrow{d^3} \cdots$$

is a strong augmented resolution of $\mathbb{R}_\pi$ by relatively injective H-modules.

Moreover the cohomology of the complex

$$0 \longrightarrow \Omega^0(\mathbb{H}^n, \mathbb{R}_\pi)^H \xrightarrow{d^1} \Omega^1(\mathbb{H}^n, \mathbb{R}_\pi)^H \xrightarrow{d^2} \Omega^2(\mathbb{H}^n, \mathbb{R}_\pi)^H \xrightarrow{d^3} \cdots$$

is canonically isomorphic to $H_c^\bullet(H, \mathbb{R}_\pi)$.

Finally, we get by Proposition II.1.17 the following connection to ordinary singular cohomology.

Corollary II.3.3. *Let $\Gamma < G$ be a discrete and torsion-free subgroup and $M = \Gamma\backslash\mathbb{H}^n$ the resulting quotient manifold.*
Then $H_c^\bullet(\Gamma, \mathbb{R}) \cong H^\bullet(M)$, where $\mathbb{R}$ denotes the trivial Γ-module.

Proof. Note that $\mathbb{H}^n$ is contractible and Γ acts freely and properly discontinuously on $\mathbb{H}^n$ by Proposition I.4.3 and Proposition I.4.4. Thus by Proposition I.4.2 the quotient map $\pi : \mathbb{H}^n \to \Gamma\backslash\mathbb{H}^n = M$ is a covering and the hypothesis of Proposition II.1.17 is fulfilled. Hence $H_c^\bullet(\Gamma, \mathbb{R}) \cong H^\bullet(M)$. □

The van Est Isomorphism

Let us fix the notation of the previous two corollaries for this subsection. By the homological algebra approach to continuous cohomology we know, that there exists a H-morphism between the two strong augmented resolutions $(\epsilon, \Omega^\bullet(\mathbb{H}^n, \mathbb{R}_\pi))$ and $(\epsilon, C((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\pi))$ extending the identity morphism $\mathrm{id} : \mathbb{R}_\pi \to \mathbb{R}_\pi$ and inducing the isomorphism in cohomology (cf. Lemma II.1.9). However it is apriori not clear what such a morphism would look like at the cochain level. Gladly our geometric situation enables us to define concrete maps, which will constitute the desired extension. This morphism is usually called the van Est isomorphism in the literature.

Remark II.3.4. *It is also possible to construct such an extension in the more abstract setting of Proposition II.1.16. For details we refer to [Gui80, Chp. III, no 7.3., p. 227].*

We will work in the hyperboloid model $H^n \cong \mathbb{H}^n$. Consider for $(q+1)$ points $x_0, \dots, x_q \in H^n$ the *straight simplex* $\mathrm{str}(x_0, \dots, x_q)$ *with vertices* $x_0, \dots, x_q$.

$$\begin{aligned} \mathrm{str} : \Delta^q &\to H^n \\ (t_1, \dots, t_q) &\mapsto \frac{(1 - \sum_{i=1}^q t_i)x_0 + t_1 x_1 + \cdots t_q x_q}{\|(1 - \sum_{i=1}^q t_i)x_0 + t_1 x_1 + \cdots + t_q x_q\|} \end{aligned}$$

We will adopt the abbreviation $t_0 = 1 - \sum_{i=1}^n t_i$ in the following. Observe that $\mathrm{str}(x_0, \dots, x_q)$ is a smooth singular q-simplex (cf. Appendix D) and that

$$\begin{aligned} (g_* \mathrm{str}(x_0, \dots x_q))(t_1, \dots, t_q) &= g(\mathrm{str}(x_0, \dots, x_q)(t_1, \dots, t_q)) = g \frac{t_0 x_0 + \dots t_q x_q}{\|t_0 x_0 + t_q x_q\|} \\ &= \frac{g(t_0 x_0 + \dots t_q x_q)}{\|g(t_0 x_0 + t_q x_q)\|} = \frac{t_0 \cdot g x_0 + \dots t_q \cdot g x_q}{\|t_0 \cdot g x_0 + t_q \cdot g x_q\|} \\ &= \mathrm{str}(g x_0, \dots g x_q)(t_1, \dots, t_q) \end{aligned}$$

for all $g \in H$, $(t_1, \dots, t_q) \in \Delta^q$.

We now define the van Est isomorphism $\Phi : \Omega^q(\mathbb{H}^n, \mathbb{R}_\pi) \to C((\mathbb{H}^n)^{q+1}, \mathbb{R}_\pi)$ at the cochain level by integrating over straight simplices

$$\Phi(\omega)(x_0, \dots, x_q) := \int_{\mathrm{str}(x_0, \dots, x_q)} \omega$$

for all $\omega \in \Omega^q(\mathbb{H}^n, \mathbb{R}_\pi)$, $x_0, \dots, x_q \in \mathbb{H}^n$. In degree zero the integral over the simplex $\mathrm{str}(x_0)$ is to be understood as evaluating the smooth function $\omega \in \Omega^0(\mathbb{H}^n, \mathbb{R}_\pi) \cong C^\infty(\mathbb{H}^n, \mathbb{R}_\pi)$ at x_0. It is not hard to see that $\Phi(\omega)$ is indeed a continuous function by applying Lebesgue's dominated convergence theorem.

Clearly Φ is linear and one readily checks that it is also continuous.

It is also an H-morphism as we have

$$\begin{aligned}\Phi(g\cdot\omega)(x_0,\dots,x_q) &= \int_{\mathrm{str}(x_0,\dots,x_q)} g\cdot\omega = \int_{\mathrm{str}(x_0,\dots,x_q)} \pi(g)\cdot(g^{-1})^*\omega \\ &= \pi(g)\cdot\int_{g_*^{-1}\mathrm{str}(x_0,\dots,x_q)}\omega = \pi(g)\cdot\int_{\mathrm{str}(g^{-1}x_0,\dots,g^{-1}x_q)}\omega \\ &= \pi(g)\cdot\Phi(\omega)(g^{-1}x_0,\dots,g^{-1}x_q) = (g\cdot\Phi(\omega))(x_0,\dots,x_q)\end{aligned}$$

for all $g \in H$, $\omega \in \Omega^q(\mathbb{H}^n,\mathbb{R}_\pi)$, $x_0,\dots,x_q \in \mathbb{H}^n$.

Finally we need to see that it is indeed an extension of $\mathrm{id} : \mathbb{R}_\pi \to \mathbb{R}_\pi$ to the resolutions $(\epsilon, \Omega^\bullet(\mathbb{H}^n,\mathbb{R}_\pi))$ and $(\epsilon, C((\mathbb{H}^n)^{\bullet+1},\mathbb{R}_\pi))$. First observe that for the i-th face inclusion $F_i : \Delta^{q-1} \to \Delta^q$ we have

$$\mathrm{str}(x_0,\dots,x_q)\circ F_i = \mathrm{str}(x_0,\dots,\hat{x_i},\dots,x_q)$$

for all $x_0,\dots,x_q \in \mathbb{H}^n$. Hence by Stoke's Theorem for smooth cochains (cf. Theorem D.4.1) we get

$$\begin{aligned}\Phi(d\omega)(x_0,\dots,x_q) &= \int_{\mathrm{str}(x_0,\dots,x_q)} d\omega = \int_{\partial\mathrm{str}(x_0,\dots,x_q)}\omega \\ &= \sum_{i=0}^q (-1)^i \int_{\mathrm{str}(x_0,\dots,x_q)\circ F_i}\omega = \sum_{i=0}^q(-1)^i\int_{\mathrm{str}(x_0,\dots,\hat{x_i},\dots,x_q)}\omega \\ &= \sum_{i=0}^q(-1)^i\Phi(\omega)(x_0,\dots,\hat{x_i},\dots,x_q) = (d\Phi(\omega))(x_0,\dots,x_q)\end{aligned}$$

for all $\omega \in \Omega^q(\mathbb{H}^n,\mathbb{R}_\pi)$, $x_0,\dots,x_q \in \mathbb{H}^n$, i.e. Φ commutes with the coboundary operators. It is also compatible with the augmentations, as we have

$$\Phi(\epsilon(t))(x_0) = \int_{\mathrm{str}(x_0)} t = t = \epsilon(\mathrm{id}(t))(x_0)$$

for all $t \in \mathbb{R}$ and $x_0 \in \mathbb{H}^n$. Hence Φ is indeed an extension of $\mathrm{id} : \mathbb{R}_\pi \to \mathbb{R}_\pi$ and thus induces an isomorphism at the cohomology level

$$\Phi : H^\bullet(\Omega^\bullet(\mathbb{H}^n,\mathbb{R}_\pi)^H) \to H^\bullet(C((\mathbb{H}^n)^{\bullet+1},\mathbb{R}_\pi)^H)$$

The Isomorphism $p_K^* : H^\bullet(C((\mathbb{H}^n)^{\bullet+1},\mathbb{R}_\pi)^H) \to H^\bullet(C(H^{\bullet+1},\mathbb{R}_\pi)^H)$

We also want to give a concrete isomorphism $H^\bullet(C((\mathbb{H}^n)^{\bullet+1},\mathbb{R}_\pi)^H) \to H^\bullet(C(H^{\bullet+1},\mathbb{R}_\pi)^H)$, where – as in the previous section – $H = G$ or G^+ and $(\pi,\mathbb{R})$ is an H-module. Further let $x \in \mathbb{H}^n$ be an arbitrary point and K its stabilizer in H. We may now consider the quotient map $p_K : H \to H/K \cong \mathbb{H}^n, g \mapsto gx$. This map induces a map $p_K^* : C((\mathbb{H}^n)^{\bullet+1},\mathbb{R}_\pi) \to C(H^{\bullet+1},\mathbb{R}_\pi)$ via precomposition

$$(p_K^* f)(g_0,\dots,g_q) := f(p_K(g_0),\dots,p_K(g_q)) = f(g_0x,\dots,g_qx)$$

for every $f \in C((\mathbb{H}^n)^{q+1},\mathbb{R}_\pi)$, $(g_0,\dots,g_q) \in H^{q+1}$, $q \in \mathbb{N}_0$. In fact this map induces an isomorphism in cohomology.

Proposition II.3.5. *Keep the above notation. The map*

$$p_K^* : C((\mathbb{H}^n)^{\bullet+1},\mathbb{R}_\pi) \to C(H^{\bullet+1},\mathbb{R}_\pi)$$

is a G-morphism of complexes extending the identity $\mathrm{id} : \mathbb{R}_\pi \to \mathbb{R}_\pi$ *to the strong augmented resolutions $(\epsilon, C((\mathbb{H}^n)^{\bullet+1},\mathbb{R}_\pi))$ and $(\epsilon, C(H^{\bullet+1},\mathbb{R}_\pi))$. In particular it induces an isomorphism at cohomology*

$$p_K^* : H^\bullet(C((\mathbb{H}^n)^{\bullet+1},\mathbb{R}_\pi)^H) \to H^\bullet(C(H^{\bullet+1},\mathbb{R}_\pi)^H)$$

Proof. One easily verifies that $p_K^* : C((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\pi) \to C(H^{\bullet+1}, \mathbb{R}_\pi)$ is continuous and linear. It is also H-equivariant, since

$$\begin{aligned}(h \cdot (p_K^* f))(g_0, \dots, g_q) &= \pi(h)(p_K^* f)(h^{-1}g_0, \dots, h^{-1}g_q) \\ &= \pi(h) f(h^{-1}g_0 x, \dots, h^{-1}g_q x) \\ &= p_K^*(h \cdot f)(g_0, \dots, g_q)\end{aligned}$$

for every $f \in C((\mathbb{H}^n)^{q+1}, \mathbb{R}_\pi)$, $(g_0, \dots, g_q) \in H^{q+1}$, $q \in \mathbb{N}_0$.

It remains to check, that it is indeed an extension of $\mathrm{id} : \mathbb{R}_\pi \to \mathbb{R}_\pi$. Let us first consider the coefficient inclusion. Then

$$p_K^*(\epsilon(t))(g_0) = \epsilon(t)(g_0 x) = t = \epsilon(t)(g_0)$$

for every $t \in \mathbb{R}$ and every $g_0 \in H$, i.e. $p_K^* \circ \epsilon = \epsilon \circ \mathrm{id}$. Finally, considering the homogeneous coboundary operators we get

$$\begin{aligned}d(p_K^* f)(g_0, \dots, g_{q+1}) &= \sum_{i=0}^{q+1} (-1)^i p_K^* f(g_0, \dots, \hat{g}_i, \dots, g_{q+1}) \\ &= \sum_{i=0}^{q+1} (-1)^i f(p_K(g_0), \dots, p_K(g_{i-1}), p_K(g_{i+1}), \dots, p_K(g_{q+1})) \\ &= p_K^*(df)(g_0, \dots, g_{q+1})\end{aligned}$$

for every $f \in L^\infty((\mathbb{H}^n)^{q+1}, \mathbb{R}_\pi)$, $g_0, \dots, g_{q+1} \in H$, $q \in \mathbb{N}_0$, i.e. $d \circ p_K^* = p_K^* \circ d$.

This concludes the proof. □

II.3.2. Continuous Bounded Cohomology and Hyperbolic Geometry

As for continuous cohomology in the previous section we now put its bounded counterpart in the setting of hyperbolic geometry. Let $\Gamma < G^+$ be a lattice subgroup in the following.

First of all applying Corollary II.2.26 to the trivial compact subgroup $K = \{1\}$ yields the following:

Corollary II.3.6. *Let $H = \Gamma$, G^+ or G, and let $(\pi, \mathbb{R})$ be a Banach H-module. Then*

$$0 \longrightarrow \mathbb{R}_\pi \xrightarrow{\epsilon} C_b(G, \mathbb{R}_\pi) \xrightarrow{d^1} C_b(G^2, \mathbb{R}_\pi) \xrightarrow{d^2} C_b(G^3, \mathbb{R}_\pi) \xrightarrow{d^3} \cdots$$

is a strong augmented resolution of $\mathbb{R}_\pi$ by relatively injective Banach H-modules.

Moreover the cohomology of the complex

$$0 \longrightarrow C_b(G, \mathbb{R}_\pi)^H \xrightarrow{d^1} C_b(G^2, \mathbb{R}_\pi)^H \xrightarrow{d^2} C_b(G^3, \mathbb{R}_\pi)^H \xrightarrow{d^3} \cdots$$

is canonically isometrically isomorphic to $H^\bullet_{cb}(H, \mathbb{R}_\pi)$.

Proof. Clearly Γ, G^+ and G are all closed subgroups of G, such that Corollary II.2.26 applies. □

Identifying $\mathbb{H}^n \cong G/K$ where K is again the stabilizer of one point in $\mathbb{H}^n$ we get by Corollary II.2.26 the following resolutions.

Corollary II.3.7. *Let $H = \Gamma$, G^+ or G, and let $(\pi, \mathbb{R})$ be a Banach H-module. Then*

$$0 \longrightarrow \mathbb{R}_\pi \xrightarrow{\epsilon} C_b(\mathbb{H}^n, \mathbb{R}_\pi) \xrightarrow{d^1} C_b((\mathbb{H}^n)^2, \mathbb{R}_\pi) \xrightarrow{d^2} C_b((\mathbb{H}^n)^3, \mathbb{R}_\pi) \xrightarrow{d^3} \cdots$$

is a strong augmented resolution of $\mathbb{R}_\pi$ by relatively injective Banach H-modules.

Moreover the cohomology of the complex

$$0 \longrightarrow C_b(\mathbb{H}^n, \mathbb{R}_\pi)^H \xrightarrow{d^1} C_b((\mathbb{H}^n)^2, \mathbb{R}_\pi)^H \xrightarrow{d^2} C_b((\mathbb{H}^n)^3, \mathbb{R}_\pi)^H \xrightarrow{d^3} \cdots$$

is canonically isometrically isomorphic to $H^\bullet_{cb}(H, \mathbb{R}_\pi)$.

Proof. Clearly Γ, G^+ and G are all closed subgroups of G, such that Corollary II.2.26 applies. □

We now turn to resolutions by L^∞-spaces. Since the trivial subgroup $\{1\}$ is clearly amenable we get by Corollary II.2.28 the following resolution.

Corollary II.3.8. *Let $H = \Gamma$, G^+ or G, and let $(\pi, \mathbb{R})$ be a Banach H-module. Then*

$$0 \longrightarrow \mathbb{R}_\pi \xrightarrow{\epsilon} L^\infty(G, \mathbb{R}_\pi) \xrightarrow{d^1} L^\infty(G^2, \mathbb{R}_\pi) \xrightarrow{d^2} L^\infty(G^3, \mathbb{R}_\pi) \xrightarrow{d^3} \cdots$$

is a strong augmented resolution of $\mathbb{R}_\pi$ by relatively injective Banach H-modules and the cohomology of the complex

$$0 \longrightarrow L^\infty(G, \mathbb{R}_\pi)^H \xrightarrow{d^1} L^\infty(G^2, \mathbb{R}_\pi)^H \xrightarrow{d^2} L^\infty(G^3, \mathbb{R}_\pi)^H \xrightarrow{d^3} \cdots$$

is canonically isometrically isomorphic to $H^\bullet_{cb}(H, \mathbb{R}_\pi)$.

This is in case of $H = G$ a generalization of our standard resolution by the complex of continuous bounded functions to essentially bounded function classes.

Because compact groups are amenable we get by the identification $\mathbb{H}^n = G/K$ as above the following resolution.

Corollary II.3.9. *Let $H = \Gamma$, G^+ or G, and let $(\pi, \mathbb{R})$ be a Banach H-module. Then*

$$0 \longrightarrow \mathbb{R}_\pi \xrightarrow{\epsilon} L^\infty(\mathbb{H}^n, \mathbb{R}_\pi) \xrightarrow{d^1} L^\infty((\mathbb{H}^n)^2, \mathbb{R}_\pi) \xrightarrow{d^2} L^\infty((\mathbb{H}^n)^3, \mathbb{R}_\pi) \xrightarrow{d^3} \cdots$$

is a strong augmented resolution of $\mathbb{R}_\pi$ by relatively injective Banach H-modules and the cohomology of the complex

$$0 \longrightarrow L^\infty(\mathbb{H}^n, \mathbb{R}_\pi)^H \xrightarrow{d^1} L^\infty((\mathbb{H}^n)^2, \mathbb{R}_\pi)^H \xrightarrow{d^2} L^\infty((\mathbb{H}^n)^3, \mathbb{R}_\pi)^H \xrightarrow{d^3} \cdots$$

is canonically isometrically isomorphic to $H^\bullet_{cb}(H, \mathbb{R}_\pi)$.

A particularly nice feature of continuous bounded cohomology is, that it allows us to compute the cohomology on the boundary of $\mathbb{H}^n$ as we will show now. Recall that G acts transitively on the boundary $\partial\mathbb{H}^n$ and we may hence identify $\partial\mathbb{H}^n \cong G/P$ where P is the stabilizer of a boundary point. We have already seen in Lemma I.2.15, that P is amenable.

We may now apply Corollary II.2.28 and get the following resolutions.

Corollary II.3.10. *Let $H = \Gamma$, G^+ or G, and let $(\pi, \mathbb{R})$ be a Banach H-module. Then*

$$0 \longrightarrow \mathbb{R}_\pi \xrightarrow{\epsilon} L^\infty(\partial\mathbb{H}^n, \mathbb{R}_\pi) \xrightarrow{d^1} L^\infty((\partial\mathbb{H}^n)^2, \mathbb{R}_\pi) \xrightarrow{d^2} L^\infty((\partial\mathbb{H}^n)^3, \mathbb{R}_\pi) \xrightarrow{d^3} \cdots$$

is a strong augmented resolution of $\mathbb{R}_\pi$ by relatively injective Banach H-modules and the cohomology of the complex

$$0 \longrightarrow L^\infty(\partial\mathbb{H}^n, \mathbb{R}_\pi)^H \xrightarrow{d^1} L^\infty((\partial\mathbb{H}^n)^2, \mathbb{R}_\pi)^H \xrightarrow{d^2} L^\infty((\partial\mathbb{H}^n)^3, \mathbb{R}_\pi)^H \xrightarrow{d^3} \cdots$$

is canonically isometrically isomorphic to $H^\bullet_{cb}(H, \mathbb{R}_\pi)$.

As a Corollary to Proposition II.2.30 we get the following strong resolution:

Corollary II.3.11. *Let $H = \Gamma$, G^+ or G, and let $(\pi, \mathbb{R})$ be a Banach H-module. Then the complex*

$$0 \longrightarrow \mathbb{R}_\pi \xrightarrow{\epsilon} \mathcal{B}^\infty(\partial\mathbb{H}^n, \mathbb{R}_\pi) \xrightarrow{d^1} \mathcal{B}^\infty((\partial\mathbb{H}^n)^2, \mathbb{R}_\pi) \xrightarrow{d^2} \mathcal{B}^\infty((\partial\mathbb{H}^n)^3, \mathbb{R}_\pi) \xrightarrow{d^3} \cdots$$

is a strong augmented resolution of $\mathbb{R}_\pi$ by (not necessarily relatively injective) Banach H-modules.

Proof. We have only to check, that $\partial\mathbb{H}^n$ is a measurable space with a measurable G-action. This is immediate for $\partial\mathbb{H}^n$ with its Borel σ-algebra as the G-action is not only measurable but even continuous. □

Finally we want to make the previously mentioned connection to singular bounded cohomology more concrete in the setting of hyperbolic geometry. The key observation here is, that if Γ is torsion-free, it acts freely and properly discontinuously on $\mathbb{H}^n$ (cf. section I.4). Therefore taking the quotient by this action induces the universal covering $p : \mathbb{H}^n \to \Gamma\backslash\mathbb{H}^n =: M$, such that M is an (oriented) hyperbolic manifold (cf. Proposition I.4.2). Then the group of Deck transformations of p is just Γ. Because $\mathbb{H}^n$ is a smooth manifold and thus in particular a countable CW-complex we can apply Proposition II.2.31 and Lemma II.2.32, which yields the following result.

Corollary II.3.12. *The complex*

$$0 \longrightarrow \mathbb{R} \xrightarrow{\epsilon} S^0_b(\mathbb{H}^n, \mathbb{R}) \xrightarrow{\delta^1} S^1_b(\mathbb{H}^n, \mathbb{R}) \xrightarrow{\delta^2} S^2_b(\mathbb{H}^n, \mathbb{R}) \xrightarrow{\delta^3} \cdots$$

is a strong augmented resolution of $\mathbb{R}$ (as the trivial Γ-module) by relatively injective Banach Γ-modules.

Moreover the cohomology of the complex

$$0 \longrightarrow S^0_b(\mathbb{H}^n, \mathbb{R})^\Gamma \xrightarrow{\delta^1} S^1_b(\mathbb{H}^n, \mathbb{R})^\Gamma \xrightarrow{\delta^2} S^2_b(\mathbb{H}^n, \mathbb{R})^\Gamma \xrightarrow{\delta^3} \cdots$$

is canonically isometrically isomorphic to $H^\bullet_{cb}(\Gamma, \mathbb{R})$. Further the cohomology is isometrically isomorphic to the bounded singular cohomology $H^\bullet_b(M)$ of $M = \Gamma\backslash\mathbb{H}^n$ via the pullback along $p : \mathbb{H}^n \to \Gamma\backslash\mathbb{H}^n$.

Some Concrete Isomorphisms

We now want to establish some isomorphisms between the above cohomologies of complexes, on which the continuous bounded cohomology can be computed. We will adopt the same notation as before, i.e. let $H = \Gamma$, G^+ or G and let $(\pi, \mathbb{R})$ be a Banach H-module. As for continuous cohomology and the van Est isomorphism the idea here is to find an H-morphisms between the resolutions extending the identity morphism $\mathrm{id} : \mathcal{C}\mathbb{R}_\pi \to \mathcal{C}\mathbb{R}_\pi$ (cf. Lemma II.2.15).

We will use as a "connecting resolution" $(\epsilon, L^\infty(G^{\bullet+1}, \mathbb{R}_\pi))$, that is, we will give an H-morphism extending the identity from every resolution to this particular one. In analogy to continuous cohomology these maps will be given in terms of pullbacks along suitable quotient maps.

First, we consider the resolution $(\epsilon, L^\infty((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\pi))$ and the quotient map $p_K : G \to G/K \cong \mathbb{H}^n, g \mapsto gx$ where x is some arbitrary point in $\mathbb{H}^n$ and K is its stabilizer. We define a morphism

$$p_K^* : L^\infty((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\pi) \to L^\infty(G^{\bullet+1}, \mathbb{R}_\pi)$$

via precomposition by

$$(p_K^* f)(g_0, \ldots, g_q) := f(p_K(g_0), \ldots, p_K(g_q)) = f(g_0 x, \ldots, g_q x)$$

for every $f \in L^\infty((\mathbb{H}^n)^{q+1}, \mathbb{R}_\pi)$, $g_0, \ldots, g_q \in G$, $q \in \mathbb{N}_0$.

Second, we consider the resolution $(\epsilon, L^\infty((\partial\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\pi))$ and the quotient map $p_P^* : G \to G/P \cong \partial\mathbb{H}^n, g \mapsto g\xi$ where ξ is some arbitrary point in $\partial\mathbb{H}^n$ and P is its stabilizer. We define a morphism

$$p_P^* : L^\infty((\partial\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\pi) \to L^\infty(G^{\bullet+1}, \mathbb{R}_\pi)$$

via precomposition by

$$(p_P^* f)(g_0, \ldots, g_q) := f(p_P(g_0), \ldots, p_P(g_q)) = f(g_0 \xi, \ldots, g_q \xi)$$

for every $f \in L^\infty((\partial\mathbb{H}^n)^{q+1}, \mathbb{R}_\pi)$, $g_0, \ldots, g_q \in G$, $q \in \mathbb{N}_0$.

Now the following holds.

Proposition II.3.13. *Let $H = \Gamma$, G^+ or G. With the above notation we get:*

(i) The inclusion $\iota : C_b(G^{\bullet+1}, \mathbb{R}_\pi) \to L^\infty(G^{\bullet+1}, \mathbb{R}_\pi)$ is an H-morphism of complexes extending the identity $\mathrm{id} : \mathcal{C}\mathbb{R}_\pi \to \mathcal{C}\mathbb{R}_\pi$ and induces in particular an isometric isomorphism in cohomology

$$\iota : H^\bullet(C_b(G^{\bullet+1}, \mathbb{R}_\pi)^H) \to H^\bullet(L^\infty(G^{\bullet+1}, \mathbb{R}_\pi)^H)$$

(ii) The inclusion $\iota' : C_b((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\pi) \to L^\infty((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\pi)$ is an H-morphism of complexes extending the identity $\mathrm{id} : \mathcal{C}\mathbb{R}_\pi \to \mathcal{C}\mathbb{R}_\pi$ and induces in particular an isometric isomorphism in cohomology

$$\iota' : H^\bullet(C_b((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\pi)^H) \to H^\bullet(L^\infty((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\pi)^H)$$

(iii) The map

$$p_K^* : L^\infty((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\pi) \to L^\infty(G^{\bullet+1}, \mathbb{R}_\pi)$$

is an H-morphism of complexes extending the identity $\mathrm{id} : \mathcal{C}\mathbb{R}_\pi \to \mathcal{C}\mathbb{R}_\pi$ and induces in particular an isometric isomorphism in cohomology

$$p_K^* : H^\bullet(L^\infty((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\pi)^H) \to H^\bullet(L^\infty(G^{\bullet+1}, \mathbb{R}_\pi)^H)$$

Furthermore it restricts to

$$p_K^* : C_b((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\pi) \to C_b(G^{\bullet+1}, \mathbb{R}_\pi)$$

which is also an extension of the identity and hence induces an isometric isomorphism in cohomology as well

$$p_K^* : H^\bullet(C_b((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\pi)^H) \to H^\bullet(C_b(G^{\bullet+1}, \mathbb{R}_\pi)^H)$$

(iv) The map

$$p_P^* : L^\infty((\partial\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\pi) \to L^\infty(G^{\bullet+1}, \mathbb{R}_\pi)$$

is an H-morphism of complexes extending the identity $\mathrm{id} : \mathcal{C}\mathbb{R}_\pi \to \mathcal{C}\mathbb{R}_\pi$ *and induces in particular an isometric isomorphism in cohomology*

$$p_P^* : H^\bullet(L^\infty((\partial\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\pi)^H) \to H^\bullet(L^\infty(G^{\bullet+1}, \mathbb{R}_\pi)^H)$$

Proof. By Lemma II.2.22 it will suffice to show that the above maps are extensions of the identity $\mathrm{id} : \mathcal{C}\mathbb{R}_\pi \to \mathcal{C}\mathbb{R}_\pi$.

(i) and (ii) are trivial to verify. (iii) and (iv) can be proven completely analogously to the corresponding Proposition II.3.5 for continuous cohomology. That p_K^* restricts to

$$p_K^* : C_b((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\pi) \to C_b(G^{\bullet+1}, \mathbb{R}_\pi)$$

is also immediate by direct inspection and in view of Proposition II.3.5. □

II.3.3. The Volume Class

We now turn to the cohomology class, that will be the most important to us in the following, the volume class. It will play a central role in the definition of the volume of a representation. Our objective in this subsection is to introduce the volume class and to recognize different cocycles in the different complexes representing it.

Before we can define it, we have to set the stage and introduce a non-trivial (Banach) G-module structure on $\mathbb{R}$. Consider the quotient map $\varepsilon : G = \mathrm{Isom}(\mathbb{H}^n) \to G/G^+$. Since G^+ is a subgroup of index two, we may identify $G/G^+ \cong \{\pm 1\}$ such that

$$\varepsilon(g) = \begin{cases} +1, & \text{if } g \text{ is orientation preserving} \\ -1, & \text{if } g \text{ is not orientation preserving} \end{cases}$$

for every $g \in G$. Clearly $\varepsilon : G \to \{\pm 1\} \cong \mathrm{Iso}(\mathbb{R})$ is a homomorphism, such that $(\varepsilon, \mathbb{R})$ (or $\mathbb{R}_\varepsilon$ for short) becomes a Banach G-module. This representation is clearly (jointly) continuous and hence $\mathbb{R}_\varepsilon$ is even a continuous Banach G-module. Therefore it is also a G-module in the sense of continuous cohomology (cf. Remark B.2.9). In the following we will frequently pull back this structure by continuous homomorphisms $H \to G^+ < G$ with image in the orientation preserving isometries. It is clear that the resulting pullback structure on $\mathbb{R}$ is the trivial one.

Let us first have a look at the continuous cohomology $H^\bullet(G, \mathbb{R}_\varepsilon)$. By Corollary II.3.2 we have $H^\bullet(G, \mathbb{R}_\varepsilon) \cong H^\bullet(\Omega^\bullet(\mathbb{H}^n, \mathbb{R}_\varepsilon)^G)$. It turns out that the hyperbolic volume form $\omega_n \in \Omega^n(\mathbb{H}^n, \mathbb{R}_\varepsilon)$ is G-equivariant. Indeed, the orientation on $\mathbb{H}^n$ is given by the volume form and an isometry $g \in G$ is by definition orientation preserving resp. reversing, if and only if $g^*\omega_n = \omega_n$ resp. $g^*\omega_n = -\omega_n$. That is $g^*\omega_n = \varepsilon(g) \cdot \omega_n$ or equivalently

$$g \cdot \omega_n = \varepsilon(g) \cdot (g^{-1})^*\omega_n = \varepsilon(g) \cdot \varepsilon(g^{-1}) \cdot \omega_n = \omega_n$$

for every $g \in G$. As ω_n is in top-degree, it is a cocycle for trivial reasons and hence defines a class in cohomology.

We may now apply the van Est isomorphism $\Phi : \Omega^n(\mathbb{H}^n, \mathbb{R}_\varepsilon)^G \to C((\mathbb{H}^n)^{n+1}, \mathbb{R}_\varepsilon)^G$ and get a continuous function

$$\mathrm{Vol}_n(x_0, \ldots, x_n) := \Phi(\omega_n)(x_0, \ldots, x_n) = \int_{\mathrm{str}(x_0,\ldots,x_n)} \omega_n, \qquad \forall (x_0, \ldots, x_n) \in \mathbb{H}^n$$

which yields another representative of the same cohomology class in the complex $C((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\varepsilon)^G$. By abuse of notation we refer to both cocycles Vol_n and ω_n as *the volume cocycle* and to the corresponding cohomology classes $[\omega_n]$ and $[\mathrm{Vol}_n]$ as *the volume class*, identifying $H^n(C((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\varepsilon)^G) \cong H^n(\Omega^\bullet(\mathbb{H}^n, \mathbb{R}_\varepsilon)^G) \cong H^n_c(G, \mathbb{R}_\varepsilon)$. If there is no ambiguity we shall also denote by ω_n the volume class.

We will now see that Vol_n may be interpreted as the signed volume of the convex hull of the points $x_0, \ldots, x_n \in \mathbb{H}^n$.

Lemma II.3.14. *Let $v_0, \ldots, v_n \in D^n \cong \mathbb{H}^n$. Then*

$$\mathrm{Vol}_n(v_0, \ldots, v_n) = \mathrm{sgn}(D(v_0, \ldots, v_n)) \cdot \mathrm{vol}(\mathrm{conv}(v_0, \ldots, v_n))$$

where

$$D(v_0, \ldots, v_n) = \det \begin{pmatrix} | & & | \\ v_1 - v_0 & \cdots & v_n - v_0 \\ | & & | \end{pmatrix}$$

is the determinant of the matrix with column vectors $v_1 - v_0, \ldots, v_n - v_0$ and

$$\mathrm{sgn}(t) = \begin{cases} +1 & , t > 0 \\ 0 & , t = 0 \\ -1 & , t < 0 \end{cases}$$

for every $t \in \mathbb{R}$. In particular

$$|\mathrm{Vol}_n(v_0, \ldots, v_n)| = \mathrm{vol}(\mathrm{conv}(v_0, \ldots, v_n))$$

and Vol_n is alternating, i.e. for any permutation $\sigma \in \mathfrak{S}_{n+1}$ we have that

$$\mathrm{Vol}_n(v_{\sigma(0)}, \ldots, v_{\sigma(n)}) = \mathrm{sgn}(\sigma) \cdot \mathrm{Vol}_n(v_0, \ldots, v_n)$$

Remark II.3.15. *In order to prove the lemma we want to introduce a different notion of a straight simplex in the projective disk model D^n. For any $v_0, \ldots, v_n \in D^n \cong \mathbb{H}^n$ define the projective straight simplex $\mathrm{str}'(v_0, \ldots, v_n) : \Delta^n \to D^n$ via*

$$\mathrm{str}'(v_0, \ldots, v_n)(t_1, \ldots, t_n) := \left(1 - \sum_{i=1}^n t_i\right) \cdot v_0 + t_1 \cdot v_1 + \cdots + t_n \cdot v_n$$

for every $(t_1, \ldots, t_n) \in \Delta^n$. Recall that in the projective disk model geodesics are just straight lines and a set is convex with respect to the hyperbolic metric if and only if it is convex in the standard euclidean sense. Therefore the image of $\mathrm{str}'(v_0, \ldots, v_n)$ is the convex hull $\mathrm{conv}(v_0, \ldots, v_n)$. It is worth noting, that when we compare the projective disk model with the hyperboloid model for hyperbolic n-space $\mathbb{H}^n$ by applying the gnomonic projection the two notions of straight simplices do not coincide. However this is almost the case, since they have the same image and are just different parametrizations which also induce the same orientation. Because when integrating a differential form over a straight simplex only the orientation and not the concrete parametrization matters, we may replace our original definition of a straight simplex with the new one. We will do that in the following without any further comment.

Proof of Lemma II.3.14. Let $v_0, \ldots, v_n \in D^n \cong \mathbb{H}^n$ and consider $\mathrm{str}(v_0, \ldots, v_n) : \Delta^n \to D^n$. Then the Jacobian matrix of $\mathrm{str}(v_0, \ldots, v_n)$ is given by

$$J_{\mathrm{str}(v_0, \ldots, v_n)}(t_1, \ldots, t_n) = \begin{pmatrix} (v_1 - v_0)^T \\ \vdots \\ (v_n - v_0)^T \end{pmatrix}$$

where the rows are just the (transposed) vectors $v_1 - v_0, \ldots, v_n - v_0$. Note that therefore

$$D(v_0, \ldots, v_n) = \det\left(J_{\mathrm{str}(v_0,\ldots,v_n)}(t_1, \ldots, t_n)\right)$$

for every $(t_1, \ldots, t_n) \in \Delta^n$. Further recall, that the volume form in the projective disk model is given by

$$\omega_n = f \cdot dx_1 \wedge \ldots \wedge dx_n$$

where

$$f(x) = \frac{1}{(1 - |x|^2)^{(n+1)/2}}$$

for every $x \in D^n \subset \mathbb{R}^n$ (cf. [Rat06]).

Now we compute

$$\begin{aligned}
\mathrm{Vol}_n(v_0, \ldots, v_n) &= \int_{\mathrm{str}(v_0,\ldots,v_n)} \omega_n \\
&= \int_{\Delta^n} \mathrm{str}(v_0, \ldots, v_n)^* \omega_n \\
&= \int_{\Delta^n} f \circ \mathrm{str}(v_0, \ldots, v_n) \cdot \mathrm{str}(v_0, \ldots, v_n)^*(dx_1 \wedge \ldots \wedge dx_n) \\
&= \int_{\Delta^n} f \circ \mathrm{str}(v_0, \ldots, v_n) \cdot \det\left(J_{\mathrm{str}(v_0,\ldots,v_n)}\right) \cdot dt_1 \wedge \ldots \wedge dt_n \\
&= \mathrm{sgn}(D(v_0, \ldots, v_n)) \cdot \int_{\Delta^n} f(\mathrm{str}(v_0, \ldots, v_n)(t_1, \ldots, t_n)) |D(v_0, \ldots, v_n)| \, dt_1 \ldots dt_n \\
&= \mathrm{sgn}(D(v_0, \ldots, v_n)) \cdot \int_{\mathrm{conv}(v_0,\ldots,v_n)} f(x_1, \ldots, x_n) \, dx_1 \ldots dx_n \\
&= \mathrm{sgn}(D(v_0, \ldots, v_n)) \cdot \mathrm{vol}(\mathrm{conv}(v_0, \ldots, v_n))
\end{aligned}$$

where we have used the transformation rule for Lebesgue integration in the second last line and the fact, that we have for the hyperbolic measure μ in the projective disk model

$$\frac{d\mu}{d\lambda} = f$$

where λ denotes the Lebesgue measure on $D^n \subset \mathbb{R}^n$, in the last line. □

Recall that by Theorem I.7.4 the volume of a n-simplex in $\mathbb{H}^n$, that is the convex hull of $n+1$ points in $\mathbb{H}^n$, is bounded by the volume of a regular ideal n-simplex. Therefore the above lemma shows, that Vol_n is in fact a bounded continuous function in $C_b((\mathbb{H}^n)^{n+1}, \mathbb{R}_\varepsilon)^G \subset C((\mathbb{H}^n)^{n+1}, \mathbb{R}_\varepsilon)^G$. Hence Vol_n represents a class in bounded cohomology, which we will also call *the volume class* and denote it by $\omega_n^b \in H_{cb}^n(G, \mathbb{R}_\varepsilon)$. Corollary II.2.45 now readily implies, that the comparison map sends ω_n^b to ω_n, i.e. $c(\omega_n^b) = \omega_n$.

Our next objective is to show, that we can think of Vol_n as a cocycle in $L^\infty((\partial\mathbb{H}^n)^{n+1}, \mathbb{R}_\varepsilon)^G$ too. Since both $\mathrm{sgn}(D(v_0, \ldots, v_n))$ and $\mathrm{vol}(\mathrm{conv}(v_0, \ldots, v_n))$ are also defined on $(\overline{D}^n)^{n+1} \cong (\overline{\mathbb{H}}^n)^{n+1}$, we can use the previous lemma to enlarge the domain of $\mathrm{Vol}_n(\cdot)$ to $\overline{\mathbb{H}}^n$ by setting

$$\mathrm{Vol}_n(v_0, \ldots, v_n) := \mathrm{sgn}(D(v_0, \ldots, v_n)) \cdot \mathrm{vol}(\mathrm{conv}(v_0, \ldots, v_n))$$

for every $(v_0, \ldots, v_n) \in (\overline{D}^n)^{n+1} \cong (\overline{\mathbb{H}}^n)^{n+1}$.

We will now use Theorem I.7.7 and Theorem I.7.8 to deduce some regulartiy properties of $\mathrm{Vol}_n : (\overline{\mathbb{H}}^n)^{n+1} \to \mathbb{R}$.

Proposition II.3.16. *We have*

(i) $\mathrm{Vol}_2 : (\mathbb{H}^2)^3 \cup (\overline{\mathbb{H}}^2)^{[3]} \to \mathbb{R}$ *is continuous, where* $(\overline{\mathbb{H}}^2)^{[3]}$ *denotes the set of all triples of points which are not contained in a proper hyperbolic subspace (cf. Theorem I.7.8).*

(ii) Let $n \geq 3$*. Then the map* $\mathrm{Vol}_n : (\overline{\mathbb{H}}^n)^{n+1} \to \mathbb{R}$ *is continuous.*

Proof. We will prove (ii) first. Let $n \geq 3$, let $(v_0, \dots, v_n) \in (\overline{D}^n)^{n+1} \cong (\overline{\mathbb{H}}^n)^{n+1}$ and let $\{(v_{0j}, \dots, v_{nj})\}_{j \in \mathbb{N}} \subset (\overline{D}^n)^{n+1}$ be a sequence converging to it. Consider the open set

$$\begin{aligned}(\overline{D}^n)^{[n+1]} &:= \{(v_0, \dots, v_n) \in (\overline{D}^n)^{n+1} : v_0, \dots, v_n \text{ are not contained in a proper hyperbolic subspace } \} \\ &= \{(v_0, \dots, v_n) \in (\overline{D}^n)^{n+1} : D(v_0, \dots, v_n) \neq 0\}\end{aligned}$$

Assume first, that $(v_0, \dots, v_n) \in (\overline{D}^n)^{n+1} - (\overline{D}^n)^{[n+1]}$. Then $\mathrm{Vol}_n(v_0, \dots, v_n) = 0$ and

$$\begin{aligned}\lim_{j\to\infty} |\mathrm{Vol}_n(v_{0j}, \dots, v_{nj}) - \mathrm{Vol}_n(v_0, \dots, v_n)| &= \lim_{j\to\infty} |\mathrm{Vol}_n(v_{0j}, \dots, v_{nj})| \\ &= \lim_{j\to\infty} \mathrm{vol}(\mathrm{conv}(v_{0j}, \dots, v_{nj})) = 0\end{aligned}$$

by Theorem I.7.7.

Now if $(v_0, \dots, v_n) \in (\overline{D}^n)^{[n+1]}$, then by continuity of $D(\cdot)$

$$\lim_{j\to\infty} D(v_{0j}, \dots, v_{nj}) = D(v_0, \dots, v_n) \neq 0$$

such that for j large enough $\mathrm{sgn}(D(v_{0j}, \dots, v_{nj})) = \mathrm{sgn}(D(v_0, \dots, v_n))$. Therefore

$$\begin{aligned}\lim_{j\to\infty} \mathrm{Vol}_n(v_{0j}, \dots, v_{nj}) &= \mathrm{sgn}(D(v_0, \dots, v_n)) \cdot \lim_{j\to\infty} \mathrm{vol}(\mathrm{conv}(v_{0j}, \dots, v_{nj})) \\ &= \mathrm{sgn}(D(v_0, \dots, v_n)) \cdot \mathrm{vol}(\mathrm{conv}(v_0, \dots, v_n)) = \mathrm{Vol}_n(v_0, \dots, v_n)\end{aligned}$$

by Theorem I.7.7.

The proof of (i) is essentially the same. One just has to take into account the discontinuities of $\mathrm{vol}(\mathrm{conv}(\cdot))$ in dimension 2. Let $(v_0, v_1, v_2) \in (D^2)^3 \cup (\overline{D}^2)^{[3]} \cong (\mathbb{H}^2)^3 \cup (\overline{\mathbb{H}}^2)^{[3]}$ and let $\{(v_{0j}, v_{1j}, v_{2j})\}_{j\in\mathbb{N}} \subset (D^2)^3 \cup (\overline{D}^2)^{[3]}$ be a sequence converging to it.

If $(v_0, v_1, v_2) \in (D^2)^3$ then also $(v_{0j}, v_{1j}, v_{2j}) \in (D^2)^3$ for j large enough. If $D(v_0, v_1, v_2) = 0$ then also $\mathrm{Vol}_2(v_0, v_1, v_2) = 0$ and

$$\begin{aligned}\lim_{j\to\infty} |\mathrm{Vol}_2(v_{0j}, v_{1j}, v_{2j}) - \mathrm{Vol}_2(v_0, v_1, v_2)| &= \lim_{j\to\infty} |\mathrm{Vol}_2(v_{0j}, v_{1j}, v_{2j})| \\ &= \lim_{j\to\infty} \mathrm{vol}(\mathrm{conv}(v_{0j}, v_{1j}, v_{2j})) = 0\end{aligned}$$

by Theorem I.7.8. If $D(v_0, v_1, v_2) \neq 0$ then as $D(v_{0j}, v_{1j}, v_{2j}) \to D(v_0, v_1, v_2)$ we have $\mathrm{sgn}(D(v_{0j}, v_{1j}, v_{2j})) = \mathrm{sgn}(D(v_0, v_1, v_2))$ for j large enough. Thus

$$\begin{aligned}\lim_{j\to\infty} \mathrm{Vol}_2(v_{0j}, v_{1j}, v_{2j}) &= \mathrm{sgn}(D(v_0, v_1, v_n)) \cdot \lim_{j\to\infty} \mathrm{vol}(\mathrm{conv}(v_{0j}, v_{1j}, v_{2j})) \\ &= \mathrm{sgn}(D(v_0, v_1, v_2)) \cdot \mathrm{vol}(\mathrm{conv}(v_0, v_1, v_2)) = \mathrm{Vol}_2(v_0, v_1, v_2)\end{aligned}$$

by Theorem I.7.8.

If $(v_0, v_1, v_2) \in (\overline{D}^2)^{[3]}$, then $D(v_0, v_1, v_2) \neq 0$. Since $D(v_{0j}, v_{1j}, v_{2j}) \to D(v_0, v_1, v_2)$, we have for j large enough $\mathrm{sgn}(D(v_{0j}, v_{1j}, v_{2j})) = \mathrm{sgn}(D(v_0, v_1, v_2))$. Thus again by Theorem I.7.8

$$\begin{aligned}\lim_{j\to\infty} \mathrm{Vol}_2(v_{0j}, v_{1j}, v_{2j}) &= \mathrm{sgn}(D(v_0, v_1, v_n)) \cdot \lim_{j\to\infty} \mathrm{vol}(\mathrm{conv}(v_{0j}, v_{1j}, v_{2j})) \\ &= \mathrm{sgn}(D(v_0, v_1, v_2)) \cdot \mathrm{vol}(\mathrm{conv}(v_0, v_1, v_2)) = \mathrm{Vol}_2(v_0, v_1, v_2)\end{aligned}$$

□

Next we need to see that Vol_n is G-equivariant.

Proposition II.3.17. $\mathrm{Vol}_n : (\overline{\mathbb{H}}^n)^{n+1} \to \mathbb{R}_\varepsilon$ *is G-equivariant, i.e.*

$$\varepsilon(g) \cdot \mathrm{Vol}_n(g^{-1}x_0, \dots, g^{-1}x_n) = \mathrm{Vol}_n(x_0, \dots, x_n)$$

for every $g \in G$ and $(x_0, \dots, x_n) \in (\overline{\mathbb{H}}^n)^{n+1}$.

Remark II.3.18. *Note that $n \geq 2$ in the whole of this section if not otherwise stated.*

Proof. Assume first that $n \geq 3$.

Let $g \in G$, $(x_0, \dots, x_n) \in (\overline{\mathbb{H}}^n)^{n+1}$ and $\{(x_0^{(j)}, \dots, x_n^{(j)})\}_{j\in\mathbb{N}} \subset (\mathbb{H}^n)^{n+1}$ a sequence converging to it. Then $\{(g^{-1}x_0^{(j)}, \dots, g^{-1}x_n^{(j)})\}_{j\in\mathbb{N}} \subset (\mathbb{H}^n)^{n+1}$ converges to $(g^{-1}x_0, \dots, g^{-1}x_n)$ and we get by continuity of $\mathrm{Vol}_n : (\overline{\mathbb{H}}^n)^{n+1} \to \mathbb{R}$

$$\begin{aligned}
\varepsilon(g) \cdot \mathrm{Vol}_n(g^{-1}x_0, \dots, g^{-1}x_n) &= \lim_{j\to\infty} \varepsilon(g) \cdot \mathrm{Vol}_n(g^{-1}x_0^{(j)}, \dots, g^{-1}x_n^{(j)}) \\
&= \varepsilon(g) \cdot \mathrm{Vol}_n(g^{-1}x_0, \dots, g^{-1}x_n) \\
&= \mathrm{Vol}_n(x_0, \dots, x_n)
\end{aligned}$$

Now consider the case $n = 2$. Again we want to use the continuity of Vol_n. Let $g \in G$ and let $(x_0, x_1, x_2) \in (\overline{D}^2)^3 - (\overline{D}^2)^{[3]}$. Then x_0, x_1, x_2 are contained in a proper hyperbolic subspace and so are $g^{-1}x_0, g^{-1}x_1, g^{-1}x_2$. Therefore

$$\varepsilon(g) \cdot \mathrm{Vol}_2(g^{-1}x_0, g^{-1}x_1, g^{-1}x_2) = 0 = \mathrm{Vol}_2(x_0, x_1, x_2)$$

Let $(x_0, x_1, x_2) \in (\overline{D}^2)^{[3]}$ and let $\{(x_0^{(j)}, x_1^{(j)}, x_2^{(j)})\}_{j\in\mathbb{N}} \subset (D^2)^3$ be a sequence converging to it. Then as before by continuity

$$\begin{aligned}
\varepsilon(g) \cdot \mathrm{Vol}_2(g^{-1}x_0, g^{-1}x_1, g^{-1}x_2) &= \lim_{j\to\infty} \varepsilon(g) \cdot \mathrm{Vol}_2(g^{-1}x_0^{(j)}, g^{-1}x_1^{(j)}, g^{-1}x_2^{(j)}) \\
&= \lim_{j\to\infty} \varepsilon(g) \cdot \mathrm{Vol}_2(g^{-1}x_0^{(j)}, g^{-1}x_1^{(j)}, g^{-1}x_2^{(j)}) \\
&= \mathrm{Vol}_2(x_0, x_1, x_2)
\end{aligned}$$

□

It is now immediate that restricting Vol_n to $(\partial\mathbb{H}^n)^{n+1}$ yields a G-equivariant bounded measurable function, i.e. $\mathrm{Vol}_n \in \mathcal{B}^\infty((\partial\mathbb{H}^n)^{n+1}, \mathbb{R}_\varepsilon)^G \subset L^\infty((\partial\mathbb{H}^n)^{n+1}, \mathbb{R}_\varepsilon)^G$. Indeed for $n \geq 3$ it is continuous and for $n = 2$ it is continuous on the full measure subset $(\partial\mathbb{H}^2)^{[3]} = (\partial\mathbb{H}^2)^{(3)}$ of triples of distinct boundary points. Note that Vol_2 is actually locally constant on $(\partial\mathbb{H}^2)^{(3)}$, since $\mathrm{Isom}(\mathbb{H}^2)$ acts 3-transitively on its boundary, such that any two ideal triangles are congruent.

The next proposition asserts, that it is in fact a cocycle and hence represents a cohomology class in $H^n(L^\infty((\partial\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\varepsilon)^G) \cong H^n_{cb}(G, \mathbb{R}_\varepsilon)$.

Proposition II.3.19. $\mathrm{Vol}_n \in \mathcal{B}^\infty((\partial\mathbb{H}^n)^{n+1}, \mathbb{R}_\varepsilon)^G \subset L^\infty((\partial\mathbb{H}^n)^{n+1}, \mathbb{R}_\varepsilon)^G$ *is a cocycle, i.e.*

$$d\mathrm{Vol}_n \equiv 0$$

Proof. Consider $(\xi_0, \dots, \xi_{n+1}) \in (\partial\mathbb{H}^n)^{n+2} - (\partial\mathbb{H}^n)^{(n+2)}$. Then $\xi_j = \xi_k$ for some $j \neq k$. Because

$$(d\mathrm{Vol}_n)(\xi_0, \dots, \xi_{n+1}) = \sum_{i=0}^{n+1} \mathrm{Vol}_n(\xi_0, \dots, \hat{\xi}_i, \dots, \xi_{n+1})$$

and Vol_n is alternating we may assume without loss of generality, that $\xi_0 = \xi_1$. Thus

$$\sum_{i=0}^{n+1} \mathrm{Vol}_n(\xi_0, \ldots, \hat{\xi}_i, \ldots, \xi_{n+1}) = \mathrm{Vol}_n(\xi_1, \xi_2, \ldots, \xi_{n+1}) - \mathrm{Vol}_n(\xi_0, \xi_2, \ldots, \xi_{n+1}) = 0$$

For $(\xi_0, \ldots, \xi_{n+1}) \in (\partial\mathbb{H}^n)^{(n+2)}$ let $\{(x_0^{(j)}, \ldots, x_{n+1}^{(j)})\}_{j\in\mathbb{N}} \subset (\mathbb{H}^n)^{n+2}$ be a sequence converging to it. Then

$$\begin{aligned}
(d\mathrm{Vol}_n)(\xi_0, \ldots, \xi_{n+1}) &= \sum_{i=0}^{n+1} (-1)^i \mathrm{Vol}_n(\xi_0, \ldots, \hat{\xi}_i, \ldots, \xi_{n+1}) \\
&= \sum_{i=0}^{n+1} (-1)^i \lim_{j\to\infty} \mathrm{Vol}_n(x_0^{(j)}, \ldots, \hat{x}_i^{(j)}, \ldots, x_{n+1}^{(j)}) \\
&= \lim_{j\to\infty} \sum_{i=0}^{n+1} (-1)^i \mathrm{Vol}_n(x_0^{(j)}, \ldots, \hat{x}_i^{(j)}, \ldots, x_{n+1}^{(j)}) \\
&= \lim_{j\to\infty} (d\mathrm{Vol}_n)(x_0^{(j)}, \ldots, x_{n+1}^{(j)}) = 0
\end{aligned}$$

and the assertion is proven. □

We have now encountered several different cocycles all coming from $\mathrm{Vol}_n : \left(\overline{\mathbb{H}}^n\right)^{n+1} \to \mathbb{R}$:

- $V_1 = [\mathrm{Vol}_n] \in H^n_{cb}(G, \mathbb{R}_\varepsilon)$ where $\mathrm{Vol}_n \in C_b((\mathbb{H}^n)^{n+1}, \mathbb{R}_\varepsilon)^G$
- $V_2 = [\mathrm{Vol}_n] \in H^n_{cb}(G, \mathbb{R}_\varepsilon)$ where $\mathrm{Vol}_n \in L^\infty((\mathbb{H}^n)^{n+1}, \mathbb{R}_\varepsilon)^G$
- $V_3 = [\mathrm{Vol}_n] \in H^n_{cb}(G, \mathbb{R}_\varepsilon)$ where $\mathrm{Vol}_n \in L^\infty((\partial\mathbb{H}^n)^{n+1}, \mathbb{R}_\varepsilon)^G$

Although they all come from restrictions of the same function, it is apriori not clear, that they all represent the same cohomology class in $H^n_{cb}(G, \mathbb{R}_\varepsilon)$. We will now see, that they in fact do. It is easy to see that $V_1 = V_2$ as the isomorphism $H^\bullet(C_b((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\varepsilon)^G) \to H^\bullet(L^\infty((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\varepsilon)^G)$ is induced by the inclusion $\iota : C_b((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\varepsilon)^G \to L^\infty((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\varepsilon)^G$ at the cochain level (cf. subsection II.3.2).

The following proposition asserts, that also $V_2 = V_3$. By further abuse of notation we will therefore always speak of *the volume class* ω_n^b and mean the cohomology class represented by one of the above cocycles. If the corresponding complex is understood we shall also speak of *the volume cocycle.*

Proposition II.3.20. *Let $x \in \mathbb{H}^n$ and $\xi \in \partial\mathbb{H}^n$ and denote by $K = G_x$ the stabilizer of x and by $P = G_\xi$ the stabilizer of ξ. Further let*

$$p_K^* : H^\bullet(L^\infty((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\varepsilon)^G) \to H^\bullet(L^\infty(G^{\bullet+1}, \mathbb{R}_\varepsilon)^G)$$

and

$$p_P^* : H^\bullet(L^\infty((\partial\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\varepsilon)^G) \to H^\bullet(L^\infty(G^{\bullet+1}, \mathbb{R}_\varepsilon)^G)$$

be the corresponding isomorphisms from subsection II.3.2. Then

$$p_K^*([\mathrm{Vol}_n]) = p_P^*([\mathrm{Vol}_n])$$

and therefore $V_2 = V_3$ with the above notation.

In particular the volume class in $H^n(L^\infty(G^{\bullet+1}, \mathbb{R}_\varepsilon)^G)$ is represented by the cocycle

$$\begin{aligned} V_y : G^{n+1} &\to \mathbb{R} \\ (g_0, \dots, g_n) &\mapsto \mathrm{Vol}_n(g_0 y, \dots, g_n y) \end{aligned}$$

for any basepoint $y \in \overline{\mathbb{H}}^n$.

Proof. Let $x \in \mathbb{H}^n$ and $\xi \in \partial\mathbb{H}^n$. Recall that the above isomorphisms are given at the cochain level by precomposition with the continuous maps

$$p_K : G \to G/K \cong \mathbb{H}^n, g \mapsto gx$$

and

$$p_P : G \to G/P \cong \partial\mathbb{H}^n, g \mapsto g\xi$$

Thus $p_K^*([\mathrm{Vol}_n])$ is represented by

$$(g_0, \dots, g_n) \mapsto \mathrm{Vol}_n(g_0 x, \dots, g_n x)$$

and $p_P^*([\mathrm{Vol}_n])$ is represented by

$$(g_0, \dots, g_n) \mapsto \mathrm{Vol}_n(g_0 \xi, \dots, g_n \xi)$$

Consider the function $f : G^n \to \mathbb{R}$ given by

$$f(g_0, \dots, g_{n-1}) := \sum_{i=0}^{n-1} (-1)^i \mathrm{Vol}_n(g_0\xi, \dots, g_i\xi, g_i x, \dots, g_{n-1}x)$$

for every $(g_0, \dots, g_{n-1}) \in G^n$. Clearly $f \in L^\infty(G^n, \mathbb{R}_\varepsilon)^G$. We claim that

$$\mathrm{Vol}_n(g_0 x, \dots, g_n x) - \mathrm{Vol}_n(g_0\xi, \dots, g_n\xi) = (df)(g_0, \dots, g_n)$$

for almost every $(g_0, \dots, g_n) \in G^{n+1}$.

In an intermediate step let us first see that for every $x' \in \mathbb{H}^n$ the function $f' : G^n \to \mathbb{R}$ given by

$$f'(g_0, \dots, g_{n-1}) := \sum_{i=0}^{n-1} (-1)^i \mathrm{Vol}_n(g_0 x', \dots, g_i x', g_i x, \dots, g_{n-1}x) \tag{II.7}$$

satisfies

$$\mathrm{Vol}_n(g_0 x, \dots, g_n x) - \mathrm{Vol}_n(g_0 x', \dots, g_n x') = (df')(g_0, \dots, g_n) \tag{II.8}$$

for every $(g_0, \dots, g_n) \in G^{n+1}$. The claim will then follow by discussing the cases $n \geq 3$ and $n = 2$ separately, and using a by now familiar continuity argument. A coboundary construction similar to (II.7) will be important later.

Because $\mathrm{Vol}_n \in C((\mathbb{H}^n)^{n+1}, \mathbb{R}_\varepsilon)^G$ is a cocycle, we have that

$$0 = (d\mathrm{Vol}_n)(x_0, \dots, x_{n+1}) = \sum_{i=0}^{n+1} (-1)^i \mathrm{Vol}_n(x_0, \dots, \hat{x}_i, \dots, x_{n+1})$$

for all $x_0, \ldots, x_{n+1} \in \mathbb{H}^n$. Thus

$$\begin{aligned}
0 &= \sum_{i=0}^{n} (-1)^i (d\mathrm{Vol}_n)(g_0 x', \ldots, g_i x', g_i x, \ldots, g_n x) \\
&= \sum_{i=0}^{n} (-1)^i \Bigg\{ \sum_{j \le i-1} (-1)^j \mathrm{Vol}_n(g_0 x', \ldots, g_{j-1} x', g_{j+1} x', \ldots, g_i x', g_i x, \ldots, g_n x) \\
&\quad + (-1)^i \mathrm{Vol}_n(g_0 x', \ldots, g_{i-1} x', g_i x, \ldots, g_n x) \\
&\quad + (-1)^{i+1} \mathrm{Vol}_n(g_0 x', \ldots, g_i x', g_{i+1} x, \ldots, g_n x) \\
&\quad + \sum_{i+2 \le j} (-1)^j \mathrm{Vol}_n(g_0 x', \ldots, g_i x', g_i x, \ldots, g_{j-2} x, g_j x, \ldots, g_n x) \Bigg\} \\
&= \sum_{i=0}^{n} \Bigg\{ \mathrm{Vol}_n(g_0 x', \ldots, g_{i-1} x', g_i x, g_{i+1} x, \ldots, g_n x) \\
&\quad - \mathrm{Vol}_n(g_0 x', \ldots, g_i x', g_{i+1} x, \ldots, g_n x) \Bigg\} \\
&+ \sum_{i=0}^{n} (-1)^i \Bigg\{ \sum_{j \le i-1} (-1)^j \mathrm{Vol}_n(g_0 x', \ldots, g_{j-1} x', g_{j+1} x', \ldots, g_i x', g_i x, \ldots, g_n x) \\
&\quad + \sum_{i+2 \le j} (-1)^j \mathrm{Vol}_n(g_0 x', \ldots, g_i x', g_i x, \ldots, g_{j-2} x, g_j x, \ldots, g_n x) \Bigg\} \\
&= \mathrm{Vol}_n(g_0 x, \ldots, g_n x) - \mathrm{Vol}_n(g_0 x', \ldots, g_n x') \\
&+ \sum_{i=0}^{n} (-1)^i \Bigg\{ \sum_{j \le i-1} (-1)^j \mathrm{Vol}_n(g_0 x', \ldots, g_{j-1} x', g_{j+1} x', \ldots, g_i x', g_i x, \ldots, g_n x) \\
&\quad + \sum_{i+2 \le j} (-1)^j \mathrm{Vol}_n(g_0 x', \ldots, g_i x', g_i x, \ldots, g_{j-2} x, g_j x, \ldots, g_n x) \Bigg\}
\end{aligned}$$

for every $(g_0, \ldots, g_n) \in G^{n+1}$, i.e.

$$\begin{aligned}
&\mathrm{Vol}_n(g_0 x, \ldots, g_n x) - \mathrm{Vol}_n(g_0 x', \ldots, g_n x') \\
&= -\sum_{i=0}^{n} (-1)^i \Bigg\{ \sum_{j \le i-1} (-1)^j \mathrm{Vol}_n(g_0 x', \ldots, g_{j-1} x', g_{j+1} x', \ldots, g_i x', g_i x, \ldots, g_n x) \\
&\quad + \sum_{i+2 \le j} (-1)^j \mathrm{Vol}_n(g_0 x', \ldots, g_i x', g_i x, \ldots, g_{j-2} x, g_j x, \ldots, g_n x) \Bigg\} \\
&= \sum_{i=0}^{n} (-1)^i \Bigg\{ \sum_{0 \le j \le i-1} (-1)^{j+1} \mathrm{Vol}_n(g_0 x', \ldots, g_{j-1} x', g_{j+1} x', \ldots, g_i x', g_i x, \ldots, g_n x) \\
&\quad + \sum_{i+1 \le j \le n} (-1)^j \mathrm{Vol}_n(g_0 x', \ldots, g_i x', g_i x, \ldots, g_{j-1} x, g_{j+1} x, \ldots, g_n x) \Bigg\}
\end{aligned}$$

On the other hand

$$\begin{aligned}
(df')(g_0, \ldots, g_n) &= \sum_{j=0}^{n} (-1)^j f'(g_0, \ldots, g_{j-1}, g_{j+1}, \ldots, g_n) \\
&= \sum_{j=0}^{n} (-1)^j \Bigg\{ \sum_{0 \le i \le j-1} (-1)^i \mathrm{Vol}_n(g_0 x', \ldots, g_i x', g_i x, \ldots, g_{j-1} x, g_{j+1} x, \ldots, g_n x)
\end{aligned}$$

$$+ \sum_{j\le i\le n-1} (-1)^i \mathrm{Vol}_n(g_0x', \dots, g_{j-1}x', g_{j+1}x', \dots, g_{i+1}x', g_{i+1}x, \dots, g_nx) \Big\}$$
$$= \sum_{j=0}^{n} (-1)^j \Big\{ \sum_{0\le i\le j-1} (-1)^i \mathrm{Vol}_n(g_0x', \dots, g_ix', g_ix, \dots, g_{j-1}x, g_{j+1}x, \dots, g_nx)$$
$$+ \sum_{j+1\le i\le n} (-1)^{i+1} \mathrm{Vol}_n(g_0x', \dots, g_{j-1}x', g_{j+1}x', \dots, g_ix', g_ix, \dots, g_nx) \Big\}$$

for every $(g_0, \dots, g_n) \in G^{n+1}$. By comparing both double sums it is immediate that they consist of the same terms with the same signs and hence are equal. Therefore we have in fact, that relation (II.8) holds.

In order to prove our claim let us first consider the case $n \ge 3$. Let $(g_0, \dots, g_n) \in G^{n+1}$ and $(x_j)_{j\in\mathbb{N}} \subset \mathbb{H}^n$ a sequence of points converging to $\xi \in \partial\mathbb{H}^n$. Consider the sequence of functions $f_j : G^n \to \mathbb{R}$ given by

$$f_j(g_0, \dots, g_{n-1}) := \sum_{i=0}^{n-1} (-1)^i \mathrm{Vol}_n(g_0x_j, \dots, g_ix_j, g_ix, \dots, g_{n-1}x)$$

for every $(g_0, \dots, g_{n-1}) \in G^n$. As we have just shown

$$\mathrm{Vol}_n(g_0x, \dots, g_nx) - \mathrm{Vol}_n(g_0x_j, \dots, g_nx_j) = (df_j)(g_0, \dots, g_n)$$

for all $j \in \mathbb{N}$. By definition of $(f_j)_{j\in\mathbb{N}}$ and by continuity of $\mathrm{Vol}_n : (\overline{\mathbb{H}}^n)^{n+1} \to \mathbb{R}$ we get

$$\begin{aligned}
\mathrm{Vol}_n(g_0x, \dots, g_nx) - \mathrm{Vol}_n(g_0\xi, \dots, g_n\xi) &= \lim_{j\to\infty} \{\mathrm{Vol}_n(g_0x, \dots, g_nx) - \mathrm{Vol}_n(g_0x_j, \dots, g_nx_j)\} \\
&= \lim_{j\to\infty} (df_j)(g_0, \dots, g_n) \\
&= \lim_{j\to\infty} \sum_{i=0}^{n} (-1)^i \mathrm{Vol}_n(g_0x_j, \dots, g_ix_j, g_ix, \dots, g_nx) \\
&= \sum_{i=0}^{n} (-1)^i \lim_{j\to\infty} \mathrm{Vol}_n(g_0x_j, \dots, g_ix_j, g_ix, \dots, g_nx) \\
&= \sum_{i=0}^{n} (-1)^i \mathrm{Vol}_n(g_0\xi, \dots, g_i\xi, g_ix, \dots, g_nx) \\
&= (df)(g_0, \dots, g_n)
\end{aligned}$$

which proves our claim for $n \ge 3$.

Let us now turn to the case of $n = 2$. Let $(g_0, g_1, g_2) \in G^3$. We want to use again the above continuity argument. However since Vol_2 is only continuous on $(\mathbb{H}^2)^3 \cup (\overline{\mathbb{H}}^2)^{[3]}$ and

$$\begin{aligned}
(df)(g_0, g_1, g_2) &= f(g_1, g_2) - f(g_0, g_2) + f(g_0, g_2) \\
&= \{\mathrm{Vol}_2(g_1\xi, g_1x, g_2x) - \mathrm{Vol}_2(g_1\xi, g_2\xi, g_2x)\} \\
&\quad - \{\mathrm{Vol}_2(g_0\xi, g_0x, g_2x) - \mathrm{Vol}_2(g_0\xi, g_2\xi, g_2x)\} \\
&\quad + \{\mathrm{Vol}_2(g_0\xi, g_0x, g_1x) - \mathrm{Vol}_2(g_0\xi, g_1\xi, g_1x)\}
\end{aligned}$$

this will only work for triples $(g_0, g_1, g_2) \in G^3$ such that all of the following triples are in $(\overline{\mathbb{H}^2})^{[3]}$:

$$\begin{array}{ll} (g_0\xi, g_1\xi, g_2\xi), & \\ (g_1\xi, g_1x, g_2x), & (g_1\xi, g_2\xi, g_2x) \\ (g_0\xi, g_0x, g_2x), & (g_0\xi, g_2\xi, g_2x) \\ (g_0\xi, g_0x, g_1x), & (g_0\xi, g_1\xi, g_1x) \end{array}$$

Denote the set of these triples by F. Then

$$F = A^c \cap (B_0^c \cap C_0^c) \cap (B_1^c \cap C_1^c) \cap (B_2^c \cap C_2^c)$$

where A denotes the set of triples (g_0, g_1, g_2) such that $g_0\xi, g_1\xi, g_2\xi$ are contained in a proper hyperbolic subspace (i.e. are not pairwise distinct), B_i denotes the set of triples (g_0, g_1, g_2) such that $g_j\xi, g_jx, g_kx$ ($j \neq i$ and $k \neq j, i$) are contained in a proper hyperbolic subspace, and C_i denotes the set of triples (g_0, g_1, g_2) such that $g_j\xi, g_j\xi, g_kx$ ($j \neq i$ and $k \neq j, i$) are contained in a proper hyperbolic subspace. We claim that A, B_i and C_i ($i = 0, 1, 2$) are null sets in G^3, such that F has full measure.

Let P denote the stabilizer of $\xi \in \partial\mathbb{H}^2$. Clearly we have $A = A_0 \cup A_1 \cup A_2$ where

$$\begin{aligned} A_0 &= \{(g_0, g_1, g_2) \in G^3 : g_2^{-1}g_1 \in P\} \\ A_1 &= \{(g_0, g_1, g_2) \in G^3 : g_1^{-1}g_0 \in P\} \\ A_2 &= \{(g_0, g_1, g_2) \in G^3 : g_2^{-1}g_0 \in P\} \end{aligned}$$

As an example we will show that A_0 is a null set. One may show mutatis mutandis that A_1 and A_2 are null sets too. The characteristic function of A_0 is

$$\chi_{A_0}(g_0, g_1, g_2) = \chi_P(g_2^{-1}g_1)$$

for every $(g_0, g_1, g_2) \in G^3$ where $\chi_P : G \to \mathbb{R}$ is the characteristic function of P. Let $\mu^{\otimes 3}$ denote the product measure of some Haar measure μ on G. Then we compute

$$\begin{aligned} \mu^{\otimes 3}(A_0) &= \int_{G^3} \chi_{A_0}(g_0, g_1, g_2)\, d\mu^{\otimes 3}(g_0, g_1, g_2) \\ &= \int_G \int_G \int_G \chi_P(g_2^{-1}g_1)\, d\mu(g_1)\, d\mu(g_2)\, d\mu(g_0) \\ &= \int_G \int_G \int_G \chi_P(g_1)\, d\mu(g_1)\, d\mu(g_2)\, d\mu(g_0) = 0 \end{aligned}$$

since P is a proper closed subgroup in G hence a proper submanifold and thus a null set (cf. Proposition A.3.7). Analogously for A_1 and A_2 such that all in all $\mu^{\otimes 3}(A) = 0$.

As an example we will now show that B_0 is a null set. Analogous arguments can be carried out for B_1 and B_2. Clearly $(g_0, g_1, g_2) \in B_0$ if and only if $g_1^{-1}g_2x$ is on the geodesic $\gamma : \mathbb{R} \to \mathbb{H}^2$ from x to ξ. Let $\mu_{\mathbb{H}}$ denote the hyperbolic volume measure on $\mathbb{H}^2$ and μ a Haar measure on G and $p : G \to \mathbb{H}^2, g \mapsto gx$. Then since $\mu_{\mathbb{H}}(\gamma) = 0$ we get that also $\mu(p^{-1}(\gamma)) = 0$ where we have confused γ with its image (cf. Proposition A.4.13 and Lemma I.4.10). Denote by χ_γ the characteristic function of $p^{-1}(\gamma) \subset G$. Then we have for the characteristic function of B_0

$$\chi_{B_0}(g_0, g_1, g_2) = \chi_\gamma(g_1^{-1}g_2)$$

and thus

$$\begin{aligned}\int_{G^3} \chi_{B_0}(g_0,g_1,g_2)\, d\mu^{\otimes 3}(g_0,g_1,g_2) &= \int_G\int_G\int_G \chi_\gamma(g_1^{-1}g_2)\, d\mu(g_2)\, d\mu(g_1)\, d\mu(g_0)\\ &= \int_G\int_G\int_G \chi_\gamma(g_2)\, d\mu(g_2)\, d\mu(g_1)\, d\mu(g_0)\\ &= 0\end{aligned}$$

for every $(g_0,g_1,g_2)\in G^3$, i.e. B_0 is a null set in G^3. Skipping the analogous arguments for B_1 and B_2 we hence have that B is a null set.

Finally we will now show as an example that C_0 is a null set. Similarly this can be proven for C_1 and C_2. Now $(g_0,g_1,g_2)\in G^3$ is in C_0 if and only if $g_2^{-1}g_1\xi$ is one of the endpoints ξ,ξ' of the geodesic γ through x and ξ. The argument is now basically the same as for B_0. The set $\{\xi,\xi'\}$ is a null set in $\partial\mathbb{H}^2$ and thus $\mu(p^{-1}(\{\xi,\xi'\}))=0$ where $p: G\to\partial\mathbb{H}^2, g\mapsto g\xi$. Let $\chi'_\gamma: G\to\mathbb{R}$ denote the characteristic function of $p^{-1}(\{\xi,\xi'\})$. Then

$$\begin{aligned}\int_{G^3} \chi_{C_0}(g_0,g_1,g_2)\, d\mu^{\otimes 3}(g_0,g_1,g_2) &= \int_G\int_G\int_G \chi'_\gamma(g_2^{-1}g_1)\, d\mu(g_1)\, d\mu(g_2)\, d\mu(g_0)\\ &= \int_G\int_G\int_G \chi'_\gamma(g_2)\, d\mu(g_1)\, d\mu(g_2)\, d\mu(g_0)\\ &= 0\end{aligned}$$

for every $(g_0,g_1,g_2)\in G^3$, i.e. C_0 is a null set in G^3. Skipping the analogous arguments for C_1 and C_2 we hence have that C is a null set. This finishes the proof of the claim that F has full measure.

Now let $(x_j)_{j\in\mathbb{N}}\subset\mathbb{H}^2$ a sequence of points converging to $\xi\in\partial\mathbb{H}^2$ and as in the case of $n\geq 3$

$$f_j(g_0,g_1) := \mathrm{Vol}_2(g_0x_j, g_0x, g_1x) - \mathrm{Vol}_2(g_0x_j, g_1x_j, g_1x)$$

for every $(g_0,g_1,g_2)\in G^3$. Then

$$\begin{aligned}\mathrm{Vol}_2(g_0x,g_1x,g_2x) - \mathrm{Vol}_2(g_0\xi,g_1\xi,g_2\xi) &= \lim_{j\to\infty}\{\mathrm{Vol}_2(g_0x,g_1x,g_2x) - \mathrm{Vol}_2(g_0x_j,g_1x_j,g_2x_j)\}\\ &= \lim_{j\to\infty}(df_j)(g_0,g_1,g_2)\\ &= \lim_{j\to\infty}\Big\{[\mathrm{Vol}_2(g_1x_j,g_1x,g_2x) - \mathrm{Vol}_2(g_1x_j,g_2x_j,g_2x)]\\ &\quad - [\mathrm{Vol}_2(g_0x_j,g_0x,g_2x) - \mathrm{Vol}_2(g_0x_j,g_2x_j,g_2x)]\\ &\quad + [\mathrm{Vol}_2(g_0x_j,g_0x,g_1x) - \mathrm{Vol}_2(g_0x_j,g_1x_j,g_1x)]\Big\}\\ &= [\mathrm{Vol}_2(g_1\xi,g_1x,g_2x) - \mathrm{Vol}_2(g_1\xi,g_2\xi,g_2x)]\\ &\quad - [\mathrm{Vol}_2(g_0\xi,g_0x,g_2x) - \mathrm{Vol}_2(g_0\xi,g_2\xi,g_2x)]\\ &\quad + [\mathrm{Vol}_2(g_0\xi,g_0x,g_1x) - \mathrm{Vol}_2(g_0\xi,g_1\xi,g_1x)]\\ &= (df)(g_0,g_1,g_2)\end{aligned}$$

for every $(g_0,g_1,g_2)\in F$, i.e.

$$p_K^*\mathrm{Vol}_2 - p_P^*\mathrm{Vol}_2 = df$$

in $L^\infty(G^3,\mathbb{R}_\varepsilon)^G$. This concludes the proof. □

We have now seen several representations of the volume class $\omega_n^b\in H^n_{cb}(G,\mathbb{R}_\varepsilon)$. Adding to the ambiguity of the terms "volume cocycle" and "volume class" we can also pull them back to $H^n_{c(b)}(G^+,\mathbb{R})$ and will refer to their images also as *volume cocycle* and *volume class* respectively. The next proposition justifies this as it shows that one gets the pullback of the volume cocycle by simply forgetting its G-equivariance and interpreting it as only a G^+-invariant cocycle.

Proposition II.3.21. *(i) In continuous cohomology the pullback $i^* : H_c^\bullet(G, \mathbb{R}_\varepsilon) \to H_c^\bullet(G^+, \mathbb{R})$ along $i : G^+ \to G$ is given at the cochain level by the inclusions*

$$\iota_1 : \Omega^\bullet(\mathbb{H}^n, \mathbb{R}_\varepsilon)^G \hookrightarrow \Omega^\bullet(\mathbb{H}^n, \mathbb{R})^{G^+}$$

and

$$\iota_2 : C((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\varepsilon)^G \hookrightarrow C((\mathbb{H}^n)^{\bullet+1}, \mathbb{R})^{G^+}$$

under the respective canonical isomorphisms. In particular thinking of ω_n and Vol_n as G^+-invariant cocycles yields their image under the pullback i^* (under the respective canonical isomorphisms).

(ii) In bounded cohomology the pullback $i^* : H_{cb}^\bullet(G, \mathbb{R}_\varepsilon) \to H_{cb}^\bullet(G^+, \mathbb{R})$ along $i : G^+ \to G$ is given at the cochain level by the inclusions

$$\iota_3 : C_b((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\varepsilon)^G \hookrightarrow C_b((\mathbb{H}^n)^{\bullet+1}, \mathbb{R})^{G^+},$$
$$\iota_4 : L^\infty((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\varepsilon)^G \hookrightarrow L^\infty((\mathbb{H}^n)^{\bullet+1}, \mathbb{R})^{G^+}$$

and

$$\iota_5 : L^\infty((\partial\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\varepsilon)^G \hookrightarrow L^\infty((\partial\mathbb{H}^n)^{\bullet+1}, \mathbb{R})^{G^+}$$

under the respective canonical isomorphisms. In particular thinking of Vol_n as a G^+-invariant cocycle in one of the respective cochain complexes yields its image under the pullback i^* (via the respective canonical isomorphisms).

Proof. There are two ways of proving this result both of which are equally evident. First, one can use the previously described isomorphisms: van Est and p_K^* for continuous cohomology; p_K^*, p_P^* for continuous bounded cohomology. All computations are trivial and we leave them out. The assertion then follows by the naive definition of the pullback map for continuous cohomology and Corollary II.2.35 for bounded cohomology.

Second, one can see this on a more conceptual level and show, that all inclusions at the level of resolutions are extensions of the identity $\mathrm{id} : \mathbb{R} \to \mathbb{R}$ and hence the result follows from Proposition II.2.33 and Remark II.2.34. □

Finally we want to understand the pullback $\rho^*(\omega_n^b)$ of the volume class, since this will be very important in the proof of the volume rigidity theorem later on. We will do so by applying Corollary II.2.42 and Corollary II.2.41.

Corollary II.3.22. *Let $H < G$ be a closed subgoup (e.g. a lattice) and $\rho : H \to G$ a continuous homomorphism. Then the pullback of the volume class $\rho^*\omega_n^b$ is represented in $H^n(L^\infty(H^{\bullet+1}, \mathbb{R}_{\varepsilon\rho})^H) \cong H_{cb}^n(H, \mathbb{R}_{\varepsilon\rho})$ by the cocycle*

$$\begin{aligned} \rho^* V_y &: H^{n+1} \to \mathbb{R} \\ (h_0, \ldots, h_n) &\mapsto \mathrm{Vol}_n(\rho(h_0)y, \ldots, \rho(h_n)y) \end{aligned}$$

for any $y \in \overline{\mathbb{H}}^n$.

Proof. Due to Proposition II.3.20 the volume class ω_n^b is represented in $L^\infty(G^{n+1}, \mathbb{R}_\varepsilon)$ by the cocycle

$$\begin{aligned} V_y &: G^{n+1} \to \mathbb{R} \\ (g_0, \ldots, g_n) &\mapsto \mathrm{Vol}_n(g_0 y, \ldots, g_n y) \end{aligned}$$

for any $y \in \overline{\mathbb{H}}^n$. Note that $V_y : G^{n+1} \to \mathbb{R}$ is actually in $\mathcal{B}^\infty(G^{n+1}, \mathbb{R})$. Hence by Corollary II.2.42 we know, that $\rho^*\omega_n^b$ is represented in $H^n(L^\infty(H^{\bullet+1}, \mathbb{R}_{\varepsilon\rho})^H)$ by the cocycle $\rho^* V_y$ as asserted. □

Corollary II.3.23. *Let $H < G$ be a closed subgroup (e.g. a lattice) and $\rho : H \to G$ a continuous group homomorphism. Moreover let $\varphi : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ be an a.e.-ρ-equivariant boundary map (cf. Corollary II.2.41). Then the pullback of the volume class $\rho^*\omega_n^b$ is represented in $H^n(L^\infty((\partial\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_{\varepsilon\rho})^H) \cong H^n_{cb}(H, \mathbb{R}_{\varepsilon\rho})$ by the cocycle*

$$\begin{aligned} \varphi^*\mathrm{Vol}_n : (\partial\mathbb{H}^n)^{n+1} &\to \mathbb{R} \\ (\xi_0, \dots, \xi_n) &\mapsto \mathrm{Vol}_n(\varphi(\xi_0), \dots, \varphi(\xi_n)) \end{aligned}$$

Proof. Recall that $\partial\mathbb{H}^n = G/P$ is an amenable regular G-space, since P – the stabilizer of some boundary point – is amenable (cf. Proposition C.2.7). Now by Proposition II.3.19 Vol_n is already in $\mathcal{B}^\infty((\partial\mathbb{H}^n)^{n+1}, \mathbb{R}_\varepsilon)^G$. Therefore by Corollary II.2.41 the assertion follows. ☐

II.3.4. Some Computations

Using the previously introduced volume class ω_n we will be able to compute some cohomology groups in this subsection. First we shall compute $H^\bullet_c(G^+, \mathbb{R})$. In particular we will see, that $H^n_c(G^+, \mathbb{R})$ is generated by the pullback of the volume class ω_n. After this computation we will see, that $H^\bullet_{c(b)}(G^+, \mathbb{R}) \cong H^\bullet_{c(b)}(G, \mathbb{R}) \oplus H^\bullet_{c(b)}(G, \mathbb{R}_\varepsilon)$, which yields that $H^n_c(G, \mathbb{R}_\varepsilon)$ is generated by the volume class ω_n. Finally we will use a simple lemma to deduce that $c : H^n_{cb}(G, \mathbb{R}_\varepsilon) \to H^n_c(G, \mathbb{R}_\varepsilon)$ is in fact an isomorphism; recall that $G = \mathrm{Isom}(\mathbb{H}^n)$.

The next proposition gives us $H^\bullet_c(G^+, \mathbb{R})$.

Proposition II.3.24. *Let $q \in \mathbb{N}_0$. Then*

$$H^q_c(G^+, \mathbb{R}) \cong \begin{cases} \mathbb{R}, & \text{if } q = 0 \\ \mathbb{R} \cong \langle[\omega_n]\rangle \cong \langle[\mathrm{Vol}_n]\rangle, & \text{if } q = n \\ 0, & \text{else} \end{cases}$$

where $[\omega_n]$ denotes the cohomology class of the volume cocycle in $H^n(\Omega^\bullet(\mathbb{H}^n, \mathbb{R})^{G^+})$ which is also represented by the cohomology class of the volume cocycle $[\mathrm{Vol}_n] \in H^n(C((\mathbb{H}^n)^{\bullet+1}, \mathbb{R})^{G^+} \cong H^n(\Omega^\bullet(\mathbb{H}^n, \mathbb{R})^{G^+})$ via the van Est isomorphism (cf. subsection II.3.3).

Proof. For brevity we will simply write $\Omega^\bullet(\mathbb{H}^n)$ instead of $\Omega^\bullet(\mathbb{H}^n, \mathbb{R})$.

We know that

$$H^\bullet_c(G^+, \mathbb{R}) \cong H^\bullet(\Omega^\bullet(\mathbb{H}^n)^{G^+})$$

where the right hand side denotes the cohomology of the cocomplex of G^+-invariant differential forms on $\mathbb{H}^n$. We shall first prove that

$$\Omega^q(\mathbb{H}^n)^{G^+} = \{0\}$$

for all $0 < q < n$. Hence let $0 < q < n$ and $\omega \in \Omega^q(\mathbb{H}^n)^{G^+}$. Because G^+ operates transitively on $\mathbb{H}^n$ and ω is G^+-invariant, it suffices to show that $\omega_p \equiv 0$ for one $p \in \mathbb{H}^n$. We shall see that for $p = 0$ in the Poincaré ball model B^n. Because of the multilinearity of ω_0 it is enough to see that for an arbitrary basis $v_1, \dots, v_n \in T_0B^n \cong \mathbb{R}^n$ the expression $\omega_0(v_{i_1}, \dots, v_{i_q})$ vanishes for every subcollection $\{v_{i_1}, \dots, v_{i_q}\}$ of the basis. For simplicity we take the standard basis $e_1, \dots, e_n \in T_0B^n \cong \mathbb{R}^n$. Now for any subcollection $e_{i_1}, \dots, e_{i_q}$ and some $j \in \{1, \dots, n\} - \{i_1, \dots, i_q\}$ consider the matrix given by $A(e_i) = e_i$ for every $i \neq i_1, j$ and $A(e_{i_1}) = -e_{i_1}$, $A(e_j) = -e_j$. Then $A \in O(n)$ and because

$$\det A = (-1)^2 = 1$$

we have that $A \in SO(n)$, i.e. $A \in G_0^+$. Again by the G^+-invariance we get

$$\begin{aligned}\omega_0(e_{i_1}, \dots, e_{i_q}) &= \omega_0((dA)_0(e_{i_1}), \dots, (dA)_0(e_{i_q})) = \omega_0(A(e_{i_1}), \dots, A(e_{i_q}))\\ &= \omega_0(-e_{i_1}, e_{i_2}, \dots, e_{i_q}) = -\omega_0(e_{i_1}, \dots, e_{i_q})\end{aligned}$$

and hence $\omega_0(e_{i_1}, \dots, e_{i_q}) = 0$. Thus $\omega \equiv 0$ and $\Omega^q(\mathbb{H}^n)^{G^+} = \{0\}$ as asserted.

It follows immediately that for every $0 < q < n$

$$H^q(\Omega^\bullet(\mathbb{H}^n)^{G^+}) = 0$$

For $q = 0$ we have

$$\begin{aligned}H^0(\Omega^\bullet(\mathbb{H}^n)^{G^+}) &= \ker\{d : \Omega^0(\mathbb{H}^n)^{G^+} \to \Omega^1(\mathbb{H}^n)^{G^+}\}\\ &= \{f : \mathbb{H}^n \to \mathbb{R} \text{ constant}\} = \mathbb{R}\end{aligned}$$

For $q = n$:

$$H^n(\Omega^\bullet(\mathbb{H}^n)^{G^+}) = \frac{\ker(d)}{\mathrm{im}(d)} = \ker(d) = \Omega^n(\mathbb{H}^n)^{G^+}$$

Clearly the volume form in $\Omega^n(\mathbb{H}^n)$ is G^+-invariant. It remains to see that $\Omega^n(\mathbb{H}^n)^{G^+}$ is one dimensional, i.e. for any two $\omega, \omega' \in \Omega^n(\mathbb{H}^n)^{G^+}$ there is a $\lambda \in \mathbb{R}$ such that $\omega \equiv \lambda\omega'$. It suffices to find a $\lambda \in \mathbb{R}$ such that for every $p \in \mathbb{H}^n$ and some basis $v_1, \dots, v_n \in T_p\mathbb{H}^n$

$$\omega_n(v_1, \dots, v_n) = \lambda\omega(v_1, \dots, v_n)$$

Because of $\omega_0, \omega_0' \in \mathrm{Alt}^n(T_0\mathbb{H}^n, \mathbb{R}) \cong \mathbb{R}$ we find a $\lambda \in \mathbb{R}$ such that

$$\omega_0(e_1, \dots, e_n) = \lambda\omega_0'(e_1, \dots, e_n)$$

For any other $p \in \mathbb{H}^n$ there is a $g \in G^+$ such that $g(0) = p$ and hence

$$\begin{aligned}\omega_p((dg)_0(e_1), \dots, (dg)_0(e_n)) &= \omega_0(e_1, \dots, e_n) = \lambda\omega_0'(e_1, \dots, e_n)\\ &= \lambda\omega_p'((dg)_0(e_1), \dots (dg)_0(e_n))\end{aligned}$$

Since g is a diffeomorphism $\{(dg)_0(e_1), \dots (dg)_0(e_n)\}$ is a basis of $T_p\mathbb{H}^n$ and the assertion follows. $\square$

The next proposition establishes a link between the equivariant (bounded) cohomology $H^\bullet_{c(b)}(G, \mathbb{R}_\varepsilon)$ and the invariant (bounded) cohomology $H^\bullet_{c(b)}(G^+, \mathbb{R})$. As a corollary we will get, that in fact $H^n_c(G^+, \mathbb{R}) \cong H^n_c(G, \mathbb{R}_\varepsilon) = \langle\omega_n\rangle$.

Proposition II.3.25 (cf. [BBI13, Proposition 1, p. 7])**.** *Let $\tau \in G - G^+$ be an orientation reversing isometry. Then the map*

$$(p, \bar{p}) : H^\bullet_{c(b)}(G^+, \mathbb{R}) \to H^\bullet_{c(b)}(G, \mathbb{R}) \oplus H^\bullet_{c(b)}(G, \mathbb{R}_\varepsilon)$$

is an isomorphism, where at the cochain level the maps

$$\begin{aligned}p\colon C_{(b)}((\mathbb{H}^n)^{q+1}, \mathbb{R})^{G^+} &\to C_{(b)}((\mathbb{H}^n)^{q+1}, \mathbb{R})^G\\ \bar{p}\colon C_{(b)}((\mathbb{H}^n)^{q+1}, \mathbb{R})^{G^+} &\to C_{(b)}((\mathbb{H}^n)^{q+1}, \mathbb{R}_\varepsilon)^G\end{aligned}$$

are given by

$$\begin{aligned}p(f)(x_0, \dots, x_q) &\coloneqq \frac{1}{2}(f(x_0, \dots, x_q) + f(\tau x_0, \dots, \tau x_q))\\ \bar{p}(f)(x_0, \dots, x_q) &\coloneqq \frac{1}{2}(f(x_0, \dots, x_q) - f(\tau x_0, \dots, \tau x_q))\end{aligned}$$

for every $f \in C_{(b)}((\mathbb{H}^n)^{q+1}, \mathbb{R})^{G^+}$, $x_0, \dots, x_q \in \mathbb{H}^n$. Moreover these maps do not depend on the choice of $\tau \in G - G^+$.

Proof. Observe that we have with respect to the left regular representation $p(f) = 1/2\,(f + \tau^{-1} \cdot f)$ and $\bar{p}(f) = 1/2\,(f - \tau^{-1} \cdot f)$ for every $f \in C_{(b)}((\mathbb{H}^n)^{q+1}, \mathbb{R})$.

First of all we need to check that p and $\bar{p}$ are well defined, i.e. the image of p resp. $\bar{p}$ is in $C_{(b)}((\mathbb{H}^n)^{q+1}, \mathbb{R})^G$ resp. $C_{(b)}((\mathbb{H}^n)^{q+1}, \mathbb{R}_\varepsilon)^G$. If f is continuous (and bounded), then so is $p(f)$ and $\bar{p}(f)$. It is also clear from the definition that p and $\bar{p}$ are linear.

Thus we have to verify, that $p(f)$ is G-invariant and $\bar{p}(f)$ is G-equivariant. We shall only show the simple calculation for the latter. If $g \in G$ is orientation preserving, we have

$$\begin{aligned} g \cdot \bar{p}(f) &= \frac{1}{2}\,(g \cdot f - g\tau^{-1} \cdot f) = \frac{1}{2}\,(f - \tau^{-1}(\tau g \tau^{-1}) \cdot f) \\ &= \frac{1}{2}\,(f - \tau^{-1} \cdot f) = \varepsilon(g)\,\bar{p}(f) \end{aligned}$$

Similarly for $g \in G$ orientation reversing

$$\begin{aligned} g \cdot \bar{p}(f) &= \frac{1}{2}\,(g \cdot f - g\tau^{-1} \cdot f) = \frac{1}{2}\,(\tau^{-1}(\tau g) \cdot f - f) \\ &= -\frac{1}{2}\,(f - \tau^{-1} \cdot f) = \varepsilon(g)\,\bar{p}(f) \end{aligned}$$

Hence $\bar{p}(f)$ is G-equivariant.

Now we want to see, that p and $\bar{p}$ do not depend on the choice of $\tau \in G \backslash G^+$. If $\rho \in G\backslash G^+$ is another orientation reversing isometry and we define p' and $\bar{p}'$ exactly like p and $\bar{p}$ resp. with τ replaced by ρ, we calculate (again only for $\bar{p}'$)

$$\begin{aligned} \bar{p}'(f) &= \frac{1}{2}\,(f - \rho^{-1} \cdot f) = \frac{1}{2}\,(f - \tau^{-1}(\tau\rho^{-1}) \cdot f) \\ &= \frac{1}{2}(f - \tau^{-1} \cdot f) = \bar{p}(f) \end{aligned}$$

So our definition is independent of the choice of τ.

Next we need to show, that p and $\bar{p}$ are morphisms of complexes, i.e. they commute with the homogeneous coboundary operator. Again we shall only verify this by a simple calculation for $\bar{p}(f)$

$$\begin{aligned} d(\bar{p}(f)) &= \frac{1}{2}\,(df - d(\tau^{-1} \cdot f)) \\ &= \frac{1}{2}\,(df - \tau^{-1} \cdot df) = \bar{p}(d(f)) \end{aligned}$$

since the G-action commutes with the homogeneous coboundary operator. Thus p and $\bar{p}$ induce a map at the cohomology level in every degree

$$(p, \bar{p})\colon H^\bullet_{c(b)}(G^+, \mathbb{R}) \to H^\bullet_{c(b)}(G, \mathbb{R}) \oplus H^\bullet_{c(b)}(G, \mathbb{R}_\varepsilon)$$

We will now show the bijectivity of the above map by giving an inverse at the cochain level. First observe that

$$C_{(b)}((\mathbb{H}^n)^{q+1}, \mathbb{R})^G \subset C_{(b)}((\mathbb{H}^n)^{q+1}, \mathbb{R})^{G^+} \quad \text{and} \quad C_{(b)}((\mathbb{H}^n)^{q+1}, \mathbb{R}_\varepsilon)^G \subset C_{(b)}((\mathbb{H}^n)^{q+1}, \mathbb{R})^{G^+}$$

because G^+ is a subgroup of G and $\varepsilon|_{G^+} \equiv 1$. Hence the map

$$\Phi : C_{(b)}((\mathbb{H}^n)^{q+1}, \mathbb{R})^G \oplus C_{(b)}((\mathbb{H}^n)^{q+1}, \mathbb{R}_\varepsilon)^G \to C_{(b)}((\mathbb{H}^n)^{q+1}, \mathbb{R})^{G^+}$$

given by $\Phi(f_1, f_2) \coloneqq f_1 + f_2$ is well defined and is clearly a morphism of complexes. For all $f_1 \in C_{(b)}((\mathbb{H}^n)^{q+1}, \mathbb{R})^G$, $f_2 \in C_{(b)}((\mathbb{H}^n)^{q+1}, \mathbb{R}_\varepsilon)^G$ we have

$$p(f_1 + f_2) = \frac{1}{2}\left(f_1 + f_2 + \tau^{-1} \cdot f_1 + \tau^{-1} \cdot f_2\right) = \frac{1}{2}\left(f_1 + f_2 + f_1 - f_2\right) = f_1$$

$$\bar{p}(f_1 + f_2) = \frac{1}{2}\left(f_1 + f_2 - (\tau^{-1} \cdot f_1 + \tau^{-1} \cdot f_2)\right) = \frac{1}{2}\left(f_1 + f_2 - f_1 + f_2\right) = f_2$$

So $(p, \bar{p})(\Phi(f_1, f_2)) = (f_1, f_2)$. Further it is clear that $p(f) + \bar{p}(f) = f$ for all $f \in C_{(b)}((\mathbb{H}^n)^{q+1}, \mathbb{R})^{G^+}$ and thus $\Phi(p(f), \bar{p}(f)) = f$. $\square$

Remark II.3.26. *Although we did not mention transfer maps, we want to remark, that $\bar{p}$ can be interpreted as the transfer map*

$$\mathrm{trans} : H^\bullet_{cb}(G^+, \mathbb{R}) \to H^\bullet_{cb}(G, \mathbb{R}_\varepsilon)$$

which is a left inverse of the pullback map

$$i^* : H^\bullet_{cb}(G, \mathbb{R}_\varepsilon) \to H^\bullet_{cb}(G^+, \mathbb{R})$$

We will encounter transfer maps in a different situation in the next chapter. For more details we refer to [Mon01, Proposition 8.6.2, p. 106].

Corollary II.3.27. *Let $q \in \mathbb{N}_0$. We have*

$$H^q_c(G^+, \mathbb{R}) \cong \begin{cases} H^0_c(G, \mathbb{R}) \cong \mathbb{R}, & \text{if } q = 0 \\ H^n_c(G, \mathbb{R}_\varepsilon) \cong \mathbb{R} \cong \langle [\mathrm{Vol}_n] \rangle, & \text{if } q = n \\ 0, & \text{else} \end{cases}$$

where $[\mathrm{Vol}_n]$ denotes the cohomology class of the volume cocycle in $H^n(C((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\varepsilon)^G)$.

Proof. First of all the cases of $q \neq n$ follow immediately from Proposition II.3.25 and the computation of $H^\bullet_c(G^+, \mathbb{R})$ in Proposition II.3.24.

Applying the isomorphism from Proposition II.3.25

$$(p, \bar{p}) : H^\bullet_{c(b)}(G^+, \mathbb{R}) \to H^\bullet_{c(b)}(G, \mathbb{R}) \oplus H^\bullet_{c(b)}(G, \mathbb{R}_\varepsilon)$$

to the volume cocycle $\mathrm{Vol}_n \in C((\mathbb{H}^n)^{n+1}, \mathbb{R})^{G^+}$ we get

$$p(\mathrm{Vol}_n)(x_0, \dots, x_n) = \frac{1}{2}(\mathrm{Vol}_n(x_0, \dots, x_n) + \mathrm{Vol}_n(\tau x_0, \dots \tau x_n)) = 0$$

and

$$\bar{p}(\mathrm{Vol}_n)(x_0, \dots, x_n) = \frac{1}{2}(\mathrm{Vol}_n(x_0, \dots, x_n) - \mathrm{Vol}_n(\tau x_0, \dots \tau x_n)) = \mathrm{Vol}_n(x_0, \dots, x_n)$$

for every $(x_0, \dots, x_n) \in (\mathbb{H}^n)^{n+1}$. In view of the above remark this is not surprising since $\mathrm{Vol}_n \in C((\mathbb{H}^n)^{n+1}, \mathbb{R})^{G^+}$ is by definition the pullback via $i^* : H^\bullet_{cb}(G, \mathbb{R}_\varepsilon) \to H^\bullet_{cb}(G^+, \mathbb{R})$ of the volume cocycle $\mathrm{Vol}_n \in C((\mathbb{H}^n)^{n+1}, \mathbb{R}_\varepsilon)^G$. This settles the case of $q = n$ and the assertion follows. $\square$

The next lemma asserts, that all G-equivariant cochains vanish in degree less than n. It will then be easy to deduce that the comparison map is an isomorphism.

Lemma II.3.28 (cf. [BBI13, Lemma 1, p. 8]). *For $q < n$ we have*

$$C_{(b)}((\mathbb{H}^n)^{q+1}, \mathbb{R}_\varepsilon)^G = 0,$$
$$L^\infty((\mathbb{H}^n)^{q+1}, \mathbb{R}_\varepsilon)^G = 0,$$
$$L^\infty((\partial\mathbb{H}^n)^{q+1}, \mathbb{R}_\varepsilon)^G = 0.$$

Proof. Let f be in one of the above function spaces (the argument will be the same for all of them). Now observe that any $q+1 \leq n$ points $x_0, \ldots, x_q$ in $\mathbb{H}^n$ or $\partial\mathbb{H}^n$ either lie on a proper hyperbolic subspace $P \subset \mathbb{H}^n$ or on the boundary of such a proper hyperbolic subspace. Taking $\tau \in G$ as the reflection along this subspace τ is orientation reversing and fixes every point $x_0, \ldots, x_q$ on P. Since f is G-equivariant, it follows that

$$f(x_0, \ldots, x_q) = -f(\tau x_0, \ldots, \tau x_q) = -f(x_0, \ldots, x_q)$$

Hence $f \equiv 0$. □

Proposition II.3.29 (cf. [BBI13, Proposition 2, p. 8]). *The comparison map induces an isomorphism in top degree*

$$c\colon H^n_{cb}(G, \mathbb{R}_\varepsilon) \longrightarrow H^n_c(G, \mathbb{R}_\varepsilon)$$

In conjunction with our previous computation of $H^n_c(G, \mathbb{R}_\varepsilon)$ this means, that $H^n_{cb}(G, \mathbb{R}_\varepsilon)$ is generated by the (bounded) volume class ω^b_n.

Proof. Due to Lemma II.3.28 there are no cochains in degree $n-1$ and so there are no coboundaries in degree n. Thus the cohomology groups $H^n_{cb}(G, \mathbb{R}_\varepsilon)$ and $H^n_c(G, \mathbb{R}_\varepsilon)$ are equal to their corresponding spaces of cocycles, i.e. the kernels of the homogeneous coboundary operator. So we obtain the following commutative diagram

$$\begin{tikzcd}
H^n_{cb}(G, \mathbb{R}_\varepsilon) \cong H^n(C_b((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\varepsilon)^G) \arrow[r, equal] \arrow[d, "\mathrm{c}"] & \ker\{d\colon C_b((\mathbb{H}^n)^{n+1}, \mathbb{R}_\varepsilon)^G \to C_b((\mathbb{H}^n)^{n+2}, \mathbb{R}_\varepsilon)^G\} \arrow[d, hook] \\
\mathbb{R} \cong \langle \omega_n \rangle = H^n(C((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\varepsilon)^G) \arrow[r, equal] & \ker\{d\colon C((\mathbb{H}^n)^{n+1}, \mathbb{R}_\varepsilon)^G \to C((\mathbb{H}^n)^{n+2}, \mathbb{R}_\varepsilon)^G\}
\end{tikzcd}$$

Clearly the map on the right is an inclusion and hence also c is injective. Further c is surjective since ω_n is represented by Vol_n and Vol_n is also bounded, i.e. $\omega_n = c[\mathrm{Vol}_n]$. Recall that $c: H^n_{cb}(G, \mathbb{R}_\varepsilon) \to H^n_c(G, \mathbb{R}_\varepsilon)$ is given by the above inclusion due to Corollary II.2.45. □

Because there are no coboundaries in degree n due to Lemma II.3.28, the norm $\|\omega^b_n\|$ is equal to the norm of the volume cocycle which represents it (cf. [BBI13, Corollary 3, p. 9]). As we have mentioned in chapter I this is equal to the volume of a regular ideal simplex in $\mathbb{H}^n$ (cf. Theorem I.7.4).

III. Volume Rigidity of Hyperbolic Lattice Representations

In this final chapter we will prove the volume rigidity theorem. In an attempt of being motivational we first state the volume rigidity theorem and deduce two classical versions of the Mostow rigidity theorem from it in section III.1. Since the volume rigidity theorem relies on the notion of the volume of a representation, we will introduce it in section III.2. The rest of the section will then elaborate on transfer maps in bounded cohomology in order to finally deduce some important properties of the volume of a representation. The last section of this chapter is then devoted to the proof of the volume rigidity theorem. The first assertion of the volume rigidity theorem will follow directly from the properties of the volume of a representation, that we have deduced in section III.2. The last assertion is much harder to prove, whence its proof is divided into three steps.

III.1. The Volume Rigidity Theorem

Let us state the main result of [BBI13] here; the volume rigidity theorem. The theorem uses the notion of the volume of a representation which we will give in section III.2.

Theorem III.1.1 (Volume Rigidity Theorem; cf. [BBI13, Theorem 1, p. 4]). *Let $n \geq 3$. Let $i\colon \Gamma \hookrightarrow \mathrm{Isom}^+(\mathbb{H}^n)$ be a lattice embedding and let $\rho\colon \Gamma \to \mathrm{Isom}^+(\mathbb{H}^n)$ be any representation. Then:*

$$|\mathrm{Vol}(\rho)| \leq |\mathrm{Vol}(i)| = \mathrm{Vol}(M)$$

with equality, if and only if ρ is conjugated to i by an isometry. Here M denotes the quotient $i(\Gamma)\backslash\mathbb{H}^n$.

For the rest of this chapter we shall fix the notation of Theorem III.1.1.

Taking in particular ρ to be another lattice embedding of Γ, we immediately recover Mostow's rigidity theorem for hyperbolic lattices:

Corollary III.1.2 (Mostow Rigidity – Algebraic Version; cf. [BBI13, Corollary 1, p. 4]). *Let $\Gamma_1, \Gamma_2 < \mathrm{Isom}^+(\mathbb{H}^n)$ be two isomorphic lattices. Then there exists an isometry $g \in \mathrm{Isom}(\mathbb{H}^n)$ conjugating Γ_1 to Γ_2.*

This may be translated to the following geometric version of Mostow's rigidity theorem:

Corollary III.1.3 (Mostow Rigidity – Geometric Version). *Let $n \geq 3$ and let M_1, M_2 be two finite volume hyperbolic n-manifolds with respective fundamental groups $\Gamma_i = \pi_1(M_i)$ $(i = 1, 2)$. If the fundamental groups Γ_1 and Γ_2 are isomorphic, then M_1 and M_2 are already isometric.*

Proof. Note that we can identify $\Gamma_i\backslash\mathbb{H}^n \cong M_i$ $(i = 1, 2)$. With this identification $\Gamma_i < \mathrm{Isom}^+(\mathbb{H}^n)$ $(i = 1, 2)$ is a lattice (cf. section I.4) and both lattices are isomorphic by hypothesis $\Gamma_1 \cong \Gamma_2$. Thus by the previous corollary there is an isometry $h \in \mathrm{Isom}(\mathbb{H}^n)$ conjugating both, i.e. $h \cdot \Gamma_1 \cdot h^{-1} \cong \Gamma_2$. In particular $h : \mathbb{H}^n \to \mathbb{H}^n$ induces a map $\bar{h} : \Gamma_1\backslash\mathbb{H}^n \cong M_1 \to \Gamma_2\backslash\mathbb{H}^n \cong M_2$ given by $\bar{h}(\pi_1(x)) = \pi_2(h(x))$ for every $x \in \mathbb{H}^n$, where $\pi_i : \mathbb{H}^n \to \Gamma_i\backslash\mathbb{H}^n$ denotes the universal covering.

It is easy to check that $\bar{h} : M_1 \to M_2$ is well-defined and bijective. It is even an isometry, since it commutes with π_1 and π_2 which are local isometries. Hence M_1 and M_2 are isometric. □

III.2. The Volume of a Representation

Let us turn to the volume of a lattice representation $\mathrm{Vol}(\cdot)$ as it occurs in the volume rigidity theorem. In subsection III.2.1 we give the definition of $\mathrm{Vol}(\cdot)$. Before we can deduce some of its important properties in subsection III.2.3 we need to introduce transfer maps in cohomology and relate them to ideas of relative cohomology in subsection III.2.2. The properties of $\mathrm{Vol}(\cdot)$ will then imply the first assertion of the volume rigidity theorem concerning the inequality as we will discuss more thoroughly in section III.3.

III.2.1. Definition

We will define the volume $\mathrm{Vol}(\rho)$ of the representation $\rho : \Gamma \to G^+$ in several steps. First suppose that Γ is torsion-free. This has the advantage, that $M = i(\Gamma)\backslash\mathbb{H}^n$ is indeed a manifold, since Γ now acts freely on $\mathbb{H}^n$.

We have already seen that the (continuous) cohomology of Γ is isomorphic to the singular cohomology of M (cf. Corollary II.3.3). Further if M is compact, by Poincaré duality the cohomology groups $H_c^n(\Gamma, \mathbb{R}) \cong H^n(M)$ in top degree are canonically isomorphic to $\mathbb{R}$ and the isomorphism is given by the evaluation on the fundamental class $[M] \in H_n(M)$ (cf. section D.1). We define the volume $\mathrm{Vol}(\rho)$ by

$$\mathrm{Vol}(\rho) = \langle \rho^*(\omega_n), [M] \rangle$$

where $\rho^* : H_c^n(G^+, \mathbb{R}) \to H_c^n(\Gamma, \mathbb{R}) \cong H^n(M)$ denotes the pullback via ρ and $\omega_n \in H_c^n(G^+, \mathbb{R})$ is the volume class as discussed in subsection II.3.3.

However if M is non-compact, the above definition fails, since $H_n(M) = 0$ such that there is no fundamental class to evaluate on (cf. [Hat02, Proposition 3.29, p. 239]). We can fix this defect by considering bounded cohomology. This is in a way natural, since $\omega_n = c(\omega_n^b)$, where $\omega_n^b \in H_{cb}^n(G, \mathbb{R}_\varepsilon)$ is the (bounded) volume class. Again we may pull it back along $\rho : \Gamma \to G^+$ and get $\rho^*(\omega_n^b) \in H_{cb}^n(\Gamma, \mathbb{R})$. As before – but for bounded cohomology – we have $H_{cb}^n(\Gamma, \mathbb{R}) \cong H_b^n(M)$ (cf. Corollary II.3.12), such that we may think of $\rho^*(\omega_n^b)$ as in $H_b^n(M)$. In analogy to the case of compact M we would like to interpret $\rho^*(\omega_n^b)$ as a class on some sensible compact subspace of M. Because M is a finite volume hyperbolic manifold, it admits a compact core $N \subset M$ which is the complement of the disjoint union of finitely many cusps $E_1, \ldots, E_k$ (cf. Corollary I.6.5).

We now get the following long exact sequence in bounded cohomology (cf. section D.3 in the appendix)

$$\ldots \to H_b^{\bullet-1}(M - N) \to H_b^\bullet(M, M - N) \to H_b^\bullet(M) \to H_b^\bullet(M - N) \to \ldots$$

Our goal is to establish, that the map in the middle $H_b^\bullet(M, M - N) \to H_b^\bullet(M)$ is an isomorphism allowing us to interpret $\rho^*(\omega_n^b) \in H_{cb}^n(\Gamma, \mathbb{R}) \cong H_b^n(M)$ as a class in $H_b^n(N, \partial N) \cong H_b^n(M, M - N)$; recall that singular bounded cohomology is a homotopy invariant (cf. section D.3). We have

$$M - N = \bigsqcup_{i=1}^{k} E_i$$

and each cusp E_i is homeomorphic to $V_i \times \mathbb{R}$, where V_i is a compact euclidean manifold. By Bieberbach's Theorem I.6.7 each V_i is finitely covered by a torus and hence has a virtually abelian and therefore amenable fundamental group $\pi_1(V_i)$ (cf. Proposition C.1.7). Again by Corollary II.3.12 we have $H_b^\bullet(E_i) \cong H_b^\bullet(\pi_1(V_i \times \mathbb{R})) \cong H_b^\bullet(\pi_1(V_i)) = 0$ and hence

$$H_b^\bullet(M - N) = H_b^\bullet(\sqcup_{i=1}^k E_i) \cong \bigoplus_{i=1}^{k} H_b^\bullet(E_i) = 0.$$

Therefore we really get an isomorphism in the middle and may interpret $\rho^*(\omega_n^b) \in H_b^n(N, \partial N)$.

Applying the comparison map $c : H_b^n(N, \partial N) \to H^n(N, \partial N)$ and evaluating on the relative fundamental class $[N, \partial N] \in H_n(N, \partial N)$ then gives the definition of the volume of ρ

$$\mathrm{Vol}(\rho) = \langle c(\rho^*(\omega_n^b)), [N, \partial N]\rangle.$$

Since any two compact cores are homotopically equivalent this definition does not depend on the choice of N. It clearly coincides with the one for compact M, since the middle map is then simply the identity due to $N = M$.

The case of torsion-free $\Gamma < G^+$ being settled, we only need to define $\mathrm{Vol}(\rho)$ in the case that Γ has torsion. By Selberg's Lemma we can find a torsion-free subgroup $\Lambda < \Gamma$ with finite index $|\Gamma : \Lambda| < \infty$ (cf. Proposition I.4.21). We now set

$$\mathrm{Vol}(\rho) := \frac{\mathrm{Vol}(\rho|_\Lambda)}{|\Gamma : \Lambda|}.$$

However this is apriori not well-defined, since it seems to depend on the choice of the finite-index torsion-free subgroup $\Lambda < \Gamma$. The next lemma will show, that the definition is indeed independent of the choice of Λ.

Lemma III.2.1. *Let $i : \Lambda' \hookrightarrow \Gamma$ be a torsion-free subgroup and $j : \Lambda \hookrightarrow \Lambda'$ a torsion-free subgroup of Λ'. Then*

$$\mathrm{Vol}(\rho \circ i \circ j) = |\Lambda : \Lambda'| \cdot \mathrm{Vol}(\rho \circ i)$$

It follows that $\mathrm{Vol}(\rho)$ *is well-defined also for non-torsion-free* Γ.

Proof. By definition we have

$$\mathrm{Vol}(\rho \circ i \circ j) = \langle (c \circ j^* \circ i^* \circ \rho^*)(\omega_n^b), [N_\Lambda, \partial N_\Lambda]\rangle$$

where $[N, \partial N]$ is the fundamental class of a compact core N of $M = \Lambda \backslash \mathbb{H}^n$.

Let $p : \Lambda\backslash\mathbb{H}^n = M \to \Lambda'\backslash\mathbb{H}^n = M'$ be the canonical $|\Lambda : \Lambda'|$-sheeted covering map. As we have already seen the pullback $j^* : H_{cb}^n(\Lambda', \mathbb{R}) \to H_{cb}^n(\Lambda, \mathbb{R})$ is realized in singular bounded cohomology by the pullback via p, that is $p^* : H_b^n(\Lambda'\backslash\mathbb{H}^n, \mathbb{R}) \to H_b^n(\Lambda\backslash\mathbb{H}^n, \mathbb{R})$ (cf. Corollary II.2.36). Further the pullback commutes with the comparison map by definition and hence

$$\langle (c \circ p^* \circ i^* \circ \rho^*)(\omega_n^b), [N, \partial N]\rangle = \langle (p^* \circ c \circ i^* \circ \rho^*)(\omega_n^b), [N, \partial N]\rangle.$$

Using the fact that $p : M \to M'$ is a locally isometric covering one easily verifies, that $p^{-1}(M'_{[\varepsilon,\infty)}) \subset M_{[\varepsilon,\infty)}$, where ε is smaller than the n-th Margulis constant (cf. section I.6). This implies, that $M'_{[\varepsilon,\infty)} \subset p(M_{[\varepsilon,\infty)})$. Let us further choose $0 < \varepsilon \leq \varepsilon_n$ so small that a compact core of M resp. M' deformation retracts to $M_{[\varepsilon,\infty)}$ resp. $M'_{[\varepsilon,\infty)}$. We may now find two compact cores $M_{[\varepsilon,\infty)} \subset N \subset M$ and $M'_{[\varepsilon,\infty)} \subset N' \subset M'$, such that

$$M'_{[\varepsilon,\infty)} \subset p(M_{[\varepsilon,\infty)}) \subset p(N) \subset N'.$$

For such N and N' we get a homotopy equivalence of pairs $(p(N), p(\partial N)) \to (N', \partial N')$ and thus we may think of p as a map from $(N, \partial N)$ to $(N', \partial N')$.

We now compute

$$\begin{aligned}
\langle (c \circ j^* \circ i^* \circ \rho^*)(\omega_n^b), [N, \partial N]\rangle &= \langle (c \circ p^* \circ i^* \circ \rho^*)(\omega_n^b), [N, \partial N]\rangle \\
&= \langle (p^* \circ c \circ i^* \circ \rho^*)(\omega_n^b), [N, \partial N]\rangle \\
&= \langle (c \circ i^* \circ \rho^*)(\omega_n^b), p_*([N, \partial N])\rangle \\
&= \langle (c \circ i^* \circ \rho^*)(\omega_n^b), |\Lambda : \Lambda'| \cdot [N', \partial N']\rangle,
\end{aligned}$$

where we have used that the degree of a covering map is the number of its sheets. The last line is equal to $|\Lambda : \Lambda'| \cdot \mathrm{Vol}(\rho \circ i)$ and we have proven the assertion

$$\mathrm{Vol}(\rho \circ i \circ j) = |\Lambda : \Lambda'| \cdot \mathrm{Vol}(\rho \circ i).$$

Let us now conclude that $\mathrm{Vol}(\rho)$ is well-defined. For that let $i_1 : \Lambda_1 \hookrightarrow \Gamma$ and $i_2 : \Lambda_2 \hookrightarrow \Gamma$ be two finite-index torsion-free subgroups. It is a basic algebraic fact, that we can find a common torsion-free subgroup of finite index $\Lambda < \Lambda_1, \Lambda_2$ with canonical embeddings $j_i : \Lambda \hookrightarrow \Lambda_i (i = 1, 2)$. Since Λ is a common subgroup we clearly have that $i_1 \circ j_1 = i_2 \circ j_2$. Using the just proven relation we obtain

$$\begin{aligned}
\frac{\mathrm{Vol}(\rho|_{\Lambda_1})}{|\Lambda_1 : \Gamma|} &= \frac{\mathrm{Vol}(\rho \circ i_1)}{|\Lambda_1 : \Gamma|} = \frac{\mathrm{Vol}(\rho \circ i_1 \circ j_1)}{|\Lambda : \Lambda_1| \cdot |\Lambda_1 : \Gamma|} \\
&= \frac{\mathrm{Vol}(\rho \circ i_2 \circ j_2)}{|\Lambda : \Lambda_2| \cdot |\Lambda_2 : \Gamma|} = \frac{\mathrm{Vol}(\rho \circ i_2)}{|\Lambda_2 : \Gamma|} \\
&= \frac{\mathrm{Vol}(\rho|_{\Lambda_2})}{|\Lambda_2 : \Gamma|}
\end{aligned}$$

We used in the equality from the first to the second line the fact that

$$|\Lambda : \Lambda_1| \cdot |\Lambda_1 : \Gamma| = |\Lambda : \Gamma| = |\Lambda : \Lambda_2| \cdot |\Lambda_2 : \Gamma|$$

□

III.2.2. Transfer Maps and Relative Cohomology

Before we can show some properties of $\mathrm{Vol}(\cdot)$ we need to introduce the so called transfer maps in cohomology. There will occur several of these maps in the following depending on which cochain complex we are considering, but they all follow essentially the same idea of orbit averaging. The upshot is, that they all "behave well" and commute in a sense that is specified in Proposition III.2.6 and its proof.

In an exception to the rest of this chapter we keep the notation of the definition of $\mathrm{Vol}(\cdot)$ here. That means $\Gamma < G^+$ is a **torsion-free** lattice subgroup and $\rho : \Gamma \to G^+$ a representation, i.e. a (continuous) homomorphism. Further we denote by $M = \Gamma\backslash\mathbb{H}^n$ the finite volume hyperbolic quotient manifold and by $N \subset M$ a compact core of M. That is N is a compact submanifold of M with boundary and its complement $M - N$ is the disjoint union of finitely many cusps $E_1, \dots, E_k$.

The Transfer Map $\mathrm{trans}_\Gamma \colon H^\bullet_{cb}(\Gamma, \mathbb{R}) \to H^\bullet_{cb}(G, \mathbb{R}_\varepsilon)$

We define the transfer map at the cochain level as a map

$$\mathrm{trans}_\Gamma \colon F_q^\Gamma \to F_q^G$$

where F_q is one of $C_b((\mathbb{H}^n)^{q+1}, \mathbb{R})$, $L^\infty((\mathbb{H}^n)^{q+1}, \mathbb{R})$, $L^\infty((\partial\mathbb{H}^n)^{q+1}, \mathbb{R})$ or $L^\infty(G^{q+1}, \mathbb{R})$ (the definition will be the same in all cases). For $c \in F_q^\Gamma$ we set

$$\mathrm{trans}_\Gamma(c)(x_0, \dots, x_q) := \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot c(\dot{g}x_0, \dots, \dot{g}x_q)\ d\mu(\dot{g}) \tag{III.1}$$

where μ is the invariant measure on $\Gamma\backslash G$ normalized such that $\mu(\Gamma\backslash G) = 1$. Note that this construction is only possible because Γ is a lattice and hence admits such an invariant probability measure.

Proposition III.2.2. *The map* trans_Γ *as above is well-defined and induces a map in cohomology*

$$\mathrm{trans}_\Gamma \colon H^\bullet_{cb}(\Gamma, \mathbb{R}) \to H^\bullet_{cb}(G, \mathbb{R}_\varepsilon)$$

which is norm non-increasing and a left-inverse of $i^* \colon H^\bullet_{cb}(G, \mathbb{R}_\varepsilon) \to H^\bullet_{cb}(\Gamma, \mathbb{R})$, *i.e.*

$$\mathrm{trans}_\Gamma \circ i^* = \mathrm{id}$$

Remark III.2.3. *Transfer maps play also a role in the broader context of continuous bounded cohomology as left-inverses of pullbacks. For more details we refer to [Mon01].*

Proof. Let $q \in \mathbb{N}_0$. We check that trans_Γ is indeed well-defined. This encompasses several steps. First, we need to see, that trans_Γ indeed maps into the prescribed spaces. We will only show this for $\mathrm{trans}_\Gamma : C_b((\mathbb{H}^n)^{q+1}, \mathbb{R})^\Gamma \to C_b((\mathbb{H}^n)^{q+1}, \mathbb{R}_\varepsilon)^G$. The other cases can be treated similarly.

Let $c \in C_b((\mathbb{H}^n)^{q+1}, \mathbb{R})^\Gamma$. We want to show, that $\mathrm{trans}_\Gamma(c)$ is indeed continuous. This will follow from the theorem on parameter integrals as it is treated in any textbook on Lebesgue integration. For that consider the map

$$F : (\mathbb{H}^n)^{q+1} \times \Gamma\backslash G \to \mathbb{R}$$

given by

$$F((x_0, \dots, x_q), \dot{g}) = \varepsilon(\dot{g}^{-1}) \cdot c(\dot{g}x_0, \dots, \dot{g}x_q)$$

for every $(x_0, \dots, x_q) \in (\mathbb{H}^n)^{q+1}$, $\dot{g} \in \Gamma\backslash G$. This function is clearly well-defined, since c is Γ-invariant and $\Gamma < G^+$ such that also ε is Γ-invariant. Moreover one readily checks, that it is measurable in the second argument for every $(x_0, \dots, x_q) \in (\mathbb{H}^n)^{q+1}$ and continuous in the first argument for every $\dot{g} \in \Gamma\backslash G$. Finally

$$|F((x_0, \dots, x_q), \dot{g})| = |c(\dot{g}x_0, \dots, \dot{g}x_q)| \leq \|c\| < \infty$$

for every $(x_0, \dots, x_q) \in (\mathbb{H}^n)^{q+1}$, $\dot{g} \in \Gamma\backslash G$. This shows that it is also integrable for every $(x_0, \dots, x_q) \in (\mathbb{H}^n)^{q+1}$ as $\Gamma\backslash G$ has finite measure and that there is an integrable upper bound given by the constant function $h : \Gamma\backslash G \to \mathbb{R}, \dot{g} \mapsto \|c\|$. From the theorem on parameter integrals we deduce that trans_Γ is indeed continuous.

$\mathrm{trans}_\Gamma(c)$ is also bounded as one computes directly:

$$\left| \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot c(\dot{g}x_0, \dots, \dot{g}x_q)\, d\mu(\dot{g}) \right| \leq \int_{\Gamma\backslash G} \|c\|\, d\mu(\dot{g}) = \|c\| \tag{III.2}$$

for every $(x_0, \dots, x_q) \in (\mathbb{H}^n)^{q+1}$, because of the normalization $\mu(\Gamma\backslash G) = 1$.

We now check, that $\mathrm{trans}_\Gamma(c)$ is in fact G-equivariant, i.e. $\mathrm{trans}_\Gamma(c) \in C_b((\mathbb{H}^n)^{q+1}, \mathbb{R}_\varepsilon)^G$. This follows from the invariance of the measure μ as we may compute

$$\begin{aligned}
(g \cdot \mathrm{trans}_\Gamma(c))(x_0, \dots, x_q) &= \varepsilon(g) \cdot \mathrm{trans}_\Gamma(g^{-1}x_0, \dots, g^{-1}x_q) \\
&= \int_{\Gamma\backslash G} \varepsilon(g)\varepsilon(\dot{g}^{-1}) \cdot c(\dot{g}g^{-1}x_0, \dots, \dot{g}g^{-1}x_q)\, d\mu(\dot{g}) \\
&= \int_{\Gamma\backslash G} \varepsilon((\dot{g}g^{-1})^{-1}) \cdot c(\dot{g}g^{-1}x_0, \dots, \dot{g}g^{-1}x_q)\, d\mu(\dot{g}) \\
&= \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot c(\dot{g}x_0, \dots, \dot{g}x_q)\, d\mu(\dot{g}) \\
&= \mathrm{trans}_\Gamma(c)(x_0, \dots, x_q)
\end{aligned}$$

for every $(x_0, \dots, x_q) \in (\mathbb{H}^n)^{q+1}$, $g \in G$. Hence trans_Γ is indeed G-equivariant.

Let us now show, that $\mathrm{trans}_\Gamma : F_q^\Gamma \to F_q^G$ is indeed a map of complexes, i.e. commutes with the homogeneous coboundary operator. This follows from the next direct computation.

$$\begin{aligned}(d\,\mathrm{trans}_\Gamma)(x_0,\dots,x_{q+1}) &= \sum_{i=0}^{q+1}(-1)^i \mathrm{trans}_\Gamma(c)(x_0,\dots,\hat{x}_i,\dots,x_{q+1}) \\ &= \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1})\cdot \sum_{i=0}^{q+1}(-1)^i c(\dot{g}x_0,\dots,\dot{g}x_{i-1},\dot{g}x_{i+1},\dots,\dot{g}x_{q+1})\, d\mu(\dot{g}) \\ &= \mathrm{trans}_\Gamma(dc)(x_0,\dots,x_{q+1})\end{aligned}$$

We need to see that all these definitions for the various possible F_q result in the same map in cohomology. We will use the concrete isomorphisms from Proposition II.3.13 to $H^\bullet(L^\infty(G^{q+1},\mathbb{R}_\pi)^H)$ where $H=\Gamma$ or G and $\pi\equiv 1$ or ε respectively. As an example we will show, that the following diagram commutes

$$\begin{array}{ccc} L^\infty(G^{\bullet+1},\mathbb{R})^\Gamma & \xrightarrow{\mathrm{trans}_\Gamma} & L^\infty(G^{\bullet+1},\mathbb{R}_\varepsilon)^G \\ \uparrow p_K^* & & \uparrow p_K^* \\ L^\infty((\mathbb{H}^n)^{\bullet+1},\mathbb{R})^\Gamma & \xrightarrow{\mathrm{trans}_\Gamma} & L^\infty((\mathbb{H}^n)^{\bullet+1},\mathbb{R}_\varepsilon)^G \end{array}$$

All other different possibilities for F_q are treated in the very same way and the computations are almost identical. Let $c\in L^\infty((\mathbb{H}^n)^{q+1})^\Gamma$, $(g_0,\dots,g_q)\in G^{q+1}$ and $x\in\mathbb{H}^n$. Then we have

$$\begin{aligned}\mathrm{trans}_\Gamma(p_K^*(c))(g_0,\dots,g_q) &= \int_{\Gamma\backslash G}\varepsilon(\dot{g}^{-1})\cdot p_K^*(c)(\dot{g}g_0,\dots,\dot{g}g_q)\, d\mu(\dot{g}) \\ &= \int_{\Gamma\backslash G}\varepsilon(\dot{g}^{-1})\cdot c(\dot{g}g_0x,\dots,\dot{g}g_qx)\, d\mu(\dot{g})\end{aligned}$$

On the other hand

$$\begin{aligned}p_K^*(\mathrm{trans}_\Gamma(c))(g_0,\dots,g_q) &= \mathrm{trans}_\Gamma(c)(g_0x,\dots,g_qx) \\ &= \int_{\Gamma\backslash G}\varepsilon(\dot{g})\cdot c(\dot{g}g_0x,\dots,\dot{g}g_qx)\, d\mu(\dot{g})\end{aligned}$$

which shows that they coincide and that the above diagram is commutative. Hence all definitions of trans_Γ result in the same map in cohomology.

Finally we need to check, that trans_Γ is norm non-increasing and a left-inverse of the pullback $i^*\colon H^\bullet_{cb}(G,\mathbb{R}_\varepsilon)\to H^\bullet_{cb}(\Gamma,\mathbb{R})$. Because all discussed resolutions yield isometrically isomorphic cohomologies, we may work in any of them to check, that the map trans_Γ is norm non-increasing. We use $C_b((\mathbb{H}^n)^{\bullet+1},\mathbb{R})$, because we have already computed the estimate (III.2), which shows that the transfer map $\mathrm{trans}_\Gamma : C_b((\mathbb{H}^n)^{\bullet+1},\mathbb{R})^\Gamma \to C_b((\mathbb{H}^n)^{\bullet+1},\mathbb{R}_\varepsilon)^G$ is norm non-increasing at the cochain level. It now follows for the cohomology

$$\begin{aligned}\|\mathrm{trans}_\Gamma([c])\| &= \inf\{\|f\| : f\in[\mathrm{trans}_\Gamma(c)]\} \\ &\le \inf\{\|\mathrm{trans}_\Gamma(c')\| : c'\in[c]\} \\ &\le \inf\{\|c'\| : c'\in[c]\} \\ &= \|[c]\|\end{aligned}$$

for every $[c] \in H^\bullet(C_b((\mathbb{H}^n)^{\bullet+1}, \mathbb{R})^\Gamma)$, i.e. trans_Γ is also at the cohomology level norm non-increasing.

The pullback $i^*\colon H^\bullet_{cb}(G, \mathbb{R}_\varepsilon) \to H^\bullet_{cb}(\Gamma, \mathbb{R})$ corresponding to the canonical inclusion $i : \Gamma \to G$ is given at the cochain level by the inclusion

$$\iota : L^\infty(G^{\bullet+1}, \mathbb{R}_\varepsilon)^G \to L^\infty(G^{\bullet+1}, \mathbb{R})^\Gamma$$

as we have already shown in Corollary II.2.35. Let $c \in L^\infty(G^{\bullet+1}, \mathbb{R}_\varepsilon)^G$. Then

$$\begin{aligned}
\mathrm{trans}_\Gamma(\iota(c))(g_0, \ldots, g_q) &= \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot c(\dot{g}g_0, \ldots, \dot{g}g_q)\, d\mu(\dot{g}) \\
&= \int_{\Gamma\backslash G} c(g_0, \ldots, g_q)\, d\mu(\dot{g}) \\
&= c(g_0, \ldots, g_q)
\end{aligned}$$

for every $(g_0, \ldots, g_q) \in G^{q+1}$, because c is G-equivariant. This already shows at the cochain level, that trans_Γ is a left-inverse of the pullback. $\square$

The Transfer Map $\tau_{\mathrm{dR}}\colon H^\bullet(N, \partial N) \to H^\bullet_c(G, \mathbb{R}_\varepsilon)$

A similar transfer map can be given from $H^\bullet(N, \partial N)$ to $H^\bullet_c(G, \mathbb{R}_\varepsilon)$. Recall that by homotopy invariance $H^\bullet(N, \partial N) \cong H^\bullet(M, M - N)$ and by de Rham's theorem $H^\bullet(M, M-N) \cong H^\bullet_{\mathrm{dR}}(M, M-N)$ (cf. Theorem D.4.2). By lifting forms along the universal covering $\pi : \mathbb{H}^n \to \Gamma\backslash\mathbb{H}^n \cong M$ we can further identify $H^\bullet_{\mathrm{dR}}(M, M-N) \cong H^\bullet(\Omega^\bullet(\mathbb{H}^n, U)^\Gamma)$, where $U = \pi^{-1}(M - N)$ is a disjoint union of horoballs.

Let $q \in \mathbb{N}_0$. We define the transfer map at the cochain level by sending a differential q-form $\alpha \in \Omega^q(\mathbb{H}^n, U)^\Gamma$ to the form $\mathrm{trans}_{\mathrm{dR}}(\alpha) \in \Omega^\bullet(\mathbb{H}^n, \mathbb{R}_\varepsilon)^G$ which is given pointwise by

$$\mathrm{trans}_{\mathrm{dR}}(\alpha)(v_1, \ldots, v_q) \coloneqq \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot (\dot{g}^*\alpha)(v_1, \ldots, v_q)\, d\mu(\dot{g})$$

for every $x \in \mathbb{H}^n$ and $v_1, \ldots, v_q \in T_x\mathbb{H}^n$. Here μ is again as in (III.1).

Remark III.2.4. *In fact one can use a more abstract integration theory for functions with values in a LCTVS to define the transfer map directly as*

$$\mathrm{trans}_{\mathrm{dR}}(\alpha) = \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot (\dot{g}^*\alpha)\, d\mu(\dot{g})$$

However the result would be the same as for our pointwise definition as above, which is why we prefer our basic approach.

Proposition III.2.5. *The map* $\mathrm{trans}_{\mathrm{dR}} : \Omega^\bullet(\mathbb{H}^n, U)^\Gamma \to \Omega^\bullet(\mathbb{H}^n, \mathbb{R}_\varepsilon)^G$ *is well-defined and induces a map in cohomology* τ_{dR} *via the following diagram*

$$\begin{tikzcd}
H^\bullet(N, \partial N) \arrow[rr, dashed, "\tau_{\mathrm{dR}}"] & & H^\bullet_c(G, \mathbb{R}_\varepsilon) \\
H^\bullet(\Omega^\bullet(\mathbb{H}^n, U)^\Gamma) \arrow[u, "\cong"] \arrow[r, "\mathrm{trans}_{\mathrm{dR}}"] & H^\bullet(\Omega^\bullet(\mathbb{H}^n, \mathbb{R}_\varepsilon)^G) \arrow[r, "="] & \Omega^\bullet(\mathbb{H}^n, \mathbb{R}_\varepsilon)^G \arrow[u, "\cong"']
\end{tikzcd}$$

where the left vertical arrow is the previously discussed identification and the right vertical arrow denotes the van Est isomorphism.

Moreover we have

$$\mathrm{trans}_{\mathrm{dR}}(\omega_{N,\partial N}) = \omega_n \in \Omega^n(\mathbb{H}^n, \mathbb{R}_\varepsilon)^G \cong H^n_c(G, \mathbb{R}_\varepsilon)$$

where $\omega_{N,\partial N} \in H^n(M, M-N)$ is the unique class with $\langle \omega_{N,\partial N}, [N, \partial N]\rangle = \mathrm{Vol}(M)$ and ω_n the hyperbolic volume form. In particular τ_{dR} is an isomorphism in top degree.

Proof. First we want to see that the map is well defined, i.e. the integral exists and defines a G-equivariant differential q-form.

Let $q \in \mathbb{N}_0$, $\alpha \in \Omega^q(\mathbb{H}^n, U)^\Gamma \subset \Omega^q(\mathbb{H}^n)$ and let $V_1, \ldots, V_q \in \mathcal{V}(\mathbb{H}^n)$ be smooth vectorfields on $\mathbb{H}^n$. Denote by $F : \Gamma\backslash G \times \mathbb{H}^n \to \mathbb{R}$ the map given by

$$F(\dot g, x) = \varepsilon(\dot g^{-1}) \cdot (\dot g^* \alpha)(V_1(x), \ldots, V_q(x))$$

for every $\dot g \in \Gamma\backslash G$, $x \in \mathbb{H}^n$. Note that this map is well-defined, since α is Γ-invariant and ε is too as $\Gamma < G^+$.

Clearly the map is smooth, since it lifts via the covering $G \times \mathbb{H}^n \to \Gamma\backslash G \times \mathbb{H}^n$ to a smooth function; note that the given map is a covering, since Γ acts freely and properly discontinuously on $G \times \mathbb{H}^n$. We want to apply the theorem on differentiable paramter integrals to deduce that the integral exists and defines in fact a smooth function. Let $U_i \subset \mathbb{H}^n$ be a sequence of open subsets such that

$$\overline{U}_i \subset U_{i+1} \text{ is compact for every } i \in \mathbb{N} \text{ and } \bigcup_{i\in\mathbb{N}} U_i = \mathbb{H}^n.$$

We will show, that $\mathrm{trans}_{\mathrm{dR}}(\alpha)(V_1, \ldots, V_q)$ is smooth on U_i for every $i \in \mathbb{N}$. Let $i \in \mathbb{N}$. By identifying $\Omega^\bullet(M, M-N) \cong \Omega^\bullet(\mathbb{H}^n, U)^\Gamma$ we see that $F(\dot g, x) = 0$ if $\pi(gx) \notin N \subset M$. By Proposition I.4.8 the map $\psi : \Gamma\backslash G \times \mathbb{H}^n \to M \times \mathbb{H}^n$, $(\dot g, x) \mapsto (\pi(gx), x)$ is proper and thus the set

$$C_i = \{\dot g \in \Gamma\backslash G : \exists x \in \overline{U}_i \text{ s.t. } \pi(gx) \in N\} = \mathrm{pr}_1 \psi^{-1}(N \times \overline{U}_i)$$

is compact. Therefore the integral is already realized by integration over C_i for every $x \in \overline{U}_i$:

$$\mathrm{trans}_{\mathrm{dR}}(\alpha)(V_1(x), \ldots, V_q(x)) = \int_{C_i} F(\dot g, x)\, d\mu(\dot g)$$

By smoothness of F it now follows easily, that every derivative is uniformly bounded on $U_i \subset \overline{U}_i$. The theorem on differentiable parameter integrals implies, that $\mathrm{trans}_{\mathrm{dR}}(\alpha)(V_1, \ldots, V_q)$ is indeed smooth on U_i. Because $i \in \mathbb{N}$ was arbitrary, we get that $\mathrm{trans}_{\mathrm{dR}}(\alpha)(V_1, \ldots, V_q)$ is smooth on $\mathbb{H}^n$.

It is clear, from the definition that $\mathrm{trans}_{\mathrm{dR}}(\alpha)_x \in \mathrm{Alt}^q(T_x\mathbb{H}^n)$ for every $x \in \mathbb{H}^n$. In summary $\mathrm{trans}_{\mathrm{dR}}(\alpha)$ is a differential q-form.

An easy computation shows, that it is also G-equivariant. Indeed, let $x \in \mathbb{H}^n$, $v_1, \ldots, v_q \in T_x\mathbb{H}^n$ and $g \in G$, then

$$\begin{aligned}
(g \cdot \mathrm{trans}_{\mathrm{dR}}(\alpha))(v_1, \ldots, v_q) &= \varepsilon(g) \cdot ((g^{-1})^* \mathrm{trans}_{\mathrm{dR}}(\alpha))(v_1, \ldots, v_q) \\
&= \varepsilon(g) \cdot \mathrm{trans}_{\mathrm{dR}}(\alpha)(dg^{-1}(v_1), \ldots, dg^{-1}(v_q)) \\
&= \int_{\Gamma\backslash G} \varepsilon((\dot g g^{-1})^{-1}) \cdot ((\dot g g^{-1})^* \alpha)(v_1, \ldots, v_q)\, d\mu(\dot g) \\
&= \int_{\Gamma\backslash G} \varepsilon(\dot g) \cdot (\dot g^* \alpha)(v_1, \ldots, v_q)\, d\mu(\dot g) \\
&= \mathrm{trans}_{\mathrm{dR}}(\alpha)(v_1, \ldots, v_q).
\end{aligned}$$

Moreover $\mathrm{trans}_{\mathrm{dR}}$ is a morphism of complexes, i.e. commutes with exterior differentiation. The key point is that we may differentiate under the integral by the theorem on differentiable parameter integrals. We will use the invariant formula for the exterior derivative of a q-form

$$\begin{aligned} d\alpha(V_0,\dots,V_q) = &\sum_{i=0}^{q}(-1)^i V_i(\alpha(V_0,\dots,\hat{V}_i,\dots,V_q)) \\ &+\sum_{i<j}(-1)^{i+j}\alpha([V_i,V_j],V_0,\dots,\hat{V}_i,\dots,\hat{V}_j,\dots,V_q) \end{aligned}$$

for all $V_0,\dots,V_q \in \mathcal{V}(\mathbb{H}^n)$, where a hat indicates omission of the variable underneath (cf. [Lee13, Proposition 14.32, p. 370]).

$$\begin{aligned} (d\mathrm{trans}_{\mathrm{dR}}(\alpha))(V_0,\dots,V_q) &= \sum_i(-1)^i V_i\left(\mathrm{trans}_{\mathrm{dR}}(\alpha)(V_0,\dots,\hat{V}_i,\dots,V_q)\right) \\ &\quad+\sum_{i<j}(-1)^{i+j}\mathrm{trans}_{\mathrm{dR}}(\alpha)([V_i,V_j],V_0,\dots,\hat{V}_i,\dots,\hat{V}_j,\dots,V_q) \\ &= \sum_i(-1)^i V_i\left(\int_{\Gamma\backslash G}\varepsilon(\dot{g}^{-1})(\dot{g}^*\alpha)(V_0,\dots,\hat{V}_i,\dots,V_q)\,d\mu(\dot{g})\right) \\ &\quad+\sum_{i<j}(-1)^{i+j}\left(\int_{\Gamma\backslash G}\varepsilon(\dot{g}^{-1})(\dot{g}^*\alpha)([V_i,V_j],V_0,\dots,\hat{V}_i,\dots,\hat{V}_j,\dots,V_q)\,d\mu(\dot{g})\right) \\ &= \int_{\Gamma\backslash G}\left(\varepsilon(\dot{g}^{-1})\sum_i(-1)^i V_i\left((\dot{g}^*\alpha)(V_0,\dots,\hat{V}_i,\dots,V_q)\right)\right. \\ &\quad\left.+\sum_{i<j}(-1)^{i+j}\varepsilon(\dot{g}^{-1})(\dot{g}^*\alpha)([V_i,V_j],V_0,\dots,\hat{V}_i,\dots,\hat{V}_j,\dots,V_q)\right)d\mu(\dot{g}) \end{aligned}$$

On the other hand

$$\begin{aligned} (\mathrm{trans}_{\mathrm{dR}}(d\alpha))(V_0,\dots,V_q) &= \int_{\Gamma\backslash G}\varepsilon(\dot{g}^{-1})\cdot\dot{g}^*(d\alpha)(V_0,\dots,V_q)\,d\mu(\dot{g}) \\ &= \int_{\Gamma\backslash G}\varepsilon(\dot{g}^{-1})\cdot d(\dot{g}^*\alpha)(V_0,\dots,V_q)\,d\mu(\dot{g}) \\ &= \int_{\Gamma\backslash G}\left(\varepsilon(\dot{g}^{-1})\sum_i(-1)^i V_i\left((\dot{g}^*\alpha)(V_0,\dots,\hat{V}_i,\dots,V_q)\right)\right. \\ &\quad\left.+\sum_{i<j}(-1)^{i+j}\varepsilon(\dot{g}^{-1})(\dot{g}^*\alpha)([V_i,V_j],V_0,\dots,\hat{V}_i,\dots,\hat{V}_j,\dots,V_q)\right)d\mu(\dot{g}) \end{aligned}$$

This shows that $d\mathrm{trans}_{\mathrm{dR}} = \mathrm{trans}_{\mathrm{dR}}d$ as asserted.

Finally we need to show, that $\mathrm{trans}_{\mathrm{dR}}(\omega_{N,\partial N}) = \omega_n$ where $\omega_{N,\partial N}$ is the unique singular cohomology class such that $\langle\omega_{N,\partial N},[N,\partial N]\rangle = \mathrm{Vol}(M)$. First of all we may represent $\omega_{N,\partial N}$ by the cohomology class of a top form in $\Omega^n(M, M-N)$. The volume form v_n on M is a nowhere vanishing top form on M and hence we may write the representative of $\omega_{N,\partial N}$ as $f\cdot v_n$, where $f: M\to\mathbb{R}$ is a smooth function vanishing on $M-N$. Lifting this form via $\pi:\mathbb{H}^n\to M=\Gamma\backslash\mathbb{H}^n$ yields

$\pi^*(f \cdot v_n) = f \circ \pi \cdot \omega_n$. Therefore we are left with the following computation. Let $x \in \mathbb{H}^n$ and $v_1, \ldots, v_n \in T_x\mathbb{H}^n$.

$$\begin{aligned}
\mathrm{trans}_{\mathrm{dR}}(f \circ \pi \cdot \omega_n)(v_1, \ldots, v_n) &= \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot \dot{g}^*(f \circ \pi \cdot \omega_n)(v_1, \ldots, v_n)\, d\mu(\dot{g}) \\
&= \int_{\Gamma\backslash G} f(\pi(\dot{g}x)) \cdot \varepsilon(\dot{g}^{-1}) \cdot (\dot{g}^*\omega_n)(v_1, \ldots, v_n)\, d\mu(\dot{g}) \\
&= \int_{\Gamma\backslash G} f(\pi(\dot{g}x)) \cdot \omega_n(v_1, \ldots, v_n)\, d\mu(\dot{g}) \\
&= \int_{\Gamma\backslash G} f(\pi(\dot{g}x))\, d\mu(\dot{g}) \cdot \omega_n(v_1, \ldots, v_n)
\end{aligned}$$

where we have used the fact, that $\omega_n \in \Omega^n(\mathbb{H}^n, \mathbb{R}_\varepsilon)^G$ is G-equivariant.

Denote by $K = G_x$ the stabilizer of $x \in \mathbb{H}^n$. Note that we can normalize the Haar measure κ on K such that

$$\int_{\Gamma\backslash G} \varphi(\dot{g})\, d\mu(\dot{g}) = \int_{\Gamma\backslash G/K} \int_K \varphi(\bar{g}k)\, d\kappa(k)\, dv(\bar{g})$$

for every $\varphi \in C_c(\Gamma\backslash G)$, where v corresponds to the measure on $\Gamma\backslash G/K$ induced by the volume form v_n on M under the identification $\Gamma\backslash G/K \to M, \bar{g} \mapsto \pi(\bar{g}x)$ (cf. Lemma A.4.11 and Proposition I.4.11, or [Rat06, p. 574]). Because $\mu(\Gamma\backslash G) = 1$ and $v(\Gamma\backslash G/K) = \mathrm{Vol}(M)$, we compute the normalization constant via

$$1 = \int_{\Gamma\backslash G} 1\, d\mu(\dot{g}) = \int_{\Gamma\backslash G/K} \int_K 1\, d\kappa(k)\, d\mu(\bar{g}) = \kappa(K) \cdot \mathrm{Vol}(M)$$

Hence we have to choose the unique Haar measure on K, which gives $\kappa(K) = \mathrm{Vol}(M)^{-1}$.

Now we compute

$$\begin{aligned}
\int_{\Gamma\backslash G} f(\pi(\dot{g}x))\, d\mu(\dot{g}) &= \int_M \int_K f(\pi(\bar{g}kx))\, d\kappa(k)\, dv(\bar{g}) \\
&= \kappa(K) \cdot \int_M f(\pi(\bar{g}x))\, dv(\bar{g}) \\
&= \mathrm{Vol}(M)^{-1} \cdot \int_M f(y)\, dv(y) \\
&= \mathrm{Vol}(M)^{-1} \cdot \int_M f \cdot v_n \\
&= \mathrm{Vol}(M)^{-1} \cdot \int_N f \cdot v_n \\
&= \mathrm{Vol}(M)^{-1} \cdot \langle \omega_{N,\partial N}, [N, \partial N] \rangle \\
&= 1
\end{aligned}$$

where we have used Lemma D.4.3.

This shows, that indeed

$$\mathrm{trans}_{\mathrm{dR}}(\omega_{N,\partial N})(v_1, \ldots, v_n) = \omega_n(v_1, \ldots, v_n)$$

Because $x \in \mathbb{H}^n$ and $v_1, \ldots, v_n \in T_x\mathbb{H}^n$ were arbitrary, the assertion follows. $\square$

Commutativity of the Transfer Maps

The following proposition shows, that the previously defined transfer maps actually commute with the comparison map.

Proposition III.2.6 (cf. [BBI13, Proposition 3, p. 12]). *We have the following commutative diagram*

$$\begin{tikzcd}
H^\bullet_{cb}(\Gamma,\mathbb{R}) \arrow[drr, "\mathrm{trans}_\Gamma"] & & \\
H^\bullet_b(N,\partial N) \arrow[u, "\cong"] \arrow[d, "c"'] & & H^\bullet_{cb}(G,\mathbb{R}_\varepsilon) \arrow[d, "c"] \\
H^\bullet(N,\partial N) \arrow[rr, "\tau_{\mathrm{dR}}"] & & H^\bullet_c(G,\mathbb{R}_\varepsilon)
\end{tikzcd}$$

Proof. In order to prove the commutativity of the above diagram we will decompose it into smaller diagrams. We will try to motivate this approach. A sensible first step would be to understand the composition

$$H^q_b(N,\partial N) \cong H^q_b(M,M-N) \xrightarrow{\cong} H^q_b(M) \xrightarrow{\cong} H^q_{cb}(\Gamma,\mathbb{R}) \xrightarrow{\mathrm{trans}_\Gamma} H^q_{cb}(G,\mathbb{R}_\varepsilon)$$

Here the first isomorphism $H^\bullet_b(M,M-N) \cong H^\bullet_b(M)$ is given by inclusion at the cochain level. We already have several concrete expressions for trans_Γ such that we are left to understand $H^\bullet_b(M) \cong H^\bullet_{cb}(\Gamma,\mathbb{R})$. Recall that $H^\bullet_b(M) \cong H^\bullet(S^\bullet_b(\mathbb{H}^n,\mathbb{R})^\Gamma)$ via the pullback along $\pi : \mathbb{H}^n \to M$ (cf. Corollary II.3.12). Our goal is to construct an isomorphism

$$H^\bullet(S^\bullet_b(\mathbb{H}^n,\mathbb{R})^\Gamma) \to H^\bullet(L^\infty((\mathbb{H}^n)^{\bullet+1},\mathbb{R})^\Gamma)$$

at the cochain level. Because both $(\epsilon, S^\bullet_b(\mathbb{H}^n,\mathbb{R}))$ and $(\epsilon, L^\infty((\mathbb{H}^n)^{\bullet+1},\mathbb{R})$ are strong augmented resolutions of the trivial module $\mathbb{R}$ by relatively injective Banach Γ-modules, it will be enough to give an extension of the identity $\mathrm{id} : \mathbb{R} \to \mathbb{R}$ between those resolutions (cf. Lemma II.2.15).

For $1 \le j \le k$, pick a point $b_j \in E_j$ in each cusp of M and $b_0 \in N$ in the compact core. We define the map $\beta' : M \to \{b_0, b_1, \ldots, b_k\}$ as the (measurable) map sending every point in N to b_0 and every point in each cusp E_i to b_i. Now we lift β' to a map $\beta : \mathbb{H}^n \longrightarrow \pi^{-1}(\{b_0,b_1,\ldots,b_k\}) \subset \mathbb{H}^n$ as follows. Choose points $\tilde{b}_j \in \pi^{-1}(b_j)$ for every $j = 0,1,\ldots,k$. Further choose a (measurable) fundamental set $\mathcal{D}_j \ni \tilde{b}_j$ for the Γ-action on $\pi^{-1}(E_j)$ for each $j = 1,\ldots,k$ and – similarly – choose a fundamental set $\mathcal{D}_0 \ni \tilde{b}_0$ for the Γ-action on $\pi^{-1}(N)$. Now define $\beta(\gamma\mathcal{D}_j) := \gamma\tilde{b}_j$ for every $\gamma \in \Gamma$. In particular β maps each horoball $\pi^{-1}(E_i)$ into itself.

This gives rise to the map

$$\beta^* : S^q_b(\mathbb{H}^n,\mathbb{R}) \longrightarrow L^\infty((\mathbb{H}^n)^{q+1},\mathbb{R})$$

defined by

$$\beta^*(c)(x_0,\ldots,x_q) := c(\mathrm{str}(\beta(x_0),\ldots,\beta(x_q)))$$

for every $(x_0,\ldots,x_q) \in (\mathbb{H}^n)^{q+1}$. In fact β^* is the extension we were looking for as the next lemma asserts.

Lemma III.2.7. *The above map β^* is a Γ-morphism of complexes extending* $\mathrm{id} : \mathbb{R} \to \mathbb{R}$ *between the resolutions* $(\epsilon, S_b^\bullet(\mathbb{H}^n, \mathbb{R}))$ *and* $(\epsilon, L^\infty((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}))$. *In particular it induces the isomorphism* $H^\bullet(S_b^\bullet(\mathbb{H}^n, \mathbb{R})^\Gamma) \cong H^\bullet(L^\infty((\mathbb{H}^n)^{\bullet+1}, \mathbb{R})^\Gamma)$.

Proof. Let $q \in \mathbb{N}_0$. It is immediate, that β^* is linear and continuous. In fact $\|\beta^*(c)\| \leq \|c\|$ for every $c \in S_b^q(\mathbb{H}^n, \mathbb{R})$.

Further it is a Γ-morphism because

$$\begin{aligned}\gamma \cdot (\beta^* c)(x_0, \dots, x_q) &= (\beta^* c)(\gamma^{-1}x_0, \dots, \gamma^{-1}x_q)\\ &= c(\mathrm{str}(\beta(\gamma^{-1}x_0), \dots, \beta(\gamma^{-1}x_q)))\\ &= c(\mathrm{str}(\gamma^{-1}\beta(x_0), \dots, \gamma^{-1}\beta(x_q)))\\ &= c(\gamma_*^{-1}\mathrm{str}(\beta(x_0), \dots, \beta(x_q)))\\ &= ((\gamma^{-1})^* c)(\mathrm{str}(\beta(x_0), \dots, \beta(x_q)))\\ &= \beta^*(\gamma \cdot c)(x_0, \dots, x_q)\end{aligned}$$

for every $c \in S_b^q(\mathbb{H}^n, \mathbb{R})$ and all $(x_0, \dots, x_q) \in (\mathbb{H}^n)^{q+1}$.

Finally it is completely straight forward to check, that it is indeed a morphism of complexes and an extension of $\mathrm{id} : \mathbb{R} \to \mathbb{R}$. We leave out the details here and refer to a similar computation we made in section II.3.1 for the van Est isomorphism. $\square$

Due to what we have said before the composition

$$H_b^q(N, \partial N) \xrightarrow{\cong} H_{cb}^q(\Gamma, \mathbb{R}) \xrightarrow{\mathrm{trans}_\Gamma} H_{cb}^q(G, \mathbb{R}_\varepsilon)$$

is given at the cochain level by the map

$$\mathrm{trans}_b = \mathrm{trans}_\Gamma \circ \beta^* : S_b^q(\mathbb{H}^n, U)^\Gamma \longrightarrow L^\infty((\mathbb{H}^n)^{q+1}, \mathbb{R}_\varepsilon)^G$$

That is

$$\mathrm{trans}_b(c)(x_0, \dots, x_q) = \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot (\beta^* c)(\dot{g}x_0, \dots, \dot{g}x_q)\, d\mu(\dot{g})$$

for every $c \in S_b^q(\mathbb{H}^n, U)^\Gamma$ and all $(x_0, \dots, x_q) \in (\mathbb{H}^n)^{q+1}$.

The above transfer map is also defined on the ordinary singular cochain complex

$$\mathrm{trans} : S^q(\mathbb{H}^n, U)^\Gamma \to C((\mathbb{H}^n)^{q+1}, \mathbb{R}_\varepsilon)^G$$

by the very same formula, namely

$$\mathrm{trans}(c)(x_0, \dots, x_q) := \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot (\beta^* c)(\dot{g}x_0, \dots, \dot{g}x_q)\, d\mu(\dot{g})$$

for every $c \in S^q(\mathbb{H}^n, U)^\Gamma$ and all $(x_0, \dots, x_q) \in (\mathbb{H}^n)^{q+1}$.

Lemma III.2.8. *The above map* $\mathrm{trans} : S^q(\mathbb{H}^n, U)^\Gamma \longrightarrow C((\mathbb{H}^n)^{q+1}, \mathbb{R}_\varepsilon)^G$ *is well defined and induces a map in cohomology. Moreover* $\mathrm{trans}_b : S_b^q(\mathbb{H}^n, U)^\Gamma \longrightarrow L^\infty((\mathbb{H}^n)^{q+1}, \mathbb{R}_\varepsilon)^G$ *ranges actually in* $C_b((\mathbb{H}^n)^{q+1}, \mathbb{R}_\varepsilon)^G$ *and we have the following commutative diagram*

$$\begin{tikzcd} S_b^q(\mathbb{H}^n, U)^\Gamma \arrow[r, "\mathrm{trans}_b"] \arrow[d] & C_b((\mathbb{H}^n)^{q+1}, \mathbb{R}_\varepsilon)^G \arrow[d] \\ S^q(\mathbb{H}^n, U)^\Gamma \arrow[r, "\mathrm{trans}"] & C((\mathbb{H}^n)^{q+1}, \mathbb{R}_\varepsilon)^G \end{tikzcd}$$

where both vertical arrows are the canonical inclusions, i.e. induce the comparison map in cohomology.

Proof. Let $c \in S^q(\mathbb{H}^n, U)^\Gamma$.

Intuitively speaking the reason for our gain in regularity is the fact that $\beta^* c$ is locally constant almost everywhere. Indeed, observe that $\beta : \mathbb{H}^n \to \mathbb{H}^n$ is locally constant on $\pi^{-1}(M - \partial N)$, such that also

$$\beta^*(c)(x_0, \ldots, x_q) = c(\operatorname{str}(\beta(x_0), \ldots, \beta(x_q)))$$

is locally constant for $(x_0, \ldots, x_q) \in \pi^{-1}(M - \partial N)$.

Now let $(x_0, \ldots, x_q) \in (\mathbb{H}^n)^{q+1}$ and let $\{(x_0^{(j)}, \ldots, x_q^{(j)})\}_{j \in \mathbb{N}} \subset (\mathbb{H}^n)^{q+1}$ be a sequence converging to it. Consider the set

$$\begin{aligned}\mathcal{L} :=& \{\dot g \in \Gamma \backslash G : \pi(\dot g x_i) \in M - \partial N \quad \forall i = 0, \ldots, q\} \\ & \cup \{\dot g \in \Gamma \backslash G : \pi(\dot g x_i^{(j)}) \in M - \partial N \quad \forall j \in \mathbb{N} \, \forall i = 0, \ldots, q\}\end{aligned}$$

We claim that it has full measure. The complement is

$$\begin{aligned}\mathcal{L}^c =& \bigcup_{i=0}^{q} \{\dot g \in \Gamma \backslash G : \pi(\dot g x_i) \in \partial N\} \\ & \cup \bigcup_{i=0}^{q} \left(\bigcup_{j \in \mathbb{N}} \{\dot g \in \Gamma \backslash G : \pi(\dot g x_i^{(j)}) \in \partial N\} \right) \\ =& \bigcup_{i=0}^{q} q_{x_i}^{-1}(\partial N) \cup \bigcup_{i=0}^{q} \left(\bigcup_{j \in \mathbb{N}} q_{x_i^{(j)}}^{-1}(\partial N) \right)\end{aligned}$$

where $q_y : \Gamma \backslash G \to M, \dot g \mapsto \pi(\dot g y)$ for some $y \in \mathbb{H}^n$. By Proposition I.4.11 a subset $A \subset M$ is a null set if and only if $q_y^{-1}(A) \subset \Gamma \backslash G$ is a null set. Therefore $\mathcal{L}^c$ is indeed a null set as the countable union of such; recall that ∂N is the finite union of codimension one submanifolds and hence a null set.

In summary we get, that the functions

$$F_j(\dot g) := \varepsilon(\dot g^{-1}) \cdot (\beta^* c)(\dot g x_0^{(j)}, \ldots, \dot g x_q^{(j)})$$

converge pointwise to the function

$$F(\dot g) := \varepsilon(\dot g^{-1}) \cdot (\beta^* c)(\dot g x_0, \ldots, \dot g x_q)$$

for every $\dot g \in \mathcal{L}$, i.e. almost everywhere.

In order to see, that also

$$\int_{\Gamma \backslash G} F_j(\dot g) \, d\mu(\dot g) \to \int_{\Gamma \backslash G} F(\dot g) \, d\mu(\dot g)$$

as $j \to \infty$, which is nothing but the continuity of $\operatorname{trans}(c)$ at $(x_0, \ldots, x_n)$, we want to apply Lebesgue's dominated convergence theorem. Therefore it will be sufficient to find an uniform upper bound on the values of $|F_j|$ for all $j \in \mathbb{N}$. In fact we will show, that they adopt only finitely many values.

Set
$$D := \max\{\sup_{j\in\mathbb{N}} \max_{i=1,\dots,q} d(x_0^{(j)}, x_i^{(j)}),\ \max_{i=1,\dots,q} d(x_0, x_i)\} < \infty$$
and consider the compact D-neighborhood N_D of N in M. Let $j \in \mathbb{N}$, $\dot g \in \Gamma\backslash G$ and suppose that $\pi(\dot g x_0^{(j)}) \in E_l - N_D$ is in one of the cusps outside the D-neighborhood of N. Then also $\pi(\dot g x_i^{(j)}) \in E_l$ for all $i = 1,\dots,q$. Furthermore for every $\gamma \in \Gamma$ the points $\gamma g x_0^{(j)},\dots,\gamma g x_q^{(j)}$ are all in the same horoball over E_l. This implies that also $\gamma\,\beta(g x_0^{(j)}),\dots,\gamma\beta(g x_q^{(j)})$ are in the same horoball over E_l. Because horoballs are convex, the entire straight simplex $\gamma\,\mathrm{str}(\beta(g x_0^{(j)}),\dots,\beta(g x_q^{(j)}))$ lies in the same horoball over E_l which is in turn contained in U. Thus $F_j(\dot g)$ vanishes for all $j\in\mathbb{N}$, $\dot g \in \Gamma\backslash G$ with $\pi(\dot g x_0^{(j)}) \in M - N_D$. This implies that
$$\begin{aligned}\bigcup_{j\in\mathbb{N}}\{\dot g \in \Gamma\backslash G : F_j(\dot g)\neq 0\} &\subseteq \bigcup_{j\in\mathbb{N}}\{\dot g\in\Gamma\backslash G : \pi(\dot g x_0^{(j)}) \in N_D\}\\ &\subseteq \mathrm{pr}_1\psi^{-1}(N_D\times C)\end{aligned}$$
where $C := \bigcup_{j\in\mathbb{N}}\{x_0^{(j)}\}\cup\{x_0\} \subset \mathbb{H}^n$ is compact and $\psi : \Gamma\backslash G\times\mathbb{H}^n \longrightarrow M\times\mathbb{H}^n$, $(\dot g, y)\mapsto(\pi(gy), y)$ is a proper map (cf. Proposition I.4.8). Because $N_D\times C$ is compact so is $K := \mathrm{pr}_1\psi^{-1}(N_D\times C)\subseteq\Gamma\backslash G$.

Because the quotient map $r : G\to\Gamma\backslash G$ is a covering and K is compact there is a compact set $K'\subset G$ such that $r(K') = K$. Consider now the compact set $C' = \bigcup_{j\in\mathbb{N}}\{(x_0^{(j)},\dots,x_q^{(j)})\}\cup\{(x_0,\dots,x_q)\}$ and the continuous map $A : G\times(\mathbb{H}^n)^{q+1}\to(\mathbb{H}^n)^{q+1}$ given by the diagonal action of G on $(\mathbb{H}^n)^{q+1}$. By Γ-invariance of c and the definition of F_j it is clear that all possible values of all $|F_j|$ are at most all the possible values of $|\beta^* c|$ on the compact set $Q := A(K'\times C')\subset(\mathbb{H}^n)^{q+1}$. By definition of β^*c the number of these values is bounded from above by the number of intersections of Q with the elements of the decomposition
$$\{\gamma_0\mathcal{D}_{i_0}\times\cdots\times\gamma_q\mathcal{D}_{i_q} : i_0,\dots,i_q\in\{0,\dots,k\}, \gamma_0,\dots,\gamma_q\in\Gamma\}$$
of $(\mathbb{H}^n)^{q+1}$. It is easy to see that these can only by finitely many because Q is compact.

The maximum of all these finitely many values is now an uniform upper bound for all $|F_j|$. By Lebesgue's dominated convergence theorem the continuity of $\mathrm{trans}(c)$ follows.

It follows immediately from the G-invariance of μ on $\Gamma\backslash G$, that $\mathrm{trans}(c)$ is G-equivariant as we have already seen in the case of the two previous transfer maps. Further it is easy to check that $\mathrm{trans} : S^\bullet(\mathbb{H}^n, U)^\Gamma \to C((\mathbb{H}^n)^{\bullet+1}, \mathbb{R}_\varepsilon)^G$ is indeed a morphism of complexes, i.e. commutes with the coboundary operators, and thus induces a map in cohomology. We omit the straightforward computations.

Now let $c \in S_b^q(\mathbb{H}^n, U)^\Gamma$ be a bounded invariant singular cochain. Because $S_b^q(\mathbb{H}^n, U)^\Gamma \subset S^q(\mathbb{H}^n, U)^\Gamma$, it follows that $\mathrm{trans}_b(c) = \mathrm{trans}(c)$ is continuous. All that remains to be checked is that $\mathrm{trans}_b(c)$ is indeed bounded. We do so by the following short computation.
$$\begin{aligned}|\mathrm{trans}_b(c)(x_0,\dots,x_q)| &\le \int_{\Gamma\backslash G}|(\beta^*c)(\dot g x_0,\dots,\dot g x_q)|\,d\mu(\dot g)\\ &\le \int_{\Gamma\backslash G}|c(\mathrm{str}(\beta(\dot g x_0),\dots,\beta(\dot g x_q)))|\,d\mu(\dot g)\\ &\le \|c\|\end{aligned}$$

This concludes the proof. □

However we already have another transfer map $\tau_{\mathrm{dR}} : H^\bullet(N, \partial N) \to H^\bullet_c(G, \mathbb{R}_\varepsilon)$ by Proposition III.2.5. Summarizing this in a diagram in cohomology we get

$$\begin{tikzcd}
H^\bullet_{cb}(\Gamma, \mathbb{R}) \arrow[drr, "\mathrm{trans}_\Gamma"] & & \\
H^\bullet_b(N, \partial N) \arrow[u, "\cong"] \arrow[rr, "\mathrm{trans}_b"] \arrow[d, "c"'] & & H^\bullet_{cb}(G, \mathbb{R}_\varepsilon) \arrow[d, "c"] \\
H^\bullet(N, \partial N) \arrow[rr, bend left, "\mathrm{trans}"] \arrow[rr, bend right, "\tau_{\mathrm{dR}}"'] & & H^\bullet_c(G, \mathbb{R}_\varepsilon) \\
H^\bullet(\Omega^\bullet(\mathbb{H}^n, U)^\Gamma) \arrow[u, "\Psi"] \arrow[rr, "\mathrm{trans}_{\mathrm{dR}}"] & & H^\bullet(\Omega^\bullet(\mathbb{H}^n, \mathbb{R}_\varepsilon)^G) \arrow[u, "\Phi"']
\end{tikzcd}$$

where Ψ is the de Rham isomorphism and Φ is the van Est isomorphism.

All that is left to be proven, is that trans and τ_{dR} are the same map in cohomology.

Lemma III.2.9. *The maps* $\mathrm{trans} : H^\bullet(N, \partial N) \to H^\bullet_c(G, \mathbb{R}_\varepsilon)$ *and* $\tau_{\mathrm{dR}} : H^\bullet(N, \partial N) \to H^\bullet_c(G, \mathbb{R}_\varepsilon)$ *are identical.*

Proof. Let $q \in \mathbb{N}_0$.

By definition of τ_{dR} in Proposition III.2.5 we have that $\Phi \circ \mathrm{trans}_{\mathrm{dR}} = \tau_{\mathrm{dR}} \circ \Psi$, i.e. the lower square in the above diagram commutes for τ_{dR}. Because Φ and Ψ are isomorphisms we are done, if we can show that also $\Phi \circ \mathrm{trans}_{\mathrm{dR}} = \mathrm{trans} \circ \Psi$.

First we will investigate both sides at the cochain level. Let $\alpha \in \Omega^q(\mathbb{H}^n, U)^\Gamma$ and $x_0, \dots, x_q \in \mathbb{H}^n$. Then

$$\begin{aligned}
\mathrm{trans}(\Psi(\alpha))(x_0, \dots, x_q) &= \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot \Psi(\alpha)(\mathrm{str}(\beta(\dot{g}x_0), \dots, \beta(\dot{g}x_q)))\, d\mu(\dot{g}) \\
&= \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot \left(\int_{\mathrm{str}(\beta(\dot{g}x_0), \dots, \beta(\dot{g}x_q))} \alpha \right) d\mu(\dot{g})
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\Phi(\mathrm{trans}_{\mathrm{dR}}(\alpha))(x_0, \dots, x_q) = \int_{\mathrm{str}(x_0, \dots, x_q)} \mathrm{trans}_{\mathrm{dR}}(\alpha) \\
&= \int_{\mathrm{str}(x_0, \dots, x_q)} \left(\int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot (\dot{g}^*\alpha)\, d\mu(\dot{g}) \right) \\
&= \int_{\Delta^q} \left(\int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot (\mathrm{str}(x_0, \dots, x_q)^* \dot{g}^*\alpha)(\partial/\partial t_1, \dots, \partial/\partial t_q)\, d\mu(\dot{g}) \right) dt_1 \dots dt_q \\
&= \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot \left(\int_{\Delta^q} (\mathrm{str}(x_0, \dots, x_q)^* \dot{g}^*\alpha)(\partial/\partial t_1, \dots, \partial/\partial t_q)\, dt_1 \dots dt_q \right) d\mu(\dot{g})
\end{aligned}$$

$$= \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot \left(\int_{\dot{g}_* \mathrm{str}(x_0,\ldots,x_q)} \alpha \right) d\mu(\dot{g})$$
$$= \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot \left(\int_{\mathrm{str}(\dot{g}x_0,\ldots,\dot{g}x_q)} \alpha \right) d\mu(\dot{g})$$

where we have used Fubini's theorem from line three to line four.

Let us now assume that $d\alpha = 0$, i.e. α is a cocycle representing a cohomology class. We claim that the function $f : (\mathbb{H}^n)^q \to \mathbb{R}$ given by

$$f(x_0,\ldots,x_{q-1}) := \sum_{i=0}^{q-1} (-1)^i \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot \left(\int_{\mathrm{str}(\beta(\dot{g}x_0),\ldots,\beta(\dot{g}x_i),\dot{g}x_i,\ldots,\dot{g}x_{q-1})} \alpha \right) d\mu(\dot{g})$$

for every $(x_0,\ldots,x_{q-1}) \in (\mathbb{H}^n)^q$ is in $C((\mathbb{H}^n)^q, \mathbb{R}_\varepsilon)^G$ and for its coboundary we have

$$df = \Phi(\mathrm{trans}_{\mathrm{dR}}(\alpha)) - \mathrm{trans}(\Psi(\alpha)) \tag{III.3}$$

This will readily imply our assertion, that trans and τ_{dR} give the same map in cohomology.

One can easily modify the argument used in Lemma III.2.8 to show that f is indeed continuous. We leave out the details here. It is also easy to show, that f is G-equivariant. Indeed

$$\begin{aligned}
&(g \cdot f)(x_0,\ldots,x_{q-1}) \\
&= \sum_{i=0}^{q-1} (-1)^i \int_{\Gamma\backslash G} \varepsilon((\dot{g}g^{-1})^{-1}) \cdot \left(\int_{\mathrm{str}(\beta(\dot{g}g^{-1}x_0),\ldots,\beta(\dot{g}g^{-1}x_i),\dot{g}g^{-1}x_i,\ldots,\dot{g}g^{-1}x_{q-1})} \alpha \right) d\mu(\dot{g}) \\
&= \sum_{i=0}^{q-1} (-1)^i \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot \left(\int_{\mathrm{str}(\beta(\dot{g}x_0),\ldots,\beta(\dot{g}x_i),\dot{g}x_i,\ldots,\dot{g}x_{q-1})} \alpha \right) d\mu(\dot{g}) \\
&= f(x_0,\ldots,x_{q-1})
\end{aligned}$$

for every $g \in G$, $x_0,\ldots,x_{q-1} \in \mathbb{H}^n$, where we have used the right-invariance of μ again.

Thus we are left to prove relation (III.3). The proof of this is analogous to our proof of relation (II.8) in Proposition II.3.20.

We will adopt the following abbreviated notation

$$T(y_0,\ldots,y_q) = \Phi(\alpha)(y_0,\ldots,y_q) = \int_{\mathrm{str}(y_0,\ldots,y_q)} \alpha$$

for all $y_0,\ldots,y_q \in \mathbb{H}^n$. Using this notation we can write

$$\begin{aligned}
\Phi(\mathrm{trans}_{\mathrm{dR}}(\alpha))(x_0,\ldots,x_q) &= \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot T(\dot{g}x_0,\ldots,\dot{g}x_q)) d\mu(\dot{g}) \\
\mathrm{trans}(\Psi(\alpha))(x_0,\ldots,x_q) &= \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot T(\beta(\dot{g}x_0),\ldots,\beta(\dot{g}x_q))\, d\mu(\dot{g}) \\
f(x_0,\ldots,x_{q-1}) &= \sum_{i=0}^{q-1} (-1)^i \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot T(\beta(\dot{g}x_0),\ldots,\beta(\dot{g}x_i),\dot{g}x_i,\ldots,\dot{g}x_{q-1}))\, d\mu(\dot{g})
\end{aligned}$$

for every $x_0,\ldots,x_q \in \mathbb{H}^n$. Note that

$$(dT)(y_0,\ldots,y_{q+1}) = d\Phi(\alpha)(y_0,\ldots,y_{q+1}) = \Phi(d\alpha)(y_0,\ldots,y_q) = 0$$

for all $y_0, \ldots, y_{q+1} \in \mathbb{H}^n$, since α is closed. Thus

$$
\begin{aligned}
0 &= \sum_{i=0}^{q} (-1)^i \int_{\Gamma\backslash G} \varepsilon(\dot g^{-1}) \cdot (dT)(\beta(\dot g x_0), \ldots, \beta(\dot g x_i), \dot g x_i, \ldots, \dot g x_q)\, d\mu(\dot g) \\
&= \sum_{i=0}^{q} (-1)^i \int_{\Gamma\backslash G} \varepsilon(\dot g^{-1}) \cdot \Big\{ \sum_{j \le i-1} (-1)^j T(\beta(\dot g x_0), \ldots, \beta(\dot g x_{j-1}), \beta(\dot g x_{j+1}), \ldots, \beta(\dot g x_i), \dot g x_i, \ldots, \dot g x_q) \\
&\quad + (-1)^i T(\beta(\dot g x_0), \ldots, \beta(\dot g x_{i-1}), \dot g x_i, \ldots, \dot g x_q) \\
&\quad + (-1)^{i+1} T(\beta(\dot g x_0), \ldots, \beta(\dot g x_i), \dot g x_{i+1}, \ldots, \dot g x_q) \\
&\quad + \sum_{i+2 \le j} (-1)^j T(\beta(\dot g x_0), \ldots, \beta(\dot g x_i), \dot g x_i, \ldots, \dot g x_{j-2}, \dot g x_j, \ldots, \dot g x_q) \Big\}\, d\mu(\dot g) \\
&= \int_{\Gamma\backslash G} \varepsilon(\dot g^{-1}) \cdot \sum_{i=0}^{q} \Big\{ T(\beta(\dot g x_0), \ldots, \beta(\dot g x_{i-1}), \dot g x_i, \dot g x_{i+1}, \ldots, \dot g x_q) \\
&\quad - T(\beta(\dot g x_0), \ldots, \beta(\dot g x_i), \dot g x_{i+1}, \ldots, \dot g x_q) \Big\} \\
&\quad + \sum_{i=0}^{q} (-1)^i \Big\{ \sum_{j \le i-1} (-1)^j T(\beta(\dot g x_0), \ldots, \beta(\dot g x_{j-1}), \beta(\dot g x_{j+1}), \ldots, \beta(\dot g x_i), \dot g x_i, \ldots, \dot g x_q) \\
&\quad + \sum_{i+2 \le j} (-1)^j T(\beta(\dot g x_0), \ldots, \beta(\dot g x_i), \dot g x_i, \ldots, \dot g x_{j-2}, \dot g x_j, \ldots, \dot g x_q) \Big\}\, d\mu(\dot g) \\
&= \int_{\Gamma\backslash G} \varepsilon(\dot g^{-1}) \cdot T(\dot g x_0, \ldots, \dot g x_q)\, d\mu(\dot g) - \int_{\Gamma\backslash G} \varepsilon(\dot g^{-1}) \cdot T(\beta(\dot g x_0), \ldots, \beta(\dot g x_q))\, d\mu(\dot g) \\
&\quad + \int_{\Gamma\backslash G} \varepsilon(\dot g^{-1}) \cdot \sum_{i=0}^{q} (-1)^i \Big\{ \sum_{j \le i-1} (-1)^j T(\beta(\dot g x_0), \ldots, \beta(\dot g x_{j-1}), \beta(\dot g x_{j+1}), \ldots, \beta(\dot g x_i), \dot g x_i, \ldots, \dot g x_q) \\
&\quad + \sum_{i+2 \le j} (-1)^j T(\beta(\dot g x_0), \ldots, \beta(\dot g x_i), \dot g x_i, \ldots, \dot g x_{j-2}, \dot g x_j, \ldots, \dot g x_q) \Big\}\, d\mu(\dot g)
\end{aligned}
$$

for all $x_0, \ldots, x_q \in \mathbb{H}^n$, i.e.

$$
\begin{aligned}
&\Phi(\mathrm{trans}_{\mathrm{dR}}(\alpha))(x_0, \ldots, x_q) - \mathrm{trans}(\Psi(\alpha))(x_0, \ldots, x_q) \\
&= - \int_{\Gamma\backslash G} \varepsilon(\dot g^{-1}) \cdot \sum_{i=0}^{q} (-1)^i \Big\{ \sum_{j \le i-1} (-1)^j T(\beta(\dot g x_0), \ldots, \beta(\dot g x_{j-1}), \beta(\dot g x_{j+1}), \ldots, \beta(\dot g x_i), \dot g x_i, \ldots, \dot g x_q) \\
&\quad + \sum_{i+2 \le j} (-1)^j T(\beta(\dot g x_0), \ldots, \beta(\dot g x_i), \dot g x_i, \ldots, \dot g x_{j-2}, \dot g x_j, \ldots, \dot g x_q) \Big\}\, d\mu(\dot g) \\
&= \int_{\Gamma\backslash G} \varepsilon(\dot g^{-1}) \cdot \sum_{i=0}^{q} (-1)^i \Big\{ \sum_{j \le i-1} (-1)^{j+1} T(\beta(\dot g x_0), \ldots, \beta(\dot g x_{j-1}), \beta(\dot g x_{j+1}), \ldots, \beta(\dot g x_i), \dot g x_i, \ldots, \dot g x_q) \\
&\quad + \sum_{i+1 \le j} (-1)^j T(\beta(\dot g x_0), \ldots, \beta(\dot g x_i), \dot g x_i, \ldots, \dot g x_{j-1}, \dot g x_{j+1}, \ldots, \dot g x_q) \Big\}\, d\mu(\dot g)
\end{aligned}
$$

On the other hand

$$
(df)(x_0, \ldots, x_q) = \sum_{j=0}^{q} (-1)^j f(x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_q)
$$

$$
\begin{aligned}
&= \int_{\Gamma\backslash G} \varepsilon(\dot g^{-1}) \cdot \sum_{j=0}^{q} (-1)^j \Big\{ \sum_{0\le i\le j-1} (-1)^i T(\beta(\dot g x_0), \dots, \beta(\dot g x_i), \dot g x_i, \dots, \dot g x_{j-1}, \dot g x_{j+1}, \dots, \dot g x_q) \\
&\quad + \sum_{j\le i\le q-1} (-1)^i T(\beta(\dot g x_0), \dots, \beta(\dot g x_{j-1}), \beta(\dot g x_{j+1}), \dots, \beta(\dot g x_{i+1}), \dot g x_{i+1}, \dots, \dot g x_q) \Big\}\, d\mu(\dot g) \\
&= \int_{\Gamma\backslash G} \varepsilon(\dot g^{-1}) \cdot \sum_{j=0}^{q} (-1)^j \Big\{ \sum_{0\le i\le j-1} (-1)^i T(\beta(\dot g x_0), \dots, \beta(\dot g x_i), \dot g x_i, \dots, \dot g x_{j-1}, \dot g x_{j+1}, \dots, \dot g x_q) \\
&\quad + \sum_{j+1\le i\le q} (-1)^{i+1} T(\beta(\dot g x_0), \dots, \beta(\dot g x_{j-1}), \beta(\dot g x_{j+1}), \dots, \beta(\dot g x_i), \dot g x_i, \dots, \dot g x_q) \Big\}\, d\mu(\dot g)
\end{aligned}
$$

for all $x_0, \dots, x_q \in \mathbb{H}^n$.

A quick comparison of both double sums under the integral shows, that they contain precisely the same summands with the same sign. This concludes the proof. $\square$

Therefore we have also proven Proposition III.2.6.

$\square$

III.2.3. Properties of $\mathrm{Vol}(\cdot)$

After these preparations we can now deduce some properties of the volume of a representation. We will from now on work with the hypothesis of the volume rigidity theorem III.1.1. These are the same as in the previous subsection except that Γ is not necessarily torsion-free anymore. In order to remedy this, we will frequently use the fact, that we may find a torsion-free subgroup of finite index in Γ (cf. Proposition I.4.21), and that the volume of a representation is multiplicative with respect to taking finite index subgroups.

Let us start with a "normalization" lemma. It shows, that for lattice embeddings the volume is equal to the volume of the corresponding quotient (orbifold).

Lemma III.2.10 (cf. [BBI13, Lemma 2, p. 15]). *Let $i : \Gamma \hookrightarrow G^+ < G$ be a lattice embedding. Then*

$$\mathrm{Vol}(i) = \mathrm{Vol}(M)$$

where $M = i(\Gamma)\backslash\mathbb{H}^n$ is the quotient and $\mathrm{Vol}(M)$ refers to the quotient measure ν/μ_Γ.

Proof. Both sides are multiplicative with respect to finite index subgroups (cf. Theorem I.4.20), so we may suppose without loss of generality that Γ is torsion-free. By definition we have

$$
\begin{aligned}
\mathrm{Vol}(M) &= \langle \omega_{N,\partial N}, [N, \partial N] \rangle \\
\mathrm{Vol}(i) &= \langle (c \circ i^*)(\omega_n^b), [N, \partial N] \rangle
\end{aligned}
$$

The desired equality would thus clearly follow from $\omega_{N,\partial N} = (c \circ i^*)(\omega_n^b)$. Because the transfer map $\tau_{\mathrm{dR}} : H^n(N, \partial N) \to H_c^n(G)$ is an isomorphism in top degree it is enough to check that

$$\tau_{\mathrm{dR}}(\omega_{N,\partial N}) = (\tau_{\mathrm{dR}} \circ c \circ i^*)(\omega_n^b)$$

But as we have already seen in Proposition III.2.5 $\tau_{\mathrm{dR}}(\omega_{N,\partial N}) = \omega_n$. Using the commutativity of the diagram in Proposition III.2.6 and the fact that trans_Γ is a left inverse of i^* we calculate the right-hand-side

$$(\tau_{\mathrm{dR}} \circ c \circ i^*)(\omega_n^b) = (c \circ \mathrm{trans}_\Gamma \circ i^*)(\omega_n^b) = c(\omega_n^b) = \omega_n$$

This concludes the proof. $\square$

We know by Proposition II.3.29, that $H^n_{cb}(G, \mathbb{R}_\varepsilon)$ is generated by ω^b_n. Further the transfer map $\mathrm{trans}_\Gamma : H^n_{cb}(\Gamma, \mathbb{R}) \to H^n_{cb}(G, \mathbb{R}_\varepsilon)$ is a left-inverse of the pullback $i^* : H^n_{cb}(G, \mathbb{R}_\varepsilon) \to H^n_{cb}(\Gamma, \mathbb{R})$ induced by the canonical inclusion $i : \Gamma \to G^+ < G$, i.e. $\mathrm{trans}_\Gamma \circ i^* = \mathrm{id}$. But what happens if we replace i^* with the pullback ρ^* along some other representation $\rho : \Gamma \to G^+ < G$? The next proposition gives an answer to that question.

Proposition III.2.11 (cf. [BBI13, Proposition 4, p. 15])**.** *The composition*

$$\mathbb{R} \cong H^n_{cb}(G, \mathbb{R}_\varepsilon) \xrightarrow{\rho^*} H^n_{cb}(\Gamma, \mathbb{R}) \xrightarrow{\mathrm{trans}_\Gamma} H^n_{cb}(G, \mathbb{R}_\varepsilon) \cong \mathbb{R}$$

is equal to $\lambda \cdot \mathrm{id}$ *with*

$$\lambda = \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}$$

and $|\lambda| \leq 1$.

Proof. As the quotient

$$\frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}$$

is invariant by passing to finite index subgroups (cf. Theorem I.4.20), we can without loss of generatliy suppose that Γ is torsion-free. Since $H^n_{cb}(G, \mathbb{R}_\varepsilon)$ is generated by ω^b_n we get a real number $\lambda \in \mathbb{R}$ such that

$$(\mathrm{trans}_\Gamma \circ \rho^*)(\omega^b_n) = \lambda \cdot \omega^b_n \tag{III.4}$$

Applying the comparison map c we get

$$(c \circ \mathrm{trans}_\Gamma \circ \rho^*)(\omega^b_n) = \lambda \cdot c(\omega^b_n) = \lambda \cdot \omega_n = \lambda \cdot \tau_{\mathrm{dR}}(\omega_{N,\partial N})$$

By the commutativity of the diagram in Proposition III.2.6 the first expression is equal to $(\tau_{\mathrm{dR}} \circ c \circ \rho^*)(\omega^b_n)$. Since τ_{dR} is injective in top degree, we get

$$(c \circ \rho^*)(\omega^b_n) = \lambda \cdot \omega_{N,\partial N}$$

Evaluating on the fundamental class we obtain

$$\mathrm{Vol}(\rho) = \langle (c \circ \rho^*)(\omega^b_n), [N, \partial N] \rangle = \lambda \cdot \langle \omega_{N,\partial N}, [N, \partial N] \rangle = \lambda \cdot \mathrm{Vol}(M)$$

In order to estimate the absolute value of λ we simply take norms in equation (III.4) and get

$$|\lambda| = \frac{\|(\mathrm{trans}_\Gamma \circ \rho^*)(\omega^b_n)\|}{\|\omega^b_n\|}$$

Since the maps trans_Γ and ρ^* are norm non-increasing we finally get as desired $|\lambda| \leq 1$.

□

A good definition of the volume of a representation should indicate, when a representation is "not very complicated". As we have already seen in section I.3 elementary subgroups of G have a quite simple structure, such that a representation is certainly "not very complicated", if its image is elementary. The next proposition shows, that our definition of the volume of a representation reflects this "simplicity" by being zero for representations with elementary image.

Proposition III.2.12. $\mathrm{Vol}(\rho) = 0$ *if* ρ *has elementary image.*

Proof. Without loss of generality we may assume that Γ is torsion-free. Let $M = \Gamma\backslash G$ and $N \subset M$ a compact core as in the definition of $\mathrm{Vol}(\rho)$.

We will distinguish three cases according to which type of elementary group $\rho(\Gamma)$ is. Recall that the pullback of the volume class $\rho^*\omega_n^b$ is represented in $H^n(L^\infty(\Gamma^{\bullet+1}, \mathbb{R})^\Gamma)$ by the cocycle $\rho^* V_y$ for any $y \in \overline{\mathbb{H}}^n$ (cf. Corollary II.3.22).

1) Assume that $\rho(\Gamma)$ is of elliptic type. Then $\rho(\Gamma)$ fixes a point $x \in \mathbb{H}^n$. Hence

$$\rho^*(V_x)(\gamma_0, \ldots, \gamma_n) = \mathrm{Vol}_n(\rho(\gamma_0)x, \ldots, \rho(\gamma_n)x) = \mathrm{Vol}_n(x, \ldots, x) = 0$$

for all $\gamma_0, \ldots, \gamma_n \in \Gamma$. Thus $\rho^*\omega_n^b = [\rho^* V_x] = 0$ and

$$\mathrm{Vol}(\rho) = \langle c(\rho^*(\omega_n^b)), [N, \partial N]\rangle = 0$$

2) Assume that $\rho(\Gamma)$ is of parabolic type. Then $\rho(\Gamma)$ fixes a point $\xi \in \partial\mathbb{H}^n$. Hence as before

$$\rho^*(V_\xi)(\gamma_0, \ldots, \gamma_n) = \mathrm{Vol}_n(\rho(\gamma_0)\xi, \ldots, \rho(\gamma_n)\xi) = \mathrm{Vol}_n(\xi, \ldots, \xi) = 0$$

for all $\gamma_0, \ldots, \gamma_n \in \Gamma$ and $\mathrm{Vol}(\rho) = 0$.

3) Finally assume that $\rho(\Gamma)$ is of hyperbolic type. Then $\rho(\Gamma)$ preserves a set $\{\xi_1, \xi_2\} \subset \partial\mathbb{H}^n$. Then $\mathrm{conv}(\rho(\gamma_0)\xi_1, \ldots, \rho(\gamma_n)\xi_1)$ is contained in the geodesic between ξ_1 and ξ_2. Because $n \geq 2$, we have again

$$\rho^*(V_{\xi_1})(\gamma_0, \ldots, \gamma_n) = \mathrm{Vol}_n(\rho(\gamma_0)\xi_1, \ldots, \rho(\gamma_n)\xi_1) = 0$$

for all $\gamma_0, \ldots, \gamma_n \in \Gamma$ and thus $\mathrm{Vol}(\rho) = 0$. □

Finally, the volume of a representation behaves well with respect to conjugation.

Lemma III.2.13. *Let $g \in G$ be an isometry. Then we have for the conjugated representation $g \cdot \rho \cdot g^{-1} : \Gamma \to G^+ < G$*

$$\mathrm{Vol}(g \cdot \rho \cdot g^{-1}) = \varepsilon(g) \cdot \mathrm{Vol}(\rho)$$

Proof. Let $g \in G$ and $y \in \overline{\mathbb{H}}^n$ be fixed by g.

Due to Proposition II.3.20 the volume class in $H^n(L^\infty(G^{\bullet+1}, \mathbb{R}_\varepsilon)^G)$ is represented by the cocycle $V_y : G^{n+1} \to \mathbb{R}, (g_0, \ldots, g_n) \mapsto \mathrm{Vol}_n(g_0 y, \ldots, g_n y)$

By Corollary II.3.22 the pullback $(g \cdot \rho \cdot g^{-1})^*(\omega_n^b)$ of the volume class is represented by

$$\begin{aligned}
(g \cdot \rho \cdot g^{-1})^* V_y(\gamma_0, \ldots, \gamma_n) &= V_y(g \cdot \rho(\gamma_0) \cdot g^{-1}, \ldots, g \cdot \rho(\gamma_n) \cdot g^{-1}) \\
&= \mathrm{Vol}_n(g\rho(\gamma_0)g^{-1}y, \ldots, g\rho(\gamma_n)g^{-1}y) \\
&= \varepsilon(g) \cdot \mathrm{Vol}_n(\rho(\gamma_0)y, \ldots, \rho(\gamma_n)y) \\
&= \varepsilon(g) \cdot \rho^* V_y(\gamma_0, \ldots, \gamma_n)
\end{aligned}$$

for every $\gamma_0, \ldots, \gamma_n \in \Gamma^{n+1}$. Hence $(g \cdot \rho \cdot g^{-1})^*(\omega_n^b) = \varepsilon(g) \cdot \rho^*(\omega_n^b)$.

The assertion now follows from the definition of $\mathrm{Vol}(\rho)$.

□

III.3. Proof of the Volume Rigidity Theorem

The first part of Theorem III.1.1 about the inequality now follows from the properties of $\mathrm{Vol}(\cdot)$. Indeed by Lemma III.2.10 we have that $\mathrm{Vol}(i) = \mathrm{Vol}(M)$ and by Proposition III.2.11 we know that in particular

$$\left|\frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\right| \leq 1$$

Thus

$$|\mathrm{Vol}(\rho)| \leq |\mathrm{Vol}(i)| = \mathrm{Vol}(M)$$

as asserted.

Hence all that remains to be proven is, that if $|\mathrm{Vol}(\rho)| = \mathrm{Vol}(M)$ then ρ is conjugated to i by an isometry. Note that we may assume without loss of generality, that $\rho : \Gamma \to G^+$ has non-elementary image. Indeed if ρ had elementary image, then $\mathrm{Vol}(\rho)$ would vanish due to Proposition III.2.12 and hence it could not be maximal.

We will do this in three steps. As in other proofs of the Mostow Rigidity Theorem, e.g. [Thu] or [BP92], we will construct an equivariant boundary map in the first step. Step two of the proof will then show, that any such equivariant boundary map sends regular simplices to regular simplices by proving Theorem III.3.6. The proof of this theorem is quite technical and is hence subdivided into several lemmas. In step three we simply apply what we have said about boundary maps in section I.8 and conclude that φ must be induced by some isometry conjugating the two representations.

III.3.1. Step 1: The Equivariant Boundary Map

The following quite general lemma provides us with a preliminary boundary map.

Lemma III.3.1 (cf. [BBI13, Lemma 3, p. 17])**.** *Let G be a locally compact group, $\Gamma < G$ a lattice and P an amenable subgroup. Let X be a compact metrizable space with a Γ-action by homeomorphisms. Then there is a Γ-equivariant measurable boundary map $\varphi' : G/P \to \mathcal{M}^1(X)$*

Proof. Let $C(X)$ denote the space of continuous real valued functions on X. The space

$$L^1_\Gamma(G, C(X)) := \left\{ f : G \to C(X) : f \text{ is measurable, } \Gamma\text{-equivariant and } \int_{\Gamma\backslash G} \|f(\dot{g})\| d\mu(\dot{g}) < \infty \right\}$$

is a separable Banach space whose dual is the space $L^\infty_\Gamma(G, C(X)^*)$ of measurable Γ-equivariant essentially bounded maps from G to $C(X)^*$ (the dual space of $C(X)$); for details concerning this duality we refer to [Bou04a, No. 6, §2, VI.32]. Observe that since $C(X)$ is separable the notion of measurability of a function $G \to C(X)^*$ is the same as to whether $C(X)^*$ is endowed with the weak-* topology or the norm topology (cf. [Mon01, Lemma 3.3.3, p. 29]). Using Corollary A.2.13 it is easy to verify that $L^\infty_\Gamma(G, \mathcal{M}^1(X))$ is a convex compact subset (with respect to the weak-* topology) of the unit ball of $L^\infty(G, C(X)^*)$ that is right P-invariant. Since P is amenable, there exists a P-fixed point, that is nothing but the map $\varphi' : G/P \to \mathcal{M}^1(X)$. □

As we have already seen in Lemma I.2.15 the stabilizer of a boundary point P is amenable. If we let Γ operate on $X = \partial\mathbb{H}^n$ via the representation $\rho : \Gamma \to G^+ < G$ then the hypothesis of the previous lemma is fulfilled and we get an a.e.-ρ-equivariant measurable map $\varphi' : \partial\mathbb{H}^n \to \mathcal{M}^1(\partial\mathbb{H}^n)$ identifying $\partial\mathbb{H}^n \cong G/P$ as usual.

Nevertheless that is not quite what we want. We are actually interested in an a.e.-ρ-equivariant boundary map $\varphi : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ (cf. Definition II.2.38). However we can get such a map from φ' by mapping $x \in \partial\mathbb{H}^n$ to the "point of highest concentration" of $\varphi'(x)$. The following lemma makes this idea precise.

Before we delve into the details we need some terminology:

Definition III.3.2. Let $\mathcal{A}_{\geq 1/2} \subset \mathcal{M}^1(\partial\mathbb{H}^n)$ denote the subset of all probability measures which have exactly one atom of mass $\geq 1/2$. Further let $\mathcal{A}_{=1/2}$ denote the subset of all probability measures with two distinct atoms of mass equal to 1/2. Finally let $\mathcal{A}_{<1/2}$ denote the subset of all probability measures with no atom of mass $\geq 1/2$, that is, every measure has only atoms of mass $< 1/2$ whence the notation.

Observe that we thus get the following decomposition into disjoint subsets

$$\mathcal{M}^1(\partial\mathbb{H}^n) = \mathcal{A}_{\geq 1/2} \dot\cup \mathcal{A}_{=1/2} \dot\cup \mathcal{A}_{<1/2}$$

Lemma III.3.3. *Let $\varphi' : \partial\mathbb{H}^n \to \mathcal{M}^1(\partial\mathbb{H}^n)$ be an a.e.-ρ-equivariant measurable map as before. Then:*

(i) $\mathcal{A}_{\geq 1/2}, \mathcal{A}_{=1/2}, \mathcal{A}_{<1/2} \subset \mathcal{M}^1(\partial\mathbb{H}^n)$ are (Borel) measurable and G-invariant.

(ii) $\varphi'(x) \in \mathcal{A}_{\geq 1/2}$ for almost every $x \in \partial\mathbb{H}^n$.

(iii) The map

$$\psi : \mathcal{A}_{\geq 1/2} \to \partial\mathbb{H}^n, \mu \mapsto \textit{the unique atom of mass } \geq 1/2 \textit{ of } \mu$$

is continuous and G-equivariant; in particular measurable.

Proof. To (i): Because G acts by homeomorphisms on $\partial\mathbb{H}^n$ it is easy to see that $\mathcal{A}_{\geq 1/2}, \mathcal{A}_{=1/2}, \mathcal{A}_{<1/2} \subset \mathcal{M}^1(\partial\mathbb{H}^n)$ are G-invariant. Since we have that $\mathcal{M}^1(\partial\mathbb{H}^n)$ is the disjoint union of $\mathcal{A}_{\geq 1/2}, \mathcal{A}_{=1/2}, \mathcal{A}_{<1/2}$ it suffices to show that $\mathcal{A}_{\geq 1/2}$ and $\mathcal{A}_{=1/2}$ are measurable.

Clearly we have that

$$\mathcal{A}_{\geq 1/2} = \mathcal{A}'_{\geq 1/2} - \mathcal{A}_{=1/2}$$

where

$$\mathcal{A}'_{\geq 1/2} := \{\mu \in \mathcal{M}^1(\partial\mathbb{H}^n) : \exists 1/2 \leq \lambda \leq 1, \nu \in \mathcal{M}(\partial\mathbb{H}^n), x \in \partial\mathbb{H}^n \text{ s.t. } \mu = \lambda\delta_x + \nu\}$$

that is the set of all probability measures with one atom of mass $\geq 1/2$ (not necessarily exactly one!). Obviously $\mathcal{A}_{=1/2}$ can be written as

$$\mathcal{A}_{=1/2} = \{1/2\,\delta_x + 1/2\,\delta_y : x, y \in \partial\mathbb{H}^n, x \neq y\}$$

We shall now prove the following claims:

(a) $\mathcal{A}'_{\geq 1/2}$ is closed in $\mathcal{M}^1(\partial\mathbb{H}^n)$

(b) $\mathcal{A}_{=1/2} \dot\cup \{\delta_x : x \in \partial\mathbb{H}^n\} = \overline{\mathcal{A}_{=1/2}}$.

To see (a) let $\mu_n = \lambda_n\delta_{x_n} + \nu_n$ with $\lambda_n \in [1/2, 1]$, $x_n \in \partial\mathbb{H}^n$ and $\nu_n \in \mathcal{M}(\partial\mathbb{H}^n)$ be a sequence in $\mathcal{A}'_{\geq 1/2}$ converging to some $\mu \in \mathcal{M}^1(\partial\mathbb{H}^n)$. We need to see that its limit can also be written as $\mu = \lambda\delta_x + \nu$ for some $\lambda \in [1/2, 1]$, $x \in \partial\mathbb{H}^n$ and $\nu \in \mathcal{M}(\partial\mathbb{H}^n)$. Since $[1/2, 1] \subset \mathbb{R}$ and $\partial\mathbb{H}^n$ are compact, there is a subsequence (μ_{n_k}) such that $\lambda_{n_k} \to \lambda$ and $x_{n_k} \to x$ as $k \to \infty$ for some $\lambda \in [1/2, 1]$ and $x \in \partial\mathbb{H}^n$. Define $\nu = \mu - \lambda\delta_x$. Obviously $\lambda_{n_k}\delta_{x_{n_k}} \to^* \lambda\delta_x$ in $\mathcal{M}(\partial\mathbb{H}^n)$ as $k \to \infty$. We get

$$\nu_{n_k} = \mu_{n_k} - \lambda_{n_k}\delta_{x_{n_k}} \to^* \mu - \lambda\delta_x \quad (k \to \infty)$$

and because $\mathcal{M}(\partial\mathbb{H}^n)$ is closed in $C(\partial\mathbb{H}^n)^*$ with the weak-* topology we have that $\nu \in \mathcal{M}(\partial\mathbb{H}^n)$. Thus $\mu_{n_k} \to^* \lambda\delta_x + \nu$ as $k \to \infty$. Because (μ_n) converges to μ also every subsequence converges to μ and

$$\mu = \lim_{k\to\infty} \mu_{n_k} = \lim_{k\to\infty} (\lambda_{n_k}\delta_{x_{n_k}} + \nu_{n_k}) = \lambda\delta_x + \nu$$

So μ has the required form.

Now to (b): Let $\mu \in \overline{\mathcal{A}_{=1/2}}$ and $\mu_n = 1/2\,\delta_{x_n} + 1/2\,\delta_{y_n}$ be a sequence in $\mathcal{A}_{=1/2}$ converging to it in $\mathcal{M}(\partial\mathbb{H}^n)$. Since $\partial\mathbb{H}^n$ is compact there is again a subsequence (μ_{n_k}) such that $x_{n_k} \to x$ and $y_{n_k} \to y$ as $k \to \infty$ for some $x, y \in \partial\mathbb{H}^n$. Let $f \in C_c(\partial\mathbb{H}^n) = C(\partial\mathbb{H}^n)$. Then

$$\int f\, d\mu_{n_k} = 1/2\, f(x_{n_k}) + 1/2\, f(y_{n_k}) \to 1/2\, f(x) + 1/2\, f(y) = \int f\, d(1/2\,\delta_x + 1/2\,\delta_y) \quad (k \to \infty)$$

and hence $\mu_{n_k} \to^* 1/2\,\delta_x + 1/2\,\delta_y = \mu$ as $k \to \infty$. If $x = y$ then $\mu = \delta_x$, else $\mu = 1/2\,\delta_x + 1/2\,\delta_y$. This proves (b).

Due to (b) we have that $\mathcal{A}_{=1/2} = \overline{\mathcal{A}_{=1/2}} - \{\delta_x : x \in \partial\mathbb{H}^n\}$. Clearly $\{\delta_x : x \in \partial\mathbb{H}^n\}$ is also closed and hence $\mathcal{A}_{=1/2}$ is measurable. Therefore also $\mathcal{A}_{\geq 1/2}$ is.

To (ii): Let $A_{\geq 1/2}, A_{=1/2}, A_{<1/2} \subset \partial\mathbb{H}^n$ denote the preimages of $\mathcal{A}_{\geq 1/2}, \mathcal{A}_{=1/2}, \mathcal{A}_{<1/2} \subset \mathcal{M}^1(\partial\mathbb{H}^n)$ under $\varphi' : \partial\mathbb{H}^n \to \mathcal{M}^1(\partial\mathbb{H}^n)$ respectively. Then $A_{\geq 1/2} \dot\cup A_{=1/2} \dot\cup A_{<1/2} = \partial\mathbb{H}^n$ and each of them is Γ-invariant since φ' is equivariant. Because Γ acts (doubly) ergodically on $\partial\mathbb{H}^n$ one of them has full measure. We shall see that it will lead to a contradiction if $A_{=1/2}$ or $A_{<1/2}$ has full measure.

First assume that $A_{<1/2}$ has full measure in $\partial\mathbb{H}^n$. Then we are in a position to apply Douady-Earle's barycenter construction for every $x, y \in A_{<1/2}$ (cf. appendix E) and consider the points $\mathrm{bary}(\varphi'(x))$ and $\mathrm{bary}(\varphi'(y))$. Due to the G-equivariance of Douady-Earle's barycenter construction and the G-invariance of the hyperbolic metric $d : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ we have that the map

$$A_{<1/2} \times A_{<1/2} \to \mathbb{R}, \quad (x, y) \mapsto d(\mathrm{bary}(\varphi'(x)), \mathrm{bary}(\varphi'(y)))$$

is Γ-invariant and clearly measurable. Because Γ acts ergodically on $A_{<1/2} \times A_{<1/2} \subset \partial\mathbb{H}^n \times \partial\mathbb{H}^n$ there is a full measure subset $A \subset A_{<1/2} \times A_{<1/2}$ such that $D = d(\mathrm{bary}(\varphi'(x)), \mathrm{bary}(\varphi'(y)))$ is constant for every $(x, y) \in A$. We want to see that there are points $(x, y) \in A$ such that for every $\gamma \in \Gamma$ also $(x, \gamma y) \in A$. First observe that because of Fubini's theorem and the fact that $A \subset \partial\mathbb{H}^n \times \partial\mathbb{H}^n$ has full measure, there has to be a point $x \in \partial\mathbb{H}^n$ such that $A_{[x]} = \{y \in \partial\mathbb{H}^n : (x, y) \in A\}$ has full measure in $\partial\mathbb{H}^n$. Now consider the set

$$A_{[x]}^{\Gamma} = \bigcap_{\gamma \in \Gamma} \gamma A_{[x]} \subset A_{[x]}$$

Recall that Γ is countable, since it is a discrete subgroup of the second countable group $G^+ = \mathrm{Isom}^+(\mathbb{H}^n)$, and the fact that the Γ action on $\partial\mathbb{H}^n$ preserves null sets. Therefore $A_{[x]}^{\Gamma}$ is Γ-invariant and has also full measure in $\partial\mathbb{H}^n$. In particular it is non-empty so there is a $y \in A_{[x]}^{\Gamma}$. Because of the Γ-invariance of $A_{[x]}^{\Gamma}$ also $\gamma y \in A_{[x]}^{\Gamma} \subset A_{[x]}$ and thus $(x, \gamma y) \in A$ for every $\gamma \in \Gamma$. Hence

$$D = d(\mathrm{bary}(\varphi'(x)), \mathrm{bary}(\varphi'(\gamma x))) = d(\mathrm{bary}(\varphi'(x)), \rho(\gamma)\mathrm{bary}(\varphi'(y)))$$

for every $\gamma \in \Gamma$. Hence the orbit $\rho(\Gamma)\mathrm{bary}(\varphi'(y))$ is bounded and the limit set $L(\rho(\Gamma))$ is empty. By Theorem I.3.8 this implies that $\rho(\Gamma)$ is elementary of elliptic type; a contradiction!

Now let us assume that $A_{=1/2} \subset \partial\mathbb{H}^n$ has full measure. Here we are going to distinguish three cases. We consider the sets

$$\begin{aligned} A_2 &:= \{(x, y) \in A_{=1/2} : |\mathrm{supp}\ (\varphi'(x)) \cap \mathrm{supp}\ (\varphi'(y))| = 2\} \\ A_1 &:= \{(x, y) \in A_{=1/2} : |\mathrm{supp}\ (\varphi'(x)) \cap \mathrm{supp}\ (\varphi'(y))| = 1\} \\ A_0 &:= \{(x, y) \in A_{=1/2} : |\mathrm{supp}\ (\varphi'(x)) \cap \mathrm{supp}\ (\varphi'(y))| = 0\} \end{aligned}$$

Obviously they give a decomposition of $A_{=1/2}$ into disjoint subsets and due to the equivariance of φ' they are also Γ-invariant. We claim that they are measurable.

Indeed, it is easy to see that $A_2 = \{(x, y) \in A_{=1/2} : \varphi'(x) = \varphi'(y)\} = (\varphi' \times \varphi')^{-1}(\Delta)$ where $\Delta \subset \mathcal{M}^1(\partial\mathbb{H}^n) \times \mathcal{M}^1(\partial\mathbb{H}^n)$ denotes the diagonal which is closed since $\mathcal{M}^1(\partial\mathbb{H}^n)$ is Hausdorff. For A_1 we see that $A_1 = \{(x, y) \in A_{=1/2} : \varphi'(x) - \varphi'(y) \in \mathcal{C}\}$ where $\mathcal{C} = \{1/2\,\delta_x - 1/2\,\delta_y : x, y \in \partial\mathbb{H}^n, x \neq y\}$. Completely analogously to the case of $\mathcal{A}_{=1/2}$ one verifies that $\mathcal{C}$ is measurable and therefore A_1 is measurable. Finally because $A_{=1/2}$ is the disjoint union of A_2, A_1, A_0 also A_0 is measurable.

Now we can again use the double ergodicity of the diagonal action of Γ on $\partial\mathbb{H}^n \times \partial\mathbb{H}^n$ to deduce that one of these sets must have full measure. Let us consider the following cases.

1) A_2 has full measure: As before we find $x \in \partial\mathbb{H}^n$ such that $A_{2,[x]} = \{y \in \partial\mathbb{H}^n : (x, y) \in A_2\}$ has full measure and observe that $A^{\Gamma}_{2,[x]} = \bigcap_{\gamma\in\Gamma} \gamma A_{2,[x]} \subset A_{2,[x]}$ is a Γ-invariant subset of full measure. Hence there is again a $y \in \partial\mathbb{H}^n$ such that $(x, \gamma y) \in A_2$ for all $\gamma \in \Gamma$. If $\{x_1, x_2\} = \text{supp}\,(\varphi'(x))$ then

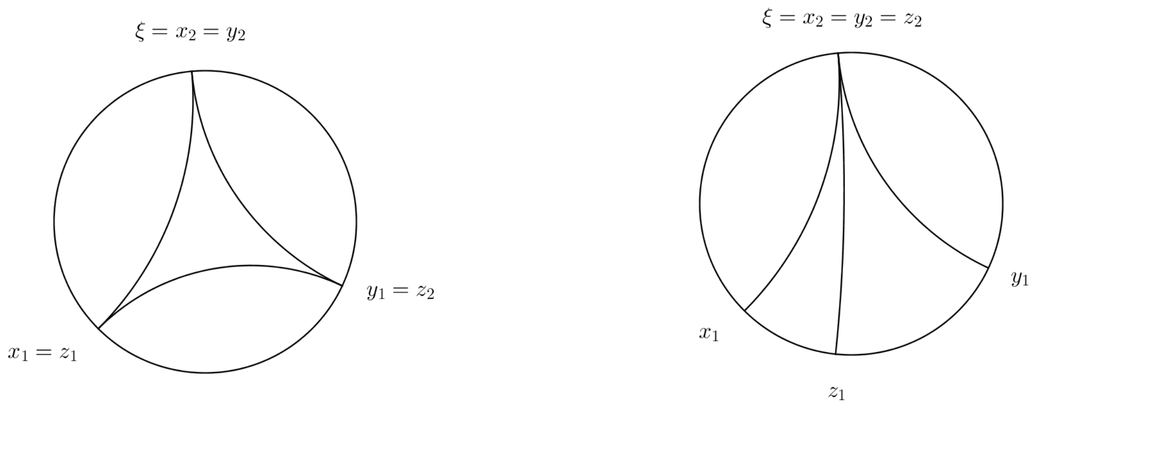
$$\begin{aligned}\{x_1, x_2\} &= \text{supp}\,(\varphi'(x)) = \text{supp}\,(\varphi'(\gamma y)) = \rho(\gamma)\text{supp}\,(\varphi'(y))\\ &= \rho(\gamma)\text{supp}\,(\varphi'(x)) = \rho(\gamma)\{x_1, x_2\}\end{aligned}$$

for every $\gamma \in \Gamma$. Therefore $\rho(\Gamma)x_1$ is a finite orbit in $\overline{\mathbb{H}}^n$ in contradiction to $\rho(\Gamma)$ being non-elementary.

2) A_1 has full measure: Then we find again $x \in \partial\mathbb{H}^n$ such that $A_{1,[x]} = \{y \in \partial\mathbb{H}^n : (x, y) \in A_1\}$ has full measure. Then also $A'_{1,[x]} = A_{1,[x]} \times A_{1,[x]} \cap A_1$ has full measure. Iterating the previous argument one more time we can find $y \in \partial\mathbb{H}^n$ such that $A'_{1,[x],[y]} = \{z \in \partial\mathbb{H}^n : (y, z) \in A'_{1,[x]}\}$ has full measure. Observe that

$$\begin{aligned}z \in A'_{1,[x],[y]} &\iff (y, z) \in A'_{1,[x]} = A_{1,[x]} \times A_{1,[x]} \cap A_1\\ &\iff (y, z) \in A_1 \text{ and } (x, y) \in A_1 \text{ and } (x, z) \in A_1\end{aligned}$$

If we denote $\text{supp}\,(\varphi'(x)) = \{x_1, x_2\}$ and $\text{supp}\,(\varphi'(y)) = \{y_1, y_2\}$ with – say – $\xi := x_2 = y_2$, then we either have $\text{supp}\,(\varphi'(z)) = \{x_1, y_1\}$ or $\text{supp}\,(\varphi'(z)) \cap \text{supp}\,(\varphi'(x)) \cap \text{supp}\,(\varphi'(y)) = \{\xi\}$ for every $z \in A'_{1,[x],[y]}$ (cf. Figure III.1).



(a) $\text{supp}\,(\varphi'(z)) = \{x_1, y_1\}$ (b) $\text{supp}\,(\varphi'(z)) \cap \text{supp}\,(\varphi'(x)) \cap \text{supp}\,(\varphi'(y)) = \{\xi\}$

Figure III.1.: The two different cases for the position of $\text{supp}\,(\varphi'(z)) = \{z_1, z_2\}$

Now we consider once more $A''_{1,[x],[y]} = A'_{1,[x],[y]} \times A'_{1,[x],[y]} \cap A_1$, which has also full measure. Consider $(z, z') \in A''_{1,[x],[y]}$ and denote $\text{supp}\,(\varphi'(z)) = \{z_1, z_2\}$, $\text{supp}\,(\varphi'(z')) = \{z'_1, z'_2\}$ with $\zeta := z_2 = z'_2$.

If supp $(\varphi'(z)) = \{x_1, y_1\}$, then clearly $\zeta = x_1$ or $\zeta = y_1$. If $\zeta = x_1$ then z_1' has to be in supp $(\varphi'(y)) = \{y_1, \xi\}$, since $z' \in A'_{1,[x],[y]}$ and thus $(y, z') \in A_1$. But this means, that either supp $(\varphi'(z')) =$ supp $(\varphi'(z))$ or supp $(\varphi'(z')) =$ supp $(\varphi'(y))$; both result in a contradiction. The case of $\zeta = y_1$ can be treated in exactly the same way (just exchange x and y in the previous argument).

Hence we must have supp $(\varphi'(z)) \cap$ supp $(\varphi'(x)) \cap$ supp $(\varphi'(y)) = \{\xi\}$. We shall argue that the only possible position for supp $(\varphi'(z'))$ is one such that $\zeta = \xi$. If $\zeta \neq \xi$ then either $\zeta = x_1$ or $\zeta = y_1$. Without loss of generality we may assume that $\zeta = x_1$ (the other case can again be treated analogously). Now it is easy to see that $z_1 = y_1$ must hold and $z_1' = \xi$ or $z_1' = y_1$. In both cases we have again a contradiction, because then supp $(\varphi'(z')) =$ supp $(\varphi(x))$ or supp $(\varphi'(z')) =$ supp $(\varphi(z))$ respectively.

Thus we have seen that $\{\xi\} = \{\zeta\} =$ supp $(\varphi(z)) \cap$ supp $(\varphi(z'))$ for all $(z, z') \in A''_{1,[x],[y]}$. This also holds for the full measure Γ-invariant subset $B := \bigcap_{\gamma \in \Gamma} \gamma A''_{1,[x],[y]}$. Thus for $(z, z') \in B$, also $(\gamma \cdot z, \gamma \cdot z') \in B$ and we obtain

$$\{\xi\} = \text{supp } (\varphi'(\gamma \cdot z)) \cap \text{supp } (\varphi'(\gamma \cdot z')) = \rho(\gamma) \cdot \text{supp } (\varphi'(z)) \cap \text{supp } (\varphi'(z')) = \{\rho(\gamma) \cdot \zeta\}$$

for all $\gamma \in \Gamma$; in contradiction to $\rho(\Gamma)$ being non-elementary.

3) A_0 has full measure: Let g_x, g_y denote the geodesics with endpoints supp $\varphi'(x)$, supp $\varphi'(y)$ respectively. By double ergodicity we get, that the distance $D := d(g_x, g_y)$ is constant for almost every $(x, y) \in A_0$.

If $D = 0$ then g_x and g_y intersect in at least one point. If they intersect in more than one point, then $g_x = g_y$ which would imply that supp $(\varphi'(x)) =$ supp $(\varphi'(y))$; a contradiction. Let us now denote by $\theta(x, y) \in (0, \pi/2]$ the unique acute angle in which g_x and g_y meet. It is easy to see that $\theta(x, y)$ depends measurably on $(x, y) \in A_0$, since the angle depends continuously on the respective geodesics. By double ergodicity one gets again, that $\theta(x, y)$ is essentially constant, say $\theta(x, y) = \theta$ for almost every $(x, y) \in A_0$. Just as several times before we may now find four points $x_1, \ldots, x_4 \in \partial\mathbb{H}^n$ with respective geodesics $g_1 := g_{x_1}, \ldots, g_4 := g_{x_4}$ such that they meet pairwise in the acute angle θ. Because every geodesic meets every other geodesic in one point they all have to lie in a two dimensional hyperbolic subspace of $\mathbb{H}^n$, such that we may assume without loss of generality, that they are all in $\mathbb{H}^2$. Further we may assume, that g_1 is the imaginary axis in the upper half plane model. Now g_2, g_3, g_4 intersect g_1 at heights y_2, y_3, y_4 and at least two of them meet g_1 in the same oriented angle (θ or $\pi - \theta$), say these two are g_2 and g_3 (cf. Figure III.2). But by construction g_2 has to intersect g_3 as well, which is only possible for $y_2 = y_3$. That in turn implies however $g_2 = g_3$; a contradiction.

If $D > 0$, let $\gamma \in \rho(\Gamma)$ be a hyperbolic element whose fixed points are not the endpoints of g_x or g_y. Then iterates of γ send any geodesic g into an arbitrarily small neighborhood of its attractive fixed point. Contradicting that g_x is at fixed distance from g_y. The existence of such a hyperbolic element is guaranteed by Proposition I.3.9.

To (iii): The G-equivariance is clear by definition, so we only need to show the continuity. Let $\mu_n = \lambda_n \delta_{x_n} + \nu_n$ be a sequence in $\mathcal{A}_{\geq 1/2}$ converging to $\mu = \lambda\delta_x + \nu \in \mathcal{A}_{\geq 1/2}$, where $\lambda_n, \lambda \in [1/2, 1]$, $\nu_n, \nu \in \mathcal{M}(\partial\mathbb{H}^n)$ and $x_n, x \in \partial\mathbb{H}^n$ $(n \in \mathbb{N})$. Then $\psi(\mu_n) = x_n$ and $\psi(\mu) = x$. Assume that (x_n) does not converge to x, i.e. there is $\varepsilon > 0$ and a subsequence (x_{n_k}) such that $d(x_{n_k}, x) \geq \varepsilon$ for every $k \in \mathbb{N}$.

Because $\partial\mathbb{H}^n$ and $[1/2, 1]$ are compact we can find subsubsequences, that we will also denote by (x_{n_k}) and (λ_{n_k}), such that $x_{n_k} \to \hat{x} \in \partial\mathbb{H}^n$ and $\lambda_{n_k} \to \hat{\lambda} \geq 1/2$ as $k \to \infty$. Then clearly $\lambda_{n_k}\delta_{x_{n_k}} \to^* \hat{\lambda}\delta_{\hat{x}}$ as $k \to \infty$ and if we set $\hat{\nu} := \mu - \hat{\lambda}\delta_{\hat{x}}$

$$\nu_{n_k} = \mu_{n_k} - \lambda_{n_k}\delta_{x_{n_k}} \to^* \mu - \hat{\lambda}\delta_{\hat{x}} = \hat{\nu} \qquad (k \to \infty)$$

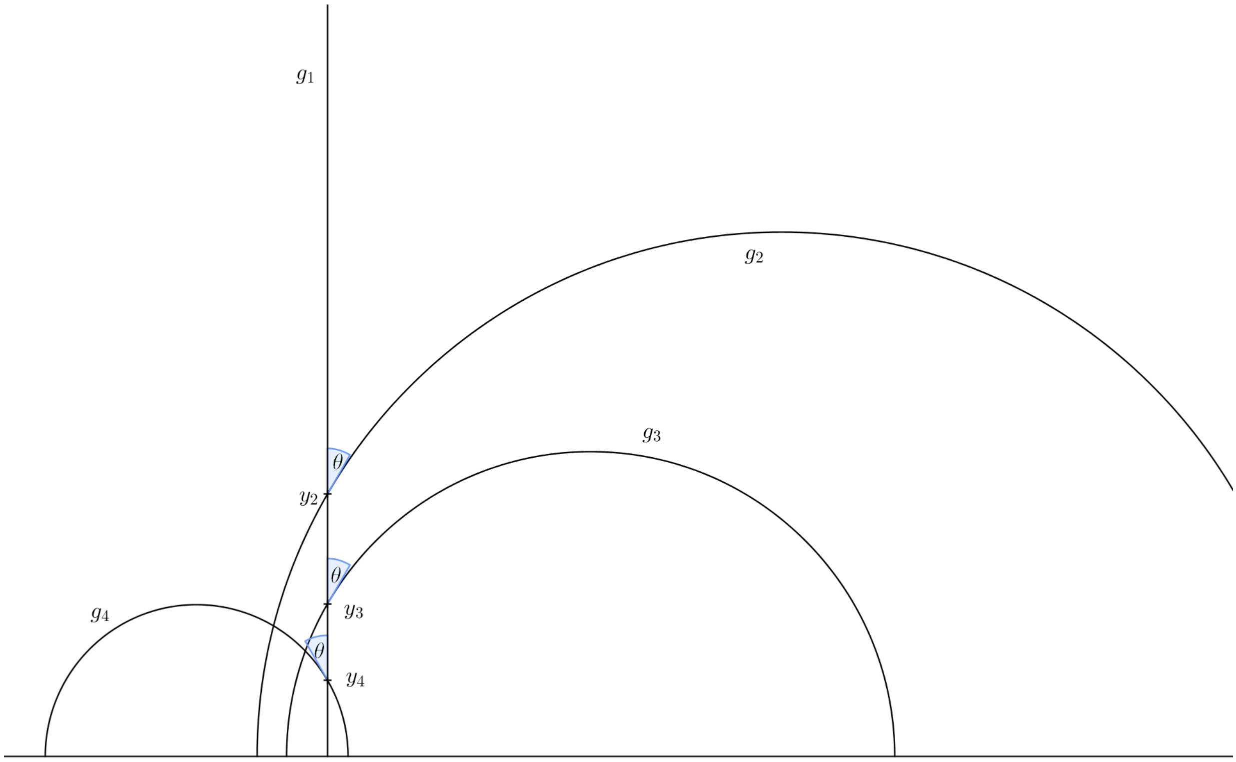


Figure III.2.: The case $D = 0$ with geodesics $g_1, \ldots, g_4$ meeting each other at an acute angle θ.

Hence

$$\mu = \lim_{n\to\infty} \mu_n = \lim_{k\to\infty} \mu_{n_k} = \lim_{k\to\infty} (\lambda_{n_k}\delta_{x_{n_k}} + \nu_{n_k}) = \hat{\lambda}\delta_{\hat{x}} + \hat{\nu}$$

That is $\hat{x}$ is also an atom of μ of mass $\hat{\lambda} \geq 1/2$. By uniqueness we must have that $x = \hat{x}$ but that contradicts $d(x_{n_k}, x) \geq \varepsilon$ for all $k \in \mathbb{N}$! $\square$

As an immediate consequence we get that the map $\varphi := \psi \circ \varphi' : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ is a measurable a.e.-ρ-equivariant boundary map. Thus we have proven the following proposition.

Proposition III.3.4. *Let $\Gamma < G^+$ be a lattice and $\rho : \Gamma \to G^+$ a representation with non-elementary image. Then there is a measurable a.e.-ρ-equivariant boundary map $\varphi : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$.*

Finally we want to prove, that such a boundary map $\varphi : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ is injective almost everywhere. We will use this fact in the second step of our proof. However the proof fits better into our current context, such that we prepone it.

Proposition III.3.5. *Let $\varphi : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ be a measurable a.e.-ρ-equivariant boundary map. Then $\varphi(x) \neq \varphi(y)$ for almost every $(x, y) \in \partial\mathbb{H}^n \times \partial\mathbb{H}^n$.*

Proof. Let us consider the measurable set $A := \{(x, y) \in \partial\mathbb{H}^n \times \partial\mathbb{H}^n : \varphi(x) = \varphi(y)\}$. Then clearly A is Γ-invariant and due to the double ergodicity of the Γ-action it has either full measure or measure zero. If A had full measure then we could again find $x \in \partial\mathbb{H}^n$ such that $A_{[x]} = \{y \in \partial\mathbb{H}^n : (x, y) \in A\}$ had full measure. Hence also $A^\Gamma_{[x]} = \bigcap_{\gamma\in\Gamma} \gamma A_{[x]}$ had full measure and were therefore non-empty. We could thus find a $y \in A^{\Gamma}_{[x]}$ and for such y the full orbit Γy were in A by construction. That is

$$\varphi(x) = \varphi(\gamma y) = \rho(\gamma)\varphi(y)$$

for all $\gamma \in \Gamma$ and thus $\rho(\Gamma)$ were elementary; a contradiction.

Therefore A must have measure zero, i.e. $\varphi(x) \neq \varphi(y)$ for almost every $(x, y) \in \partial\mathbb{H}^n \times \partial\mathbb{H}^n$. □

III.3.2. Step 2: Mapping Regular Simplices to Regular Simplices

Before we proceed let us observe, that we may assume without loss of generality that $\mathrm{Vol}(\rho) \geq 0$. Indeed, in order to change the sign of $\mathrm{Vol}(\rho)$ we may simply conjugate it by some orientation reversing isometry $\tau \in G - G^+$ (cf. Lemma III.2.13).

The next theorem will enable us to prove, that an a.e.-ρ-equivariant boundary map sends regular simplices to regular simplices.

Theorem III.3.6 (cf. [BBI13, Theorem 2, p. 4])**.** *Let $i : \Gamma \hookrightarrow G^+ < G$ be a lattice embedding and let $\rho : \Gamma \to \mathrm{Isom}^+(\mathbb{H}^n) = G^+$ be any representation with non-elementary image. Further let $\varphi : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ be an a.e.-ρ-equivariant measurable map. Then for every $(n+1)$-tuple of points $\xi_0, \dots, \xi_n \in \partial\mathbb{H}^n$*

$$\int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1})\mathrm{Vol}_n(\varphi(\dot{g}\xi_0), \dots, \varphi(\dot{g}\xi_n))\, d\mu(\dot{g}) = \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\mathrm{Vol}_n(\xi_0, \dots, \xi_n) \tag{III.5}$$

where μ is the invariant probability measure on $\Gamma\backslash G$.

Remark III.3.7. *The above formula is actually a very concrete version of formula (2.12) in [BI09, Proposition 2.44, p. 27]. As we have already mentioned in the introduction Burger and Iozzi succeed in proving Mostow's rigidity theorem in dimension* 3 *by applying their formula in [BI09, Section 3.1., pp. 29].*

Because the proof of this theorem is quite technical, we want to prove the following important corollary first.

Corollary III.3.8. *If in the notation of Theorem III.3.6 ρ has maximal volume, that is* $\mathrm{Vol}(\rho) = \mathrm{Vol}(M)$ *with $M = \Gamma\backslash\mathbb{H}^n$, then φ sends the vertices of almost every regular ideal simplex to the vertices of a regular ideal simplex of the same orientation.*

Proof. By assumption we have for every $(\xi_0, \dots, \xi_n) \in (\partial\mathbb{H}^n)^{n+1}$

$$\begin{aligned}\mathrm{Vol}_n(\xi_0, \dots, \xi_n) &= \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\mathrm{Vol}_n(\xi_0, \dots, \xi_n) \\ &= \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1})\mathrm{Vol}_n(\varphi(\dot{g}\xi_0), \dots, \varphi(\dot{g}\xi_n))\, d\mu(\dot{g}) \\ &= \int_{\mathcal{D}} \varepsilon(g^{-1})\mathrm{Vol}_n(\varphi(g\xi_0), \dots, \varphi(g\xi_n))\, d\mu_G(g)\end{aligned}$$

where $\mathcal{D}$ is a measurable fundamental set for the left action of Γ on G and μ_G is a Haar measure on G (cf. Theorem A.4.20 and Proposition A.4.21). If we choose $\bar{\eta} = (\eta_0, \dots, \eta_n) \in (\partial\mathbb{H}^n)^{(n+1)}$ to be the vertices of a positively oriented regular (ideal) simplex, then we know, that $\mathrm{Vol}_n(\eta_0, \dots, \eta_n) \geq 0$ is maximal (cf. Theorem I.7.4) and hence

$$\mathrm{Vol}_n(\eta_0, \dots, \eta_n) \geq \varepsilon(g^{-1})\mathrm{Vol}_n(\varphi(g\eta_0), \dots, \varphi(g\eta_n))$$

for all $g \in \mathcal{D}$. Thus we must have

$$\mathrm{Vol}_n(\eta_0, \ldots, \eta_n) = \varepsilon(g^{-1})\mathrm{Vol}_n(\varphi(g\eta_0), \ldots, \varphi(g\eta_n))$$

for all $g \in \mathcal{L} \subset \mathcal{D}$, where $\mathcal{L}$ is a set of full measure.

Now observe that if $g \in \mathcal{L}$ and $\gamma \in \Gamma$ then

$$\begin{aligned}\varepsilon((\gamma g)^{-1})\mathrm{Vol}_n(\varphi(\gamma g\eta_0), \ldots, \varphi(\gamma g\eta_n)) &= \varepsilon(g^{-1})\underbrace{\varepsilon(\gamma^{-1})}_{=1}\mathrm{Vol}_n(\rho(\gamma)\varphi(g\eta_0), \ldots, \rho(\gamma)\varphi(g\eta_n))\\ &= \varepsilon(g^{-1})\underbrace{\varepsilon(\rho(\gamma))}_{=1}\mathrm{Vol}_n(\varphi(g\eta_0), \ldots, \varphi(g\eta_n))\\ &= \mathrm{Vol}_n(\eta_0, \ldots, \eta_n)\end{aligned}$$

Thus equality holds for every $g \in \Gamma\mathcal{L}$. But $\Gamma\mathcal{L}$ has full measure, because

$$\mu_G(G - \Gamma\mathcal{L}) = \mu_G(\Gamma(\mathcal{D} - \mathcal{L})) \leq \sum_{\gamma \in \Gamma} \mu_G(\gamma(\mathcal{D} - \mathcal{L})) = \sum_{\gamma \in \Gamma} \mu_G(\mathcal{D} - \mathcal{L}) = 0$$

Using the identification $\Phi_{\bar{\eta}} : G \to T, g \mapsto (g\eta_0, \ldots, g\eta_n)$, where T denotes as in chapter I the set of all regular ideal simplices, we can conclude that for almost every $(\xi_0, \ldots, \xi_n) \in T$

$$\mathrm{Vol}_n(\eta_0, \ldots, \eta_n) = \varepsilon(\Phi_{\bar{\eta}}^{-1}(\xi_0, \ldots, \xi_n))\mathrm{Vol}_n(\varphi(\xi_0), \ldots, \varphi(\xi_n)) \tag{III.6}$$

We already see, that $(\varphi(\xi_0), \ldots, \varphi(\xi_n))$ is a regular ideal simplex since these are exactly the simplices, that achieve maximal volume such as $(\eta_0, \ldots, \eta_n)$. The only thing left to show, is that the orientation is also preserved.

Observe that

$$\varepsilon(\Phi_{\bar{\eta}}^{-1}(\xi_0, \ldots, \xi_n)) = \begin{cases} +1, & \text{if } (\xi_0, \ldots, \xi_n) \text{ has the same orientation as } (\eta_0, \ldots, \eta_n) \\ -1, & \text{if } (\xi_0, \ldots, \xi_n) \text{ has the opposite orientation as } (\eta_0, \ldots, \eta_n) \end{cases}$$

for every $(\xi_0, \ldots, \xi_n) \in T$.

Hence by equation (III.6), if $(\xi_0, \ldots \xi_n)$ is positively oriented (as $(\eta_0, \ldots, \eta_n)$ is), then

$$\mathrm{Vol}_n(\eta_0, \ldots, \eta_n) = \mathrm{Vol}_n(\varphi(\xi_0), \ldots, \varphi(\xi_n))$$

and $(\varphi(\xi_0), \ldots, \varphi(\xi_n))$ is a positively oriented regular ideal simplex; if $(\xi_0, \ldots \xi_n)$ is negatively oriented, then

$$\mathrm{Vol}_n(\eta_0, \ldots, \eta_n) = -\mathrm{Vol}_n(\varphi(\xi_0), \ldots, \varphi(\xi_n))$$

and $(\varphi(\xi_0), \ldots, \varphi(\xi_n))$ is a negatively oriented regular ideal simplex. Thus $\varphi : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ maps the vertices of almost every positively (resp. negatively) oriented regular ideal simplex to the vertices of a positively (resp. negatively) oriented regular ideal simplex. $\square$

Let us now turn to the proof of Theorem III.3.6. It will be easy to deduce the integral equality (III.5) for **almost every $(n + 1)$-tuple of points** $\xi_0, \ldots, \xi_n \in \partial\mathbb{H}^n$ from Proposition III.2.11. However note that the subset of vertices of regular ideal simplices $T \subset (\partial\mathbb{H}^n)^{n+1}$ has measure zero, such that we really need equation (III.5) to hold for all tuples $(\xi_0, \ldots, \xi_n) \in (\partial\mathbb{H}^n)^{n+1}$.

Proof of Theorem III.3.6. We shall first see that the equality holds almost everywhere. The theorem will then follow from Proposition III.3.9, which states, that this suffices in order to conclude that, the equality holds everywhere.

By Proposition III.2.11 we know that at the cohomology level the equality

$$\mathrm{trans}_\Gamma \circ \rho^* = \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)} \cdot \mathrm{id} \tag{III.7}$$

holds. Because all cochains vanish in degree $< n$ by Lemma II.3.28 this equality is actually an equality for cocycles in degree n. By Corollary II.3.23 $\rho^*(\omega_n^b)$ is represented by $\varphi^*\mathrm{Vol}_n$ in $L^\infty((\partial\mathbb{H}^n)^{n+1}, \mathbb{R}_\varepsilon)^G$. Therefore

$$\begin{aligned}\frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\mathrm{Vol}_n(\xi_0, \ldots, \xi_n) &= \mathrm{trans}_\Gamma(\varphi^*(\mathrm{Vol}_n))(\xi_0, \ldots, \xi_n)\\ &= \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1})\varphi^*(\mathrm{Vol}_n)(\dot{g}\xi_0, \ldots, \dot{g}\xi_n)\, d\mu(\dot{g})\\ &= \int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1})\mathrm{Vol}_n(\varphi(\dot{g}\xi_0), \ldots, \varphi(\dot{g}\xi_n))\, d\mu(\dot{g})\end{aligned}$$

for almost every $(\xi_0, \ldots, \xi_n) \in (\partial\mathbb{H}^n)^{n+1}$, since this equality holds only in $L^\infty((\partial\mathbb{H}^n)^{n+1}, \mathbb{R}_\varepsilon)^G$. □

Proposition III.3.9 (cf. [BBI13, Proposition 5, p. 19]). *Let $i\colon \Gamma \to G^+$ be a lattice embedding, $\rho\colon \Gamma \to G^+$ a representation and $\varphi\colon \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ an a.e.-ρ-equivariant measurable map. If*

$$\int_{\Gamma\backslash G} \varepsilon(\dot{g}^{-1}) \cdot \mathrm{Vol}_n(\varphi(\dot{g}\xi_0), \ldots, \varphi(\dot{g}\xi_n))\, d\mu(\dot{g}) = \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)} \mathrm{Vol}_n(\xi_0, \ldots, \xi_n) \tag{III.8}$$

holds for almost every $(\xi_0, \ldots, \xi_n) \in (\partial\mathbb{H}^n)^{n+1}$, then the equality holds everywhere.

As several times before we will use the following notation. $(\partial\mathbb{H}^n)^{(n+1)}$ denotes the G-invariant open subset of $(\partial\mathbb{H}^n)^{n+1}$ consisting of all $(n+1)$-tuples of pairwise distinct points $(\xi_0, \ldots, \xi_n)$. Because any ideal simplex contained in a proper hyperbolic subspace has no volume, we see that the volume cocycle Vol_n vanishes on $(\partial\mathbb{H}^n)^{n+1} - (\partial\mathbb{H}^n)^{(n+1)}$, such that equation (III.8) holds on this set trivially.

Proof of Proposition III.3.9. Identifying $\partial\mathbb{H}^n \cong S^{n-1} \subset \mathbb{R}^n$, let us consider the function $\varphi : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ as a function $\varphi : \partial\mathbb{H}^n \to \mathbb{R}^n$ and denote by φ_j its coordinates for $j = 1, \ldots, n$. Since $\partial\mathbb{H}^n \cong G/P$, where P is a minimal parabolic, let ν be the quasi-invariant measure on $\partial\mathbb{H}^n$ obtained from the decomposition of the Haar measure μ_G on G with respect to the Haar measure μ_P on P, i.e. there is a strictly positive continuous function $q : G \to \mathbb{R}^+$ such that

$$\int_G f(g)q(g)\, d\mu_G(g) = \int_{\partial\mathbb{H}^n} \left(\int_P f(\dot{g}\xi)\, d\mu_P(\xi) \right) d\nu(\dot{g})$$

for every integrable function f on G (cf. Theorem A.4.16).

By applying Lusin's Theorem A.2.6 to φ_j for every $j = 1, \ldots, n$ we find for every $\delta > 0$ a measurable set $B_{j,\delta} \subset \partial\mathbb{H}^n$ with measure $\nu(B_{j,\delta}) \leq \delta$ and a continuous function $f'_{j,\delta} : \partial\mathbb{H}^n \to \mathbb{R}$ such that $\varphi_j \equiv f'_{j,\delta}$ on $\partial\mathbb{H}^n - B_{j,\delta}$. Set $f'_\delta := (f_{1,\delta}, \ldots, f_{n,\delta}) : \partial\mathbb{H}^n \to \mathbb{R}^n$ and consider the composition $f_\delta := r \circ f'_\delta$ with the retraction $r : \mathbb{R}^n \to \overline{B^n}$ to the closed unit ball $\overline{B^n}$ in $\mathbb{R}^n$. Then, by setting $B_\delta := \bigcup_{j=1}^n B_{j,\delta}$, φ coincides on $\partial\mathbb{H}^n - B_\delta$ with the continuous function $f_\delta : \partial\mathbb{H}^n \to \overline{B^n}$ and $\nu(B_\delta) \leq n\delta$.

Let $\mathcal{D} \subset G$ be a measurable fundamental set for the action of Γ on G (cf. Theorem A.4.20 and Proposition A.4.21). For every measurable subset $E \subset \mathcal{D}$, any measurable map $\psi : \partial\mathbb{H}^n \to \overline{B^n}$ and any point $(\xi_0, \ldots, \xi_n) \in (\partial\mathbb{H}^n)^{n+1}$, we use the notation

$$\mathcal{J}(\psi, E, (\xi_0, \ldots, \xi_n)) := \int_E \varepsilon(g^{-1})\mathrm{Vol}_n(\psi(g\xi_0), \ldots, \psi(g\xi_n))\, d\mu_G(g)$$

Thus our goal is to show that if

$$\mathcal{J}(\varphi, \mathcal{D}, (\xi_0, \dots, \xi_n)) = \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)} \mathrm{Vol}_n(\xi_0, \dots, \xi_n) \tag{III.9}$$

for almost every $(\xi_0, \dots, \xi_n) \in (\partial \mathbb{H}^n)^{n+1}$, then the equality holds everywhere. As we have pointed out before it is in fact enough to show equation (III.9) only for all $(\xi_0, \dots, \xi_n) \in (\partial \mathbb{H} n)^{(n+1)}$.

Fix $\varepsilon > 0$ and let $K_\varepsilon \subset \mathcal{D}$ be a compact subset such that $\mu_G(\mathcal{D} - K_\varepsilon) < \varepsilon$. The rest of the proof is broken up in two lemmas, that we state and use, but whose proof we postpone.

Replacing φ with f_δ in equation (III.9), we get an estimate for the made error by the following lemma.

Lemma III.3.10 (cf. [BBI13, Lemma 5, p. 21]). *With the notations as above, there exists a function $M_\varepsilon(\delta)$ with the property $\lim_{\delta \to 0} M_\varepsilon(\delta) = 0$, such that*

$$|\mathcal{J}(\varphi, K_\varepsilon, (\xi_0, \dots, \xi_n)) - \mathcal{J}(f_\delta, K_\varepsilon, (\xi_0, \dots \xi_n))| \leq M_\varepsilon(\delta) \tag{III.10}$$

for all $(\xi_0, \dots, \xi_{n+1}) \in (\partial \mathbb{H}^n)^{n+1}$.

Observe that by definition

$$\begin{aligned}
&|\mathcal{J}(\varphi, \mathcal{D}, (\xi_0, \dots, \xi_n)) - \mathcal{J}(\varphi, K_\varepsilon, (\xi_0, \dots, \xi_n))| \\
&= \left| \int_{\mathcal{D}} \varepsilon(g^{-1}) \mathrm{Vol}_n(\varphi(g\xi_0), \dots, \varphi(g\xi_n)) \, d\mu(g) - \int_{K_\varepsilon} \varepsilon(g^{-1}) \mathrm{Vol}_n(\varphi(g\xi_0), \dots, \varphi(g\xi_n)) \, d\mu(g) \right| \\
&\leq \int_{\mathcal{D} - K_\varepsilon} |\mathrm{Vol}_n(\varphi(g\xi_0), \dots, \varphi(g\xi_n))| d\mu(g) \leq \mu(\mathcal{D} - \mathcal{K}_\varepsilon) \|\mathrm{Vol}_n\| = \varepsilon \|\mathrm{Vol}_n\|
\end{aligned}$$

holds for all (!) $(\xi_0, \dots, \xi_n) \in (\partial \mathbb{H}^n)^{[n+1]}$. Now we only need a suitable estimate for

$$\left| \mathcal{J}(f_\delta, K_\varepsilon, (\xi_0, \dots, \xi_n)) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)} \mathrm{Vol}_n(\xi_0, \dots, \xi_n) \right|$$

This is achieved by the following lemma.

Lemma III.3.11 (cf. [BBI13, Lemma 6, p. 22]). *There exists a function $L(\varepsilon, \delta)$ such that*

$$\lim_{\varepsilon \to 0} \lim_{\delta \to 0} L(\varepsilon, \delta) = 0$$

and

$$\left| \mathcal{J}(f_\delta, K_\varepsilon, (\xi_0, \dots, \xi_n)) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)} \mathrm{Vol}_n(\xi_0, \dots, \xi_n) \right| \leq L(\varepsilon, \delta) \tag{III.11}$$

for all $(\xi_0, \dots, \xi_{n+1}) \in (\partial \mathbb{H}^n)^{(n+1)}$.

Putting these ”everywhere-estimates” together we get

$$\begin{aligned}
&\left| \mathcal{J}(\varphi, \mathcal{D}, (\xi_0, \dots, \xi_n)) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)} \mathrm{Vol}_n(\xi_0, \dots, \xi_n) \right| \\
&\leq |\mathcal{J}(\varphi, \mathcal{D}, (\xi_0, \dots, \xi_n)) - \mathcal{J}(\varphi, K_\varepsilon, (\xi_0, \dots, \xi_n))| \\
&+ |\mathcal{J}(\varphi, K_\varepsilon, (\xi_0, \dots, \xi_n)) - \mathcal{J}(f_\delta, K_\varepsilon, (\xi_0, \dots, \xi_n))| \\
&+ \left| \mathcal{J}(f_\delta, \mathcal{K}_\varepsilon, (\xi_0, \dots, \xi_n)) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)} \mathrm{Vol}_n(\xi_0, \dots, \xi_n) \right| \\
&\leq \varepsilon \|\mathrm{Vol}_n\| + M_\varepsilon(\delta) + L(\varepsilon, \delta)
\end{aligned}$$

for all $(\xi_0, \dots, \xi_n) \in (\partial\mathbb{H}^n)^{(n+1)}$. Because $\varepsilon, \delta > 0$ were arbitrary we can consider the limit

$$\begin{aligned}
&\left|\mathcal{J}(\varphi, \mathcal{D}, (\xi_0, \dots, \xi_n)) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\mathrm{Vol}_n(\xi_0, \dots, \xi_n)\right| \\
&= \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\left|\mathcal{J}(\varphi, \mathcal{D}, (\xi_0, \dots, \xi_n)) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\mathrm{Vol}_n(\xi_0, \dots, \xi_n)\right| \\
&\leq \lim_{\varepsilon\to 0}\lim_{\delta\to 0}(\varepsilon\|\mathrm{Vol}_n\| + M_\varepsilon(\delta) + L(\varepsilon, \delta)) \\
&= \lim_{\varepsilon\to 0}(\varepsilon\|\mathrm{Vol}_n\| + \lim_{\delta\to 0} L(\varepsilon, \delta)) = 0
\end{aligned}$$

and assuming the unproven lemmas the assertion follows.

$\square$

We shall now prove the previously used lemmas. However we need yet another technical lemma to proceed:

Lemma III.3.12 (cf. [BBI13, Lemma 4, p. 21])**.** *With the above notations,*

$$\mu_G(\{g \in K_\varepsilon :\ g\xi \in B_\delta\}) \leq \sigma_\varepsilon(\delta) \tag{III.12}$$

where $\sigma_\varepsilon(\delta)$ does not depend on $\xi \in \partial\mathbb{H}^n$ and $\sigma_\varepsilon(\delta) \to 0$ for $\delta \to 0$.

Proof of Lemma III.3.12. Recall that $\partial\mathbb{H}^n = G/P$ where P is the stabilizer of one point at the boundary. As we have shown in Lemma I.2.16 there is a measurable section $\eta : G/P \to G$ of the canonical projection $\pi : G \to G/P$ such that $F := \eta(G/P)$ is relatively compact. Now let $\tilde{B}_\delta := \eta(B_\delta)$ and for $\xi \in B_\delta$ set $\tilde{\xi} := \eta(\xi) \in \tilde{B}_\delta$. We claim that

$$\{g \in K_\varepsilon : g\xi \in B_\delta\} = \{g \in K_\varepsilon : \text{there exists } p \in C_\varepsilon \text{ with } g\tilde{\xi}p \in \tilde{B}_\delta\}$$

where $C_\varepsilon := P \cap F^{-1}(K_\varepsilon)^{-1}F$.

First consider the $\supseteq$ inclusion. Let $g \in K_\varepsilon$ such that $g\tilde{\xi}p \in \tilde{B}_\delta$ for some $p \in C_\varepsilon$. Then

$$g\xi = g\pi(\tilde{\xi}) = \pi(g\tilde{\xi}p) \in \pi(\tilde{B}_\delta) = B_\delta$$

Now consider the other inclusion $\subseteq$. Let $g \in K_\varepsilon$ such that $g\xi \in B_\delta$. Then

$$\pi(g\tilde{\xi}) = g\pi(\tilde{\xi}) = g\xi = \pi(\eta(g\xi))$$

and there is a $p \in P$ such that $\eta(g\xi) = g\tilde{\xi}p$, i.e. $p \in P \cap F^{-1}(K_\varepsilon)^{-1}F = C_\varepsilon$, because $\eta(g\xi), \tilde{\xi} \in F = \eta(G/P)$. Hence the claim is proven and we get

$$\begin{aligned}
&\{g \in K_\varepsilon : g\xi \in B_\delta\} = \{g \in K_\varepsilon : \text{there exists } p \in C_\varepsilon \text{ with } g\tilde{\xi}p \in \tilde{B}_\delta\} \\
&= \{g \in K_\varepsilon \cap \tilde{B}_\delta p^{-1}\tilde{\xi}^{-1} \text{ for some } p \in C_\varepsilon\} \subset K_\varepsilon \cap \tilde{B}_\delta C_\varepsilon^{-1}\tilde{\xi}^{-1}
\end{aligned}$$

Thus

$$\mu_G(\{g \in K_\varepsilon : g\xi \in B_\delta\}) \leq \mu_G(K_\varepsilon\tilde{\xi} \cap \tilde{B}_\delta C_\varepsilon^{-1}) \leq \mu_G(\tilde{B}_\delta C_\varepsilon^{-1})$$

where we have also used the fact that G is unimodular in the first inequality (cf. Proposition I.2.17).

Recall that we have for every integrable function f on G

$$\int_G f(g)q(g)\, d\mu_G(g) = \int_{\partial\mathbb{H}^n}\left(\int_P f(g\xi)\, d\mu_P(\xi)\right) d\nu(\dot{g})$$

for some strictly positive continuous function $q : G \to \mathbb{R}^+$ and a positive measure ν on $\partial\mathbb{H}^n$.

We may assume that $\mu_G(\tilde{B}_\delta C_\varepsilon^{-1}) \neq 0$ (otherwise we are done). Then, since q is continuous and strictly positive and the integral is on a relatively compact set, there exists a constant $0 < \alpha < \infty$ such that

$$\alpha\mu_G(\tilde{B}_\delta C_\varepsilon^{-1}) = \int_{\partial\mathbb{H}^n} \left(\int_P \chi_{\tilde{B}_\delta C_\varepsilon^{-1}}(g\xi)\, d\mu_P(\xi) \right) d\nu(\dot{g})$$

By construction we have, that if $g \in \tilde{B}_\delta$, then $g\xi \in \tilde{B}_\delta C_\varepsilon^{-1}$ if and only if $\xi \in C_\varepsilon^{-1}$. Indeed if $\xi \in C_\varepsilon^{-1}$ this is obvious. If $g\xi \in \tilde{B}_\delta C_\varepsilon^{-1}$ then $g\xi = \eta(g_1)s$ where $g_1 \in B_\delta$, $s \in C_\varepsilon^{-1} \subset P$. Since $g \in \tilde{B}_\delta$ we have $g = \eta(g_0)$ for some $g_0 \in B_\delta$. Now

$$\begin{aligned} g\xi = \eta(g_0)\xi = \eta(g_1)s \implies \pi(\eta(g_0)\xi) = \pi(\eta(g_1)s) \implies \pi(\eta(g_0)) = \pi(\eta(g_1)) \\ \implies g_0 = g_1 \implies \xi = s \in C_\varepsilon^{-1} \end{aligned}$$

Thus

$$\int_P \chi_{\tilde{B}_\delta C_\varepsilon^{-1}}(\dot{g}\xi)\, d\mu_P(\xi) = \mu_P(C_\varepsilon^{-1})$$

and hence

$$\alpha\mu_G(\tilde{B}_\delta C_\varepsilon^{-1}) = \nu(B_\delta)\mu_P(C_\varepsilon^{-1})$$

Since $\nu(B_\delta) < n\delta$, inequality (III.12) is proven with

$$\sigma_\varepsilon(\delta) := \frac{1}{\alpha}\mu_P(C_\varepsilon^{-1})n\delta$$

□

Proof of Lemma III.3.10. Let us fix $(\xi_0, \dots, \xi_n) \in (\partial\mathbb{H}^n)^{n+1}$. Then we have

$$\begin{aligned} &|\mathcal{J}(\varphi, K_\varepsilon, (\xi_0, \dots, \xi_n)) - \mathcal{J}(f_\delta, K_\varepsilon, (\xi_0, \dots \xi_n)| \\ &\leq |\mathcal{J}(\varphi, K_{\varepsilon,0}, (\xi_0, \dots, \xi_n)) - \mathcal{J}(f_\delta, K_{\varepsilon,0}, (\xi_0, \dots, \xi_n))| \\ &+ |\mathcal{J}(\varphi, K_{\varepsilon,1}, (\xi_0, \dots, \xi_n)) - \mathcal{J}(f_\delta, K_{\varepsilon,1}, (\xi_0, \dots, \xi_n))| \end{aligned}$$

where

$$K_{\varepsilon,0} := \bigcap_{j=0}^{n} \{g \in K_\varepsilon : g\xi_j \in (\partial\mathbb{H}^n - B_\delta)\} \qquad \text{and} \qquad K_{\varepsilon,1} = K_\varepsilon - K_{\varepsilon,0}$$

However

$$\mathcal{J}(\varphi, K_{\varepsilon,0}, (\xi_0, \dots, \xi_n)) = \mathcal{J}(f_\delta, K_{\varepsilon,0}, (\xi_0, \dots, \xi_n))$$

since $f_\delta(x) = \varphi(x)$ for all $x \in \partial\mathbb{H}^n - B_\delta$ and $g\xi_j \in \partial\mathbb{H}^n - B_\delta$ for all $g \in K_{\varepsilon,0}$ and $j = 0, \dots, n$ by definition. Further

$$\mu_G(K_{\varepsilon,1}) = \mu_G\left(K_\varepsilon \cap \bigcup_{j=0}^{n} \{g \in K_\varepsilon : g\xi_j \in B_\delta\} \right) \leq (n+1)\sigma_\varepsilon(\delta)$$

Now

$$\begin{aligned} &|\mathcal{J}(\varphi, K_{\varepsilon,1}, (\xi_0, \dots, \xi_n)) - \mathcal{J}(f_\delta, K_{\varepsilon,1}, (\xi_0, \dots, \xi_n))| \\ &\leq |\mathcal{J}(\varphi, K_{\varepsilon,1}, (\xi_0, \dots, \xi_n))| + |\mathcal{J}(f_\delta, K_{\varepsilon,1}, (\xi_0, \dots, \xi_n))| \\ &\leq 2\|\mathrm{Vol}_n\|\mu_G(K_{\varepsilon,1}) \leq 2\|\mathrm{Vol}_n\|(n+1)\sigma_\varepsilon(\delta) =: M_\varepsilon(\delta) \end{aligned}$$

and inequality (III.10) is proven with the above definition of $M_\varepsilon(\delta)$. □

Proof of Lemma III.3.11. Observe that by Lemma III.3.10 and the almost everywhere validity of equation (III.9)

$$\begin{aligned}
&\left|\mathcal{J}(f_\delta, K_\varepsilon, (\xi_0, \ldots, \xi_n)) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\mathrm{Vol}_n(\xi_0, \ldots, \xi_n)\right| \\
&\leq |\mathcal{J}(f_\delta, K_\varepsilon, (\xi_0, \ldots, \xi_n)) - \mathcal{J}(\varphi, K_\varepsilon, (\xi_0, \ldots, \xi_n))| \\
&+ |\mathcal{J}(\varphi, K_\varepsilon, (\xi_0, \ldots, \xi_n)) - \mathcal{J}(\varphi, \mathcal{D}, (\xi_0, \ldots, \xi_n))| \\
&+ \left|\mathcal{J}(\varphi, \mathcal{D}, (\xi_0, \ldots, \xi_n)) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\mathrm{Vol}_n(\xi_0, \ldots, \xi_n)\right| \\
&\leq M_\varepsilon(\delta) + \varepsilon\|\mathrm{Vol}_n\|
\end{aligned}$$

for almost every $(\xi_0, \ldots, \xi_n) \in (\partial\mathbb{H}^n)^{(n+1)}$. We shall call the set of full measure where the above estimate holds $\mathcal{L}$ for later reference.

We want to use the continuity of Vol_n on $(\partial\mathbb{H}^n)^{(n+1)}$ to prove the inequality (cf. Proposition II.3.16). In order to estimate the error, that we make by only concerning these tuples, we need to estimate for every $(\xi_0, \ldots, \xi_n) \in (\partial\mathbb{H}^n)^{(n+1)}$ the measure of the set

$$\mathcal{E}(\xi_0, \ldots, \xi_n) := \{g \in K_\varepsilon : f_\delta(g\xi_0), \ldots, f_\delta(g\xi_n) \text{ are pairwise distinct}\}$$

By Proposition III.3.5 we know that the set $\{(\xi, \xi') \in \partial\mathbb{H}^n \times \partial\mathbb{H}^n : \varphi(\xi) = \varphi(\xi')\}$ has measure zero. Because f_δ coincides with φ on $\partial\mathbb{H}^n - B_\delta = B_\delta^c$ the set $F := \{(\xi, \xi') \in B_\delta^c \times B_\delta^c : f_\delta(\xi) = f_\delta(\xi')\}$ has measure zero. Therefore also the set $\{g \in G : g(\xi, \xi') \in F\}$ has μ_G-measure zero for arbitrary $(\xi, \xi') \in \partial\mathbb{H}^n \times \partial\mathbb{H}^n$ (cf. Proposition A.4.13). Additionally

$$\begin{aligned}
K_\varepsilon - \mathcal{E}(\xi_0, \ldots, \xi_n) &= \{g \in K_\varepsilon : \exists i \neq j \text{ s.t. } f_\delta(g\xi_i) = f_\delta(g\xi_j)\} \\
&= \{g \in K_\varepsilon : \exists i \neq j \text{ s.t. } f_\delta(g\xi_i) = f_\delta(g\xi_j) \text{ and } (g\xi_i, g\xi_j) \in B_\delta^c \times B_\delta^c\} \\
&\cup \{g \in K_\varepsilon : \exists i \neq j \text{ s.t. } f_\delta(g\xi_i) = f_\delta(g\xi_j) \text{ and } (g\xi_i \in B_\delta \text{ or } g\xi_j \in B_\delta)\} \\
&\subset \bigcup_{i \neq j} \{g \in K_\varepsilon : g(\xi_i, \xi_j) \in F\} \cup \bigcup_{j=0}^{n} \{g \in K_\varepsilon : g\xi_j \in B_\delta\}
\end{aligned}$$

We thus get the estimate

$$\begin{aligned}
\mu_G(K_\varepsilon - \mathcal{E}(\xi_0, \ldots, \xi_n)) &\leq \sum_{i \neq j} \mu_G(\{g \in K_\varepsilon : g(\xi_i, \xi_j) \in F\}) + \sum_{j=0}^{n} \mu_G(\{g \in K_\varepsilon : g\xi_j \in B_\delta\}) \\
&= \sum_{j=0}^{n} \mu_G(\{g \in K_\varepsilon : g\xi_j \in B_\delta\}) \leq (n+1)\sigma_\varepsilon(\delta)
\end{aligned}$$

where we have also used Lemma III.3.12.

Now back to our above set $\mathcal{L}$. Because $\mathcal{L}$ has full measure it is dense in $(\partial\mathbb{H}^n)^{(n+1)}$ (cf. Corollary A.4.17 and Proposition A.3.3). Hence for an arbitrary $(\xi_0, \ldots, \xi_n) \in (\partial\mathbb{H}^n)^{(n+1)}$ there is a sequence of points $(\xi_0^{(k)}, \ldots, \xi_n^{(k)}) \in \mathcal{L}$ converging to it. Then for every $g \in \mathcal{E}(\xi_0, \ldots, \xi_n)$

$$\lim_{k \to \infty} \mathrm{Vol}_n(f_\delta(g\xi_0^{(k)}), \ldots, f_\delta(g\xi_n^{(k)})) = \mathrm{Vol}_n(f_\delta(g\xi_0), \ldots, f_\delta(g\xi_n)).$$

If we apply the dominated convergenc theorem to the sequence $h_k(g) := \mathrm{Vol}_n(f_\delta(g\xi_0^{(k)}), \ldots, f_\delta(g\xi_n^{(k)}))$, we get

$$\lim_{k \to \infty} \mathcal{J}(f_\delta, \mathcal{E}(\xi_0, \ldots, \xi_n), (\xi_0^{(k)}, \ldots, \xi_n^{(k)})) = \mathcal{J}(f_\delta, \mathcal{E}(\xi_0, \ldots, \xi_n), (\xi_0, \ldots, \xi_n))$$

Now we are in a position to put everything together and get

$$
\begin{aligned}
&\left|\mathcal{J}(f_\delta, K_\varepsilon, (\xi_0,\dots,\xi_n)) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\mathrm{Vol}_n(\xi_0,\dots,\xi_n)\right| \\
&\leq \left|\mathcal{J}(f_\delta, K_\varepsilon, (\xi_0,\dots,\xi_n)) - \mathcal{J}(f_\delta, \mathcal{E}(\xi_0,\dots,\xi_n), (\xi_0,\dots,\xi_n))\right| \\
&+ \left|\mathcal{J}(f_\delta, \mathcal{E}(\xi_0,\dots,\xi_n), (\xi_0,\dots,\xi_n)) - \mathcal{J}(f_\delta, \mathcal{E}(\xi_0,\dots,\xi_n), (\xi_0^{(k)},\dots,\xi_n^{(k)}))\right| \\
&+ \left|\mathcal{J}(f_\delta, \mathcal{E}(\xi_0,\dots,\xi_n), (\xi_0^{(k)},\dots,\xi_n^{(k)})) - \mathcal{J}(f_\delta, K_\varepsilon, (\xi_0^{(k)},\dots,\xi_n^{(k)}))\right| \\
&+ \left|\mathcal{J}(f_\delta, K_\varepsilon, (\xi_0^{(k)},\dots,\xi_n^{(k)})) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\mathrm{Vol}_n(\xi_0^{(k)},\dots,\xi_n^{(k)})\right| \\
&+ \left|\frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\mathrm{Vol}_n(\xi_0^{(k)},\dots,\xi_n^{(k)}) - \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}\mathrm{Vol}_n(\xi_0,\dots,\xi_n)\right|
\end{aligned}
$$

for all $(\xi_0,\dots,\xi_n) \in (\partial\mathbb{H}^n)^{(n+1)}$.

The first and third lines after the inequality sign are each $\leq (n+1)\|\mathrm{Vol}_n\|\sigma_\varepsilon(\delta)$ as we have shown above; the second line is less than δ if k is large enough; the fourth line is $\leq M_\varepsilon(\delta) + \varepsilon\|\mathrm{Vol}_n\|$ since $(\xi_0^{(k)},\dots,\xi_n^{(k)}) \in \mathcal{L}$ for all $k \in \mathbb{N}$ and finally the last line is also less than δ if k is large enough.

Hence the assertion is proven with

$$L(\varepsilon, \delta) := 2\delta + 2(n+1)\|\mathrm{Vol}_n\|\sigma_\varepsilon(\delta) + M_\varepsilon(\delta) + \varepsilon\|\mathrm{Vol}_n\|$$

□

III.3.3. Step 3: The Boundary Map is an Isometry

In the last step we can now piece together what we have proven before. By Corollary III.3.8 we know that $\varphi : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ maps almost every regular ideal simplex to a regular ideal simplex with the same orientation. By Proposition I.8.3 we know, that $\varphi : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ is hence essentially equal to an isometry $h \in \mathrm{Isom}(\mathbb{H}^n)$ on $\partial\mathbb{H}^n$, i.e.

$$\varphi(\xi) = h(\xi)$$

for almost every $\xi \in \partial\mathbb{H}^n$. Because φ is a.e.-ρ-equivariant the same holds for h and we get

$$h(i(\gamma)\cdot\xi) = \rho(\gamma)\cdot h(\xi) \tag{III.13}$$

for almost every $\xi \in \partial\mathbb{H}^n$ and every $\gamma \in \Gamma$. Recall that isometries act via homeomorphisms on the boundary $\partial\mathbb{H}^n$ such that in equation (III.13) all maps and actions are continuous. Because equation (III.13) holds on a full measure subset and every full measure subset of $\partial\mathbb{H}^n$ is dense, we get by a simple continuity argument that the equality holds for every $\xi \in \partial\mathbb{H}^n$.

Since isometries are completely determined by their action on the boundary we get

$$h \cdot i(\gamma) = \rho(\gamma) \cdot h$$

for every $\gamma \in \Gamma$, which is nothing but

$$h \cdot i(\gamma) \cdot h^{-1} = \rho(\gamma)$$

Therefore we have found an isometry h that conjugates i and ρ and the proof of Theorem III.1.1 is finished.

A. Measure Theory

A.1. General Results

We assume that the reader is already familiar with the basic notions of measures and related theorems and topics, e.g. Fubini's Theorem, Dominated Convergenc Theorem, Fatou's Lemma, L^p-spaces etc. There are many good textbooks on these topics such as [Rud09], [AE01]. Therefore we will just focus on less commonly treated results in standard lectures on measure theory and analysis, and introduce some conventions.

We want to stress here again, that we follow [Bou89] in the definition of a locally compact space, i.e. it is automatically Hausdorff.

Definition A.1.1. Let $(X, \mathfrak{A}, \mu)$ be a measure space. A subset $N \subset X$ is called a *null set* or *(μ-)negligible*, if there is a measurable set $N' \in \mathfrak{A}$ such that $N \subset N'$ and $\mu(N') = 0$. A subset $A \subset X$ is called *conull*, if its complement is a null set.

Definition A.1.2. A measure space $(X, \mathfrak{A}, \mu)$ is called *complete*, if every subset of a μ-negligible set is already contained in $\mathfrak{A}$.

The following theorem puts us in the comfortable position, that we may assume without loss of generality that every measure space is complete.

Theorem A.1.3 (Completion of measures)**.** *Let $(X, \mathfrak{A}, \mu)$ be a measure space and let $\mathfrak{N}$ the system of all μ-negligible sets. Set*

$$\mathfrak{A}^* = \{A \cup N : A \in \mathfrak{A}, N \in \mathfrak{N}\}$$

and define $\mu^* : \mathfrak{A}^* \to [0, \infty]$ by

$$\mu^*(A \cup N) := \mu(A)$$

for every $A \in \mathfrak{A}$, $N \in \mathfrak{N}$.

Then:

(i) $\mathfrak{A}^*$ is a σ-algebra, μ^* is well-defined and $(X, \mathfrak{A}^*, \mu^*)$ is a complete measure space. μ^* is the only extension of μ to a content on $\mathfrak{A}^*$.

(ii) Every extension ρ of μ is an extension of μ^*.

Proof. See [Els11, 6.3 Satz, p. 64]. □

Recall the notion of a measure space being σ-finite:

Definition A.1.4 (σ-finite)**.** A measure space $(X, \mathfrak{A}, \mu)$ is called *σ-finite*, if there is a sequence $\{E_n\}_{n \in \mathbb{N}}$ of sets $E_n \in \mathfrak{A}$ such that $\mu(E_n) < \infty$ and $\bigcup_{n=1}^{\infty} E_n = X$.

Let us now prove a slight generalization of the dominated convergence theorem.

Theorem A.1.5 (General Lebesgue Dominated Convergence Theorem)**.** *Let $(X, \mathcal{A}, \mu)$ be a σ-finite complete measure space. Let (f_n) be a sequence of measurable functions on X that converge a.e. pointwise to some function f. Suppose there is a sequence (g_n) of integrable functions on X*

that converge pointwise a.e. to an integrable function g such that $|f_n| \leq g_n$ for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} \int_X g_n = \int_X g$, then

$$\lim_{n\to\infty} \int_X |f_n - f| = 0$$

In particular f is integrable and

$$\lim_{n\to\infty} \int_X f_n = \int_X f$$

This is [RF10, Theorem 19, p. 89]. Since the theorem is only stated in the real version and there is no proof given in the book, we transfer it to the realm of more general measure spaces and give a proof based on the well known Fatou Lemma.

Proof. Since $|f_n| \leq |g_n|$ pointwise a.e. for all $n \in \mathbb{N}$, we have in the limit $|f| \leq g$ pointwise almost everywhere. This implies

$$|f_n - f| \leq g_n + g$$

pointwise a.e. for every $n \in \mathbb{N}$. We can now apply Fatou's Lemma to the non-negative function $g_n + g - |f_n - f| \geq 0$ and get

$$\liminf_{n\to\infty} \int g_n + g - |f_n - f|\, d\mu \geq \int \liminf_{n\to\infty} (g_n + g - |f_n - f|)\, d\mu$$

The right-hand-side is equal to $2 \int g\, d\mu$ by hypothesis. The left-hand-side can be computed to

$$\begin{aligned}\liminf_{n\to\infty} \int g_n + g - |f_n - f| d\mu &= \liminf_{n\to\infty} \int g_n\, d\mu + \int g\, d\mu - \limsup_{n\to\infty} \int |f_n - f|\, d\mu \\ &= 2 \int g\, d\mu - \limsup_{n\to\infty} \int |f_n - f|\, d\mu\end{aligned}$$

This in turn implies that

$$\limsup_{n\to\infty} \int |f_n - f|\, d\mu \leq 0$$

such that

$$0 \leq \liminf_{n\to\infty} \int |f_n - f|\, d\mu \leq \limsup_{n\to\infty} \int |f_n - f|\, d\mu \leq 0$$

Hence the limit exists and we have

$$\lim_{n\to\infty} |f_n - f|\, d\mu = 0$$

which concludes the proof. □

Definition A.1.6 (Absolute continuity)**.** Let $(X, \mathfrak{A})$ be a measurable space with measures μ and ν on $\mathfrak{A}$. Then μ is said to be *absolutely continuous with respect to* ν if $\mu(A) = 0$ for every set $A \in \mathfrak{A}$ such that $\nu(A) = 0$. We will denote the relation of absolute continuity by "$\ll$", i.e.

$$\mu \ll \nu \iff (\forall A \in \mathfrak{A} : \nu(A) = 0 \implies \mu(A) = 0)$$

Definition A.1.7 (Equivalence)**.** Let $(X, \mathfrak{A})$ be a measurable space with measures μ and ν on $\mathfrak{A}$. μ and ν are said to be *equivalent* if $\mu \ll \nu$ and $\nu \ll \mu$, i.e. if they are absolutely continuous with respect to each other. Any equivalence class of measures is then called a *measure class*.

A.2. Measures on Topological Spaces

Definition A.2.1 (Borel measurable)**.** Let X be a topological space. The σ-algebra generated by all open sets of X is called the *Borel σ-algebra of X*. We denote it by $\mathfrak{B}(X)$ or sometimes only $\mathfrak{B}$ if there is no ambiguity. Elements of $\mathfrak{B}$ are called *(Borel) measurable*.

Let X, Y be topological spaces and $f : X \to Y$ be a map. Then f is called *(Borel) measurable* if it is a measurable map between the measurable spaces $(X, \mathfrak{B}(X))$, $(Y, \mathfrak{B}(Y))$, i.e. for every $B \in \mathfrak{B}(Y)$ is $f^{-1}(B) \in \mathfrak{B}(X)$.

Remark A.2.2. *Note that our notion of a measurable function as simply being a Borel measurable function is not always the most natural or effective one. Problems arise with Borel measurable functions when the target space is too large. Thus [Bou04a] and [RS00] take a different and more technical approach to circumvent these issues.*

However in a geometric situation where all topological spaces are locally compact and second countable both notions coincide (cf. [Bou04a, Proposition 1, No. 1 §5 IV.59] and Lusin's Theorem A.2.6 below).

In the following let X be a Hausdorff space and let $\mathfrak{O}$, $\mathfrak{C}$, $\mathfrak{K}$ denote the systems of open resp. closed resp. compact subsets of X. Further we set $\mathfrak{B} = \mathfrak{B}(X)$ as before.

Definition A.2.3. Let $\mathfrak{A} \supset \mathfrak{B}$ be a σ-algebra and $\mu : \mathfrak{A} \to [0, \infty]$ a measure.

(i) μ is called *locally finite* if for every $x \in X$ there is an open neighborhood U about x such that $\mu(U) < \infty$. A locally finite measure $\mu : \mathfrak{B} \to [0, \infty]$ is called a *Borel measure*.

(ii) μ is called *inner regular*, if

$$\mu(A) = \sup\{\mu(K) : K \subset A, K \in \mathfrak{K}\}$$

for every $A \in \mathfrak{A}$.

(iii) μ is called a *Radon measure*, if it is an inner regular Borel measure.

(iv) μ is called *outer regular*, if

$$\mu(A) = \inf\{\mu(U) : U \supset A, U \in \mathfrak{O}\}$$

for every $A \in \mathfrak{A}$.

(v) μ is called *regular*, if it is inner and outer regular.

Remark A.2.4. *The above definitions are not consistently used in the literature! For example in [Rud09] a Borel measure is simply a measure on $\mathfrak{B}(X)$ without any further properties. However we follow here [Els11, 1.1 Definition, p. 313].*

Lemma A.2.5 (Regularity of Borel measures)**.** *Let X be a locally compact second countable (Hausdorff) space. Then every Borel measure on $\mathfrak{B}(X)$ is regular.*

Proof. See [Els11, 1.12 Korollar, p. 319]. □

Theorem A.2.6 (Lusin's Theorem)**.** *Let X, Y be Hausdorff spaces, let Y be further second countable, let $\mu : \mathfrak{B}(X) \to [0, \infty]$ be a σ-finite regular Borel measure and let $f : X \to Y$ be a map. Then the following are equivalent:*

(i) There is a (Borel) measurable function $g : X \to Y$ such that $f = g$ μ-a.e.

(ii) *For every $U \subset X$ open with $\mu(U) < \infty$ and for every $\delta > 0$ there is $K \subset U$ compact such that $\mu(U - K) < \delta$ and f restricted to K is continuous.*

(iii) *For every $A \in \mathfrak{B}(X)$ with $\mu(A) < \infty$ and every $\delta > 0$ there is $K \subset A$ compact such that $\mu(A - K) < \delta$ and f restricted to K is continuous.*

(iv) *For every $T \subset X$ compact and every $\delta > 0$ there is $K \subset T$ compact such that $\mu(T - K) < \delta$ and f restricted to K is continuous.*

Proof. See [Els11, 1.18 Satz, p. 323]. □

Definition A.2.7. Let X be a topological space. A linear form $I : C_c(X) \to \mathbb{R}$ is called *positive*, if

$$f \geq 0 \implies I(f) \geq 0$$

for every $f \in C_c(X)$. Here $C_c(X)$ denotes the space of continuous real-valued functions on X with compact support.

Theorem A.2.8 (Riesz representation theorem)**.** *Let X be a locally compact (Hausdorff) topological space and $I : C_c(X) \to \mathbb{R}$ a positive linear form. Then there is exactly one Radon measure $\mu : \mathfrak{B}(X) \to [0, \infty]$, such that*

$$I(f) = \int_X f \, d\mu$$

for every $f \in C_c(X)$. Further we have that

$$\mu(K) = \inf\{I(f) : f \in C_c(X), f \geq \chi_K\}, \qquad \forall K \in \mathfrak{K}$$

Proof. See [Els11, 2.5 Darstellungssatz von F. Riesz, p. 335]. □

Recall that $C_c(X)$ is a locally convex topological vector space with its topology given by uniform convergence on compact subsets.

Theorem A.2.9. *Let X be a locally compact space. Then every positive linear form $I : C_c(X) \to \mathbb{R}$ is continuous.*

Proof. Let $I : C_c(X) \to \mathbb{R}$ be a positive linear form and let $K \subset X$ be compact. Then there is a continuous mapping $f_0 \in C_c(X, [0, 1])$ such that $f_0(x) = 1$ for every $x \in K$. Thus we have for every continuous function $g : X \to \mathbb{R}$ with support in K

$$-\|g\| \cdot f_0 \leq g \leq \|g\| \cdot f_0$$

and hence $|I(g)| \leq \|g\| \cdot I(f_0)$ which proves the theorem. □

This theorem in conjunction with the Riesz representation theorem A.2.8 allows us to identify all Radon measures on a locally compact space with the subset $\mathcal{M}(X) \subset C_c(X)^*$ of all *continuous* positive linear forms on $C_c(X)$. Indeed, every Radon measure μ on X induces a positive linear form $I : C_c(X) \to \mathbb{R}$ by integration $I(f) = \int_X f \, d\mu$ ($f \in C_c(X)$). Note that not every measure is so well behaved, that continuous functions with compact support are integrable. On the other hand by the Riesz representation theorem every (continuous) positive linear form $I : C_c(X) \to \mathbb{R}$ amounts to a Radon measure.

Remark A.2.10. *The above identification is implicitly used in [Bou04a] as they define a (positive) real measure as a positive continuous linear form on $C_c(X)$.*

Because we are only concerned with geometric situations in which all the occuring spaces are locally compact we come to the following convention.

Convention A.2.11. From now on we will always mean by a measure on a locally compact (Hausdorff) space X a Radon measure μ and identify it with an element of $\mathcal{M}(X) \subset C_c(X)^*$. Thus we will sometimes write $\mu(f)$ instead of $\int_X f\, d\mu$ ($f \in C_c(X), \mu \in \mathcal{M}(X)$). Further we will always complete the resulting measure space (cf. Theorem A.1.3).

Recall that $C_c(X)^*$ – as the dual of $C_c(X)$ – can be equipped with the weak-* topology. Because $\mathcal{M}(X) \subset C_c(X)^*$ this induces a topology on the space of all measures on X. We call this topology the weak-* topology or the vague topology on $\mathcal{M}(X)$. A sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}(X)$ converges to $\mu \in \mathcal{M}(X)$ if and only if

$$\int_X f\, d\mu_n = \mu_n(f) \to \mu(f) = \int_X f\, d\mu \quad (n \to \infty)$$

for every $f \in C_c(X)$.

This topology has a very neat property. For that recall the following result from linear functional analysis.

Theorem A.2.12 (Banach-Alaoglu)**.** *Let E be a normed linear space. Then the unit ball in E^* with respect to the norm topology on E^* is compact with respect to the weak-* topology.*

Proof. See [Zim90, Theorem 1.1.28, p. 22]. □

If X is a compact metric space then $C_c(X) = C(X)$ and $C(X)$ is a normed space. Hence the unit ball in $C_c(X)^* = C(X)^*$ is compact with respect to the weak-* topology. Let $\mathcal{M}^1(X) \subset \mathcal{M}(X)$ denote the set of all probability (Radon) measures on X. We will now see that this space is in fact compact with respect to the weak-* topology on $\mathcal{M}(X)$.

Corollary A.2.13. *Let X be a compact metric space. Then the space of all probability measures $\mathcal{M}^1(X)$ is compact with respect to the weak-* topology on $\mathcal{M}(X) \subset C(X)^*$.*

Proof. It will be sufficient to show, that $\mathcal{M}^1(X)$ is weak-* closed in the normed unit ball $C(X)^*_1$ of $C(X)^*$. However we have that

$$|\mu(f)| = \left|\int_X f\, d\mu\right| \leq \int_X |f|\, d\mu \leq \mu(X) \cdot \|f\| = \|f\|$$

for every $f \in C(X)$ and every $\mu \in \mathcal{M}^1(X)$, i.e. $\mathcal{M}^1(X) \subset C(X)^*_1$.

Further

$$\mu \in \mathcal{M}^1(X) \iff (\forall f \in C(X) : f \geq 0 \implies \mu(f) \geq 0) \text{ and } \mu(1) = 1$$

Thus

$$\mathcal{M}^1(X) = \{\lambda \in C(X)^* : \lambda(1) = 1\} \cap \bigcap_{f\in C(X), f\geq 0} \{\lambda \in C(X)^* : \lambda(f) \geq 0\}$$

which is clearly closed in the weak-* topology. □

Definition A.2.14 (Dirac measure)**.** Let X be a locally compact space and $x \in X$. The measure induced by the positive linear form $\delta_x : C_c(X) \to \mathbb{R}, f \mapsto f(x)$ is called the *Dirac measure at x*.

Definition A.2.15 (Atom)**.** Let X be a locally compact space. A measure $\mu \in \mathcal{M}(X)$ is said to have an *atom at $x \in X$ with weight* $\lambda > 0$, if there is a measure $\nu \in \mathcal{M}(X)$ such that $\nu(\{x\}) = 0$ and $\mu = \lambda \cdot \delta_x + \nu$ (cf. [Bou04a, No. 10 §6 V]).

Definition A.2.16 (Support)**.** If μ is a measure on a locally compact space X, one defines the support of μ, denoted by supp (μ), to be the closed set complementary to the largest of the open sets in X on which the restriction of μ is zero.

A.3. The Canonical Measure Class on an Oriented Smooth Manifold

Let M be an oriented smooth manifold with or without boundary and let $\omega \in \Omega^n(M)$ be an orientation form for M (cf. [Lee13, p. 381]). We can now define a positive linear form $\mu : C_c(M) \to \mathbb{R}$ by

$$\mu_\omega(f) = \int_M f \cdot \omega.$$

for every $f \in C_c(M)$. Obviously μ is linear and, since ω induces the orientation of M, μ is also positive. Hence μ defines a measure on M. Note that this construction turns M into a σ-finite measure space equipped with a regular Borel measure (cf. Lemma A.2.5).

Example A.3.1. In case of an oriented Riemannian manifold M one may take for ω the volume form. In this way we get the *hyperbolic volume measure* ν on $\mathbb{H}^n$ for instance.

If $\tilde{\omega}$ is another orientation form on M, then $\omega = \alpha \cdot \tilde{\omega}$ for some strictly positive smooth function $\alpha : M \to \mathbb{R}$. This shows, that the measure class of μ_ω does not depend on the orientation form ω and we can speak of null sets in M without any specification of ω. Whenever we have an oriented smooth manifold with or withour boundary we will think of it as equipped with this **canonical measure class**.

The following lemma gives a neat characterization of the null sets of this canonical measure class.

Lemma A.3.2. *Let $N \subset M$ be measurable. Then N is a null set if and only if for every coordinate chart (U, φ) of M its image $\varphi(N \cap U) \subset \mathbb{R}^n$ is a Lebesgue null set. Further $N \subset M$ is a null set, if $\varphi_i(U_i \cap N) \subset \mathbb{R}^n$ is a Lebesgue null set for every $i \in I$, where $\{(U_i, \varphi_i)\}_{i \in I}$ is a covering of M by coordinate charts.*

Proof. Without loss of generality we may assume, that every coordinate chart is orientation preserving. Indeed, if a chart is not orientation preserving we may compose it with a reflection and consider the resulting chart. This is admissible, since Lebesgue null sets in $\mathbb{R}^n$ remain null sets after applying a reflection. We shall choose an orientation form ω of M for the rest of the proof.

Let $\varphi : U \to V \subset \mathbb{R}^n$ be an oriented coordinate chart of M. The measure μ_ω restricted to U is given by $\mu_{\omega|U}$. The image measure $\varphi_*(\mu_{\omega|U})$ is given by

$$\begin{aligned} \int_{\mathbb{R}^n} f \, d\varphi_*(\mu_{\omega|U}) &= \int_U f \circ \varphi \cdot \omega \\ &= \int_V (\varphi^{-1})^*(f \circ \varphi \cdot \omega|U) \\ &= \int_{\mathbb{R}^n} f \cdot (\varphi^{-1})^*(\omega|U) \end{aligned}$$

for every $f \in C_c(V)$. Then $(\varphi^{-1})^*\omega|U = \alpha \cdot dx_1 \wedge \ldots \wedge dx_n$ for some smooth strictly positive function $\alpha : V \to \mathbb{R}$. Hence

$$\int_{\mathbb{R}^n} f \, d\varphi_*(\mu_{\omega|U}) = \int_{\mathbb{R}^n} f \cdot \alpha \, dx_1 \ldots dx_n$$

for every $f \in C_c(V)$, which shows that $\varphi_*(\mu_{\omega|U})$ is equivalent to the Lebesgue measure on $V \subset \mathbb{R}^n$.

Thus if $\mu_\omega(N) = 0$ then also $0 = \mu_\omega|_U(N) = \mu_{\omega|U}(N \cap U) = \varphi_*(\mu_{\omega|U})(\varphi(U \cap N))$, such that $\varphi(U \cap N)$ is a Lebesgue null set by the equivalence of $\varphi_*(\mu_{\omega|U})$ and the Lebesgue measure.

Conversely let $\varphi(U \cap N)$ be a Lebesgue null set for every coordinate chart $\varphi : U \to \mathbb{R}^n$ and choose a countable covering $\{U_i\}_{i\in\mathbb{N}}$ of M by oriented coordinate charts $\varphi_i : U_i \to V_i \subset \mathbb{R}^n$. Then we compute

$$0 \leq \mu_\omega(N) \leq \sum_{i\in\mathbb{N}} \mu_{\omega|U_i}(U_i \cap N) = \sum_{i\in\mathbb{N}} (\varphi_i)_*(\mu_\omega|U_i)(\varphi_i(U_i \cap N)) = 0$$

such that N is a null set for μ_ω. The last assertion now follows from what we have shown so far, since one may always choose a countable subcover from any covering $\{U_i\}_{i\in I}$ by Lindelöf's Theorem. □

The above lemma shows, that our notion of null sets on a smooth manifold (with or without boundary) coincides with the notion of "sets of measure zero" in [Lee13, pp. 125].

Proposition A.3.3. *Let M be a smooth manifold with or without boundary and $A \subset M$ a null set in M. Then $M - A$ is dense in M.*

Proof. This is [Lee13, Proposition 6.8, p. 128]. □

Theorem A.3.4. *Let M and N be smooth manifolds with or without boundary, $F : M \to N$ a smooth map, and $A \subset M$ a null set. Then $F(A) \subset N$ is a null set.*

Proof. This is [Lee13, Theorem 6.9, p. 128]. □

The following corollary is immediate.

Corollary A.3.5. *Let $F : M \to M$ be a diffeomorphism and let $A \subset M$ be a null set. Then also $F^{-1}(A)$ and $F(A)$ are null sets.*

Thus the canonical measure class of a smooth manifold (with or without boundary) is invariant under diffeomorphisms.

Corollary A.3.5 will become important, when we consider quasi-invariant measures on homogeneous spaces in the next section.

Proposition A.3.6. *Let M and N be smooth manifolds with or without boundary and $F : M \to N$ a smooth submersion. Then:*

(i) If $A \subset N$ is a null set then, $F^{-1}(A)$ is a null set.

(ii) If $A \subset N$ is conull, then $F^{-1}(A)$ is conull.

Proof. First of all (i) implies (ii), since

$$M - F^{-1}(A) = F^{-1}(N) - F^{-1}(A) = F^{-1}(N - A)$$

Let us turn to (i). Let $m = \dim M \geq \dim N = n$. By the rank theorem (cf. [Lee13, Theorem 4.12 (Rank Theorem), p. 81]), for each point $p \in M$, there exist smooth coordinate charts (U, φ) for M centered at p and (V, ψ) for N centered at $F(p)$ such that $F(U) \subset V$, in which F has a coordinate representation of the form

$$\hat{F}(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m) = (x^1, \ldots, x^n)$$

Since $A \subset N$ is a null set, $\psi(A \cap V) \subset \mathbb{R}^n$ is a Lebesgue null set. Because $\varphi(F^{-1}(A) \cap U) = \varphi(F^{-1}(A \cap V)) = \hat{F}^{-1}(\psi(A \cap V))$ and $\hat{F}$ is just the projection on the first m coordinates it follows (by Fubini's Theorem), that also $\varphi(F^{-1}(A) \cap U)$ is a Lebesgue null set in $\mathbb{R}^m$.

Because M may be covered by such charts $F^{-1}(A)$ is a null set. □

Finally the next statement is a consequence of Sard's Theorem (cf. [Lee13, Theorem 6.10, p. 129]).

Proposition A.3.7. *Let M be a smooth manifold with or without boundary and $S \subset M$ an immersed submanifold with or without boundary. If $\dim S < \dim M$, then S is a null set in M.*

Proof. See [Lee13, Corollary 6.12, p. 131]. □

Remark A.3.8. *What we have developed for oriented smooth manifolds may be generalized to arbitrary smooth manifolds by considering densities instead of differential forms (cf. [Lee13, p. 427]).*

A.4. Invariant Measures

Our main reference for this section is [Bou04b, Chapter VII]. Recall that [Bou04b] uses a slightly different definition of measurable functions as we have already mentioned in Remark A.2.2. However both notions coincide in all of our geometric applications.

A.4.1. Basic Definitions

Let G be a topological group operating continuously on the left in a locally compact space X; we write for the action of $s \in G$ on $x \in X$ simply sx. We denote by $\gamma_X(s)$, or $\gamma(s)$ the homeomorphism of X onto X defined by

$$\boldsymbol{\gamma}(s)x = sx$$

We have

$$\boldsymbol{\gamma}(st) = \boldsymbol{\gamma}(s)\boldsymbol{\gamma}(t) \qquad \forall s, t \in G$$

If f is a function defined on X, $\boldsymbol{\gamma}(s)f$ will be defined by the left regular representation

$$(\boldsymbol{\gamma}(s)f)(x) = f(s^{-1}x)$$

If $\mu \in \mathcal{M}(X)$ is a measure, we define $\boldsymbol{\gamma}(s)\mu$ or sometimes $s_*\mu$ by

$$(s_*\mu)f = (\boldsymbol{\gamma}(s)\mu)f = \mu(\boldsymbol{\gamma}(s^{-1})f)$$

i.e.

$$\int_X f\, d(\boldsymbol{\gamma}(s)\mu)(x) = \int_X f(sx)\, d\mu(x)$$

for every $s \in G$, $f \in C_c(X)$.

If A is a measurable set, then $s^{-1}A$ is measurable and

$$(\boldsymbol{\gamma}(s)\mu)(A) = \mu(s^{-1}A)$$

The measure $\boldsymbol{\gamma}(s)\mu$ may also be defined as the *image* or *pushforward* of μ under $\boldsymbol{\gamma}(s)$.

Instead of writing $d(\boldsymbol{\gamma}(s)\mu)(x)$, it is sometimes useful to write $d\mu(s^{-1}x)$, which then yields

$$\int_X f(x)\, d\mu(s^{-1}x) = \int_X f(sx)\, d\mu(x)$$

for every $s \in G$, $f \in C_c(X)$.

Definition A.4.1 (invariance)**.** Let μ be a measure on X.

(i) μ is said to be *invariant* under G if $\boldsymbol{\gamma}(s)\mu = \mu$ for every $s \in G$.

(ii) μ is said to be *relatively invariant* under G if $\boldsymbol{\gamma}(s)\mu$ is propotional to μ for every $s \in G$

(iii) μ is said to be *quasi-invariant* under G if $\boldsymbol{\gamma}(s)\mu$ is equivalent to μ for every $s \in G$.

Remark A.4.2. *If μ is quasi-invariant and μ' is another measure on X equivalent to μ, then $\boldsymbol{\gamma}(s)\mu'$ is equivalent to $\boldsymbol{\gamma}(s)\mu$, hence to μ, hence to μ', and so μ' is quasi-invariant. To say that μ is quasi-invariant under G therefore means that the* measure class *of μ is invariant under G.*

If μ is quasi-invariant, then the support of μ is invariant under G.

The above can be analogously transferred to the case of a right action of G on X. Thus let G be a topological group operating continuously on the right in a locally compact space X; we write for the action of $s \in G$ on $x \in X$ simply xs. We denote by $\boldsymbol{\delta}_X(s)$, or $\boldsymbol{\delta}(s)$, the homeomorphism of X defined by

$$\boldsymbol{\delta}(s)x = xs^{-1}$$

We have

$$\boldsymbol{\delta}(st) = \boldsymbol{\delta}(s)\boldsymbol{\delta}(t) \qquad \forall s, t \in G$$

As before we define for every $s \in G$, $x \in X$, $f \in C_c(X)$ and measurable set $A \subset X$.

$$\begin{aligned}
(\boldsymbol{\delta}(s)f)(x) &= f(xs) \\
(\boldsymbol{\delta}(s)\mu)(f) &= \mu(\boldsymbol{\delta}(s^{-1})f) \\
\int_X f(x)\, d(\boldsymbol{\delta}(s)\mu)(x) &= \int_X f(xs^{-1})\, d\mu(x) \\
(\boldsymbol{\delta}(s)\mu)(A) &= \mu(As)
\end{aligned}$$

We agree to write $d\mu(xs)$ in place of $d(\boldsymbol{\delta}(s)\mu)(x)$ which then yields

$$\int_X f(x)\, d\mu(xs) = \int_X f(xs^{-1})\, d\mu(x)$$

A.4.2. Haar Measure and Modulus

Let G be a locally compact group. It operates on itself by left an right translation, according to the formulas $\boldsymbol{\gamma}(s)x = sx$, $\boldsymbol{\delta}(s)x = xs^{-1}$ for all $x, s \in G$. Then

$$\boldsymbol{\gamma}(s)\boldsymbol{\delta}(t) = \boldsymbol{\delta}(t)\boldsymbol{\gamma}(s) \qquad \forall s, t \in G$$

All of the foregoing is applicable here, thus we have on G the concepts of measures that are left-invariant, right-invariant, relatively left-invariant, relatively right-invariant, left quasi-invariant, right quasi-invariant.

Definition A.4.3 (Haar measure)**.** Let G be a locally compact group. A nonzero (positive) measure on G that is left (resp. right) invariant is called a *left (resp. right) Haar measure on G*.

Theorem A.4.4. *On every locally compact group, there exists a left (resp. right) Haar measure, and, up to a constant factor, there exists only one.*

Proof. See [Bou04b, Theorem 1, VII.6 §1]. □

Let μ be a left Haar measure on G For every $s \in G$, $\boldsymbol{\delta}(s)\mu$ is also left invariant, therefore there exists a unique number $\Delta_G(s) > 0$ such that $\boldsymbol{\delta}(s)\mu = \Delta_G(s)\mu$. This number is independent of the choice of μ by Theorem A.4.4.

Definition A.4.5 (Modular function Δ_G)**.** The function Δ_G on G is called the *modulus* or *modular function* of G. If $\Delta_G \equiv 1$, the group G is said to be unimodular.

Corollary A.4.6. *The modular function $\Delta_G : G \to \mathbb{R}_+^*$ is a continuous representation.*

Proof. See [Bou04b, No. 3 §1 VII.10]. □

Proposition A.4.7. *Let G be unimodular locally compact group. Then:*

(i) If f is μ-integrable on G, then the functions $x \mapsto f(sx)$, $x \mapsto f(xs)$ and $x \mapsto f(x^{-1})$ are all μ-integrable and their integrals coincide:

$$\int f(sx)\, d\mu(x) = \int f(xs)\, d\mu(x) = \int f(x^{-1})\, d\mu(x) = \int f(x)\, d\mu(x)$$

(ii) If A is a measurable subset of G, then sA, As and A^{-1} are measurable and they have the same measure:

$$\mu(sA) = \mu(As) = \mu(A^{-1}) = \mu(A)$$

Proof. See [Bou04b, 4), VII.12 §1]. □

Proposition A.4.8. *If G is discrete, compact or abelian then G is unimodular.*

Proof. See [Bou04b, Corollary, VII.12 §1]. □

In case of a discrete group G a Haar measure is clearly given, by the measure, which assigns each point of G the mass 1. This particular Haar measure is then called the *normalized* Haar measure on G. Similarly if G is compact every Haar measure is finite and there is only one Haar measure μ on G such that $\mu(G) = 1$. Again this particular Haar measure is then called the *normalized* Haar measure on G. One immediately notices, that both definitions do not coincide in case of a compact discrete (i.e. finite) group G. Thus we will always explicitly specify what is meant by normalized Haar measure in this situation.

A.4.3. Invariant Measures on Quotients X/H

Now we turn to measures on quotients by group actions. We follow here [Bou04b, VII §2]. Let X be a locally compact space in which a locally compact group H operates on the right continuously and properly. Then X/H is Hausdorff and we denote by $\pi : X \to X/H$ the canonical quotient map. Let us further fix a left Haar measure β on H.

Let f be a continuous numerical function on X whose support has compact intersection with the saturation of every compact subset of X. The formula

$$f^\flat(\pi(x)) = \int_H f(x\xi)\, d\beta(\xi)$$

defines a continuous function $f^\flat$ on X/H. If $f \in C_c(X)$, then $f^\flat \in C_c(X/H)$. The mapping $f \mapsto f^\flat$ of $C_c(X)$ into $C_c(X/H)$ is linear and the image of $C_c(X)$ is $C_c(X/H)$.

Proposition A.4.9 (Quotient measures)**.** *(i) Let λ be a measure on X/H. There exists one and only one measure $\lambda^\sharp$ on X such that*

$$\int_{X/H} f^\flat\, d\lambda = \int_X f\, d\lambda^\sharp \tag{A.1}$$

for all $f \in C_c(X)$. One has $\boldsymbol{\delta}(\xi)\lambda^\sharp = \Delta_H(\xi)\lambda^\sharp$ for all $\xi \in H$.

(ii) Conversely, let μ be a measure on X such that $\boldsymbol{\delta}(\xi)\mu = \Delta_H(\xi)\mu$ for all $\xi \in H$. Then there exists one and only one measure λ on X/H such that $\mu = \lambda^\sharp$.

Proof. See [Bou04b, Proposition 4, VII.31 §2]. □

Definition A.4.10. With hypotheses and notations as in Proposition A.4.9, λ is called the quotient of μ by β and is denoted $\frac{\mu}{\beta}$ or μ/β. Further whenever λ is a measure on X/H we will denote by $\lambda^\sharp$ the unique measure from Proposition A.4.9. We will call $\lambda^\sharp$ the *lifted measure* corresponding to λ.

The formula (A.1) may, by analogy with the usual notation for double integrals, be written as

$$\int_X f(x)\, d\lambda^\sharp(x) = \int_{X/H} \int_H f(x\xi)\, d\beta(\xi)\, d\lambda(\dot{x}), \qquad (\dot{x} = \pi(x)) \tag{A.2}$$

This involves an abuse of notation, the integral $\int_H f(x\xi)\, d\beta(\xi)$ being regarded as a function of $\dot{x}$ and not of x; this manner of writing will be used frequently provided no confusion can arise.

Lemma A.4.11. *Assume that μ is a measure on X, such that $\boldsymbol{\delta}(\xi)\mu = \Delta_H(\xi)\mu$ for every $\xi \in H$. Then the quotient measures $(\alpha\cdot\mu)/\beta$ and $\mu/(\alpha\cdot\beta)$ exist and the following formulas hold:*

$$\frac{(\alpha\cdot\mu)}{\beta} = \alpha\cdot\frac{\mu}{\beta} \quad \textit{and} \quad \frac{\mu}{\alpha\cdot\beta} = \alpha^{-1}\cdot\frac{\mu}{\beta}$$

Proof. In order to show, that the quotient measures above exist, we need to check that $\boldsymbol{\delta}(\xi)(\alpha\cdot\mu) = \Delta_H(\xi)(\alpha\cdot\mu)$. Note that we already know that $\boldsymbol{\delta}(\xi)\mu = \Delta_H(\xi)\mu$ for every $\xi \in H$, such that the quotient measure $\mu/(\alpha\cdot\beta)$ exists. Now we compute

$$\boldsymbol{\delta}(\xi)(\alpha\cdot\mu) = \alpha\cdot\boldsymbol{\delta}(\xi)\mu = \alpha\Delta_H(\xi)\mu = \Delta_H(\xi)(\alpha\cdot\mu)$$

for every $\xi \in H$.

Let $f \in C_c(X)$. We compute

$$\int_X f(x)\, d(\alpha\cdot\mu)(x) = \int_{X/H}\int_H f(x\xi)\, d\beta(\xi)\, d(\alpha\cdot\mu/\beta)(\dot{x})$$

But also

$$\begin{aligned}\int_X f(x)\, d(\alpha\cdot\mu)(x) &= \alpha\cdot\int_X f(x)\, d\mu(x)\\ &= \alpha\cdot\int_{X/H}\int_H f(x\xi)\, d\beta(\xi)\, d(\mu/\beta)(\dot{x})\\ &= \int_{X/H}\int_H f(x\xi)\, d\beta(\xi)\, d(\alpha\cdot\mu/\beta)(\dot{x})\end{aligned}$$

By the uniqueness of quotient measures we get as asserted $(\alpha\cdot\mu)/\beta = \alpha\cdot(\mu/\beta)$.

Analogously

$$\int_X f(x)\, d\mu(x) = \int_{X/H}\int_H f(x\xi)\, d(\alpha\cdot\beta)(\xi)\, d(\mu/(\alpha\cdot\beta))(\dot{x})$$

and

$$\begin{aligned}\int_X f(x)\, d\mu(x) &= \int_{X/H}\int_H f(x\xi)\, d\beta(\xi)\, d(\mu/\beta)(\dot{x})\\ &= \int_{X/H}\int_H f(x\xi)\, d(\alpha\cdot\beta)(\xi)\, d(\alpha^{-1}\cdot(\mu/\beta))(\dot{x})\end{aligned}$$

Again by uniqueness, we get $\mu/(\alpha\cdot\beta) = \alpha^{-1}\cdot(\mu/\beta)$ as asserted. □

The following proposition gives formula (A.1) for the general case of integrable functions.

Proposition A.4.12. *Let λ be a (positive) measure on X/H and let f be a $\lambda^\sharp$-integrable function on X, with values in a Banach space or in $\overline{\mathbb{R}}$. Then the set of $\dot{x} \in X/H$ such that $\xi \mapsto f(x\xi)$ is not β-integrable is λ-negligible; the function $f^\flat$ on X/H defined almost everywhere by the formula*

$$f^\flat(\dot{x}) = \int_H f(x\xi)\, d\beta(\xi), \qquad (\dot{x} = \pi(x)) \tag{A.3}$$

is λ-integrable, and

$$\int_{X/H} f^\flat \, d\lambda = \int_X f \, d\lambda^\sharp \tag{A.4}$$

and

$$\int_{X/H} |f^\flat| \, d\lambda \leq \int_X |f| \, d\lambda^\sharp \tag{A.5}$$

Proof. This is c) in [Bou04b, Proposition 5, VII.33 §2]. □

As for formula (A.1) we will write formula (A.4) as

$$\int_X f(x)\, d\lambda^\sharp(x) = \int_{X/H} \int_H f(x\xi)\, d\beta(\xi)\, d\lambda(\dot{x}), \qquad (\dot{x} = \pi(x)) \tag{A.6}$$

by abuse of notation.

The next proposition gives useful criteria for negligible sets and measurable resp. integrable functions on the quotient X/H.

Proposition A.4.13. *Let λ be a positive measure on X/H. Then:*

(i) Let N be a subset of X/H. For N to be locally λ-negligible, it is necessary and sufficient that $\pi^{-1}(N)$ is locally $\lambda^\sharp$-negligible.

(ii) Let g be a function on X/H, with values in a Banach space or in $\overline{\mathbb{R}}$. For g to be measurable, it is necessary and sufficient that $g \circ \pi$ be measurable.

(iii) Let h be a function on X/H, with values in a Banach space or in $\overline{\mathbb{R}}$. For h to be λ-integrable, it is necessary and sufficient that $g \circ \pi$ be $\lambda^\sharp$-integrable.

Proof. This is c) in [Bou04b, Proposition 6, VII.34 §2]. □

Recall the following definition of locally negligible sets (cf. [Bou04a, Definition 3, IV.61 §5]).

Definition A.4.14 (locally negligible)**.** Let X be a locally compact space with a measure μ. A set $A \subset X$ is said to be *locally negligible* (or *locally μ-negligible*) if for every $x \in X$ there exists a neighborhood V of x such that $V \cap A$ is negligible.

Note that in the applications we consider X is always second countable, such that every locally negligible set is automatically negligible and vice versa.

A.4.4. Quasi-invariant Measures on Homogeneous Spaces G/H

We will further need some results on homogeneous spaces; we follow [Bou04b, No. 5 §2 VII]. Therefore let G be a locally compact group and H a closed subgroup of G. Consider the homogeneous space G/H of left cosets with respect to H, on which G acts coninuously on the left. We are going to show, that there is one and only one class of nonzero quasi-invariant measures on G/H.

Note that H operates on G continuously and properly by right translation; and the quotient space G/H is paracompact.

Theorem A.4.15. *Let G be a locally compact group, H a closed subgroup of G.*

Then any two nonzero quasi-invariant measures on G/H are equivalent; the subsets of G/H locally negligible for these measures are those whose inverse image in G is locally negligible for a Haar measure.

Proof. This is a) of [Bou04b, Theorem 1, VII.40 §2]. □

Theorem A.4.16. *Let G be a locally compact group, H a closed subgroup of G, μ a left Haar measure on G, and β a left Haar measure on H. Then:*

(i) There exist functions q continuous and > 0 on G, such that

$$q(x\xi) = \frac{\Delta_H(\xi)}{\Delta_G(\xi)} q(x)$$

for all $x \in G$ and $\xi \in H$.

(ii) Given such a function q, one can form the measure $\lambda = (q \cdot \mu)/\beta$ on G/H (cf. Definition A.4.10), and λ is a nonzero measure quasi-invariant under G.

(iii) Let f be a $q \cdot \mu$-integrable function on G, with values in a Banach space or in $\overline{\mathbb{R}}$. Then, the set of $\dot{x} \in G/H$ such that $\xi \mapsto f(x\xi)$ is not β-integrable is λ-negligible; the function $\dot{x} \mapsto \int_H f(x\xi)\, d\beta(\xi)$ is λ-integrable and

$$\int_G f(x)q(x)\, d\mu(x) = \int_{G/H} \int_H f(x\xi)\, d\beta(\xi)\, d\lambda(\dot{x})$$

Proof. (i) resp. (ii) are a) resp. b) of [Bou04b, Theorem 2, VII.41 §2]. (iii) is c) of [Bou04b, VII.42 §2]. □

Corollary A.4.17. *Let G be a Lie group and H a closed subgroup of G. Then the quotient G/H is a smooth manifold such that G acts smoothly from the left on it. The canonical measure class on G/H as a smooth manifold is the unique quasi-invariant measure class on G/H given by Theorem A.4.16.*

Proof. By Corollary A.3.5 the canonical measure class on G/H is preserved by diffeomorphisms. Because G acts via diffeomorphisms on G/H also the canonical measure class is quasi-invariant under G. By Theorem A.4.15 any two quasi-invariant measures on G/H are equivalent such that the canonical measure class on G/H coincides with the measure class given by Theorem A.4.16. □

Remark A.4.18. *It is worth noting, that what we have stated so far in subsections A.4.3 and A.4.4 is equally true (mutatis mutandis) for left actions instead of right actions.*

A.4.5. Integration on a Fundamental Set

In case of a discrete group action we can realize the integration on the coset space by integration on a measurable fundamental set. We follow here [Bou04b, No. 10 §2 VII], but consider left actions instead of right actions as we will only be interested in this case. Let X be a locally compact space and H a discrete group operating on the left continuously and properly in X. Let π be the canonical mapping of X onto the right coset space $H\backslash X$. For every $x \in X$, we denote by H_x the stabilizer of x in H; this is a finite subgroup of H; its order will be denoted by $n(x)$. For every $s \in H$, $H_{sx} = sH_x s^{-1}$, therefore $n(sx) = n(x)$. There exists an open neighborhood U of x such that $U \cap sU = \emptyset$ for $s \notin H_x$; for $y \in U$, one has $H_y \subset H_x$; thus the function n on X is upper semi-continuous. When X is second countable, H is countable; for, let $K_1, K_2, \dots$ be a covering of X by a sequence of compact subsets, and let $x_0 \in X$; the set of $s \in H$ such that $sx_0 \in K_i$ is finite, whence our assertion.

Definition A.4.19 (fundamental set)**.** Let $F \subset X$. One says that F is a *fundamental set* (for H) if the restriction of π to F is a bijection of F onto $H\backslash X$. In other words, F is a system of representatives for the equivalence relation defined by H.

Theorem A.4.20. *Let X be a locally compact space that is σ-compact, H a discrete group operating continuously and properly on the left in X, π the canonical mapping of X onto $H\backslash X$, μ a measure on X invariant under H, β the normalized Haar measure of H, and $\lambda = \mu/\beta$. Further let F be a measurable fundamental set and let k be a function on $H\backslash X$. Then:*

For k to be measurable (resp. λ-integrable), it is necessary and sufficient that $n^{-1} \cdot \chi_F \cdot (k \circ \pi)$ be measurable (resp. μ-integrable); and, if k is λ-integrable then

$$\int_{H\backslash X} k \, d\lambda = \int_F n^{-1} \cdot (k \circ \pi) \, d\mu$$

Proof. This is c) in [Bou04b, Theorem 4, VII.52 §3]. □

Recall that a locally compact space X is said to be σ-compact or countable at infinity if it is a countable union of compact subsets; in particular every second countable locally compact space is countable at infinity.

Proposition A.4.21. *Let G be a locally compact second countabel group with a discrete subgroup Γ. Then there exists a measurable fundamental set for the left action of Γ on G.*

We give the proof of [Tor]:

Proof. The canonical projection $\pi : G \to \Gamma\backslash G$ is a local homeomorphism. Combined with second-countability, this implies the existence of an open cover $\{U_n\}_{n\in\mathbb{N}}$ of G such that $\pi : U_n \to \pi(U_n)$ is a homeomorphism for every $n \in \mathbb{N}$. We set $F_1 = U_1$ and define

$$F_n = U_n - U_n \cap \pi^{-1}\left(\pi\left(\bigcup_{k<n} U_k\right)\right)$$

Then $F := \bigcup_{n\in\mathbb{N}} F_n$ is a measurable fundamental set for the left action of Γ on G. □

B. G-modules and Banach G-modules

We want to give a brief exposition of the basic notions of G-modules and Banach G-modules here. These are the central objects of continuous cohomology and continuous bounded cohomology as discussed in chapter II.

Let H, G be topological groups. Every vector space will be over the field of real numbers $\mathbb{R}$.

B.1. G-modules

We follow essentially [Gui80].

B.1.1. Basics

Definition B.1.1. A *locally convex topological vector space (LCTVS)* is a real vector space E such that its topology is given by a family $\{\|.\|_\alpha\}_{\alpha \in A}$ of seminorms, i.e. a subbasis for the topology is given by all $\|.\|_\alpha$-balls $B_{\alpha,r}(x) = \{y \in E : \|x - y\|_\alpha < r\}$ with $x \in E$, $r > 0$, $\alpha \in A$.

A continuous linear map $\alpha : E \to F$ of two LCTVS is called a *morphism of LCTVS* or simply a *morphism*.

Remark B.1.2. *Indeed, locally convex topological vector spaces form a category with continuous linear maps between them, such that the above terminology is justified. However we will not use the language of category theory too much in the following.*

Definition B.1.3. A *G-module* is a pair (π, E), where E is a LCTVS and $\pi : G \to \mathrm{Aut}(E)$ is a group homomorphism into the group of continuous linear automorphisms of E, such that the structure map

$$\begin{aligned} G \times E &\to E \\ (g, v) &\mapsto \pi(g)v \end{aligned}$$

is (jointly) continuous. If π is understood, we shall frequently omit it and refer to (π, E) just by E. We then simply write $g \cdot v$ or gv instead of $\pi(g)v$ for all $g \in G$, $v \in E$.

Let E and F be G-modules and $\alpha : E \to F$ a morphism. We call α a *G-morphism*, if it is *G-equivariant*, i.e.

$$\alpha(g \cdot v) = g \cdot \alpha(v)$$

for all $g \in G$ and $v \in E$.

Example B.1.4. The most basic but yet important example is $\mathbb{R}$ as a trivial G-module (via the trivial representation).

Definition B.1.5. Let E be a G-module. The *subspace of invariants* is the subspace

$$E^G := \{v \in E : g \cdot v = v\}$$

Lemma B.1.6. *Let $\alpha : E \to F$ be a G-morphism of G-modules. Then α restricts to a morphism between the subspaces of invariants $\alpha : E^G \to F^G$*

Proof. This follows immediately from the definitions. ☐

B.1.2. Pullback Structure

Let (π, E) be a Banach G-module and $\rho : H \to G$ a continuous group homomorphism. We can think of E as a Banach H-module via the representation $\pi \circ \rho : H \to G \to \mathrm{Aut}(E)$. This structure is called the *pullback structure* on E and we denote the resulting H-module by $\rho^* E$.

If H is a subgroup of G this allows us to think of any Banach G-module E via the pullback structure induced by the inclusion $i : H \hookrightarrow G$. In this case, however, we will also write E instead of $i^* E$ for simplicity and just speak of E as a Banach H-module.

Lemma B.1.7. *Let (π_1, E) and (π_2, F) be two G-modules and $\alpha : E \to F$ a G-morphism. Then $\alpha : \rho^* E \to \rho^* F$ is an H-morphism, i.e. compatible with the pullback structure.*

Proof. This is a simple calculation. We have

$$\alpha(\pi_1(\rho(h))v) = \pi_2(\rho(h))\alpha(v)$$

for all $v \in E$ and $h \in H$, since α is a G-morphism and $\rho(h) \in G$. □

B.2. Banach G-modules

We follow essentially [Mon01].

B.2.1. Basics

Definition B.2.1. A real vector space E with a norm $\|.\|$ is called a *Banach space*, if it is complete with respect to $\|.\|$.

A continuous linear map $\alpha : E \to F$ of Banach spaces is called a *morphism of Banach spaces* or simply a *morphism*. The space $\mathcal{L}(E, F)$ of all continuous linear maps becomes itself a Banach space with the operator norm

$$\|\alpha\| = \sup_{v \in E, v \neq 0} \frac{\|\alpha(v)\|}{\|v\|}, \qquad \alpha \in \mathcal{L}(E, F)$$

A morphism is called *isometric*, if it preserves the norm.

Remark B.2.2. *Indeed, Banach spaces form a category with continuous linear maps between them, such that the above terminology is justified. However we will not use the language of category theory too much in the following.*

Definition B.2.3. A *Banach G-module* is a pair (π, E), where E is a Banach space and $\pi : G \to \mathrm{Iso}(E)$ is a group homomorphism into the group of isometric linear automorphisms of E. If π is understood, we shall frequently omit it and refer to (π, E) just by E. We then simply write $g \cdot v$ or gv instead of $\pi(g)v$ for all $g \in G$, $v \in E$.

Let E and F be Banach G-modules and $\alpha : E \to F$ a morphism (of Banach spaces, i.e. a continuous linear map). We call α a *G-morphism*, if it is *G-equivariant*, i.e.

$$\alpha(g \cdot v) = g \cdot \alpha(v)$$

for all $g \in G$ and $v \in E$.

Remark B.2.4. *We stress, that there is apriori no continuity assumption on the homomorphism $\pi : G \to \mathrm{Iso}(E)$!*

Example B.2.5. The most basic but yet important example is $\mathbb{R}$ as a trivial Banach G-module (via the trivial representation).

Definition B.2.6. Let E be a Banach G-module. The *subspace of invariants* is the subspace

$$E^G := \{v \in E : g \cdot v = v\}$$

Lemma B.2.7. *Let $\alpha : E \to F$ be a G-morphism of Banach G-modules. Then α restricts to a morphism between the subspaces of invariants $\alpha : E^G \to F^G$*

Proof. This follows immediately from the definitions. $\square$

Definition B.2.8. The Banach G-module (π, E) is *continuous* if the structure map

$$\begin{aligned} G \times E &\to E \\ (g, v) &\mapsto \pi(g)v \end{aligned}$$

is continuous, where $G \times E$ is endowed with the product topology (whence the occasional use of the expression *jointly continuous*).

Remark B.2.9. *It is clear from the definition, that a continuous Banach G-module is also a G-module.*

The following lemma shows that due to the isometric action of G on E it actually suffices to consider the orbit maps in order to check for continuity.

Lemma B.2.10. *A Banach G-module (π, E) is continuous if and only if its orbit maps*

$$\begin{aligned} G &\to E \\ g &\mapsto \pi(g)v \end{aligned}$$

are continuous at $e \in G$ (the neutral element of G) for every $v \in E$.

Proof. See [Mon01, Lemma 1.1.1, p. 10]. $\square$

Definition B.2.11. Let (π, E) be a Banach G-module. We define its *maximal continuous submodule*

$$\mathcal{C}_\pi E := \{v \in E : G \to E, g \mapsto gv \text{ is continuous}\}$$

If the representation π is understood, we shall drop the subscript and simply denote it by $\mathcal{C}E$.

The terminology is justified by the following lemma.

Lemma B.2.12. *The Banach G-module E induces on the set $\mathcal{C}E$ the structure of a continuous Banach G-module. Moreover, $\mathcal{C}E$ contains all continuous Banach G-submodules of E.*

Proof. See [Mon01, Lemma 1.2.3, p. 15]. $\square$

Lemma B.2.13. *Any G-morphism $\alpha : E \to F$ of Banach G-modules restricts to $\mathcal{C}E \to \mathcal{C}F$.*

Proof. See [Mon01, Lemma 1.2.4, p. 15]. $\square$

Lemma B.2.14. *Let E be a Banach G-module. Then*

$$E^G \subseteq \mathcal{C}E.$$

In particular $E^G = \mathcal{C}E^G$.

Proof. For every $v \in E^G$ we have that the orbit map $g \mapsto gv$ is constant and therefore continuous, i.e. $v \in \mathcal{C}E$. $\square$

B.2.2. Pullback Structure

Let (π, E) be a Banach G-module and $\rho : H \to G$ a continuous group homomorphism. We can think of E as a Banach H-module via the representation $\pi \circ \rho : H \to G \to \mathrm{Aut}(E)$. This structure is called the *pullback structure* on E and we denote the resulting H-module by $\rho^* E$.

If H is a subgroup of G this allows us to think of any Banach G-module E via the pullback structure induced by the inclusion $i : H \hookrightarrow G$. In this case, however, we will also write E instead of $i^* E$ for simplicity and just speak of E as a Banach H-module.

Lemma B.2.15. *Let (π_1, E) and (π_2, F) be two Banach G-modules and $\alpha : E \to F$ a G-morphism. Then $\alpha : \rho^* E \to \rho^* F$ is an H-morphism, i.e. compatible with the pullback structure.*

Proof. The proof for oridnary G-modules works verbatim. □

Lemma B.2.16. *For any Banach G-module (π, E) we have*

$$\mathcal{C}_\pi E \subset \mathcal{C}_{\pi\rho} E$$

Proof. By definition

$$\mathcal{C}_\pi E = \{v \in E : g \mapsto \pi(g)v \text{ is continuous}\}$$

and

$$\mathcal{C}_{\pi\rho} E = \{v \in E : h \mapsto \pi(\rho(h))v \text{ is continuous}\}$$

If $G \to E, g \mapsto \pi(g)v$ is continuous, then so is $H \to E, h \mapsto \pi(\rho(h))v$ and the asserted inclusion follows. □

C. Amenability

We want to give a brief introduction to amenable groups and amenable actions here. They play an important role in the study of continuous bounded cohomology; in particular in the context of L^∞-resolutions.

We follow in essence [Mon01, II.5]. In the following G is a locally compact second countable topological group.

C.1. Amenable Groups

Definition C.1.1 (Amenable Group)**.** G is called *amenable* if one of the following equivalent conditions is satisfied (cf. [Mon01, II.5, p. 46]):

(i) *(fixed point property)* For every (jointly) continuous linear G-action on a Hausdorff locally convex topological vector space F and every non-empty compact convex G-invariant subset $K \subset F$, there is a G-fixed point in K.

(ii) *(invariant mean property)* There is an *invariant mean* on $L^\infty(G, \mathbb{R})$, that is a norm one G-morphism $\mathfrak{m} : L^\infty(G, \mathbb{R}) \to \mathbb{R}$ such that $\mathfrak{m}(\mathbf{1}_G) = 1$, where $\mathbb{R}$ is the trivial Banach G-module and $\mathbf{1}_G$ denotes the constant function with value 1 on G. Here G is equipped with a Haar measure.

Proposition C.1.2. *Every compact group is amenable.*

Proof. Recall that if G is a compact group, then every function in $L^\infty(G, \mathbb{R})$ is integrable, since the Haar measure μ on G is finite. We can normalize μ, such that $\mu(G) = 1$.

An invariant mean $\mathfrak{m} : L^\infty(G, \mathbb{R}) \to \mathbb{R}$ is now given by integration

$$\mathfrak{m}(f) := \int_G f(g)\, d\mu(g), \qquad \forall f \in L^\infty(G, \mathbb{R})$$

as one readily checks. □

Proposition C.1.3. *Every finite group is amenable.*

Proof. This follows from the previous proposition, since every finite group is compact with respect to its discrete topology. □

Proposition C.1.4. *Every abelian group is amenable.*

Proof. This is a consequence of the above fixed point property and the Markov-Kakutani fixed point theorem. For details see [Pat88, (0.14) Proposition, p. 13]. □

Proposition C.1.5. *Let $N \lhd G$ be a normal subgroup. Then G is amenable if and only if N and G/N are amenable.*

Proof. See [Pat88, (1.13) Proposition, p. 31]. □

Proposition C.1.6. *Every locally compact solvable group is amenable.*

Proof. Let G be a locally compact solvable group. Then it admits a finite derived series

$$1 = G^{(k)} \lhd G^{(k-1)} \lhd \cdots \lhd G^{(1)} \lhd G^{(0)} := G$$

where $G^{(i+1)} = [G^{(i)}, G^{(i)}]$, i.e. the quotient group $G^{(i)}/G^{(i+1)}$ is abelian, for every $i = 0, \ldots, k-1$. We get the following short exact sequences

$$1 \to G^{(i)} \to G^{(i-1)} \to G^{(i-1)}/G^{(i)} \to 1$$

for every $i = 1, \ldots, k$. Starting from $i = k$, where $G^{(k)} = 1$ is clearly amenable one deduce succesively that every $G^{(i)}$ is amenable by Proposition C.1.5. Hence $G^{(0)} = G$ is amenable too. □

Proposition C.1.7. *Let $H < G$ be an amenable subgroup with finite index $[G : H] = m < \infty$. Then G is also amenable, i.e. any virtually amenable group is amenable.*

Proof. Let F be a Hausdorff LCTVS on which G acts jointly continuous and linear. Further let $K \subset F$ be a G-invariant compact convex subset. By restricting the action of G to H we get a jointly continuous linear action of H on F and K remains H-invariant. Thus by the amenability of H there is a fixed point f_H of the H-action in K.

Now consider a representational system $\{g_i : i = 1, \ldots, m\}$ of the left cosets of H in G, i.e.

$$G = \bigsqcup_{i=1}^{m} g_i H$$

We set

$$f_G := \sum_{i=1}^{m} \frac{1}{m} \cdot g_i f_H$$

and claim that f_G is indeed a fixed point of the G-action in K. As K is convex and $\sum_{i=1}^{m} \frac{1}{m} = 1$, we have that f_G is indeed in K.

Now to any $g \in G$ and $i \in \{1, \ldots, m\}$ there is a unique $\varphi_g(i) \in \{1, \ldots, m\}$ such that $gg_i \in g_{\varphi_g(i)}H$. We denote this bijection by $\varphi_g : \{1, \ldots, m\} \to \{1, \ldots, m\}$. Thus there is also a unique $h(i, g) \in H$ such that $gg_i = g_{\varphi_g(i)}h(i, g)$. Hence for every $g \in G$ we get

$$gf_G = \sum_{i=1}^{m} \frac{1}{m} gg_i f_H = \sum_{i=1}^{m} \frac{1}{m} g_{\varphi_g(i)} h(i, g) f_H = \sum_{i=1}^{m} \frac{1}{m} g_{\varphi_g(i)} f_H = f_G$$

i.e. f_G is indeed a G-fixed point in K.

Therefore G is amenable by the fixed point property. □

C.2. Amenable Actions

Definition C.2.1 (Standard Borel space)**.** A measurable space $(X, \mathcal{A})$ is called *standard* if $(X, \mathcal{A})$ is isomorphic to some compact metric space with the Borel σ-algebra.

Definition C.2.2 (Regular G-space)**.** A *regular G-space* is a standard Borel space S on which G acts measurably, together with a G-invariant measure class with the following property:

The measure class contains a probability measure μ turning (S, μ) in a standard probability space such that the natural isometric G-action $\lambda^\flat$ on $L^1(\mu)$

$$(\lambda^\flat(g)\varphi)(s) = \varphi(g^{-1}s)\frac{dg^{-1}\mu}{d\mu}(s), \qquad (\varphi \in L^1(\mu), s \in S)$$

is continuous, where $dg^{-1}\mu/d\mu$ is the Radon-Nikodym derivative.

Example C.2.3. The following are some examples given in [Mon01, Example 2.1.2, p. 18]:

(i) G itself as a locally compact second countable topological group is a regular G-space with its Haar measure class. More generally, for any closed subgroup $H < G$ the homogeneous space G/H with the class of the natural quasi-invariant measures is a regular G-space.

(ii) The product of finitely many or countably many regular G-spaces is still a regular G-space when endowed with the product structure.

(iii) If S is a regular G-space, H another topological group and $H \to G$ a continuous homomorphism, then by pullback S is a regular H-space.

Definition C.2.4 (Amenable Action, Amenable G-space)**.** Let S be a regular G-space. The G-action on S is called *amenable* if there is a G-equivariant conditional expectation $L^\infty(G \times S) \to L^\infty(S)$ (cf. [Mon01, Theorem 5.3.2, p. 48]). In this case we call S also an *amenable regular G-space.*

Definition C.2.5 (Conditional Expectation)**.** A *conditional expectation* $\mathfrak{m} : L^\infty(G \times S) \to L^\infty(S)$ is a norm one linear continuous map such that

(i) $\mathfrak{m}(\mathbf{1}_{G\times S}) = \mathbf{1}_S$

(ii) for all $f \in L^\infty(G \times S)$ and each measurable subset $A \subset S$ one has $\mathfrak{m}(f \cdot \mathbf{1}_{G\times A}) = \mathfrak{m}(f) \cdot \mathbf{1}_A$

Remark C.2.6. *There is also a definition of amenable action more related to the fixed point property of amenable groups in Definition C.1.1. However we do not need it here and refer to [Mon01, pp. 48] for more details.*

We now state some properties of such amenable G-spaces

Proposition C.2.7. *Let $H < G$ be a closed subgroup. Then the G-action on G/H is amenable if and only if H is an amenable group.*

Proof. See [Zim84, Proposition 4.3.2, p. 78]. □

Proposition C.2.8. *If S, T are regular G-spaces and the G-action on S is amenable, then the diagonal G-action on $S \times T$ is amenable.*

Proof. See [Zim84, Proposition 4.3.4, p. 79]. □

Lemma C.2.9. *Let $H < G$ be a closed subgroup and S an amenable regular G-space. The restriction to H of the action on S is amenable.*

Proof. See [Mon01, Lemma 5.4.3, p. 53]. □

D. Classical Cohomology

We will assume that the reader is already familiar with classical (co)homology theories such as singular (co)homology and de Rham cohomology. The objective of this appendix is to fix some notation and to recall some results from topology. However we also give an introduction to singular bounded cohomology and its relative version, which admits a long exact sequence just as ordinary singular cohomology. Finally we define relative de Rham cohomology and prove a relative version of de Rham's theorem, that we were unable to find in the standard literature.

There are already many good books on algebraic and differential topology. Our main references are [Hat02] and [Lee13].

D.1. Singular Homology

Let $n \in \mathbb{N}_0$. We define the *standard n-simplex* Δ^n in $\mathbb{R}^n$ as the subset

$$\Delta^n := \left\{ (t_1, \dots, t_n) \in \mathbb{R}^n : 0 \leq t_i \leq 1 \text{ and } \sum_{i=1}^{n} t_i \leq 1 \right\}$$

For any $(n+1)$ points $v_0, \dots, v_n$ in some euclidean space $\mathbb{R}^m$ we can now define the affine map

$$[v_0, \dots, v_n] : \Delta^n \longrightarrow \mathbb{R}^m$$

given by

$$[v_0, \dots, v_n](t_1, \dots, t_n) := \left(1 - \sum_{i=1}^{n} t_i \right) v_0 + t_1 v_1 + \dots + t_n v_n$$

for all $(t_1, \dots, t_n) \in \Delta^n$.

With this notation the boundary faces of Δ^n can be parametrized via the maps

$$F_{i,n} = [e_0, \dots, \hat{e_i}, \dots, e_n] : \Delta^{n-1} \longrightarrow \Delta^n$$

where $e_0 = 0 \in \mathbb{R}^n$, $e_1, \dots, e_n \in \mathbb{R}^n$ is the standard basis and as usual a hat over a variable indicates its omission.

Let M be a topological space. A *singular n-simplex* is a continuous map $\sigma : \Delta^n \to M$. We denote by $S_n(M)$ the free abelian group generated by all singular n-simplices and call it the *singular chain group*. Actually it is customary to define the singular chain groups with coefficients in an arbitrary ring and not only $\mathbb{Z}$-coefficients, but we will not need this here.

We can define *boundary maps*

$$\partial : S_n(M) \to S_{n-1}(M)$$

via

$$\partial \sigma = \sum_{i=0}^{n} (-1)^i \cdot (\sigma \circ F_{i,n})$$

for every singular n-simplex $\sigma : \Delta^n \to M$.

Because $\partial \circ \partial = 0$. This gives us a chain complex

$$\cdots \longleftarrow S_{n-1}(M) \xleftarrow{\partial} S_n(M) \xleftarrow{\partial} S_{n+1}(M) \longleftarrow \cdots$$

The *singular n-cycles* are the elements of

$$Z_n(M) = \ker\{\partial : S_n(M) \to S_{n-1}(M)\}$$

and the *singular n-boundaries* are the elements of

$$B_n(M) = \mathrm{im}\{\partial : S_{n-1}(M) \to S_n(M)\}$$

The *n-th singular homology group* is the quotient

$$H_n(M) = \frac{Z_n(M)}{B_n(M)}$$

It is also possible to define *relative singular homology*. For that consider a subspace $N \subset M$ and set

$$S_n(M,N) = S_n(M)/S_n(N)$$

We call these groups the *relative singular chain groups*. We get the following short exact sequence of chain complexes. Observe that one can identify $S_n(M,N)$ with the free abelian group generated by all singular n-simplices in M whose image is not entirely contained in N. Therefore we get the following split short exact sequence.

$$0 \longrightarrow S_\bullet(N) \xrightarrow{\iota_*} S_\bullet(M) \longrightarrow S_\bullet(M,N) \longrightarrow 0$$

where $i_* : S_n(N) \to S_n(M)$ is the map induced by the inclusion $i : N \hookrightarrow M$.

The quotient

$$H_n(M,N) = \frac{\ker\{\partial : S_n(M,N) \to S_{n-1}(M,N)\}}{\mathrm{im}\{\partial : S_{n+1}(M,N) \to S_n(M,N)\}}$$

is then called the *n-th relative singular homology group*. It is worth noting, that $H_\bullet(M,\emptyset) = H_\bullet(M)$.

By the familiar snake lemma one gets a natural long exact sequence in homology

$$\cdots \longrightarrow H_n(N) \xrightarrow{i_*} H_n(M) \longrightarrow H_n(M,N) \xrightarrow{\partial_*} H_{n-1}(N) \longrightarrow \cdots$$

with connecting homomorphisms $\partial_* : H_\bullet(M,N) \to H_{\bullet-1}(N)$.

The (relative) singular homology groups enjoy many nice properties, e.g. homotopy invariance, excision etc...; for details we refer to [Hat02].

We will adopt the following notation for another subspace $A \subset M$

$$H_n(M|A) = H_n(M, M-A)$$

and for a point in $x \in M$ we simply write

$$H_n(M|x) = H_n(M|\{x\}) = H_n(M, M-\{x\})$$

Now assume that M is a closed oriented topological n-manifold. Then there is an element $[M] \in H_n(M)$ such that the restriction $H_n(M) \to H_n(M|x)$ maps $[M]$ to the given orientation at each point $x \in M$ (cf. [Hat02, Theorem 3.26, p. 236]). This element is called the *fundamental class of M*.

Similarly if M is a compact oriented topological n-manifold with boundary ∂M, there is an element $[M, \partial M] \in H_n(M, \partial M)$ restricting to the given orientation at each pont $x \in M - \partial M$ (cf. [Hat02, p. 253]). Then this element is also called the *(relative) fundamental class of M.*

If M admits a triangulation, it is possible to give a concrete depiction of the fundamental class. Indeed, let $\{\sigma_i : \Delta^n \to M\}$ be the set of all characterstic functions of the triangulation of M. Then the fundamental class of M is represented by the sum $\sum_i k_i \sigma_i$, where $k_i = +1$ if $\sigma_i : \Delta^n \to M$ is orientation preserving, and $k_i = -1$ if it is not (cf. [Hat02, p. 238], [BP92, Proposition C.3.1., p. 104]). This is for example the case if M is a smooth manifold with or without boundary (see [Mun66, 10.6 Theorem, p. 103]).

In view of the de Rham theorem we are about to state later it will be important to observe, that we can compute the singular homology groups of a smooth manifold M on the chain complex of smooth simplices. We follow essentially [Lee13, Chapter 18], but we generalize the results to the case of a smooth manifold with boundary. The fact that a smooth manifold M with boundary ∂M is homotopy equivalent to its interior $\mathrm{int} M = M - \partial M$ as the following theorem states, will be of frequent use for our generalizations.

Theorem D.1.1. *Let M be a smooth manifold with nonempty boundary and let $i : \mathrm{int} M \to M$ denote the inclusion. There exists a proper smooth embedding $R : M \to \mathrm{int} M$ such that both $i \circ R : M \to M$ and $R \circ i : \mathrm{int} M \to \mathrm{int} M$ are smoothly homotopic to the respective identity maps. Therefore i is a homotopy equivalence.*

Proof. The proof uses the familiar collar theorem. For details see [Lee13, Theorem 9.26., p. 223]. □

Let M be a smooth manifold with or without boundary in the following.

A map $\sigma : \Delta^n \to M$ is called a *smooth singular n-simplex* if it is a smooth mapping between smooth manifolds with corners, i.e. if every point of Δ^n admits an open neighborhood on which σ has a smooth extension. For more details on smooth manifolds with corners we refer to [Lee13, Chapter 16, pp. 415].

Let us denote by $S_n^\infty(M)$ the set of all smooth singular n-simplices on M. Clearly the usual boundary operator $\partial : S_n(M) \to S_{n-1}(M)$ restricts to $\partial : S_n^\infty(M) \to S_{n-1}^\infty(M)$, thus giving us a chain complex

$$\cdots \longleftarrow S_{n-1}^\infty(M) \xleftarrow{\partial} S_n^\infty(M) \xleftarrow{\partial} S_{n+1}^\infty(M) \longleftarrow \cdots$$

We will call the n-th homology group of this chain complex

$$H_n^\infty(M) = \frac{\ker\{\partial : S_n^\infty(M) \to S_{n-1}^\infty(M)\}}{\mathrm{im}\{\partial : S_{n+1}^\infty(M) \to S_n^\infty(M)\}}$$

the *n-th smooth singular homology group.* Note that the canonical inclusion $\iota : S_\bullet^\infty(M) \to S_\bullet(M)$ of chain complexes commutes with the boundary maps and hence induces a map in homology.

It is also possible to define relative smooth singular homology groups. For that let $i : N \hookrightarrow M$ be an embedded submanifold and set

$$S_n^\infty(M, N) = S_n^\infty(M)/S_n^\infty(N)$$

We call these groups the *relative smooth singular chain groups.* We get the following short exact sequence of chain complexes. Observe that one can identify $S_n^\infty(M, N)$ with the free abelian group generated by all smooth singular n-simplices in M whose image is not entirely contained in N. Therefore we get the following split short exact sequence.

$$0 \longrightarrow S_\bullet^\infty(N) \xrightarrow{i_*} S_\bullet^\infty(M) \longrightarrow S_\bullet^\infty(M,N) \longrightarrow 0$$

where $i_* : S_n^\infty(N) \to S_n^\infty(M)$ is the map induced by the (smooth) inclusion $i : N \hookrightarrow M$.

The quotient

$$H_n^\infty(M,N) = \frac{\ker\{\partial : S_n^\infty(M,N) \to S_{n-1}^\infty(M,N)\}}{\operatorname{im}\{\partial : S_{n+1}^\infty(M,N) \to S_n^\infty(M,N)\}}$$

is then called the *n-th relative smooth singular homology group*. It is worth noting, that $H_\bullet^\infty(M,\emptyset) = H_\bullet^\infty(M)$.

By the familiar snake lemma one gets a natural long exact sequence in homology

$$\cdots \longrightarrow H_n^\infty(N) \xrightarrow{i_*} H_n^\infty(M) \longrightarrow H_n^\infty(M,N) \xrightarrow{\partial_*} H_{n-1}^\infty(N) \longrightarrow \cdots$$

with connecting homomorphisms $\partial_* : H_\bullet^\infty(M,N) \to H_{\bullet-1}^\infty(N)$.

It turns out that (relative) smooth and singular homology groups coincide as the next theorem states.

Theorem D.1.2. *Let M be a smooth manifold with or without boundary and $i : N \hookrightarrow M$ an embedded submanifold. Denote by $\iota : S_\bullet^\infty(M,N) \hookrightarrow S_\bullet(M,N)$ the canonical inclusion of chain complexes.*

Then the map induced by the inclusion of chain complexes $\iota_* : H_\bullet^\infty(M,N) \to H_\bullet(M,N)$ is an isomorphism.

Proof. In case of non-relative homology groups and a smooth manifold without boundary this is precisely [Lee13, Theorem 18.7, p. 474]. We will generalize this result following a usual pattern in algebraic topology. First we prove it in the non-relative case for smooth manifolds with boundary using Theorem D.1.1. Then we generalize it further to the relative case by using the long exact sequence and the familiar five lemma.

Let us denote by $j : \operatorname{int}M \hookrightarrow M$ the canonical inclusion. By Theorem D.1.1 this is a (smooth) homotopy equivalence and hence induces an isomorphism in both smooth and singular homology $j_* : H_\bullet^{(\infty)}(\operatorname{int}M) \to H_\bullet^{(\infty)}(M)$. The inclusion of chain complexes ι and the map i_* induced by the previous inclusion commute at the chain level and hence give a commutative diagram of the homology complexes

$$\begin{tikzcd}
H_\bullet^\infty(M) \arrow[r, "\iota_*"] & H_\bullet(M) \\
H_\bullet^\infty(\operatorname{int}M) \arrow[u, "\cong"] \arrow[r, "\iota_*"] & H_\bullet(\operatorname{int}M) \arrow[u, "\cong"]
\end{tikzcd}$$

This settles the case of M having a boundary.

Now the inclusion of $\iota : S_\bullet^\infty(M,N) \to S_\bullet(M,N)$ constitute a morphism of short exact sequences of complexes

$$\begin{tikzcd}
0 \arrow[r] & S_\bullet^\infty(N) \arrow[r, "i_*"] \arrow[d, "\iota_*"] & S_\bullet^\infty(M) \arrow[r] \arrow[d, "\iota_*"] & S_\bullet^\infty(M,N) \arrow[r] \arrow[d, "\iota_*"] & 0 \\
0 \arrow[r] & S_\bullet(N) \arrow[r, "i_*"] & S_\bullet(M) \arrow[r] & S_\bullet(M,N) \arrow[r] & 0
\end{tikzcd}$$

which yield a long exact sequence in homology by the snake lemma.

$$\begin{tikzcd}
\cdots \arrow[r] & H_n^\infty(N) \arrow[r] \arrow[d, "\cong"'] & H_n^\infty(M) \arrow[r] \arrow[d, "\cong"'] & H_n^\infty(M,N) \arrow[r] \arrow[d, "\iota_*"'] & H_{n-1}^\infty(N) \arrow[r] \arrow[d, "\cong"'] & H_{n-1}^\infty(M) \arrow[r] \arrow[d, "\cong"'] & \cdots \\
\cdots \arrow[r] & H_n(N) \arrow[r] & H_n(M) \arrow[r] & H_n(M,N) \arrow[r] & H_{n-1}(N) \arrow[r] & H_{n-1}(M) \arrow[r] & \cdots
\end{tikzcd}$$

By the five lemma the map in the middle is then also an isomorphism and the assertion follows. □

D.2. Singular Cohomology

We will only consider singular cohomology with real coefficients here. Again let M be a topological space.

By dualizing the singular chain groups $S_n(M)$ we get the so called *singular cochain groups*

$$S^n(M,\mathbb{R}) = S_n(M)^* = \operatorname{Hom}(S_n(M),\mathbb{R})$$

In what follows we will not mention the coefficient ring $\mathbb{R}$ and simply write $S^n(M)$.

By taking adjoints we get coboundary maps $\delta = \partial^* : S^n(M) \to S^{n+1}(M)$, i.e.

$$(\delta\alpha)(\sigma) = \alpha(\partial\sigma)$$

for every $\alpha \in S^n(M)$ and every singular $(n+1)$-simplex $\sigma : \Delta^{n+1} \to M$. By these coboundary maps we get the singular cochain complex

$$\cdots \longrightarrow S^{n-1}(M) \xrightarrow{\delta} S^n(M) \xrightarrow{\delta} S^{n+1}(M) \longrightarrow \cdots$$

The n-th cohomology group of this cochain complex

$$H^n(M) = \frac{\ker\{\delta : S^n(M) \to S^{n+1}(M)\}}{\operatorname{im}\{\delta : S^{n-1}(M) \to S^n(M)\}}$$

is called the *n-th singular cohomology group.*

Just as for singular homology we can define relative singular cohomology groups. For that consider a subspace $N \subset M$. By dualizing the split short exact sequence for relative singular chain complexes we get the following split short exact sequence of cochain complexes.

$$0 \longrightarrow S^\bullet(M,N) \longrightarrow S^\bullet(M) \xrightarrow{\iota^*} S^\bullet(N) \longrightarrow 0$$

where $\iota^* : S^n(M) \to S^n(M)$ is the map induced by the inclusion $\iota : N \to M$ and $S^n(M,N) \subset S^n(M)$ is generated by the singular n-cochains that vanish on singular n-simplices $\sigma : \Delta^n \to M$ whose image is completely contained in N.

The $S^n(M,N)$ are called *relative singular cochain groups.* The quotient

$$H^n(M,N) = \frac{\ker\{\delta : S^n(M,N) \to S^{n+1}(M,N)\}}{\operatorname{im}\{\delta : S^{n-1}(M,N) \to S^n(M,N)\}}$$

is then called the *n-th relative singular cohomology group.*

By the familiar snake lemma one gets a natural long exact sequence in cohomology

$$\cdots \longrightarrow H^n(M,N) \longrightarrow H^n(M) \xrightarrow{\iota^*} H^n(N) \xrightarrow{\delta_*} H^{n+1}(M,N) \longrightarrow \cdots$$

with connecting homomorphisms $\delta_* : H^\bullet(M,N) \to H^{\bullet+1}(N)$. It is worth noting that $H^\bullet(M,\emptyset) = H^\bullet(M)$.

Just as singular homology also singular cohomology enjoys many nice properties, e.g. homotopy invariance, excision etc...; for details see [Hat02].

We can define a product $\langle .,. \rangle : S^n(M,N) \times S_n(M,N) \to \mathbb{R}$ by evaluation, i.e.

$$\langle \alpha, c \rangle := \alpha(c)$$

It is easy to see, that this product is well-defined and even induces a product

$$\langle .,. \rangle : H^n(M,N) \times H_n(M,N) \to \mathbb{R}$$

This is called the *Kronecker product.* The universal coefficient theorem ([Hat02, Theorem 3.2, p. 195]) implies, that this product in fact induces an isomorphism

$$H^n(M,N) \to \operatorname{Hom}(H_n(M,N), \mathbb{R})$$

since we are only concerned with real coefficients.

Now let M be a smooth manifold with or without boundary and $i : N \hookrightarrow M$ an embedded submanifold. We can also get a smooth version of singular cohomology by dualizing the smooth singular chain complex. Set

$$S^\bullet_\infty(M,N) = S^\infty_\bullet(M,N)^* = \operatorname{Hom}(S^\infty_\bullet(M,N), \mathbb{R})$$

and define $S^\bullet_\infty(M) = S^\bullet_\infty(M,\emptyset)$. These are called the *(relative) smooth singular cochain groups.* They constitute the *(relative) smooth singular cochain complex* via the usual coboundary maps and its n-th cohomology group

$$H^n_\infty(M,N) = \frac{\ker\{\delta : S^n_\infty(M,N) \to S^{n+1}_\infty(M,N)\}}{\operatorname{im}\{\delta : S^{n-1}_\infty(M,N) \to S^n_\infty(M,N)\}}$$

is called the *n-th (relative) smooth singular cohomology group.*

Completely analogously one gets the familiar short exact sequence at the cochain level inducing a long exact sequence in cohomology. We also get by the same definition as for singular cohomology a Kronecker product

$$\langle .,. \rangle : H^n_\infty(M,N) \times H^\infty_n(M,N) \to \mathbb{R}$$

which in turn induces by the universal coefficient theorem of homological algebra an isomorphism

$$H^n_\infty(M,N) \to \operatorname{Hom}(H^\infty_n(M,N), \mathbb{R})$$

By naturality of the short exact sequence in the universal coefficient theorem and Theorem D.1.2 one immediately deduces, that the adjoint map $\iota^* : S^\bullet(M,N) \to S^\bullet_\infty(M,N)$ induces an isomorphism $\iota^* : H^\bullet(M,N) \to H^\bullet_\infty(M,N)$.

D.3. Singular Bounded Cohomology

As before let M be a topological space. Note that we can think of $S^n(M)$ as the set of real valued functions on all singular n-simplices $\sigma : \Delta^n \to M$. Thus we can define a norm on $S^n(M)$ by setting

$$\|\alpha\| = \sup\{|\alpha(\sigma)| : \sigma \text{ singular } n\text{-simplex}\}$$

for every $\alpha \in S^n(M)$.

The subspace of all cochains, which are bounded with respect to this norm is called the *singular bounded cochain group*

$$S_b^n(M) = \{\alpha \in S^n(M) : \|\alpha\| < \infty\}$$

It is immediate, that the usual singular coboundary maps $\delta : S^n(M) \to S^{n+1}(M)$ restrict to $\delta : S_b^n(M) \to S_b^{n+1}(M)$. Thus we get the *singular bounded cochain complex*

$$\cdots \longrightarrow S_b^{n-1}(M) \xrightarrow{\delta} S_b^n(M) \xrightarrow{\delta} S_b^{n+1}(M) \longrightarrow \cdots$$

The n-th cohomology group of this cochain complex

$$H_b^n(M) = \frac{\ker\{\delta : S_b^n(M) \to S_b^{n+1}(M)\}}{\operatorname{im}\{\delta : S_b^{n-1}(M) \to S_b^n(M)\}}$$

is called the *n-th singular bounded cohomology group.* We can now define the quotient seminorm on cohomology via

$$\|[\alpha]\| = \inf\{\|\beta\| : \beta \in [\alpha]\}$$

for every cohomology class $[\alpha] \in H_b^n(M)$.

For a subspace $N \subset M$ we can again define a relative version of singular bounded cohomology. Indeed consider the subspace $S_b^n(M, N)$ of all *bounded* singular cochains that vanish on simplices completely contained in N. Again we get a short exact sequence of cochain complexes

$$0 \longrightarrow S_b^\bullet(M, N) \longrightarrow S_b^\bullet(M) \xrightarrow{i^*} S_b^\bullet(N) \longrightarrow 0$$

where $i^* : S^n(M) \to S^n(M)$ is the map induced by the inclusion $i : N \to M$. By the snake lemma from homological algebra we get a long exact sequence in bounded cohomology

$$\cdots \longrightarrow H_b^n(M, N) \longrightarrow H_b^n(M) \xrightarrow{\iota^*} H_b^n(N) \xrightarrow{\delta_*} H_b^{n+1}(M, N) \longrightarrow \cdots$$

with connecting homomorphisms $\delta_* : H_b^\bullet(M, N) \to H_b^{\bullet+1}(N)$.

Observe that the canonical inclusion of complexes $i^\bullet : S_b^\bullet(M) \to S^\bullet(M)$ is in fact a morphism of complexes, i.e. commutes with the coboundary maps. The induced map

$$c : H_b^n(M) \to H^n(M)$$

is called the *comparison map.* Further note that the inclusion $S_b^n(M) \to S^n(M)$ also restricts to $S_b^n(M, N) \to S^n(M, N)$ and $S_b^n(N) \to S^n(N)$. This gives us a morphism of short exact sequences of cochain complexes

$$\begin{tikzcd}
0 \arrow[r] & S_b^\bullet(M, N) \arrow[r] \arrow[d] & S_b^\bullet(M) \arrow[r, "i^*"] \arrow[d] & S_b^\bullet(N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S^\bullet(M, N) \arrow[r] & S^\bullet(M) \arrow[r, "i^*"] & S^\bullet(N) \arrow[r] & 0
\end{tikzcd}$$

By naturality of the connecting homomorphisms we get that the comparison maps constitute a morphism of long exact sequences in cohomology

$$\begin{tikzcd}
\cdots \arrow[r] & H_b^n(M,N) \arrow[r] \arrow[d, "c"'] & H_b^n(M) \arrow[r, "i^*"] \arrow[d, "c"'] & H_b^n(N) \arrow[r, "\delta_*"] \arrow[d, "c"'] & H_b^{n+1}(M,N) \arrow[r] \arrow[d, "c"'] & \cdots \\
\cdots \arrow[r] & H^n(M,N) \arrow[r] & H^n(M) \arrow[r, "i^*"] & H^n(N) \arrow[r, "\delta_*"] & H^{n+1}(M,N) \arrow[r] & \cdots
\end{tikzcd}$$

It is important to note, that singular bounded (relative) cohomology is a homotopy invariant. Indeed, let $f, g : M_1 \to M_2$ be two homotopic continuous maps between topological spaces. The proof of homotopy invariance for usual singular (co)homology uses a prism operator $P : S^\bullet(M_2) \to S^{\bullet-1}(M_1)$ providing a cochain homotopy $f^* - g^* = \delta P + P\delta$, where P is the dual of the prism operator in [Hat02, Theorem 2.10, p. 111]. It is not hard to see, that this prism operator restricts to the bounded cochain complex (see also [Iva87, 1. Introduction, p. 1091]).

D.4. De Rham Cohomology

Let M be a smooth manifold with or without boundary. Recall that the de Rham cohomology of M is defined by the cohomology of the cochain complex of differential forms

$$\cdots \longrightarrow \Omega^{n-1}(M) \xrightarrow{d} \Omega^n(M) \xrightarrow{d} \Omega^{n+1}(M) \longrightarrow \cdots$$

where the coboundary maps $d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ are the exterior derivatives. We denote the *n-th de Rham cohomology group* by

$$H^n_{\mathrm{dR}}(M) = \frac{\ker\{d : \Omega^n(M) \to \Omega^{n+1}(M)\}}{\mathrm{im}\{d : \Omega^{n-1}(M) \to \Omega^n(M)\}}$$

Now let $i : N \hookrightarrow M$ be an embedded submanifold. Define

$$\Omega^n(M,N) := \ker\{\iota^* : \Omega^n(M) \to \Omega^n(N)\} \subset \Omega^n(M)$$

which is the set of all n-forms vanishing when restricted to the submanifold N. It is immediate, that the coboundary maps restric to $d : \Omega^n(M,N) \to \Omega^{n+1}(M,N)$ and hence we get the *relative de Rham cochain complex*

$$\cdots \longrightarrow \Omega^{n-1}(M,N) \xrightarrow{d} \Omega^n(M,N) \xrightarrow{d} \Omega^{n+1}(M,N) \longrightarrow \cdots$$

Its n-th cohomology group

$$H^n_{\mathrm{dR}}(M,N) = \frac{\ker\{d : \Omega^n(M,N) \to \Omega^{n+1}(M,N)\}}{\mathrm{im}\{d : \Omega^{n-1}(M,N) \to \Omega^n(M,N)\}}$$

is called the *n-th relative de Rham cohomology group.*

Further we get the following short exact sequence of cochain complexes

$$0 \longrightarrow \Omega^\bullet(M,N) \longrightarrow \Omega^\bullet(M) \xrightarrow{i^*} \Omega^\bullet(N) \longrightarrow 0$$

Indeed i^* is surjective, since one can extend any differential form on N via a covering of submanifold charts for N in M and a smooth partition of unity to a differential form in M restricting to the given one in N.

The snake lemma from homological algebra yields a long exact sequence in de Rham cohomology

$$\cdots \longrightarrow H^n_{\mathrm{dR}}(M,N) \longrightarrow H^n_{\mathrm{dR}}(M) \xrightarrow{i^*} H^n_{\mathrm{dR}}(N) \xrightarrow{d_*} H^{n+1}_{\mathrm{dR}}(M,N) \longrightarrow \cdots$$

with connecting homomorphisms $d_* : H^n_{\mathrm{dR}}(N) \to H^{n+1}_{\mathrm{dR}}(M,N)$. It is worth noting that $H^\bullet_{\mathrm{dR}}(M,\emptyset) = H^\bullet(M)$.

D.4.1. The de Rham Isomorhism

As before let M be a smooth manifold with or without boundary and $i : N \hookrightarrow M$ an embedded submanifold.

Before we will define the de Rham isomorphism in the following, we need to understand how to integrate a smooth n-simplex Let $\sigma : \Delta^n \to M$ be a smooth n-simplex and $\omega \in \Omega^n(M)$ a smooth n-form. The integral of ω over σ is now defined as

$$\int_\sigma \omega = \int_{\Delta^n} \sigma^*\omega$$

in the sense of integration theory for smooth manifolds with corners (cf. [Lee13]). We extend this definition linearly to all smooth singular n-cochains $c = \sum_{i=1}^k c_i\sigma_i \in S^\infty_n(M)$, i.e. the integral of ω over c is defined as

$$\int_c \omega = \sum_{i=1}^k c_i \int_{\sigma_i} \omega$$

The following chain version of Stokes' Theorem holds.

Theorem D.4.1. *Let $c \in S^\infty_n(M)$ and $\omega \in \Omega^{n-1}(M)$. Then*

$$\int_{\partial c} \omega = \int_c d\omega$$

Proof. The proof for a smooth manifold without boundary given in [Lee13, Theorem 18.12, p. 481] works verbatim for a smooth manifold with boundary. □

This theorem enables us to define a map of cochain complexes $\Psi : \Omega^n(M) \to S^n_\infty(M)$ via

$$\Psi(\omega)(c) = \int_c \omega$$

for every $\omega \in \Omega^n(M)$ and $c \in S^\infty_n(M)$. Ψ clearly commutes with the coboundary maps of the respective cochain complexes by Theorem D.4.1 and hence induces a map in cohomology

$$\Psi : H^\bullet_{\mathrm{dR}}(M) \longrightarrow H^\bullet_\infty(M) \cong H^\bullet(M)$$

By standard homological algebra Ψ also induces a map in relative cohomology

$$\begin{tikzcd}
0 \arrow[r] & \Omega^\bullet(M,N) \arrow[r] \arrow[d, dashed, "\Psi"'] & \Omega^\bullet(M) \arrow[r, "i^*"] \arrow[d, "\Psi"'] & \Omega^\bullet(N) \arrow[r] \arrow[d, "\Psi"'] & 0 \\
0 \arrow[r] & S^\bullet_\infty(M,N) \arrow[r] & S^\bullet_\infty(M) \arrow[r, "i^*"] & S^\bullet_\infty(N) \arrow[r] & 0
\end{tikzcd}$$

The map $\Psi : H^\bullet_{\mathrm{dR}}(M,N) \longrightarrow H^\bullet_\infty(M,N) \cong H^\bullet(M,N)$ is then called the *de Rham isomorphism* due to the next theorem.

Theorem D.4.2. *Let M be a smooth manifold with or without boundary and $i : N \hookrightarrow M$ an embedded submanifold. Then the map*

$$\Psi : H^\bullet_{\mathrm{dR}}(M,N) \longrightarrow H^\bullet_\infty(M,N) \cong H^\bullet(M,N)$$

is a natural isomorphism.

Proof. Naturality follows as in the proof of [Lee13, Proposition 18.13, p. 482].

By [Lee13, Theorem 18.14, p. 484] (and the universal coefficient theorem) we know that $\Psi : H^\bullet_{\mathrm{dR}}(M) \to H^\bullet_\infty(M)$ is an isomorphism if M has empty boundary. We will generalize this following the same pattern as in the proof of Theorem D.1.2.

Let us first proof it for non-relative cohomology if M has a boundary. Denote by $j : \mathrm{int}M \hookrightarrow M$ the canonical inclusion. Clearly Ψ commutes with the induced pullback maps j^* and hence we get the following commutative diagram in cohomology

$$\begin{tikzcd}
H^\bullet_{\mathrm{dR}}(M) \arrow[r, "\Psi"] \arrow[d, "j^*"'] & H^\bullet_\infty(M) \arrow[d, "j^*"'] \\
H^\bullet_{\mathrm{dR}}(\mathrm{int}M) \arrow[r, "\Psi"] & H^\bullet_\infty(\mathrm{int}M)
\end{tikzcd}$$

Because j is a smooth homotopy equivalence it induces isomorphisms in cohomology. Thus $\Psi : H^\bullet_{\mathrm{dR}}(M) \to H^\bullet_\infty(M)$ is also an isomorphism if M has a boundary.

Turning to the relative case we get by the previous diagram of short exact sequences, that Ψ induces a map between the two long exact cohomology sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H^n_{\mathrm{dR}}(M) \arrow[r] \arrow[d, "\cong"'] & H^n_{\mathrm{dR}}(M) \arrow[r] \arrow[d, "\cong"'] & H^n_{\mathrm{dR}}(M,N) \arrow[r] \arrow[d, "\Psi"'] & H^{n+1}_{\mathrm{dR}}(M) \arrow[r] \arrow[d, "\cong"'] & H^{n+1}_{\mathrm{dR}}(N) \arrow[r] \arrow[d, "\cong"'] & \cdots \\
\cdots \arrow[r] & H^n_\infty(M) \arrow[r] & H^n_\infty(M) \arrow[r] & H^n_\infty(M,N) \arrow[r] & H^{n+1}_\infty(M) \arrow[r] & H^{n+1}_\infty(N) \arrow[r] & \cdots
\end{tikzcd}$$

By the five lemma the map in the middle is then also an isomorphism and the assertion follows. □

Finally the next lemma shows, that for an oriented compact smooth manifold with (or without) boundary M evaluation on the fundamental class $[M, \partial M]$ corresponds to integration over M via the de Rham isomorphism.

Lemma D.4.3. *Let M be an oriented compact smooth manifold with or without boundary of dimension n and let $[M, \partial M] \in H_n(M, \partial M)$ be its fundamental class. Further let $[\omega] \in \Omega^n(M, \partial M)$ be a cohomology class in top degree. Then*

$$\langle \Psi([\omega]), [M, \partial M] \rangle = \int_M \omega$$

Proof. As we have already mentioned before M admits a smooth triangulation by simplices $\sigma_i : \Delta^n \to M$. Further its fundamental class $[M, \partial M]$ is represented at the chain level by the sum

$$c = \sum_{i=1}^{k} k_i \sigma_i$$

where $k_i = +1$ if σ_i is orientation preserving and $k_i = -1$ if it is not.

Therefore we can compute directly

$$\begin{aligned}
&\langle \Psi([\omega]), [M, \partial M] \rangle \\
&= \Psi(\omega)(c) = \int_c \omega \\
&= \sum_{i=1}^{k} k_i \int_{\sigma_i} \omega = \sum_{i=1}^{k} \int_{\Delta^n} k_i \sigma_i^* \omega \\
&= \int_M \omega
\end{aligned}$$

where the last equality follows from [Lee13, Proposition 16.8, p. 408]. □

E. Douady-Earle's Barycenter Construction

The objective of this chapter is to introduce Douady-Earle's barycenter construction. This construction associates equivariantly to every "nice" probability measure on the boundary $\partial\mathbb{H}^n$ its "barycenter" in $\mathbb{H}^n$. Furthermore the barycenter depends continuously on the measure, when the subspace of all "nice" proabability measures in $\mathcal{M}^1(\partial\mathbb{H}^n)$ is equipped with the induced weak-* topology.

Historically Douady and Earle introduced this construction first in their joint paper [DE86] for dimension two. Later Besson, Courtois and Gallot generalized it to universal covering spaces of compact Riemannian manifolds with strictly negative curvature and diffuse boundary measures in [BCG95, Appendice A, p. 781]. Although the barycenter construction is stated in [FK06] and [BCG99] for probability measures with no atom of mass greater than 1/2 on the boundary of hyperbolic n-space, we were unable to find an appropriate reference. A possible reason might be, that we were unable to access [BCG96]. However the result for non-atomic measures is not sufficient for our purposes, such that we rediscover a proof here. We hope that some of the readers find this convenient.

Since the proof makes heavy use of Busemann functions we will first introduce them and demonstrate some of their properties.

E.1. Busemann Functions

Definition E.1.1. For $x \in \mathbb{H}^n$ and $\gamma : \mathbb{R} \to \mathbb{H}^n$ a unit speed geodesic we define the Busemann function

$$b(x,\gamma) := \lim_{t\to\infty}(d(\gamma(t),x) - t)$$

where $d(\cdot,\cdot)$ denotes the hyperbolic distance.

Intuitively the Busemann function measures the relative distance from x to $\gamma(\infty)$.

Remark E.1.2. *The Busemann function $b(x,\gamma)$ is well defined. This is due to the fact that $t \mapsto d(\gamma(t),x) - t)$ is bounded*

$$d(\gamma(t),x) - t \leq \underbrace{d(\gamma(t),\gamma(0))}_{=t} + d(\gamma(0),x) - t = d(\gamma(0),x) < \infty$$

and monotone

$$d(\gamma(t+s),x) - (t+s) - (d(\gamma(t),x) - t) \geq \underbrace{d(\gamma(t+s),\gamma(t))}_{=s} - s \geq 0$$

Hence the limit as $t \to \infty$ exists.

The Busemann function has the following transformation behaviour.

Lemma E.1.3. *For all $x \in \mathbb{H}^n$, $\gamma : \mathbb{R} \to \mathbb{H}^n$ a unit speed geodesic and $g \in \mathrm{Isom}(\mathbb{H}^n)$ we have*

$$b(g(x), g \circ \gamma) = b(x,\gamma)$$

Proof. Let $x \in \mathbb{H}^n$, $\gamma : \mathbb{R} \to \mathbb{H}^n$ a unit speed geodesic and $g \in \mathrm{Isom}(\mathbb{H}^n)$. Then by definition

$$b(g(x), g \circ \gamma) = \lim_{t\to\infty} (d(g(\gamma(t)), g(x)) - t) = \lim_{t\to\infty} (d(\gamma(t), x) - t) = b(x, \gamma)$$

$\square$

By fixing a certain point o in $\mathbb{H}^n$ as "the origin" we can define for $x \in \mathbb{H}^n$ and $\theta \in \partial\mathbb{H}^n$

$$b_o(x, \theta) := \lim_{t\to\infty} (d(\gamma(t), x) - t)$$

where $\gamma : \mathbb{R} \to \mathbb{H}^n$ is now the unique unit speed geodesic with $\gamma(0) = o$ and $\gamma(\infty) = \theta$. In this case we shall call $b_o(x, \theta)$ also a Busemann function. The above transformation behaviour then translates naturally to

$$b_o(x, \theta) = b_{g(o)}(g(x), g(\theta))$$

for every $x \in \mathbb{H}^n$, $\theta \in \partial\mathbb{H}^n$ and $g \in \mathrm{Isom}(\mathbb{H}^n)$. For the upper half space model U^n and the Poincaré ball model B^n we shall take as origins $e_n \in U^n$ and $0 \in B^n$ respectively and denote the corresponding Busemann functions with $b_{U^n}(x, \theta)$ and $b_{B^n}(x, \theta)$. Note that the Cayley transform exchanges e_n and 0 such that this choice is in a way consistent.

Next we will compute $b_{U^n}(x, \theta)$ for arbitrary $x \in U^n$ and $\theta \in \partial U^n = \mathbb{R}^{n-1} \times \{0\} \cup \{\infty\}$.

Proposition E.1.4. *We have:*

(i) For all $x = (x_1, \dots, x_n) \in U^n$

$$b_{U^n}(x, \infty) = -\ln(x_n) = -\ln(\langle x, e_n \rangle)$$

(ii) For all $x \in U^n$, $\theta \in \mathbb{R}^{n-1} \times \{0\}$

$$b_{U^n}(x, \theta) = -\ln\left(\frac{1 + |\theta|^2}{|x - \theta|^2}\, x_n\right)$$

Proof. To (i): Let $x = (x_1, \dots, x_n) \in U^n$. Recall for the hyperbolic distance in the upper half space model

$$d_{U^n}(x, y) = \operatorname{arccosh}\left(1 + \frac{|x - y|^2}{2x_n y_n}\right) \quad \forall x, y \in U^n$$

(cf. [Rat06]). We shall also use the following formula

$$\operatorname{arccosh}(z) = \ln(z + \sqrt{z^2 - 1}) \quad \forall z \in \mathbb{R}$$

Since we have fixed $o = e_n$ as our origin and the unique unit speed geodesic $\gamma : \mathbb{R} \to U^n$ with $\gamma(0) = e_n$ and $\gamma(\infty) = \infty$ is $\gamma(t) = e^t e_n$, we need to calculate

$$b_{U^n}(x, \infty) = \lim_{t\to\infty} (d_{U^n}(\gamma(t), x) - t) = \lim_{t\to\infty} \left(\operatorname{arccosh}\left(1 + \frac{|x - \gamma(t)|^2}{2x_n \gamma_n(t)}\right) - t\right)$$

Observe that

$$\begin{aligned}
\frac{|x-\gamma(t)|^2}{2x_n\gamma_n(t)} &= \frac{x_1^2+\ldots+x_{n-1}^2}{2x_n\gamma_n(t)} + \frac{(x_n-\gamma_n(t))^2}{2x_n\gamma_n(t)} \\
&= \frac{x_1^2+\ldots+x_{n-1}^2}{2x_n\gamma_n(t)} + \frac{x_n^2}{2x_n\gamma_n(t)} - \frac{2x_n\gamma_n(t)}{2x_n\gamma_n(t)} + \frac{\gamma_n(t)^2}{2x_n\gamma_n(t)} \\
&= \underbrace{\frac{x_1^2+\ldots+x_{n-1}^2}{2x_n\gamma_n(t)} + \frac{x_n}{2\gamma_n(t)}}_{=:R(t)\to 0\ (t\to\infty)} + \frac{\gamma_n(t)}{2x_n} - 1 \\
&= \underbrace{R(t) + \frac{\gamma_n(t)}{2x_n}}_{=:T(t)\to\infty\ (t\to\infty)} -1
\end{aligned}$$

Hence

$$\begin{aligned}
\operatorname{arccosh}\left(1+\frac{|x-\gamma(t)|^2}{2x_n\gamma_n(t)}\right) - t &= \operatorname{arccosh}(T(t)) - t \\
&= \ln(T(t)+\sqrt{T(t)^2-1}) - t \\
&= \ln(T(t)) + \underbrace{\ln(1+\sqrt{1-T(t)^{-2}})}_{=:S(t)\to\ln(2)\ (t\to\infty)} - t \\
&= \ln(e^t/2x_n) - \ln(e^t/2x_n) + \ln(e^t/2x_n + R(t)) + S(t) - t \\
&= t - \ln(2x_n) + \underbrace{\int_{e^t/2x_n}^{e^t/2x_n+R(t)} \frac{1}{x}\,dx}_{=:I(t)\to 0\ (t\to\infty)} + S(t) - t \\
&= -\ln(x_n) - \ln(2) + I(t) + S(t)
\end{aligned}$$

Thus as asserted

$$\begin{aligned}
b_{U^n}(x,\infty) &= \lim_{t\to\infty}\left(\operatorname{arccosh}\left(1+\frac{|x-\gamma(t)|^2}{2x_n\gamma_n(t)}\right) - t\right) \\
&= -\ln(x_n) - \ln(2) + \lim_{t\to\infty} I(t) + \lim_{t\to\infty} S(t) \\
&= -\ln(x_n) - \ln(2) + \ln(2) = -\ln(x_n)
\end{aligned}$$

To (ii): Let $x \in U^n$, $\theta \in \mathbb{R}^{n-1}$. We shall use (i) and the transformation behaviour of b_{U^n}. Let's consider the inversion

$$i_{\theta,\alpha}(x) = \alpha\frac{x-\theta}{|x-\theta|^2} + \theta$$

where $\alpha = \|\theta\|^2 + \|e_n\|^2 = \|\theta\|^2 + 1$. Then $i_{\theta,\alpha}$ exchanges ∞ with θ and leaves the sphere centered at θ with radius $\sqrt{\alpha}$ invariant; in particular $i_{\theta,\alpha}(e_n) = e_n$. Let $\gamma_\theta : \mathbb{R} \to U^n$ denote the unit speed geodesic with $\gamma_\theta(0) = e_n$ and $\gamma_\theta(\infty) = \theta$ and $\gamma_\infty(t) = e^t e_n$ the unit speed geodesic from e_n to ∞. Then $\gamma_\theta = i_{\theta,\alpha}\circ\gamma_\infty$ and hence by the transformation behaviour (and $i_{\theta,\alpha}^2 = \mathrm{id}$)

$$\begin{aligned}
b_{U^n}(x,\theta) &= b_{U^n}(i_{\theta,\alpha}^2(x), (i_{\theta,\alpha}(\infty)) = -\ln(\langle i_{\theta,\alpha}(x), e_n\rangle) \\
&= -\ln\left(\frac{1+|\theta|^2}{|x-\theta|^2}\,x_n\right)
\end{aligned}$$

where we have used in the last equality that $\langle\theta, e_n\rangle = 0$, since $\theta \in \mathbb{R}^{n-1}\times\{0\}$. $\square$

We can use this formula to investigate what happens when we change our base point from e_n to some $o = (o_1, \ldots, o_{n-1}, o_n) \in U^n$.

Lemma E.1.5. *Let $o = (o_1, \ldots, o_{n-1}, o_n) \in U^n$.*
Then

$$b_o(x, \infty) - b_{e_n}(x, \infty) = \ln(o_n)$$

for every $x \in U^n$, and

$$b_o(x, \theta) - b_{e_n}(x, \theta) = -\ln(o_n) + \ln\left(\frac{1 + |\theta|^2}{1 + o_n^{-2}|\theta - p(o)|^2}\right)$$

for every $x \in U^n, \theta \in \mathbb{R}^{n-1}$, where $p : U^n \to \mathbb{R}^{n-1}$ denotes the canonical projection on the first $(n-1)$-coordinates. In particular these differences do not depend on x!

Proof. This will follow from the transformation behaviour and the explicit formulas from the previous proposition. Note that the map $g : U^n \to U^n, x \mapsto o_n \cdot x + p(o)$ is an isometry taking e_n to o, that fixes θ. Its inverse is $g^{-1}(x) = o_n^{-1} \cdot x - o_n^{-1} \cdot p(o)$ as one readily checks.

Let $x \in U^n$. We compute

$$\begin{aligned} b_o(x, \infty) - b_{e_n}(x, \infty) &= b_{e_n}(g^{-1}(x), \infty) - b_{e_n}(x, \infty) \\ &= -\ln(\langle o_n^{-1} \cdot x - o_n^{-1} \cdot p(o), e_n\rangle) + \ln(x_n) \\ &= \ln(o_n) \end{aligned}$$

Now let $\theta \in \mathbb{R}^{n-1} \times \{0\}$. Then

$$\begin{aligned} b_o(x, \infty) - b_{e_n}(x, \infty) &= b_{e_n}(g^{-1}(x), g^{-1}(\theta)) - b_{e_n}(x, \theta) \\ &= -\ln\left(\frac{1 + |\theta|^2}{|g^{-1}(x) - g^{-1}(\theta)|^2}\langle g^{-1}(x), e_n\rangle\right) + \ln\left(\frac{1 + |\theta|^2}{|x - \theta|^2}x_n\right) \\ &= \ln\left(\frac{1 + |\theta|^2}{1 + |g^{-1}(x)|^2} \cdot \frac{|g^{-1}(x) - g^{-1}(\theta)|^2}{|x - \theta|^2} \cdot \frac{x_n}{\langle g^{-1}(x), e_n\rangle}\right) \\ &= \ldots = -\ln(o_n) + \ln\left(\frac{1 + |\theta|^2}{1 + o_n^{-2}|\theta - p(o)|^2}\right) \end{aligned}$$

□

Now we want to see the Busemann functions also in the Ball model B^n. Recall that we can switch between the upper half space model U^n and the Poincaré ball model B^n by inversion $i = i_{-e_n,2}$ at the circle of radius $\sqrt{2}$ with center $-e_n$; the (inverse) Cayley transform. Therefore we get the following formula.

Proposition E.1.6. *For every $x \in B^n$ and $\theta \in \partial B^n = S^{n-1}$*

$$b_{B^n}(x, \theta) = -\ln\left(\frac{1 - |x|^2}{|x - \theta|^2}\right)$$

Proof. By applying the transformation behaviour to the isometry $i = i_{-e_n,2}$ we get for every $x \in B^n$ and every $\theta \in \partial B^n$

$$b_{B^n}(x, \theta) = b_{U^n}(i(x), i(\theta))$$

Note that the respective fixed origins are mapped to each other by i. We shall only verify the above identity for $i(\theta) \in \mathbb{R}^{n-1} \times \{0\} \subset \partial U^n$. The special case for $i(\theta) = \infty$ (i.e. $\theta = -e_n$) is easily verified in the very same way. Let $x \in B^n$ and $\theta \in S^{n-1} \backslash \{-e_n\}$. We have

$$b_{U^n}(i(x), i(\theta)) = -\ln\left(\frac{1 + |i(\theta)|^2}{|i(x) - i(\theta)|^2}\langle i(x), e_n\rangle\right)$$

Computing each occuring term separately

$$\begin{aligned}
1+|i(\theta)|^2 &= 1+\left|2\frac{\theta+e_n}{|\theta+e_n|^2}-e_n\right|^2 \\
&= 1+\frac{4}{|\theta+e_n|^2}-4\frac{\langle\theta+e_n,e_n\rangle}{|\theta+e_n|^2}+1 \\
&= 2-4\frac{\langle\theta,e_n\rangle}{|\theta+e_n|^2} = 2|\theta+e_n|^{-2}(|\theta+e_n|^2-2\langle\theta,e_n\rangle) \\
&= 4|\theta+e_n|^{-2}
\end{aligned}$$

$$\begin{aligned}
|i(x)-i(\theta)|^2 &= \left|2\frac{x+e_n}{|x+e_n|^2}-e_n-\left(2\frac{\theta+e_n}{|\theta+e_n|^2}-e_n\right)\right|^2 \\
&= 4|x+e_n|^{-4}|\theta+e_n|^{-4}\left|(x+e_n)|\theta+e_n|^2-(\theta+e_n)|x+e_n|^2\right|^2 \\
&= 4|x+e_n|^{-2}|\theta+e_n|^{-2}\left(|x+e_n|^2-\langle x+e_n,\theta+e_n\rangle+|x+e_n|^2\right) \\
&= 4|x+e_n|^{-2}|\theta+e_n|^{-2}|x-\theta|^2
\end{aligned}$$

$$\begin{aligned}
\langle i(x),e_n\rangle &= \left\langle e_n, 2\frac{x+e_n}{|x+e_n|^2}-e_n\right\rangle \\
&= -1+2\frac{\langle x+e_n,e_n\rangle}{|x+e_n|^2} \\
&= |x+e_n|^{-2}(-|x|^2-2\langle x,e_n\rangle-1+2+2\langle x,e_n\rangle) \\
&= |x+e_n|^{-2}(1-|x|^2)
\end{aligned}$$

Hence as asserted

$$b_{B^n}(x,\theta) = -\ln\left(\frac{1-|x|^2}{|x-\theta|^2}\right)$$

□

Remark E.1.7. *It is now easy to see, that the Busemann function $b_{B^n}(x,\theta)$ is smooth in x and continuous in θ. By the transformation behaviour the same is true for b_{U^n}.*

The Busemann function has the additional nice property that it is geodesically convex.

Proposition E.1.8. *Let $\gamma:\mathbb{R}\to U^n$ be a geodesic and $\theta\in\partial U^n$. Then*

$$\partial_t^2 b_{U^n}(\gamma(t),\theta)\geq 0 \quad \forall t\in\mathbb{R}$$

and equality holds if and only if $\theta\in\{\gamma(\infty),\gamma(-\infty)\}$.

Proof. Let $\varphi\in\mathrm{Isom}(U^n)$ be an isometry such that $\varphi(\gamma(t))=e^{\lambda t}e_n$ for some $\lambda>0$ and every $t\in\mathbb{R}$. Then we have

$$b_{U^n}(\gamma(t),\theta) = b_{U_n}(\varphi(\gamma(t)),\varphi(\theta)) = b_{U_n}(e^{\lambda t}e_n,\varphi(\theta))$$

First consider the case $\varphi(\theta)=\infty=(\varphi\circ\gamma)(\infty)$, that is $\theta=\gamma(\infty)$. We get

$$b_{U^n}(e^{\lambda t}e_n,\infty) = -\ln(e^{\lambda t}) = -\lambda t$$

and hence

$$\partial_t^2 b_{U^n}(\gamma(t),\theta)=0 \quad \forall t\in\mathbb{R}$$

Otherwise if $\varphi(\theta) \in \mathbb{R}^{n-1} \times \{0\}$ we get

$$\begin{aligned} b_{U^n}(e^{\lambda t}e_n, \varphi(\theta)) &= -\ln\left(\frac{1+|\varphi(\theta)|^2}{|e^{\lambda t}e_n - \varphi(\theta)|^2}\, e^{\lambda t}\right) \\ &= -\lambda t - \ln(1+|\varphi(\theta)|^2) + \ln\left(e^{2\lambda t} + |\varphi(\theta)|^2\right) \end{aligned}$$

Thus

$$\partial_t b_{U^n}(e^{\lambda t}e_n, \varphi(\theta)) = -\lambda + \frac{2\lambda e^{2\lambda t}}{e^{2\lambda t} + |\varphi(\theta)|^2}$$

and

$$\begin{aligned} \partial_t^2 b_{U^n}(e^{\lambda t}e_n, \varphi(\theta)) &= \frac{4\lambda^2 e^{2\lambda t}\left(e^{2\lambda t} + |\varphi(\theta)|^2\right) - 2\lambda e^{2\lambda t}\left(2\lambda e^{2\lambda t}\right)}{\left(e^{2\lambda t} + |\varphi(\theta)|^2\right)^2} \\ &= \frac{4\lambda^2 e^{2\lambda t}|\varphi(\theta)|^2}{\left(e^{2\lambda t} + |\varphi(\theta)|^2\right)^2} \geq 0 \end{aligned}$$

for every $t \in \mathbb{R}$. The last inequality is only an equality if $\varphi(\theta) = 0 = (\varphi \circ \gamma)(-\infty)$ that is $\theta = \gamma(-\infty)$. $\square$

E.2. The Barycenter Construction

After our discussion of the Busemann function in the previous section we are now ready to introduce Douady-Earle's barycenter construction. This will allow us to associate G-equivariantly to each "nice" measure μ on $\partial\mathbb{H}^n$ a unique point in $\mathbb{H}^n$ called the barycenter of μ. Recall that the action of $\mathrm{Isom}(\mathbb{H}^n)$ on the measures on the boundary $\partial\mathbb{H}^n$ is given by

$$(g \cdot \mu)(A) = g_*(\mu)(A) = \mu(g^{-1}A)$$

for every $g \in \mathrm{Isom}(\mathbb{H}^n)$, $\mu \in \mathcal{M}(\partial\mathbb{H}^n)$ and $A \subseteq \partial\mathbb{H}^n$ measurable. For a concise exposition of the measure theory we are going to need, we refer to appendix A.

A natural way to construct such a map would be to take the point in $\mathbb{H}^n$ which has in average the minimal distance to every point of $\partial\mathbb{H}^n$, where we average according to our boundary measure. However by definition every point of $\mathbb{H}^n$ has infinite distance to the boundary. That is why we have to consider Busemann functions, which intuitively measure the relative distance of a point to the boundary given an origin. The averaged relative distance is then given by the following function in the upper half space model U^n

$$\mathcal{B}_\mu(x) := \int_{\partial U^n} b_{U^n}(x, \theta)\, d\mu(\theta)$$

for every $\mu \in \mathcal{M}^1(\partial U^n)$ and $x \in U^n$. It is now easy to see using the theorem about parameter integrals, that $\mathcal{B}_\mu$ is in fact a smooth function. Due to the ease of computations in the upper half space model U^n we shall stick to it for the rest of this section.

The next theorem tells us that $\mathcal{B}_\mu$ has in fact a unique minimum, if the boundary measure is not too concentrated.

Theorem E.2.1 (Barycenter Construction)**.** *Let $\mu \in \mathcal{M}^1(\partial U^n)$ be a probability measure that has no atoms of mass $\geq 1/2$. Then $\mathcal{B}_\mu$ is geodesically strictly convex and has a unique minimum* bary(μ) *in U^n called the* barycenter of μ.

Remark E.2.2. *It is trivial, to see that one may in fact define the barycenter for any finite (positive) measure on $\partial\mathbb{H}^n$ by normalization. The barycenter is not affected by scaling the respective measure.*

Proof. Strict convexity: It suffices to show that for every geodesic $\gamma : \mathbb{R} \to U^n$

$$\partial_t^2 \mathcal{B}_\mu(\gamma(t)) > 0$$

for every $t \in \mathbb{R}$.

Let $t \in \mathbb{R}$. By Proposition E.1.8 we have

$$\begin{aligned}\partial_t^2 \mathcal{B}_\mu(\gamma(t)) &= \partial_t^2 \int_{\partial U^n} b_{U^n}(\gamma(t),\theta)\, d\mu(\theta) \\ &= \int_{\partial U^n} \partial_t^2 b_{U^n}(\gamma(t),\theta)\, d\mu(\theta) \\ &= \int_{\partial U^n \backslash \{\gamma(\infty),\gamma(-\infty)\}} \partial_t^2 b_{U^n}(\gamma(t),\theta)\, d\mu(\theta)\end{aligned}$$

Now

$$\mu(\partial U^n \backslash \{\gamma(\infty),\gamma(-\infty)\}) = \mu(\partial U^n) - \underbrace{\mu(\{\gamma(\infty)\})}_{<1/2} - \underbrace{\mu(\{\gamma(-\infty)\})}_{<1/2} > 0$$

Further $\partial_t^2 b_{U^n}(\gamma(t),\theta) > 0$ for every $\theta \in \partial U^n \backslash \{\gamma(\infty),\gamma(-\infty)\}$ and $\partial_t^2 b_{U^n}(\gamma(t),\theta)$ is continuous in θ. Thus there is a compact set $V \subset \partial U^n \backslash \{\gamma(\infty),\gamma(-\infty)\}$ by inner regularity and $\varepsilon > 0$ such that $\mu(V) > 0$ and $\partial_t^2 b_{U^n}(\gamma(t),\theta) \geq \varepsilon$ for every $\theta \in V$. Therefore we get

$$\partial_t^2 \mathcal{B}_\mu(\gamma(t)) \geq \int_V \partial_t^2 b_{U^n}(\gamma(t),\theta)\, d\mu(\theta) \geq \mu(V) \cdot \varepsilon > 0$$

and $\mathcal{B}_\mu$ is geodesically strictly convex. Hence if a minimum of $\mathcal{B}_\mu$ exists it is unique.

Existence of a minimum: Observe that $b_{U^n}(e_n,\theta) = 0$ for all $\theta \in \partial U^n$ and hence also

$$\mathcal{B}_\mu(e_n) = \int_{\partial U^n} b_{U^n}(e_n,\theta)\, d\mu(\theta) = 0$$

If there is a minimum it thus has to be contained in the geodesically convex closed set $A := \{x \in U^n : \mathcal{B}_\mu(x) \leq 0\} \ni e_n$. We need to see that A is also bounded.

It will be sufficient to show, that $B_\mu(\gamma(t)) \to \infty$ as $t \to \infty$ where $\gamma : \mathbb{R} \to U^n$ is any geodesic with $\gamma(0) = e_n$. Indeed if A is unbounded there is a geodesic $\gamma : \mathbb{R} \to U^n$ starting from o such that $\gamma(t) \in A$ for all $t \geq 0$. But then $B_\mu(\gamma(t)) \leq 0$ for all $t \geq 0$ in contradiction to $B_\mu(\gamma(t)) \to \infty$ as $t \to \infty$.

Let $\gamma : \mathbb{R} \to U^n$ be a geodesic with $\gamma(0) = e_n$. By applying an isometry $\varphi \in \mathrm{Stab}(e_n)$ we can assume that $\gamma(t) = e^t e_n$ since

$$\begin{aligned}B_\mu(\varphi(\gamma(t))) &= \int_{\partial U^n} b_{U^n}(\varphi(\gamma(t)),\theta)\, d\mu(\theta) = \int_{\partial U^n} b_{U^n}(\gamma(t),\varphi^{-1}(\theta))\, d\mu(\theta) \\ &= \int_{\partial U^n} b_{U^n}(\varphi(\gamma(t)),\theta)\, d\varphi_*\mu(\theta)\end{aligned}$$

and $\varphi_*\mu$ is also in $\mathcal{M}^1(\partial U^n)$ with no atom of mass $\geq 1/2$.

For $x = (0, \ldots, 0, x_n) \in \mathbb{R}\, e_n \subset U^n$ we have

$$\begin{aligned}
B_\mu(x) &= -\ln(x_n)\, \mu(\{\infty\}) + \int_{\mathbb{R}^{n-1}} -\ln\left(\frac{1+|\theta|^2}{|x-\theta|^2}\, x_n\right) d\mu(\theta) \\
&= -\ln(x_n)\, \mu(\{\infty\}) + \int_{\mathbb{R}^{n-1}} \ln\left(\frac{x_n + \frac{|\theta|^2}{x_n}}{1+|\theta|^2}\right) d\mu(\theta) \\
&= -\ln(x_n)\, \mu(\{\infty\}) + \int_{B(0,R)^c} \underbrace{\ln\left(\frac{x_n + \frac{|\theta|^2}{x_n}}{1+|\theta|^2}\right)}_{\geq -\ln(x_n)} d\mu(\theta) + \int_{B(0,R)} \ln\left(\frac{x_n + \frac{|\theta|^2}{x_n}}{1+|\theta|^2}\right) d\mu(\theta)
\end{aligned}$$

where $R > 0$ is arbitrary, $B(0,R) \subset \mathbb{R}^{n-1}$ denotes the euclidean ball of radius R and center 0 and the estimate for the first integral can be easily verified by some fairly standard analysis of the integrand. Since $\mu(\{\infty\}) < 1/2$ there is $R > 0$ and $0 < \alpha < 1$ such that

$$\alpha \cdot \mu(B(0,R)) - \mu(B(0,R)^c) - \mu(\{\infty\}) > 0$$

Thus

$$B_\mu(x) \geq -\ln(x_n)\left(\mu(\{\infty\}) + \mu(B(0,R)^c)\right) + \int_{B(0,R)} \ln\left(\frac{x_n + \frac{|\theta|^2}{x_n}}{1+|\theta|^2}\right) d\mu(\theta)$$

Further

$$\ln\left(\frac{x_n + \frac{|\theta|^2}{x_n}}{1+|\theta|^2}\right) = \alpha\, \ln(x_n) \cdot \ln\left(\frac{x_n^{1-\alpha} + x_n^{-1-\alpha}|\theta|^2}{1+|\theta|^2}\right)$$

and for every $\theta \in B(0,R)$

$$\ln\left(\frac{x_n^{1-\alpha} + x_n^{-1-\alpha}|\theta|^2}{1+|\theta|^2}\right) \geq \ln\left(\frac{x_n^{1-\alpha}}{1+R^2}\right) \to \infty \quad (x_n \to \infty)$$

Thus – if x_n is large enought – we may assume that this last expression is ≥ 1.

Piecing this together we get

$$\begin{aligned}
B_\mu(x) &\geq -\ln(x_n)\left(\mu(\{\infty\}) + \mu(B(0,R)^c)\right) + \alpha \ln(x_n) \cdot \mu(B(0,R)) \\
&= \ln(x_n) \underbrace{\left(\alpha \cdot \mu(B(0,R)) - \mu(\{\infty\}) - \mu(B(0,R)^c\right)}_{>0} \to \infty
\end{aligned}$$

as $x_n \to \infty$ and we are done. □

Remark E.2.3. *Note that we have proven that the set $\{x \in U^n : \mathcal{B}_\mu(x) \leq 0\}$ is compact for every prabability measure μ on ∂U^n with no atoms of mass greater than $1/2$. From now on we shall denote this set of measures by $\mathcal{A}_{<1/2}$. We will use this notation also in chapter III, when we are going to proof the volume rigidity theorem.*

The next example shows, that $\mathcal{B}_\mu$ does not need to have a unique minimum if $\mu \notin \mathcal{A}_{<1/2}$.

Example E.2.4. Let $\mu = 1/2\,\delta_0 + 1/2\,\delta_\infty \in \mathcal{M}^1(\partial U^n)$, which has two atoms: one at 0 and one at ∞. We compute

$$\mathcal{B}_\mu(x) = -\frac{1}{2}\left(\ln(x_n) + \ln\left(\frac{x_n}{|x|^2}\right)\right) = -\ln\left(\frac{x_n}{|x|}\right).$$

Since $x_n \le |x|$, we get that $\mathcal{B}_\mu(x) \ge 0$. However for every $x \in U^n$ with $x = x_n \cdot e_n$ for some $x_n > 0$ we have $x_n = |x|$ such that $\mathcal{B}_\mu(x) = 0$. Hence $\mathcal{B}_\mu(x)$ assumes its minimum at every point of the positive n-th coordinate axis, such that there is no unique minimum and therefore no barycenter.

Also observe that $\mathcal{B}_\mu(x) = -\ln(x_n/|x|)$ is invariant under scaling by some $\lambda > 0$ and rotations around the n-th coordinate axis. This already hints towards the G-equivariance of the barycenter construction, that we are going to prove in the next proposition.

Finally, the barycenter map is indeed G-equivariant and continuous with respect to the induced weak-* topology on $\mathcal{A}_{<1/2}$ as the next proposition asserts.

Proposition E.2.5. *The map*

$$\mathrm{bary} : \mathcal{A}_{<1/2} \to U^n, \mu \mapsto \mathrm{bary}(\mu)$$

is G-equivariant and continuous, where $\mathcal{A}_{<1/2} \subset C(\partial U^n)^*$ is equipped with the induced weak- topology.*

Proof. First, $\mathrm{bary}(\mu) : \mathcal{A}_{<1/2} \to U^n$ is G-equivariant. Indeed, let $\mu \in \mathcal{A}_{<1/2}$, $g \in G$ and $x = \mathrm{bary}(\mu)$, i.e. the unique minimum of $\mathcal{B}_\mu(x) = \int_{\partial \mathbb{H}^n} b_{e_n}(x,\theta)\, d\mu(\theta)$ in U^n. Then by the transformation behaviour of the Busemann function b we get

$$\begin{aligned}
\mathcal{B}_{g_*\mu}(gy) &= \int_{\partial U^n} b_{e_n}(gy,\theta)\, dg_*\mu(\theta) \\
&= \int_{\partial U^n} b_{e_n}(gy, g\theta)\, d\mu(\theta) \\
&= \int_{\partial U^n} b_{g^{-1}e_n}(y,\theta)\, d\mu(\theta) \\
&= \int_{\partial U^n} b_{e_n}(y,\theta) + C(g,\theta)\, d\mu(\theta) \\
&= \int_{\partial U^n} b_{e_n}(y,\theta)\, d\mu(\theta) + C_\mu(g) \\
&= \mathcal{B}_\mu(y) + C_\mu(g)
\end{aligned}$$

for every $y \in \mathbb{H}^n$, where $C(g,\theta) = b_{e_n}(x,\theta) - b_{g^{-1}e_n}(x,\theta)$ is the difference between the two Busemann functions, which does not depend on x (cf. Lemma E.1.5), and $C_\mu(g) = \int_{\partial U^n} C(g,\theta)\, d\mu(\theta)$.

Thus we have

$$\mathcal{B}_{g_*\mu}(gx) = \mathcal{B}_\mu(x) + C_\mu(g) \le \mathcal{B}_\mu(g^{-1}y) + C_\mu(g) = \mathcal{B}_{g_*\mu}(y)$$

for every $y \in U^n$, which shows that gx is the unique minimum of $\mathcal{B}_{g_*\mu}$. That is

$$\mathrm{bary}(g_*\mu) = gx = g\,\mathrm{bary}(\mu)$$

as asserted.

Now to continuity:

Let $(\mu_n)_{n\in\mathbb{N}} \in \mathcal{A}_{<1/2}^{\mathbb{N}}$ be a sequence converging to some $\mu \in \mathcal{A}_{<1/2}$ in the weak-* topology. Hence for every $x \in \mathbb{H}^n$

$$\mathcal{B}_{\mu_n}(x) = \int_{\partial U^n} b(x,\theta) d\mu_n(\theta) \to \int_{\partial U^n} b(x,\theta) d\mu(\theta) = \mathcal{B}_\mu(x) \quad (n \to \infty)$$

pointwise.

Let $x_n := \mathrm{bary}(\mu_n)$ and $x := \mathrm{bary}(\mu)$. As we have already seen $\mathcal{B}_\nu(e_n) = 0$ and the set $\{x \in U^n : \mathcal{B}_\nu(x) \leq 0\} \subset U^n$ is compact and convex for every $\nu \in \mathcal{A}_{<1/2}$. Therefore the set

$$C := \bigcap_{n\in\mathbb{N}} \{x \in U^n : \mathcal{B}_{\mu_n}(x) \leq 0\} \cap \{x \in U^n : \mathcal{B}_\mu(x) \leq 0\}$$

is non-empty, compact, convex and contains the respective minima $\{x_n : n \in \mathbb{N}\} \cup \{x\} \subset C$. Now we shall see that the family $\{\mathcal{B}_{\mu_n} : n \in \mathbb{N}\}$ has a uniform Lipschitz bound on C. Let $x, y \in C$ and $\gamma : [0, d(x, y)] \to C$ the geodesic segment joining x and y. We get

$$\begin{aligned}
|\mathcal{B}_{\mu_n}(x) - \mathcal{B}_{\mu_n}(y)| &\leq \int_{\partial U^n} |b(x,\theta) - b(y,\theta)|\, d\mu_n(\theta) \\
&= \int_{\partial U^n} \left| \int_0^{d(x,y)} \frac{d}{ds} b(\gamma(s),\theta)\, ds \right| d\mu_n(\theta) \\
&= \int_{\partial U^n} \left| \int_0^{d(x,y)} \langle \mathrm{grad}_x b(\gamma(s),\theta), \dot\gamma(s)\rangle\, ds \right| d\mu_n(\theta) \\
&\leq \int_{\partial U^n} \int_0^{d(x,y)} \|\mathrm{grad}_x b(\gamma(s),\theta)\|_{\gamma(s)}\, ds\, d\mu_n(\theta) \\
&\leq \int_{\partial U^n} d(x,y) \cdot M(C)\, d\mu_n(\theta) = d(x,y) \cdot M(C)
\end{aligned}$$

where $M(C) = \sup\{\|\mathrm{grad}_x b(x',\theta)\|_{x'} : x' \in C, \theta \in \partial U^n\} < \infty$.

Now we argue by contradiction. Assume that the sequence (x_n) does not converge to x. Then there is $\varepsilon > 0$ and a subsequence (x_{n_k}) such that $d(x_{n_k}, x) \geq \varepsilon$ for every $k \in \mathbb{N}$. Since $C \backslash B(x, \varepsilon)$ is compact there is a subsequence of this subsequence, which we shall again denote by (x_{n_k}), such that (x_{n_k}) converges to some $y \in C\backslash B(x,\varepsilon)$. Because of the uniform Lipschitz bound we get

$$\begin{aligned}
|\mathcal{B}_{\mu_{n_k}}(x_{n_k}) - \mathcal{B}_\mu(y)| &\leq |\mathcal{B}_{\mu_{n_k}}(x_{n_k}) - \mathcal{B}_{\mu_{n_k}}(y)| + |\mathcal{B}_{\mu_{n_k}}(y) - \mathcal{B}_\mu(y)| \\
&\leq M(C)\, d(x_{n_k}, y) + |\mathcal{B}_{\mu_{n_k}}(y) - \mathcal{B}_\mu(y)|
\end{aligned}$$

and thus $\lim_{k\to\infty} \mathcal{B}_{\mu_{n_k}}(x_{n_k}) = \mathcal{B}_\mu(y)$. But because all (x_{n_k}) are the respecitve minima of $\mathcal{B}_{\mu_{n_k}}$ we also have for every $\xi \in U^n$

$$\mathcal{B}_{\mu_{n_k}}(x_{n_k}) \leq \mathcal{B}_{\mu_{n_k}}(\xi)$$

Taking the limit $k \to \infty$ on both sides we obtain

$$\mathcal{B}_\mu(y) \leq \mathcal{B}_\mu(\xi)$$

for every $\xi \in U^n$. Thus y must be the unique minimum x of $\mathcal{B}_\mu$ which is a contradiction to $y \in C\backslash B(x,\varepsilon)$. □

E.3. Visualization of $\mathcal{B}_\mu$

In this last section we want to give some plots of $\mathcal{B}_\mu$ for different probability measures $\mu \in \mathcal{M}^1(\partial B^2)$. For the plots we used the Python script in Listing E.3 using Python version 2.7.9, Matplotlib version 1.3.1 and NumPy version 1.8.1.

Let us assume that $\mu \in \mathcal{M}^1(\partial B^2)$ is absolutely continuous with respect to the angle measure λ on $S^1 \cong \partial B^2$. Therefore we can write $d\mu = \hat f\, d\lambda$ for some positive $\hat f \in L^1_{\mathrm{loc}}(S^1)$ by the Radon-Nikodym Theorem. Note that

$$\int_{S^1} \hat f\, d\lambda = 1$$

since μ is supposed to be a probability measure.

In order to compute

$$\mathcal{B}_\mu(x) = \int_{\partial B^2} b_{B^2}(x, \theta)\, d\mu(\theta)$$

for $x \in B^2$, we further identify $S^1 \cong [0, 2\pi)$ as measure spaces via the restricted exponential mapping $\exp : [0, 2\pi) \to S^1, t \mapsto \exp(it)$, whence

$$\mathcal{B}_\mu(x) = \int_0^{2\pi} b_{B^2}(x, \exp(it)) \cdot f(t)\, dt$$

where $f(t) = \hat{f}(\exp(it))$ is some "density function" on $[0, 2\pi)$ (cf. Listing E.3). Thus given such an f we can compute values of $\mathcal{B}_\mu$ using the concrete formula for Busemann functions in Proposition E.1.6.

We will first use the following family of density functions

$$f_a(t) := \begin{cases} \frac{1}{a \cdot 2\pi}, & \text{if } \pi - a \cdot \pi \leq t \leq \pi + a \cdot \pi \\ 0, & \text{else} \end{cases}$$

depending on some parameter $0 < a \leq 1$ (cf. Listing E.3, line 27). These are normalized step functions with just one bump of diameter $a \cdot 2\pi$ around π. Note that we get for $a = 1$ the equidistributional measure on S^1, i.e. $\lambda/2\pi$. Figure E.1, Figure E.2 and Figure E.3 depict the different shapes of $\mathcal{B}_\mu$ and f_a for $a = 1.0$, $a = 0.5$ and $a = 0.1$ respectively. As one would expect the "denser" it gets at $-1 \in S^1 \subset \mathbb{C}$ the more the barycenter tends towards -1.

The next family of density functions we want to consider are stepfunctions with two bumps of respective diameter $a \cdot \pi$ around $\frac{\pi}{2}$ and $\frac{3\pi}{2}$

$$f_a(t) := \begin{cases} \frac{1}{a \cdot 2\pi}, & \text{if } \frac{\pi}{2} - a \cdot \frac{\pi}{2} \leq t \leq \frac{\pi}{2} + a \cdot \frac{\pi}{2} \\ \frac{1}{a \cdot 2\pi}, & \text{if } \frac{3\pi}{2} - a \cdot \frac{\pi}{2} \leq t \leq \frac{3\pi}{2} + a \cdot \frac{\pi}{2} \\ 0, & \text{else} \end{cases}$$

depending on some parameter $0 < a \leq 1$ (cf. Listing E.3, just replace the previous definition of `f` with the commented definition in line 35). Again for $a = 1$ we recover the equidistributional measure on S^1. Figure E.4 and Figure E.5 show the cases for $a = 0.5$ and $a = 0.1$ respectively.

As $a \to 0$ it is easy to see, that the measures induced by f_a converge to $\mu = 1/2\,\delta_i + 1/2\,\delta_{-i}$. However μ admits no unique minimum, i.e. has no barycenter (cf. Figure E.6). Observe that this is Example E.2.4 transferred from the upper half plane model U^2 to the Poincaré ball model B^2. This is no contradiction to the continuity of $\text{bary} : \mathcal{A}_{<1/2} \to \mathbb{H}^n$, because the limit measure $\mu = 1/2\,\delta_i + 1/2\,\delta_{-i}$ is not in $\mathcal{A}_{<1/2}$!

Finally, we depict in Figure E.7 the shape of $\mathcal{B}_\mu$ corresponding to a more exoticly shaped density function

$$f(t) = \frac{1}{6\pi}(\sin(3t) + \sin(2t) + \sin(t) + \cos(3t) + \cos(2t) + \cos(t) + 3).$$

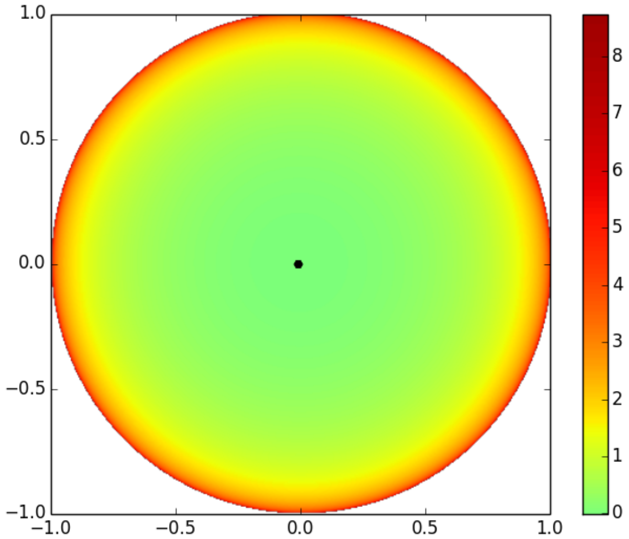
Listing E.1: Code to plot $\mathcal{B}_\mu$ in ball model B^2

```python
import numpy as np
import matplotlib.pyplot as plt
import pickle

from matplotlib.colors import Normalize

# Define a new normalizer for a nicer color output:
```

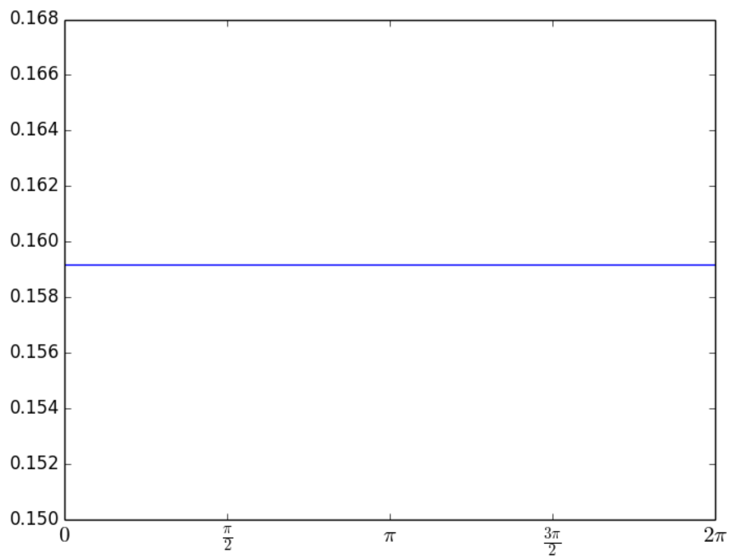
```python

class SigmoidNormalize(Normalize):
    def __init__(self, alpha=1.0, vmin=None, vmax=None, clip=False):
        self.alpha=alpha
        Normalize.__init__(self,vmin,vmax,clip)

    def sigmoid(self,t):
        x=np.exp(-self.alpha*t)
        x=1/(1+x)
        return x

    def __call__(self,value,clip=None):
        return np.ma.masked_array(self.sigmoid(value))


# Define a density function f:

def f(x):
    a=1.0  # scaling factor
    if (np.pi - np.pi*a <= x) and (x <= np.pi + np.pi*a) :
        return (1.0/(a*2*np.pi))
    else:
        return 0

"""
def f(x):
    a=1.0   #scaling factor
    if (np.pi/2 - a*np.pi/2 <= x) and (x < np.pi/2 + a*np.pi/2):
        return 1.0/(a*2*np.pi)
    elif (3*np.pi/2 - a*np.pi/2 <= x) and (x < 3*np.pi/2 + a*np.pi/2):
        return 1.0/(a*2*np.pi)
    else:
        return 0.0
"""

"""
def f(x):
    return (np.sin(3*x) + np.sin(2*x) + np.sin(x) +
        np.cos(3*x) + np.cos(2*x) + np.cos(x) + 3)/(6.0*np.pi)
"""

# Define the Busemann function in the disk model

def buse(x,y,thetax,thetay):
    return np.log(((x-thetax)**2+(y-thetay)**2)/(1-x**2-y**2))

# Integrate the busemann function with
# respect to the given density in order to
# get the function B(.,.):

def B(x,y):
    M=100    # number of steps for the numerical integration
    delta=2*np.pi/M

    X=np.arange(0.0, 2*np.pi, delta)

    # Define a vectorial version of the integran:
    busexy = np.vectorize(lambda t: f(t)*buse(x,y,np.cos(t),np.sin(t)))
    Y=busexy(X)

    # We use the trapez rule for numerical integration:
```

```
    return np.trapz(Y,X)

'''
# Integrate the Busemann function with respect to the measure
# $\mu = 0.5 \delta_{\theta_1} + 0.5 \delta_{\theta_2}$
# where $\theta_1 = (\cos(\pi/2), \sin(\pi/2))$ and
# $\theta_2 = (\cos(3 \pi/2), \sin(3 \pi/2))$

def B(x,y):
    theta1x = np.cos(np.pi/2)
    theta1y = np.sin(np.pi/2)

    theta2x = np.cos(3.0*np.pi/2)
    theta2y = np.sin(3.0*np.pi/2)

    return 0.5*buse(x,y,theta1x,theta1y) + 0.5*buse(x,y,theta2x,theta2y)
'''

##################      Main Script      ##################

# file name for temporary storage of the graph of B(.,.)
PATH_DATA='bary_data'

# indicates whether the graph of B(.,.) needs to be recomputed
RECOMPUTE=True

# number of intermediate steps for each axis, i.e. N^2= "number of pixels"
N=400




if RECOMPUTE:
    # compute the graph of B(.,.):

    delta=2.0/N     # Step size
    grid = np.ma.zeros((N,N))   # raw image

    # Print how many % of the computation is already done:
    print "Progress:␣"
    l=0

    # Compute the value of B(.,.) at each pixel:
    for i in range(0,N):
        for j in range(0,N):
            x=-1+i*delta
            y=-1+j*delta
            # B(.,.) takes by definition only values in the disk
            if x**2+y**2<1:
                grid[j,i]=B(x,y)
            else:
                grid[j,i]=np.ma.masked
                # mask other pixels outside the disk
            l=l+1
            print 100.0*l/(N**2)    # print progress

    # Store image in temp file:
    datafile=open(PATH_DATA,'w')
    p=pickle.Pickler(datafile)
    p.dump(grid)
    datafile.close()
```

```
else:
    # Don't compute but load image from temp file:

    datafile=open(PATH_DATA,'r')
    up = pickle.Unpickler(datafile)
    grid = up.load()
    datafile.close()

# Find minimum aka "the barycenter"
j0,i0 = np.unravel_index(grid.argmin(),grid.shape)

minx = 2*float(i0)/N - 1
miny = 2*float(j0)/N - 1

#Open figure for plot of B(.,.)
fig = plt.figure()
ax = fig.add_subplot(111)

#Plot B(.,.)
im=ax.imshow(grid,
        norm=SigmoidNormalize(alpha=0.4),
        extent=(-1,1,-1,1),
        origin='lower')

#Plot barycenter as black little hexagon
ax.plot(minx,miny,marker='H',color='black')

#Plot colorbar
fig.colorbar(im)

#Open new figure to plot density function f(.)
fig = plt.figure()
ax=fig.add_subplot(111)

T = np.arange(0,2*np.pi,2*np.pi/N)
vf = np.vectorize(f, otypes=[np.float])
Y = vf(T)

#specify ticks on x-axis as fractions of pi
unit = np.pi/2
l=2*np.pi/unit
x_tick = np.arange(0, 5)
x_label = [r"$0$", r"$\frac{\pi}{2}$", r"$\pi$",
r"$\frac{3␣\pi}{2}$",r"$2␣\pi$"]

ax.set_xticks(x_tick*unit)
ax.set_xticklabels(x_label,fontsize=15)


ax.plot(T,Y)

plt.show()
```

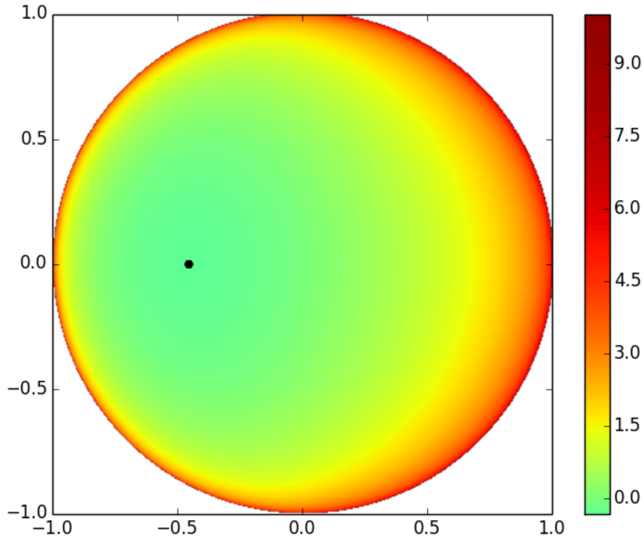


(a) Plot of $\mathcal{B}_\mu$

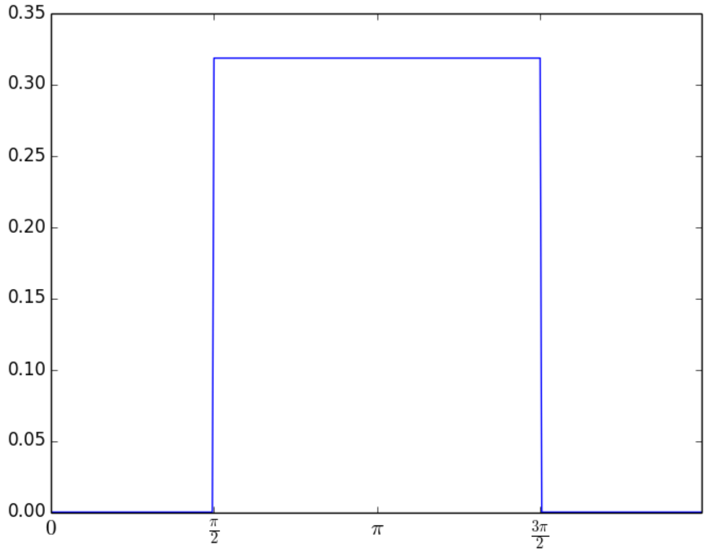


(b) Plot of f_a

Figure E.1.: Case $a = 1$: Equidistribution. The barycenter is depicted as a little black hexagon.

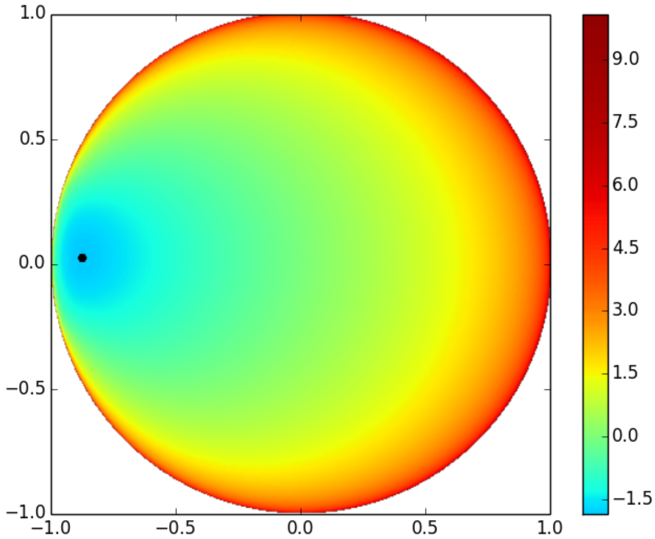


(a) Plot of $\mathcal{B}_\mu$

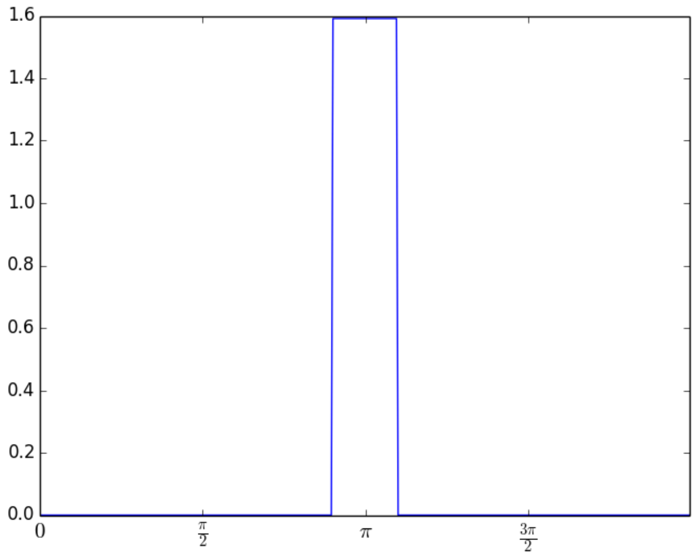


(b) Plot of f_a

Figure E.2.: Case $a = 0.5$: One bump of diameter $a \cdot 2\pi$. The barycenter is depicted as a little black hexagon.

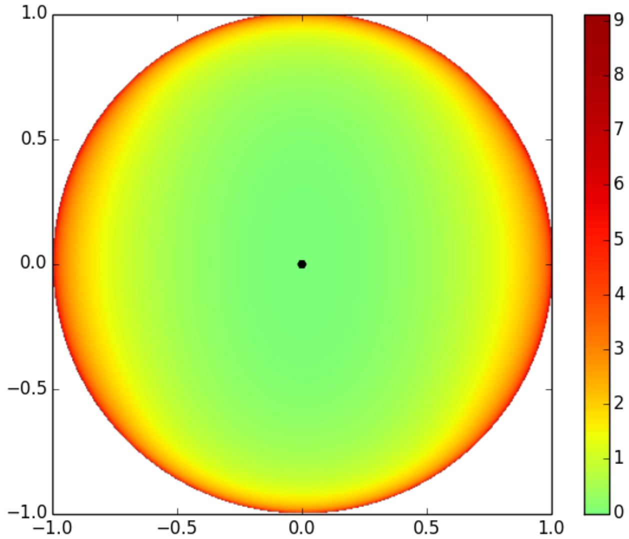


(a) Plot of $\mathcal{B}_\mu$

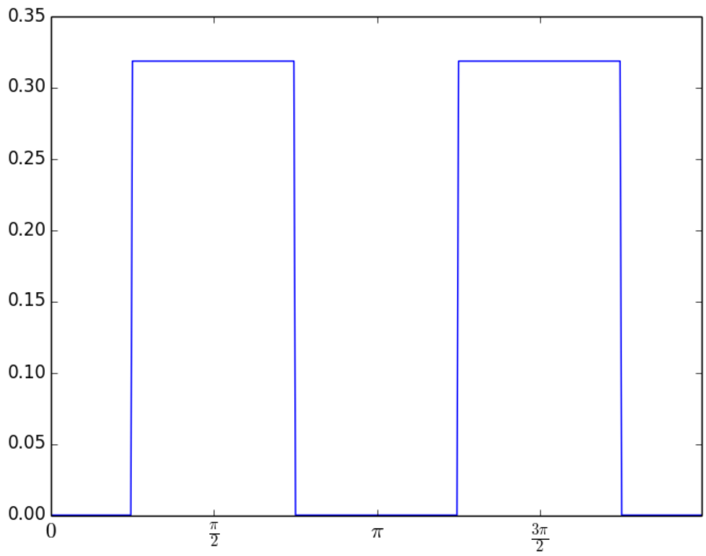


(b) Plot of f_a

Figure E.3.: Case $a = 0.1$: One bump of diameter $a \cdot 2\pi$. The barycenter is depicted as a little black hexagon.

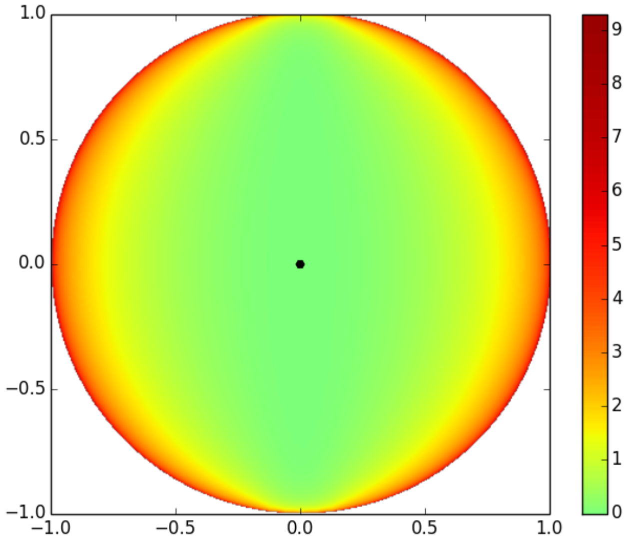


(a) Plot of $\mathcal{B}_\mu$

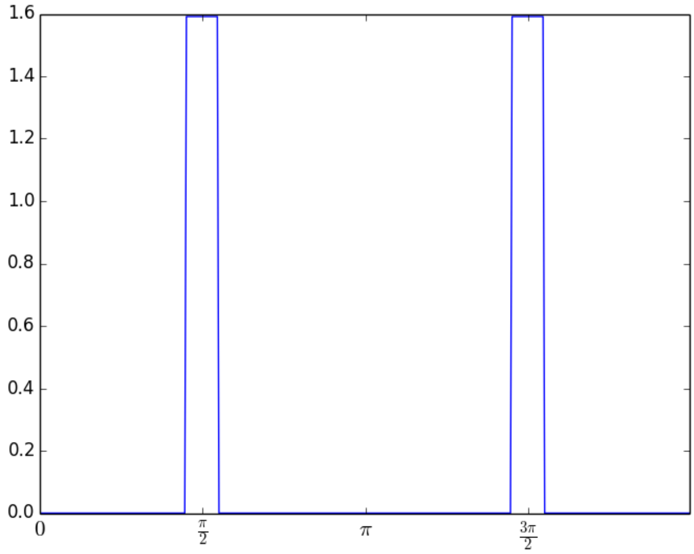


(b) Plot of f_a

Figure E.4.: Case $a = 0.5$: Two bumps of resp. diameter $a \cdot \pi$. The barycenter is depicted as a little black hexagon.



(a) Plot of $\mathcal{B}_\mu$



(b) Plot of f_a

Figure E.5.: Case $a = 0.1$: Two bumps of resp. diameter $a \cdot \pi$. The barycenter is depicted as a little black hexagon.

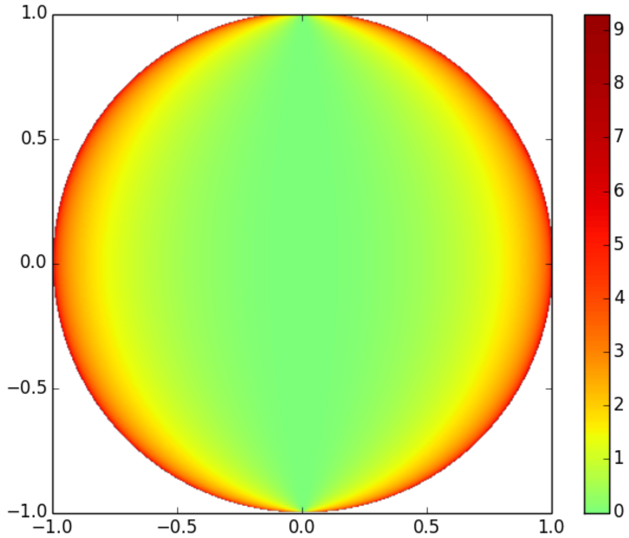
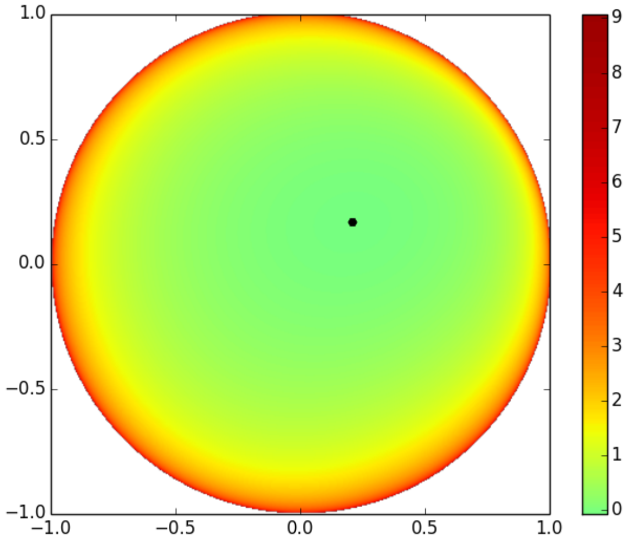
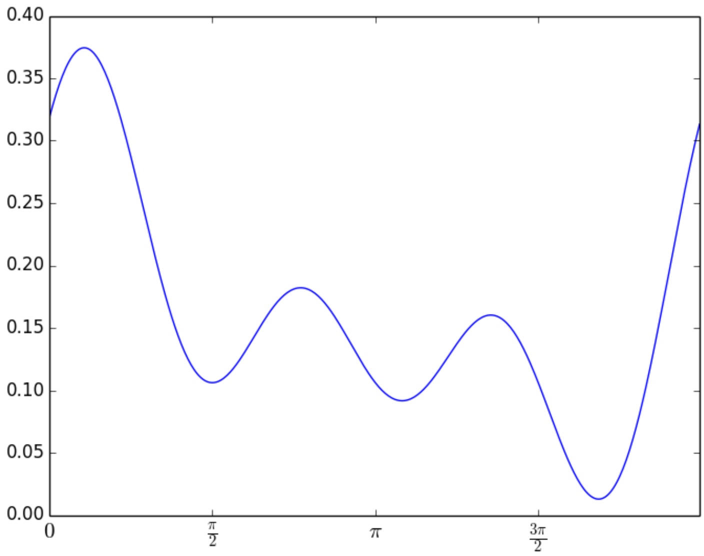


Figure E.6.: Plot of $\mathcal{B}_\mu$ for $\mu = 1/2\,\delta_i + 1/2\,\delta_{-i}$. No Barycenter!



(a) Plot of $\mathcal{B}_\mu$



(b) Plot of f

Figure E.7.: A more involved density function $f(t) = \frac{1}{6\pi}\,(\sin(3t)+\sin(2t)+\sin(t)+\cos(3t)+\cos(2t)+\cos(t)+3)$.

Bibliography

[AE01] H. Amann and J. Escher. *Analysis III.* Birkhäuser Verlag, Basel - Boston - Berlin, first edition, 2001.

[BBI13] M. Bucher, M. Burger, and A. Iozzi. A dual interpretation of the Gromov-Thurston proof of Mostow rigidity and volume rigidity for representations of hyperbolic lattices. In *Trends in harmonic analysis*, volume 3 of *Springer INdAM Ser.*, pages 47–76. Springer, Milan, 2013.

[BCG95] G. Besson, G. Courtois, and S. Gallot. Entropies et rigidités des espaces localement symétriques de courbure strictement négative. *Geom. Funct. Anal.*, 5(5):731–799, 1995.

[BCG96] G. Besson, G. Courtois, and S. Gallot. Minimal entropy and Mostow's rigidity theorems. *Ergodic Theory Dynam. Systems*, 16(4):623–649, 1996.

[BCG99] G. Besson, G. Courtois, and S. Gallot. Lemme de Schwarz réel et applications géométriques. *Acta Math.*, 183(2):145–169, 1999.

[BI02] M. Burger and A. Iozzi. Boundary maps in bounded cohomology. Appendix to: "Continuous bounded cohomology and applications to rigidity theory" [Geom. Funct. Anal. **12** (2002), no. 2, 219–280; MR1911660 (2003d:53065a)] by Burger and N. Monod. *Geom. Funct. Anal.*, 12(2):281–292, 2002.

[BI07] Marc Burger and Alessandra Iozzi. Bounded differential forms, generalized Milnor-Wood inequality and an application to deformation rigidity. *Geom. Dedicata*, 125:1–23, 2007.

[BI09] M. Burger and A. Iozzi. A useful formula from bounded cohomology. In *Géométries à courbure négative ou nulle, groupes discrets et rigidités*, volume 18 of *Sémin. Congr.*, pages 243–292. Soc. Math. France, Paris, 2009.

[BIW03] M. Burger, A. Iozzi, and A. Wienhard. Surface group representations with maximal Toledo invariant. *C. R. Math. Acad. Sci. Paris*, 336(5):387–390, 2003.

[BIW10] M. Burger, A. Iozzi, and A. Wienhard. Surface group representations with maximal Toledo invariant. *Ann. of Math. (2)*, 172(1):517–566, 2010.

[BM00] M.B. Bekka and M. Mayer. *Ergodic theory and topological dynamics of group actions on homogeneous spaces*, volume 269 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 2000.

[BM02] M. Burger and N. Monod. Continuous bounded cohomology and applications to rigidity theory. *Geom. Funct. Anal.*, 12(2):219–280, 2002.

[Bou89] N. Bourbaki. *General Topology Chapters 1-4.* Springer, second edition, 1989.

[Bou04a] N. Bourbaki. *Integration Chapters 1-6.* Springer, 2004.

[Bou04b] N. Bourbaki. *Integration Chapters 7-9.* Springer, 2004.

[Bow93] B.H. Bowditch. Geometrical finiteness for hyperbolic groups. *Journal of Functional Analysis*, (113):245–317, 1993.

[BP92] R. Benedetti and C. Petronio. *Lectures on hyperbolic geometry.* Universitext. Springer-Verlag, Berlin, 1992.

[Bre93] G.E. Bredon. *Topology and Geometry.* Graduate Texts in Mathematics. Springer-Verlag, 1993.

[dC92] M.P. do Carmo. *Riemannian Geometry.* Mathematics: Theory & Applications. Birkhäuser, second edition, 1992.

[DE86] A. Douady and C.J. Earle. Conformally natural extension of homeomorphisms of the circle. *Acta Math.*, 157(1-2):23–48, 1986.

[Dun99] N.M. Dunfield. *Cyclic surgery, degrees of maps of character curves, and volume rigidity for hyperbolic manifolds.* ProQuest LLC, Ann Arbor, MI, 1999. Thesis (Ph.D.)–The University of Chicago.

[Els11] J. Elstrodt. *Maß- und Integrationstheorie.* Springer, 7., korrigierte und aktualisierte auflage edition, 2011.

[FK06] S. Francaviglia and B. Klaff. Maximal volume representations are Fuchsian. *Geom. Dedicata*, 117:111–124, 2006.

[Gro82] M. Gromov. Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.*, (56):5–99 (1983), 1982.

[Gui80] A. Guichardet. *Cohomologie des groupes topologiques et des algèbres de Lie*, volume 2 of *Textes Mathématiques [Mathematical Texts].* CEDIC, Paris, 1980.

[Hat02] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[Iva87] N.V. Ivanov. Foundations of the theory of bounded cohomology. *J. of Soviet Mathematics*, 37:1090–1115, 1987.

[Kap09] M. Kapovich. *Hyperbolic manifolds and discrete groups.* Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2009. Reprint of the 2001 edition.

[Lee13] J.M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics.* Springer, New York, second edition, 2013.

[Lü10] A. Lücker. Approaches to mostow rigidity in hyperbolic space. `http://wiki.epfl.ch/grtr/documents/lucker2010.pdf`, 2010. Accessed: 23/11/2015.

[Mon01] N. Monod. *Continuous bounded cohomology of locally compact groups*, volume 1758 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2001.

[Mun66] J.R. Munkres. *Elementary differential topology*, volume 1961 of *Lectures given at Massachusetts Institute of Technology, Fall.* Princeton University Press, Princeton, N.J., 1966.

[Pat88] A.L.T. Paterson. *Amenability*, volume 29 of *Mathematical Surveys and Monographs.* American Mathematical Society, 1988.

[Rat06] J.G. Ratcliffe. *Foundations of hyperbolic manifolds*, volume 149 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2006.

[RF10] H.L. Royden and P.M. Fitzpatrick. *Real Analysis*. Pearson, fourth edition, 2010.

[RS00] H. Reiter and J.D. Stegeman. *Classical Harmonic Analysis and Locally Compact Groups*. Oxford University Press, second edition, 2000.

[Rud09] W. Rudin. *Reelle und Komplexe Analysis*. Oldenbourg Verlag München, second edition, 2009.

[Thu] W.P. Thurston. The geometry and topology of three-manifolds. `http://library.msri.org/books/gt3m/`. Accessed: 07/04/2015.

[Thu97] W.P. Thurston. *Three-Dimensional Geometry and Topology*, volume 1 of *Princeton Mathematical Series*. Princeton University Press, 1997.

[Tor] S. Tornier. Haar measures. `http://www.math.ethz.ch/~torniers/download/2014/haar_measures.pdf`. Accessed: 21/09/2015.

[Wie04] A. Wienhard. Bounded cohomology and geometry. http://arxiv.org/abs/math/0501258, 2004. Accessed: 22/11/2015.

[Zim84] R.J. Zimmer. *Ergodic theory and semisimple groups*, volume 81 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.

[Zim90] R.J. Zimmer. *Essential Results of Functional Analysis*. The University of Chicago Press, 1990.